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Circle actions, quantum cohomology, and the Fukaya category of Fano toric varieties

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We define a class of noncompact Fano toric manifolds which we call *admissible toric manifolds*, for which Floer theory and quantum cohomology are defined. The class includes Fano toric negative line bundles, and it allows blow-ups along fixed point sets.

We prove closed-string mirror symmetry for this class of manifolds: the Jacobian ring of the superpotential is the symplectic cohomology (not the quantum cohomology). Moreover, $SH^*(M)$ is obtained from $QH^*(M)$ by localizing at the toric divisors. We give explicit presentations of $SH^*(M)$ and $QH^*(M)$, using ideas of Batyrev, McDuff and Tolman.

Assuming that the superpotential is Morse (or a milder semisimplicity assumption), we prove that the wrapped Fukaya category for this class of manifolds satisfies the toric generation criterion, ie is split-generated by the natural Lagrangian torus fibers of the moment map taken with suitable holonomies. In particular, the wrapped category is compactly generated and cohomologically finite.

We prove a generic generation theorem: a generic deformation of the monotone toric symplectic form defines a local system for which the twisted wrapped Fukaya category satisfies the toric generation criterion. This theorem, together with a limiting argument about continuity of eigenspaces, are used to prove the untwisted generation results.

We prove that for any closed Fano toric manifold, and a generic local system, the twisted Fukaya category satisfies the toric generation criterion. If the superpotential is Morse (or assuming semisimplicity), also the untwisted Fukaya category satisfies the criterion.

The key ingredients are nonvanishing results for the open-closed string map, using tools from the paper by Ritter and Smith; we also prove a conjecture from that paper that any monotone toric negative line bundle contains a nondisplaceable monotone Lagrangian torus. The above presentation results require foundational work: we extend the class of Hamiltonians for which the maximum principle holds for symplectic manifolds conical at infinity, thus extending the class of Hamiltonian circle actions for which invertible elements can be constructed in $SH^*(M)$. Computing $SH^*(M)$ is notoriously hard and there are very few known examples beyond the cases of cotangent bundles and subcritical Stein manifolds. So this computation is significant in itself, as well as being the key ingredient in proving the above results in homological mirror symmetry.

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1 Introduction

1A Circle actions, generators and relations in Floer cohomology

The goal of this paper is to develop techniques to produce generators and relations in the quantum cohomology $QH^*(M)$ and in the symplectic cohomology $SH^*(M)$ of monotone symplectic manifolds conical at infinity, by exploiting Hamiltonian circle actions, and to deduce generation results for the Fukaya category $\mathcal{F}(M)$ and the wrapped Fukaya category $\mathcal{W}(M)$. To clarify the big picture, we recall that by foundational work of Ritter and Smith [37] there is a commutative diagram, called the *acceleration diagram*, relating all these invariants via the open–closed string map $\mathcal{OC}$ on Hochschild homology:

$$\begin{tikzcd} \mathrm{HH}_*(\mathcal{F}(M)) \arrow[r] \arrow[d, "\mathcal{OC}"'] & \mathrm{HH}_*(\mathcal{W}(M)) \arrow[d, "\mathcal{OC}"] \\ QH^*(M) \arrow[r, "c^*"] & SH^*(M) \end{tikzcd} \tag{1}$$

Recall that in the compact Fukaya category $\mathcal{F}(M)$ one only allows *closed* monotone orientable Lagrangian submanifolds, and one works with pseudoholomorphic maps, whereas for the wrapped Fukaya category $\mathcal{W}(M)$ one also allows *noncompact* Lagrangians which are Legendrian at infinity, and the definition of the morphism spaces involves wrapping one Lagrangian around the other by using a Hamiltonian flow (more precisely, a direct limit construction is needed just like for $SH^*(M)$, where one increases the growth of the Hamiltonian at infinity).

The tool for producing generators and relations in $QH^*(M)$ and $SH^*(M)$ comes from our foundational paper [35], which constructs a commutative diagram of ring homomorphisms:

$$\begin{tikzcd} & SH^*(M) \\ \widetilde{\pi}_1(\mathrm{Ham}_{\ell\geq 0}(M,\omega)) \arrow[ur, "\mathcal{R}"] \arrow[r, "r"] & QH^*(M) \arrow[u, "c^*"'] \end{tikzcd} \tag{2}$$

Namely, from loops of Hamiltonian diffeomorphisms on M we construct elements in $QH^*(M)$ and $SH^*(M)$, by generalizing the ideas of the Seidel representation [38] to the noncompact setting. The noncompact setting brings with it technical difficulties, such as the necessity of a *maximum principle* which prevents holomorphic curves and Floer solutions from escaping to infinity. These seemingly technical constraints are responsible for interesting algebro-homological consequences: for example, the elements above are invertible in $SH^*(M)$ but not in $QH^*(M)$ (whereas in the compact setting, Seidel elements are always invertible in quantum cohomology). Further down the line, this phenomenon will also imply that for many noncompact monotone toric manifolds,

$$SH^*(M) \cong \mathrm{Jac}(W_M) \tag{3}$$

recovers the Jacobian ring of the Landau–Ginzburg superpotential W_M (a purely combinatorial invariant associated to the moment polytope of M), whereas one may have first imagined that the isomorphism $QH^*(C) \cong \mathrm{Jac}(W_C)$ for closed monotone toric manifolds C (see Batyrev [5] and Givental [21; 20]) should persist also in the noncompact setting (see Example 3.6).

The result (3) is also noteworthy because it is a proof of closed-string mirror symmetry.

Convention *In this paper, we always restrict $SH^*(M) \equiv SH^*_0(M)$ to the component of contractible loops. For simply connected M, eg any toric manifold, this is all of $SH^*(M)$. Recall also that $SH^*_0(M)$ contains the unit $c^*(1)$, so if $SH^*_0(M) = 0$ then all of $SH^*(M) = 0$.*

As an illustration of diagram (2) we recall the following application from [35], which exploited the circle action given by rotation in the fibers of a line bundle.

Theorem 1.1 *Let $\pi\colon E \to B$ be a complex line bundle over a closed symplectic manifold (B, ω_B). Assume E is* negative, *meaning $c_1(E) = -k[\omega_B]$ for some $k > 0$. Then*

$$SH^*(E) \cong QH^*(E)_{\mathrm{PD}[B]} \tag{4}$$

are isomorphic algebras, where $QH^*(E)$ has been localized at the zero section.

The negativity assumption is needed to ensure that the total space is a symplectic manifold conical at infinity, so Floer theory is defined. The theorem was not phrased in terms of localization in [35] so we comment on this in Section 4B. More generally, by [35], we have:

Theorem 1.2 *$SH^*(M)$ is a localization of $QH^*(M)$ whenever there is a Hamiltonian S^1–action on M which corresponds to the Reeb flow at infinity.*

As a vector space, $QH^*(M)$ is just the ordinary cohomology, however it is a nontrivial task to determine its ring structure, especially in the noncompact setting. For example, the nontriviality of $SH^*(E)$ above is equivalent to $\pi^*c_1(E) = \mathrm{PD}[B]$ being nonnilpotent in $QH^*(E)$, and it is not known whether this holds in general (we will prove it for toric E).

Even in the deceptively simple case of negative line bundles, $QH^*(E) \cong QH^*(B)$ is false. The moduli spaces of holomorphic spheres are trapped inside B by the maximum principle, but the dimensions are all wrong as they don't take into account the new fiber direction of E. Even for $\mathcal{O}(-k) \to \mathbb{C}P^m$, although we succeeded in [35] to compute $QH^*(E)$ for $1 \le k \le m/2$ by difficult virtual localization techniques, these calculations became intractable for $k > m/2$.

This reality check, that determining $QH^*(M)$ is a difficult problem, and that it is an important first step towards understanding the Fukaya category $\mathcal{F}(M)$, motivates our work. Although the high-profile route would be to express the Gromov–Witten invariants in terms of obstruction bundle invariants, packaged up inside a generating function, such approaches would still leave us with the task of actually calculating the obstruction bundle invariants. So in practice, strictly speaking, one hasn't computed the invariants and, for example, this route would not help in understanding how $QH^*(M)$ decomposes into eigenspaces of the $c_1(TM)$–action, which is a key step towards determining generators of $\mathcal{F}(M)$.

A key issue in constructing the representations $r, \mathcal{R}$ from diagram (2) is to ensure that the Hamiltonian S^1–actions are generated by Hamiltonians H which satisfy a certain maximum principle. In [35], we proved this holds provided $H = \text{constant} \cdot R$ at infinity, where R is the radial coordinate on the conical end. This is the same strong constraint on the class of Hamiltonians that arises in the direct limit construction of $SH^*(M)$. The existence of such an S^1–action implies that the Reeb flow at infinity needs to arise from a circle action on M. Unfortunately, already for toric negative line bundles, we prove in Section 4C that the natural rotations around the toric divisors are generated by Hamiltonians

$$H(y, R) = f(y) R \tag{5}$$

which depend on the coordinate y of the contact hypersurface Σ which defines the conical end (eg for line bundles, Σ is a sphere subbundle). In Section 2C via Appendix C we prove:

Lemma 1.3 (extended maximum principle) *The maximum principle applies to Hamiltonians of the form* (5) *for which* $f\colon \Sigma \to \mathbb{R}$ *is invariant under the Reeb flow.*

This also enlarges the class of Hamiltonians for which $SH^*(M)$ can be defined. Although it is a mild generalization, this result makes the crucial difference between whether or not the Floer-machinery can be jump-started.

1B Floer theory for noncompact toric varieties

We now describe the noncompact Fano toric varieties M for which we can define and compute $SH^*(M)$ and $QH^*(M)$. We omit from this introduction the discussion of NEF toric varieties (Section 4K) and the discussion of how our machinery applies to blow-ups of noncompact Kähler manifolds (Sections 3E–3F). The applicability is limited only by our knowledge of whether a maximum principle applies.

We will describe a family of noncompact monotone toric manifolds M for which (3) will hold, and for which the analogue of (4) is

$$SH^*(M) \cong QH^*(M)_{\mathrm{PD}[D_1],\dots,\mathrm{PD}[D_r]},$$

where we localize at the generators corresponding to the toric divisors $D_i \subset M$. Moreover we obtain a combinatorial description of $QH^*(M)$ whose localization is the Jacobian ring.

Conjecture More speculatively, our work suggests that in situations where a toric divisor is removed from a toric variety M, and one allows the Hamiltonians which define Floer cohomology to grow to infinity near the divisor, then this should correspond to localizing the ring at the r–element corresponding to rotation about the divisor.

Definition 1.4 A noncompact toric manifold (X, ω_X) is *admissible* if

(1) X is conical at infinity (see Section C1);

(2) X is monotone, so $c_1(TX) = \lambda_X [\omega_X]$ for some $\lambda_X > 0$;

(3) the rotations g_i about the toric divisors D_i (see Section 3B) are generated by Hamiltonians H_i with nonnegative slope satisfying the maximum principle (Theorem C.2 for $f \geq 0$);

(4) at least one of the H_i has strictly positive slope at infinity ($f > 0$ in Theorem C.2).

The conditions required are, essentially, what is necessary to make sense of Floer theory with current technology. The prototypical examples of admissible manifolds are Fano toric negative line bundles, which we discuss in detail later. In Sections 3E and 3F we prove that the above class of manifolds allows for blow-ups along fixed point sets.

Theorem 1.5 *Let (X, ω_X) be any admissible toric manifold, and let $\mathcal{J}$ denote the ideal generated by the linear relations* (19) *and the quantum Stanley–Reisner relations* (21)*. Then:*

(1) *There is an algebra isomorphism*

$$QH^*(X, \omega_X) \cong \Lambda[x_1, \dots, x_r]/\mathcal{J}, \quad \mathrm{PD}[D_i] \mapsto x_i.$$

(2) *The symplectic cohomology is the localization of the above ring at all* $x_i = \mathrm{PD}[D_i]$,

$$SH^*(X,\omega_X) \cong \Lambda[x_1^{\pm 1},\dots,x_r^{\pm 1}]/\mathcal{J}, \quad c^*(\mathrm{PD}[D_i]) \mapsto x_i,$$

and $c^*\colon QH^*(X) \to SH^*(X)$ *equals the canonical localization map sending* $x_i \mapsto x_i$.

(3) *The moment polytope of* X*, given by* $\Delta = \{y \in \mathbb{R}^n : \langle y, e_i\rangle \geq \lambda_i\}$*, determines an isomorphism*

$$\psi\colon QH^*(X,\omega) \to R_X/(\partial_{z_1}W,\dots,\partial_{z_n}W), \quad x_i \mapsto t^{-\lambda_i} z^{e_i},$$

which sends $c_1(TX) = \sum x_i \mapsto W = \sum t^{-\lambda_i} z^{e_i}$*, where* W *is the superpotential (Section 3C) and the ring* R_X *is described in Definition 3.8.*

(4) *The moment polytope also determines the following algebra isomorphism (see Definition 3.5 for* $\mathrm{Jac}(W)$*):*

$$SH^*(X,\omega_X) \cong \mathrm{Jac}(W), \quad x_i \mapsto t^{-\lambda_i} z^{e_i}, \quad c^*(c_1(TX)) \mapsto W.$$

(5) *The canonical map* $c^*\colon QH^*(X) \to SH^*(X)$ *corresponds to the localization map*

$$R_X/(\partial_{z_1}W,\dots,\partial_{z_n}W) \to \mathrm{Jac}(W), \quad z_i \mapsto z_i,$$

which is the quotient homomorphism by the ideal generated by the generalized 0*–eigenvectors of multiplication by all* z^{e_i} *(equivalently of all* z_i*, since* X *is smooth).*

Part of the proof involves adapting the McDuff and Tolman proof [30, Section 5] of the Batyrev presentation of $QH^*(C)$ for closed monotone toric manifolds C (reviewed in Sections 3A–3B) to the noncompact setting. Recall that the toric divisors D_i are the $\mathrm{codim}_{\mathbb{C}} = 1$ complex torus orbits and the linear relations are the classical relations they satisfy in $H_{2(\dim_{\mathbb{C}} C - 1)}(C,\mathbb{Z})$. The key observation of McDuff and Tolman is that the remaining necessary relations for the algebra $QH^*(C)$ come from applying the Seidel representation $\pi_1\mathrm{Ham}(C) \to QH^*(C)^\times$ to the relations satisfied by the natural circle rotation actions g_i about the toric divisors. In our noncompact setting, the Seidel representation is replaced by the representations $r, \mathcal{R}$ from diagram (2), and some care needs to be taken in picking "lifts" $g_i^\wedge$ of the rotations g_i because one now works with an extension $\widetilde{\pi}_1\mathrm{Ham}$ of $\pi_1\mathrm{Ham}$; Sections 2B, 4D and 5D discuss this technical issue. A key step (Lemma 4.7) is to compute the r–element for the rotations g_i,

$$r(g_i^\wedge) = x_i = \mathrm{PD}[D_i] \in QH^2(X) \quad \text{and} \quad \mathcal{R}(g_i^\wedge) = c^*\,\mathrm{PD}[D_i] \in SH^2(X)^\times.$$

This is a consequence of a more general computation which we prove in Section 2C, as follows.

Lemma 1.6 *Let (M,ω) be a monotone Kähler manifold of dimension $\dim_{\mathbb{C}} M = m$. Suppose that $g \in \pi_1 \operatorname{Ham}_{\ell\geq 0}(M)$ acts holomorphically on M, and the fixed point set*

$$D = \operatorname{Fix}(g) \subset M$$

is a complex submanifold of dimension $d = \dim_{\mathbb{C}} \operatorname{Fix}(g)$. Hence dg_t acts as a unitary map on a unitary splitting $T_x M = T_x D \oplus \mathbb{C}^{m-d}$, for $x \in D$. Suppose further that the eigenvalues of $dg_t\colon \mathbb{C}^{m-d} \to \mathbb{C}^{m-d}$ define degree 1 paths $S^1 \to S^1$; more generally, the clutching construction using dg_t defines a rank $m-d$ holomorphic vector bundle over $\mathbb{P}^1$, which by a theorem of Grothendieck splits as $\oplus\mathcal{O}(n_i)$, and for this lemma it is enough to assume that all $n_i = -1$.

Let $g^\wedge$ be the lift of g which maps the constant disc (c_x, x) to itself, for $x \in D$. Then the Maslov index $I(g^\wedge) = m-d$, and

$$r(g^\wedge) = \mathrm{PD}[\operatorname{Fix}(g)] + (\textit{higher order } t \textit{ terms}) \in QH^{2(m-d)}(M). \tag{6}$$

If the fixed point set has codimension $m-d=1$, then

$$r(g^\wedge) = \mathrm{PD}[\operatorname{Fix}(g)] \in QH^2(M). \tag{7}$$

1C Toric negative line bundles

As an illustration of the general theory of Theorem 1.5, we will now determine $QH^*(E)$, $SH^*(E)$ and the wrapped Fukaya category $\mathcal{W}(E)$ for monotone (ie Fano) toric negative line bundles $\pi\colon E \to B$, in terms of the analogous invariants for the base B (in Section 4K we also comment on the case where the line bundle is Calabi–Yau). Heuristically this is the symplectic analogue of the *quantum Lefschetz hyperplane theorem* [28]: the invariants of the hyperplane section $B \subset E$ are recovered from invariants of the ambient E and a quantum multiplication operation by an Euler class.

To clarify, E is any complex line bundle over a monotone toric manifold (B,ω_B) with

$$c_1(E) = -k[\omega_B], \quad \text{where } 0 < k < \lambda_B.$$

Recall B is monotone if $c_1(TB) = \lambda_B[\omega_B]$ for $\lambda_B > 0$, and we may assume $[\omega_B] \in H^2(B,\mathbb{Z})$ is primitive. The case $k = \lambda_B$ would make the total space Calabi–Yau, but in that case (and more generally for large k) it turns out that $SH^*(E) = 0$ by [35], also $\mathcal{W}(E)$ is not so interesting since it is homologically trivial being a module over $SH^*(E)$.

We appreciate that some may view such E as a rather basic geometric setup. We hope those readers will nevertheless appreciate that these are the first steps in the study of noncompact symplectic manifolds which are not exact, so we are trying to move beyond the case of cotangent bundles and Stein manifolds which are well-studied in the symplectic literature.

Theorem 1.7 *$QH^*(E)$ is generated by the toric divisors $x_i = \mathrm{PD}[\pi^{-1}(D_i^B)]$, and they satisfy the same linear relations and quantum Stanley–Reisner relations that the generators $x_i = \mathrm{PD}[D_i^B]$ of $QH^*(B)$ satisfy, except for a change in the Novikov variable (defined in Section 2A),*

$$t \mapsto t\,(\pi^* c_1(E))^k, \quad \text{where } \pi^* c_1(E) = -\frac{k}{\lambda_B} \sum x_i.$$

Moreover, $SH^*(E) \cong \mathrm{Jac}(W_E)$ and it is obtained by localizing $QH^*(E)$ at $\pi^* c_1(E)$.

Although the above follows from Theorem 1.5, some substantial work is involved:

- We explicitly compute the Hamiltonians generating the rotations about the toric divisors (Theorem 4.5) to verify that the maximum principle in Lemma 1.3 applies.
- We verify that the Hamiltonians involved in the quantum Stanley–Reisner (SR) relations have positive slope at infinity (a condition required for r–elements in QH^* to be defined, whereas $\mathcal{R}$–elements in SH^* do not require this); see Lemma 4.12.
- We compute the moment polytope of E in terms of the moment polytope of B (see Appendix A) to be able to compare the presentations of $QH^*(B)$ and $QH^*(E)$. We compare the linear and quantum SR relations for B and E in Corollary 4.10.
- The fact that the quantum SR relations of B and E differ by the above change of Novikov variable is proved in Theorem 4.17.

Given that the presentations of $QH^*(B)$ and $QH^*(E)$ are the same up to a change in Novikov parameter, one might hope that there is a natural homomorphism $QH^*(B) \to QH^*(E)$. However this is neither natural nor entirely correct. In the monotone (Fano) regime, using a formal Novikov variable t is not strictly necessary as there are no convergence issues in the definition of the quantum product. However, in the absence of t, the Novikov change of variables becomes meaningless. Secondly, over the usual Novikov ring Λ from Section 2A, t is invertible, but under the change of variables the image $t\,(\pi^* c_1(E))^k \in QH^*(E)$ is never invertible [35]; this is an instance of the general fact that $\mathcal{R}$–elements are invertible in SH^* but r–elements may not be invertible in QH^*. Provided we restrict the Novikov ring, we show in Theorem 4.18 that there are ring homomorphisms

$$\begin{aligned}\varphi\colon\ & QH^*(B;\mathbb{K}[t]) \to QH^*(E;\mathbb{K}[t]) \to SH^*(E;\mathbb{K}[t]),\\ \varphi\colon\ & QH^*(B;\mathbb{K}[t,t^{-1}]) \to SH^*(E;\mathbb{K}[t,t^{-1}]),\end{aligned} \tag{8}$$

defined by $x_i \mapsto x_i$ and $t \mapsto t(\pi^* c_1(E))^k$. These φ are useful in deducing relations in E from relations in B. For example, the characteristic polynomial of $[\omega_B]$ maps via φ to the characteristic polynomial of $[\omega_E] = \pi^*[\omega_B]$.

In Section 4G we compute QH^* and SH^* for $\mathcal{O}(-k) \to \mathbb{P}^m$ and $\mathcal{O}(-1,-1) \to \mathbb{P}^1 \times \mathbb{P}^1$. Recall $QH^*(\mathbb{P}^m) = \Lambda[x]/(x^{1+m} - t)$, where $x^{1+m} - t$ is the characteristic polynomial of $x = [\omega_{\mathbb{P}^m}]$; here Λ is the Novikov ring (Section 2A). Applying φ to $x^{1+m} - t$ we effortlessly deduce that

$$\begin{aligned}QH^*(\mathcal{O}_{\mathbb{P}^m}(-k)) &= \Lambda[x]/(x^{1+m} - t(-kx)^k),\\ SH^*(\mathcal{O}_{\mathbb{P}^m}(-k)) &= \Lambda[x]/(x^{1+m-k} - t(-k)^k),\end{aligned} \tag{9}$$

in the monotone regime, so $1 \le k \le m$. This is a vast improvement over the difficulty of the virtual localization methods of [35] which were only feasible for $k \le m/2$.

The φ are also crucial in Section 4J to determine the splitting of $QH^*(E)$ into eigensummands of $c_1(TE) = \lambda_E[\omega_E]$ in terms of the splitting of $QH^*(B)$ into eigensummands of $c_1(TB) = \lambda_B[\omega_B]$, assuming now that the underlying field we work over is algebraically closed. This decomposition plays a key role later in decomposing the Fukaya category of E.

Theorem 1.8 (assuming char($\mathbb{K}$) does not divide k) *The eigensummand decomposition for $QH^*(B)$, as a $\Lambda[x]$–module with x acting by multiplication by $[\omega_B]$, has the form*

$$QH^*(B) \cong \frac{\Lambda[x]}{(x^{\lambda_B} - \mu_1^{\lambda_B} t)^{d_1}} \oplus \cdots \oplus \frac{\Lambda[x]}{(x^{\lambda_B} - \mu_p^{\lambda_B} t)^{d_p}} \oplus \frac{\Lambda[x]}{x^{d_{p+1}}} \oplus \cdots \oplus \frac{\Lambda[x]}{x^{d_q}}. \tag{10}$$

It determines the following $\Lambda[x]$–module isomorphisms, where x acts as $[\omega_E] = \pi^*[\omega_B]$:

$$\begin{aligned}SH^*(E) &\cong \frac{\Lambda[x]}{[x^{\lambda_B-k} - (-k)^k \mu_1^{\lambda_B} t]^{d_1}} \oplus \cdots \oplus \frac{\Lambda[x]}{[x^{\lambda_B-k} - (-k)^k \mu_p^{\lambda_B} t]^{d_p}},\\ QH^*(E) &\cong SH^*(E) \oplus \ker(x^{\mathrm{large}}).\end{aligned} \tag{11}$$

In Ritter and Smith [36] (version 1) we conjectured that $SH^*(E)$ was nonzero for all monotone toric negative line bundles E, and we proved this for $\mathcal{O}_{\mathbb{P}^m}(-k)$, for $1 \le k \le m/2$. For more general E, this is not an immediate consequence of the explicit presentation in Theorem 1.7; presenting a ring does not mean it is easy to determine whether an element vanishes or not. By Theorem 1.1, $SH^*(E) \neq 0$ is equivalent to the condition that $\pi^* c_1(E)$ is not nilpotent in $QH^*(E)$. By Theorem 1.8, $\pi^* c_1(E) = -k\pi^*[\omega_B]$ is nonnilpotent in $QH^*(E)$ precisely if $c_1(B)$ is nonnilpotent in $QH^*(B)$. In [36] (version 1) we conjectured that for closed Fano toric varieties, $c_1(TB) \in QH^*(B)$ is always nonnilpotent, but despite the explicit Batyrev presentation of $QH^*(B)$ this was not known. In Section 6I we explain how a recent result of Galkin [18] implies this conjecture. Thus:

Corollary 1.9 *Let E be any monotone toric negative line bundle. Then $c_1(TE) \in QH^*(E)$ is nonnilpotent and $SH^*(E) \neq 0$.*

This result has significant geometric implications: we will see below that it implies E always contains a nondisplaceable monotone Lagrangian torus (Section 6I).

1D Generation results for the Fukaya category and the wrapped category

We now come back to the discussion of diagram (1). Recall that the direct limit construction which defines the morphism spaces of the wrapped category $\mathcal{W}(M)$ is typically responsible for making the Hochschild homology $\mathrm{HH}_*(\mathcal{W}(M))$ infinite-dimensional, just like $SH^*(M)$ is typically expected to be infinite-dimensional. This is in contrast to the surprising outcome in Theorem 1.5 that $SH^*(M)$ is finite-dimensional for admissible toric manifolds. The goal of the following section is to prove the open-string analogue of this (at least under a genericity assumption on the superpotential), namely cohomological finiteness for the wrapped category $\mathcal{W}(M)$. We will show that $\mathcal{W}(M)$ is split-generated by the unique compact monotone Lagrangian toric fiber L of the moment map (taken together with finitely many choices of holonomy data, determined by the superpotential), and that L is nondisplaceable.

Remark Before proceeding, it may be good to clarify what was legitimately to be expected and what was not. Applications to the existence of nondisplaceable Lagrangian toric fibers were to be expected, given the substantial work in the case of closed toric manifolds by Fukaya, Oh, Ohta and Ono [16]. In fact for negative line bundles $E \to B$, although messy, one can even explicitly compute the critical points of the superpotential of E in terms of those of B (carried out in Ritter and Smith [37]), and then general machinery due to Cho and Oh [10] implies the nondisplaceability of the Lagrangian fibers corresponding to the critical points. The novelty of this paper is another: we prove that these toric fibers actually split-generate the whole wrapped category (and not just for negative line bundles). This is surprising in the noncompact setting because the wrapped category admits noncompact Lagrangians and in general it is expected to be either cohomologically infinite or trivial, just like $SH^*(M)$. Secondly, our approach bypasses the messy computations involving the superpotential thanks to our closed-string mirror symmetry result $SH^*(M) \cong \mathrm{Jac}(W_M)$ in Theorem 1.5.

We now discuss the generation theorems. Diagram (1), in the monotone setting, is not entirely precise. The Lagrangians are typically *obstructed*: for example, in the toric case, there are always holomorphic Maslov 2 discs bounding toric Lagrangians, and these cause $\partial^2 \neq 0$ at the Floer chain level, thus Floer theory becomes problematic [15]. In fact, with current technology, one must restrict the categories $\mathcal{F}_\lambda(M), \mathcal{W}_\lambda(M)$: they are labeled by the eigenvalues λ of the action of $c_1(TM)$ on $QH^*(M)$ by quantum multiplication, and one only allows objects L with a fixed m_0–value $\lambda = m_0(L)$. Following [15] (see also [3; 37]), $m_0(L)$ is the count of holomorphic Maslov index 2 discs bounding L passing through a generic point of L:

$$\mathfrak{m}_0(L) = m_0(L)\,[L] = \lambda\,[L] \quad \text{for } L \in \mathrm{Ob}(\mathcal{F}_\lambda(M)) \text{ or } \mathrm{Ob}(\mathcal{W}_\lambda(M)).$$

Remark We emphasize that we will not undertake the onerous task of generalizing these constructions to a curved A_∞–category $\mathcal{F}(M)$, $\mathcal{W}(M)$ which mixes different m_0–values, although this task will perhaps be within reach after forthcoming work of Abouzaid, Fukaya, Oh, Ohta and Ono.

The generation criterion of Abouzaid [1], adapted to the monotone setup by Ritter and Smith [37], loosely speaking states that Lagrangians $L_1, \dots, L_m$ split-generate the category if $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{W}_\lambda(M)) \to SH^*(M)$ hits the unit when restricted to the subcategory generated by those L_j (the analogous statement holds for $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda(M)) \to QH^*(M)$).

In Ritter and Smith [37] we showed that $\mathcal{OC}$ is an $SH^*(M)$–module map (a property which was independently observed in the exact case by Ganatra [19]). Analogously, $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda(M)) \to QH^*(M)$ is a $QH^*(M)$–module map. In [37] we also proved that $\mathcal{OC}$ preserves eigensummands, so the acceleration diagram becomes

$$\begin{tikzcd} \mathrm{HH}_*(\mathcal{F}_\lambda(M)) \arrow[r] \arrow[d, "\mathcal{OC}"'] & \mathrm{HH}_*(\mathcal{W}_\lambda(M)) \arrow[d, "\mathcal{OC}"] \\ QH^*(M)_\lambda \arrow[r, "c^*"] & SH^*(M)_\lambda \end{tikzcd} \tag{12}$$

where $QH^*(M)_\lambda$ and $SH^*(M)_\lambda$ are the generalized λ–eigensummands for multiplication by $c_1(TM)$ and $c^*(c_1(TM))$ respectively. We proved in [37] that the generation criterion for the λ–piece of the category holds provided that $\mathcal{OC}$ hits an invertible element in the λ–eigensummand.

In the toric setup, since toric Lagrangians are compact and since $c^*(1) = 1$ in the acceleration diagram, $\mathcal{W}_\lambda(M)$ is split-generated by toric Lagrangians if one can show that

$$\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda(M)) \to QH^*(M)_\lambda$$

hits an invertible, so we reduce to studying holomorphic discs bounding Lagrangians. We will say that the *toric generation criterion holds for* λ if the composite

$$\mathcal{CO} \circ \mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}^{\text{toric}}_\lambda(M)) \to QH^*(M) \to HF^*(K,K)$$

hits the unit $[K] \in HF^*(K,K)$ for any Lagrangian $K \in \mathrm{Ob}(\mathcal{F}_\lambda(M))$. Here, $\mathcal{F}^{\text{toric}}_\lambda(M)$ is the subcategory of $\mathcal{F}_\lambda(M)$ generated by the toric Lagrangians (with holonomy data), and $\mathcal{CO}$ is the closed–open string map. This is in fact enough to prove split-generation; see [1; 37].

