CHUNG-TYPE LAW OF THE ITERATED LOGARITHM AND EXACT MODULI OF CONTINUITY FOR A CLASS OF ANISOTROPIC GAUSSIAN RANDOM FIELDS

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Abstract. We establish a Chung-type law of the iterated logarithm and the exact local and uniform moduli of continuity for a large class of anisotropic Gaussian random fields with a harmonizable-type integral representation and the property of strong local nondeterminism. Compared with the existing results in the literature, our results do not require the assumption of stationary increments and provide more precise upper and lower bounds for the limiting constants. The results are applicable to the solutions of a class of linear stochastic partial differential equations driven by a fractional-colored Gaussian noise, including the stochastic heat equation.

1. Introduction

The purpose of this paper is to establish a general framework that is useful for studying the regularity properties of sample functions of anisotropic Gaussian random fields and can be directly applied to the solutions of linear SPDEs. This is mainly motivated by [6] and [27]. We consider a class of Gaussian random fields $\{v(x), x \in \mathbb{R}^k\}$ that satisfy Assumption 2.1 in [6] (see Assumption 2.1 below) and the property of strong local nondeterminism, or strong LND for short, with respect to an anisotropic metric (see Assumption 2.2 below). For these Gaussian random fields, we prove some limit theorems that provide precise information about the oscillation behavior of the sample function $x \mapsto v(x)$.

The main results of this paper are as follows. We prove a Chung-type law of the iterated logarithm (LIL) in Theorem 4.4, the exact local and uniform moduli of continuity in Theorems 5.2 and 6.1, respectively. Our strategy is to first prove a zero–one law for each of the limit theorems (see Lemma 3.1), showing that the limit is equal to a constant in $[0, \infty]$ almost surely. Then, we prove that the constant is in fact positive and finite by establishing a finite upper bound and positive lower bound for the limit, and therefore, the corresponding modulus function in the limit theorem is sharp. We give an application of the main results to the solutions of a class of linear SPDEs

$$\frac{\partial}{\partial t} u(t,x) = \mathscr{L} u(t,x) + \dot{W}(t,x)$$

driven by a fractional-colored Gaussian noise, including the stochastic heat equation [3, 7]. It is also a notable result of this paper that $u(t,x)$ satisfies the strong LND property (see Lemma 7.3), which strengthens a result of [7].

In general, there are different ways to describe the sample path variation of random fields. The Chung-type LIL characterizes the lower envelope (lim inf) for the local oscillations of the sample functions at a fixed point. The local modulus of continuity at a fixed point is, for many Gaussian random fields, given by the ordinary Khinchin-type LIL, which complements the Chung-type LIL by characterizing the upper envelope (lim sup) for the local oscillations at a fixed point.

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On the other hand, the uniform modulus of continuity specifies the maximum oscillation of the sample functions over certain sets such as a compact interval.

The Chung-type LIL for a class of isotropic and anisotropic Gaussian random fields with stationary increments has been studied by Li and Shao [14] (see also [18, 25]) and Luan and Xiao [15]. The exact local and uniform moduli of continuity for a class of anisotropic Gaussian random fields have been studied by Meerschaert et al. [19]. The novelty of the present paper is a general framework based on a harmonizable-type representation and the strong LND property of a Gaussian random field that may not have stationary increments, which is employed to extend and improve some of the results of [14, 15, 19], and can be directly applied to the solutions of SPDEs. In particular, with a harmonizable-type representation, we are able to decompose the random field and create independence, making it possible to establish general zero–one laws (Lemma 3.1) which can be strengthened to prove the Chung-type LIL as well as the exact local and uniform moduli of continuity. The independence structure from the harmonizable-type representation also allows the use of the second Borel–Cantelli lemma, which facilitates a simple proof of one of the bounds for the Chung-type LIL and the exact local modulus of continuity.

Let us summarize the major differences and improvements in our results compared to the existing results in the literature. The Chung-type LIL results in [14], [18], [25] and [15] were proved for Gaussian random fields with stationary increments, meaning that for any $h \in \mathbb{R}^k$,

$$\{v(x+h) - v(h), x \in \mathbb{R}^k\} \stackrel{d}{=} \{v(x) - v(0), x \in \mathbb{R}^k\},$$

and, in particular, it is enough for them to consider the Chung-type LIL at the origin. Our Theorem 4.4 applies to a wider class of Gaussian random fields that may not necessarily have stationary increments, and we prove a Chung-type LIL at any fixed point x_0. Moreover, our Theorem 4.4 gives explicit upper and lower bounds for the constant in Chung's LIL in terms of the constants that appear in the small ball probability estimates. This implies that the limiting constant in Chung's LIL is given in terms of the small ball constant provided it exists, see (4.8) below. We remark that the connection between the bounds on the limiting constant in Chung's LIL and the small ball estimates is also given in Theorem 7.1 of [14], but not explicitly stated in Theorem 1.1 of [15].

For exact local and uniform moduli of continuity of Gaussian processes with stationary increments, some general theory has been established by Marcus and Rosen [17]. Also, Meerschaert et al. [19] have used the sectorial LND property and the Fernique-type inequalities to prove the exact uniform modulus of continuity of anisotropic Gaussian random fields. Especially, [19] provides an effective way to prove the lower bound for the uniform modulus of continuity, which is usually a more difficult task than proving the upper bound. Our Theorems 5.2 and 6.1 improve the results in [19]. We prove exact local and uniform moduli of continuity under two metrics respectively: one is the canonical metric d defined in (2.4), and the other one is the metric Δ defined in (2.3), which is comparable to d under Assumptions 2.1 and 2.3. For the local modulus of continuity in Theorem 5.2, under the canonical metric d, we are able to prove that the exact constant in the LIL is $\sqrt{2}$. This is an improvement to Theorem 5.6 of [19], which only shows that the constant is at least $\sqrt{2}$ (see Remark 5.3 below). We achieve this sharper result by using a tail probability estimate due to Talagrand [22], which is stated in Lemma 5.1 below.

For the uniform modulus of continuity, the strong LND assumption in our Theorem 6.1 is stronger than the condition in Theorem 4.1 of [19], but we obtain better upper and lower bounds for the limiting constant (see Remark 6.2 below). Our approach is to start with a crude upper bound and then optimize it using an approximation argument based on anisotropic lattice points. We also refine the the proof in [19] based on the strong LND property and a conditioning argument to get a sharper lower bound.

The rest of the paper is organized as follows. In Section 2, we state the assumptions for the Gaussian random fields to be considered in this paper and give some remarks about the assumptions. We also briefly discuss the difference in the LND properties between stochastic heat and wave equations. In Section 3, we prove zero–one laws which will be useful for establishing the Chung-type LIL and the exact local and uniform moduli of continuity. In Section 4, we establish small ball probability estimates and Chung's LIL. In Sections 5 and 6, we prove the exact local and uniform moduli of continuity, respectively. In Section 7, we consider as an application a class of linear SPDEs driven by a fractional-colored Gaussian noise [3, 7]. We establish harmonizable-type representations and strong LND property for the solutions, and apply our results to obtain Chung's LIL and exact local and uniform moduli of continuity. These results improve significantly those in [7, 24]. Finally, in Section 8, we provide another example of anisotropic Gaussian random fields that do not have stationary increments and satisfy Assumptions 2.1 and 2.2 of the present paper.

2. Assumptions

Consider a real-valued continuous centered Gaussian random field $v = \{v(x), x \in \mathbb{R}^k\}$. Let T be a compact rectangle in $\mathbb{R}^k$. We introduce some assumptions for v. Notice that Assumption 2.1 is from [6] and Assumption 2.2 is from [27].

Assumption 2.1. There exists a centered Gaussian random field $\{v(A, x), A \in \mathscr{B}(\mathbb{R}_+), x \in T\}$, where $\mathscr{B}(\mathbb{R}_+)$ is the Borel σ-algebra on $\mathbb{R}_+ := [0, \infty)$, such that the following properties hold:

(a) For every $x \in T$, $A \mapsto v(A, x)$ is an independently scattered Gaussian noise such that $v(\mathbb{R}_+, x) = v(x)$ and the processes $v(A, \cdot)$ and $v(B, \cdot)$ are independent whenever A and B are disjoint.

(b) There exist constants $c_0 > 0$, $a_0 \geq 0$, and $\gamma_j > 0$, $j = 1, \dots, k$, such that for all $a_0 \leq a < b \leq \infty$ and $x, y \in T$,

$$\big\|v([a,b),x) - v(x) - v([a,b),y) + v(y)\big\|_{L^2} \leq c_0\bigg(\sum_{j=1}^k a^{\gamma_j}|x_j - y_j| + b^{-1}\bigg) \tag{2.1}$$

and

$$\big\|v([0,a_0),x) - v([0,a_0),y)\big\|_{L^2} \leq c_0\sum_{j=1}^k |x_j - y_j|. \tag{2.2}$$

In the above, $\|X\|_{L^2} := [\mathbb{E}(X^2)]^{1/2}$ for a random variable X.

Define α_j $(j = 1, \dots, k)$ by the relation $\gamma_j = \alpha_j^{-1} - 1$, that is, $\alpha_j = (\gamma_j + 1)^{-1}$. Note that $0 < \alpha_j < 1$. The parameters α_j characterize the Hölder regularity of v (see Lemma 2.4 below). Let $Q = \sum_{j=1}^k \alpha_j^{-1}$ and define the metric Δ by

$$\Delta(x,y) := \sum_{j=1}^k |x_j - y_j|^{\alpha_j}, \quad x, y \in \mathbb{R}^k. \tag{2.3}$$

We will also use the canonical metric $d = d_v$ associated with v. It is defined by

$$d(x,y) = d_v(x,y) := \big\|v(x) - v(y)\big\|_{L^2}, \quad x, y \in \mathbb{R}^k. \tag{2.4}$$

Assumption 2.2. There exists a constant $c_2 > 0$ such that for all integers $n \geq 1$, for all $x, x^1, \dots, x^n \in T$,

$$\mathrm{Var}\big(v(x)|v(x^1), \dots, v(x^n)\big) \geq c_2 \min_{0 \leq i \leq n} \Delta^2(x, x^i),$$

where $x^0 = 0$.

Assumption 2.3. There exists a constant $c_3 > 0$ such that for all $x, y \in T$,

$$\big\|v(x) - v(y)\big\|_{L^2} \geq c_3 \Delta(x, y).$$

The following are some remarks about these assumptions. Assumption 2.1 above is the same as Assumption 2.1 in [6] and [5], and is satisfied by many Gaussian random fields that have a spectral or harmonizable-type representation, or more generally, a stochastic integral representation. For example, it is shown in [6] that the solutions of linear stochastic heat and wave equations admit harmonizable-type representations and satisfy Assumption 2.1. The same is true for fractional Brownian sheets [5]. Assumption 2.1 implies an upper bound for the increments of v in L^2-norm in terms of the metric Δ:

Lemma 2.4. *Under Assumption 2.1, there exist constants $\varepsilon_1 > 0$ and c_1 such that for all $x, y \in T$ with $\Delta(x, y) \leq \varepsilon_1$,*

$$\|v(x) - v(y)\|_{L^2} \leq c_1 \Delta(x, y). \tag{2.5}$$

Proof. This is a consequence of Proposition 2.2 of [6] where $\varepsilon_1 = \min\{a_0^{-1}, 1\}$ and $c_1 = 4c_0$. □

Assumption 2.2 is known as the property of strong local nondeterminism (strong LND) with respect to the metric Δ, which has found various applications in studying probabilistic, analytic and fractal properties of Gaussian random fields (cf. [26, 27]). For Gaussian random fields with stationary increments, [16] provides sufficient conditions in terms of their spectral measures for them to have the property of strong LND. In Sections 7 and 8, we will show that the solutions of a class of linear SPDEs driven by a fractional-colored Gaussian noise and a class of Gaussian random fields with non-stationary increments also have the property of strong LND.

Assumption 2.2 implies the lower bound in Assumption 2.3 if $T \subset \mathbb{R}^k \backslash \{0\}$ is compact: for all $x, y \in T \subset \mathbb{R}^k \backslash \{0\}$,

$$\|v(x) - v(y)\|_{L^2} \geq \sqrt{c_2}\, \Delta(x, y). \tag{2.6}$$

The strong LND property (Assumption 2.2) will be used to prove optimal bounds for the small ball probability, which is the main ingredient of the proof of Chung's LIL (Theorem 4.4). This property will also be needed in the proof of the lower bound for the exact uniform modulus of continuity (Theorem 6.1). Assumption 2.3 is much weaker than Assumption 2.2. To establish the exact local modulus of continuity, or the ordinary LIL (Theorem 5.2), we will use Assumption 2.3, but not Assumption 2.2.

In Lemma 7.3 of this paper, we will prove that the solutions of a class of linear SPDEs with a fractional-colored Gaussian noise, including the stochastic heat equation, satisfy the strong LND property. We remark that, on the other hand, the solution of the linear stochastic wave equation does not satisfy the strong LND property, but satisfies a different form of LND [13, 12]. For this reason, the Chung-type LIL for the stochastic wave equation has a different form than the stochastic heat equation; see [11]. The uniform modulus of continuity for the stochastic wave equation is also established in [13, 12]. For the local modulus of continuity, our Theorem 5.2 still applies to the stochastic wave equation because it does not require the strong LND property. It also applies to fractional Brownian sheets.

3. Zero–one laws

In Lemma 3.1 below, we establish zero–one laws for the Chung-type LIL and the local and uniform moduli of continuity, showing that the limit in each of these laws is equal to a constant almost surely. At this stage, we do not rule out the possibility that the constant could be zero or infinity. Later in our main theorems, we will strengthen these zero–one laws and prove that the limiting constants are indeed positive and finite.

Lemma 3.1. *The following statements hold under Assumption 2.1.*

(i) *For any fixed $x_0 \in T$, there exists a constant $0 \le \kappa_1 \le \infty$ which may depend on x_0 such that*

$$\liminf_{r\to 0+} \sup_{x\in T:\, \Delta(x,x_0)\le r} \frac{|v(x)-v(x_0)|}{r(\log\log(1/r))^{-1/Q}} = \kappa_1 \quad a.s. \tag{3.1}$$

(ii) *For any fixed $x_0 \in T$, there exists a constant $0 \le \kappa_2 \le \infty$ which may depend on x_0 such that*

$$\lim_{r\to 0+} \sup_{x\in T:\, 0<\Delta(x,x_0)\le r} \frac{|v(x)-v(x_0)|}{\Delta(x,x_0)\sqrt{\log\log(\Delta(x,x_0)^{-1})}} = \kappa_2 \quad a.s. \tag{3.2}$$

(iii) *There exists a constant $0 \le \kappa_3 \le \infty$ such that*

$$\lim_{r\to 0+} \sup_{x,y\in T:\, 0<\Delta(x,y)\le r} \frac{|v(x)-v(y)|}{\Delta(x,y)\sqrt{\log(\Delta(x,y)^{-1})}} = \kappa_3 \quad a.s. \tag{3.3}$$

Moreover, under Assumptions 2.1 and 2.3, (3.2) *and* (3.3) *also hold when Δ is replaced by the canonical metric d, with possibly different constants.*

Proof. By Assumption 2.1, $v(x)$ can be represented as the infinite sum

$$v(x) = \sum_{n=0}^{\infty} v_n(x), \tag{3.4}$$

where $v_n(x) = v([n, n+1), x)$ and $v_n = \{v_n(x), x \in T\}$ $(n = 0, 1, \dots)$ is a sequence of independent Gaussian random fields. Let $\mathscr{F}_n$ be the σ-algebra generated by the processes $\{v_m, m \ge n\}$ and the null events, and let $\mathscr{F}_\infty = \bigcap_{n=0}^\infty \mathscr{F}_n$ be the σ-algebra of all tail events. By Kolmogorov's zero–one law, $\mathbb{P}(A) = 0$ or 1 for $A \in \mathscr{F}_\infty$.