The practical reasons (see Appendix A) for wanting to work in the Fano toric setup are:

- There is machinery due to Cho and Oh [10] for producing Lagrangians: there is a nondisplaceable toric Lagrangian L_p with holonomy data for each critical point $p \in \mathrm{Crit}(W)$ of the superpotential W, and it satisfies $\lambda = m_0(L) = W(p)$.
- One has that $p \in \mathrm{Crit}(W)$ if and only if the point class $[\mathrm{pt}] \in HF^*(L_p, L_p)$ is a cycle [10; 3]. In fact it then follows by a spectral sequence argument that there is an isomorphism of vector spaces [10, Theorem 10.1]

$$HF^*(L_p, L_p) \cong H^*(L_p; \Lambda). \tag{13}$$

  This can be described explicitly and one can even recover the ring structure [8], a Clifford algebra determined by W.
- The nonconstant holomorphic discs bounding toric Lagrangians have Maslov index at least 2, and those of index 2 are combinatorially determined by the moment polytope [10] (in fact, the superpotential W can be interpreted as a count of those discs [3]).
- By Ostrover and Tyomkin [32], the Jacobian ring $\mathrm{Jac}(W)$ of a closed Fano toric variety has a field summand for each nondegenerate critical point of W (see Section 6D).

Our original goal in [37] was to prove that toric Lagrangians split-generate $\mathcal{W}(E)$, but we only succeeded in doing this when E is $\mathcal{O}(-k) \to \mathbb{P}^m$ for $1 \le k \le m/2$, as this was the only regime where $QH^*(E)$ and $SH^*(E)$ were explicitly known [35]. The steps in [37] were as follows:

- The ring $SH^*(\mathcal{O}_{\mathbb{P}^m}(-k))$ in (9) decomposes into 1–dimensional eigensummands.
- A combinatorial exercise with the superpotential W of $\mathcal{O}_{\mathbb{P}^m}(-k)$ shows that all the eigenvalues $\lambda = W(p)$ required by that decomposition do in fact arise.
- One can check that L_p is the monotone Lagrangian torus in the sphere bundle of $\mathcal{O}_{\mathbb{P}^m}(-k)$ lying over the Clifford torus in $\mathbb{P}^m$.
- Observe that the simplest component of $\mathcal{OC}$, namely

$$\mathcal{OC}^{0|0}\colon HF^*(L, L) \to QH^*(M) \tag{14}$$

  is nonzero on the point class $[\mathrm{pt}] \in HF^*(L_p, L_p)$, since there is an obvious Maslov index 2 disc bounding L_p and living in the fiber determined by the given generic point in L_p, and this disc determines the nonzero leading term of $\mathcal{OC}^{0|0}([\mathrm{pt}])$.
- Since $\mathcal{OC}$ respects eigensummands, $\mathcal{OC}$ hits a nonzero, and thus invertible, element in the 1–dimensional eigensummand $SH^*(\mathcal{O}_{\mathbb{P}^m}(-k))_\lambda$, thus split-generation holds.

Remark Given the importance of (14), we mention that this map was first studied by Albers [2]; the fact that this map is a $QH^*(M)$–module map for closed monotone manifolds M first appeared in Biran and Cornea [7]; more generally, string maps first appeared in Seidel [39].

In [37], we could not run the same argument above for more general monotone toric negative line bundles $E \to B$, due to the following issues:

(1) We needed an explicit decomposition for $SH^*(E)$, Theorem 1.1 was not enough.

(2) It is computationally unfeasible to check that the critical points of the superpotential W_E recover *all* eigenvalues arising in the decomposition for $SH^*(E)$.

(3) Ideally, one wants a condition on the superpotential W_B of base B to ensure generation for E, rather than a condition on E.

(4) Even under the conjectural assumption that $SH^*(E) \cong \mathrm{Jac}(W_E)$, and assuming W_E is Morse (so by Ostrover and Tyomkin [32], $SH^*(E)$ is then a direct sum of fields), it may still happen that an eigensummand $SH^*(E)_\lambda$ consists

of several field summands. So the above trick of "nonzero element implies invertible" would fail. This argument would only work if we also assumed that all eigenvalues were distinct, ie simple (Section 6D).

By Theorem 1.7, we solve issues (1) and (2), since

$$SH^*(E) \cong \mathrm{Jac}(W_E), \quad c^*(c_1(TE)) \mapsto W$$

ensures that the eigensummand decompositions agree. Theorem 1.8 solves issue (3); in particular, if the superpotential W_B is Morse then W_E is Morse. Issue (4) requires a general perturbation argument (Appendix B), which we also use in the general argument for admissible toric manifolds, and we explain this later. We first clarify why, in the absence of the condition that W be Morse, the generation problem is out of reach by current technology.

1E The motivation for requiring the superpotential to be Morse

In Sections 6B and 6H, we inspect the generation condition more closely. We show that $\mathcal{CO}\colon QH^*(M) \to HF^*(K,K)$, with $\lambda = m_0(K)$, will vanish on the eigensummands for eigenvalues different from λ. Moreover, $\mathcal{CO}$ vanishes on multiples of $c_1(TM)-\lambda$, thus it will vanish on multiples of $x-\lambda$ in a generalized eigensummand $\Lambda[x]/(x-\lambda)^d$ of $QH^*(M)_\lambda$. This shows that when the eigensummands of $QH^*(M)_\lambda$ are not 1–dimensional (so $d \geq 2$), then the toric generation criterion will fail if the image of $\mathcal{OC}$ consists of λ–eigenvectors. Finally, in Theorem 6.19, by a deformation argument which we explain later, we show that the map in (14) always consists of λ–eigenvectors, thus we get:

Corollary 1.10 *For closed Fano toric manifolds M and for (noncompact) admissible toric manifolds M, the map $\mathcal{OC}^{0|0}\colon HF^*(L,L) \to QH^*(M)$ lands in the $m_0(L)$–eigenspace (not just the generalized eigenspace). So if the superpotential is not Morse, then proving the toric generation criterion requires knowing higher-order components of $\mathcal{OC}$ rather than just $\mathcal{OC}^{0|0}$.*

It is for this reason that our paper will assume that the superpotential W is Morse (or, more mildly, λ–*semisimplicity*, meaning the nondegeneracy assumption at critical points p of W with eigenvalue $\lambda = W(p)$). Although this is a generic condition, it does imply the strong consequence that $QH^*(M)$ is semisimple; that is, a direct sum of fields [32]. For some time, it was conjectured that $QH^*(M)$ was always semisimple for closed toric Fano manifolds, but this was shown to be false by Ostrover and Tyomkin [32]. In the noncompact setting, one can see from (9) that $QH^*(\mathcal{O}_{\mathbb{P}^m}(-k))$ for $k \geq 2$ is not semisimple due to the generalized 0–eigensummand. In fact more generally, for monotone toric negative line bundles E, the zero-eigensummand $QH^*(E)_0$ is still rather mysterious. The toric generation criterion cannot hold for $\mathcal{F}_0(E)$ since there are no toric Lagrangians for $\lambda = 0$. This is because by (3), the superpotential does not have critical value zero; W corresponds to a nonzero multiple of $c^*(\pi^* c_1(E)) \in SH^*(E)$, which is invertible by Theorem 1.1, and $SH^*(E) \neq 0$ by Corollary 1.9.

Corollary 1.11 *The toric generation criterion fails for $\mathcal{F}_0(E)$, ie $\lambda = 0$, for any monotone toric negative line bundle.*

The perturbation theory in Appendix B, which we explain below, in fact suggests that generating a generalized eigensummand of type $\Lambda[x]/(x-\lambda)^d$ of $QH^*(M)_\lambda$ will require a limit of d eigenvectors of a deformation of W (which makes W_{deformed} Morse, and d Lagrangians with different m_0 values are involved in the limit). Thus, knowledge of $\mathcal{OC}$ up to degree d would be required. This is out of reach by current technology, because we cannot make sense of a Fukaya category which mixes different m_0–values (the $\partial^2 \neq 0$ problem). The semisimplicity assumption on λ is not problematic at the linear algebra level: Theorem B.5 shows that the deformation theory still predicts (conjecturally!) the surjectivity of the full map

$$\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}^{\text{toric}}(M)) \to QH^*(M) \to SH^*(M),$$

provided one could make sense of a Fukaya category in which Lagrangians with different m_0–values co-exist.

An explicit example for closed Fano toric manifolds, where $\mathcal{OC}^{0|0}$ is not enough to prove the toric generation criterion, is the smooth Fano 4–fold called U_8, number 116, in Batyrev's classification [6]. This is the example of Ostrover and Tyomkin [32] having nonsemisimple quantum cohomology. The eigensummand for eigenvalue $W(p) = -6$ has the form $\Lambda[x]/(x+6)^d$ for $d \geq 2$, but the only torus L_p available for split-generation for that summand has $\mathcal{OC}^{0|0}$ landing in the eigenspace by Corollary 1.10. Finally, as remarked above, $\mathcal{CO}$ vanishes on that eigenspace as these vectors are multiples of $x+6$.

1F Twisting the Fukaya category, generation via a deformation theory argument

We recall (Section 5A) that for any Fano toric variety X, near the preferred *monotone* toric symplectic form ω_X, there is a family of nonmonotone toric symplectic forms ω_F depending on the parameters $\lambda_i = F(e_i)$ which define the moment polytope $\Delta_X = \{y \in \mathbb{R}^n : \langle y, e_i \rangle \geq \lambda_i\}$ (having fixed the edges e_i of the fan of X). We show in

Sections 5C and 6A that using ω_F in Floer theory is the same as using ω_X together with a local system of coefficients determined by ω_F. A key observation is that ω_X and ω_F are both Kähler for the integrable complex structure J, so we can still use ω_X to control energies in the Floer theory for ω_F.

It is well-established that for generic ω_F (ie for generic perturbation parameters $\lambda_i = F(e_i)$), the twisted superpotential W_F is Morse (see [32], [25, Corollary 5.12], [16, Proposition 8.8]). We need to strengthen this to ensure also simplicity (ie distinct eigenvalues), in Section 6E:

Lemma 1.12 *For a generic choice of toric symplectic form ω_F (meaning, after a generic small perturbation of the values $F(e_i) \in \mathbb{R}$), W_F is Morse and the critical values are all distinct and nonzero.*

Example The twisting can have nontrivial consequences in Floer cohomology. An interesting example is $E = \mathcal{O}(-1,-1)$ over $B = \mathbb{P}^1 \times \mathbb{P}^1$ (Section 4G). The 0–eigenvectors in $QH^*(B)_0$ yield 0–eigenvectors in $QH^*(E)_0$ which get quotiented out in $SH^*(E)$. But, for generic twistings, B will not have a 0–eigensummand, so the twisted $SH^*(E)$ will have larger rank than in the untwisted case (contrary to expectations from classical Novikov homology, where twisting can only make the rank drop). See Remark 6.13.

In Sections 5E–5F we prove the twisted analogue of Theorem 1.7. Running the generation argument outlined above implies (see Section 6D):

Theorem 1.13 (generic generation theorem) *Let $E \to B$ be a monotone toric negative line bundle. Perturb ω_B to a nearby generic toric symplectic form, inducing a (nonmonotone) toric symplectic form ω_F on E having a simple Morse twisted superpotential. Then the ω_F–twisted wrapped Fukaya category $\mathcal{W}_\lambda(E)_{\omega_F}$ is split-generated by the unique monotone Lagrangian torus taken with suitable holonomies. The same holds for $\mathcal{F}_\lambda(E)_{\omega_F}$ provided $\lambda \neq 0$.*

The aim now is to use this generic generation result to obtain, in the limit as we undo the deformation, a generation result in the undeformed case. We only sketch the argument here (see Section 6F for details). For a small deformation of the parameters F, the critical points of W move continuously. Each critical point p of the undeformed W gives rise to the unique monotone Lagrangian torus fiber $L = L_p$ of the moment map together with a choice of holonomy determined by p. When we deform to W_F, the new L'_p is a Lagrangian torus fiber of the moment map close to the original monotone L, and the holonomy is also deformed. The key observation is that the holomorphic torus action of the toric variety identifies all the moduli spaces involved in the Floer theory of L'_p with those involved for L_p. So analytically, we are counting the same PDE solutions, except we count them with different holonomy weights. Since $p \in \mathrm{Crit}(W)$, the vector subspaces $\Lambda \cdot [\mathrm{pt}] \subset HF^*(L'_p, L'_p)_{\omega_F}$ can be identified even as we vary F, so in this sense the vector space map

$$\mathcal{OC}\colon \Lambda \cdot [\mathrm{pt}] \subset HF^*(L'_p, L'_p)_{\omega_F} \to QH^*(E) \cong H^*(E) \otimes \Lambda$$

can be viewed as a linear map between fixed vector spaces, which depends holomorphically on some parameter data F. Appendix B develops the necessary matrix perturbation theory to tackle this problem. The upshot is:

Theorem 1.14 *Let W_E be Morse, or more generally assume that λ is a* semisimple *eigenvalue of W (ie critical points of W_E with critical value λ are nondegenerate). Then the sum of the images of the deformed $\mathcal{OC}^{0|0}$–maps, summing over eigenvalues $\lambda' = W_F(p')$ which converge to λ, will converge to the image of the undeformed $\mathcal{OC}^{0|0}$–map. By Theorem 1.13, the sum of the deformed images is $\oplus\, QH^*(E)_{\lambda'}$ and (by Appendix B) the vector space $\oplus\, QH^*(E)_{\lambda'}$ converges to $QH^*(E)_\lambda$ in the Grassmannian. Thus the undeformed $\mathcal{OC}^{0|0}$ surjects onto $QH^*(E)_\lambda$, so it hits an invertible element, so the toric generation criterion holds.*

Remark 1.15 One cannot easily apply the above to $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda(E)) \to QH^*(E)$. The issue in this deformation theory, is that chain-level expressions which are cycles in the undeformed case are typically not cycles in the deformed case, due to the different holonomy weights. In other words, we have no analogue of (13) for $\mathrm{HH}_*(\mathrm{A}_\infty\text{–algebra for } L_p)$.

Combining Theorem 1.14 with Theorem 1.8 we deduce (see Theorem 6.17):

Corollary 1.16 *Let $E \to B$ be a monotone toric negative line bundle. If the superpotential W_B of B is Morse then the toric generation criterion holds for $\mathcal{W}(E)$. The split-generator is a nondisplaceable monotone Lagrangian torus in the sphere bundle taken with finitely many choices of holonomy data. Thus $\mathcal{W}(E)$ is compactly generated and cohomologically finite.*

More generally, if W_B is not Morse, but it has a semisimple eigenvalue $\lambda \neq 0$, then the toric generation criterion holds for $\mathcal{W}_\lambda(E)$ and $\mathcal{F}_\lambda(E)$.

We finally discuss how these generation arguments apply more generally, first for closed Fano toric varieties and then, in the noncompact case, for admissible toric manifolds.

1G Generation theorems for closed Fano toric manifolds

For closed Fano toric varieties C, considering Equation (14) on the point class $[\mathrm{pt}] \in HF^*(L_p, L_p)$ for $p \in \mathrm{Crit}(W)$, the constant disc contributes the leading term in

$$\mathcal{OC}^{0|0}([\mathrm{pt}]) = \mathrm{PD}([\mathrm{pt}]) + (\text{higher order } t \text{ terms}) \neq 0 \in QH^*(C) \tag{15}$$

(see Lemma 6.8). This does not help in the noncompact case as $\mathrm{PD}([\mathrm{pt}]) = [\mathrm{vol}_M] = 0 \in H^{\dim_{\mathbb{R}} M}(M) = 0$. The same arguments outlined in Section 1F therefore imply the following.

Theorem 1.17 *Let (C, ω_C) be a closed monotone toric manifold. If we twist quantum cohomology $QH^*(C)_\eta$ by a generic $\eta \in H^2(C; \mathbb{R})$ close to $[\omega_C]$, then $QH^*(C)_\eta$ is a direct sum of 1–dimensional eigensummands and the twisted superpotential is Morse. So the toric generation criterion applies to the twisted Fukaya category $\mathcal{F}_\lambda(C)_\eta$.*

Corollary 1.18 *For closed monotone toric manifolds (C, ω_C), if the superpotential is Morse then the toric generation criterion holds. More generally, the criterion holds for $\mathcal{F}(C)_\lambda$ for any semisimple critical value λ of the superpotential.*

Remark Forthcoming work of Abouzaid, Fukaya, Oh, Ohta and Ono aims to prove mirror symmetry for closed toric manifolds (without the Fano assumption). In particular, they will show that the toric generation criterion always applies for closed toric Fano manifolds (ie even if W is not Morse). We emphasize that our generation results in the closed Fano case are a much more modest project by comparison. Since we assume monotonicity, the technical difficulties in defining the Fukaya category are rather mild, and we have tools, such as eigensummand splittings, which would not be available in their general setup.

1H Generation results for (noncompact) admissible toric manifolds

The arguments outlined above hold more generally for admissible toric manifolds M (Definition 1.4), since $SH^*(M) \cong \mathrm{Jac}(W_M)$ by Theorem 1.5. The only issue is the calculation of $\mathcal{OC}^{0|0}([\mathrm{pt}])$ in (14). The key result from Section 6C is the following.

Lemma 1.19 *For an admissible toric manifold M, $c_1(TM)$ admits a* compact *cycle representative $C = \mathrm{PD}(c_1(TM))$. After a choice of basis of cycles (which affects the other terms in the expansion below), C determines an lf-cycle $C^\vee$ which is intersection-dual to C. Then*

$$\mathcal{OC}^{0|0}([\mathrm{pt}]) = m_0(L)\ \mathrm{PD}(C^\vee) + (\textit{linearly independent terms}) \in QH^*(M).$$

In particular, for $\lambda = m_0(L) \neq 0$, we deduce $\mathcal{OC}^{0|0}([\mathrm{pt}]) \neq 0$.

At the heart of this is a fact due to Kontsevich, Seidel and Auroux [3]: for monotone toric manifolds M, and a toric Lagrangian L which does not intersect a representative of $c_1(TM)$, the holomorphic Maslov index 2 discs bounding L passing through a generic point of L will hit $c_1(TM)$ once at an interior point. This fact is responsible for the equation $\mathcal{CO}(c_1(TM)) = c_1(TM) * [L] = m_0(L)\,[L]$. The count of discs in this equation is in fact the same as the count involved in the coefficient of $\mathrm{PD}(C^\vee)$ above (so, succinctly, $\langle \mathcal{OC}([\mathrm{pt}]), C\rangle = \langle [\mathrm{pt}], \mathcal{CO}(C)\rangle = m_0(L)$). It is crucial that $c_1(TM)$ has a compact cycle representative C, and not just a locally finite cycle (in the noncompact setting, one struggles to obtain a duality relationship between $\mathcal{OC}$ and $\mathcal{CO}$ precisely because cycles and lf-cycles are generally unrelated). We prove in Lemma 6.4 that for orientable Lagrangian submanifolds L, one can always find a representative of $c_1(TM)$ disjoint from L (this fails for nonorientable Lagrangians, such as $\mathbb{RP}^2 \subset \mathbb{CP}^2$, but our Fukaya categories only allow orientable Lagrangians, following [37]).

The deformation argument, which ensures that the twisted superpotential becomes Morse with simple eigenvalues, combined with the above lemma implies:

Theorem 1.20 *Let M be any (noncompact) admissible toric manifold M (as in Definition 1.4), and let ω_F be a generic toric (nonmonotone) symplectic form close to the monotone form ω_M. For any nonzero eigenvalue λ of $c_1(TM) \in QH^*(M, \omega_M)$ the ω_F–twisted Fukaya categories $\mathcal{W}_\lambda(M)_{\omega_F}$ and $\mathcal{F}_\lambda(M)_{\omega_F}$ are split-generated by the unique monotone Lagrangian torus taken with suitable holonomies determined by the superpotential W_F.*

The matrix perturbation argument from Section 6F then implies:

Corollary 1.21 *Let M be any (noncompact) admissible toric manifold. If W_M is Morse and $\lambda \neq 0$, then the toric generation criterion holds for $\mathcal{W}_\lambda(M)$ and $\mathcal{F}_\lambda(M)$. In particular, $\mathcal{W}_\lambda(M)$ is compactly generated and cohomologically finite. This also holds for non-Morse W_M provided λ is a semisimple eigenvalue.*

1I Mirror symmetry

Finally we comment on (3) in the light of mirror symmetry. Ganatra [19] showed that for exact symplectic manifolds X conical at infinity, if $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{W}(X)) \to SH^*(X)$ hits a unit, then the closed–open string map $\mathcal{CO}\colon SH^*(X) \to \mathrm{HH}^*(\mathcal{W}(X))$ is an isomorphism of rings. Significant work would be involved in showing that this also holds for monotone X, but let us suppose it does for the sake of argument. Then, for admissible toric manifolds M with Morse superpotential W_M, it would follow that

$$SH^*(M) \cong \mathrm{HH}^*(\mathcal{W}(M)) \cong \mathrm{HH}^*(\mathcal{F}^{\mathrm{toric}}(M)),$$

where the latter is the subcategory of the (compact) Fukaya category generated by the toric Lagrangians. The assumption that W_M is Morse allows one to identify (noncanonically) the semisimple categories $\mathcal{F}^{\text{toric}}(M)$ and $\mathrm{Mat}(W_M)$, the category of matrix factorizations of W_M; in fact, more generally, the work of Abouzaid, Fukaya, Oh, Ohta and Ono will construct a canonical A_∞–functor $\mathcal{F}(C) \to \mathrm{Mat}(W_C)$ for closed toric varieties C. In particular,

$$\mathrm{HH}^*(\mathcal{F}^{\text{toric}}(M)) \cong \mathrm{HH}^*(\mathrm{Mat}(W_M)).$$

For Morse W_M, it follows by Dycherhoff [13] that $\mathrm{HH}^*(\mathrm{Mat}(W_M)) \cong \mathrm{Jac}(W_M)$. Thus, glossing over these speculative identifications, mirror symmetry confirms that $SH^*(M) \cong \mathrm{Jac}(W_M)$.

Acknowledgement I thank Mohammed Abouzaid for spotting the subtle issue about cycles explained in Remark 1.15, which fixed a mistake in the original version of this paper.

2 Construction of invertible elements in the symplectic cohomology

2A The Novikov ring

We will always work over the field

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i t^{n_i} : a_i \in \mathbb{K},\, n_i \in \mathbb{R},\, \lim n_i = \infty \right\},$$

called the *Novikov ring*, where $\mathbb{K}$ is some chosen ground field, and t is a formal variable. We will mostly be concerned with *monotone* symplectic manifolds (M,ω), meaning

$$c_1(TM)[u] = \lambda_M \omega[u]$$

for spheres $u \in \pi_2(M)$. For toric manifolds $\pi_1(M) = 1$, so the condition is equivalent to requiring $c_1(TM) = \lambda_M[\omega]$. For monotone manifolds, the Novikov ring is graded by placing t in (real) degree $|t| = 2\lambda_M$. We often also use a Novikov variable T in degree $|T| = 2$, so

$$T = t^{1/\lambda_M}.$$

In all Floer constructions, solutions are counted with Novikov weights. For example, in the definition of the quantum cohomology, holomorphic spheres u are counted with weight

$$t^{\omega[u]} = T^{c_1(TM)[u]}$$

lying in degree $2\lambda_M \omega[u] = 2c_1(TM)[u]$.

2B Review of the construction of invertibles in symplectic cohomology

Let M be a symplectic manifold conical at infinity (see Section C1), satisfying *weak+ monotonicity*; that is, at least one of the following conditions holds:

(1) $\omega(\pi_2(M)) = 0$ or $c_1(TM)(\pi_2(M)) = 0$.

(2) M is monotone: $\exists\, \lambda > 0$ with $\omega(A) = \lambda c_1(TM)(A)$, $\forall A \in \pi_2(M)$.

(3) The minimal Chern number $|N| \geq \dim_{\mathbb{C}} M - 1$.

That one of these conditions holds is equivalent to the condition:

$$\text{If } \omega(A) > 0, \text{ then } 2 - \dim_{\mathbb{C}} M \leq c_1(TM)(A) < 0 \text{ is false.}$$

One can further weaken this assumption by only requiring this for *effective* classes A, that is, those which are represented by a J–holomorphic sphere (see [35]). Weak+ monotonicity ensures Floer theory is well-behaved by "classical" arguments (see Hofer and Salamon [24]).

In what follows, $SH^*(M)$ denotes the symplectic cohomology of M (see Section C5), restricting to only contractible loops (which is everything when $\pi_1(M) = 1$, eg for toric M).

When working in the monotone setting, we work over the Novikov ring from Section 2A. Otherwise, we work over the Novikov ring $\mathfrak{R}$ from Section 5B, or over the Novikov ring defined in [35] (see the technical remark in Section 5C).

Theorem 2.1 (Ritter [35]) *We can construct a homomorphism*

$$\mathcal{S}\colon \widetilde{\pi}_1 \operatorname{Ham}_\ell(M,\omega) \to SH^*(M)^\times, \quad \widetilde{g} \mapsto \mathcal{S}_{\widetilde{g}}(1)$$

into invertible elements of the symplectic cohomology, where Ham_ℓ *refers to Hamiltonian diffeomorphisms generated by Hamiltonians* $K = K_t$ *which are linear in* R *for large* R*:*

$$K_t = \kappa_t R$$

(negative slopes $\kappa_t < 0$ *are also allowed).*

The symbol $\widetilde{\pi}_1$ refers to an extension of the usual π_1–group by the group $\Gamma = \pi_2(M)/\pi_2(M)_0$, where $\pi_2(M)_0$ is the subgroup of spheres on which both ω and $c_1(TM)$ vanish. As this will be important in the application in Section 4C, we now explain this (see [38] for details). A loop $g\colon S^1 \to \operatorname{Ham}_\ell(M,\omega)$ acts on the space $\mathcal{L}_0 M$ of contractible free loops by $(g \cdot x)(t) = g_t \cdot x(t)$.

Technical remark (Note that for M simply connected, eg toric M, this issue doesn't arise.) The proof in [38, Lemma 2.2] that $g \cdot x$ is contractible used the Arnol'd conjecture. In our case, if this failed for a (time-dependent!) Hamiltonian K linear at infinity, then $SH^*(M) = 0$ as the unit $c^*(1)$ would vanish ($c^*\colon QH^*(M) \to HF^*(K) = 0 \to SH^*(M)$ factorizes). One can check this is consistent with the construction of the $r, \mathcal{R}$–element for g in diagram (2), which turns out to be zero; this is because the construction would factorize through a map $CF^*(g^*H_0) \to CF^*(H_0)$ where one restricts to loops in the class of noncontractible loops $g^{-1} \cdot \mathcal{L}_0(M)$, but here H_0 is a C^2–small Hamiltonian so its 1–orbits are contractible. In most applications this does not arise: the S^1–action g will typically have fixed points, therefore g must preserve the connected component $\mathcal{L}_0 M \subset \mathcal{L}M$ of contractible loops.

Let $\widetilde{\mathcal{L}}_0 M$ denote the (connected) cover of $\mathcal{L}_0 M$ which is given by pairs (v, x) where $v\colon \mathbb{D}^2 \to M$ is a smooth disc with boundary $x \in \mathcal{L}_0 M$, where we identify two pairs if both ω and $c_1(TM)$ vanish on the sphere obtained by gluing together the discs. Then the action of g on $\mathcal{L}_0 M$ lifts to a continuous map

$$\widetilde{g}\colon \widetilde{\mathcal{L}}_0 M \to \widetilde{\mathcal{L}}_0 M$$

which is uniquely determined by the image of a constant disc (c_x, x) at a constant loop $x \in M$. Any two such lifts differ by an element in the deck group Γ. The above $\widetilde{\pi}_1$ Ham group is $\pi_0 \widetilde{G}$ of the group $\widetilde{G}$ of these choices of lifts $\widetilde{g}$.

The Λ–module automorphism $\mathcal{S}_{\widetilde{g}} \in \mathrm{Aut}(SH^*(M))$, which turns out to be the pair-of-pants product by $\mathcal{S}_{\widetilde{g}}(1) \in SH^*(M)^\times$, arises from the chain level isomorphism

$$\mathcal{S}_{\widetilde{g}}\colon HF^*(H, J, \omega) \to HF^{*+2I(\widetilde{g})}(g^*H, g^*J, \omega), \quad x \mapsto \widetilde{g}^{-1} \cdot x$$

given by pulling back all the Floer data by $\widetilde{g}$, where $I(\widetilde{g})$ is a Maslov index (see [35]) and

$$g^*H(m,t) = H(g_t \cdot m, t) - K(g_t \cdot m, t), \quad g^*J = dg_t^{-1} \circ J_t \circ dg_t.$$

The direct limit of the maps $\mathcal{S}_{\widetilde{g}}$ as the slope of H at infinity is increased yields a Λ–module automorphism $\mathcal{S}_{\widetilde{g}}\colon SH^*(M) \to SH^*(M)$ with inverse $\mathcal{S}_{\widetilde{g}^{-1}}$.

The analogous construction for closed symplectic manifolds (C, ω) is a well-known argument due to Seidel [38]. For closed C, the Floer cohomologies are all isomorphic to $QH^*(C)$, and so the above chain isomorphism turns into a quantum product by a quantum invertible. The above homomorphism would then be the well-known *Seidel representation*

$$\widetilde{\pi}_1 \,\mathrm{Ham}(C, \omega) \to QH^*(C)^\times.$$

In the noncompact setup, however, the situation is quite different. The groups $HF^*(H)$ depend dramatically on the slope of H at infinity. Only when the slope of K at infinity is positive is it possible by [35] to obtain a typically *noninvertible* element in $QH^*(M)$ as follows. Write $\mathrm{Ham}_{\ell\geq 0}$ to mean that we impose additionally the condition that the slope $\kappa_t \geq 0$ above. Any $g\colon S^1 \to \mathrm{Ham}_{\ell\geq 0}(M,\omega)$ gives rise to an endomorphism

$$\mathcal{R}_{\widetilde{g}} = \varphi_H \circ \mathcal{S}_{\widetilde{g}}\colon HF^*(H,J) \to HF^{*+2I(\widetilde{g})}(H,J), \tag{16}$$

where φ_H is the continuation map

$$\varphi_H\colon HF^*(g^*H, g^*J,\omega) \to HF^*(H,J,\omega).$$

In the direct limit, as the slope of H grows to infinity, this yields the Λ–module automorphism

$$\mathcal{R}_{\widetilde{g}} = \mathcal{S}_{\widetilde{g}}\colon SH^*(M) \to SH^{*+2I(\widetilde{g})}(M).$$

The reason for the positive slope condition on K is that g^*H will have smaller slope than H, which ensures that the continuation map φ_H exists, in particular the continuation map for $H = H_0$ of very small slope at infinity will exist.

Recall that for such H_0 of small slope, $QH^*(M) \cong HF^*(H_0)$ and there is a canonical Λ–algebra homomorphism

$$c^*\colon QH^*(M) \cong HF^*(H_0) \to \varinjlim HF^*(H) = SH^*(M).$$

Taking $H = H_0$ in (16) determines a Λ–module homomorphism

$$r_{\widetilde{g}}\colon QH^*(M) \to QH^{*+2I(\widetilde{g})}(M),$$

which turns out to be the quantum cup product by $r_{\widetilde{g}}(1)$.

Theorem 2.2 (Ritter [35]) *The homomorphism*

$$r\colon \widetilde{\pi}_1(\mathrm{Ham}_{\ell\geq 0}(M,\omega)) \to QH^*(M), \quad \widetilde{g} \mapsto r_{\widetilde{g}}(1)$$

fits into the commutative diagram (2).

The key observation is that when the slope $\kappa_t > 0$, the map $r_{\widetilde{g}}$ in fact determines $SH^*(M)$.

Theorem 2.3 (Ritter [35]) *Given $g\colon S^1 \to \mathrm{Ham}_{\ell>0}(M,\omega)$, the canonical map $c^*\colon QH^*(M) \to SH^*(M)$ induces an isomorphism of Λ–algebras*

$$SH^*(M) \cong QH^*(M)/(\text{generalized } 0\text{–eigenspace of } r_{\widetilde{g}}),$$

where we recall that the generalized 0–eigenspace is $\ker r_{\widetilde{g}}^d$ for any $d \geq \operatorname{rank} H^*(M)$.

This follows from the previous theorem as follows. Given a Hamiltonian H_0 with very small slope at infinity, one defines

$$H_k = (g^{-k})^* H_0,$$

which has slope proportional to k at infinity: $\mathrm{slope}(H_0) + k \cdot \mathrm{slope}(K)$. Here we use that we work in $\mathrm{Ham}_{\ell>0}$ so that $\mathrm{slope}(K) > 0$. Notice that $g^* H_k = H_{k-1}$.

The maps $\mathcal{S}_{\widetilde{g}}^k$ identify $HF^*(H_k) \cong HF^{*+2kI(\widetilde{g})}(H_0)$; more precisely,

$$S_{\widetilde{g}}^k = S_{\widetilde{g}^k}\colon HF^*((g^{-k})^* H_0, (g^{-k})^* J) \xrightarrow{\cong} HF^{*+2kI(\widetilde{g})}(H_0, J),$$

and recall that $HF^*(H, J)$ only depends on the slope of H and it does not depend on the choice of ω–compatible almost complex structure J of contact type at infinity.

By a naturality argument, this implies that the continuation map $HF^*(H_{k-1}) \to HF^*(H_k)$ can be identified with $\mathcal{S}_{\widetilde{g}}^{-k} \circ \varphi_{H_0} \circ \mathcal{S}_{\widetilde{g}}^k$. So, after identifying

$$HF^{*-2kI(\widetilde{g})}(H_k) \cong HF^*(H_0) \cong QH^*(M),$$

all continuation maps $HF^*(H_k) \to HF^*(H_{k+1})$ are identified with the map

$$r_{\widetilde{g}}\colon QH^*(M) \to QH^{*+2I(\widetilde{g})}(M).$$

The theorem then follows by linear algebra.

2C Invertibles in the symplectic cohomology for a larger class of Hamiltonians

In Appendix C we will strengthen the maximum principle:

Theorem 2.4 (extended maximum principle) *Let $H\colon M \to \mathbb{R}$ have the form*

$$H(y, R) = f(y)R$$

for large R, where $(y, R) \in \Sigma \times (1, \infty)$ are the collar coordinates, and $f\colon \Sigma \to \mathbb{R}$ satisfies

- *f is invariant under the Reeb flow (that is, $df(Y) = 0$ for the Reeb vector field Y);*
- *$f \geq 0$.* [this condition can be omitted if $d\beta = 0$]

Let J be an ω–compatible almost complex structure of contact type at infinity. Then the maximum principle holds for the R–coordinate of any Floer solution.

The above also holds for the parametrized maximum principle (Theorem C.11) so it allows us to compute $SH^*(M)$ for a larger class of Hamiltonians (Corollary C.13):

Theorem 2.5 *Let $H_j\colon M \to \mathbb{R}$ be Hamiltonians of the form $H_j = m_j R$ for large R, whose slopes $m_j \to \infty$, and let J be of contact type at infinity. Suppose g_t is the flow for a Hamiltonian of the form $K = f(y)R$ for large R, with $f\colon \Sigma \to \mathbb{R}$ invariant under the Reeb flow. Then the pairs $(g^* H_j, g^* J)$ compute symplectic cohomology:*

$$SH^*(M) = \lim_{j\to\infty} HF^*(g^* H_j, g^* J)$$

Theorem 2.6 *Theorems 2.1, 2.2 and 2.3 also hold when we enlarge $\mathrm{Ham}_\ell(M)$, $\mathrm{Ham}_{\ell\geq 0}(M)$, $\mathrm{Ham}_{\ell>0}(M)$ to allow for generating Hamiltonians K of the form*

$$K_t(y,R) = f_t(y)R$$

for large R, where $(y,R)\in\Sigma\times[1,\infty)$ are the collar coordinates, and where $f_t\colon \Sigma\to\mathbb{R}$ is invariant under the Reeb flow. In the cases $\ell\geq 0$, $\ell>0$ we require $f_t\geq 0$, $f_t>0$ respectively.

Proof The construction of $\mathcal{S}_{\tilde{g}}\colon HF^*(H_k, J, \omega) \to HF^{*+2I(\tilde{g})}(g^* H_k, g^* J, \omega)$ is never an issue, since it is an identification at the chain level. However, to obtain a map $\mathcal{S}_{\tilde{g}}\colon SH^*(M) \to SH^{*+2I(\tilde{g})}(M)$ from this, by a direct limit argument, it is necessary that the pairs $(g^* H_k, g^* J)$ can be used to compute $SH^*(M)$ when we let the slopes $k\to\infty$. The previous two theorems enable us to do this. The proof of Theorem 2.1 then follows by the same arguments as in [35].

Theorem 2.2 follows similarly: for $f\geq 0$ the slope $m-f(y)$ of $g^* H_m = (m-f(y))R$ is smaller than m, so the maps φ_H still exist (see Section C5). Finally, the argument described under Theorem 2.3 can still be carried out, since the Hamiltonians $H_k = (g^{-k})^* H_0$ have slope proportional to k at infinity: $\mathrm{slope}(H_0) + k\cdot\mathrm{slope}(K)$, where $\mathrm{slope}(K) = f(y) > 0$. □

Proof of Lemma 1.6 We mimic the argument in [35, Section 10]. By [35], $r_{g^\wedge}(1)\in QH^*(M)$ is a count of $(j,\hat{J})$–holomorphic sections of the bundle $E_g\to\mathbb{P}^1$ whose fiber is M and whose transition map over the equator of $\mathbb{P}^1$ is g, where $\hat{J}$ is an almost complex structure on E_g related to J. The moduli spaces $\mathcal{S}(j,\hat{J},\gamma+S_{g^\wedge})$ of such sections break up according to certain equivalence classes $\gamma+S_{g^\wedge}$ indexed by $\gamma\in\pi_2(M)/\pi_2(M)_0$.

Technical remark The class $\mathcal{S}_{g^\wedge}$ is represented by the constant section at $x\in\mathrm{Fix}(g)$, as it is obtained by gluing the following two sections on the upper/lower hemispheres [35]: $s^+_{g^\wedge}(z) = c_x(z) = x$, $s^-_{g^\wedge}(z) = (g^\wedge\cdot c_x)(z) = c_x(z) = x$, using the hypothesis $g^\wedge\cdot c_x = c_x$.

The virtual dimension of the moduli space is

$$\dim_{\mathbb{C}} \mathcal{S}(j, \widehat{J}, \gamma + S_{g^\wedge}) = \dim_{\mathbb{C}} M - I(g^\wedge) + c_1(TM)(\gamma). \tag{17}$$

As M is monotone, $c_1(TM)(\gamma) \geq 0$, so the first term in Equation (6) arises as the locally finite cycle swept by the constant sections (so $\gamma = 0$). Indeed, the constant sections must lie in $\mathrm{Fix}(g)$ otherwise the transition map g would make them nonconstant. If we can show that the constant sections are regular, then Equation (6) follows.

To prove regularity of the constant sections we mimic [35, Section 10]. For a constant section $u\colon \mathbb{P}^1 \to E_g$, the vertical tangent space is

$$(u^* T^v E_g)_z = TM_{u(z)} \cong T_{u(z)} D \oplus \mathbb{C}^{m-d}.$$

Here we use that dg_t is complex linear and g_t is symplectic to deduce that dg_t is unitary, so it preserves the unitary complement $\mathbb{C}^{m-d}$ of the fixed subspace TD. As we vary $z \in \mathbb{P}^1$, the transition g along the equator acts identically on TD and it acts by dg_t on $\mathbb{C}^{m-d}$. By the assumption on the eigenvalues of dg_t, the $\mathbb{C}^{m-d}$ summand gives rise to the bundle $\mathcal{O}(-1)^{\oplus d} \to \mathbb{P}^1$. Essentially by definition, using that $g^\wedge$ fixes (c_x, x), one has that $I(g^\wedge) = \deg(t \mapsto \det(dg_t\colon T_x M \to T_x M))$, which is the sum $m-d$ of the degrees of the eigenvalues. Thus $u^* T^v E_g \cong \mathcal{O}^d \oplus \mathcal{O}(-1)^{\oplus m-d}$. So the obstruction bundle is, using Dolbeault's theorem and Serre duality,

$$\operatorname{coker} \overline{\partial} = H^1(\mathbb{P}^1, \mathcal{O}^d \oplus \mathcal{O}(-1)^{\oplus m-d}) \cong H^0(\mathbb{P}^1, \mathcal{O}(-2)^{\oplus d} \oplus \mathcal{O}(-1)^{\oplus m-d})^\vee = 0.$$

For $I(g^\wedge) = m-d = 1$ we have $\dim_{\mathbb{C}} \mathcal{S}(j, \widehat{J}, \gamma + S_{g^\wedge}) \geq \dim_{\mathbb{C}} M - 1$, but by the maximum principle the locally finite pseudocycle $\mathcal{S}(j, \widehat{J}, \gamma + S_{g^\wedge})$ cannot sweep the unbounded manifold M. This forces $c_1(TM)(\gamma) = 0$ and so, by monotonicity, that $\gamma = 0$. So Equation (7) follows.

We remark that in [35] a larger Novikov ring was used than here, which kept track of the class γ in the count of sections. But in the monotone case it suffices to keep track of $\omega(\gamma)$ and we count sections with weight $t^{\omega(\gamma)}$. For constant sections, this weight is 1. □

3 The quantum cohomology of noncompact monotone toric manifolds

3A Review of the Batyrev–Givental presentation of QH*(M) for closed toric monotone manifolds M

In this section, M will be a *closed* monotone toric manifold. Let Δ be a moment polytope for M, and let F be the corresponding fan (we recall some of this terminology in Appendix A). Then there are correspondences

$$\begin{aligned}\{\text{edges } e_i \text{ of } F\} &= \{\text{inward normals } e_i \text{ to facets of } \Delta\}\\ &\leftrightarrow \{\text{facets } F_i \text{ of } \Delta\}\\ &\leftrightarrow \{\text{toric divisors } D_i\}\\ &\leftrightarrow \{\text{homogeneous coordinates } x_i\},\end{aligned}$$

and in particular $D_i = \{x_i = 0\}$. Recall that *facet* means $\operatorname{codim}_{\mathbb{R}} = 1$ face.

Example For $\mathbb{CP}^2$ one has $\Delta = \{y \in \mathbb{R}^2 : y_1 \geq 0, y_2 \geq 0, -y_1 - y_2 \geq -1\}$, the fan F has edges $e_1 = (1,0)$, $e_2 = (0,1)$, $e_3 = (-1,-1)$, and the cones are the $\mathbb{R}_{\geq 0}$–span of proper subsets of the edges.

We work over a field $\mathbb{K}$ of characteristic zero. Then the classical cohomology is

$$H^*(M) = \mathbb{K}[x_1, \ldots, x_r] \Big/ \begin{pmatrix}\text{linear relations,}\\ \text{classical Stanley–Reisner relations}\end{pmatrix}. \tag{18}$$

The linear relations are

$$\sum \langle \xi, e_i \rangle \, x_i = 0, \tag{19}$$

where ξ runs over the standard basis of $\mathbb{R}^n$ and $\langle \cdot, \cdot \rangle$ is the standard inner product on $\mathbb{R}^n$.

Example 3.1 For $\mathbb{CP}^2$ one has $x_1 - x_3 = 0$, $x_2 - x_3 = 0$, so this identifies all x_i with a variable x.

Definition 3.2 (primitive subset) A subset of indices $I = \{i_1, \ldots, i_a\}$ for the edges is called *primitive* if the subset $\{e_{i_1}, \ldots, e_{i_a}\}$ does not define a cone of F but any proper subset does; equivalently, if $F_{i_1} \cap \cdots \cap F_{i_a} = \varnothing$ but for any proper subset of I the intersection is nonempty; equivalently, if $D_{i_1} \cap \cdots \cap D_{i_a} = \varnothing$ but for any proper subset of I the intersection is nonempty.

The classical SR relations are $x_{i_1} \cdot x_{i_2} \cdots x_{i_a} = 0$ for primitive I.

Example For $\mathbb{CP}^2$ the subset $\{e_1, e_2, e_3\}$ is primitive, so $x_1 x_2 x_3 = 0$. Therefore $H^*(\mathbb{CP}^2) = \mathbb{K}[x]/(x^3)$.

The quantum cohomology is

$$QH^*(M) = \mathbb{K}[x_1, \ldots, x_r] \Big/ \begin{pmatrix}\text{linear relations,}\\ \text{quantum Stanley–Reisner relations}\end{pmatrix}. \tag{20}$$

Batyrev [5] showed that to each primitive subset $I = \{i_1, \dots, i_a\}$ there corresponds a unique disjoint subset $j_1, \dots, j_b$ of indices, yielding a linear dependence relation

$$e_{i_1} + \ldots + e_{i_a} = c_1 e_{j_1} + \cdots + c_b e_{j_b}$$

for (nonzero) positive $c_1, \dots, c_b \in \mathbb{Z}$.

Remark This uses the compactness assumption, that the cones cover $\mathbb{R}^n$. Thus $\sum e_{i_k}$ lies in a cone and so is uniquely a nonnegative linear combination of the edges e_{j_ℓ} defining that cone.

The quantum version of the SR relation is then

$$x_{i_1} x_{i_2} \cdots x_{i_a} = t^{\omega(\beta_I)} x_{j_1}^{c_1} \cdots x_{j_b}^{c_b}, \tag{21}$$

where the $\beta_I \in H_2(M)$ arising in the exponent of the Novikov variable t is the class obtained from $e_{i_1} + \cdots + e_{i_a} - c_1 e_{j_1} - \cdots - c_b e_{j_b} = 0$ under the identification

$$(\mathbb{Z}\text{–linear relations among the } e_i) \longleftrightarrow H_2(M, \mathbb{Z}),$$

where explicitly, $\sum n_i e_i = 0$ corresponds to the $\beta \in H_2(M, \mathbb{Z})$ whose intersection products with the toric divisors are $\beta \cdot D_i = n_i$. In particular, Batyrev showed that $\omega(\beta_I) > 0$.

Example For $\mathbb{CP}^2$ one has $e_1 + e_2 + e_3 = 0$ and $\beta_{1,2,3} = [\mathbb{CP}^1]$, so $x_1 x_2 x_3 = t$. So $QH^*(\mathbb{CP}^2) = \mathbb{K}[x]/(x^3 - t)$.

Recall that the polytope is defined by the edges e_i of the fan and by real parameters λ_i,

$$\Delta = \{y \in \mathbb{R}^n : \langle y, e_i \rangle \geq \lambda_i\}. \tag{22}$$

Since $c_1(TM) = \sum \mathrm{PD}[D_i]$ and $[\omega] = -\sum \lambda_i \, \mathrm{PD}[D_i]$, via (20) we have

$$c_1(TM) = \sum x_i, \quad [\omega] = -\sum \lambda_i = x_i,$$

from which it follows that

$$\begin{aligned} c_1(TM)(\beta_I) &= a - c_1 - \cdots - c_b, \\ [\omega](\beta_I) &= -\lambda_{i_1} - \cdots - \lambda_{i_a} + c_1 \lambda_{j_1} + \cdots + c_b \lambda_{j_b}. \end{aligned} \tag{23}$$

Example For $\mathbb{CP}^2$,

$$(\lambda_1, \lambda_2, \lambda_3) = (0, 0, -1), \quad c_1(TM) = 3x, \quad [\omega] = x$$

in $H^*(\mathbb{CP}^2) = \mathbb{K}[x]/(x^3)$.

3B Review of the McDuff–Tolman proof of the presentation of $QH^*(M)$

For each tuple $(n_1, \dots, n_r) \in \mathbb{Z}^r$ there is an associated Hamiltonian S^1–action given in homogeneous coordinates by $x_i \mapsto e^{2\pi\sqrt{-1}n_i t} x_i$. The generators, corresponding to the standard basis of $\mathbb{Z}^r$, are the natural rotations g_i about the toric divisors D_i, yielding correspondences

$\{\text{edges } e_i\} \leftrightarrow \{\text{toric divisors } D_i\} \leftrightarrow \{\text{homogeneous coordinates } x_i\} \leftrightarrow \{\text{rotations } g_i\}$.

In particular, $D_i = (x_i = 0) = \mathrm{Fix}(g_i)$ are the fixed points of g_i.

The Hamiltonians H_i for the g_i are given in terms of the moment map $\mu\colon M \to \mathbb{R}^n$ (tacitly identifying $\mathbb{R}^n = \mathfrak{u}(1)^n$) and the data which defines Δ:

$$H_i(x) = \langle \mu(x), e_i \rangle - \lambda_i = \tfrac{1}{2}|x_i|^2, \tag{24}$$

where the constant λ_i ensures that $H_i \geq 0$ and $D_i = H_i^{-1}(0)$. We emphasize that the final equality in (24) holds only for $x \in f^{-1}(0)$, using the notation of Section A3.

Example For $\mathbb{CP}^2$ with $[x_1 : x_2 : x_3]$ the usual coordinates,

- g_i is the standard rotation in the i^{th} entry,
- the moment map is $\mu = (|x_1|^2, |x_2|^2)/\sum |x_j|^2$, and
- $H_i = |x_i|^2/\sum |x_j|^2$.

In the notation of Section A3, $x \in f^{-1}(0)$ imposes $f(x) = \frac{1}{2}\sum |x_j|^2 - 1 = 0$, so then $H_i = \frac{1}{2}|x_i|^2$.

For a closed monotone toric manifold (M, ω), the Seidel representation [38]

$$\mathcal{S}\colon \pi_1 \mathrm{Ham}(M, \omega) \to QH^*(M)^\times$$

is a homomorphism into the even degree invertible elements of $QH^*(M)$.

Theorem 3.3 (McDuff and Tolman [30]) *For a closed monotone toric manifold M, the rotations g_i yield Seidel elements*

$$\mathcal{S}(g_i) = t^{\lambda_i} x_i.$$

Via the correspondences

$$\begin{aligned}
(\text{primitive } I) &\leftrightarrow \{\text{relations among edges } e_{i_1}+\cdots+e_{i_a}=c_1e_{j_1}+\cdots+c_be_{j_b}\}\\
&\leftrightarrow \left\{\begin{matrix}\text{relations among Hamiltonians}\\ H_{i_1}+\cdots+H_{i_a}=c_1H_{j_1}+\cdots+c_bH_{j_b}+\text{const.}\end{matrix}\right\}\\
&\leftrightarrow \{\text{relations among rotations } g_{i_1}\cdots g_{i_a}=g_{j_1}^{c_1}\cdots g_{j_b}^{c_b}\}\\
&\leftrightarrow \{\text{quantum SR relations } x_{i_1}x_{i_2}\cdots x_{i_a}=t^{\omega(\beta_I)}x_{j_1}^{c_1}\cdots x_{j_b}^{c_b}\},
\end{aligned}$$

the Seidel homomorphism $\mathcal{S}$ recovers the quantum SR relations (21) using (23).

The final ingredient of the argument of McDuff and Tolman is the following useful algebraic trick. The lemma in fact does not involve toric geometry and is stated in greater generality in [30]. It holds for any closed monotone symplectic manifold (M,ω) taking $\varphi(x_i)$ to be a choice of algebraic generators for $H^*(M)$, and it also holds in the noncompact monotone setup if one can construct quantum cohomology. The argument only relies on the t–adic valuation for the Novikov ring and the fact that quantum corrections occur with strictly positive t–power.

Lemma 3.4 (McDuff and Tolman [30]) *Consider the algebra homomorphisms*

$$\varphi\colon \mathbb{K}[x_1,\dots,x_r]\to H^*(M;\mathbb{K}) \quad \textit{and} \quad \psi\colon \mathbb{K}[x_1,\dots,x_r]\otimes\Lambda\to QH^*(M)$$

determined by $\varphi(x_i)=\mathrm{PD}[D_i]$ *and* $\psi(x_i)=\mathrm{PD}[D_i]\otimes 1$ *(using quantum multiplication to define* ψ *). By construction* φ *is surjective.*

Then ψ *is surjective. Moreover, suppose* $p_1,\dots,p_r$ *generate* $\ker\varphi$ *and suppose there exist*

$$P_j=p_j+q_j\in\ker\psi\subset\mathbb{K}[x_1,\dots,x_r]\otimes\Lambda$$

with t–valuations $\mathrm{val}_t(q_j)>0$. *Then the* P_j *generate* $\ker\psi$, *thus*

$$\begin{aligned}
H^*(M;\mathbb{K})&=\mathbb{K}[x_1,\dots,x_r]/(p_1,\dots,p_r),\\
QH^*(M)&=\Lambda[x_1,\dots,x_r]/(P_1,\dots,P_r).
\end{aligned}$$

3C Batyrev's argument: from the presentation of QH^* to Jac(W)

Let X be a monotone toric manifold of dimension $\dim_{\mathbb{C}} X=n$. We have defined the superpotential $W=\sum t^{-\lambda_i}z^{e_i}$ of X in Definition A.9, with e_i,λ_i as in (22).

Definition 3.5 The Jacobian ring of W is

$$\mathrm{Jac}(W)=\Lambda[z_1^{\pm1},\dots,z_n^{\pm1}]/(\partial_{z_1}W,\dots,\partial_{z_n}W).$$

In the definition above, $\Lambda[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$ should be thought of as the coordinate ring of the complex torus $(\mathbb{C}^*)^n \subset X$ whose compactification is X.

Following Batyrev [5, Theorem 8.4], consider the homomorphism

$$\psi\colon \Lambda[x_1, \dots, x_r] \to \mathrm{Jac}(W), \quad x_i \mapsto t^{-\lambda_i} z^{e_i},$$

sending the i^{th} homogeneous coordinate to the i^{th} summand of the superpotential W. In $\mathrm{Jac}(W)$ the quotient by the derivatives of W corresponds to the linear relations (19). Indeed, imposing $z_j \partial W/\partial z_j = 0$ is equivalent to

$$\sum_i e_i^j t^{-\lambda_i} z^{e_i} = 0,$$

where e_i^j is the j^{th} entry of e_i, which corresponds to the linear relation $\sum e_i^j x_i = \sum \langle \xi, e_i \rangle x_i = 0$ when ξ is the j^{th} standard basis vector of $\mathbb{R}^n$.

When X is compact, the union of the cones of the fan for X covers all of $\mathbb{R}^n$ [5, Definition 2.3(iii)], which immediately implies that the above map ψ is surjective. This fails in the noncompact case, as the following example shows.

Example 3.6 In Appendix A we show that for a monotone toric negative line bundle E the fan does not cover $-e_f$, where e_f is the edge corresponding to the fiber direction. Multiplication by $x_f = \pi^* c_1(E) = \mathrm{PD}[B] \in QH^*(E)$ is never invertible by [35], even though its image $\psi(x_f) = z_f$ is tautologically invertible since z_f^{-1} is a generator of $\mathrm{Jac}(W)$. Thus $QH^*(E) \cong \mathrm{Jac}(W)$ fails; in fact we will prove that $SH^*(E) \cong \mathrm{Jac}(W)$.

From a Floer-theoretic perspective, the reason x_f^{-1} exists in $SH^*(E)$ but does not exist in $QH^*(E)$ is that the representation $\mathcal{S}_{\tilde{g}_f^{-1}}(1) = x_f^{-1}$ is defined in $SH^*(E)$ (see Section 2B), since negative slope Hamiltonians are not problematic for $SH^*(E)$, whereas the representation $r_{\tilde{g}_f^{-1}}(1) = x_f^{-1} \in QH^*(E)$ cannot be defined due to a failure of the maximum principle.

Now consider in general the kernel of ψ. By construction, the kernel is generated by the relations $\prod_p x_p^{a_p} = T^{\sum a_p - \sum c_q} \prod_q x_q^{c_q}$ for each relation amongst edges of the form

$$\sum a_p e_p = \sum c_q e_q,$$

where $a_p, c_q \geq 0$ are integers (the power of T is explained in Lemma 4.8). Batyrev proved [5, Theorem 5.3] that the ideal generated by these relations has a much smaller set of generators, namely those arising from relations $\sum e_{i_p} = \sum c_q e_{j_q}$ for primitive subsets $I = \{i_1, i_2, \dots\}$ (see Definition 3.2). These give rise to the quantum SR relations.

Remark 3.7 We remark that the notation $\exp(\varphi(v_i))$ of [5] corresponds to our $t^{-\lambda_i}$, so that

$$T^{c_1(TX)[\beta_I]} = T^{|I|-\sum c_q} = t^{-\sum \lambda_i + \sum c_q \lambda_{j_q}} = t^{[\omega_X](\beta_I)} \tag{25}$$

since $c_1(TX) = \sum \mathrm{PD}[D_i] = \lambda_X[\omega_X]$ by monotonicity. Here $[\omega_X] = \sum -\lambda_i \,\mathrm{PD}[D_i]$ corresponds to the φ of [5, Section 5] (see [5, Definition 5.8]), and $T = t^{1/\lambda_X}$ (see Section 2A).

From now on, let (X^{2n}, ω) be a (possibly noncompact) toric manifold with moment polytope $\Delta = \{y \in \mathbb{R}^n : \langle y, e_i\rangle \geq \lambda_i,\ i = 1, \dots, r\}$ and superpotential $W = \sum t^{-\lambda_i} z^{e_i}$.

Definition 3.8 The above data for X determines an algebra over Λ,

$$R_X = \Big\{\sum \lambda_e z^e : \text{finite sum, } \lambda_e \in \Lambda,\ e \in \mathrm{span}_{\mathbb{Z}_{\geq 0}}(e_i)\Big\} \subset \Lambda[z_1^{\pm 1}, \dots, z_n^{\pm 1}].$$

When X is compact, the fan of X is complete, so $R_X = \Lambda[z_1^{\pm 1}, \dots, z_n^{\pm 1}]$.

Corollary 3.9 *Suppose $QH^*(X, \omega) \cong \Lambda[x_1, \dots, x_r]/\mathcal{J}$, where $\mathcal{J}$ is the ideal generated by the linear relations and the quantum SR relations determined by Δ. Then there is an isomorphism*

$$\psi\colon QH^*(X, \omega) \to R_X/(\partial_{z_1} W, \dots, \partial_{z_n} W), \quad x_i \mapsto t^{-\lambda_i} z^{e_i},$$

which sends $c_1(TX) = \sum x_i \mapsto W = \sum t^{-\lambda_i} z^{e_i}$. When X is closed, this is the well-known isomorphism $QH^*(X, \omega) \cong \mathrm{Jac}(W) = \Lambda[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/(\partial_{z_1} W, \dots, \partial_{z_n} W)$.

Proof Observe that the image of ψ is precisely R_X, by construction. Moreover, ψ descends to the quotient

$$\bar{\psi}\colon \Lambda[x_1, \dots, x_r]/(\text{quantum SR relations}) \to R_X.$$

By Batyrev's argument, $\bar{\psi}$ is injective (the quotient on the left is the same as quotienting by all positive integral relations amongst edges, as mentioned above, and then it is clear that the map is injective). The image under $\bar{\psi}$ of the linear relations is precisely the ideal $(\partial_{z_1} W, \dots, \partial_{z_r} W)$. The claim follows by quotienting both sides by that ideal. □

Corollary 3.10 *Keeping the assumption of Corollary 3.9, there is a quotient map*

$$QH^*(X, \omega) \to \mathrm{Jac}(W), \quad x_i \mapsto t^{-\lambda_i} z^{e_i},$$

which sends $c_1(TX) \mapsto W$ and corresponds to the natural map

$$\begin{aligned} c^*\colon R_X/(\partial_{z_1} W, \dots, \partial_{z_n} W) &\to R_X[w_1, \dots, w_r]/(\partial_i W, w_i z^{e_i} - 1 : i = 1, \dots, r) \\ &\cong \mathrm{Jac}(W), \end{aligned}$$

which is the canonical localization at the variables $z_1, \dots, z_r$.

Proof The only thing left to prove is the very last isomorphism. This follows by Section 4B: the w_i formally invert the z^{e_i}, which corresponds to taking the quotient of $R_X/(\partial_{z_1}W,\dots,\partial_{z_n}W)$ by the generalized 0–eigenspace of z^{e_i} acting on $R_X/(\partial_{z_1}W,\dots,\partial_{z_n}W)$, and after quotienting there is a multiplicative inverse for z^{e_i}. The w_i correspond to z^{-e_i} in $\mathrm{Jac}(W)$, and the smoothness of X ensures that the $\mathbb{Z}$–span of the e_i is $\mathbb{Z}^n$ (see the comment in Section A1 about smoothness). This implies the surjectivity of the map into $\mathrm{Jac}(W)$. □

Conjecture 3.11 *For noncompact monotone toric manifolds X for which $SH^*(X)$ can be defined (eg for X conical at infinity), we expect that*

$$c^*\colon QH^*(X)\cong R_X/(\partial_{z_1}W,\dots,\partial_{z_n}W)\to \mathrm{Jac}(W)\cong SH^*(X)$$

is the canonical map $c^*\colon QH^*(X)\to SH^*(X)$. In particular, c^* is the canonical localization map, localizing at the variables $x_i=\mathrm{PD}[D_i]$ corresponding to the toric divisors D_i.

We now prove the conjecture for admissible toric manifolds (Definition 1.4), and in particular in Section 4I we prove that this applies to monotone toric negative line bundles.

3D Admissible noncompact toric manifolds

Proof of Theorem 1.5 Definition 1.4 parts (1) and (2) ensure that $QH^*(M)$ and $SH^*(M)$ are well-defined (see [35]). Part (2) means we can work over the Novikov ring Λ (in particular, Lemma 4.8 will hold). Part (3) is the analogue of Corollary 4.6 for negative line bundles: it ensures by Theorem 2.6 that the representation from [35],

$$r\colon \widetilde{\pi}_1(\mathrm{Ham}_{\ell\geq 0}(X,\omega_X))\to QH^*(X,\omega_X),\quad \widetilde{g}\mapsto r_{\widetilde{g}}(1),$$

is defined on the S^1–actions g_i given by rotation around the toric divisors, lifting to $g_i^\wedge$ as in Lemma 1.6 so that $r_{g_i^\wedge}(1)=x_i=\mathrm{PD}[D_i]$ (using Lemma 1.6 and $\mathrm{Fix}(g_i)=[D_i]$). The algebraic trick in Lemma 3.4 then ensures that the generators x_i of $QH^*(X,\omega_X)$ satisfy precisely the relations in $\mathcal{J}$; in particular, the quantum SR relations arise from the relations amongst the rotations g_i, the fact that the representation r is a homomorphism, and the fact that these relations only involve positive slope Hamiltonians (see the proof of Lemma 4.12).