To prove (i), we will show that for any fixed $x_0 \in T$, the random variable

$$X := \liminf_{r\to 0+} \sup_{x\in T:\, \Delta(x,x_0)\le r} \frac{|v(x)-v(x_0)|}{r(\log\log(1/r))^{-1/Q}}$$

is measurable with respect to the σ-algebra $\mathscr{F}_\infty$. For any $n \ge 1$ and $x \in T$, let

$$Y_n(x) = \sum_{m=0}^{n-1} v_m(x) \quad \text{and} \quad Z_n(x) = \sum_{m=n}^{\infty} v_m(x).$$

Note that $v(x) = Y_n(x) + Z_n(x)$ and $Y_n(x) = v([0, n), x)$. Consider $n \ge a_0$, where a_0 is the constant in Assumption 2.1. Then by (2.1) with $a = n$ and $b = \infty$, for all $x, y \in T$, we have

$$\|Y_n(x) - Y_n(y)\|_{L^2} \le c_0 \sum_{j=1}^{k} n^{\gamma_j} |x_j - y_j|.$$

Since Y_n is Gaussian, this implies that for any $p \ge 2$, there is a finite constant C which depends on n such that for all $x, y \in T$,

$$\mathbb{E}(|Y_n(x) - Y_n(y)|^p) \le C|x-y|^p.$$

Then, by Kolmogorov's continuity theorem, for any $0 < \beta < 1$, with probability one, $x \mapsto Y_n(x)$ is β-Hölder continuous on T. If we choose β such that $\max\{\alpha_1, \dots, \alpha_k\} < \beta < 1$, then for a.e. ω, there exists $C = C(\omega, n) < \infty$ such that for all $x, y \in T$,

$$|Y_n(x) - Y_n(y)| \le C \sum_{j=1}^{k} |x_j - y_j|^\beta \tag{3.5}$$

This implies that for any $x_0 \in T$,

$$\lim_{r\to 0+} \sup_{x\in T:\, \Delta(x,x_0)\le r} \frac{|Y_n(x)-Y_n(x_0)|}{r(\log\log(1/r))^{-1/Q}} = 0 \quad \text{a.s.}$$

Since $v = Y_n + Z_n$, we have

$$X = \liminf_{r\to 0+} \sup_{x\in T:\, \Delta(x,x_0)\le r} \frac{|Z_n(x)-Z_n(x_0)|}{r(\log\log(1/r))^{-1/Q}} \quad \text{a.s.}$$

This means that X is an $\mathscr{F}_n$-measurable random variable, and this is true for arbitrary $n \ge a_0$. Therefore, X is $\mathscr{F}_\infty$-measurable. By Kolmogorov's zero–one law, this implies (i).

For (ii) and (iii), notice that the limits on the left-hand side of (3.2) and (3.3) both exist by monotonicity. Moreover, similarly to the above arguments, for any $n \ge a_0$, by (3.5), we have

$$\lim_{r\to 0+} \sup_{x\in T:\, 0<\Delta(x,x_0)\le r} \frac{|Y_n(x)-Y_n(x_0)|}{\Delta(x,x_0)\sqrt{\log\log(\Delta(x,x_0)^{-1})}} = 0 \quad \text{a.s.} \tag{3.6}$$

and

$$\lim_{r\to 0+} \sup_{x,y\in T:\, 0<\Delta(x,y)\le r} \frac{|Y_n(x)-Y_n(y)|}{\Delta(x,y)\sqrt{\log(\Delta(x,y)^{-1})}} = 0 \quad \text{a.s.} \tag{3.7}$$

It follows that the left-hand side of (3.2) and (3.3) are $\mathscr{F}_\infty$-measurable random variables and therefore are constants a.s. by Kolmogorov's zero–one law.

Finally, to see that (3.2) and (3.3) also hold when Δ is replaced by d, note that for each $n \ge a_0$, by (3.5) and Assumption 2.3, for a.e. ω, there exists $C = C(\omega, n) < \infty$ such that for all $x, y \in T$ with $d(x,y) \le r$, we have

$$|Y_n(x) - Y_n(y)| \le C r^{\beta-\alpha^*} d(x,y),$$

where $\alpha^* = \max\{\alpha_1, \dots, \alpha_k\}$. Therefore, (3.6) and (3.7) hold with Δ being replaced by d, and the desired result follows from the fact that $v = Y_n + Z_n$ and Kolmogorov's zero–one law. □

4. Chung-type law of the iterated logarithm

This section is devoted to proving the Chung-type LIL. It is well known that the small ball probability is a key step in establishing Chung's LIL ([18, 14]). The following lemma is a reformulation of Talagrand's lower bound for small ball probabilities of Gaussian processes [23, Lemma 2.2]. See Ledoux [9, p.257] for a proof.

Lemma 4.1. *Let $\{X(t), t \in S\}$ be a separable, real-valued, mean-zero Gaussian process indexed by a bounded set S with canonical metric $d_X(s,t) = \|X(s)-X(t)\|_{L^2}$. Let $N(S, d_X, \varepsilon)$ denote the smallest number of d_X-balls of radius ε needed to cover the set S. Suppose there is a decreasing function $\psi : (0,\delta) \to (0,\infty)$ such that $N(S, d_X, \varepsilon) \le \psi(\varepsilon)$ for all $\varepsilon \in (0,\delta)$ and there are constants $a_2 \ge a_1 > 1$ such that for all $\varepsilon \in (0,\delta)$,*

$$a_1\psi(\varepsilon) \le \psi(\varepsilon/2) \le a_2\psi(\varepsilon). \tag{4.1}$$

Then, there is a finite constant K depending only on a_1 and a_2 such that for all $u \in (0,\delta)$,

$$\mathbb{P}\left\{ \sup_{s,t\in S} |X(s)-X(t)| \le u \right\} \ge \exp\left(-K\psi(u)\right). \tag{4.2}$$

Recall the metric Δ defined in (2.3). Let $B_\Delta(x,r) = \{y \in \mathbb{R}^k : \Delta(x,y) \le r\}$ be the closed Δ-ball centered at x of radius r. In the proposition below, we prove optimal bounds for the small ball probability of v around a fixed point x_0, which generalizes Theorem 5.1 in [27] (or Lemma 2.2 of [15]), where the case of $x_0 = 0$ and $r = 1$ was considered.

Proposition 4.2. *Under Assumptions 2.1 and 2.2, there exist positive finite constants* C_1, C_2 *and* $r_0 > 0$ *small such that for any* $0 < u < r \le r_0$ *and* $x_0 \in T$ *with* $B_\Delta(x_0, r) \subset T$*, we have*

$$\exp\left(-C_1(r/u)^Q\right) \le \mathbb{P}\left\{\sup_{x \in B_\Delta(x_0,r)} |v(x) - v(x_0)| \le u\right\} \le \exp\left(-C_2(r/u)^Q\right). \tag{4.3}$$

Proof. We first prove the lower bound in (4.3). Consider the Gaussian random field $\{v(x), x \in T\}$ and the canonical metric $d_v(x,y) = \|v(x) - v(y)\|_{L^2}$. By Assumption 2.1 and Lemma 2.4, we can find some small $r_0 > 0$ such that $d_v(x,y) \le c_1 \Delta(x,y)$ for all $x, y \in T$ with $\Delta(x,y) \le r$ and $0 < r \le r_0$. This implies $N(B_\Delta(x_0,r), d_v, \varepsilon) \le C_0(r/\varepsilon)^Q$ for all $\varepsilon > 0$ small, where C_0 does not depend on r or ε. Take $S = B_\Delta(x_0, r)$ and $\psi(\varepsilon) = C_0(r/\varepsilon)^Q$. Then ψ satisfies (4.1) with $a_1 = a_2 = 2^Q > 1$. Hence, the lower bound in (4.3) follows from Lemma 4.1.

The proof of the upper bound in (4.3) is based on Assumption 2.2 and a conditioning argument. Suppose $0 < u < r \le r_0$ and $B_\Delta(x_0, r) \subset T$. Notice that, for $x_0 = (x_{0,1}, \dots, x_{0,k})$, the rectangle $I := \prod_{j=1}^k [x_{0,j},\, x_{0,j} + (k^{-1}r)^{1/\alpha_j}]$ is contained in $B_\Delta(x_0, r)$. For simplicity, we consider the case where x_0 lies in the orthant $[0,\infty)^k$, so that the interior of I does not contain the origin (otherwise, in order to retain this latter property for I, we can modify the definition of I by using the interval $[x_{0,j} - (k^{-1}r)^{1/\alpha_j},\, x_{0,j}]$ for $x_{0,j} < 0$ and the rest of the proof is similar). It suffices to prove that

$$\mathbb{P}\left\{\sup_{x \in I} |v(x) - v(x_0)| \le u\right\} \le \exp\left(-C_2(r/u)^Q\right). \tag{4.4}$$

Since $r/u > 1$, we can find an integer $n \ge 2$ such that $n - 1 < r/u \le n$ (in particular, $n/2 < r/u$). Divide I into sub-rectangles of side lengths $\big(r/(kn)\big)^{1/\alpha_j}$ $(j = 1, \dots, k)$. The number of sub-rectangles is $N \sim n^Q$. Let x_i $(1 \le i \le N)$ denote the upper-right vertices of the sub-rectangles in any order. For each $1 \le j \le N$, let

$$A_j = \left\{\max_{1 \le i \le j} |v(x_i) - v(x_0)| \le u\right\}.$$

Then by conditioning,

$$\mathbb{P}(A_j) = \mathbb{E}\left[\mathbf{1}_{A_{j-1}}\, \mathbb{P}\left\{|v(x_j) - v(x_0)| \le u \,\middle|\, v(x_i) : 0 \le i \le j-1\right\}\right]. \tag{4.5}$$

By Assumption 2.2, the property that the x_i's are separated by a Δ-distance of at least $r/(kn)$ and that the interior of I does not contain the origin, we have

$$\operatorname{Var}\big(v(x_j) \big| v(x_i) : 0 \le i \le j-1\big) \ge c_2 \min_{0 \le i \le j-1} \Delta^2(x_j, x_i) \ge c_2 (r/(kn))^2. \tag{4.6}$$

Since the random field v is Gaussian, the conditional distribution of $v(x_j)$ given all the $v(x_i)$, with $0 \le i \le j-1$, is a Gaussian distribution with conditional variance $\operatorname{Var}(v(x_j)|v(x_i) : 0 \le i \le j-1)$. Then, (4.6) and Anderson's inequality [2, Theorem 2] imply that

$$\mathbb{P}\left\{|v(x_j) - v(x_0)| \le u \,\middle|\, v(x_i) : 0 \le i \le j-1\right\} \le \mathbb{P}\left\{|Z| \le \frac{u}{\sqrt{c_2}(r/(kn))}\right\} \le \exp(-C), \tag{4.7}$$

where Z is a standard Gaussian random variable and the last inequality holds for some constant $C > 0$ since $k \le knu/r \le 2k$. Then, based on (4.5) and (4.7), we can use induction to deduce that

$$\mathbb{P}\left\{\max_{1 \le i \le N} |v(x_i) - v(x_0)| \le u\right\} = \mathbb{P}(A_N) \le \exp(-CN).$$

Since $N \sim n^Q$ and $n - 1 < r/u \le n$, this implies (4.4) and completes the proof. $\square$

The next lemma is an isoperimetric inequality for general Gaussian processes.

Lemma 4.3. [10, p.302] *There is a universal constant K_0 such that the following statement holds. Let S be a bounded set and $\{X(s), s \in S\}$ be a separable Gaussian process. Let $D = \sup\{d_X(s,t) : s,t \in S\}$ be the diameter of S in metric d_X. Then for any $u > 0$,*

$$\mathbb{P}\bigg\{ \sup_{s,t\in S} |X(s) - X(t)| \ge K_0\Big(u + \int_0^D \sqrt{\log N(S, d_X, \varepsilon)}d\varepsilon\Big)\bigg\} \le \exp\left(-\frac{u^2}{D^2}\right).$$

Now, we are ready to prove the Chung-type LIL.

Theorem 4.4. *Under Assumptions 2.1 and 2.2, for any fixed $x_0 \in T$, there exists a positive finite constant κ which may depend on x_0 such that*

$$\liminf_{r\to 0+} \sup_{x\in T:\Delta(x,x_0)\le r} \frac{|v(x) - v(x_0)|}{r(\log\log(1/r))^{-1/Q}} = \kappa^{1/Q} \quad a.s.$$

and $C_2 \le \kappa \le C_1$, where C_1 and C_2 are the constants in Proposition 4.2. In particular, κ coincides with the following limit, which is called the small ball constant of v on $\{x \in T : \Delta(x, x_0) \le r\}$, if it exists:

$$\kappa = -\lim_{r\to 0, u/r\to 0} \left(\frac{u}{r}\right)^Q \log \mathbb{P}\bigg\{ \sup_{x\in T:\Delta(x,x_0)\le r} |v(x) - v(x_0)| \le u\bigg\}. \tag{4.8}$$

Proof of Theorem 4.4. Fix $x_0 \in T$. To simplify notations, define $h(r) := r(\log\log(1/r))^{-1/Q}$ and

$$L(r) := \sup_{x\in T:\Delta(x,x_0)\le r} \frac{|v(x) - v(x_0)|}{h(r)}.$$

By Lemma 3.1, $\liminf_{r\to 0+} L(r) = \kappa_1$ a.s. for some constant $0 \le \kappa_1 \le \infty$. To prove the theorem, we will show that

$$\liminf_{r\to 0+} L(r) \ge C_2^{1/Q} \quad \text{a.s.} \tag{4.9}$$

and

$$\liminf_{r\to 0+} L(r) \le C_1^{1/Q} \quad \text{a.s.} \tag{4.10}$$

We first prove the lower bound (4.9). Let $a > 1$ be a constant. For each $n \ge 1$, let $r_n = a^{-n}$. Consider a constant K such that $0 < K < a^{-1}C_2^{1/Q}$ and consider the event

$$A_n = \bigg\{ \sup_{x\in T:\Delta(x,x_0)\le r_n} |v(x) - v(x_0)| \le Kh(r_{n-1})\bigg\}.$$

By the small ball probability estimates in Proposition 4.2,

$$\begin{aligned}\mathbb{P}(A_n) &\le \exp\left(-C_2(aK)^{-Q} \log\log(1/r_{n-1})\right)\\ &= ((n-1)\log a)^{-C_2(aK)^{-Q}}.\end{aligned}$$

Then $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ since $C_2(aK)^{-Q} > 1$. By using the Borel–Cantelli lemma and letting $K \uparrow a^{-1}C_2^{1/Q}$ along a rational sequence, we get that

$$\liminf_{n\to\infty} \sup_{x\in T:\Delta(x,x_0)\le r_n} \frac{|v(x) - v(x_0)|}{h(r_{n-1})} \ge a^{-1}C_2^{1/Q} \quad \text{a.s.} \tag{4.11}$$

Note that h is increasing for $r > 0$ small. For any $r > 0$ small, we can find n large enough such that $r_n \le r \le r_{n-1}$ and $h(r) \le h(r_{n-1})$. Then, by (4.11), we have

$$\liminf_{r\to 0+} L(r) \ge a^{-1}C_2^{1/Q} \quad \text{a.s.,}$$

which implies (4.9) since $a > 1$ is arbitrary.

Now, we turn to the proof of the upper bound (4.10). It relies on Assumption 2.1 which allows us to create independence. Fix $\delta > 0$ small. For any $n \ge 1$, let $\rho_n = \exp(-(n^\delta + n^{1+\delta}))$ and $b_n = \exp(n^{1+\delta})$. For any $x \in T$, let $v_n(x) = v([b_n, b_{n+1}), x)$ and $\widetilde{v}_n(x) = v(\mathbb{R}_+ \setminus [b_n, b_{n+1}), x)$ so that $v(x) = v_n(x) + \widetilde{v}_n(x)$. By Assumption 2.1(a), the processes $v_1, v_2, \dots$ are independent, and for each $n \ge 1$, v_n and $\widetilde{v}_n$ are also independent.