By Theorem 2.6, the representation

$$\mathcal{S}\colon \widetilde{\pi}_1(\mathrm{Ham}_\ell(X,\omega_X))\to SH^*(X,\omega_X)^\times,\quad \widetilde{g}\mapsto \mathcal{S}_{\widetilde{g}}(1)$$

from [35] yields invertible elements $\mathcal{S}(g_i^\wedge)(1) = x_i \in SH^*(E)^\times$. Note that the inverse $x_i^{-1} = \mathcal{S}((g_i^{-1})^\wedge)(1)$ exists since the Hamiltonian is allowed to have negative slope for the representation $\mathcal{S}$.

The fact that SH^* is the quotient of QH^*, and hence is the localization at all x_i, follows by Theorem C.14 and Section 4B, using the assumption in Definition 1.4 part (4). The fact that $c^*(r_{g_i^\wedge}(1)) = \mathcal{S}_{g_i^\wedge}(1)$ follows by Theorem 2.2. The rest follows by Section 3C. □

3E Blow-up at a point

We briefly recall some generalities about blow-ups, following Guillemin [23, Chapter 1]. For a complex manifold X of dimension $\dim_{\mathbb{C}} X = n$, the blow-up $\pi\colon \widetilde{X} \to X$ at a point $p \in X$ is a holomorphic map, which is a biholomorphism outside of the exceptional divisor $E = \pi^{-1}(p)$, and E can naturally be identified with $\mathbb{CP}^{n-1}$ viewed as the projectivization of the normal bundle of $p \in X$. Suppose, in addition, that (X, ω_X) is symplectic. Then $\widetilde{X}$ carries a symplectic form $\omega_{\widetilde{X}}$ such that $\omega_{\widetilde{X}} - \pi^*\omega_X$ is compactly supported near E and restricts to $\varepsilon\omega_{FS}$ on $E \cong \mathbb{CP}^{n-1}$. Here ω_{FS} is the normalized Fubini–Study form, and $\varepsilon > 0$ is called the blow-up parameter (namely, the symplectic area of degree one spheres in E). When there is a Hamiltonian G–action on X for a compact group G (such as a torus in the case of toric X), and p is a fixed point of G, then $\omega_{\widetilde{X}}$ can be chosen to be G–equivariant and the action will lift to a G–Hamiltonian action on $\widetilde{X}$.

By Griffiths and Harris [22, Chapter 4, Section 7] the cohomology of the blow-up is

$$H^*(\widetilde{X}) = \pi^* H^*(X) \oplus \bar{H}^*(E), \tag{26}$$

where the pull-back π^* is injective, and $\bar{H}^*(E) = \bar{H}^*(\mathbb{CP}^{n-1})$ is the reduced cohomology (ie quotiented by $H^*(\text{point})$). The generator $-\omega_{FS} \in H^*(E)$ corresponds to $\mathrm{PD}[E]$ since the normal bundle to $E \subset \widetilde{X}$ has Chern class $-\omega_{FS}$. Moreover,

$$c_1(T\widetilde{X}) = \pi^* c_1(TX) - (n-1)\,\mathrm{PD}[E].$$

Thus, if X is monotone, so $c_1(TX) = \lambda_X[\omega_X]$ with $\lambda_X > 0$, and we want $(\widetilde{X}, \omega_{\widetilde{X}})$ to be monotone, then we require

$$\varepsilon = \frac{n-1}{\lambda_X}.$$

This ensures that $c_1(T\widetilde{X}) = \lambda_X[\omega_{\widetilde{X}}]$. The same condition is required if by monotonicity we just require $c_1(TX) = \lambda_X[\omega_X]$ to hold when integrating over spheres, by observing that $c_1(T\widetilde{X})$ and $\omega_{\widetilde{X}}$ integrate respectively to $n-1$ and $\varepsilon\lambda_X$ on degree one spheres in E.

The above condition on ε cannot always be achieved, as there may not be a sufficiently large Darboux neighborhood around p to apply the local model of a blow-up of $\mathbb{C}^n$ at 0 with area parameter ε (see [23, Theorem 1.10]).

More generally, if X satisfies weak+ monotonicity (see Section 2B) then so does $\widetilde{X}$, since the holomorphic spheres $A \subset E$ satisfy $c_1(T\widetilde{X})(A) = (n-1)\omega_{FS}(A) \geq 0$.

Let (X, ω_X) be a noncompact Kähler manifold (a complex manifold with a compatible symplectic form ω_X) with a Hamiltonian S^1–action. Suppose also that X is conical at infinity, satisfies weak+ monotonicity, and the Hamiltonian generating the S^1–action on X has the form prescribed by the extended maximum principle in Theorem C.2 with $f > 0$.

Theorem 3.12 *Under the above assumptions, the blow-up $(\widetilde{X}, \omega_{\widetilde{X}})$ of X at a fixed point of the action is a Kähler manifold conical at infinity, satisfying weak+ monotonicity, whose lifted Hamiltonian S^1–action satisfies Theorem C.2 with $f > 0$. In particular, $SH^*(\widetilde{X})$ is determined as a quotient of $QH^*(\widetilde{X})$ by Theorem C.14.*

Proof Away from the exceptional divisor, X and $\widetilde{X}$ are biholomorphic so the Hamiltonians generating the S^1–actions agree. The claim follows. □

In applications, via Lemma 1.6, $r_{g^\wedge}(1) \in QH^*(\widetilde{X})$ will typically be PD$[E]$.

For a toric variety X, with a choice of toric symplectic form ω_X, a fixed point $p \in X$ of the torus action corresponds to a vertex $p \in \Delta$ of the moment polytope (22). The moment polytope $\widetilde{\Delta}$ of the one-point blow-up is then obtained by chopping off a standard simplex from Δ at p corresponding to $\varepsilon\Delta_{\mathbb{CP}^n}$ [23, Theorem 1.12]. Thus, p is replaced by the opposite facet in this simplex, which is a copy of $\varepsilon\Delta_{\mathbb{CP}^{n-1}}$. Explicitly,

$$\widetilde{\Delta} = \Delta \cap \{y \in \mathbb{R}^n : \langle y, e_0 \rangle \geq \lambda_0\},$$

where $e_0 = e_{i_1} + \cdots + e_{i_n}$ is the sum of those edges of the fan for X which are the inward normals to the facets of Δ which meet at p, and $\lambda_0 = \varepsilon + \lambda_{i_1} + \cdots + \lambda_{i_n}$.

When (X, ω_X) is monotone, we can normalize so that $[\omega_X] = c_1(TX)$ (so the monotonicity constant $\lambda_X = 1$), and by Section A4 one can pick all $\lambda_i = -1$. So if we require $(\widetilde{X}, \omega_{\widetilde{X}})$ to be monotone, we need $\lambda_0 = -1$ which agrees with the above monotonicity constraint for $\widetilde{X}$,

$$\varepsilon = n - 1.$$

Since the Euclidean length of the edges in Δ through the vertex p corresponds to the symplectic areas of the spheres corresponding to those edges [23, Chapter 2], that condition on ε can be satisfied if the edges have length strictly greater than $n-1$. For example, for $X=\mathbb{CP}^2$ with $\omega_X=c_1(TX)=3\omega_{FS}$, the edges have length 3 and the condition is $\varepsilon=1$. So $\mathbb{CP}^2$ can be blown up in a monotone way at a vertex (indeed, even at all three vertices) of $\Delta_{\mathbb{CP}^2}$.

Theorem 3.13 *Let (X,ω_X) be an admissible toric manifold (Definition 1.4). Suppose X admits a monotone blow-up $(\widetilde{X},\omega_{\widetilde{X}})$ at a toric fixed point (or several such points). Then $(\widetilde{X},\omega_{\widetilde{X}})$ is admissible, and in particular Theorem 1.5 holds.*

Proof Part (1) of Definition 1.4 follows since X is conical and $\omega_{\widetilde{X}}$ agrees with ω_X away from E (where X and $\widetilde{X}$ can be identified). Part (3) of Definition 1.4 follows as in the proof of Theorem 3.12 for the edges e_i, and for the new edge $e_0=e_{i_1}+\cdots+e_{i_n}$ the rotation $g_0=g_{i_1}g_{i_2}\cdots g_{i_n}$ is generated by the sum of the Hamiltonians for the g_i, so again it has nonnegative slope. □

The passage from $QH^*(X,\omega_X)$ to $QH^*(\widetilde{X},\omega_{\widetilde{X}})$ at the level of vector spaces is already known by Equation (26), although in terms of the linear relations some care is needed: $\sum\langle\xi,e_i\rangle x_i=0$ now contains the additional term $\langle\xi,e_0\rangle x_0\equiv\langle\xi,e_{i_1}+\cdots+e_{i_n}\rangle x_0$. The new edge e_0 gives rise to the new generator

$$r_{g_0^{\wedge}}(1)=\mathrm{PD}[E]$$

of $\bar{H}^*(E)$ in (26). The only new quantum SR relation required is

$$x_{i_1}x_{i_2}\cdots x_{i_n}=T^{n-1}x_0,$$

corresponding to the relation $e_{i_1}+\cdots+e_{i_n}=e_0$ among edges. Indeed, suppose there was another "new" relation among edges involving e_0. Substituting $e_0=e_{i_1}+\cdots+e_{i_n}$ shows that this relation was detected and thus generated by the original relations among edges e_i, $i\neq 0$.

Example We conclude with a basic example: $X=\mathbb{C}^{n+1}$ blown up at the origin. Then $\widetilde{X}$ is the total space of $\mathcal{O}(-1)\to\mathbb{CP}^n$, where $E=\mathbb{CP}^n$ is the base. The edges for X are $e_1=(1,0,\dots,0)$, $e_2=(0,1,0,\dots,0)$, …, $e_{n+1}=(0,\dots,0,1)$. The new edge for $\widetilde{X}$ is $e_0=(1,1,\dots,1)$. The linear relations $x_i=0$ for X become $x_i=-x_0$. The new (and only) quantum SR relation is $x_1\cdots x_{n+1}=T^nx_0$. Letting $x=-x_0$, we deduce

$$QH^*(\widetilde{X})\cong\Lambda[x]/(x^{n+1}+T^nx),$$

which agrees with the computation of $QH^*(\mathcal{O}_{\mathbb{CP}^n}(-1))$, indeed x represents $-\mathrm{PD}[E]$ which is the pull-back of $\omega_{\mathbb{CP}^n}$ via $\mathcal{O}(-1)\to\mathbb{CP}^n$.

3F Blow-up along a closed complex submanifold

Similarly, one can blow up a complex manifold X along a closed complex submanifold $S \subset X$. The exceptional divisor of the blow-up $\pi\colon \widetilde{X} \to X$ is $E = \pi^{-1}(S)$, which can be identified with the projectivization of the normal bundle $\nu_{S\subset X}$. A neighborhood of $E \subset \widetilde{X}$ is modeled on the tautological bundle over $\mathbb{P}(\nu_{S\subset X})$. The cohomology is [22, Chapter 4, Section 7]

$$H^*(\widetilde{X}) = \pi^* H^*(X) \oplus \bar{H}^*(E), \tag{27}$$

where π^* is injective, and now $\bar{H}^*(E)$ denotes $H^*(E)$ quotiented by the pull-back of $H^*(S)$ via $E \equiv \mathbb{P}(\nu_{S\subset X}) \to S$. One can explicitly describe $\bar{H}^*(E)$ [22, Chapter 4, Section 7], in particular it is generated as an $H^*(X)$–algebra by $\mathrm{PD}[E]$. Moreover,

$$c_1(T\widetilde{X}) = \pi^* c_1(TX) - (\operatorname{codim}_{\mathbb{C}} X - 1)\,\mathrm{PD}[E].$$

If, in addition, (X, ω_X) is symplectic, then one can construct $\omega_{\widetilde{X}}$ with analogous properties as in Section 3E. In particular, $\omega_{\widetilde{X}} - \pi^*\omega_X$ is compactly supported near E, and restricts to $\varepsilon\omega_E$ on E, where ω_E is a symplectic form such that $[\omega_E] = -\mathrm{PD}[E]$. If X is monotone, then monotonicity of $\widetilde{X}$ holds if

$$\varepsilon = \frac{\operatorname{codim}_{\mathbb{C}} X - 1}{\lambda_X}.$$

Just as for one-point blow-ups, it is not always possible to pick such an ε, but if we only require weak+ monotonicity then that will hold for $\widetilde{X}$ if it holds for X. Theorem 3.12 becomes:

Theorem 3.14 *Under the assumptions above Theorem 3.12, the blow-up $(\widetilde{X}, \omega_{\widetilde{X}})$ of X along a closed complex submanifold $S \subset X$ fixed by the action is a Kähler manifold conical at infinity, satisfying weak+ monotonicity, whose lifted Hamiltonian S^1–action satisfies Theorem C.2 with $f > 0$. Also, $SH^*(\widetilde{X})$ is determined as a quotient of $QH^*(\widetilde{X})$ by Theorem C.14.*

In applications, via Lemma 1.6, $r_{g^\wedge}(1) \in QH^*(\widetilde{X})$ will typically be $\mathrm{PD}[E]$.

For X toric, McDuff and Tolman [31] describe the blow-up of X along the complex submanifold corresponding to a face F_I of the moment polytope Δ of X, of complex codimension at least 2. The face $F_I = F_{i_1} \cap \cdots \cap F_{i_a}$ is an intersection of the facets $F_j = \{y \in \Delta : \langle y, e_j\rangle = \lambda_j\}$ for a collection $I = \{i_1, \ldots, i_a\}$, and the codimension is $|I|$. The new polytope is

$$\widetilde{\Delta} = \Delta \cap \{y \in \mathbb{R}^n : \langle y, e_0\rangle \ge \lambda_0\},$$

where $e_0=\sum_{i\in I}e_i$ and $\lambda_0=\varepsilon+\sum_{i\in I}\lambda_i$. This holds for all small parameters $\varepsilon>0$ as long as the new facet $F_0=\{y\in\Delta:\langle y,e_0\rangle=\lambda_0\}$ stays bounded away from the vertices of $\Delta\setminus F_I$. For monotone X, taking $\lambda_i=-1$, the blow-up $\widetilde{X}$ is monotone provided that $\lambda_0=-1$, which agrees with the above condition, $\varepsilon=|I|-1$.

Theorem 3.15 *Let (X,ω_X) be an admissible toric manifold (Definition 1.4). Suppose that X admits a monotone blow-up $(\widetilde{X},\omega_{\widetilde{X}})$ along a face F_I of complex codimension $|I|\geq 2$ (or several such faces). Then $(\widetilde{X},\omega_{\widetilde{X}})$ is admissible, in particular Theorem 1.5 holds.*

Proof The same proof as in Theorem 3.13 applies. In particular, for the new edge $e_0=\sum_{i\in I}e_i$ the rotation $g_0=\prod_{i\in I}g_i$ is generated by the sum of the Hamiltonians for the g_i, so again it has nonnegative slope. □

The passage from $QH^*(X,\omega_X)$ to $QH^*(\widetilde{X},\omega_{\widetilde{X}})$ at the level of vector spaces is already known by Equation (27). Just as at the end of Section 3E, the linear relations $\sum\langle\xi,e_i\rangle x_i=0$ contain the new term $\langle\xi,e_0\rangle x_0\equiv\langle\xi,\sum_{i\in I}e_i\rangle x_0$. The new edge e_0 produces the new generator $r_{\hat{g}_0}(1)=\mathrm{PD}[E]$ of $\overline{H}^*(E)$ in (27). The only new quantum SR relation required is

$$\prod_{i\in I}x_i=T^{|I|-1}x_0.$$

4 Presentation of QH^*, SH^* for toric negative line bundles

4A Definition

A complex line bundle $\pi\colon E\to B$ over a closed symplectic manifold (B,ω_B) is called *negative* if $c_1(E)=-k[\omega_B]$ for $k>0$. As shown for example in [35, Section 7], there is a Hermitian metric on E which determines a norm $r\colon E\to\mathbb{R}_{\geq 0}$ and which determines a symplectic form

$$\omega=\pi^*\omega_B+\boldsymbol{\pi}\Omega$$

such that $[\omega]=\pi^*[\omega_B]\in H^2(E)$, where we temporarily use the bold symbol $\boldsymbol{\pi}$ for the mathematical constant to avoid confusion with the pullback map π^*. There is a Hermitian connection whose curvature $\mathcal{F}$ satisfies $(1/2\boldsymbol{\pi}i)\pi^*\mathcal{F}=k\pi^*\omega_B$. Outside of the zero section of E, there is an angular 1–form θ on E (which vanishes on horizontal vectors and satisfies $\theta_w(w)=0$ and $\theta_w(iw)=1/2\boldsymbol{\pi}$ in the fiber directions) such that

$$d\theta=k\pi^*\omega_B\quad\text{and}\quad d(r^2\theta)=\Omega.$$

So outside of the zero section, ω is exact: $\omega = d((1/k)\theta + \boldsymbol{\pi} r^2\theta)$. Moreover, ω has the block form

$$\begin{pmatrix} (1+k\boldsymbol{\pi} r^2)\pi^*\omega_B & 0 \\ 0 & \omega_{\text{standard}} \end{pmatrix}$$

in the decomposition $TE \cong T^{\text{horiz}}E \oplus E$; it is the standard form in the vertical $\mathbb{C}$–fibers.

Technical remark In [35, Section 7] we considered $\omega = d\theta + \varepsilon\Omega = k\pi^*\omega_B + \varepsilon\Omega$, so $[\omega] = k\pi^*[\omega_B]$. It turns out that for toric E a more natural choice is to rescale that form by $1/k$ and to choose $\varepsilon = k\boldsymbol{\pi}$, as we did above. The Floer theory is not affected by this rescaling.

In this decomposition, we define an ω–compatible almost complex structure

$$J = \begin{pmatrix} J_B & 0 \\ 0 & i \end{pmatrix}$$

in terms of an ω_B–compatible almost complex structure J_B for (B, ω_B); in particular, fiberwise it is just multiplication by $i = \sqrt{-1}$.

It is shown for example in [35] that the radial coordinate is

$$R = \frac{1+k\boldsymbol{\pi} r^2}{1+k\boldsymbol{\pi}}.$$

The Liouville vector field is

$$Z = \frac{1+k\boldsymbol{\pi} r^2}{k\boldsymbol{\pi} r^2}\frac{w}{2},$$

and the Reeb vector field is

$$Y = X_R = \frac{2k\boldsymbol{\pi}}{1+k\boldsymbol{\pi}} i w,$$

where $w \in \mathbb{C}$ is the local fiber coordinate in a unitary frame (the extra k in Y compared to [35] is due to the rescaling mentioned in the technical remark above). Since iw is the angular vector field whose flow wraps around the fiber circle in time $2\boldsymbol{\pi}$, the S^1–action which rotates the fiber circle in time 1 is generated by the vector field $((1+k\boldsymbol{\pi})/k)Y = X_{(1+k\boldsymbol{\pi})R/k}$.

The contact type condition has the form $JZ = c(R)Y$, as in Remark C.1.

For negative line bundles $\pi\colon E \to B$, assuming weak+ monotonicity (see Section 2B), Theorem 2.3 applies to the circle action $g_t = e^{2\boldsymbol{\pi} i t}$ acting by rotation on the fibers of E. In this case $r_{\widetilde{g}}$ is quantum multiplication by $\pi^* c_1(E) = -k\pi^*[\omega_B]$.

Theorem 4.1 (Ritter [35]) *The canonical map $QH^*(E) \to SH^*(E)$ induces an isomorphism of Λ–algebras*

$$SH^*(E) = QH^*(E)/(\text{generalized } 0\text{–eigenspace of } \pi^* c_1(E)).$$

For $E = \mathrm{Tot}(\mathcal{O}(-k) \to \mathbb{P}^m)$ and $1 \le k \le m/2$, this becomes explicitly, via $x = \pi^*[\omega_B]$,

$$QH^*(\mathcal{O}_{\mathbb{P}^m}(-k)) = \Lambda[x]/(x^{1+m} - (-k)^k\, T^{1+m-k} x^k),$$
$$SH^*(\mathcal{O}_{\mathbb{P}^m}(-k)) = \Lambda[x]/(x^{1+m-k} - (-k)^k\, T^{1+m-k}).$$

4B Some remarks about localizations of rings

Recall that for a ring R and an element $f \in R$, the *localization R_f of R at f* consists of equivalence classes (r, f^n) for $n \in \mathbb{N}$ (informally thought of as fractions r/f^n) under the equivalence relation $(r, f^n) \simeq (r', f^m)$ whenever $f^k(rf^m - r'f^n) = 0$ for some $k \in \mathbb{N}$. In particular, if f is nilpotent then $R_f = 0$.

Lemma 4.2 *Localizing at f is the same as formally introducing an inverse z of f,*

$$R_f \cong R[z]/(1 - fz). \qquad \square$$

There is a *canonical localization map* $c\colon R \to R_f$, $r \mapsto (r, 1)$. Given an ideal $I \subset R$, the *saturation* $(I : f^\infty) \subset R$ is the preimage under c of the localized ideal $R_f I \subset R_f$,

$$c^{-1}(I) = (I : f^\infty) = \{r \in R : f^k r \in I \text{ for some } k \in \mathbb{N}\}.$$

Lemma 4.3 *In general, $(I : f^\infty) = I' \cap R$, where $I' \subset R[z]$ is the ideal generated by I and $1 - fz$. Recall $(I : f^\infty) \mapsto R_f I$ via the surjection $c\colon R \to R_f$. Thus the canonical localization map c determines the natural isomorphism of rings*

$$R/(I : f^\infty) \to R_f / R_f I.$$

In particular, for $I = 0$, $R/(0 : f^\infty) \cong R_f$. $\square$

Corollary 4.4 *For negative line bundles $E \to B$ satisfying weak+ monotonicity, $SH^*(E) \cong QH^*(E)_{\pi^* c_1(E)}$ is the localization of $QH^*(E)$ at $f = \pi^* c_1(E)$, and the canonical map $c^*\colon QH^*(E) \to SH^*(E)$ corresponds to the canonical localization map.*

In particular, given a presentation $QH^*(E) \cong \mathbb{K}[x_1, \dots, x_r]/\mathcal{J}$ for an ideal $\mathcal{J}$ of relations, the corresponding presentation for $SH^*(E)$ is

$$SH^*(E) \cong QH^*(E)_f \cong \mathbb{K}[x_1, \dots, x_r, z]/\langle \mathcal{J}, 1 - fz \rangle.$$

Proof In Theorem 4.1, the generalized 0–eigenspace of $f = \pi^* c_1(E)$ is precisely the saturation $(0 : f^\infty)$ of the trivial ideal $0 \subset QH^*(E)$. Thus, by the previous lemma, $SH^*(E) \cong QH^*(E)/(0 : f^\infty) \cong QH^*(E)_f$. □

4C The Hamiltonians generating the rotations around the toric divisors

Let $\pi\colon E \to B$ be a monotone toric negative line bundle, with $c_1(E) = -k[\omega_B]$. We describe these in detail in Appendix A. Since toric manifolds are simply connected, monotonicity implies

$$c_1(TE) = \lambda_E[\omega_E] = \lambda_E \pi^*[\omega_B], \quad c_1(TB) = \lambda_B[\omega_B], \quad \lambda_E = \lambda_B - k > 0.$$

(We used that $c_1(TE) = \pi^* c_1(TB) + \pi^* c_1(E)$ by splitting TE). The toric divisors are

$$D_i = \pi^{-1}(D_i^B) \quad \text{for } i = 1, \dots, r, \quad D_{r+1} = [B],$$

where D_i^B are the toric divisors in B and $[B] \subset E$ is the zero section. The Hamiltonians H_i which generate the standard rotations g_i about D_i (see Section 3B) actually depend on the radial coordinate R, despite what (24) might suggest. This is inevitable since the Hamiltonians H_i satisfy relations $H_{i_1} + \cdots + H_{i_a} = c_1 H_{j_1} + \cdots + c_b H_{j_b} + \text{constant}$, and H_{r+1} obviously depends on R. In fact, if H_i were R–independent, then g_i and g_i^{-1} are both in $\pi_1 \operatorname{Ham}_{\ell \geq 0}(M)$ so $r_{\tilde{g}_i}(1)$ would be invertible with inverse $r_{\tilde{g}_i^{-1}}(1)$. But this is not the case, for example, for $E = \mathcal{O}(-1) \to \mathbb{P}^1$, where $r_{\tilde{g}_1}(1) = x = \pi^*[\omega_{\mathbb{P}^1}]$ satisfies $x^2 + Tx = 0$ in $QH^*(E)$.

Theorem 4.5 *The Hamiltonians H_i for the rotation g_i about $D_i = \pi^{-1}(D_i^B)$ are*

$$H_i(x) = (1 + k\pi) R \cdot \pi^* H_i^B(x),$$

where by convention $H_i = 0$ on D_i and $H_i^B = 0$ on D_i^B, and the rotation g_{r+1} about the zero section $D_{r+1} = [B]$ has Hamiltonian

$$H_{r+1} = \frac{1 + k\pi}{k} R - \frac{1}{k}.$$

Proof We first clarify what the coordinate R is. By Section A3, we know a formula for the moment map $\mu_E(x)$ when x satisfies a certain quadratic equation $f_E(x) = 0$; in particular, we know the last entry, $\mu_E(x) = (\dots, \frac{1}{2}|x_{r+1}|^2)$. By Lemma A.7, the norm in the fiber is then $|w| = (1/\sqrt{2\pi})|x_{r+1}|$, so by Section 4A the radial coordinate R is

$$R = \frac{1 + k\pi |w|^2}{1 + k\pi} = \frac{1}{1 + k\pi}(1 + k\tfrac{1}{2}|x_{r+1}|^2).$$

Lifting $X_i = X_{H_i}$ from $E = (\mathbb{C}^{r+1} - Z_E)/G_E$ to $\mathbb{C}^{r+1}$ yields the vector field $X_i = \partial/\partial\theta_i$, where $x_i = |x_i|e^{2\pi\sqrt{-1}\theta_i}$ is the i^{th} factor of $\mathbb{C}^{r+1}$; indeed its flow is the rotation

$$g_i(t)\colon x_i \mapsto e^{2\pi\sqrt{-1}t}x_i,$$

fixing all other variables x_j. The projection $\pi\colon E \to B$ is the forgetful map

$$(x_1, \ldots, x_{r+1}) \mapsto (x_1, \ldots, x_r)$$

on homogeneous coordinates (this can be easily verified in a local trivialization). Therefore $d\pi \cdot (\partial/\partial\theta_i) = X_{H_i^B}$ in the zero section $B \subset E$.

By Section 4A, $\omega = (1+k\pi r^2)\pi^*\omega_B + \pi d(r^2)\wedge\theta$ outside of the zero section, where $r\colon E \to \mathbb{R}$ is the Hermitian norm in the fiber. Therefore,

$$\begin{aligned} dH_i &= \omega\Big(\cdot\,,\frac{\partial}{\partial\theta_i}\Big) \\ &= (1+k\pi r^2)\pi^*\omega_B\Big(\cdot\,,\frac{\partial}{\partial\theta_i}\Big) + \pi(d(r^2)\wedge\theta)\Big(\cdot\,,\frac{\partial}{\partial\theta_i}\Big) \\ &= (1+k\pi r^2)d(\pi^* H_i^B)(\cdot) + \pi d(r^2)(\cdot)\theta\Big(\frac{\partial}{\partial\theta_i}\Big), \end{aligned}$$

where in the last line we use the fact that r does not vary when we rotate x_i; indeed, rotating x_i does not change $|x_i|^2$, so it preserves the equation $f_E(x) = 0$, so the last entry $\frac{1}{2}|x_{r+1}|^2$ of $\mu_E(x)$ will be preserved, and this determines r.

The last term above is equal to $(1/k)d(1+k\pi r^2)(\cdot)\theta(\partial/\partial\theta_i)$, and recalling that $1+k\pi r^2 = (1+k\pi)R$, we get

$$d(H_i - (1+k\pi)R\pi^* H_i^B) = (1+k\pi)\Big[\frac{1}{k}\theta\Big(\frac{\partial}{\partial\theta_i}\Big) - \pi^* H_i^B\Big]dR.$$

Now observe that, in general, if $dH = G\,dR$ for functions H, G, R, such that R is a local coordinate, then completing $R = R_1$ to a system of local coordinates $R_1, R_2, \ldots, R_d$, implies that $G = \partial H/\partial R$ and $\partial H/\partial R_j = 0$ for $j \neq 1$. So $H = H(R)$ and $G = G(R)$ only depend on R.

In our situation above, this implies that $H_i - (1+k\pi)R\pi^* H_i^B = h_i(R)$ for some function h_i and that

$$\frac{\partial H_i}{\partial R} = \frac{1+k\pi}{k}\theta\Big(\frac{\partial}{\partial\theta_i}\Big).$$

Now evaluate $H_i = (1+k\pi)R\pi^* H_i^B + h_i(R)$ at $D_i = \pi^{-1}(D_i^B)$, using that $H_i^B = 0$ on D_i^B and that $H_i = 0$ on D_i, to deduce that $h_i(R) \equiv 0$. The first claim follows.

The second claim follows from

$$H_{r+1} = \tfrac{1}{2}|x_{r+1}|^2 = \frac{1+k\pi}{k} R - \frac{1}{k},$$

where the first equality holds since $f_E(x) = 0$. $\square$

Corollary 4.6 *The Hamiltonians H_i which define the S^1–rotations g_i about the toric divisors D_i satisfy Theorem 2.6 for* $\mathrm{Ham}_{\ell \geq 0}(E)$*. So, picking lifts $\widetilde{g}_i$ (see Section 2B), they give rise to*

$$r_{\widetilde{g}_i}(1) \in QH^*(E), \quad \mathcal{R}_{\widetilde{g}_i}(1) \in SH^*(E)^\times,$$

for $i = 1, \dots, r+1$.

Proof For $i = 1, \dots, r$, we have $H_i = f_i(y)R$ for $f_i(y) = (1+k\pi)\pi^* H_i^B(1, y)$. Notice that $\pi^* H_i^B$ does not depend on R, it only depends on the point $(1, y)$ in the sphere bundle $\Sigma = SE = \{R = 1\}$ (or rather, on the projection of $(1, y)$ via $\pi\colon SE \to B$). Notice that the $f_i(y)$ are invariant under the Reeb flow (which is rotation in the fiber). Finally, $f_i(y) \geq 0$ since $H_i^B \geq 0$, by (24). $\square$

Lemma 4.7 *Let $g_i^\wedge$ be the lift of g_i (in the sense of Section 2B) which maps the constant disc (c_x, x) to itself, for $x \in D_i$. Then*

$$r_{g_i^\wedge}(1) = \mathrm{PD}[D_i] \in QH^2(E), \quad \mathcal{R}_{g_i^\wedge}(1) = c^* \,\mathrm{PD}[D_i] \in SH^2(E)^\times.$$

Proof This follows by Lemma 1.6 since $\mathrm{Fix}(g_i) = D_i$, using the fact that $I(g_i^\wedge) = 1$ (one can explicitly compute $I(g_i^\wedge)$ as in [35, Section 7.8]). $\square$

4D The problem with relating the lifted rotations

Although these lifts $g_i^\wedge$ appear to be canonical, a relation $\prod g_i^{a_i} = \prod g_j^{b_j}$ only implies the lifted relation $\prod (g_i^\wedge)^{a_i} = \prod (g_j^\wedge)^{b_j}$ up to factor t^d corresponding to a deck transformation in $\pi_2(M)/\pi_2(M)_0$ (see Section 2B).

For closed symplectic manifolds C, this issue did not arise. In fact, for closed C one can define the Seidel representation directly on $\pi_1 \mathrm{Ham}(C)$, rather than on an extension thereof, by a work-around which involves normalization arguments for the Hamiltonians, as explained for example in [29, page 433]. For noncompact M, it is unclear to us whether a work-around exists; of course, those normalization arguments involving integration of ω^{top} over M will fail.

In any case, for monotone M this is never a problem since one can measure the discrepancy between $\prod \mathcal{R}_{g_i^\wedge}(1)^{a_i}$ and $\prod \mathcal{R}_{g_j^\wedge}(1)^{b_j}$ simply by comparing gradings, as follows.

Lemma 4.8 *A relation* $\prod g_i^{a_i} = \prod g_j^{b_j}$ *corresponds to the relation*

$$\prod x_i^{a_i} = T^{\sum a_i - \sum b_j} \prod x_j^{b_j}$$

in $SH^*(M)$, *where* $x_i = \mathrm{PD}[D_i]$.

Proof By Lemma 4.7, $\mathcal{R}^{\wedge}_{g_i}(1) = x_i$ has grading $2 = 2\,\mathrm{codim}_{\mathbb{C}}\, D_i$. The claim follows since, for M monotone, $SH^*(M)$ is $\mathbb{Z}$–graded over the (graded) Novikov ring; see Section 2A. □

4E The quantum SR relations for monotone toric negative line bundles

By Lemma A.3, E arises as $\mathcal{O}\big(\sum n_i D_i^B\big) \xrightarrow{\pi} B$ for $n_i \in \mathbb{Z}$, so $c_1(E) = \sum n_i\, \mathrm{PD}[D_i^B]$. The edges of the fan for E are

$$e_1 = (b_1, -n_1), \ \ldots\ , e_r = (b_r, -n_r),\ e_f = (0, \ldots, 0, 1) \in \mathbb{Z}^{n+1},$$

where the b_j are the edges of the fan for B. We often use the index f instead of $r+1$, so $e_f = e_{r+1}$, to emphasize that this index corresponds to the fiber coordinate $x_f = x_{r+1}$. The cones of the fan for E are $\mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_{j_1}, \ldots, e_{j_k}\}$ and $\mathrm{span}_{\mathbb{R}_{\geq 0}}\{e_{j_1}, \ldots, e_{j_k}, e_f\}$ whenever $\mathrm{span}_{\mathbb{R}_{\geq 0}}\{b_{j_1}, \ldots, b_{j_k}\}$ is a cone for B.

Corollary 4.9 *The primitive collections for* E *are those of* B, *ie* $I = I^B$.

Proof If $I^B \cup \{e_f\}$ were primitive, then I^B would determine a cone in E and hence in B, but then $I^B \cup \{e_f\}$ would be a cone for E and so would not be primitive. □

Corollary 4.10 *Recall that the relations for* B *are:*

(1) **Linear relations** $\sum \langle \xi, b_i \rangle x_i = 0$ *as* ξ *ranges over the standard basis of* $\mathbb{R}^r$.

(2) **Quantum SR relations** $\prod_{p \in I^B} x_{i_p} = T^{|I^B| - \sum c_q} \prod_q x_{j_q}^{c_q}$ *for primitive collections* I^B *(corresponding to the relation* $\sum b_{i_p} = \sum c_q b_{j_q}$ *among edges).*

Then the relations for E *are:*

(1) *the linear relations for* B;

(2) *the new linear relation* $x_f = \sum n_i x_i$;

(3) $\prod_{p \in I^B} x_{i_p} = T^{|I^B| - \sum c_q - c_f} \cdot x_f^{c_f} \cdot \prod_q x_{j_q}^{c_q}$, *where* $c_f = -\sum_{p \in I^B} n_{i_p} + \sum_q c_q \cdot n_{j_q}$ *(corresponding to the relation* $\sum_{p \in I^B} e_{i_p} = \sum_q c_q e_{j_q} + c_f e_f$ *among edges).*

Theorem 4.11 *The above linear relations and quantum SR relations hold in* $QH^*(E)$.

Proof The linear relations from B hold also in $H^*(E)$ since $H^*(E) \cong H^*(B)$. The new linear relation $x_f = \sum n_i x_i$ holds because the toric divisor $D_{r+1} = [B]$, as an lf-cycle, is Poincaré dual to $\pi^* c_1(E)$ which in turn is Poincaré dual to $\pi^*\big(\sum n_i [D_i^B]\big)$. The quantum SR relations hold in $SH^*(E)$ by Lemma 4.8. The fact that they hold in $QH^*(E)$ follows from using the representation $r\colon \widetilde{\pi}_1 \operatorname{Ham}_{\ell \geq 0}(E,\omega) \to QH^*(E)$ and Theorem 2.6, and using the following lemma, which ensures that only Hamiltonians of positive slope are involved in the quantum SR relations. □

Lemma 4.12 *The quantum SR relations for E involve only positive slope Hamiltonians.*

Proof By Corollary 4.6, any monomial involving nonnegative powers of the rotations $g_1^\wedge, \dots, g_r^\wedge, g_f^\wedge$ will be generated by a Hamiltonian of positive slope. The quantum SR relations in $QH^*(E)$ arise from comparing the values of the homomorphism $r\colon \widetilde{\pi}_1 \operatorname{Ham}_{\ell \geq 0}(E,\omega) \to QH^*(E)$ (see Theorem 2.6) on the rotations $\prod_{p \in I^B} g_{i_p}^\wedge$ and $(g_f^\wedge)^{c_f} \cdot \prod_q (g_{j_q}^\wedge)^{c_q}$. The values are well-defined because all the powers in those monomials are nonnegative: c_q are positive integers by definition, and we now prove $c_f \geq 0$.

Applying the formula for $c_1(TM)(\beta_I)$ explained at the end of Section 3A,

$$c_1(TE)(\beta_I) = |I^B| - \sum c_q - c_f = c_1(TB)(\beta_I) - c_f.$$

But $c_1(TE) = \pi^* c_1(TB) + \pi^* c_1(E)$, by splitting TE into horizontal and vertical spaces, so

$$c_f = -c_1(E)(\beta_I) = k\omega_B(\beta_I) > 0,$$

using Batyrev's result that $\omega_B(\beta_I) > 0$. (We identify $\beta_I \in H_2(E) \cong H_2(B)$ calculated in E and B as it is prescribed by the same intersection products: $\beta_I \cdot D_{i_p} = 1$, $\beta_I \cdot D_{j_q} = -c_q$.) □

4F Presentation of $QH^*(E)$ and $SH^*(E)$

Theorem 4.13 *Let $E \to B$ be a monotone toric negative line bundle. Then the presentation of the quantum cohomology of E can be recovered from the presentation for B:*

$$\begin{aligned} QH^*(B) \cong \Lambda[x_1, \dots, x_r] / \Big(&\text{linear relations in } B, \\ &\text{quantum SR relations } \prod x_{i_p} = T^{|I^B| - \sum c_q} \cdot \prod x_{j_q}^{c_q}\Big), \end{aligned}$$

$$\begin{aligned} QH^*(E) \cong \Lambda[x_1, \dots, x_r] / \Big(&\text{linear relations in } B, \\ &\prod x_{i_p} = T^{|I^B| - \sum c_q - c_f} \cdot \Big(\sum n_i x_i\Big)^{c_f} \cdot \prod x_{j_q}^{c_q}\Big), \end{aligned}$$

running over primitive relations $\sum b_{i_p} = \sum c_q b_{j_q}$ *in* B, $i_p \in I^B$, *and* $c_f = -\sum n_{i_p} + \sum c_q \cdot n_{j_q}$.

Moreover, the symplectic cohomology is the localization of the above ring at $x_f = \sum n_i x_i$,

$$SH^*(E) \cong \Lambda[x_1, \dots, x_r, z]/\big(\text{linear relations in } B, \text{quantum SR relations}, \\ z \cdot \textstyle\sum n_i x_i - 1\big),$$

and the canonical map $c^*\colon QH^*(E) \to SH^*(E)$ *is the canonical localization map sending* $x_i \mapsto x_i$ *(recall that this induces the isomorphism in Theorem 4.1).*

Proof The computation of $QH^*(E)$ follows from Theorem 4.11 and Lemma 3.4. The computation of $SH^*(E)$ then follows, using Corollary 4.4. □

4G Examples: QH^* and SH^* for $\mathcal{O}_{\mathbb{P}^m}(-k)$ and $\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1)$

Corollary 4.14 *Let* $E = \mathcal{O}_{\mathbb{P}^m}(-k)$ *be monotone (meaning* $1 \le k \le m$*). Then*

$$QH^*(E) \cong \Lambda[x]/(x^{1+m} - T^{1+m-k}(-kx)^k),$$
$$SH^*(E) \cong \Lambda[x]/(x^{1+m-k} - T^{1+m-k}(-k)^k).$$

Remark 4.15 This recovers, for $1 \le k \le m/2$, the computation from [35] (see Theorem 4.1) which was a rather difficult virtual localization computation. That approach was computationally unwieldy for $m/2 < k \le m$. So for $m/2 < k \le m$ the above result is new.

Proof The standard presentation of $QH^*(\mathbb{P}^m)$ involves variables $x_1, \dots, x_{m+1}$; the linear relations make all x_j equal, call this variable x (which represents $\omega_{\mathbb{P}^m}$); and the quantum SR relation is $x_1 \cdots x_{m+1} = T^{m+1}$. Thus $QH^*(\mathbb{P}^m) = \Lambda[x]/(x^{1+m} - T^{1+m})$.

Representing $\mathcal{O}(-k)$ as $E = \mathcal{O}(-kD_{m+1})$, apply Theorem 4.13. The quantum SR relation in E becomes $x_1 \cdots x_{m+1} = T^{1+m-k}(-kx_{m+1})^k$ since $c_f = k$. The claim follows. □

Corollary 4.16 *Let* $E = \mathrm{Tot}(\mathcal{O}(-1,1) \to \mathbb{P}^1 \times \mathbb{P}^1)$*; that is,* $B = \mathbb{P}^1 \times \mathbb{P}^1$ *with* $\omega_B = \omega_1 + \omega_2$, *and* $c_1(E) = -(\omega_1 + \omega_2)$, *where* $\omega_j = \pi_j^* \omega_{\mathbb{P}^1}$ *via the two projections* $\pi_j\colon B \to \mathbb{P}^1$. *Then*

$$QH^*(B) \cong \Lambda[x_1, x_2]/(x_1^2 - t, x_2^2 - t),$$
$$QH^*(E) \cong \Lambda[x_1, x_2]/(x_1^2 + t(x_1 + x_2), x_2^2 + t(x_1 + x_2)),$$
$$SH^*(E) \cong \Lambda[x_1, x_2]/(x_1^2 - 4t^2, x_2^2 - 4t^2, x_1 + x_2 + 4t) \cong \Lambda[x_1]/(x_1^2 - 4t^2),$$

where the $x_j = \omega_j$. In particular, using the convention that $T = t^{1/\lambda_M}$ lies in grading 2 (see Section 2A), $\omega_B = x_1 + x_2$ has

- *minimal polynomial $X^3 - 4tX = X(X-2T)(X+2T)$ and characteristic polynomial $X^4 - 4tX^2 = X^2(X-2T)(X+2T)$ in $QH^*(B)$,*
- *minimal polynomial $X^3 + 4tX^2 = X^2(X+4T)$ and characteristic polynomial $X^4 + 4tX^3 = X^3(X+4T)$ in $QH^*(E)$,*
- *minimal and characteristic polynomials both equal to $X + 4T$ in $SH^*(E)$.*

Proof The computation of $QH^*(B)$ can be obtained from the moment polytope (a square with inward normals $e_1 = (1,0)$, $-e_1$, $e_2 = (0,1)$ and $-e_2$) or by explicitly computing the quantum product $\omega_1 * \omega_2 = \omega_1 \wedge \omega_2$. The minimal polynomial of ω_B follows by computation. The matrix for multiplication by $\omega_B = x_1 + x_2$ is

$$\begin{pmatrix} 0 & t & t & 0 \\ 1 & 0 & 0 & t \\ 1 & 0 & 0 & t \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

in the basis 1, x_1, x_2, x_1x_2, and since this has rank 2 the characteristic polynomial has an extra X factor.

Observe that $k = 1$, $\lambda_B = 2$, $\lambda_E = 1$. So $t_B = T_B^2$ and $T_E = t_E$. By Theorem 4.18 for $E = \mathcal{O}(-D_1 - D_2)$, the presentation of $QH^*(E)$ follows by replacing t_B by $t_E(-x_1 - x_2)$. One obtains the characteristic polynomial of $\pi^*\omega_B$ by inspecting the new matrix

$$\begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & -t & -t & 2t^2 \\ 1 & -t & -t & 2t^2 \\ 0 & 1 & 1 & -2t \end{pmatrix}.$$

To compute $SH^*(E)$ localize at $c_1(E)$: introduce z with $1 - z(-x_1 - x_2) = 0$. For $X = x_1 + x_2$, we have $X^3(X + 4T) = 0 \in QH^*(E)$. Multiplying by z^3 we get $X = -4T$, $z = -(4T)^{-1}$. $\square$

Remarks With some effort, the calculation of $QH^*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1))$ can in fact be verified by hand (that is, by explicitly computing $\omega_i * \omega_j$). One can also check that $QH^*(\mathbb{P}^1 \times \mathbb{P}^1)$ is semisimple (a direct sum of fields as an algebra), since the 0–eigenspace splits as an algebra into two fields spanned by $2Ty \pm y^2$, where $y = x_1 - x_2$. On the other hand, $QH^*(\mathcal{O}_{\mathbb{P}^1\times\mathbb{P}^1}(-1,-1))$ is not semisimple, due to the generalized 0–eigenspace which creates zero divisors in the ring.

4H The change in presentation from $QH^*(B)$ to $QH^*(E)$ is a change in the Novikov variable

In this section we will restrict the Novikov field of Section 2A to a smaller subring $R = \mathbb{K}[t,t^{-1}]$ or $\mathbb{K}[t]$, and we write $QH^*(M;R)$ when we work over R (recall that the quantum product only involves positive integer powers of t). We put subscripts B and E on t (so $t_B = T_B^{\lambda_B}$, $t_E = T_E^{\lambda_E} = T_E^{\lambda_B - k}$) when we want to distinguish the variable t used for B and for E.

Recall that by Theorem 4.13,

$$\sum n_i x_i = c_1(E) = -k[\omega_B] = -\tfrac{k}{\lambda_B} c_1(TB) = -\tfrac{k}{\lambda_B} \sum x_i.$$

Theorem 4.17 *The presentations of $QH^*(B)$ and $QH^*(E)$ in Theorem 4.13 are identical after the change of Novikov parameter*

$$T^{\lambda_B} \mapsto T^{\lambda_B - k}\Big(\sum n_i x_i\Big)^k.$$

Proof By Equation (23) in Section 3A, for primitive $I = I^B$,

$$|I^B| - \sum c_q = c_1(TB)(\beta_I).$$

By the proof of Lemma 4.12, $-c_f = c_1(TE)(\beta_I) - c_1(TB)(\beta_I) = c_1(E)(\beta_I)$, so the quantum SR relations for $QH^*(E)$ in Theorem 4.13 become

$$\prod x_{i_p} = T^{c_1(TE)(\beta_I)} \cdot \Big(\sum n_i x_i\Big)^{-c_1(E)(\beta_I)} \cdot \prod x_{j_q}^{c_q}.$$

So the presentations of $QH^*(B)$ and $QH^*(E)$ are identical after the change of Novikov parameter

$$T^{c_1(TB)(\beta_I)} \mapsto T^{c_1(TE)(\beta_I)} \cdot \Big(\sum n_i x_i\Big)^{-c_1(E)(\beta_I)},$$

and it remains to check that this is consistent as β_I varies (varying $I = I^B$). But $c_1(TB)(\beta_I) = \lambda_B[\omega_B](\beta_I)$, $c_1(TE)(\beta_I) = (\lambda_B - k)[\omega_B](\beta_I)$, and $-c_1(E)(\beta_I) = k[\omega_B](\beta_I)$. So the above changes of Novikov parameter are all implied by $T^{\lambda_B} \mapsto T^{\lambda_B - k}\big(\sum n_i x_i\big)^k$, as claimed. □

Theorem 4.18 *There is a ring homomorphism (which is not a Λ–module homomorphism)*

$$\varphi\colon QH^*(B;\mathbb{K}[t,t^{-1}]) \to SH^*(E;\mathbb{K}[t,t^{-1}]), \tag{28}$$

$$\varphi(x_i) = x_i, \quad \varphi(t_B) = t_E c_1(E)^k = t_E\Big(\sum n_i x_i\Big)^k,$$

or equivalently, $\varphi(T_B^{\lambda_B}) = T_E^{\lambda_B - k}\left(\sum n_i x_i\right)^k$*. Over* $\mathbb{K}[t]$*,* φ *lifts to* $QH^*(E;\mathbb{K}[t])$*,*

(29) $$\begin{aligned}&\varphi\colon QH^*(B;\mathbb{K}[t]) \to QH^*(E;\mathbb{K}[t]),\\ &\varphi(x_i) = x_i, \quad \varphi(t_B) = t_E(\textstyle\sum n_i x_i)^k,\end{aligned}$$

so we obtain a factorization

$$\varphi\colon QH^*(B;\mathbb{K}[t]) \to QH^*(E;\mathbb{K}[t]) \to QH^*(E) \to SH^*(E).$$

(The restriction to $\mathbb{K}[t]$ *is necessary, as* $c_1(E)$ *is never invertible in* $QH^*(E)$ *by* [35]*.)*

Any polynomial relation $P(x, t_B, t_B^{-1}) = 0$ *in* $QH^*(B)$ *gives rise to the relation* $P(x, \varphi(t_B), \varphi(t_B)^{-1}) = 0$ *in* $SH^*(E)$*; and any polynomial relation* $P(x, t_B) = 0$ *in* $QH^*(B)$ *yields* $P(x, \varphi(t_B)) = 0$ *in* $QH^*(E)$*. (This applies to linear relations, quantum SR relations, characteristic/minimal polynomials of* $y \in H^{2d}(B) \cong H^{2d}(E)$ *but we don't claim the image is characteristic/minimal.)*

Proof The ring homomorphism φ in (28) and (29) is well-defined since the linear/quantum relations for B map to those for E. In particular, in (28) we use the fact that $\pi^* c_1(E)$ is an invertible in $SH^*(E)$ so $\varphi(t_B^{-1})$ is well-defined.

The final part of the claim would follow immediately from the existence of φ if the polynomials $P(x, t, t^{-1}), P(x, t)$ were known to vanish in the smaller rings $QH^*(B;\mathbb{K}[t, t^{-1}])$, $QH^*(B;\mathbb{K}[t])$ rather than in $QH^*(B)$. It remains to justify why the vanishing in $QH^*(B)$ implies the vanishing in those smaller rings. For monotone toric X, by exactness of localization at t,

(30) $$QH^*(X;\mathbb{K}[t, t^{-1}]) = QH^*(X;\mathbb{K}[t]) \otimes_{\mathbb{K}[t]} \mathbb{K}[t, t^{-1}].$$

Since the toric divisors and any of their intersections yield a chain complex computing the cohomology, the quantum cohomologies $QH^*(X;\mathbb{K}[t])$, $QH^*(X;\mathbb{K}[t, t^{-1}])$ are free modules over $\mathbb{K}[t]$, $\mathbb{K}[t, t^{-1}]$ respectively. Similarly, by exactness of completion in t, one can replace $\mathbb{K}[t, t^{-1}]$ in (30) by the ring $\mathbb{K}((t))$ of Laurent series. Since QH^* is $\mathbb{Z}$–graded, (30) also holds if we replace $\mathbb{K}[t, t^{-1}]$, $\mathbb{K}[t]$ by Λ, $\mathbb{K}((t))$ respectively. It follows that

$$QH^*(X) = QH^*(X;\Lambda) = QH^*(X;\mathbb{K}[t, t^{-1}]) \otimes_{\mathbb{K}[t, t^{-1}]} \Lambda,$$

and as before, the two QH^* are free modules over the relevant ring. Thus, if $P(x, t_B, t_B^{-1}) = 0$ in $QH^*(X)$ then this also holds in $QH^*(X;\mathbb{K}[t, t^{-1}])$. So in particular if $P(x, t_B) = 0$ in $QH^*(X)$, then this also holds in $QH^*(X;\mathbb{K}[t, t^{-1}])$ and hence in $QH^*(X;\mathbb{K}[t])$ by (30).

We now prove the claim about the characteristic/minimal polynomials of $y \in H^{2d}(B)$. Notice we are assuming that $y \in H^{2d}(B)$ does not involve t and lies entirely in a fixed degree. It follows that these polynomials are homogeneous with respect to the $\mathbb{Z}$–grading on $QH^*(B) = QH^*(B;\Lambda)$. Now consider the algebra $QH^*(B;\mathbb{K})$, in other words setting $t = 1$. This algebra is no longer $\mathbb{Z}$–graded like $QH^*(B)$, but it is still well-defined (in the monotone setting there are no compactness issues in defining the quantum cohomology). Observe that the minimal and characteristic polynomials for $y \in QH^*(B;\mathbb{K})$ and $y \in QH^*(B)$ are related by homogenization (ie inserting powers of t to ensure the monic polynomials are homogeneous in the $\mathbb{Z}$–grading). It follows that these polynomials only involve positive powers of t, so we may apply the map φ, using the fact that $\varphi(y) = y \in QH^{2d}(E)$ since it does not involve t. □

4I $SH^*(E)$ is the Jacobian ring

Theorem 4.19 *The Jacobian ring for E (see Definition 3.5) is isomorphic to $SH^*(E)$ via*

$$\begin{aligned} SH^*(E) \cong QH^*(E)[c]/(c\cdot\pi^*c_1(E)-1) &\to \mathrm{Jac}(W_E), \\ \mathrm{PD}[D_i] &\mapsto t^{-\lambda_i^E} z^{e_i}, \\ \pi^*c_1(E) = \mathrm{PD}[B] &\mapsto z_{n+1}, \end{aligned}$$

which is the i^{th} summand in the definition of the superpotential W (see Definition A.9). In particular, $c_1(TE)$ maps to W_E. Moreover, $t^{-\lambda_i^E} z^{e_i} = (tz_{n+1}^k)^{-\lambda_i^B} z^{(b_i,0)}$ in terms of the data b_i, λ_i^B which determines the moment polytope of B.

Proof It follows from Theorem 4.13, Corollary 3.9, Lemma A.6 and Example A.10. □

In Example A.10 of Appendix A we check that the superpotential of a negative line bundle E, with $c_1(E) = -k[\omega_B]$, is

$$\begin{aligned} W_E(z_1,\dots,z_n,z_{n+1}) &= z_{n+1} + W_B(z_1,\dots,z_n)\big|_{(t \text{ replaced by } tz_{n+1}^k)} \\ &= z_{n+1} + \varphi(W_B), \end{aligned}$$

where $\varphi(z_i) = z_i$ and $\varphi(t_B) = t_E z_{n+1}^k$ (as is consistent with Theorem 4.18). Working over $\mathbb{K}[t]$ instead of Λ, this φ defines a ring homomorphism

$$\varphi\colon \mathrm{Jac}(W_B;\mathbb{K}[t]) \to \mathrm{Jac}(W_E;\mathbb{K}[t])$$

identifiable precisely with the $\varphi\colon QH^*(B;\mathbb{K}[t]) \to SH^*(E;\mathbb{K}[t])$ from Theorem 4.18.

Remark 4.20 At this stage one could, as was done in [37], recover the eigenvalues of $c_1(TE)$ acting on $QH^*(E)$ in terms of those of $c_1(TB)$ acting on $QH^*(B)$, by working with $\mathrm{Jac}(W_E)$, $\mathrm{Jac}(W_B)$ and comparing the critical values of W_E, W_B. But this will not recover the multiplicities of the eigenvalues and it will not recover Theorem 4.22.

4J The eigenvalues of $c_1(TE)$ and the superpotential W_E

We now assume $\mathbb{K}$ is algebraically closed

in the definition of Λ in Section 2A (but we make no assumption on the characteristic). This is necessary so that we can freely speak of the eigenvalues of $c_1(TE)$ (the roots of the characteristic polynomial of quantum multiplication by $c_1(TE)$ on $QH^*(E)$). Indeed, for X monotone, $QH^*(X)$ can be defined over $\mathbb{K}$ at the cost of losing the $\mathbb{Z}$–grading, and a splitting $\prod(x-\mu_i)$ of the characteristic polynomial of $c_1(TX)$, where $\mu_i \in \mathbb{K}$, immediately yields a splitting $\prod(x-\mu_i T)$ when working over Λ instead of $\mathbb{K}$. This follows because $QH^*(X)$ is $\mathbb{Z}$–graded and x, T are both in degree 2. Observe that this factorization is not legitimate over $\mathbb{K}[t]$ in general, since $T = t^{1/\lambda_X}$ is a fractional power unless $\lambda_X = 1$. However, the following lemma (and later results) will show that the factors $x-\mu_i T$ can be collected in λ_X–families, so the resulting factorization (in higher-order factors) will be legitimate over $\mathbb{K}[t]$.

Since $c_1(TB) \in H^2(B,\mathbb{Z})$ is integral and $c_1(TB) = \lambda_B[\omega_B]$, by rescaling ω_B we may assume $[\omega_B]$ is a primitive integral class and $\lambda_B \in \mathbb{N}$ (called the index of the Fano variety B).

Lemma 4.21 *For monotone toric negative line bundles $E \to B$ with $c_1(E) = -k[\omega_B]$, we have that:*

1. *Nonzero eigenvalues $\lambda = \mu T_B$ of $c_1(TB)$ arise in λ_B–families of the form $\lambda, \xi\lambda, \dots, \xi^{\lambda_B - 1}\lambda$, where $\xi \neq 1$ is a λ_B^{th} root of unity.*
2. *There is a free action of the $(\lambda_E = \lambda_B - k)^{th}$ roots of unity on the critical points of W_E.*
3. *The nonzero eigenvalues of $c_1(TE)$ are precisely the critical values of W_E and they arise in families of size $\lambda_E = \lambda_B - k$.*
4. *The dimensions of the generalized subeigenspaces $\ker(c_1(TB) - \lambda\,\mathrm{Id})^d \subset QH^*(B)$ for $d \in \mathbb{N}$ are invariant under the action $\lambda \mapsto \xi\lambda$ by λ_B^{th} roots of unity.*
5. *The dimensions of $\ker(c_1(TE) - \lambda\,\mathrm{Id})^d \subset QH^*(E)$ are invariant under the action $\lambda \mapsto \xi\lambda$ by $(\lambda_B - k)^{th}$ roots of unity.*

Proof Recall from Section 3C that $QH^*(B) \cong \mathrm{Jac}(W_B)$, $c_1(TB) \mapsto W_B$ by Batyrev. For purely algebraic reasons (Ostrover and Tyomkin [32, Corollary 2.3 and Section 4.1]) it follows that the eigenvalues of $c_1(TB)$ acting on $QH^*(B)$ are precisely the critical values of W_B. (This result was originally proved by considering special Lagrangians in B, and is due to Kontsevich, Seidel, and Auroux [3, Section 6].) Then (1) follows by Corollary A.16.

Now run the same argument for E. Claim (2) follows by Corollary A.16. By Theorem 4.19,

$$\begin{aligned} \text{(31)} \quad & \mathrm{Jac}(W_E) \cong SH^*(E) \cong QH^*(E)/(\text{generalized 0–eigenspace of } \pi^* c_1(E)), \\ & W_E \mapsto c_1(TE). \end{aligned}$$

Again, for algebraic reasons, the first part of (3) follows, and the second part follows by (2).

Now we prove (4). Firstly, $\ker(c_1(TB) - \lambda)^d \cong \ker(W_B - \lambda)^d$ (acting on $\mathrm{Jac}(W_B)$). But

$$\ker(W_B(z) - \xi\lambda)^d = \ker \xi^d (W_B(\xi^{-1} z) - \lambda)^d \cong \ker(W_B(z) - \lambda)^d,$$

where in the first equality we use $W_B(\xi z) = \xi W_B(z)$ (Lemma A.15), and the second equality is the isomorphism $f(z) \mapsto f(\xi z)$. So (4) follows. Similarly, (5) follows from (31) by making $c_1(TE) - \lambda$ act on $SH^*(E)$ for $\lambda \neq 0$. (For $\lambda = 0$ there is nothing to prove.) □

For generation results, it will be important to know the dimensions of the generalized eigenspaces of the quantum multiplication action of $c_1(TE) = (\lambda_B - k)[\omega_E] \in QH^*(E)$ in terms of that for $c_1(TB) = \lambda_B[\omega_B] \in QH^*(B)$. The next result aims to describe the Jordan normal form (JNF) for $[\omega_E] = \pi^*[\omega_B] \in QH^*(E)$ in terms of the JNF for $[\omega_B] \in QH^*(B)$. Recall that the JNF of $[\omega_B]$ is determined by taking the primary decomposition of $QH^*(B)$ viewed as a finitely generated torsion module over the principal ideal domain (PID) $\Lambda[x]$ (using the fact that Λ is a field), where x acts by multiplication by $[\omega_B]$. Namely, each summand $\Lambda[x]/(x - \mu T)^d$ corresponds to a $d \times d$ Jordan block for $\lambda = \mu T$. By Lemma 4.21(4), for nonzero μ the factors $\Lambda[x]/(x - \mu T)^d$ arise in λ_B–families. By the Chinese Remainder Theorem such a family yields a summand $\Lambda[x]/(x^{\lambda_B} - \mu^{\lambda_B} t)^d$, corresponding to a λ_B–family of $d \times d$ Jordan blocks for the eigenvalues listed in Lemma 4.21(1). So the JNF yields the $\Lambda[x]$–module isomorphism (10).

In the following result we use the convention that for $f \in \mathbb{K}[t_B][x]$ the notation $\varphi(f)$ means we replace t_B by $t_E(-kx)^k$, as is consistent with the definition of φ in Theorem 4.18 and the x–actions. We remark that when $\mathrm{char}(\mathbb{K})$ divides k, the

SR relations in E are the classical SR relations, by Theorem 4.13, so $QH^*(E;\Lambda) \cong H^*(E) \otimes \Lambda$ as a ring, which we are not interested in.

Theorem 4.22 (assuming char($\mathbb{K}$) does not divide k) *The isomorphism in* (10) *determines the* $\Lambda[x]$*–module isomorphisms in* (11)*, with* x *acting as* $[\omega_E] = \pi^*[\omega_B]$*.*

The characteristic polynomial of $[\omega_E]$ *is the image under* φ *of the characteristic polynomial of* $[\omega_B]$*. The minimal polynomial of* $[\omega_E]$ *is, possibly after dropping some* x*–factors, the image of the minimal polynomial of* $[\omega_B]$*. The* $(\lambda_B - k)$*–family of nonzero eigenvalues* $\mu_j^E T_E$ *of* $\pi^*[\omega_B] \in QH^*(E)$ *of Lemma 4.21(1) arises from a* λ_B*–family* $\mu_j T_B$ *for* B*, via*

$$(\mu_j^E)^{\lambda_B - k} = (-k)^k \mu_j^{\lambda_B}.$$

Proof The proof is divided into steps.