Let $K := ((1+\delta)C_1)^{1/Q}$. Since $v(x) = v_n(x) + \widetilde{v}_n(x)$ and v_n and $\widetilde{v}_n$ are independent, we can apply Anderson's inequality [2, Theorem 2] to get that

$$\mathbb{P}\bigg\{\sup_{x\in T:\Delta(x,x_0)\le\rho_n} |v_n(x) - v_n(x_0)| \le Kh(\rho_n)\bigg\} \ge \mathbb{P}\bigg\{\sup_{x\in T:\Delta(x,x_0)\le\rho_n} |v(x) - v(x_0)| \le Kh(\rho_n)\bigg\}.$$

Then, by Proposition 4.2, the right-hand side is at least

$$\begin{aligned} \exp\left(-C_1K^{-Q}\log\log(1/\rho_n)\right) &= (n^\delta + n^{1+\delta})^{-C_1K^{-Q}} \\ &\ge (2n^{1+\delta})^{-C_1K^{-Q}}. \end{aligned}$$

Since $(1+\delta)C_1K^{-Q} = 1$, we have

$$\sum_{n=1}^\infty \mathbb{P}\bigg\{\sup_{x\in T:\Delta(x,x_0)\le\rho_n} |v_n(x) - v_n(x_0)| \le Kh(\rho_n)\bigg\} = \infty.$$

Since $v_1, v_2, \dots$ are independent, the second Borel–Cantelli lemma implies

$$\liminf_{n\to\infty} \sup_{x\in T:\Delta(x,x_0)\le\rho_n} \frac{|v_n(x) - v_n(x_0)|}{h(\rho_n)} \le ((1+\delta)C_1)^{1/Q} \quad \text{a.s.} \tag{4.12}$$

To complete the proof of (4.10), we claim that

$$\limsup_{n\to\infty} \sup_{x\in T:\Delta(x,x_0)\le\rho_n} \frac{|\widetilde{v}_n(x) - \widetilde{v}_n(x_0)|}{h(\rho_n)} = 0 \quad \text{a.s.} \tag{4.13}$$

We prove this by using Lemma 4.3. Consider the process $\widetilde{v}_n$ on the set $S_n := B_\Delta(x_0, \rho_n)$. By (2.1) of Assumption 2.1, for all $x, y \in S_n$,

$$\|\widetilde{v}_n(x) - \widetilde{v}_n(y)\|_{L^2} \le c_0\bigg(\sum_{j=1}^k b_n^{\gamma_j}|x_j - y_j| + b_{n+1}^{-1}\bigg). \tag{4.14}$$

Recall that $\gamma_j = \alpha_j^{-1} - 1$. Let D_n be the diameter of S_n in the metric $d_{\widetilde{v}_n}$. Then

$$D_n \le C\rho_n\bigg(\sum_{j=1}^k (b_n\rho_n)^{\alpha_j^{-1}-1} + (b_{n+1}\rho_n)^{-1}\bigg). \tag{4.15}$$

Note that $b_n\rho_n = \exp(-n^\delta)$. Also, by the mean value theorem, $(n+1)^{1+\delta} - n^{1+\delta} \ge (1+\delta)n^\delta$, which implies $b_{n+1}\rho_n \ge \exp(\delta n^\delta)$. Provided $\delta \le \min\{\alpha_1^{-1} - 1, \dots, \alpha_k^{-1} - 1\}$, we have

$$D_n \le C\rho_n \exp(-\delta n^\delta). \tag{4.16}$$

Also, by the independence of v_n and $\widetilde{v}_n$ and Lemma 2.4, for n large, for all $x, y \in S_n$,

$$\|\widetilde{v}_n(x) - \widetilde{v}_n(y)\|_{L^2} \le \|v(x) - v(y)\|_{L^2} \le c_1\Delta(x, y).$$

This implies $N(S_n, d_{\widetilde{v}_n}, \varepsilon) \le C(\rho_n/\varepsilon)^Q$ for $\varepsilon > 0$ small. Then for n large,

$$\begin{aligned}\int_0^{D_n} \sqrt{\log N(S_n, d_{\widetilde{v}_n}, \varepsilon)}\, d\varepsilon &\le C \int_0^{C\rho_n \exp(-\delta n^\delta)} \sqrt{\log(\rho_n/\varepsilon)}\, d\varepsilon \\ &= C\rho_n \int_0^{C\exp(-\delta n^\delta)} \sqrt{\log(1/\varepsilon)}\, d\varepsilon \\ &\le C\rho_n \exp(-\delta n^\delta)\sqrt{\delta n^\delta}.\end{aligned}$$

The last inequality can be verified using the change of variable $\varepsilon = e^{-u^2}$ and the elementary inequality $\int_x^\infty u^2 e^{-u^2}\, du \le Cxe^{-x^2}$ for x large. Let $\zeta > 0$. Then for n large, we have

$$2K_0\zeta h(\rho_n) \ge K_0\Big(\zeta h(\rho_n) + \int_0^{D_n} \sqrt{\log N(S_n, d_{\widetilde{v}_n}, \varepsilon)}\, d\varepsilon\Big),$$

where K_0 is the universal constant in Lemma 4.3. Then, by that lemma and (4.16), we have

$$\begin{aligned}\mathbb{P}\Big\{\sup_{x\in T:\Delta(x,x_0)\le\rho_n} |\widetilde{v}_n(x) - \widetilde{v}_n(x_0)| \ge 2K_0\zeta h(\rho_n)\Big\} &\le \exp\left(-\frac{\zeta^2 h(\rho_n)^2}{D_n^2}\right) \\ &\le \exp\left(-\frac{\zeta^2\exp(2\delta n^\delta)}{C^2(\log(n^\delta + n^{1+\delta}))^{2/Q}}\right).\end{aligned}$$

Hence

$$\sum_{n=1}^\infty \mathbb{P}\Big\{\sup_{x\in T:\Delta(x,x_0)\le\rho_n} |\widetilde{v}_n(x) - \widetilde{v}_n(x_0)| \ge 2K_0\zeta h(\rho_n)\Big\} < \infty.$$

By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \sup_{x\in T:\Delta(x,x_0)\le\rho_n} \frac{|\widetilde{v}_n(x) - \widetilde{v}_n(x_0)|}{h(\rho_n)} \le 2K_0\zeta \quad \text{a.s.}$$

Since $\zeta > 0$ is arbitrary, we get (4.13).

Finally, recall that $v(x) = v_n(x) + \widetilde{v}_n(x)$. Combining (4.12) and (4.13) yields

$$\liminf_{r\to 0+} L(r) \le ((1+\delta)C_1)^{1/Q} \quad \text{a.s.}$$

Since $\delta > 0$ is arbitrary, we get (4.10). The proof of Theorem 4.4 is complete. □

5. The exact local modulus of continuity

In this section, we are going to prove the exact local modulus of continuity, which takes the form of the ordinary LIL. First, let us recall the following result of Talagrand [22, Theorem 2.4].

Lemma 5.1. *Let $\{X(t), t \in S\}$ be a mean-zero continuous Gaussian process. Let*

$$\sigma^2 := \sup_{t\in S} \|X(t)\|_{L^2}^2.$$

Consider the canonical metric d_X on S defined by $d_X(s,t) = \|X(s) - X(t)\|_{L^2}$. Assume that for some constant $M > \sigma$, some $p > 0$ and some $0 < \varepsilon_0 \le \sigma$, we have

$$N(S, d_X, \varepsilon) \le (M/\varepsilon)^p \quad \textit{for all } \varepsilon < \varepsilon_0.$$

Then for $u > \sigma^2[(1+\sqrt{p})/\varepsilon_0]$, we have

$$\mathbb{P}\Big\{\sup_{t\in S} X(t) \ge u\Big\} \le \left(\frac{KMu}{\sqrt{p}\,\sigma^2}\right)^p \Phi\left(\frac{u}{\sigma}\right),$$

where $\Phi(x) = (2\pi)^{-1/2}\int_x^\infty e^{-y^2/2}\, dy$ and K is a universal constant.

The following Gaussian estimate is standard:

$$\frac{1}{2\sqrt{2\pi}x}e^{-x^2/2} \le \Phi(x) \le \frac{1}{\sqrt{2\pi}}e^{-x^2/2} \quad \text{for all } x \ge 1. \tag{5.1}$$

Recall that $d(x,y) = \|v(x) - v(y)\|_{L^2}$ is the canonical metric of v. The following theorem gives the exact local modulus of continuity under the metrics d and Δ, respectively. Note that the strong LND property (Assumption 2.2) is not required for this result.

Theorem 5.2. *Under Assumptions 2.1 and 2.3, for any fixed $x_0 \in T$, we have*

$$\lim_{r\to 0+} \sup_{x\in T:\, 0<d(x,x_0)\le r} \frac{|v(x)-v(x_0)|}{d(x,x_0)\sqrt{\log\log(d(x,x_0)^{-1})}} = \sqrt{2} \quad a.s. \tag{5.2}$$

and

$$\lim_{r\to 0+} \sup_{x\in T:\, 0<\Delta(x,x_0)\le r} \frac{|v(x)-v(x_0)|}{\Delta(x,x_0)\sqrt{\log\log(\Delta(x,x_0)^{-1})}} = \kappa \quad a.s. \tag{5.3}$$

for some positive finite constant κ satisfying

$$\sqrt{2}\, c_3 \le \kappa \le \sqrt{2}\, c_1,$$

where c_1 is the constant in (2.5) *and c_3 is the constant in Assumption 2.3.*

Remark 5.3. *Meerschaert et al.* [19] *have considered Gaussian random fields that have stationary increments and satisfy $d(x,y) \asymp \Delta(x,y)$, but only proved that the limit in* (5.2) *is equal to some finite constant $\kappa_1 \ge \sqrt{2}$. Our theorem does not require stationarity of increments and we obtain the exact constant $\kappa_1 = \sqrt{2}$. Meerschaert et al.* [19] *also proved another form of LIL:*

$$\limsup_{|\varepsilon|\to 0+} \sup_{s:\, |s_j|\le|\varepsilon_j|} \frac{|v(x_0+s)-v(x_0)|}{d(s,0)\sqrt{\log\log(1+\prod_{j=1}^k |s_j|^{-\alpha_j})}} = \kappa_2 \quad a.s.$$

Proof of Theorem 5.2. Fix $x_0 \in T$. For $r > 0$, define

$$L(r) := \sup_{x\in T:\, 0<d(x,x_0)\le r} \frac{|v(x)-v(x_0)|}{d(x,x_0)\sqrt{\log\log(d(x,x_0)^{-1})}}.$$

By Lemma 3.1, $\lim_{r\to 0+} L(r) = \kappa$ a.s. for some constant $0 \le \kappa \le \infty$. To prove (5.2), we claim that

$$\limsup_{r\to 0+} L(r) \le \sqrt{2} \quad \text{a.s.} \tag{5.4}$$

and

$$\limsup_{r\to 0+} L(r) \ge \sqrt{2} \quad \text{a.s.} \tag{5.5}$$

We first prove the upper bound (5.4). Let $a > 1$ and $\zeta > 0$ be constants. For each $n \ge 1$, let

$$r_n = a^{-n} \quad \text{and} \quad u_n = (1+\zeta) r_n \sqrt{2\log\log(1/r_n)}.$$

Consider the event

$$A_n = \left\{ \sup_{x\in T: d(x,x_0)\le r_n} |v(x)-v(x_0)| > u_n \right\}.$$

We are going to use Lemma 5.1 to derive an upper bound for $\mathbb{P}(A_n)$. Fix a large n. Consider $S := \{x \in T : d(x,x_0) \le r_n\}$ and $X(x) := v(x) - v(x_0)$ for $x \in S$. Then,

$$\sigma^2 := \sup_{x\in S} \|X(x)\|_{L^2}^2 = r_n^2$$

and by Lemma 2.4, for all $x, y \in S$,

$$d_X(x,y) \le c_1 \sum_{j=1}^{k} |x_j - y_j|^{\alpha_j}.$$

Then $N(S, d_X, \varepsilon) \le C_0(r_n/\varepsilon)^Q$ for $0 < \varepsilon < \sigma$, where C_0 is a constant independent of ε or n, and can be chosen such that $M := C_0^{1/Q} r_n > \sigma$. For n large enough, $u_n > r_n(1+\sqrt{Q})$. Take $\varepsilon_0 = \sigma$ and $p = Q$. Then by Lemma 5.1, we have

$$\mathbb{P}(A_n) \le 2\bigg(\frac{K C_0^{1/Q} r_n u_n}{\sqrt{Q}\, r_n^2}\bigg)^Q \Phi(u_n/r_n).$$

Using the estimate (5.1), we get that

$$\mathbb{P}(A_n) \le C(\log n)^{Q/2}\, n^{-(1+\zeta)^2}.$$

Hence, $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$. By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \sup_{x\in T:\, 0<d(x,x_0)\le r_n} \frac{|v(x) - v(x_0)|}{r_n\sqrt{\log\log(1/r_n)}} \le (1+\zeta)\sqrt{2} \quad \text{a.s.}$$

and thus

$$\limsup_{n\to\infty} \sup_{x\in T:\, r_{n+1}\le d(x,x_0)\le r_n} \frac{|v(x) - v(x_0)|}{r_{n+1}\sqrt{\log\log(1/r_n)}} \le a(1+\zeta)\sqrt{2} \quad \text{a.s.}$$

This implies that

$$\limsup_{r\to 0+} L(r) \le a(1+\zeta)\sqrt{2} \quad \text{a.s.}$$

Letting $a \downarrow 1$ and $\zeta \downarrow 0$ along rational sequences, we get (5.4).

We turn to the proof of the lower bound (5.5). Fix $0 < \varepsilon < 1$. Let $0 < \delta < 1$ be a small fixed number (depending on ε) to be determined. Write $x_0 = (x_{0,1}, \dots, x_{0,k})$. For each $n \ge 1$, let

$$x_n = \big(x_{0,1} + \rho_n^{\alpha_1^{-1}}, \dots, x_{0,k} + \rho_n^{\alpha_k^{-1}}\big),$$

where $\rho_n = \exp(-(n^\delta + n^{1+\delta}))$. With Assumption 2.1, we can write $v(x) = v_n(x) + \widetilde{v}_n(x)$, where

$$v_n(x) = v([b_n, b_{n+1}), x), \quad \widetilde{v}_n(x) = v(\mathbb{R}_+ \setminus [b_n, b_{n+1}), x),$$

and $b_n = \exp(n^{1+\delta})$. We aim to prove that

$$\limsup_{n\to\infty} \frac{|v_n(x_n) - v_n(x_0)|}{d(x_n,x_0)\sqrt{\log\log(d(x_n,x_0)^{-1})}} \ge (1-\varepsilon)\sqrt{2} \quad \text{a.s.} \tag{5.6}$$

and

$$\limsup_{n\to\infty} \frac{|\widetilde{v}_n(x_n) - \widetilde{v}_n(x_0)|}{d(x_n,x_0)\sqrt{\log\log(d(x_n,x_0)^{-1})}} \le \varepsilon \quad \text{a.s.} \tag{5.7}$$

To prove (5.6), we consider for each $n \ge 1$ the event

$$B_n = \Big\{|v_n(x_n) - v_n(x_0)| \ge (1-\varepsilon) d(x_n,x_0)\sqrt{2\log\log(d(x_n,x_0)^{-1})}\Big\}.$$

Similarly to (4.14)–(4.16) in the proof of Theorem 4.4, we can deduce from Assumption 2.1 that, provided $\delta \le \min\{\alpha_1^{-1} - 1, \dots, \alpha_k^{-1} - 1\}$,

$$\begin{aligned} \|\widetilde{v}_n(x_n) - \widetilde{v}_n(x_0)\|_{L^2} &\le c_0\Big(\sum_{j=1}^{k} b_n^{\gamma_j}|x_{n,j} - x_{0,j}| + b_{n+1}^{-1}\Big) \\ &\le K_1 \rho_n \exp(-\delta n^\delta). \end{aligned} \tag{5.8}$$

Note that $\Delta(x_n, x_0) = k\rho_n$. By Assumption 2.3 and Lemma 2.4,

$$c_3\Delta(x, x_0) \le d(x, x_0) \le c_1\Delta(x, x_0) \tag{5.9}$$

for all x in a neighbourhood of x_0. Then, for n large,

$$c_3 k\rho_n \le d(x_n, x_0) \le c_1 k\rho_n. \tag{5.10}$$

Therefore, by the triangle inequality, (5.8) and (5.10), we have

$$\begin{aligned}\|v_n(x_n) - v_n(x_0)\|_{L^2} &\ge \|v(x_n) - v(x_0)\|_{L^2} - \|\widetilde{v}_n(x_n) - \widetilde{v}_n(x_0)\|_{L^2}\\ &\ge (1 - K_2\exp(-\delta n^\delta))d(x_n, x_0).\end{aligned} \tag{5.11}$$

Now, (5.11) and (5.10) imply that for n large,

$$B_n \supset \left\{|v_n(x_n) - v_n(x_0)| \ge (1-\varepsilon/2)\|v_n(x_n) - v_n(x_0)\|_{L^2}\sqrt{2\log\log(C/\rho_n)}\right\},$$

where C is a suitable constant. Then, by the standard Gaussian estimate (5.1), we get that, for n large,

$$\mathbb{P}(B_n) \ge K(\log n)^{-1/2} n^{-(1-\varepsilon/2)^2(1+\delta)}.$$

By choosing δ small enough such that $(1-\varepsilon/2)^2(1+\delta) \le 1$, we have $\sum_{n=1}^\infty \mathbb{P}(B_n) = \infty$. Hence, by the independence among $v_1, v_2, \dots$ and the second Borel–Cantelli lemma, we get (5.6).