Step 1 Given $f(x) \in \mathbb{K}[t_E][x]$ and $v \in QH^*(B;\mathbb{K}[t])$, suppose $f(x)\varphi(v) = 0$ in $QH^*(E;\mathbb{K}[t])$, where the polynomial $f(x)$ acts on $QH^*(E;\mathbb{K}[t])$ by making x act by multiplication by $[\omega_E] = \pi^*[\omega_B]$. Then $x^{\text{large}} f(x)v = 0$ in $QH^*(B;\mathbb{K}[t])$ after the change of variables

$$t_E(-kx)^k = t_B. \tag{32}$$

Proof of Step 1 Since $f(x)\cdot\varphi(v) = 0$, $f(x)\varphi(v)$ lies in the ideal generated by the linear relations and SR relations for E. By Theorems 4.13 and 4.18, the linear relations in B and E agree and the SR relations agree up to (32). Thus, $(-kx)^{\text{large}} f(x)\varphi(v)$ lies in the ideal generated by the linear/SR relations for B, where the factor $(-kx)^{\text{large}}$ ensures that all the occurrences of t_E can be turned into constant $\cdot\, x^{\text{positive}} \cdot t_B$ via (32). Thus, $(-kx)^{\text{large}} f(x)v = 0$ in B; making the power of $(-kx)$ larger if necessary to ensure that in the expression of $f(x)$, all occurrences of t_E are again replaced via (32). Step 1 follows since k is invertible in $\mathbb{K}$.

Step 2 We claim that (10) holds for $QH^*(B;\mathbb{K}[t,t^{-1}])$ after replacing $\Lambda[x]$ by $\mathbb{K}[t,t^{-1}][x]$.

Proof of Step 2 This is not immediate since $\mathbb{K}[t,t^{-1}][x]$ is not a PID. First, (10) holds for $QH^*(B;\mathbb{K})$, with $\Lambda[x]$ replaced by $\mathbb{K}[x]$ and t replaced by 1, since $\mathbb{K}$ is a field. This decomposition yields a decomposition of the unit $1 = \sum g_j + \sum h_{p+j} \in QH^*(B;\mathbb{K})$ in terms of polynomials $g_j, h_{p+j} \in \mathbb{K}[x]$ (corresponding to the units in the various summands) which are annihilated by $(x^{\lambda_B} - \mu_j^{\lambda_B})^{d_j}$ and $x^{d_{p+j}}$ respectively (and are not annihilated by any nonzero polynomials of lower degree). Now reinsert positive powers of t so as to make everything of homogeneous degree (recall that QH^* is $\mathbb{Z}$–graded when working with a graded Novikov variable t), to obtain

$$t^{\text{positive}} = \sum g_j + \sum h_{p+j} \in QH^*(B;\mathbb{K}[t]),$$

where $g_j, h_{p+j} \in \mathbb{K}[t][x]$ are annihilated by $(x^{\lambda_B} - \mu_j^{\lambda_B} t)^{d_j}$ and $x^{d_{p+j}}$ respectively (but not by lower degree polynomials). Over $\mathbb{K}[t,t^{-1}][x]$, we can rescale by $t^{-\text{positive}}$ to obtain a decomposition of 1, and then Step 2 follows.

Step 3 Observe that for $\xi^{\lambda_B} = 1$ and $\mu \in \mathbb{K}$, the image of a λ_B–family of factors

$$(x-\mu T_B)(x-\xi\mu T_B)\cdots(x-\xi^{\lambda_B-1}\mu T_B) = x^{\lambda_B} - \mu^{\lambda_B} t_B$$

via the map φ of Theorem 4.18 is, using $\sum n_i x_i = c_1(E) = -k[\omega_B] = -kx$,

$$x^{\lambda_B} - \mu^{\lambda_B} t_E(-kx)^k = x^k(x^{\lambda_B-k} - (-k)^k \mu^{\lambda_B} t_E).$$

Step 4 We claim that there is an isomorphism of $\mathbb{K}[t,t^{-1}][x]$–modules,

$$QH^*(E;\mathbb{K}[t,t^{-1}]) \cong SH^*(E;\mathbb{K}[t,t^{-1}]) \oplus \ker(x^{\text{large}}). \tag{33}$$

In particular, $SH^*(E;\mathbb{K}[t,t^{-1}])$ is a free $\mathbb{K}[t,t^{-1}]$–module, since $QH^*(E;\mathbb{K}[t,t^{-1}])$ is free (see the proof of Theorem 4.18).

Over the PID $\Lambda[x]$, the above would follow by Theorem 4.1. To obtain the splitting over $\mathbb{K}[t,t^{-1}][x]$, it suffices to prove that there are polynomials $a(x,t), b(x,t) \in \mathbb{K}[t][x]$ satisfying the equality $a(x,t)x^{\text{large}} + b(x,t)\overline{f} = t^{\text{positive}}$ (since rescaling by $t^{-\text{positive}}$ then decomposes 1), where $f = x^{\text{large}}\overline{f}$ is the minimal polynomial of $[\omega_E]$, and where $\overline{f}$ means we remove all x–factors from f. First working over $\mathbb{K}[x]$, Bézout's lemma would yield that equality for some polynomials $a(x,1), b(x,1) \in \mathbb{K}[x]$ if we put $t=1$ in $\overline{f}$. Now reinsert positive powers of t to make those polynomials of homogeneous degree, to obtain the required $a(x,t)$, $b(x,t)$. Step 4 follows.

Step 5 Applying φ to the decomposition $1 = \sum g_j + \sum h_{p+j}$ (over $\mathbb{K}[t,t^{-1}][x]$) we obtain a decomposition of $1 = \varphi(1) = \sum \varphi(g_j)$ in $SH^*(E;\mathbb{K}[t,t^{-1}])$. Here, since x acts invertibly on SH^*, it follows that $\varphi(h_{p+j}) = 0$ and, using Step 3, $(x^{\lambda_B-k} - (-k)^k \mu_j^{\lambda_B} t_E)^{d_j}$ annihilates $\varphi(g_j)$. Conversely, if a polynomial $f(x) \in \mathbb{K}[t,t^{-1}][x]$ annihilates $\varphi(g_j)$, then $x^{\text{large}} f(x) \in \mathbb{K}[t_E][x]$ annihilates $\varphi(g_j)$. So by Step 1, $x^{\text{larger}} f(x) \in \mathbb{K}[t_B][x]$ annihilates g_j in $QH^*(B;\mathbb{K}[t_B])$, and hence it must be divisible by $(x^{\lambda_B} - \mu_j^{\lambda_B} t_B)^{d_j}$. Applying φ shows that $f(x)$ must be divisible by $(x^{\lambda_B-k} - (-k)^k \mu_j^{\lambda_B} t_E)^{d_j}$, so the latter is the minimal polynomial annihilating $\varphi(g_j)$.

Step 6 Each $\varphi(g_j)$ generates a submodule C_i, namely the span of $\varphi(g_j)$, $x\varphi(g_j)$, $x^2\varphi(g_j), \dots,$ in $SH^*(E;\mathbb{K}[t,t^{-1}])$. We claim that $\sum C_i$ is direct; note that the theorem then follows.

Proof of Step 6 Suppose $\sum p_i(x) = 0$ in $SH^*(E;\mathbb{K}[t,t^{-1}])$, where $p_i(x) \in C_i$. Then $\sum x^{\mathrm{large}} p_i(x) = 0$ in $QH^*(E;\mathbb{K}[t])$. Step 1 implies $\sum x^{\mathrm{large}} p_i(x) = 0$ in $QH^*(B;\mathbb{K}[t])$. But in $QH^*(B;\mathbb{K}[t,t^{-1}])$, the summands generated by g_j, xg_j, $x^2 g_j, \dots$, as j varies, are direct by construction. Therefore $x^{\mathrm{large}} p_i(x) = 0$ in $QH^*(B;\mathbb{K}[t,t^{-1}])$ for each i. Thus $x^{\mathrm{large}} p_i(x) = 0$ in $SH^*(E;\mathbb{K}[t,t^{-1}])$, and so $p_i(x) = 0$ since x acts invertibly in SH^*. Step 6 thus follows. □

4K The Calabi–Yau case and the NEF case

We call the condition $c_1(TM)(\pi_2(M)) = 0$ from Section 2B the *Calabi–Yau case*, which ensures that the representation in Section 2B is defined. For toric M, $\pi_1(M) = 1$, so this condition is equivalent to $c_1(TM) = 0$. More generally, one can work with *NEF* toric M, meaning that $c_1(TM)(A) \geq 0$ for all nontrivial spheres $A \in \pi_2(M)$ which have a J–holomorphic representative (the Fano case corresponds to requiring a strict inequality, $c_1(TM)(A) > 0$). For closed toric manifolds, the NEF case is studied in McDuff and Tolman [30, Example 5.4]. The key observation is the following.

Lemma 4.23 *If we replace the monotonicity assumption by NEF, then Lemma 4.7 becomes*

$$r_{g_i^{\wedge}}(1) = \mathrm{PD}[D_i] + (\textit{higher order } t) \in QH^2(E),$$
$$\mathcal{R}_{g_i^{\wedge}}(1) = c^*(r_{g_i^{\wedge}}(1)) = c^*\,\mathrm{PD}[D_i] + (\textit{higher order } t) \in SH^2(E)^{\times}.$$

Proof This follows from Lemma 1.6. □

By Lemmas 4.23, 5.3 and 3.4, it follows that in the Batyrev presentation for (possibly noncompact) NEF toric varieties M we must replace the quantum SR relations (21) with

$$\mathcal{R}_{i_1} \cdots \mathcal{R}_{i_a} = s^{\omega(\beta_I)} T^{c_1(TM)(\beta_I)} \mathcal{R}_{j_1}^{c_1} \cdots \mathcal{R}_{j_b}^{c_b},$$

where $\mathcal{R}_j = \mathcal{R}_{g_j^{\wedge}}(1)$, and we work over a modified Novikov ring $\mathfrak{R}$; see Sections 2B and 5B. The lowest order T terms on each side of the equation agree with those in (21), but the higher order T terms are hard to compute in practice.

For negative line bundles $E \to B$ satisfying weak+ monotonicity (Section 2B), $r_{g_f^{\wedge}}(1) = \pi^* c_1(E)$ may not in fact hold, where g_f is the natural rotation in the fiber. In [35] we proved that $r_{g_f^{\wedge}}(1) = (1+\lambda_+)\pi^* c_1(E)$, where $\lambda_+ \in \Lambda$ involves only strictly positive powers of t (and s^0–terms). This rescaling does not affect $SH^*(E) \cong QH^*(E)/\ker \pi^* c_1(E)^{\mathrm{large}}$ (Theorem 4.1).

The Calabi–Yau case is $k = \lambda_B$. By [35], $SH^*(E)$ is $\mathbb{Z}$–graded, finite-dimensional, and has an automorphism $\mathcal{R}_{g_f^{\wedge}}$ of degree 2. It follows that $SH^*(E) = 0$ and

$\pi^* c_1(E) \in QH^*(E)$ is nilpotent. So the wrapped Fukaya category $\mathcal{W}(E)$ (assuming it is defined) would not be interesting since it would be homologically trivial (being a module over $SH^*(E)$).

5 Twisted theory: nonmonotone toric symplectic forms

5A Toric symplectic forms

For this background section, we refer for details to Ostrover and Tyomkin [32], Fulton [17, Section 3.4], Batyrev [5] or Cox and Katz [12, Section 3.3]. Some of the terminology is also illustrated in Appendix A.

Recall that from a *fan* $\Sigma \in \mathbb{Z}^n$ one can construct a toric variety $X = X_\Sigma$; in particular, this determines a complex structure J on X_Σ. We always assume that X_Σ is smooth.

Recall that a *piecewise linear function* F on Σ means a real-valued function which is linear on each cone σ of Σ, thus $F(v) = \langle u_\sigma, v\rangle$ for some $u_\sigma \in \mathbb{R}^n$, whenever $v \in \sigma$. So F on σ is determined by linearity by the values $F(e_i) = \langle u_\sigma, e_i\rangle$ for those edges e_i of Σ which lie in σ. F is *strictly convex* if $\langle u_\sigma, v\rangle > F(v)$ for $v \notin \sigma$, equivalently if $\langle u_\sigma, e_i\rangle > F(e_i)$ for $e_i \notin \sigma$.

Choosing a piecewise linear strictly convex function F on Σ is equivalent to choosing a Kähler form ω_F satisfying

$$[\omega_F] = \sum -F(e_i)\,\mathrm{PD}[D_i],$$

where $D_i \subset X_\Sigma$ are the toric divisors. In particular, (X_Σ, ω_F) is then a toric manifold (ie the torus action is Hamiltonian) and the symplectic form ω_F on X_Σ is J–compatible. We call these the *toric symplectic forms*. The moment polytope of (X_Σ, ω_F) is

$$\Delta_F = \{y \in \mathbb{R}^n : \langle y, e_i\rangle \geq \lambda_i\},$$

where e_i are the edges of the fan (which are inward normals to the facets of Δ_F), and

$$\lambda_i = F(e_i).$$

Piecewise linear strictly convex functions F, for which all $F(e_i)$ are integers, correspond to ample divisors: the divisor is $D_F = \sum -F(e_i) D_i$. The canonical divisor is $K = -\sum D_i$.

We always assume that X_Σ is Fano; that is, the anti-canonical bundle $\Lambda^{\mathrm{top}}_{\mathbb{C}} TX_\Sigma$ is ample. Thus the ample anti-canonical divisor $-K = \sum D_i$, corresponding to $c_1(TX) = \sum \mathrm{PD}[D_i]$, corresponds to the piecewise linear function $F(e_i) = -1$ for all e_i. This corresponds to a symplectic form ω_Δ satisfying $[\omega_\Delta] = c_1(TX) \in H^2(X)$.

This is the unique symplectic form for which the corresponding moment polytope Δ is *reflexive* (see Section A4).

We will always denote by ω_X the monotone integral Kähler form obtained from rescaling ω_Δ, so that $\omega_X \in H^2(X,\mathbb{Z})$ is a primitive class. Thus,

$$c_1(TX) = [\omega_\Delta] = \lambda_X[\omega_X],$$

where the positive integer λ_X is called the *index* of the Fano variety, and it is also the monotonicity constant for the monotone symplectic manifold (X_Σ, ω_X).

5B The Novikov ring $\mathfrak{R}$ over Λ_s

In the nonmonotone case, we need to change the Novikov ring (Section 2A). Since the values of $c_1(TX)$ and ω_F on spheres may no longer be proportional, we use two formal parameters s, t instead of one. In the definition of the quantum product, defining $QH^*(X,\omega_F)$, we use weights

$$s^{\omega_F(A)} T^{c_1(TX)(A)} = s^{\omega_F(A)} t^{\omega_X(A)}$$

when counting spheres A in X (recall that $t = T^{\lambda_X}$). The Novikov ring is now defined as

$$\mathfrak{R} = \Lambda_s((T)) = \Lambda_s[T^{-1}, T],$$

that is, Laurent series in the formal variable T with coefficients in the Novikov field Λ_s,

$$\Lambda_s = \left\{ \sum_{i=0}^{\infty} a_i s^{n_i} : a_i \in \mathbb{K}, n_i \in \mathbb{R}, \lim n_i = \infty \right\},$$

where s is a new formal parameter lying in grading $|s| = 0$, and $|T| = 2$ (since $t = T^{\lambda_X}$ has $|t| = 2\lambda_X$). When F is integer-valued, one could further restrict n_i to lie in $\mathbb{Z}$.

As usual, t ensures that QH^* is $\mathbb{Z}$–graded. It is not necessary to complete in t (ie allow series in positive powers of t). At the cost of losing the $\mathbb{Z}$–grading, one could omit t altogether. We complete in t because we want $\mathfrak{R}$ to be a field (this is important in the generation results of Section 6D, although Remark 6.7 shows a work-around). If one wants $\mathfrak{R}$ to be algebraically closed, it suffices to allow arbitrary real powers of t with the growth condition as in Section 2A (alternatively, one can take the field of Puiseux series in t over Λ_s). This will not be necessary for us: we only need to factorize the characteristic polynomial of $c_1(TX)$, and to factorize this into linear factors it is enough to have the root $T = t^{1/\lambda_X}$.

Technical remark As in Section 4J, when discussing eigenvalues of $c_1(TX)$ we want Λ_s to be algebraically closed. As in 4J, the presence or absence of T is not an issue: the $\mathbb{Z}$–grading imposes how to insert powers of T into a factorization of the characteristic polynomial of $c_1(TX)$ over Λ_s. We recall that if $\mathbb{K}$ is an algebraically closed field of characteristic zero, then Λ_s is algebraically closed [16, Lemma A.1].

5C Using a nonmonotone toric form is the same as twisting

We emphasize that the moduli spaces of spheres that we count have not changed, since we are using the same complex structure J (which is compatible with both Kähler forms ω_F and the monotone ω_X). In particular, there are no compactness issues in the definition of the quantum or Floer cohomologies for ω_F because J and small perturbations of J are also tamed by the monotone form ω_X (and we can use $c_1(TX)(A) = \lambda_X \omega_X(A)$ to control indices of solutions). Observe that we are only changing the s–weights with which we count the solutions: for ω_F we use weight $s^{\omega_F(A)} T^{c_1(TX)(A)}$ whereas for the monotone form ω_X we would just use $T^{c_1(TX)(A)}$. By convention we will omit the irrelevant factor $s^{\omega_X(A)} = s^{\lambda_X c_1(TX)(A)}$ in the monotone case, since this can be recovered by formally replacing T with $s^{\lambda_X} T$. This convention ensures, as we will now explain, that the theory for ω_F is just the ω_F–twisted theory for ω_X.

We briefly recall from [33; 34] the definition of the *twisted quantum cohomology* and the *twisted Floer cohomology* for the form ω_X twisted by the 2–form ω_F. One introduces a system $\underline{\mathfrak{R}}_{\omega_F}$ of local coefficients, meaning that one counts Morse/Floer/GW solutions u with an extra weight factor $s^{[\omega_F](u)}$. Here, $[\omega_F](u)$ can be defined either by evaluating the cocycle $[\omega_F]$ on the chain u, or by integrating $u^*\omega_F$ over the domain of u. Since ω_F is a 2–cocycle, this weight is trivial (s^0) when u is a Morse trajectory, thus the twisting does not affect the vector space $QH^*(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F}) = H^*(X_\Sigma) \otimes \mathfrak{R}$; but it does affect the quantum product by inserting the weights $s^{\omega_F(A)}$ when u is a sphere in the class $A \in H_2(X_\Sigma)$.

Corollary 5.1 *The quantum cohomology for ω_F can be identified with the ω_F–twisted quantum cohomology for ω_X,*

$$QH^*(X_\Sigma, \omega_F) \cong QH^*(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F}).$$

The construction of the twisted Floer cohomology and twisted symplectic cohomology, and the proof that these have a product structure when twisting by closed two-forms η is carried out in [34]. The discussion in Sections 2B and 2C on the construction of invertibles can be carried out in the twisted case, simply by working over $\mathfrak{R}$ and inserting weights $s^{\eta[u]}$.

Technical remark In [35], for weak+ monotone manifolds M, we worked over a very large Novikov ring generated by $\pi_2(M)$ (modulo those classes on which ω_M and $c_1(TM)$ both vanished). One can just as well only keep track of the ω_M and $c_1(TM)$ values on spheres by using weights s and T as described above (using only T in the monotone case, by convention).

In particular, in the GW–section counting in [35] for the bundles $E_g \to S^2$ with fiber M, we use weights $s^{\eta(A)}$ where A is the relevant $\pi_2(M)$ class associated with the section (explicitly, in the notation of the proof of Lemma 1.6, this would be the weight $s^{\eta(\gamma)}$).

Corollary 5.2 *The symplectic cohomology for ω_F can be identified with the ω_F–twisted symplectic cohomology for ω_X,*

$$SH^*(X_\Sigma, \omega_F) \cong SH^*(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F}).$$

5D Rotations and the lifting problem

It follows that the Seidel representation for closed Fano toric varieties, and the representation defined in Section 2C for noncompact Fano toric varieties, are defined for ω_F and coincide with the ω_F–twisted representations obtained for ω_X. In particular, Lemma 4.7 still holds,

$$\mathcal{R}(g_i^\wedge) = x_i,$$

since the constant sections lie in the class $\gamma = 0$, so no s–weight appears.

The lifting problem that already occurred in the monotone case (Section 4D) is tricky in the nonmonotone case, as Lemma 4.7 is no longer available. The following resolves this issue.

Lemma 5.3 *Let (X_Σ, ω_F) be a Fano toric manifold. Any relation $\prod g_i^{a_i} = \mathrm{id}$ for $a_i \in \mathbb{Z}$ (corresponding to a relation $\sum a_i e_i = 0$ amongst the edges of Σ) yields the relation*

$$\prod (g_i^\wedge)^{a_i} = s^{-\sum F(e_i) a_i}\, T^{\sum a_i}\, \mathrm{id}.$$

Proof By construction, $\psi = \prod (g_i^\wedge)^{a_i}$ is a lift of the identity map, and so differs from the identity by an element of the deck group $\Gamma = \pi_2(M)/\pi_2(M)_0$ (see Section 2B). This deck group can be identified with the monomials $s^n T^m$. Just as in the monotone case, $m = c_1(TX)[\beta_a] = \sum a_i$, where $\beta_a \in H_2(X)$ is the homology class corresponding to the relation $\sum a_i e_i = 0$ (see the proof of Theorem 4.17). Recall from Section 3A that β_a satisfies the intersection products $\beta_a \cdot D_i = a_i$. We claim that the power of s^n above is

$$n = \omega_F(\beta_a) = \sum -F(e_i)\,\mathrm{PD}(D_i)(\beta_a) = -\sum F(e_i) a_i.$$

To prove this, it suffices to show that the image of the constant disc (c_x, x) at a point $x \in X_\Sigma \setminus \bigcup D_i$ under the action of ψ is a sphere representing the class β_a. Since ψ is a lift of the identity map $\prod g_i^{a_i}$, we know that ψ maps the constant loop x to itself, therefore it maps the constant disc c_x to a new disc bounding the constant x, so this new disc is in fact a sphere. To determine that it represents β_a it now remains to check that the sphere intersects a_i times the divisor D_i.

We may assume x has homogeneous coordinates $x_i = 1$. The image under g_1 of the path $(r, 1, \dots, 1)_{1 \geq r \geq 0}$ from $x = (1, \dots, 1)$ to the point $p_1 = (0, 1, \dots, 1) \in D_1$ is $\gamma_r(t) = (re^{2\pi i t}, 1, \dots, 1)$. The loop $\gamma_r(t)$ can be filled by the obvious disc $\Gamma_r(s,t) = (\gamma_s(t))_{r \geq s \geq 0}$. As r varies, this gives a family of discs, ending at the constant disc (c_{p_1}, p_1) at $p_1 \in D_1$. Since $g_1^\wedge(c_{p_1}) = c_{p_1}$ by definition of the lift $g_1^\wedge$, it follows that $g_1^\wedge(c_x) = \Gamma_1(s,t)$. Similarly, $(g_1^\wedge)^{a_1}(c_x)$ is represented by the obvious disc $(se^{2\pi i a_1 t}, 1, \dots, 1)$ where $0 \leq s, t \leq 1$.

Inductively, we claim that $(g_k^\wedge)^{a_k} \cdots (g_1^\wedge)^{a_1} \cdot (c_x, x)$ is represented by the disc

$$c(s,t) = (se^{2\pi i a_1 t}, \dots, se^{2\pi i a_k t}, 1, \dots, 1), \tag{34}$$

where $0 \leq s, t \leq 1$.

For the inductive step, we use the following general trick. Since $\pi_1(X) = 1$, any loop $y = y(t)$ gives rise to a disc $\overline{y} = \overline{y}(s,t)$ with $\overline{y}(1,t) = y(t)$ and $\overline{y}(0,t)$ equal to a chosen basepoint. Given a filling disc (c, y) for y, we obtain a sphere $c \# -\overline{y}$ representing some class $\lambda = s^n T^m \in \Gamma$. Thus $(c, y) = \lambda \cdot (\overline{y}, y)$ so $g_i^\wedge(c, y) = \lambda g_i^\wedge(\overline{y}, y)$. If the basepoint is chosen to lie in D_i, then $g_i^\wedge(\overline{y})$ is the disc $(g_i)_t \overline{y}(s,t)$; this follows by the same argument as in the previous paragraph, by considering the path $\overline{y}(r,t)$ of loops and the path of discs $(\overline{y}(s,t))_{r \geq s \geq 0}$.

Now, the inductive step. Apply the observation to (c, y) for $c(s,t)$ as in (34). The homotopy $\overline{y}$ from $y(t) = (e^{2\pi i a_1 t}, \dots, e^{2\pi i a_k t}, 1, 1, \dots, 1)$ to $(1, \dots, 1, 0, 1, \dots, 1) \in D_{k+1}$ first homotopes the first coordinate to 1, then homotopes the second coordinate to 1, and so forth, and finally it follows the path from $x_{k+1} = 1$ to $x_{k+1} = 0$ in that coordinate. By construction, the class of the sphere $c \# -\overline{y}$ is trivial in Γ since we produced a path of discs from c to $\overline{y}$. Applying $(g_{k+1})^{a_{k+1}}$ to the path of loops $\overline{y}(s, \cdot)$ corresponds to acting by $e^{2\pi i a_{k+1} t}$ on the x_{k+1} coordinate, and it now easily follows that the class of this image disc is the same as the class of $(se^{2\pi i a_1 t}, \dots, se^{2\pi i a_{k+1} t}, 1, \dots, 1)$, as required for the inductive step.

By induction, it follows that $\prod (g_i^{\wedge})^{a_i} \cdot (c_x, x)$ is represented by the disc

$$(se^{2\pi i a_1 t}, \dots, se^{2\pi i a_r t}),$$

which intersects $D_i = (x_i = 0)$ precisely a_i times, as required. □

5E Presentation of $QH^*(B, \omega_F^B)$, $QH^*(E, \omega_F^E)$, $SH^*(E, \omega_F^E)$ in the Fano case.

By Lemma 5.3, the presentation of $QH^*(B)$ for closed monotone $X_\Sigma = B$ (Section 3B) and that of $QH^*(E)$, $SH^*(E)$ for monotone toric negative line bundles $X_\Sigma = E$ (Theorem 4.13) holds for the nonmonotone form ω_F by twisting coefficients. In the closed case, this was proposed by Batyrev [5, Section 5], and proved by Givental [21; 20], Cieliebak and Salamon [11] and McDuff and Tolman [30, Section 5].

Corollary 5.4 *In the notation of Theorem 4.13, but now using a (nonmonotone) toric form ω_F^B on B and working over $\mathfrak{R}$, abbreviating $\lambda_i = F(e_i)$, we have:*

$$QH^*(B, \omega_F^B) \cong \mathfrak{R}[x_1, \dots, x_r] \Big/ \Big(\text{linear relations in } B, \text{ twisted SR relations: } \prod x_{i_p} = s^{-\sum \lambda_{i_p} + \sum c_q \lambda_{j_q}} T^{|I^B| - \sum c_q} \cdot \prod x_{j_q}^{c_q} \Big),$$

$$QH^*(E, \omega_F^E) \cong \mathfrak{R}[x_1, \dots, x_r] \Big/ \Big(\text{linear relations in } B, \text{ twisted SR relations after Novikov parameter change: } \prod x_{i_p} = s^{-\sum \lambda_{i_p} + \sum c_q \lambda_{j_q}} T^{|I^B| - \sum c_q - c_f} \cdot \cdot \Big(\sum n_i x_i\Big)^{c_f} \cdot \prod x_{j_q}^{c_q} \Big),$$

$$SH^*(E, \omega_F^E) \cong \mathfrak{R}[x_1, \dots, x_r, z] \Big/ \Big(z \cdot \sum n_i x_i - 1, \text{ and the same relations as for } QH^*(E, \omega_F^E) \Big),$$

where, by Lemma A.6, $F^E(e_i) = \lambda_i^E = \lambda_i$ and $F^E(e_f) = \lambda_f^E = 0$ (so $s^{c_f \lambda_f}$ does not appear). In particular, the form $\omega_F^E = \pi^* \omega_F^B + \pi \Omega$ on E arises as ω in Section 4A from $\omega_F^B \in H^2(B)$ in place of ω_B, and corresponds to the piecewise linear function F^E on the fan for E.

Analogues of Theorems 4.18–4.22 and Lemma 4.21 hold. So there is a ring homomorphism

$$\varphi \colon QH^*(B, \omega_F^B; \Lambda_s[t, t^{-1}]) \to SH^*(E, \omega_F^E; \Lambda_s[t, t^{-1}]),$$

with $\varphi(x_i) = x_i$, $\varphi(t_B) = t_E\big(\sum n_i x_i\big)^k$ *and* $\varphi(s_B) = s_E$. *Over* $\Lambda_s[t]$, *this factorizes as*

$$\varphi\colon QH^*(B, \omega_F^B; \Lambda_s[t]) \to QH^*(E, \omega_F^E; \Lambda_s[t]) \to SH^*(E, \omega_F^E; \Lambda_s[t]).$$

These can be identified with the ring homomorphisms obtained for the monotone form ω_B *but twisting coefficients using* ω_F^B *and* ω_F^E *respectively.*

5F The F–twisted superpotential W_F and $\mathrm{Jac}(W_F)$

The *F–twisted superpotential* W_F of X_Σ is the superpotential associated to (X_Σ, ω_F),

$$W_F\colon (\mathfrak{R} \setminus \{0\})^n \to \mathfrak{R}, \quad W_F(z) = \sum s^{-F(e_i)} T z^{e_i}.$$

In the case $F(e_i) = -1$, that is, using the monotone ω_Δ, this becomes $\sum s T z^{e_i}$, so we can ignore s and we obtain the untwisted superpotential for ω_Δ.

For closed toric (X_Σ, ω_F), it follows from the presentation of QH^* and Section 3C that

$$QH^*(X_\Sigma, \omega_F) \cong \mathrm{Jac}(W_F), \quad \mathrm{PD}[D_i] \mapsto s^{-F(e_i)} T z^{e_i}, \quad c_1(TX) \mapsto W_F,$$

where $\mathrm{Jac}(W_F) \equiv \mathfrak{R}[z_1^{\pm 1}, \dots, z_n^{\pm 1}]/(\partial_{z_1} W_F, \dots, \partial_{z_n} W_F)$ for $n = \dim_{\mathbb{C}} X$.

Theorem 5.5 (analogue of Theorem 4.19) *There is an isomorphism*

$$\begin{aligned} SH^*(E, \omega_F^E) \cong QH^*(E, \omega_F^E)[c]/(c \cdot \pi^* c_1(E) - 1) \to \mathrm{Jac}(W_F^E), \\ \mathrm{PD}[D_i] \mapsto s^{-\lambda_i^E} T z^{e_i}, \\ \pi^* c_1(E) = \mathrm{PD}[B] \mapsto T z_{n+1}. \end{aligned}$$

In particular, $c_1(TE)$ *maps to* W_F^E. *Moreover,* $s^{-\lambda_i^E} T z^{e_i} = (s z_{n+1}^k)^{-\lambda_i^B} T z^{(b_i, 0)}$.