For (5.7), we use (5.8) and (5.10) above to get that

$$\begin{aligned}&\mathbb{P}\left\{|\widetilde{v}_n(x_n) - \widetilde{v}_n(x_0)| \ge \varepsilon d(x_n, x_0)\sqrt{\log\log(d(x_n, x_0)^{-1})}\right\}\\ &\le \mathbb{P}\left\{|\widetilde{v}_n(x_n) - \widetilde{v}_n(x_0)| \ge K\varepsilon\|\widetilde{v}_n(x_n) - \widetilde{v}_n(x_0)\|_{L^2}\exp(\delta n^\delta)\sqrt{\log\log(C/\rho_n)}\right\}.\end{aligned}$$

This probability, by standard Gaussian estimate, is bounded above by

$$C'\exp\left(-\frac{1}{2}K^2\varepsilon^2\exp(2\delta n^\delta)\log\log(C/\rho_n)\right) \le C' n^{-2}$$

for n large. Thus, the Borel–Cantelli lemma implies (5.7). Since $v(x) = v_n(x) + \widetilde{v}_n(x)$, combining (5.6) and (5.7) yields

$$\limsup_{n\to\infty} \frac{|v(x_n) - v(x_0)|}{d(x_n, x_0)\sqrt{\log\log(d(x_n, x_0)^{-1})}} \ge (1-\varepsilon)\sqrt{2} - \varepsilon \quad \text{a.s.}$$

Since $d(x_n, x_0) \to 0$, this implies $\limsup_{r\to 0+} L(r) \ge (1-\varepsilon)\sqrt{2} - \varepsilon$ a.s. Letting $\varepsilon \downarrow 0$ along a rational sequence, we get (5.5). This completes the proof of (5.2). Finally, (5.3) is a direct consequence of Lemma 3.1, (5.2) and (5.9). □

6. The exact uniform modulus of continuity

The following theorem establishes the exact uniform modulus of continuity for v.

Theorem 6.1. *Under Assumptions 2.1, 2.2, and 2.3, we have*

$$\lim_{r\to 0+} \sup_{x,y\in T:\, 0<\Delta(x,y)\le r} \frac{|v(x) - v(y)|}{\Delta(x,y)\sqrt{\log(\Delta(x,y)^{-1})}} = \kappa \quad \textit{a.s.} \tag{6.1}$$

and

$$\lim_{r\to 0+} \sup_{x,y\in T:\, 0<d(x,y)\le r} \frac{|v(x) - v(y)|}{d(x,y)\sqrt{\log(d(x,y)^{-1})}} = \kappa' \quad \textit{a.s.} \tag{6.2}$$

for some positive finite constants κ and κ' satisfying

$$\sqrt{2Qc_2} \le \kappa \le \sqrt{2Q}\, c_1 \quad \textit{and} \quad \sqrt{2Qc_2}\, c_1^{-1} \le \kappa' \le \sqrt{2Q}, \tag{6.3}$$

where $Q = \sum_{j=1}^k \alpha_j^{-1}$, c_1 *is the constant in* (2.5) *and* c_2 *is the constant in Assumption 2.2.*

Remark 6.2. *Our Assumption 2.2 of strong LND is stronger than condition* (A2) *in Theorem 4.1 of Meerschaert et al.* [19]*, which is known as the sectorial LND property. But our estimates* (6.3) *for the constants are sharper than the estimates* (4.3) *in* [19].

Proof of Theorem 6.1. For any $r > 0$, let

$$L(r) := \sup_{x,y\in T:\, 0<\Delta(x,y)\le r} \frac{|v(x) - v(y)|}{\Delta(x,y)\sqrt{\log(\Delta(x,y)^{-1})}}.$$

By Lemma 3.1, $\lim_{r\to 0+} L(r) = \kappa$ a.s. for some constant $0 \le \kappa \le \infty$. To prove (6.1), we aim to show that

$$\lim_{r\to 0+} L(r) \le \sqrt{2Q}\, c_1 \quad \text{a.s.} \tag{6.4}$$

and

$$\lim_{r\to 0+} L(r) \ge \sqrt{2Qc_2} \quad \text{a.s.} \tag{6.5}$$

For the upper bound (6.4), we first prove that there is a finite constant C_0 such that for any fixed $b > 1$,

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, \Delta(x,y)\le 2kb^{-n}} \frac{|v(x) - v(y)|}{b^{-n}\sqrt{\log b^n}} \le C_0 \quad \text{a.s.} \tag{6.6}$$

Indeed, by Theorem 1.3.5 in [1], there exists a universal constant K_0 such that for a.e. ω, there exists $r_0 = r_0(\omega)$ such that for all $0 < r < r_0$,

$$\sup_{x,y\in T:\, d(x,y)\le r} |v(x) - v(y)| \le K_0 \int_0^r \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon,$$

where d is the canonical metric of v. By Lemma 2.4, $N(T, d, \varepsilon) \le C\varepsilon^{-Q}$ for all $\varepsilon > 0$ small, thus for $r > 0$ small,

$$\int_0^r \sqrt{\log N(T, d, \varepsilon)}\, d\varepsilon \le Cr\sqrt{\log(1/r)}.$$

Also, by Lemma 2.4, if $\Delta(x,y) \le 2kb^{-n}$ and if $n \ge n_0(\omega)$ is large enough, then $d(x,y)$ would be less than $r_0(\omega)$. Hence, (6.6) follows immediately.

Of course, (6.6) implies $\limsup_{r\to 0+} L(r) \le \kappa$ for some finite constant κ. In order to improve this and get the sharper bound (6.4), we use an approximation argument based on anisotropic lattice points. Let $\varepsilon > 0$ and $1 < a < 2$. Choose b such that $a < b < a^{1+\varepsilon/(2Q)}$. Let $n \ge 1$ be an integer. For each $i = (i_1, \dots, i_k) \in \mathbb{Z}^k$ and $m = (m_1, \dots, m_k) \in \mathbb{Z}^k$, define the anisotropic lattice points $z_{n,i}$ and $h_{n,m}$ in $\prod_{j=1}^k b^{-n/\alpha_j}\mathbb{Z}$ by

$$z_{n,i} = (i_1 b^{-n/\alpha_1}, \dots, i_k b^{-n/\alpha_k}) \quad \text{and} \quad h_{n,m} = (m_1 b^{-n/\alpha_1}, \dots, m_k b^{-n/\alpha_k}).$$

Let

$$I_n = \{i \in \mathbb{Z}^k : z_{n,i} \in T\} \quad \text{and} \quad M_n = \left\{ m \in \mathbb{Z}^k : \sum_{j=1}^k |m_j|^{\alpha_j} b^{-n} \le a^{-n} \right\}.$$

Consider the event

$$A_n = \left\{ \max_{i\in I_n} \max_{m\in M_n} |v(z_{n,i} + h_{n,m}) - v(z_{n,i})| > c_1 a^{-n}\sqrt{2(Q+\varepsilon)\log a^n} \right\}.$$

By Lemma 2.4, for $i \in I_n$, $m \in M_n$ and n large,

$$\|v(z_{n,i} + h_{n,m}) - v(z_{n,i})\|_{L^2} \le c_1 \sum_{j=1}^k |m_j|^{\alpha_j} b^{-n} \le c_1 a^{-n}.$$

Also, the cardinality of I_n is $\le Cb^{Qn}$ and that of M_n is $\le C(b/a)^{Qn}$. It follows that

$$\mathbb{P}(A_n) \le Cb^{Qn}\left(\frac{b}{a}\right)^{Qn} \max_{i\in I_n}\max_{m\in M_n} \mathbb{P}\left\{\frac{|v(z_{n,i}+h_{n,m})-v(z_{n,i})|}{\|v(z_{n,i}+h_{n,m})-v(z_{n,i})\|_{L^2}} > \sqrt{2(Q+\varepsilon)\log a^n}\right\}.$$

Then by the standard Gaussian estimate (5.1), for n large,

$$\mathbb{P}(A_n) \le C\left(\frac{b^2}{a}\right)^{Qn} \exp(-(Q+\varepsilon)\log a^n) = C\left(\frac{b^2}{a^{2+\varepsilon/Q}}\right)^{Qn}.$$

The choice of b implies that $\sum_{n=1}^\infty \mathbb{P}(A_n) < \infty$. By the Borel–Cantelli lemma,

$$\limsup_{n\to\infty} \max_{i\in I_n}\max_{m\in M_n} \frac{|v(z_{n,i}+h_{n,m})-v(z_{n,i})|}{a^{-n}\sqrt{\log a^n}} \le c_1\sqrt{2(Q+\varepsilon)} \quad \text{a.s.} \tag{6.7}$$

To prove (6.4), we consider $x, y \in T$ such that $a^{-n-1} \le \Delta(x,y) \le a^{-n}$, and approximate them by lattice points. Write $x = (x_1, \dots, x_k)$ and $y = (y_1, \dots, y_k)$. Choose $i \in I_n$ such that $z_{n,i} = z_{n,i}(x) = (i_1 b^{-n/\alpha_1}, \dots, i_k b^{-n/\alpha_k})$ is the lattice point that is closest to x. In particular, for all $j \in \{1, \dots, k\}$, we have $|x_j - i_j b^{-n/\alpha_j}| \le b^{-n/\alpha_j}$. Since $\Delta(x,y) \le a^{-n}$, we can also find $m \in M_n$ such that for all $j \in \{1, \dots, k\}$,

$$\begin{aligned} m_j b^{-n/\alpha_j} &\le y_j - x_j \le (m_j+1) b^{-n/\alpha_j} \quad \text{if } y_j - x_j \ge 0,\\ (m_j-1) b^{-n/\alpha_j} &\le y_j - x_j \le m_j b^{-n/\alpha_j} \quad \text{if } y_j - x_j < 0. \end{aligned}$$

Let $h_{n,m} = h_{n,m}(x,y) = (m_1 b^{-n/\alpha_1}, \dots, m_k b^{-n/\alpha_k})$ and write

$$v(y) - v(x) = [v(y) - v(z_{n,i}+h_{n,m})] + [v(z_{n,i}+h_{n,m}) - v(z_{n,i})] + [v(z_{n,i}) - v(x)].$$

Note that

$$\Delta(z_{n,i}, x) \le \sum_{j=1}^k |x_j - i_j b^{-n/\alpha_j}|^{\alpha_j} \le k b^{-n}$$

and

$$\begin{aligned} \Delta(y, z_{n,i}+h_{n,m}) &\le \Delta(y, x+h_{n,m}) + \Delta(x, z_{n,i})\\ &\le \sum_{j=1}^k |y_j - x_j - m_j b^{-n/\alpha_j}|^{\alpha_j} + \sum_{j=1}^k |x_j - i_j b^{-n/\alpha_j}|^{\alpha_j}\\ &\le 2kb^{-n}. \end{aligned}$$

Then, since $b > a$, (6.6) implies that

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, a^{-n-1}\le\Delta(x,y)\le a^{-n}} \frac{|v(y) - v(z_{n,i}+h_{n,m})|}{a^{-n}\sqrt{\log a^n}} = 0 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, a^{-n-1}\le\Delta(x,y)\le a^{-n}} \frac{|v(z_{n,i}) - v(x)|}{a^{-n}\sqrt{\log a^n}} = 0 \quad \text{a.s.}$$

Therefore, together with (6.7), we have

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, a^{-n-1}\le\Delta(x,y)\le a^{-n}} \frac{|v(y) - v(x)|}{a^{-n}\sqrt{\log a^n}} \le c_1\sqrt{2(Q+\varepsilon)} \quad \text{a.s.}$$

This implies that

$$\limsup_{r\to 0+} \sup_{x,y\in T:\, 0<\Delta(x,y)\le r} \frac{|v(y) - v(x)|}{\Delta(x,y)\sqrt{\log(\Delta(x,y)^{-1})}} \le ac_1\sqrt{2(Q+\varepsilon)} \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ and $a \downarrow 1$ along rational sequences, we get the upper bound (6.4).

Next, we prove the lower bound (6.5) using the strong LND property from Assumption 2.2. Let $T = \prod_{j=1}^k [t_j - s_j, t_j + s_j]$. For simplicity, we consider the case where $(t_1, \dots, t_k) \in [0, \infty)^k$ (the proof is similar for other cases). In this case, it is enough to prove (6.5) with T being replaced by $\tilde{T} = \prod_{j=1}^k [t_j, t_j + s_j]$, whose interior does not contain the origin. For each $n \ge 1$, let $r_n = 2^{-n}$. For each $i = (i_1, \dots, i_k) \in \mathbb{Z}^k$, denote

$$i - 1^* = (i_1 - 1, i_2 \dots, i_k).$$

Define the lattice points $x_{n,i}$ by

$$x_{n,i} = (i_1 2^{-n/\alpha_1}, \dots, i_k 2^{-n/\alpha_k}).$$

Let

$$I_n = \{i \in \mathbb{Z}^k : x_i \in \tilde{T} \text{ and } x_{i-1^*} \in \tilde{T}\} \quad \text{and} \quad I_n' = \{i \in \mathbb{Z}^k : x_i \in \tilde{T}\}.$$

Note that $\Delta(x_{n,i}, x_{n,i-1^*}) = r_n$ and the function $r \mapsto r\sqrt{\log(1/r)}$ is increasing for $r > 0$ small. Then

$$\begin{aligned} \lim_{r \to 0+} L(r) &\ge \lim_{n \to \infty} \sup_{x, y \in \tilde{T} :\, 0 < \Delta(x,y) \le r_n} \frac{|v(x) - v(y)|}{\Delta(x,y)\sqrt{\log(\Delta(x,y)^{-1})}} \\ &\ge \liminf_{n \to \infty} L_n, \end{aligned}$$

where

$$L_n := \max_{i \in I_n} \frac{|v(x_{n,i}) - v(x_{n,i-1^*})|}{r_n \sqrt{\log(1/r_n)}}.$$

To prove (6.5), it suffices to prove that

$$\liminf_{n \to \infty} L_n \ge \sqrt{2Qc_2} \quad \text{a.s.} \tag{6.8}$$

To this end, let $0 < K < \sqrt{2Qc_2}$. Fix a large integer n and write $x_i = x_{n,i}$ for simplicity. We claim that there is a constant C independent of n or i such that for all $i \in I_n$,

$$\mathbb{P}\left\{ \frac{|v(x_i) - v(x_{i-1^*})|}{r_n \sqrt{\log(1/r_n)}} \le K \,\middle|\, v(x_j) : j \in I_n' \setminus \{i\} \right\} \le \exp\left(-\frac{C 2^{-nK^2/(2c_2)}}{\sqrt{n}} \right). \tag{6.9}$$

Indeed, for all $i \in I_n$, by Assumption 2.2 and the property that the interior of $\tilde{T}$ does not contain the origin,

$$\operatorname{Var}\big(v(x_i) \big| v(x_j) : j \in I_n' \setminus \{i\}\big) \ge c_2 \min_{j \in I_n' \cup \{0\} \setminus \{i\}} \Delta^2(x_i, x_j) = c_2\, r_n^2. \tag{6.10}$$

The conditional distribution of $v(x_i)$ given $\{v(x_j) : j \in I_n' \setminus \{i\}\}$ is Gaussian with conditional variance $\operatorname{Var}(v(x_i) | v(x_j) : j \in I_n' \setminus \{i\})$, and $v(x_{i-1^*})$ is constant given $\{v(x_j) : j \in I_n' \setminus \{i\}\}$. Then, by Anderson's inequality [2] and (6.10), we have

$$\mathbb{P}\left\{ \frac{|v(x_i) - v(x_{i-1^*})|}{r_n \sqrt{\log(1/r_n)}} \le K \,\middle|\, v(x_j) : j \in I_n' \setminus \{i\} \right\} \le \mathbb{P}\left\{ |Z| \le K \sqrt{c_2^{-1} \log(1/r_n)} \right\},$$

where Z is a standard Gaussian random variable. Hence, we can derive (6.9) using the Gaussian estimate (5.1) and the elementary inequality $1 - x \le \exp(-x)$.

Let $N = N(n)$ be the cardinality of I_n. Order the members of I_n by $i(1), \dots, i(N)$ in a way such that the value of the first coordinate of i is nondecreasing. For each $m \in \{1, \dots, N\}$, let $I_n(m) = \{i(1), \dots, i(m)\}$ and consider the event

$$B_m = \left\{ \max_{i \in I_n(m)} \frac{|v(x_i) - v(x_{i-1^*})|}{r_n \sqrt{\log(1/r_n)}} \le K \right\}.$$

Notice that, for each $2 \le m \le N$, the event B_{m-1} depends on the value of process v at points among $\{x_{i(1)-1^*}, x_{i(1)}, \dots, x_{i(m-1)-1^*}, x_{i(m-1)}\}$, none of which coincides with $x_{i(m)}$ because of the way we order the members of I_n. Therefore, $B_{m-1} \in \sigma\{v(x_j) : j \in I_n' \setminus \{i(m)\}\}$. It follows from (6.9) that

$$\begin{aligned}\mathbb{P}(B_m) &= \mathbb{E}\left[\mathbf{1}_{B_{m-1}}\, \mathbb{P}\left\{\frac{|v(x_{i(m)}) - v(x_{i(m)-1^*})|}{r_n\sqrt{\log(1/r_n)}} \le K \,\middle|\, v(x_j) : j \in I_n' \setminus \{i(m)\}\right\}\right]\\ &\le \mathbb{P}(B_{m-1}) \exp\left(-\frac{C2^{-nK^2/(2c_2)}}{\sqrt{n}}\right).\end{aligned}$$

Note that $N \sim C2^{nQ}$. By induction, we get that

$$\mathbb{P}(L_n \le K) = \mathbb{P}(B_N) \le \exp\left(-C2^{nQ}\frac{2^{-nK^2/(2c_2)}}{\sqrt{n}}\right).$$

Since $Q - K^2/(2c_2) > 0$, we have $\sum_{n=1}^{\infty} \mathbb{P}(L_n \le K) < \infty$. Hence, by the Borel–Cantelli lemma,

$$\liminf_{n\to\infty} L_n \ge K \quad \text{a.s.}$$

Now, we let $K \uparrow \sqrt{2Qc_2}$ along a rational sequence to get (6.8). This proves (6.1).

We turn to the proof of (6.2). Let

$$\tilde{L}(r) := \sup_{x,y\in T:\, 0<d(x,y)\le r} \frac{|v(x) - v(y)|}{d(x,y)\sqrt{\log(d(x,y)^{-1})}}.$$

By Lemma 3.1, $\lim_{r\to 0+} \tilde{L}(r) = \kappa'$ a.s. for some constant $0 \le \kappa' \le \infty$. Moreover, (6.1) and Lemma 2.4 imply that

$$\lim_{r\to 0+} \tilde{L}(r) \ge \sqrt{2Qc_2}\, c_1^{-1} \quad \text{a.s.}$$

It remains to prove that

$$\lim_{r\to 0+} \tilde{L}(r) \le \sqrt{2Q} \quad \text{a.s.} \tag{6.11}$$

This can be proved by a similar argument that led to (6.4) above. In fact, the proof of (6.6) above shows that for any fixed $b > 1$,

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, d(x,y)\le b^{-n}} \frac{|v(x) - v(y)|}{b^{-n}\sqrt{\log b^n}} \le C \quad \text{a.s.} \tag{6.12}$$

Let $\varepsilon > 0$, $1 < a < 2$ and b be such that $a < b < a^{1+\varepsilon/(2Q)}$. We modify the above approximation argument as follows. For fixed n, choose any minimal cover $\{B_d(z_{n,i}, b^{-n})\}_i$ of T consisting of d-balls with centers $z_{n,i} \in T$, and define $I_n = \{z_{n,i}\}_i$. For each $z_{n,i}$, define $M_{n,i} = \{h_{n,i,m}\}_m$ such that $\{B_d(z_{n,i} + h_{n,i,m}, b^{-n})\}_m$ is a minimal cover of $B_d(z_{n,i}, (1+\varepsilon)a^{-n})$. Consider the event

$$A_n = \left\{\max_{z_{n,i}\in I_n} \max_{h_{n,i,m}\in M_{n,i}} |v(z_{n,i} + h_{n,i,m}) - v(z_{n,i})| > (1+\varepsilon)a^{-n}\sqrt{2(Q+\varepsilon)\log a^n}\right\}.$$

Since d is comparable to Δ by Assumption 2.3 and Lemma 2.4, the cardinality of I_n is $\le Cb^{nQ}$ and that of $M_{n,i}$ is $\le C(b/a)^{nQ}$. Also, $\|v(z_{n,i} + h_{n,i,m}) - v(z_{n,i})\|_{L^2} \le (1+\varepsilon)a^{-n}$. Then, as before, we can show that $\sum_{n=1}^{\infty} \mathbb{P}(A_n) < \infty$ and

$$\limsup_{n\to\infty} \max_{z_{n,i}\in I_n} \max_{h_{n,i,m}\in M_{n,i}} \frac{|v(z_{n,i} + h_{n,i,m}) - v(z_{n,i})|}{a^{-n}\sqrt{\log a^n}} \le (1+\varepsilon)\sqrt{2(Q+\varepsilon)} \quad \text{a.s.} \tag{6.13}$$

Consider $x, y \in T$ such that $a^{-n-1} \le d(x,y) \le a^{-n}$. Then, we can find $z_{n,i} = z_{n,i}(x) \in I_n$ such that $d(z_{n,i}, x) \le b^{-n}$. Since $d(z_{n,i}, y) \le d(z_{n,i}, x) + d(x,y) \le b^{-n} + a^{-n} \le (1+\varepsilon)a^{-n}$ for n large, we can also find $h_{n,i,m} = h_{n,i,m}(x,y) \in M_{n,i}$ such that $d(z_{n,i} + h_{n,i,m}, y) \le b^{-n}$. Then, by (6.12) and $b > a$, we get

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, a^{-n-1}\le d(x,y)\le a^{-n}} \frac{|v(z_{n,i}) - v(x)|}{a^{-n}\sqrt{\log a^n}} = 0 \quad \text{a.s.}$$

and

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, a^{-n-1}\le d(x,y)\le a^{-n}} \frac{|v(z_{n,i} + h_{n,i,m}) - v(y)|}{a^{-n}\sqrt{\log a^n}} = 0 \quad \text{a.s.}$$

Combining this with (6.13), we get

$$\limsup_{n\to\infty} \sup_{x,y\in T:\, a^{-n-1}\le d(x,y)\le a^{-n}} \frac{|v(y) - v(x)|}{a^{-n}\sqrt{\log a^n}} \le (1+\varepsilon)\sqrt{2(Q+\varepsilon)} \quad \text{a.s.}$$

which implies that

$$\limsup_{r\to 0+} \sup_{x,y\in T:\, 0<d(x,y)\le r} \frac{|v(y) - v(x)|}{d(x,y)\sqrt{\log(d(x,y)^{-1})}} \le (1+\varepsilon)a\sqrt{2(Q+\varepsilon)} \quad \text{a.s.}$$

Letting $\varepsilon \downarrow 0$ and $a \downarrow 1$ yields (6.11). This completes the proof of Theorem 6.1. □

7. Linear SPDEs driven by fractional-colored noise

In this section, we give an application of our main results to a class of linear SPDEs. Consider the equation

$$\frac{\partial}{\partial t} u(t,x) = \mathscr{L} u(t,x) + \dot{W}(t,x), \quad t \ge 0,\ x \in \mathbb{R}^d, \tag{7.1}$$

with zero initial condition $u(0,x) = 0$. Here, $\mathscr{L}$ is the infinitesimal generator of a symmetric Lévy process $X = \{X(t), t \ge 0\}$ taking values in $\mathbb{R}^d$, and $\dot{W}$ is a fractional-colored (or white-colored) centered Gaussian noise with Hurst index $1/2 \le H < 1$ in time and spatial covariance f, i.e.,

$$\mathbb{E}[\dot{W}(t,x)\dot{W}(s,y)] = \rho_H(t-s) f(x-y),$$

where

$$\rho_H(t-s) = \begin{cases} a_H |t-s|^{2H-2} & \text{if } 1/2 < H < 1, \\ \delta(t-s) & \text{if } H = 1/2, \end{cases}$$

and where $a_H = H(2H-1)$ and δ is the delta function. When X is a Brownian motion, $\mathscr{L}$ is the Laplace operator and (7.1) is the stochastic heat equation. Furthermore, when $H = 1/2$, (7.1) is the stochastic heat equation considered in [6].

The existence of the solution to (7.1) has been studied in [3, 7] (and in [4] for $H = 1/2$), and the space-time regularity of the solution has been studied in [24, 7]. Herrell et al. [7] used the the idea of string processes of Mueller and Tribe [20] and showed that $\{u(t,x), t \ge 0, x \in \mathbb{R}^k\}$ admits the decomposition

$$u(t,x) = U(t,x) - Y(t,x), \tag{7.2}$$

where $\{U(t,x), t \ge 0, x \in \mathbb{R}^k\}$ has stationary increments and satisfies the property of strong LND, while $\{Y(t,x), t \ge 0, x \in \mathbb{R}^k\}$ has smooth sample paths. Consequently, certain regularity properties of $u(t,x)$ can be deduced from those of $U(t,x)$. Now we can deal with $\{u(t,x), t \ge 0, x \in \mathbb{R}^k\}$ directly.

We assume that f is the Fourier transform of a tempered measure μ which is absolutely continuous with respect to the Lebesgue measure with density h, i.e., $\mu(d\xi) = h(\xi)d\xi$. A typical example is $h(\xi) = |\xi|^{-\beta}$, $0 < \beta < d$. In this case, f is called the Riesz kernel: $f(x) = C|x|^{\beta-d}$, where C is some suitable constant depending on β and d; see [21, §V].

Let $\Psi(\xi)$ be the characteristic exponent of X given by

$$\mathbb{E}[e^{i\xi\cdot X(t)}] = e^{-t\Psi(\xi)}, \quad t \ge 0,\ \xi \in \mathbb{R}^d.$$

Note that $\Psi(\xi) = \Psi(-\xi) \ge 0$ for all $\xi \in \mathbb{R}^d$ since X is assumed to be symmetric. Assume that $X(t)$ has a probability density function given by

$$p_t(x) = \frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} e^{-i\xi\cdot x} e^{-t\Psi(\xi)} d\xi, \quad t > 0,\ x \in \mathbb{R}^d. \tag{7.3}$$

Let $T_0 > 0$. Recall that the Gaussian noise W defines a linear isometry from the Hilbert space completion $\mathcal{HP}$ of the space $C_c^\infty((0,T_0)\times\mathbb{R}^d)$ of compactly supported smooth functions with respect to the inner product $\langle\cdot,\cdot\rangle_{\mathcal{HP}}$ into the Gaussian space in $L^2(\mathbb{P})$:

$$\begin{aligned} \varphi \mapsto W(\varphi) &:= \int_0^{T_0}\int_{\mathbb{R}^d} \varphi(s,y) W(ds,dy), \\ \mathbb{E}[W(\varphi)W(\psi)] &= \langle \varphi, \psi\rangle_{\mathcal{HP}}. \end{aligned} \tag{7.4}$$

For test functions φ, ψ on $(0,T_0)\times\mathbb{R}^d$, the inner product $\langle\varphi,\psi\rangle_{\mathcal{HP}}$ is defined by

$$\begin{aligned} &\langle\varphi,\psi\rangle_{\mathcal{HP}} := \int_0^{T_0}\int_0^{T_0} ds\, dr \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dy\, dz\, \varphi(s,y)\rho_H(s-r) f(y-z)\psi(r,z) \\ &= b_H \int_{\mathbb{R}} d\tau\, |\tau|^{1-2H} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} dy\, dz\, f(y-z)\mathcal{F}(\varphi(\cdot,y)\mathbf{1}_{[0,T_0]}(\cdot))(\tau)\overline{\mathcal{F}(\psi(\cdot,z)\mathbf{1}_{[0,T_0]}(\cdot))(\tau)} \\ &= c_{H,d}\int_{\mathbb{R}} d\tau\, |\tau|^{1-2H}\int_{\mathbb{R}^d} d\xi\, h(\xi)\mathcal{F}(\varphi\mathbf{1}_{[0,T_0]})(\tau,\xi)\overline{\mathcal{F}(\psi\mathbf{1}_{[0,T_0]})(\tau,\xi)}, \end{aligned} \tag{7.5}$$

where $b_H = a_H(2^{2(1-H)}\sqrt{\pi})^{-1}\Gamma(H-1/2)/\Gamma(1-H)$ and $c_{H,d} = b_H(2\pi)^{-d}$, for $1/2 < H < 1$; see [3, 7]. In fact, the equalities in (7.5) also hold for $H = 1/2$ with $b_{1/2} = (2\pi)^{-1}$. In the above, $\mathcal{F}$ denotes the Fourier transform defined, for any functions $g : \mathbb{R} \to \mathbb{C}$ and $\varphi : \mathbb{R}^{1+d} \to \mathbb{C}$, by

$$\mathcal{F}g(\tau) = \int_{\mathbb{R}} e^{-i\tau s} g(s)\, ds, \quad \mathcal{F}\varphi(\tau,\xi) = \int_{\mathbb{R}^{1+d}} e^{-i\tau s - i\xi\cdot y}\varphi(s,y)\, ds\, dy.$$

If follows from [7] and [4] (for $1/2 < H < 1$ and $H = 1/2$ respectively) that if

$$\int_{\mathbb{R}^d} \frac{\mu(d\xi)}{1+\Psi(\xi)^{2H}} < \infty, \tag{7.6}$$

then (7.1) has a random field solution on $[0, T_0]$ which is given by

$$u(t,x) = \int_0^t \int_{\mathbb{R}^d} p_{t-s}(x-y) W(ds,dy).$$

Denote $G_{t,x}(s,y) = p_{t-s}(x-y)\mathbf{1}_{[0,t]}(s)$. It follows from (7.5) that for any $a_1,\dots,a_n \in \mathbb{R}$, for any $t^1,\dots,t^n \in [0,T_0]$ and $x^1,\dots,x^n \in \mathbb{R}^d$, we have

$$\mathbb{E}\left[\left(\sum_{j=1}^n a_j u(t^j,x^j)\right)^2\right] = c_{H,d}\int_{\mathbb{R}} d\tau\, |\tau|^{1-2H}\int_{\mathbb{R}^d} d\xi\, h(\xi)\, |\mathcal{F}G(\tau,\xi)|^2, \tag{7.7}$$

where $G = \sum_{j=1}^n a_j G_{t^j,x^j}$. Note that $p_t(\cdot)$ is equal to the inverse Fourier transform of $\xi \mapsto e^{-t\Psi(\xi)}$ since $\Psi(\xi) = \Psi(-\xi)$. Hence, it can be verified that the Fourier transform of $G_{t,x}(\cdot,\cdot)$ is

$$\mathcal{F}G_{t,x}(\tau,\xi) = \frac{e^{-i\xi\cdot x}(e^{-i\tau t} - e^{-t\Psi(\xi)})}{\Psi(\xi) - i\tau}, \quad \tau\in\mathbb{R},\ \xi\in\mathbb{R}^d. \tag{7.8}$$

Dalang et al. [6] have established a harmonizable representation for the solution of the stochastic heat equation

$$\frac{\partial}{\partial t}u(t,x) = \Delta u(t,x) + \dot{W}(t,x),$$

where $\dot{W}$ is a spatially homogeneous Gaussian noise that is white in time and colored in space. In the following, we follow the approach of [6] to establish a similar representation for the solution of equation (7.1) driven by the fractional-colored Gaussian noise.