As in Section 4I, the map $\varphi\colon QH^*(B, \omega_F^B; \Lambda_s[t, t^{-1}]) \to SH^*(E, \omega_F^E; \Lambda_s[t, t^{-1}])$ corresponds to the map

$$\begin{aligned} \varphi\colon \mathrm{Jac}(W_F^B; \Lambda_s[t, t^{-1}]) \to \mathrm{Jac}(W_F^E); \Lambda_s[t, t^{-1}]), \\ \varphi(z_i) = z_i, \quad \varphi(t_B) = t_E z_{n+1}^k, \quad \varphi(s_B) = s_E. \end{aligned}$$

6 The Fukaya category and generation results

6A The Fukaya category for nonmonotone toric forms versus the twisted Fukaya category

Recall by [34] that one can twist the Lagrangian Floer complexes $CF^*(L, L')$ by a closed two-form $\eta \in H^2(M)$ which vanishes on L and L'. Namely, one introduces a system of local coefficients $\underline{\mathfrak{R}}_\eta$, and disc counts are weighted by $s^{\eta(u)}$. By Stokes' theorem, these weights are invariant under a homotopy of the disc u as long as the boundary of u moves within $L \cup L'$; the integral of η over the path within $L \cup L'$ will vanish since η vanishes on $L \cup L'$. Similarly, one can twist A_∞–operations using η, and thus define an η–twisted Fukaya category $\mathcal{F}(M; \underline{\mathfrak{R}}_\eta)$, provided that one restricts to considering only Lagrangians L on which η vanishes. In order to drop that restriction on Lagrangians, more work is required: assuming $\mathbb{K}$ has characteristic zero, one can define the bulk deformation $\mathcal{F}(M; D)$ for a divisor $D \subset M$ by work of Fukaya, Oh, Ohta and Ono [15] (see [16] in the toric setting), which corresponds to twisting by η when η is Poincaré dual to D. One can also allow $\mathbb{R}$–linear combinations of divisors $\sum -\lambda_i D_i$, in which case each positive intersection of a disc u with D_i gets counted with weight $s^{-\lambda_i}$. These are the particularly simple bulk deformations in real codimension 2 which do not involve counting new, higher-dimensional, moduli spaces; they only involve introducing weights in the original counts.

We are concerned with twisting the monotone toric manifold (X_Σ, ω_X) by ω_F, and comparing the ω_F–twisted Fukaya category $\mathcal{F}(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F})$ with the (untwisted) Fukaya category for (X_Σ, ω_F). If we restrict the Fukaya category to $\mathcal{F}^{\text{toric}}$, meaning we only allow toric Lagrangians L (the $\operatorname{codim}_{\mathbb{C}} = 1$ complex torus orbits, together with holonomy data), then the objects L are Lagrangian submanifolds both for ω_X and ω_F, provided all $F(e_i)$ are close to -1. Indeed, following [23, Sections 2.1–2.2], when $F(e_i)$ is close to -1, the moment polytopes Δ_F and Δ_X (the polytopes for ω_F and ω_X respectively) will undergo a small variation that does not change the diffeomorphism type of the toric manifolds X_{Δ_F} and X_{Δ_X} built as symplectic reductions (via the Delzant construction). One can then show [23, Theorem 2.7] that X_{Δ_F} is T^n–equivariantly symplectomorphic to (X_{Δ_X}, ω_F). In particular, ω_F will then vanish on the toric Lagrangians L in (X_Σ, ω_X), so the twisted category $\mathcal{F}^{\text{toric}}(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F})$ is well-defined.

Corollary 6.1 *When all the $F(e_i)$ are close to -1, there is a natural identification between the twisted category $\mathcal{F}^{\text{toric}}(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F})$ and $\mathcal{F}^{\text{toric}}(X_\Sigma, \omega_F)$.* □

Remark 6.2 Since $H_2(X)$ is generated by toric divisors, ω_F is Poincaré dual to an $\mathbb{R}$–linear combination D_F of toric divisors so the bulk-deformed category $\mathcal{F}(X_\Sigma, \omega_X; D_F)$

is defined and naturally contains the full subcategory $\mathcal{F}^{\mathrm{toric}}(X_\Sigma, \omega_X; \underline{\mathfrak{R}}_{\omega_F})$. However, it is less clear how to compare the bulk-deformed category with $\mathcal{F}(X_\Sigma, \omega_F)$, since the (nontoric) Lagrangians have changed when passing from ω_X to ω_F. However, for the purposes of proving that the generation criterion holds for toric Lagrangians, we anyway restrict to the subcategory of toric Lagrangians and check that the open–closed string map hits an invertible element. So for the purposes of generation, we only deal with the categories in Corollary 6.1. In particular, we bypass the technical issue of defining the Fukaya category $\mathcal{F}(X_\Sigma, \omega_F)$ in the nonmonotone setting and proving that the structural results of [37] (the $\mathcal{OC}$–map, the QH^*–module structures, the generation criterion) generalize from the case of monotone Kähler forms ω_X to the case of a (nonmonotone) Kähler form ω_F close to ω_X (so both are compatible with the complex structure J). This is a technical issue in the sense that the Kuranishi machinery of [15] is very likely to succeed in this generalization, but we will not undertake this onerous task. We emphasize that, when restricting to toric Lagrangians, this is not an issue since the Lagrangians for ω_X and ω_F are then the same and thus ω_X can be used to control J–holomorphic discs in the construction of $\mathcal{F}^{\mathrm{toric}}(X_\Sigma, \omega_F)$.

Finally, the constructions of Ritter and Smith [37] obtained for monotone symplectic manifolds apply immediately to the ω_F–twisted setting and to the bulk-deformed setting when deforming by divisors. Again, this is because we never change the moduli spaces being counted, but only the weights in these counts. In particular, the open–closed string map into twisted QH^* is defined, it is a module map over twisted QH^*, and the generation criterion holds.

6B What it means for the toric generation criterion to hold

Recall that the generation criterion of Abouzaid [1], generalized to the monotone setting by Ritter and Smith [37], states that for X a noncompact symplectic manifold conical at infinity, assuming exactness or monotonicity, a full subcategory $\mathcal{S}$ will split-generate the wrapped Fukaya category $\mathcal{W}(X)$ if $1 \in \mathcal{OC}(\mathrm{HH}_*(\mathcal{S}))$, where $\mathcal{OC}$ is the *open–closed string map*

$$\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{W}(X)) \to SH^*(X).$$

By Ritter and Smith [37], this is an SH^*–module map so it is in fact enough to hit an invertible element. The discussion, here and below, holds analogously for the compact Fukaya category $\mathcal{F}(X)$, and it also holds for closed monotone symplectic manifolds X, in which cases one considers the QH^*–module homomorphism

$$\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}(X)) \to QH^*(X).$$

A closer inspection of the argument shows that in fact a weaker condition is required, namely that for any object K of $\mathcal{W}(X)$ which one hopes to split-generate, one wants

$$\mathrm{HH}_*(\mathcal{S}) \subset \mathrm{HH}_*(\mathcal{W}(X)) \xrightarrow{\mathcal{OC}} SH^*(X) \xrightarrow{\mathcal{CO}} HW^*(K,K)$$

to hit the unit in the wrapped Lagrangian Floer cohomology $HW^*(K,K)$. Since the *closed–open string map* $\mathcal{CO}$ is a ring homomorphism, this is automatic if $\mathcal{OC}$ hits 1 since $\mathcal{CO}(1) = 1$.

In the monotone case (resp. closed monotone case) we must split the category into summands $\mathcal{W}_\lambda(X)$ (resp. $\mathcal{F}_\lambda(X)$), indexed by the m_0–value of the objects, so $\mathfrak{m}_0(L) = m_0(L)\,[L] = \lambda\,[L]$. Once we restrict $\mathcal{OC}$ to this summand, it becomes very unlikely that it will hit the unit. We now discuss the implications and the remedy to this issue.

Lemma 6.3 *If K does not intersect a representative of* $\mathrm{PD}(c_1(TX))$, *then the map* $\mathcal{CO}\colon SH^*(X) \to HW^*(K,K)$ *(resp.* $\mathcal{CO}\colon QH^*(X) \to HF^*(K,K)$*) vanishes on the following:*

1. *The generalized eigensummands* $SH^*(X)_\lambda$ *(resp.* $QH^*(X)_\lambda$*) for* $\lambda \neq m_0(K)$.
2. *The ideal* $(c_1(TX) - m_0(K)) * SH^*(X)$ *(resp.* $(c_1(TX) - m_0(K)) * QH^*(X)$*). So* $\mathcal{CO}$ *vanishes on eigenvectors of* $c_1(TX)$ *arising in Jordan blocks of size at least* 2.

Proof Multiplication by $c_1(TX) - m_0(K)$ is invertible on a generalized eigensummand G for eigenvalues $\lambda \neq m_0(K)$. So $(c_1(TX) - m_0(K))G = G$. But $\mathcal{CO}$ is a ring map, so on any multiple of $c_1(TX) - m_0$ it will vanish,

$$\begin{aligned} \mathcal{CO}((c_1(TX) - m_0(K))y) &= (\mathcal{CO}(c_1(TX)) - m_0(K)) * \mathcal{CO}(y) \\ &= (m_0(K) - m_0(K))\mathcal{CO}(y) = 0. \end{aligned}$$

Here we used the equation $\mathcal{CO}(c_1(TX)) = m_0(K)\,[K]$, due to Kontsevich, Seidel and Auroux [3] (see also the explanation in [37]). We point out that this equation is the count of holomorphic discs bounding L, through a generic point of L, which hit $\mathrm{PD}(c_1(TX))$. For index reasons (using monotonicity and the fact that K is orientable), only index 0 and 2 discs contribute. The index 0 discs are constant by monotonicity, and they would contribute $[K \cap \mathrm{PD}(c_1(TX))]$, but the assumption that K does not intersect a representative of $\mathrm{PD}(c_1(TX))$ ensures that this term does not arise. Finally, $m_0(K)$ is by definition the count of the index 2 discs.

The remark about Jordan blocks of size at least 2 also follows, since such a block arises as a summand $\mathfrak{R}[x]/(x - m_0(K))^d$ for $d \geq 2$ when we decompose SH^* (resp. QH^*) as an $\mathfrak{R}[x]$–module, with x acting by multiplication by $c_1(TX)$. The eigenvectors in this summand are multiples of $(x - m_0(K))^{d-1}$, so they vanish under $\mathcal{CO}$. □

That K does not intersect a representative of the anticanonical class $\mathrm{PD}(c_1(TX))$ holds for toric Lagrangians since they do not intersect the toric divisors. However, it holds in fact in general, after homotoping the representative, as follows. We emphasize that we only work with orientable Lagrangian submanifolds [37] (so we may use Poincaré duality arguments). The following would fail in the nonorientable setting of $\mathbb{RP}^2 \subset \mathbb{CP}^2$.

Lemma 6.4 *For any (orientable) Lagrangian submanifold $K \subset X$ in a monotone symplectic manifold (X, ω_X), K does not intersect some representative of* $\mathrm{PD}(c_1(TX))$.

Proof $D = PD[c_1(TX)]$ can be represented by the vanishing of a generic smooth section $s\colon X \to \mathcal{E}$ of a complex line bundle $\mathcal{E} \to X$ with Chern class $c_1(TX)$. Now for any Lagrangian $j\colon K \hookrightarrow X$, the first Chern class of the pull-back bundle $j^*\mathcal{E}$ is $j^*c_1(TX)$. But $c_1(TX) = \lambda_X[\omega_X]$ by monotonicity, so $j^*c_1(TX) = \lambda_X[j^*\omega_X] = 0 \in H^2(X;\mathbb{R})$ since K is Lagrangian. So $j^*\mathcal{E}$ is a trivial smooth line bundle. Moreover, the intersection $K \cap D$ (for generic s, $D = s^{-1}(0)$ will be transverse to K) can be represented by the vanishing of the pulled-back section j^*s. Since $j^*\mathcal{E}$ is trivial, we can homotope the smooth section j^*s so that it does not vanish. More precisely, in a tubular neighborhood of K (which has the same homotopy type of K, and so the restriction of $\mathcal{E}$ over that neighborhood is still trivial), we can deform s using a bump function supported near K so that s no longer vanishes near K. This is equivalent to a homotopy of D which ensures that $K \cap D$ is empty, as required. □

Theorem 6.5 (Ritter and Smith [37]) *The map $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{W}_\lambda(X)) \to SH^*(X)$ lands in the generalized λ–eigensummand $SH^*(X)_\lambda$ of $SH^*(X)$ for the $QH^*(X)$–module action of $c_1(TX)$ (resp. $\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda(X)) \to QH^*(X)$ lands in the generalized λ–eigensummand $QH^*(X)_\lambda$). Moreover, if $\mathcal{OC}$ hits an invertible element in that eigensummand, and $m_0(K) = \lambda$, then $\mathcal{CO} \circ \mathcal{OC}$ hits $1 \in HW^*(K,K)$ (resp. $1 = [K] \in HF^*(K,K)$).*

Definition 6.6 (generation criterion) Let B be a closed Fano toric manifold, with a choice of toric symplectic form ω_F. For an eigenvalue λ of the action of $c_1(TB)$ on $QH^*(B)$, we will say that *the toric generation criterion holds for* λ if the composite

$$\mathcal{CO} \circ \mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda^{\mathrm{toric}}(B)) \to QH^*(B) \to HF^*(K,K)$$

hits the unit $[K] \in HF^*(K,K)$ for any Lagrangian $K \in \mathrm{Ob}(\mathcal{F}_\lambda(B))$. Note that we restricted $\mathcal{OC}$ to the subcategory $\mathcal{F}_\lambda^{\mathrm{toric}}(B) \subset \mathcal{F}_\lambda(B)$ generated by the toric Lagrangians (with holonomy data). As remarked above, this condition holds if $\mathcal{OC}$ hits an invertible element in the generalized eigensummand $QH^*(B)_\lambda$.

We will say that *the toric generation criterion holds* if it holds for all eigenvalues λ.

The same terminology applies for the compact category $\mathcal{F}(M)$ for admissible toric manifolds (Definition 1.4). For the wrapped category $\mathcal{W}(M)$, we instead work with

$$\mathcal{CO} \circ \mathcal{OC}\colon \mathrm{HH}_*(\mathcal{W}_\lambda^{\mathrm{toric}}(M)) \to SH^*(M) \to HW^*(K,K).$$

By the acceleration diagram (1), the wrapped $\mathcal{OC}$ map can be identified with

$$\mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda^{\mathrm{toric}}(M)) \to QH^*(M) \xrightarrow{c^*} SH^*(M)$$

because the toric Lagrangians are compact objects, and so the subcategories generated by them in $\mathcal{F}(M)_\lambda$ and $\mathcal{W}(M)_\lambda$ are quasi-isomorphic via the acceleration functor. Thus, for the purposes of toric generation for the wrapped category, we reduce to working with

$$\mathcal{CO} \circ \mathcal{OC}\colon \mathrm{HH}_*(\mathcal{F}_\lambda^{\mathrm{toric}}(M)) \to QH^*(M) \to HF^*(K,K),$$

since the acceleration map $HF^*(K,K) \to HW^*(K,K)$ sends unit to unit.

By the above discussion of the generation criterion, and the comments about the twisted generalization in Section 6A, observe that when *the toric generation criterion holds for* λ, it implies that the relevant category $\mathcal{F}_\lambda(B)$, $\mathcal{F}_\lambda(M)$, $\mathcal{W}_\lambda(M)$ is split-generated by the toric Lagrangians. By Cho and Oh [10] (see also [3, Section 6]), the toric Lagrangians L (with holonomy) for which $HF^*(L,L) \neq 0$ are known: they are in bijection with the critical points of the superpotential, as explained in Appendix A, Section A7.

Note that for monotone toric negative line bundles E, the toric generation for $\lambda = 0$ for $\mathcal{F}(E)$ can never hold since $\mathcal{F}(E)_0$ does not contain toric Lagrangians with $HF^*(L,L) \neq 0$, since the critical values of W_E are all nonzero by Theorem 4.19.

Remark 6.7 We emphasize that our Novikov ring $\mathfrak{R} = \Lambda_s((T))$ defined in Section 5C is a field. This is practical when considering the generation criterion, since a nonzero element in a field is always invertible. We mention, however, that one could avoid completing in T: one can work with $\Lambda_s[t,t^{-1}]$. Indeed, one can work with the field Λ_s, temporarily losing the $\mathbb{Z}$–grading, prove invertibility, and then reinsert powers of t a posteriori to ensure the $\mathbb{Z}$–grading is respected (this may yield an inverse up to t^{positive}, but that we can invert). So invertibility of, say, $\mathcal{OC}([\mathrm{pt}])$, for $[\mathrm{pt}] \in C_*(L) = CF^*(L,L)$, will hold over Λ_s if and only if it holds over $\Lambda_s[t,t^{-1}]$. Over $\Lambda_s[t,t^{-1}]$ the later statements about the existence of a "field summand", ie a copy of $\mathfrak{R} = \Lambda_s((T))$, should then be rephrased as "a free summand over $\Lambda_s[t,t^{-1}]$".

6C Calculation of $\mathcal{OC} = \mathcal{OC}^{0|0}$: $HF(L, L) \to QH^*(X)$ on the point class

Lemma 6.8 *Let X be one of the following:*

(1) *a closed Fano toric manifold (B, ω_F);*

(2) *a Fano negative line bundle $E \to B$;*

(3) *a (noncompact) admissible toric manifold M (Definition 1.4).*

Let L be the Lagrangian with holonomy corresponding to a critical point x of the (possibly F–twisted) superpotential W. Consider the map $\mathcal{OC}$: $HF^*(L, L) \to QH^*(X)$ on the point class $[\mathrm{pt}] \in HF^*(L, L)$ (which is a cycle since x is critical [10; 3]). Then, respectively:

(1) *$\mathcal{OC}([\mathrm{pt}]) = \mathrm{PD}(\mathrm{pt}) + (\text{higher order } t) \neq 0 \in QH^*(B)$, where the leading term arises as the constant disc at the input point;*

(2) *$\mathcal{OC}([\mathrm{pt}]) = (\text{fiber holonomy}) \cdot T \cdot \mathrm{PD}([B]) + (\text{higher order } t) \neq 0 \in QH^*(E)$, where the leading term arises as the standard fiber disc through the input point;*

(3) *$\mathcal{OC}([\mathrm{pt}]) = m_0(L)\,\mathrm{PD}(C^\vee) + (\text{linearly independent terms}) \in QH^*(M)$, which is nonzero if $\lambda = m_0(L) \neq 0$, where $C = \mathrm{PD}(c_1(TM))$ is a compact cycle representative and the lf-cycle $C^\vee$ is its intersection dual (with respect to a choice of basis which affects the other terms).*

Proof Cases (1) and (2) follow directly upon inspection; in case (2) the constant disc does not contribute because $\mathrm{PD}[\mathrm{pt}] = 0 \in H^*(E)$ since $H^{\dim_{\mathbb{R}} E}(E) = 0$, and the next-order term is determined by Maslov 2 discs, but for dimension reasons $[B]$ is the only test cycle we can use up to rescaling, and only the fiber Maslov index 2 disc hits B. One can also prove (2) via (3): the base $[B]$ as an lf-cycle represents $\pi^* c_1(E) = -k[\pi^*\omega_B]$, and $c_1(TE) = \pi^* c_1(TB) + \pi^* c_1(E) = (\lambda_B - k)\pi^*\omega_B$. So we can take $C = ((\lambda_B - k)/(-k))[B]$. Then $C^\vee = -(k/\lambda_E)\text{fiber} \in H_*^{\mathrm{lf}}(E)$, so $\mathrm{PD}(C^\vee) = -(k/\lambda_E)\pi^*\mathrm{vol}_B$, where recall $\lambda_E = \lambda_B - k$. Finally,

$$\mathrm{PD}(C^\vee) = \frac{-k}{\lambda_B - k}\pi^*\omega_B^{\mathrm{top}},$$

since pull-back is Poincaré dual to taking preimages, and $[\mathrm{pt}]$ is PD to ω_B^{top} in $H^*(B)$.

For case (3), recall (eg see [37]) that $\lambda = m_0(L)$ is the count of Maslov index 2 discs which bound L and whose boundary passes through the input point in L, and that the $c_1(TM)$ action on $HF^*(L, L)$ gives $c_1(TM) * [L] = m_0(L)\,[L] = \lambda\,[L]$ because those Maslov index 2 discs automatically hit the anticanonical divisor $C = \mathrm{PD}(c_1(TM))$

once in the interior of the disc. Now $c_1(TM)$ is a multiple of $[\omega_M]$, and ω_M is exact at infinity (M is conical at infinity). Therefore $c_1(TM)$ can be represented by a closed de Rham form which is compactly supported, which in turn is Poincaré dual to a cycle $C \in H_2(M)$. It follows that there exists a *compact* lf-cycle C representing $c_1(TM)$ and, since on cohomology the disc count above does not depend on the choice of representative, we may use C. Extend C to a basis $C, C_2, C_3, \dots$ of lf-cycles for $H_*^{\mathrm{lf}}(M)$, and take the dual basis $C^\vee, C_2^\vee, C_3^\vee, \dots$ in $H_*(M)$ with respect to the intersection product; in particular, in this sense the cycle $C^\vee$ is then dual to the lf-cycle C. By definition, the disc count which yields $c_1(TM) * [L] = m_0(L)[L]$ is the same as the disc count which determines the coefficient of $\mathrm{PD}(C^\vee)$ in $\mathcal{OC}([\mathrm{pt}])$. The other terms in the expansion of $\mathcal{OC}([\mathrm{pt}])$ involve $\mathrm{PD}(C_j^\vee)$ for $j \geq 2$. These other terms cannot cancel the $\mathrm{PD}(C^\vee)$ term in the vector space $H^*(M;\Lambda)$, since $C^\vee, C_2^\vee, C_3^\vee, \dots$ are linearly independent in $H_*(M)$. Thus, computation (3) follows, where $m_0(L)$ already contains the relevant power of t, namely t^{1/λ_M}. □

6D Generation for 1–dimensional eigensummands

Theorem 6.9 (Ostrover and Tyomkin [32, Lemma 3.5]) *The Jacobian ring of a closed Fano toric variety has a field summand for each nondegenerate critical point of the superpotential.*

Theorem 6.10 *Let X be a closed Fano toric manifold (B, ω_F), or a Fano negative line bundle $E \to B$, or an admissible toric manifold M (Definition 1.4). Let W be the associated (possibly F–twisted) superpotential. If p is a nondegenerate critical point of W, and it is the only critical point with critical value $\lambda = W(p)$, then the toric generation criterion holds for $\lambda \neq 0$ (for B, also $\lambda = 0$ works).*

Proof Essentially this now follows by Lemma 6.8. Since p is nondegenerate, by Theorem 6.9 the Jacobian ring has a field summand (their argument also holds in the noncompact toric setting). Since it is the only critical point with that eigenvalue, for B we deduce that $QH^*(X)_\lambda \cong \mathrm{Jac}(W_X)_\lambda$ is a field summand, and for $X = E$ and $X = M$ we deduce $SH^*(X)_\lambda \cong \mathrm{Jac}(W_X)_\lambda$ is a field summand. By Theorem 6.5, $\mathcal{OC}(\mathrm{pt}) \in QH^*(X)_\lambda$, so $\mathcal{OC}(\mathrm{pt})$ is a nonzero element in a field, and therefore it is invertible. The claim follows. □

Even when the superpotential W is a Morse function (critical points are nondegenerate), the above argument may not apply. Indeed, if there are several critical points with the same critical value $\lambda = W(x)$, then $\mathcal{OC}(\mathrm{pt})$ is nonzero in $QH^*(X)_\lambda$, which is a sum of fields, but it could be noninvertible. We will study this problem by first considering the twisted theory, where critical values generically separate, and then reconsider the untwisted case in the limit.

6E Generation for generic toric forms

The following is due to Ostrover and Tyomkin [32, Theorem 4.1], based on arguments in Iritani [25, Corollary 5.12] and Fukaya, Oh, Ohta and Ono [16, Proposition 8.8]. Recall that an algebra is *semisimple* if it is isomorphic to a direct sum of fields.

Theorem 6.11 [25; 16; 32] *For a closed Fano toric variety B, the superpotential W_F is Morse for a generic choice of toric symplectic form ω_F, ie after a generic small perturbation of the values $F(e_i) \in \mathbb{R}$. In particular, $QH^*(B)$ is then semisimple.*

Remark 6.12 The Jordan normal form for the endomorphism of $\mathrm{Jac}(W_F)$ given by multiplication by W_F arises from the primary decomposition of the $\mathfrak{R}[x]$–module $\mathrm{Jac}(W_F)$, where x acts by W_F. The summands have the form

$$\mathfrak{R}[x]/(x-\lambda)^d,$$

where λ is an eigenvalue of multiplication by W_F, and there can be several summands with the same λ. Theorem 6.9 [32, Theorem 4.1 and Corollary 2.3] consists in showing that each critical point p of W_F with critical value $W_F(p) = \lambda$ gives rise to such a summand, and that if p is nondegenerate then $d = 1$ (so the summand is a field, a copy of $\mathfrak{R}$). One also needs to show that this decomposition respects the algebra structure, not just the module structure.

Each critical point p of W_F determines an idempotent $1_p \in \mathrm{Jac}(W_F)$, representing $1 \in \mathfrak{R}[x]/(x-\lambda)^d$ in the relevant summand, such that multiplication by 1_p corresponds to projection to the summand. These elements 1_p, as $p \in \mathrm{Crit}(W_F)$ vary, determine a decomposition of $1 = \sum 1_p \in \mathrm{Jac}(W_F)$ satisfying the "orthonormality relations" $1_p \cdot 1_q = \delta_{p,q} 1_p$, and the primary decomposition of $\mathrm{Jac}(W_F)$ corresponds to mapping $f \in \mathrm{Jac}(W_F)$ to $\sum 1_p \cdot f$.

This theorem is still not sufficient to deduce that toric generation holds generically, due to the issue of repeated critical values mentioned above. So we need to prove Lemma 1.12.

Proof of Lemma 1.12 Consider the superpotential $W(a) = \sum s^{a_i} T z^{e_i}$, where $F(e_i) = -a_i$. By Theorem 6.11, $W(a)$ is Morse for generic a. Now let the values a_i vary locally near a point A for which $W(A)$ is Morse, so $W(a)$ is also Morse. Consider the map

$$F\colon \{(a,z) \in \mathbb{R}^r \times \mathfrak{R}^n : dW(a)|_z = 0\} \to \mathfrak{R}, \quad F(a,z) = W(a)(z),$$

which assigns the critical value to the critical points. Near a given critical point (A, Z) of $W(A)$, the critical points of $W(a)$ are smoothly parametrized by a such that

$z = z(a)$; this follows by an implicit function theorem argument, using the fact that the superpotential is an analytic function and that the critical points are all simple since $W(A)$ is Morse. So the domain of F is parametrized as $(a, z(a))$ near (A, Z) for some smooth function $a \mapsto z(a)$. To see how F varies locally as we vary a, consider the partial derivatives

$$\partial_{a_i}(F(a, z(a))) = \partial_{a_i} F + \sum_j \partial_{z_j} F \cdot \partial_{a_i}(z_j(a)).$$

But $\partial_{z_j} F = \partial_{z_j} W(a) = 0$ at the critical point $z(a)$, therefore $\partial_{a_i}(F(a, z(a))) = s^{a_i} T z^{e_i}$ recovers the i^{th} term in $W(a)$. (Here we stipulated that $\partial_{a_i} s^{a_i} = s^{a_i}$. One could run the argument first by fixing a real value $s = \exp(1)$, since if critical values are distinct when putting $s = \exp(1)$ then they certainly are also for a formal variable s.)

As explained in the proof of Lemma A.15, by making an $\mathrm{SL}(n, \mathbb{Z})$ transformation of the fan one can assume that a subset $e_1, \dots, e_n$ of the edges is the standard basis of $\mathbb{R}^n$, so $W(a)$ would have the form

$$s^{a_1} T z_1 + \cdots + s^{a_n} T z_n + s^{a_{n+1}} T d_{n+1}(z) + \cdots + s^{a_r} T d_r(z),$$

where $d_j(z)$ are monomials in $z_i^{\pm 1}$. In particular, $\partial_{a_i}(F(a, z(a))) = s^{a_i} T z_i$ for $i = 1, \dots, n$.

Hence, if all partial derivatives $\partial_{a_i} F$ at two different critical points (a, z) and $(a, \tilde{z})$ are equal, then all z_i–coordinates must be equal, and so $z = \tilde{z}$. Thus, if any of the critical values are repeated, then a generic small variation of a will separate those critical values, because at different critical points z, $\tilde{z}$, at least one of the partial derivatives $\partial_{a_i} F$ is different (it is generic because fixing one of the a_i coordinates is a codimension 1 condition). □

Remark 6.13 If $c_1(TB) \in QH^*(B, \omega_B)$ has a zero eigenvalue, then by Lemma 1.12, Theorem 4.22 and Corollary 5.4, rank $SH^*(E, \omega_E)$ will jump when we generically deform ω_E to ω_F^E, since $QH^*(B, \omega_F^B)$ will have no 0–eigenspace. This applies to $\mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^1}(-1, -1)$ (see Section 4G).

Theorem 6.14 *For a closed Fano toric manifold B with a generic choice of toric symplectic form ω_F^B close to ω_B, the toric generation criterion holds for $\mathcal{F}(B, \omega_F^B)$ and for the twisted category $\mathcal{F}(B, \omega_B; \underline{\mathfrak{R}}_{\omega_F^B})$. For a Fano negative line bundle E, constructing ω_F^E from a generic ω_F^B as in Corollary 5.4, the toric generation criterion holds for $\mathcal{W}(E, \omega_F^E)$ and for the twisted category $\mathcal{W}(E, \omega_E; \underline{\mathfrak{R}}_{\pi^* \omega_F^B})$. It also holds for $\mathcal{F}(E, \omega_F^E)$ and for $\mathcal{F}(E, \omega_E; \underline{\mathfrak{R}}_{\pi^* \omega_F^B})$ for all $\lambda \neq 0$. For an admissible toric manifold M (Definition 1.4), with a generic choice of toric symplectic form ω_F*

close to the monotone ω_M, for all $\lambda \neq 0$ the toric generation criterion holds for $\mathcal{W}(M,\omega_F)$, $\mathcal{F}(M,\omega_F)$, $\mathcal{W}(M,\omega_M;\underline{\mathfrak{R}}_{\omega_F})$ and $\mathcal{F}(M,\omega_M;\underline{\mathfrak{R}}_{\omega_F})$. In these generic settings, $QH^*(B,\omega_F^B) \cong QH^*(B,\omega_B;\underline{\mathfrak{R}}_{\omega_F^B})$, $SH^*(E,\omega_F^E) \cong SH^*(E,\omega_E;\underline{\mathfrak{R}}_{\omega_F^E})$ and $SH^*(M,\omega_F) \cong SH^*(M,\omega_M;\underline{\mathfrak{R}}_{\omega_F})$ are semisimple, with each field summand corresponding to a different eigensummand.