Let $\tilde{W}_1, \tilde{W}_2$ be independent space-time Gaussian white noise on $\mathbb{R}\times\mathbb{R}^d$. Let $\tilde{W} = \tilde{W}_1 + i\tilde{W}_2$. For each $(t,x) \in [0,T_0]\times\mathbb{R}^d$, define

$$v(t,x) = c_{H,d}^{1/2}\,\mathrm{Re}\iint_{\mathbb{R}\times\mathbb{R}^d} \mathcal{F}G_{t,x}(\tau,\xi)\,|\tau|^{\frac{1-2H}{2}} h^{\frac{1}{2}}(\xi)\tilde{W}(d\tau,d\xi). \tag{7.9}$$

The following lemma verifies that $v(t,x)$ has the same law as the solution $u(t,x)$ of equation (7.1). We will call (7.9) the harmonizable representation of $u(t,x)$.

Lemma 7.1. *The Gaussian random field $v = \{v(t,x), (t,x) \in [0,T_0]\times\mathbb{R}^d\}$ has the same law as the solution $u = \{u(t,x), (t,x)\in[0,T_0]\times\mathbb{R}^d\}$ of equation* (7.1).

Proof. It is clear that v is Gaussian. By (7.5), for any $(t,x),(s,y)\in[0,T_0]\times\mathbb{R}^d$,

$$\begin{aligned}\mathbb{E}[v(t,x)v(s,y)] &= \iint_{\mathbb{R}\times\mathbb{R}^d} \mathcal{F}G_{t,x}(\tau,\xi)\,\overline{\mathcal{F}G_{s,y}(\tau,\xi)}\,|\tau|^{1-2H}h(\xi)\,d\tau\,d\xi\\ &= \langle G_{t,x}, G_{s,y}\rangle_{\mathcal{HP}}\\ &= \mathbb{E}[u(t,x)u(s,y)].\end{aligned}$$

Hence v and u have the same law. □

From now on, suppose that there exist positive finite constants c_Ψ and C_Ψ such that

$$c_\Psi|\xi|^\alpha \le \Psi(\xi) \le C_\Psi|\xi|^\alpha \text{ for all } \xi\in\mathbb{R}^d, \quad \text{where } 0<\alpha\le 2, \tag{7.10}$$

and there exist positive finite constants c_h and C_h such that

$$c_h|\xi|^{-\beta} \le h(\xi) \le C_h|\xi|^{-\beta} \text{ for all } \xi\in\mathbb{R}^d, \quad \text{where } 0<\beta<d. \tag{7.11}$$

Define

$$\theta_1 := H - \frac{d-\beta}{2\alpha} \quad \text{and} \quad \theta_2 := \alpha\theta_1 = \alpha H - \frac{d-\beta}{2}.$$

These are the Hölder exponents of $u(t,x)$ in time and space respectively. By (7.6), if $\beta > d-2\alpha H$, or equivalently, $\theta_1>0$, then (7.1) has a solution. Consider the following metric on $\mathbb{R}\times\mathbb{R}^d$:

$$\Delta((t,x),(s,y)) = |t-s|^{\theta_1} + \sum_{j=1}^d |x_j-y_j|^{\theta_2}. \tag{7.12}$$

We always have $\theta_1<1$ since $H<1$ and $\beta<d$. Furthermore, we assume the following condition:

$$\theta_1 > 0 \quad \text{and} \quad \theta_2 < 1. \tag{7.13}$$

The condition $\theta_2<1$ is equivalent to $\beta < d-2\alpha H+2$.

We now verify Assumptions 2.1 and 2.2 for the solution of (7.1) using the harmonizable representation (7.9).

Lemma 7.2. *Suppose Ψ and h satisfy conditions (7.10) and (7.11) respectively. Suppose θ_1 and θ_2 satisfy condition (7.13). Let T be a compact rectangle in $(0,\infty)\times\mathbb{R}^d$. Then the Gaussian random field $\{v(A,t,x), A\in\mathscr{B}(\mathbb{R}_+), (t,x)\in T\}$ defined by*

$$v(A,t,x) = c_{H,d}^{1/2}\,\mathrm{Re}\iint_{\{(\tau,\xi):\max(|\tau|^{\theta_1},|\xi|^{\theta_2})\in A\}} \mathcal{F}G_{t,x}(\tau,\xi)\,|\tau|^{\frac{1-2H}{2}} h^{\frac{1}{2}}(\xi)\tilde{W}(d\tau,d\xi) \tag{7.14}$$

satisfies Assumption 2.1(a). Moreover, there exists a finite constant c_0 such that for all $0\le a<b\le\infty$ and all $(t_0,x_0),(t,x)\in T$,

$$\begin{aligned}&\|v([a,b),t,x)-v(t,x)-v([a,b),t_0,x_0)+v(t_0,x_0)\|_{L^2}\\ &\le c_0\left(a^{\gamma_1}|t-t_0|+a^{\gamma_2}\sum_{j=1}^d|x_j-x_{0,j}|+b^{-1}\right),\end{aligned} \tag{7.15}$$

where $\gamma_1=\theta_1^{-1}-1$ and $\gamma_2=\theta_2^{-1}-1$. In particular, Assumption 2.1(b) is satisfied for $a_0=0$.

Proof. It is obvious that $v(A,t,x)$ satisfies part (a) of Assumption 2.1. For part (b), the proof is similar to that of Lemma 7.3 in [6]: First,

$$\begin{aligned}&v([a,b),t,x)-v(t,x)-v([a,b),t_0,x_0)+v(t_0,x_0)\\ &=v([0,a),t_0,x_0)-v([0,a),t,x)+v([b,\infty),t_0,x_0)-v([b,\infty),t,x).\end{aligned}$$

By (7.7) and (7.8),

$$\begin{aligned}&\mathbb{E}[(v([0,a),t,x)-v([0,a),t_0,x_0))^2]\\ &=C\iint_{D_1(a)}\left|\frac{(e^{-i\tau t}-e^{-t\Psi(\xi)})-e^{-i\xi\cdot(x_0-x)}(e^{-i\tau t_0}-e^{-t_0\Psi(\xi)})}{\Psi(\xi)-i\tau}\right|^2|\tau|^{1-2H}h(\xi)\,d\tau\,d\xi\\ &=C\iint_{D_1(a)}\frac{\varphi_1(t,x,\tau,\xi)^2+\varphi_2(t,x,\tau,\xi)^2}{\Psi(\xi)^2+|\tau|^2}|\tau|^{1-2H}h(\xi)\,d\tau\,d\xi\end{aligned} \tag{7.16}$$

and

$$\begin{aligned}&\mathbb{E}[(v([b,\infty),t,x)-v([b,\infty),t_0,x_0))^2]\\ &=C\iint_{D_2(b)}\frac{\varphi_1(t,x,\tau,\xi)^2+\varphi_2(t,x,\tau,\xi)^2}{\Psi(\xi)^2+|\tau|^2}|\tau|^{1-2H}h(\xi)\,d\tau\,d\xi,\end{aligned} \tag{7.17}$$

where

$$\begin{aligned}&D_1(a)=\{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^d:\max(|\tau|^{\theta_1},|\xi|^{\theta_2})<a\},\\ &D_2(b)=\{(\tau,\xi)\in\mathbb{R}\times\mathbb{R}^d:\max(|\tau|^{\theta_1},|\xi|^{\theta_2})\ge b\},\\ &\varphi_1(t,x,\tau,\xi)=\cos(\tau t)-e^{-t\Psi(\xi)}-\cos(\xi\cdot(x_0-x)+\tau t_0)+e^{-t_0\Psi(\xi)}\cos(\xi\cdot(x_0-x)),\\ &\varphi_2(t,x,\tau,\xi)=-\sin(\tau t)+\sin(\xi\cdot(x_0-x)+\tau t_0)-e^{-t_0\Psi(\xi)}\sin(\xi\cdot(x_0-x)).\end{aligned}$$

Consider (7.16). Note that $\varphi_1(t_0,x_0,\tau,\xi)=0=\varphi_2(t_0,x_0,\tau,\xi)$, and

$$|\partial_t\varphi_j|\le|\tau|+\Psi(\xi)\quad\text{and}\quad|\partial_x\varphi_j|\le2|\xi|,\quad j=1,2.$$

Then, by the mean value theorem,

$$\begin{aligned}
&\mathbb{E}[(v([0,a),t,x)-v([0,a),t_0,x_0))^2] \\
&\leq C\iint_{D_1(a)}\Big(4(|\tau|^2+\Psi(\xi)^2)|t-t_0|^2+8|\xi|^2|x-x_0|^2\Big)\frac{|\tau|^{1-2H}|\xi|^{-\beta}}{\Psi(\xi)^2+|\tau|^2}d\tau\,d\xi \\
&= 4C|t-t_0|^2\iint_{D_1(a)}|\tau|^{1-2H}|\xi|^{-\beta}d\tau\,d\xi+8C|x-x_0|^2\iint_{D_1(a)}\frac{|\tau|^{1-2H}|\xi|^{2-\beta}}{\Psi(\xi)^2+|\tau|^2}d\tau\,d\xi \\
&=: 4C|t-t_0|^2I_1+8C|x-x_0|^2I_2.
\end{aligned}$$

Using polar coordinates $r=|\xi|$,

$$\begin{aligned}
I_1 &= C\iint_{\{(\tau,r)\in\mathbb{R}\times\mathbb{R}_+:\max(|\tau|^{\theta_1},r^{\theta_2})<a\}}|\tau|^{1-2H}r^{d-\beta-1}d\tau\,dr \\
&\leq C\int_{-a^{\theta_1^{-1}}}^{a^{\theta_1^{-1}}}d\tau\,|\tau|^{1-2H}\int_0^{a^{\theta_2^{-1}}}dr\,r^{d-\beta-1} \\
&= Ca^{(2-2H)\theta_1^{-1}+(d-\beta)\theta_2^{-1}}.
\end{aligned}$$

Since $\theta_2=\alpha\theta_1$ and $\gamma_1=\theta_1^{-1}-1$, we get that $I_1\leq Ca^{2\gamma_1}$.

For I_2, we use the condition $c|\xi|^\alpha\leq\Psi(\xi)\leq C|\xi|^\alpha$, polar coordinates $r=|\xi|$, and symmetry of the integrand in τ to get that

$$I_2\leq C\iint_{\{(\tau,r)\in\mathbb{R}_+^2:\max(\tau^{\theta_1},r^{\theta_2})<a\}}\frac{\tau^{1-2H}r^{d-\beta-1}}{(c^2\wedge 1)(r^{2\alpha}+\tau^2)}d\tau\,dr.$$

Putting $\rho=r^\alpha$ and $|z|$ the Euclidean norm of $z=(\tau,\rho)$ in $\mathbb{R}^2$, we get that

$$\begin{aligned}
I_2 &\leq C\iint_{\{(\tau,\rho)\in\mathbb{R}_+^2:\max(\tau,\rho)<a^{\theta_1^{-1}}\}}\frac{\tau^{1-2H}\rho^{\alpha^{-1}(d-\beta+2)-1}}{\rho^2+\tau^2}d\tau\,d\rho \\
&\leq C\iint_{\{z\in\mathbb{R}_+^2:|z|<\sqrt{2}a^{\theta_1^{-1}}\}}|z|^{-2H+\alpha^{-1}(d-\beta+2)-2}dz \\
&= Ca^{\theta_1^{-1}(-2H+\alpha^{-1}(d-\beta+2))}=Ca^{2\gamma_2}.
\end{aligned}$$

Therefore, we have

$$\mathbb{E}[(v([0,a),t,x)-v([0,a),t_0,x_0))^2]\leq C\left(a^{2\gamma_1}|t-t_0|^2+a^{2\gamma_2}|x-x_0|^2\right). \tag{7.18}$$

For (7.17), we can use the bounds $|\varphi_1|\leq 4$ and $|\varphi_2|\leq 3$ to deduce that

$$\mathbb{E}[(v([b,\infty),t,x)-v([b,\infty),t_0,x_0))^2]\leq C\iint_{D_2(b)}\frac{|\tau|^{1-2H}|\xi|^{-\beta}}{c^2|\xi|^{2\alpha}+|\tau|^2}d\tau\,d\xi.$$

To estimate the above integral, we split $D_2(b)$ into two parts:

$$D_2(b)=\big\{|\tau|^{\theta_1}\leq|\xi|^{\theta_2},|\xi|^{\theta_2}\geq b\big\}\cup\big\{|\tau|^{\theta_1}>|\xi|^{\theta_2},|\tau|^{\theta_1}\geq b\big\}.$$

Passing to polar coordinates, we have

$$\begin{aligned}
&\iint_{D_2(b)} \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{c^2|\xi|^{2\alpha}+|\tau|^2} d\tau\, d\xi \\
&\le C \iint_{\{|\tau|^{\theta_1}\le r^{\theta_2}, r^{\theta_2}\ge b\}} \frac{|\tau|^{1-2H} r^{d-\beta-1}}{c^2 r^{2\alpha}} d\tau\, dr + C \iint_{\{|\tau|^{\theta_1}> r^{\theta_2}, |\tau|^{\theta_1}\ge b\}} \frac{|\tau|^{1-2H} r^{d-\beta-1}}{|\tau|^2} d\tau\, dr \\
&= C\int_{b^{\theta_2^{-1}}}^\infty dr\, r^{d-\beta-1-2\alpha}\int_{-r^\alpha}^{r^\alpha} d\tau\, |\tau|^{1-2H} + C\int_{b^{\theta_1^{-1}}}^\infty d\tau\, |\tau|^{-1-2H}\int_0^{|\tau|^{\alpha^{-1}}} dr\, r^{d-\beta-1} \\
&= Cb^{-2}.
\end{aligned}$$

We have shown that

$$\mathbb{E}[(v([b,\infty),t,x) - v([b,\infty),t_0,x_0))^2] \le Cb^{-2}. \tag{7.19}$$

Therefore, (7.15) follows immediately from (7.18) and (7.19). □

The following result shows that $u(t,x)$ satisfies strong LND with respect to the metric Δ defined in (7.12) above, thus verifies Assumption 2.2.

Lemma 7.3. *Suppose Ψ and h satisfy conditions* (7.10) *and* (7.11) *respectively. Suppose θ_1 and θ_2 satisfy* (7.13). *Let T be a compact rectangle in $(0,\infty)\times\mathbb{R}^d$. Then there exists a constant $c_2>0$ such that for any $n\ge 1$, for any $(t,x),(t^1,x^1),\dots,(t^n,x^n)\in T$, we have*

$$\mathrm{Var}(u(t,x)|u(t^1,x^1),\dots,u(t^n,x^n)) \ge c_2 \min_{1\le j\le n} \Delta^2((t,x),(t^j,x^j)).$$

In particular, this implies that $\|u(t,x)-u(s,y)\|_{L^2}\ge\sqrt{c_2}\Delta((t,x),(s,y))$ for all $(t,x),(s,y)\in T$.

Proof. We may assume that $T=[a,a']\times[-b,b]^d$, where $0<a<a'<\infty$ and $0<b<\infty$. It suffices to show that there exists a positive constant C such that

$$\mathbb{E}\left[\left(u(t,x)-\sum_{j=1}^n a_j u(t^j,x^j)\right)^2\right] \ge Cr^{2\theta_2},$$

for any $n\ge1$, any $(t,x),(t^1,x^1),\dots,(t^n,x^n)\in T$, and any $a_1,\dots,a_n\in\mathbb{R}$, where

$$r=\min_{1\le j\le n}(|t-t^j|^{1/\alpha}\vee|x-x^j|).$$

From (7.7) and (7.8), we see that

$$\begin{aligned}
&\mathbb{E}\left[\left(u(t,x)-\sum_{j=1}^n a_j u(t^j,x^j)\right)^2\right] \qquad (7.20)\\
&\ge K_0\int_{\mathbb{R}} d\tau \int_{\mathbb{R}^d} d\xi \left| e^{-i\xi\cdot x}(e^{-i\tau t}-e^{-t\Psi(\xi)}) - \sum_{j=1}^n a_j e^{-i\xi\cdot x^j}(e^{-i\tau t^j}-e^{-t^j\Psi(\xi)})\right|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{C_\Psi^2|\xi|^{2\alpha}+|\tau|^2},
\end{aligned}$$

where $K_0=c_{H,d}\,c_h$. Let M be a finite constant such that $|t-t'|^{1/\alpha}\vee|x-x'|\le M$ for all $(t,x),(t',x')\in T$. Let $\rho=\min\{a/M^\alpha,1\}$. Choose and fix two nonnegative smooth test functions $f:\mathbb{R}\to\mathbb{R}_+$ and $g:\mathbb{R}^d\to\mathbb{R}_+$ which vanish outside $[-\rho,\rho]$ and the unit ball respectively, and satisfy $f(0)=g(0)=1$. Let $f_r(\tau)=r^{-\alpha}f(r^{-\alpha}\tau)$, $g_r(\xi)=r^{-d}g(r^{-1}\xi)$, and denote the Fourier transforms of f_r and g_r by $\widehat{f}_r$ and $\widehat{g}_r$ respectively. Consider the integral

$$I:=\int_{\mathbb{R}} d\tau\int_{\mathbb{R}^d} d\xi\left[e^{-i\xi\cdot x}(e^{-i\tau t}-e^{-t\Psi(\xi)})-\sum_{j=1}^n a_j e^{-i\xi\cdot x^j}(e^{-i\tau t^j}-e^{-t^j\Psi(\xi)})\right]e^{i\xi\cdot x}e^{i\tau t}\widehat{f}_r(\tau)\widehat{g}_r(\xi).$$

By inverse Fourier transform and (7.3), we have

$$I = (2\pi)^{1+d}\bigg[f_r(0)g_r(0) - f_r(t)(p_t * g_r)(0) - \sum_{j=1}^n a_j\Big(f_r(t-t^j)g_r(x-x^j) - f_r(t)(p_{t^j} * g_r)(x-x^j)\Big)\bigg].$$

By the definition of r, for every j, either $|t-t^j| \ge r^\alpha$ or $|x-x^j| \ge r$, thus $f_r(t-t^j)g_r(x-x^j) = 0$. Moreover, since $t/r^\alpha \ge a/M^\alpha \ge \rho$, we have $f_r(t) = 0$ and hence

$$I = (2\pi)^{1+d} r^{-\alpha-d}. \tag{7.21}$$

On the other hand, by the Cauchy–Schwarz inequality and (7.20),

$$I^2 \le \frac{1}{K_0}\,\mathbb{E}\bigg[\Big(u(t,x) - \sum_{j=1}^n a_j u(t^j,x^j)\Big)^2\bigg] \int_{\mathbb{R}}\int_{\mathbb{R}^d} \big|\widehat{f}_r(\tau)\widehat{g}_r(\xi)\big|^2 \big(C_\Psi^2|\xi|^{2\alpha} + |\tau|^2\big)|\tau|^{2H-1}|\xi|^\beta d\tau\, d\xi.$$

Note that $\widehat{f}_r(\tau) = \widehat{f}(r^\alpha\tau)$ and $\widehat{g}_r(\xi) = \widehat{g}(r\xi)$. Then by scaling, we have

$$\begin{aligned}
&\int_{\mathbb{R}}\int_{\mathbb{R}^d} \big|\widehat{f}_r(\tau)\widehat{g}_r(\xi)\big|^2 \big(C_\Psi^2|\xi|^{2\alpha} + |\tau|^2\big)|\tau|^{2H-1}|\xi|^\beta d\tau\, d\xi\\
&= r^{-2\alpha-2\alpha H-\beta-d}\int_{\mathbb{R}}\int_{\mathbb{R}^d} \big|\widehat{f}(\tau)\widehat{g}(\xi)\big|^2 \big(C_\Psi^2|\xi|^{2\alpha} + |\tau|^2\big)|\tau|^{2H-1}|\xi|^\beta d\tau\, d\xi\\
&=: r^{-2\alpha-2\alpha H-\beta-d} C_0,
\end{aligned}$$

where C_0 is a finite constant since $\widehat{f}$ and $\widehat{g}$ are rapidly decreasing functions. It follows that

$$I^2 \le \frac{C_0}{K_0}\, r^{-2\alpha-2\alpha H-\beta-d}\,\mathbb{E}\bigg[\Big(u(t,x) - \sum_{j=1}^n a_j u(t^j,x^j)\Big)^2\bigg]. \tag{7.22}$$

Combining (7.21) and (7.22), we conclude that

$$\mathbb{E}\bigg[\Big(u(t,x) - \sum_{j=1}^n a_j u(t^j,x^j)\Big)^2\bigg] \ge \frac{(2\pi)^{2+2d}K_0}{C_0}\, r^{2\theta_2}.$$

This completes the proof of Lemma 7.3. □

Under conditions (7.10), (7.11) and (7.13), Lemmas 2.4, 7.2 and 7.3 imply that for any compact rectangle T in $(0,\infty)\times\mathbb{R}^d$, there exist positive finite constants c_1 and c_3 such that for all $(t,x),(s,y) \in T$,

$$c_3\Delta((t,x),(s,y)) \le \|u(t,x) - u(s,y)\|_{L^2} \le c_1\Delta((t,x),(s,y)). \tag{7.23}$$

See also [8, Theorem 4.1].

By applying our results, we obtain the following theorem, which strengthens the regularity results in Propositions 3.7 and 3.10 of [7] and provides more precise information and bounds for the limiting constants.

Theorem 7.4. *Suppose Ψ and h satisfy conditions* (7.10) *and* (7.11) *respectively. Suppose θ_1 and θ_2 satisfy condition* (7.13). *Then the following statements hold.*

(i) *Chung-type law of the iterated logarithm: For any fixed $(t_0,x_0) \in (0,\infty)\times\mathbb{R}^d$,*

$$\liminf_{r\to 0+} \sup_{\substack{t>0,\, x\in\mathbb{R}^d:\\ \Delta((t,x),(t_0,x_0))\le r}} \frac{|u(t,x) - u(t_0,x_0)|}{r(\log\log(1/r))^{-1/Q}} = \kappa_1^{1/Q} \quad a.s.,$$

where $Q = \frac{1}{\theta_1} + \frac{d}{\theta_2}$ *and* κ_1 *is a positive finite constant given by Theorem 4.4.*

(ii) *The exact local modulus of continuity: For any fixed* $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^d$,

$$\lim_{r\to 0+} \sup_{\substack{t>0,\, x\in\mathbb{R}^d:\\ 0<d((t,x),(t_0,x_0))\le r}} \frac{|u(t,x)-u(t_0,x_0)|}{d((t,x),(t_0,x_0))\sqrt{\log\log(d((t,x),(t_0,x_0))^{-1})}} = \sqrt{2} \quad a.s.,$$

where $d((t,x),(t_0,x_0)) = \|u(t,x)-u(t_0,x_0)\|_{L^2}$, *and*

$$\lim_{r\to 0+} \sup_{\substack{t>0,\, x\in\mathbb{R}^d:\\ 0<\Delta((t,x),(t_0,x_0))\le r}} \frac{|u(t,x)-u(t_0,x_0)|}{\Delta((t,x),(t_0,x_0))\sqrt{\log\log(\Delta((t,x),(t_0,x_0))^{-1})}} = \kappa_2 \quad a.s.$$

for some positive finite constant κ_2 *such that*

$$\sqrt{2}\, c_3 \le \kappa_2 \le \sqrt{2}\, c_1,$$

where c_1, c_3 *are constants satisfying* (7.23), *with* T *being any neighborhood of* (t_0, x_0).

(iii) *The exact uniform modulus of continuity: For any compact rectangle* T *in* $(0,\infty)\times\mathbb{R}^d$,

$$\lim_{r\to 0+} \sup_{\substack{(t,x),(s,y)\in T:\\ 0<d((t,x),(s,y))\le r}} \frac{|u(t,x)-u(s,y)|}{d((t,x),(s,y))\sqrt{\log(d((t,x),(s,y))^{-1})}} = \kappa_3 \quad a.s.$$

and

$$\lim_{r\to 0+} \sup_{\substack{(t,x),(s,y)\in T:\\ 0<\Delta((t,x),(s,y))\le r}} \frac{|u(t,x)-u(s,y)|}{\Delta((t,x),(s,y))\sqrt{\log(\Delta((t,x),(s,y))^{-1})}} = \kappa_4 \quad a.s.$$

for some positive finite constants κ_3, κ_4 *satisfying*

$$\sqrt{2Qc_2}\, c_1^{-1} \le \kappa_3 \le \sqrt{2Q} \quad \text{and} \quad \sqrt{2Qc_2} \le \kappa_4 \le \sqrt{2Q}\, c_1,$$

where $Q = \frac{1}{\theta_1} + \frac{d}{\theta_2}$, c_1 *is the constant in* (7.23) *and* c_2 *is the constant in Lemma 7.3.*

Proof. By Lemma 7.1, (i)–(iii) hold if and only if they hold for the Gaussian random field $v(t,x)$ defined in (7.9). By Lemmas 7.2 and 7.3, $v(t,x)$ satisfies Assumptions 2.1, 2.2 and 2.3. Therefore, the desired results follows from Theorems 4.4, 5.2 and 6.1. □

Remark 7.5. *The constants* κ_1 *and* κ_2 *are independent of the point* (t_0,x_0). *This is due to the decomposition* (7.2) *above and the fact that the random field* $\{U(t,x), t\ge 0, x\in\mathbb{R}^k\}$ *has stationary increments* [7].

In the theorem below, we consider the special case that $\Psi(\xi) = |\xi|^\alpha$, which corresponds to the equation (7.1) with $\mathscr{L}$ being the fractional Laplacian $-(-\Delta)^{\alpha/2}$. We are able to obtain the exact constants for the LIL in time variable and space variable respectively. This result strengthens Corollaries 3.8 and 3.9 of [7].

Theorem 7.6. *Suppose* $\Psi(\xi) = |\xi|^\alpha$ *and* $h(\xi) = |\xi|^{-\beta}$, *where* $0<\alpha\le 2$, $0<\beta<d$. *Suppose* θ_1 *and* θ_2 *satisfy condition* (7.13). *Then, for any fixed* $(t_0,x_0)\in(0,\infty)\times\mathbb{R}^d$, *almost surely,*

$$\limsup_{\delta\to 0} \frac{|u(t_0+\delta,x_0)-u(t_0,x_0)|}{|\delta|^{\theta_1}\sqrt{\log\log(1/|\delta|)}} = \left(2c_{H,d}\iint_{\mathbb{R}\times\mathbb{R}^d} |e^{-i\tau}-1|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2+|\xi|^{2\alpha}}\, d\tau\, d\xi\right)^{1/2} \tag{7.24}$$

and

$$\limsup_{|\varepsilon|\to 0} \frac{|u(t_0,x_0+\varepsilon)-u(t_0,x_0)|}{|\varepsilon|^{\theta_2}\sqrt{\log\log(1/|\varepsilon|)}} = \left(2c_{H,d}\iint_{\mathbb{R}\times\mathbb{R}^d} |e^{-i\xi_1}-1|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2+|\xi|^{2\alpha}}\, d\tau\, d\xi\right)^{1/2}. \tag{7.25}$$

Proof. Let κ_5 and κ_6 denote the quantity on the right-hand sides of (7.24) and (7.25) respectively. Fix $(t_0, x_0) \in (0,\infty) \times \mathbb{R}^d$. We claim that

$$\|(u(t_0+s, x_0) - u(t_0, x_0)\|_{L^2} = |s|^{\theta_1}\big(\kappa_5 + o(1)\big) \quad \text{as } s \to 0, \tag{7.26}$$

and

$$\|(u(t_0, x_0+y) - u(t_0, x_0)\|_{L^2} = |y|^{\theta_2}\big(\kappa_6 + o(1)\big) \quad \text{as } |y| \to 0. \tag{7.27}$$

Once the claims are proved to be true, we can consider a sequence of neighborhoods converging to the point t_0 and x_0 respectively, and apply Theorem 5.2 to the processes $\{u(t, x_0), t > 0\}$ and $\{u(t_0, x), x \in \mathbb{R}^d\}$ to obtain (7.24) and (7.25) respectively.

To prove (7.26), for $s > 0$, we use (7.7) to get that

$$\begin{aligned}
&\|u(t_0+s, x_0) - u(t_0, x_0)\|_{L^2}^2 \\
&= c_{H,d} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|(e^{-i\tau(t_0+s)} - e^{-(t_0+s)|\xi|^\alpha}) - (e^{-i\tau t_0} - e^{-t_0|\xi|^\alpha})\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi \\
&= c_{H,d} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|e^{-i\tau t_0}(e^{-i\tau s} - 1) - e^{-t_0|\xi|^\alpha}(e^{-s|\xi|^\alpha} - 1)\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi.
\end{aligned}$$

Then the change of variables $\tau \mapsto s^{-1}\tau$ and $\xi \mapsto s^{-1/\alpha}\xi$ leads to

$$\begin{aligned}
&\|u(t_0+s, x_0) - u(t_0, x_0)\|_{L^2}^2 \\
&\quad = c_{H,d}\, s^{2\theta_1} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|(e^{-i\tau} - 1) - e^{s^{-1}(i\tau t_0 - t_0|\xi|^\alpha)}(e^{-|\xi|^\alpha} - 1)\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi.
\end{aligned}$$

Similarly, for $s > 0$ small,

$$\begin{aligned}
&\|u(t_0-s, x_0) - u(t_0, x_0)\|_{L^2}^2 \\
&= c_{H,d}\, s^{2\theta_1} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|(e^{-i\tau} - 1) - e^{s^{-1}(i\tau(t_0-s) - (t_0-s)|\xi|^\alpha)}(e^{-|\xi|^\alpha} - 1)\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi.
\end{aligned}$$

Since $0 \le 1 - e^{-|\xi|^\alpha} \le \min(1, |\xi|^\alpha)$ and

$$\iint_{\mathbb{R}\times\mathbb{R}^d} \min(1, |\xi|^{2\alpha}) \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi < \infty,$$

by the dominated convergence theorem, we have $|s|^{-2\theta_1}\|u(t_0+s, x_0) - u(t_0, x_0)\|_{L^2}^2 \to \kappa_5^2$ as $s \to 0$, which is exactly (7.26).

For (7.27), we let $y \in \mathbb{R}^d \setminus \{0\}$ and use (7.7) again to get that

$$\begin{aligned}
&\|u(t_0, x_0+y) - u(t_0, x_0)\|_{L^2}^2 \\
&= c_{H,d} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|e^{-i\xi\cdot(x_0+y)}(e^{-i\tau t_0} - e^{-t_0|\xi|^\alpha}) - e^{-i\xi\cdot x_0}(e^{-i\tau t_0} - e^{-t_0|\xi|^\alpha})\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi \\
&= c_{H,d} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|(e^{-i\xi\cdot y} - 1)(1 - e^{i\tau t_0 - t_0|\xi|^\alpha})\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi.
\end{aligned}$$

By the change of variables $\tau \mapsto |y|^{-\alpha}\tau$ and $\xi \mapsto |y|^{-1}\xi$, the above expression is equal to

$$c_{H,d}\, |y|^{2\theta_2} \iint_{\mathbb{R}\times\mathbb{R}^d} \big|\big(e^{-i\xi\cdot\frac{y}{|y|}} - 1\big)\big(1 - e^{|y|^{-\alpha}(i\tau t_0 - t_0|\xi|^\alpha)}\big)\big|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi.$$

Then, by a rotation of the variable ξ which takes the unit vector $\frac{y}{|y|}$ to the basis vector $\mathbf{e}_1 = (1, 0, \dots, 0)$, we deduce that

$$\begin{aligned}&\|u(t_0, x_0 + y) - u(t_0, x_0)\|_{L^2}^2\\ &= c_{H,d}\,|y|^{2\theta_2} \iint_{\mathbb{R}\times\mathbb{R}^d} \left|\left(e^{-i\xi_1} - 1\right)\left(1 - e^{|y|^{-\alpha}(i\tau t_0 - t_0|\xi|^{\alpha})}\right)\right|^2 \frac{|\tau|^{1-2H}|\xi|^{-\beta}}{|\tau|^2 + |\xi|^{2\alpha}}\, d\tau\, d\xi.\end{aligned}$$

By the dominated convergence theorem, we get (7.27) as $y \to 0$. The proof is complete. □

8. Strongly LND anisotropic Gaussian fields with non-stationary increments

Finally, we construct a class of anisotropic Gaussian random fields that have strong LND property but do not have stationary increments. Let $f : \mathbb{R}^k \to \mathbb{R}$ be a nonnegative function such that for all $\xi \in \mathbb{R}^k$,

$$\frac{C_1}{(\sum_{j=1}^k |\xi_j|^{\alpha_j})^{Q+2}} \le f(\xi) \le \frac{C_2}{(\sum_{j=1}^k |\xi_j|^{\alpha_j})^{Q+2}}, \tag{8.1}$$

where $0 < \alpha_j < 1$, $Q = \sum_{j=1}^k \alpha_j^{-1}$ and C_1, C_2 are positive finite constants. It can be verified that f satisfies

$$\int_{\mathbb{R}^k} \min\{1, |\xi|^2\} f(\xi) d\xi < \infty.$$

Define the Gaussian random field $v = \{v(x), x \in \mathbb{R}^k\}$ by

$$v(x) = \int_{\mathbb{R}^k} \prod_{j=1}^k (e^{ix_j\xi_j} - 1) W(d\xi), \tag{8.2}$$

where W is a centered complex-valued Gaussian random measure whose control measure has density f, meaning that for all Borel sets $A, B \subset \mathbb{R}^k$,

$$\mathbb{E}[W(A)\overline{W(B)}] = \int_{A\cap B} f(\xi) d\xi, \quad \text{and} \quad W(-A) = \overline{W(A)}.$$

This implies that v is real-valued. Note that v does not have stationary increments. Still, we can verify that v satisfies Assumptions 2.1 and 2.2 in Lemmas 8.1 and 8.2 below.

Lemma 8.1. *Let T be a compact rectangle in $\mathbb{R}^k$. Then the process $\{v(A, x), A \in \mathscr{B}(\mathbb{R}_+), x \in T\}$ defined by*

$$v(A, x) = \int_{\{\max_j |\xi_j|^{\alpha_j} \in A\}} \prod_{j=1}^k (e^{ix_j\xi_j} - 1) W(d\xi)$$

satisfies Assumption 2.1(a). Moreover, there exists a finite constant c_0 such that for all $0 \le a < b \le \infty$ and all $x, y \in T$,

$$\|v(x) - v([a, b), x) - v(y) + v([a, b), y)\|_{L^2} \le c_0 \bigg(\sum_{j=1}^k a^{\gamma_j} |x_j - y_j| + b^{-1} \bigg), \tag{8.3}$$

where $\gamma_j = \alpha_j^{-1} - 1$. In particular, Assumption 2.1(b) is satisfied for $a_0 = 0$.

Proof. It is clear that $v(A, x)$ satisfies Assumption 2.1(a). For (8.3), by writing $v(x) - v(y)$ as the telescoping sum

$$\begin{aligned}&[v(x_1, \dots, x_k) - v(y_1, x_2, \dots, x_k)] + [v(y_1, x_2, \dots, x_k) - v(y_1, y_2, x_3, \dots, x_k)]\\ &\qquad + \cdots + [v(y_1, \dots, y_{k-1}, x_k) - v(y_1, \dots, y_k)]\end{aligned}$$

and similarly for $v([a,b),x) - v([a,b),y)$, it is enough to prove (8.3) for x and y that only differ in one coordinate. By re-arranging coordinates, we only need to consider the case that $x = (x_1, \dots, x_k)$ and $y = (y_1, x_2, \dots, x_k)$. Note that

$$\begin{aligned} &v(x) - v([a,b),x) - v(y) + v([a,b),y) \\ &= [v([0,a),x) - v([0,a),y)] + [v([b,\infty),x) - v([b,\infty),y)]. \end{aligned}$$

We estimate the two terms separately. By $|e^{iz} - e^{iz'}| \le |z - z'|$, $|e^{iz} - 1| \le 2$ and (8.1),

$$\begin{aligned} &\|v([0,a),x) - v([0,a),y)\|_{L^2}^2 \\ &= \int_{\{\max_j |\xi_j|^{\alpha_j} < a\}} \left| \prod_{j=1}^k (e^{ix_j\xi_j} - 1) - (e^{iy_1\xi_1} - 1) \prod_{j=2}^k (e^{ix_j\xi_j} - 1) \right|^2 f(\xi)\, d\xi \\ &\le 2^{2k-2} C_2 |x_1 - y_1|^2 \int_{\{\max_j |\xi_j|^{\alpha_j} < a\}} \frac{|\xi_1|^2}{(\sum_{j=1}^k |\xi_j|^{\alpha_j})^{Q+2}}\, d\xi. \end{aligned}$$

Note that $|\xi_1|^2 = (|\xi_1|^{\alpha_1})^{2+2(1-\alpha_1)/\alpha_1} \le (\sum_{j=1}^k |\xi_j|^{\alpha_j})^{2+2(1-\alpha_1)/\alpha_1}$. Then, by the change of variables $\xi_j \mapsto a^{\alpha_j^{-1}} \xi_j$, followed by another change $\xi_j \mapsto z_j^{2/\alpha_j}$, the last integral is equal to

$$\begin{aligned} &a^{2\alpha_1^{-1}-2} \int_{\{\max_j |\xi_j|^{\alpha_j} < 1\}} \Big(\sum_{j=1}^k |\xi_j|^{\alpha_j} \Big)^{-Q+2(1-\alpha_1)/\alpha_1} d\xi \\ &\le a^{2\alpha_1^{-1}-2} \prod_{j=1}^k (2/\alpha_j) \int_{\{\max_j |z_j|^2 < 1\}} |z|^{-k+4(1-\alpha_1)/\alpha_1} dz \\ &\le C a^{2\alpha_1^{-1}-2}, \end{aligned}$$

where C is a finite constant, so we get the estimate $\|v([0,a),x) - v([0,a),y)\|_{L^2}^2 \le C a^{2\gamma_j} |x_1 - y_1|^2$. On the other hand, by $|e^{iz} - 1| \le 2$ and (8.1),

$$\|v([b,\infty),x) - v([b,\infty),y)\|_{L^2}^2 \le 2^{2k} C_2 \int_{\{\max_j |\xi_j|^{\alpha_j} \ge b\}} \frac{1}{(\sum_{j=1}^k |\xi_j|^{\alpha_j})^{Q+2}}\, d\xi.$$

Now, by similar changes of variables $\xi_j \mapsto b^{\alpha_j^{-1}} \xi_j$ and then $\xi_j \mapsto z_j^{2/\alpha_j}$, we get that $\|v([b,\infty),x) - v([b,\infty),y)\|_{L^2}^2 \le C b^{-2}$. Combining the two estimates above finishes the proof of (8.3). □

Lemma 8.2. *Define $\Delta(x,y) = \sum_{j=1}^k |x_j - y_j|^{\alpha_j}$. Let T be a compact rectangle in $\mathbb{R}^k$ away from the axes. Then, there exists a positive finite constant c_2 such that for all $n \ge 1$, for all $x, x^1, \dots, x^n \in T$,*

$$\mathrm{Var}(v(x) | v(x^1), \dots, v(x^n)) \ge c_2 \min_{1 \le \ell \le n} \Delta^2(x, x^\ell).$$

In particular, this implies that $\|v(x) - v(y)\|_{L^2} \ge \sqrt{c_2}\, \Delta(x,y)$ for all $x, y \in T$.

Proof. We may assume that $a \le |x_j| \le b$ for all $x = (x_1, \dots, x_k) \in T$, where $0 < a < 1 < b < \infty$ are constants. It suffices to prove that there exists a positive finite constant c such that for all $n \ge 1$, for all $x, x^1, \dots, x^n \in T$ and all $a_1, \dots, a_n \in \mathbb{R}$,

$$\mathbb{E}\left[\Big(v(x) - \sum_{\ell=1}^n a_\ell v(x^\ell) \Big)^2 \right] \ge c r^2, \quad \text{where } r = \min_{1 \le \ell \le n} \max_{1 \le j \le k} |x_j - x_j^\ell|^{\alpha_j}. \tag{8.4}$$

By (8.1),

$$\begin{aligned} &\mathbb{E}\Big[\Big(v(x)-\sum_{\ell=1}^n a_\ell v(x^\ell)\Big)^2\Big] \\ &\geq C_1\int_{\mathbb{R}^k}\Big|\prod_{j=1}^k(e^{ix_j\xi_j}-1)-\sum_{\ell=1}^n a_\ell\prod_{j=1}^k(e^{ix_j^\ell\xi_j}-1)\Big|^2\frac{d\xi}{(\sum_{j=1}^k|\xi_j|^{\alpha_j})^{Q+2}}. \end{aligned} \tag{8.5}$$

Let $\rho=\min\{1,a(2b)^{-\alpha^*/\alpha_*}\}$, where $\alpha^*=\max\{\alpha_1,\dots,\alpha_k\}$ and $\alpha_*=\min\{\alpha_1,\dots,\alpha_k\}$. For each $j=1,\dots,k$, let $\phi^j:\mathbb{R}\to\mathbb{R}_+$ be a nonnegative smooth function supported on $[-\rho,\rho]$ satisfying $\phi^j(0)=1$. Let $\phi^j_r(z)=r^{-\alpha_j^{-1}}\phi^j(r^{-\alpha_j^{-1}}z)$ and let $\widehat{\phi}^j_r$ denote the Fourier transform of ϕ^j_r. Consider the integral

$$I:=\int_{\mathbb{R}^k}\Big[\prod_{j=1}^k(e^{ix_j\xi_j}-1)-\sum_{\ell=1}^n a_\ell\prod_{j=1}^k(e^{ix_j^\ell\xi_j}-1)\Big]\prod_{j=1}^k\big[e^{-ix_j\xi_j}\widehat{\phi}^j_r(\xi_j)\big]\,d\xi.$$

Then, by inverse Fourier transform,

$$I=(2\pi)^k\Big[\prod_{j=1}^k(\phi^j_r(0)-\phi^j_r(x_j))-\sum_{\ell=1}^n a_\ell\prod_{j=1}^k(\phi^j_r(x_j-x_j^\ell)-\phi^j_r(x_j))\Big].$$

Note that $\phi^j_r(0)=r^{-Q}$. Since $r\leq(2b)^{\alpha^*}$, we have $r^{-\alpha_j^{-1}}|x_j|\geq a(2b)^{-\alpha^*/\alpha_*}\geq\rho$, thus $\phi^j_r(x_j)=0$. Also, for each ℓ, by the definition of r, $\max_j|x_j-x_j^\ell|^{\alpha_j}\geq r$, so there exists some j such that $|x_j-x_j^\ell|^{\alpha_j}\geq r$. For this j, $r^{-\alpha_j^{-1}}|x_j-x_j^\ell|\geq 1\geq\rho$, and hence $\phi^j_r(x_j-x_j^\ell)=0$. This implies that for each ℓ,

$$\prod_{j=1}^k(\phi^j_r(x_j-x_j^\ell)-\phi^j_r(x_j))=0.$$

Therefore, we have

$$I=(2\pi)^k r^{-Q}. \tag{8.6}$$

On the other hand, by the Cauchy–Schwarz inequality and (8.5) above,

$$I^2\leq\frac{1}{C_1}\mathbb{E}\Big[\Big(v(x)-\sum_{\ell=1}^n a_\ell v(x^\ell)\Big)^2\Big]\times\int_{\mathbb{R}^k}\prod_{j=1}^k|\widehat{\phi}^j_r(\xi_j)|^2\Big(\sum_{j=1}^k|\xi_j|^{\alpha_j}\Big)^{Q+2}d\xi.$$

By $\widehat{\phi}^j_r(\xi_j)=\widehat{\phi}^j(r^{\alpha_j^{-1}}\xi_j)$ and the change of variables $\xi_j\mapsto r^{-\alpha_j^{-1}}\xi_j$, the integral on the right-hand side is equal to

$$r^{-2Q-2}\int_{\mathbb{R}^k}\prod_{j=1}^k|\widehat{\phi}^j(\xi_j)|^2\Big(\sum_{j=1}^k|\xi_j|^{\alpha_j}\Big)^{Q+2}d\xi=C_0r^{-2Q-2}.$$

Therefore, together with (8.6), we have

$$(2\pi)^{2k}r^{-2Q}=I^2\leq\frac{C_0}{C_1}r^{-2Q-2}\,\mathbb{E}\Big[\Big(v(x)-\sum_{\ell=1}^n a_\ell v(x^\ell)\Big)^2\Big]$$

and (8.4) follows. The proof is complete. □

Corollary 8.3. *Let T be a compact rectangle in $\mathbb{R}^k$ away from the axes. Then, Theorems 4.4, 5.2 and 6.1 can be applied to the Gaussian random field v defined by* (8.2).

We point out that even though the Gaussian random field $v = \{v(x), x \in \mathbb{R}^k\}$ defined by (8.2) and the fractional Brownian sheet with parameters $(\alpha_1, \ldots, \alpha_k) \in (0,1)^k$ share some similarity in their definitions and many sample path properties, some of their other fine properties such as Chung's LILs and exact Hausdorff measure functions are rather different (see Lee [11]). Instead, it can be proved that these latter properties of $v = \{v(x), x \in \mathbb{R}^k\}$ are similar to those in [15, 16] for Gaussian random fields with stationary increments and spectral density f that satisfies (8.1).

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