Proof This follows by Lemma 1.12 and Theorem 6.10. For E, we use Theorem 4.22 to obtain the result for the toric form ω_F^E induced on E by the generic ω_F^B. The twisted analogues are equivalent rephrasings since the twisted monotone setup and the (untwisted) nonmonotone setup are identifiable (Corollaries 5.1, 5.2 and 6.1). □

6F Generation for semisimple critical values

Definition 6.15 (semisimple critical value) A critical value λ of the superpotential W of a Fano toric manifold is called *semisimple* if all $p \in \mathrm{Crit}(W)$ with $W(p) = \lambda$ are nondegenerate critical points. We sometimes also call λ a *semisimple eigenvalue*.

Lemma 6.16 *For a semisimple critical value λ, the generalized eigenspace $QH^*(X)_\lambda$ is just the eigenspace of $c_1(TX)$ for λ.*

Proof This is Remark 6.12 rephrased. A generalized eigensummand $\mathfrak{R}[x]/(x-\lambda)^d$ with $d \geq 2$ would give rise to a nonzero nilpotent element $x-\lambda$, which is not allowed in a semisimple summand of $QH^*(X)$. So generalized eigenvectors for λ are eigenvectors. □

Theorem 6.17 (working with $\mathfrak{R}$ defined over $\mathbb{K} = \mathbb{C}$) *For a closed monotone toric manifold (B,ω_B), the toric generation criterion holds for any semisimple eigenvalue λ. For an admissible toric manifold M (Definition 1.4, for example a monotone toric negative line bundle) the toric generation criterion holds for $\mathcal{W}(M)$ and $\mathcal{F}(M)$ for any semisimple eigenvalue $\lambda \neq 0$.*

Proof By Theorem 6.14, for the ω_F–twisted category the toric generation criterion holds when the $F(e_i)$ are generic and close to -1. Recall that when $F(e_i) = -1$ we can identify the twisted with the untwisted theory.

For brevity denote by $QH_F^*(B) = QH^*(B,\omega_B;\underline{\mathfrak{R}}_{\omega_F})$ the twisted theory, and similarly write $SH_F^*(M)$, $QH_F^*(M)$. Recall by Remark 6.12 that there is a decomposition $1 = \sum 1_p$ in $QH^*(B)$ (resp. in $SH^*(M)$, since $\mathrm{Jac}(W_M) \cong SH^*(M)$), where p runs over the critical points of the relevant superpotential W. Similarly, in the twisted case $1 = \sum 1_{p,F}$ in $QH^*_F(B)$ (resp. in$SH^*_F(M)$), where p runs over the critical points of the twisted superpotential.

The summands $\mathfrak{R}[x]/(x-\lambda)^d$ in the primary decomposition mentioned in Remark 6.12 arise as the local rings $\mathcal{O}_{Z_W,p}$ of $\mathcal{O}(Z_W) = \mathrm{Jac}(W)$ for a certain scheme Z_W associated to the superpotential W [32, Corollary 2.3 and Lemma 3.4], and 1_p is the relevant unit in that local ring. As in the proof of Lemma 1.12, the nondegenerate critical points p and their critical values λ_p vary smoothly as we vary the values $F(e_i)$, since the superpotential is an analytic function. A priori, it is not clear whether (or in what sense) the decomposition of $\mathcal{O}(Z_W)$ into summands $\mathcal{O}_{Z_W,p}$ varies continuously, smoothly or analytically with the parameters $F(e_i)$ near parameter values where different eigenvalues λ_p collide. For this problem, we have developed the necessary matrix perturbation theory in Appendix B.

In the terminology of Appendix B, multiplication by 1_p is the eigenprojection P_{λ_p} onto the generalized eigenspace for the eigenvalue λ_p for the family of endomorphisms of the vector space $H^*(B;\mathfrak{R}) \cong QH^*_F(B) \cong \mathrm{Jac}(W^B_F)$ given by quantum multiplication by the superpotential W_F. Analogously for M, working with $H^*(M;\mathfrak{R}) \cong QH^*_F(M)$ and quantum multiplication by $c_1(TM)$, or working with a localization thereof, $SH^*_F(M) \cong \mathrm{Jac}(W^M_F)$ using multiplication by W^M_F. We will phrase the rest of the argument for B, as it is analogous for M.

Before applying Appendix B, we need to rephrase the problem in terms of a family $A(x) \in \mathrm{End}(\mathbb{C}^n)$, where $A(x)$ depends holomorphically on a complex parameter x near $0 \in \mathbb{C}$. So instead of working over $\mathfrak{R}$, we therefore briefly work over $\mathbb{C}$ by setting $T = 1$ and $s = e^x$ (this is legitimate since B and M are Fano: the quantum relations and quantum product only involve finitely many powers of s and T). So $W = \sum s^{-F(e_i)} T z^{e_i}$ becomes

$$A(x) = W(a,x) = \sum \exp(a_i x) z^{e_i},$$

acting by quantum multiplication as explained above, where $a_i = -F(e_i)$, and working over $\mathbb{C}$ instead of $\mathfrak{R}$. In other words, we have *specialized* the quantum cohomology by fixing a value $s = e^x$ for $x \in \mathbb{C}$ close to 0. Observe that taking $x = 0$, so that $s = 1$, recovers by definition the quantum cohomology for the monotone symplectic form (so $F(e_i) = -1$), except for having dropped T due to the substitution $T = 1$; but powers of T can be reinserted a posteriori as dictated by the $\mathbb{Z}$–grading.

By assumption, λ is a semisimple eigenvalue, so Corollary B.2 applies to the family of matrices $A(x)$. In particular, there is a continuous family of eigenvalues $\lambda_j(x) = \lambda_{p_j}$ converging to λ, which is analytic on a punctured disc around $x = 0$ and which is real-differentiable at $x = 0$. By the proof of Lemma 1.12 in Section 6E, for a generic choice of $a_i = -F(e_i)$ the $\lambda_j(x)$ are pairwise distinct for all small $x \neq 0$, and the $\lambda_j(x)$ are in fact holomorphic even at $x=0$, since $\lambda_j(x) = W(a,x)(z)|_{z=p_j(a,x)}$, where $p_j(a,x)$ are nondegenerate critical points of $W(a,x)$ for small x (since they are nondegenerate at $x=0$). Keeping track also of a, the derivatives of the $\lambda_j(a,x)$ at $x=0$ are

$$\begin{aligned}\partial_x\big|_{x=0}\lambda_j(a,x) &= \sum \frac{\partial}{\partial z_i}\bigg|_{\substack{z=p_j(a,x)\\x=0}} W(a,x)(z)\cdot \frac{\partial}{\partial x}\bigg|_{x=0}[p_j(a,x)]_i \\ &\qquad\qquad\qquad + \frac{\partial}{\partial x}\bigg|_{\substack{z=p_j(a,x)\\x=0}} W(a,x)(z) \\ &= \sum a_i[p_j(a,0)]^{e_i},\end{aligned}$$

since the first derivative vanishes, as p_j is critical. For a close to $\underline{1} = (1,\dots,1)$, the leading term is $\sum a_i[p_j(\underline{1},0)]^{e_i}$. We want to show that for generic a, these leading terms are different for different j, since then the $\partial_x\big|_{x=0}\lambda_j(a,x)$ are different, and so the second half of Corollary B.2 applies. Consider the auxiliary function $G(z) = \sum a_i z^{e_i}$. Then

$$\partial_{a_i} G\big|_{p_j(\underline{1},0)} = p_j(\underline{1},0)^{e_i}.$$

Now as in the proof of Lemma 1.12 in Section 6E, for two different choices of j the $p_j(\underline{1},0)^{e_i}$ must differ for some i since otherwise two of the critical points $p_j(\underline{1},0)$ would coincide, contradicting nondegeneracy. Therefore, varying a_i generically ensures that the values $G(p_j(\underline{1},0)) = \sum a_i[p_j(\underline{1},0)]^{e_i}$ differ for different j, as required.

By Corollary B.2 it follows that for generic a close to $\underline{1}$, the eigenspaces

$$E_j(x) = QH^*_F(B;\mathbb{C})_{\lambda_j(x)}$$

for $\lambda_j(x)$ vary holomorphically in x even at $x=0$ (continuously would have sufficed). The final step of the argument is to consider the linear map

$$\mathcal{OC}_j\colon HF^*(L_{p_j(x)}, L_{p_j(x)}) \to H^*(B;\mathbb{C}) \equiv QH^*_F(B;\mathbb{C}),$$

where $L_{p_j(x)}$ is the toric Lagrangian (together with holonomy data) corresponding to the critical point $p_j(x)$ of $W(a,x)$; we recall this correspondence in Appendix A, Section A7.

By Theorem 6.5, $\mathcal{OC}_j$ lands in $E_j(x)$. We care about the image of [pt],

$$\mathcal{OC}_j(x) = \mathcal{OC}_j[\mathrm{pt}] \in E_j(x).$$

We will show below that $\mathcal{OC}_j(x)$ depends continuously on x. Therefore, at $x=0$, $\mathcal{OC}_j(0)$ lands in $E_j(0)$ which is one of the field summands in the λ–eigenspace

$E_\lambda(A(0)) = \bigoplus E_i(0)$ of $A(0) = W(a,0)$ (the superpotential in the monotone case, working over $\mathbb{C}$). This continues to hold even if we insert the appropriate powers of T to achieve $\mathbb{Z}$–grading, so it holds for the $\mathcal{OC}_j$ map constructed in the monotone case over the usual Novikov ring of Section 2A.

Therefore, in the monotone case, $\mathcal{OC}_j[\mathrm{pt}]$, for $[\mathrm{pt}] \in HF^*(L_{p_j(0)}, L_{p_j(0)})$, can only be nonzero in a specific field summand of the λ–eigenspace of $QH^*(B;\omega_B)$ determined by the convergence of the eigenspaces of $A(x)$ described in Corollary B.2.

The final ingredient is that, as in the proof of Lemma 6.8, $\mathcal{OC}_j(0)$ is nonzero and so is invertible in that field summand $E_j(0)$. Therefore $\bigoplus \mathcal{OC}_j(0)$, summing over all j for which $\lambda_j(0) = \lambda$, is invertible in $\bigoplus E_j(0) = E_\lambda(A(0))$, so the generation criterion holds and the theorem follows.

Finally, we explain why $\mathcal{OC}(x)$ depends continuously on x. The difficulty is that not only the holonomy data but also the toric Lagrangian submanifolds $L_j = L_{p_j(a,x)}$ may vary with x. For the monotone form ω_B (so all $F(e_i) = -1$), the point $y_j \in \Delta$ in the moment polytope corresponding to L_j is always the barycenter (Section A9). But for some small deformations of F, it may happen that y_j moves away from the barycenter (eg this is shown for the one point blow-up of $\mathbb{CP}^2$ in [16]).

Nevertheless we claim that the moduli spaces of rigid discs bounding L_p counted by $\mathcal{OC}([\mathrm{pt}])$ for $[\mathrm{pt}] \in C_*(L_p) = CF^*(L_p, L_p)$ (so with suitable intersection conditions at marked points) vary in a smooth 1–family with F, when all $F(e_i)$ are close to -1.

For low Maslov index discs, that claim can be checked explicitly. Indeed, such discs intersect a low number of toric divisors (half of the Maslov index [10; 3]), and so the complement of the toric divisors that it does not intersect can be parametrized by a holomorphic chart $\mathbb{C}^n$ in which discs bounding a torus $S^1(r_1) \times \cdots \times S^1(r_n) \subset \mathbb{C}^n$ are known explicitly and they vary smoothly when varying the radii r_j. For high Maslov indices, there may not be such a chart, so instead we use the global holomorphic action of the complex torus T on the toric variety as follows (recall that a toric variety is defined in terms of a dense complex torus T, see Section A1).

Toric Lagrangians arise as orbits of the real torus $T_{\mathbb{R}} \subset T$, and we can use elements of T to map one toric Lagrangian L to another, L'. Since the action of T is holomorphic, this mapping will yield a natural bijection between moduli spaces of holomorphic discs bounding L and those bounding L'. If we impose generic intersection conditions at marked points for the discs bounding L, then provided L' is sufficiently close to L, the corresponding discs will be close, so the corresponding discs bounding L' will satisfy the same intersection conditions (at some other nearby marked points). For the sake of clarity we emphasize that, of course, the weights with which these moduli spaces are counted at the chain level are the culprit behind dramatic changes in the homology, rather than an essential change in the moduli spaces (if we generically vary y, keeping the holonomy fixed, then p will no longer be a critical point, so $HF^*(L_p, L_p)$ will vanish, but this happens because with the incorrect weights the cancellations of disc counts in the boundary operator will fail).

The upshot is that the map $\mathcal{OC}_{F,\lambda_p}\colon HF_F^*(L_p, L_p) \to QH_F^*(X)_{\lambda_p}$ on the point class in $C_*(L_p) = CF^*(L_p, L_p)$, at the chain level, is the same Laurent polynomial in the generators $x_i = \mathrm{PD}[D_i] \in C^*(X; \mathfrak{R})$ except for the coefficients in Λ_s (the power of t in the summands is determined by the grading). These coefficient functions vary smoothly as we vary the values $F(e_i)$ since they come from $s^{\omega_F(u)}$, integrating $\omega_F = \sum -F(e_i)\,\mathrm{PD}[D_i]$ over the discs u. $\square$

Remark 6.18 Theorem 6.17 also holds for the (nonmonotone) Fano toric manifold (B, ω_F) and for Fano negative line bundles over it. In this case, a perturbation of ω_F to $\omega_{F'}$ corresponds to keeping ω_F but twisting by $\omega_{F'} - \omega_F$ (such as $QH^*(B, \omega_{F'}) \cong QH^*(B, \omega_F; \underline{\mathfrak{R}}_{\omega_{F'}-\omega_F})$).

6G $\mathcal{OC}^{0|0}$: $HF^*(L_p, L_p) \to QH^*(B)_{\lambda_p}$ lands in the λ_p–eigenspace

Theorem 6.19 (working with $\mathfrak{R}$ defined over $\mathbb{K} = \mathbb{C}$) *Let X be a closed monotone toric manifold B or a (noncompact) admissible toric manifold M (Definition 1.4). For $p \in \mathrm{Crit}(W)$, the image of $\mathcal{OC}^{0|0}\colon HF^*(L_p, L_p) \to QH^*(X)_{\lambda_p} \subset QH^*(X; \omega_X)$ lands in the eigenspace (not just the generalized eigenspace).*

Proof By Theorem 6.5, working over $\mathbb{C}$, the map

$$\mathcal{OC}(x) = \mathcal{OC}^{0|0}\colon HF^*(L_{pj(a,x)}, L_{pj(a,x)}) \to QH_F^*(B; \mathbb{C})$$

lands in the eigensummand $QH_F^*(B; \mathbb{C})_{\lambda_j(x)}$ corresponding to the eigenvalue $\lambda_j(x)$ associated with $L_j(x) = L_{p_j(a,x)}$, where $p_j(a,x) \in \mathrm{Crit}(W_F)$, using the notation from the proof of Theorem 6.17. By Theorem 6.17, for generic $a_i = -F(e_i)$ near $F(e_i) = -1$ the eigenvalues are distinct so this eigensummand is 1–dimensional. So the image of $\mathcal{OC}(x)$ consists of eigenvectors (or zero).

The theorem follows if we show that in the limit $x = 0$, also the image of $\mathcal{OC}(0)$ consists of eigenvectors (observe that the critical points $p_j(a,x)$ of an analytic function will vary continuously, so they are continuous also at $x = 0$). By Lemma B.4, the 1–dimensional eigenspaces $E_j(x)$ of the $\lambda_j(x)$ vary continuously in $\mathbb{P}(\mathbb{C}^n)$ even at $x = 0$, where they converge to a 1–dimensional subspace $E_j(0)$ of the λ–eigenspace of W, where $\lambda = \lambda_j(0)$. We claim that $\mathcal{OC}(x)$ is continuous in x, therefore $\mathcal{OC}(0)$ is trapped inside the limit $E_j(0)$ as required.

That $\mathcal{OC}(x)$ is continuous in x is proved just like in the proof of Theorem 6.17, replacing the point cycle [pt] by any cycle in $C_*(L)$, and using the vector space isomorphism from (13), $H^*(L_p;\Lambda) \cong HF^*(L_p, L_p)$. More precisely, the latter isomorphism is used in the following sense. A Floer cycle corresponds to an ordinary cycle together with (finitely many) higher order t correction terms which are constructed inductively to kill off the Floer coboundary [8]. The coefficients in these correction terms depend smoothly on x since the area/holonomy weights are smooth in x (the relevant finite moduli spaces of discs vary smoothly for small x as in the proof of Theorem 6.17, so the areas/holonomies of those discs vary smoothly in x). Therefore, composing with the isomorphism $H^*(L_p;\Lambda) \cong HF^*(L_p, L_p)$ we can pretend that $\mathcal{OC}^{0|0}$ is defined as a map of vector spaces $H^*(L;\Lambda) \to H^*(X;\Lambda)$, so the machinery of Appendix B applies, just like it did for the point class in Theorem 6.17. □

Remark 6.20 For the full $\mathcal{OC}$ map defined on $\mathrm{HH}_*(\mathrm{A}_\infty\text{–algebra of } L_p)$, the above argument does not apply because there is no analogue of (13): in fact the rank of this map is expected to jump (see the discussion in Section 1E and Remark 1.15).

6H The failure of $\mathcal{OC}^{0|0}$ to detect generation for some degenerate critical values

By Ostrover and Tyomkin [32] there are closed smooth toric Fano varieties B for which $QH^*(B,\omega_B)$ is not semisimple, due to the presence of a generalized eigenspace (which is not an eigenspace) for multiplication by W on $\mathrm{Jac}(W) \cong QH^*(B)$. The example in [32, Section 5] is the smooth Fano 4–fold called U_8, number 116, in Batyrev's classification [6]. It has a superpotential W with a critical point p such that $W(p) = -6$ and $\mathrm{Hess}_p W$ is degenerate. A further simple calculation shows that this is the only critical point with value $W(p) = -6$, therefore the eigensummand of $QH^*(U_8)$ for the eigenvalue -6 is a generalized eigenspace isomorphic to $\Lambda[x]/(x+6)^d$ for some $d \geq 2$.

Corollary 6.21 (working with $\mathfrak{R}$ defined over $\mathbb{K} = \mathbb{C}$) *The map*

$$\mathcal{OC}^{0|0}\colon HF^*(L_p, L_p) \to QH^*(B)_{\lambda_p}$$

does not hit an invertible element for the closed smooth toric 4*–fold* $B = U_8$ *taken with the monotone toric symplectic form.*

Proof This follows from the above observations and Theorem 6.19, as the eigenspace of x in $\Lambda[x]/(x+6)^d$ is spanned by $(x+6)^{d-1}$ and $\mathcal{CO}\colon QH^*(B) \to HF^*(K,K)$ vanishes on multiples of $c_1(TB) + 6$ whenever $m_0(K) = -6$ by Lemma 6.3. □

6I A remark about generation, in view of Galkin's result

Theorem 6.22 (Galkin [18]) *For a closed monotone toric manifold (B, ω_B), the complex-valued superpotential $W\colon (\mathbb{C}^*)^n \to \mathbb{C}$ always has a nondegenerate critical point $p \in (\mathbb{C}^*)^n$ with strictly positive real coordinates.*

The idea of the proof is to show that a function of type $\sum a_i z^{e_i}$, with positive real $a_i > 0$, restricted to $z = \exp(u) \in (\mathbb{R}^*)^n$, has positive-definite Hessian in u_j, and that $W \to \infty$ in any direction $u \in \mathbb{R}^n$ going to infinity. The first property is a computation; the second property follows because the cones of the fan of a closed toric manifold cover $\mathbb{R}^n$ so at least one of the terms of W will grow to infinity (the other terms are positive). Thus there is a global minimum p. One then checks it satisfies Theorem 6.22.

In the noncompact setting, for admissible toric manifolds M (Definition 1.4), the first property still holds, but the second property can fail. For example, for $\mathcal{O}_{\mathbb{P}^1}(-1)$, the superpotential $W = z_1 + t z_1^{-1} z_2 + z_2$, putting $t = 1$, will not grow to infinity in the direction $u = (-1, -1)$, and the only critical point of W is $z = t \cdot (-1, 1)$, which does not have the positive coordinates predicted by Theorem 6.22.

The condition that W grow to infinity at infinity is equivalent to requiring that the fan of M not lie entirely in a half-plane (this fails for negative line bundles), since then we would have some inner product $\langle u, e_i \rangle > 0$, and thus the term $z^{e_i} = \exp\langle u, e_i \rangle$ would grow to infinity as we positively rescaled u. Subject to checking that W has this growth property for a particular M (implying Theorem 6.22 for M), the arguments below would generalize to M.

The passage between the complex-valued and the Novikov-valued superpotentials is done as follows. The monotone toric form $\omega_\Delta = \sum \mathrm{PD}[D_i] = c_1(TB)$ yields $\lambda_i = -1$ so $W = \sum t^{-\lambda_i} z^{e_i} = t \sum z^{e_i}$. Thus it suffices to study the critical points of the complex-valued Laurent polynomial $W = \sum z^{e_i}$ and then reinsert powers of T as dictated by the $\mathbb{Z}$–grading of $QH^*(B)$ (compare with the sanity check in the proof of Theorem A.18).

Corollary 6.23 *For B any closed monotone toric manifold, $c_1(TB) \in QH^*(B)$ is not nilpotent.*

Proof Observe that $W = \sum z^{e_i}$ will take a strictly positive real value on the p in Theorem 6.22, so $c_1(TB) \in QH^*(B; \mathbb{C})$ has a strictly positive real eigenvalue λ_p. The same is true for the superpotential defined over the Novikov ring Λ by reinserting powers of T as dictated by the $\mathbb{Z}$–grading of $QH^*(B)$ (the eigenvalue will be $\lambda_p T$, with T in degree 2). $\square$

Lemma 6.24 [37] *For any monotone negative line bundle $E \to B$, if $c_1(TB) \in QH^*(B)$ is nonnilpotent then $c_1(TE) \in QH^*(E)$ is nonnilpotent.*

Proof This follows by Theorem 4.22: a nonzero eigenvalue of $c_1(TB)$ gives rise to a nonzero eigenvalue of $c_1(TE)$. Since only the latter claim is needed, rather than the full Theorem 4.22, one can also prove the claim by carefully comparing the critical points of the superpotential W_E in terms of those of W_B and applying Theorem A.11, as was done in [37]. □

Corollary 6.25 [37] *For any monotone negative line bundle $E \to B$, the symplectic cohomology $SH^*(E) \neq 0$, and there is a nondisplaceable monotone Lagrangian torus $L \subset E$ with $HF^*(L,L) \cong HW^*(L,L) \neq 0$ using suitable holonomy data. By Corollary A.19, L is the unique monotone Lagrangian torus orbit in E. It lies in the sphere bundle $SE \subset E$ of radius $1/\sqrt{\pi\lambda_E}$, and it projects to the unique monotone Lagrangian torus orbit in B.*

Proof Using Lemma 6.24, this follows from $SH^*(E) \cong \mathrm{Jac}(W_E)$ (Theorem 4.19): each nonzero eigenvalue of $c_1(TE)$ must arise as a critical value of W_E, and the corresponding critical point gives rise to such an L. In [37], this was proved by computing the critical point of W_E in terms of that for W_B and then applying Theorem A.11. □

Galkin's result implies that there is a monotone toric Lagrangian $L_p \subset B$ (with holonomy data) with $HF^*(L_p, L_p) \neq 0$, corresponding to the nondegenerate critical point p of W_B of Theorem 6.22. (In fact, $L_p \subset B$ is the toric fiber over the barycenter of the moment polytope; see Section A9.) If the real positive number $W_B(p)$ is a semisimple critical value, then by Theorem 6.17 the toric generation criterion holds for $\lambda = W_B(p)$ (and if p is the only critical point with critical value $\lambda = W_B(p)$, then L_p split-generates $\mathcal{F}(B)_\lambda$ by Theorem 6.10). However, L_p is unlikely to split-generate all of $\mathcal{F}(B)_\lambda$ if $\dim QH^*(B)_\lambda \geq 2$: one can only expect L_p to split-generate the subcategory of those $K \in \mathrm{Ob}(\mathcal{F}(B)_\lambda)$ for which

$$1_p * [K] = [K],$$

where 1_p is the unit in the field summand of $QH^*(B)_\lambda$ corresponding to p. This is true for the following reason. Let 1_q denote the units in the summands of the primary decomposition of $QH^*(B)_\lambda$ as a $\Lambda[x]$–module, where x acts by $c_1(TB)*$ (that is, $q \in \mathrm{Crit}(W_B)$ with $W_B(q) = \lambda$). Then $1 = \sum 1_q$ and the "orthonormality relations" $1_q * 1_p = \delta_{q,p}\, 1_p$ hold by Remark 6.12. Since $\mathcal{CO}\colon QH^*(B) \to HF^*(K,K)$ is a ring map, $\mathcal{CO}\big(\sum 1_q\big) = [K]$. Therefore the condition $1_p * [K] = [K]$ implies that $\mathcal{CO}(1_p) = [K]$. Finally, Lemma 6.8 implies that $\mathcal{OC}\colon HF^*(L_p, L_p) \to QH^*(B)$

hits 1_p so the generation criterion applies to Lagrangians $K \in \mathrm{Ob}(\mathcal{F}(B)_\lambda)$ with $1_p * [K] = [K]$. However, in general $\mathcal{CO}(1_p) = 1_p * [K]$ need not equal $[K]$, so more Lagrangians than just L_p are needed to split-generate $\mathcal{F}(B)_\lambda$.

Appendix A: The moment polytope of a toric negative line bundle

A1 The fan of a line bundle over a toric variety

For basics on toric geometry, we refer the reader to Fulton [17], Guillemin [23] and Cox and Katz [12, Chapter 3]. There are also useful summaries contained in Batyrev [5] and Ostrover and Tyomkin [32].

A *toric variety* X of complex dimension n is a normal variety which contains a complex torus $T = (\mathbb{C}^*)^n$ as a dense open subset, together with an action of T on X extending the natural action of T on itself by multiplication. We always assume that X is nonsingular.

A toric variety is described by a *fan*, which is a certain collection of cones inside $\mathbb{Z}^n$. The n–dimensional cones correspond to affine open sets in X and their arrangement prescribes the way in which these glue together to yield X. More globally, from the fan one can explicitly write down a *homogeneous coordinate ring* for X, with coordinates $x_1, \dots, x_r$ indexed by the edges $e_1, \dots, e_r \in \mathbb{Z}^n$ of the fan. Namely,

$$X = (\mathbb{C}^n \setminus Z)/G \quad \text{with torus} \quad T = (\mathbb{C}^*)^r/G.$$

Here Z is the union of the vanishing sets of those x_i corresponding to subsets of edges which do not span a cone (so Z is the union of the vanishing sets $\bigcap_{i \in I}(x_i = 0)$ for primitive subsets of indices $I = \{i_1, \dots, i_a\}$, see Definition 3.2); and G is the kernel of the homomorphism

$$\begin{aligned} &\mathrm{Exp}(\beta)\colon (\mathbb{C}^*)^r \to (\mathbb{C}^*)^n, \\ &(t_1, \dots, t_r) \mapsto (t_1^{e_{1,1}} t_2^{e_{2,1}} \cdots t_r^{e_{r,1}},\ t_1^{e_{1,2}} t_2^{e_{2,2}} \cdots t_r^{e_{r,2}}, \dots), \end{aligned}$$

determined by the coordinates of the edges $e_i = (e_{i,1}, \dots, e_{i,n})$. More directly, G will contain $(t^{a_1}, \dots, t^{a_r})$ precisely if $\sum a_i e_i = 0$ is a $\mathbb{Z}$–linear dependence relation amongst the edges.

Example A.1 For $X = \mathrm{Tot}(\mathcal{O}(-k) \to \mathbb{P}^m)$, the edges in $\mathbb{Z}^{m+1}$ are $e_1 = (1, 0, \dots, 0)$, $\dots$, $e_m = (0, \dots, 1, 0)$, $e_{m+1} = (-1, \dots, -1, k)$, $e_{m+2} = (0, \dots, 0, 1)$. The cones are the $\mathbb{R}_{\geq 0}$–span of any subset of the edges, except for the two subsets $\{e_1, \dots, e_{m+1}\}$

and $\{e_1,\dots,e_{m+2}\}$. Those exceptional subsets determine $Z=(x_1=\cdots=x_{m+1}=0)$. Since $\mathrm{Exp}(\beta)\colon (\mathbb{C}^*)^{m+2}\to(\mathbb{C}^*)^{m+1}$ sends $t\mapsto(t_1t_{m+1}^{-1},t_2t_{m+1}^{-1},\dots,t_{m+1}^k t_{m+2})$ it follows that $G=\{(t,\dots,t,t^{-k}):t\in\mathbb{C}^*\}$, which corresponds to the linear relation $e_1+\cdots+e_{m+1}-ke_{m+2}=0$. Thus the toric variety $X=(\mathbb{C}^{m+2}\setminus Z)/G$, with torus $T=(\mathbb{C}^*)^{m+2}/G\cong(\mathbb{C}^*)^{m+1}$, has homogenous coordinates $x_1,\dots,x_{m+2}$. Then $[x_1:\cdots:x_{m+1}]$ defines the projection $X\to\mathbb{P}^m$ to the base, and the zero-section is $(x_{m+2}=0)$. Over the coordinate patches $U_1=(x_1\neq0)$ and $U_2=(x_2\neq0)$ we can assume $x_1=1$ and $x_2=1$ respectively, and on the overlap $U_1\cap U_2$ we can identify $(1,x_2,\dots,x_{m+2})$ with $(x_2^{-1},1,x_2^{-1}x_3,\dots,x_2^{-1}x_{m+1},x_2^kx_{m+2})$ using the G–action. So the transition $U_1\to U_2$ multiplies by $\varphi=1/x_2$ in the base coordinates and by φ^{-k} in the fiber coordinate. In general, $g_{ij}=(x_i/x_j)^k$ is the transition in the fiber going from U_j to U_i, as expected for $\mathcal{O}_{\mathbb{P}^m}(-k)$.

We now describe the general construction of $\mathrm{Tot}(\pi\colon E\to B)$ for a line bundle over a toric variety B. We will always assume that B is a smooth closed manifold, which imposes constraints on what fans can arise:

(1) Compactness is equivalent to the condition that the cones of the fan of B cover $\mathbb{R}^n$ (in particular, the $\mathbb{R}_{\geq0}$–span of the edges $b_1,\dots,b_r\in\mathbb{Z}^n$ is $\mathbb{R}^n$).

(2) Smoothness is equivalent to the condition that each subset of edges which generates a cone extends to a $\mathbb{Z}$–basis of $\mathbb{Z}^n$ (in particular, b_i is *primitive*: it is the first $\mathbb{Z}^n$–point on the ray $\mathbb{R}_{\geq0}b_i\subset\mathbb{R}^n$).

The edges of B correspond to homogeneous coordinates x_i for B. They give rise to divisors

$$D_i=(x_i=0)\subset B,$$

called *toric divisors*. These are precisely the irreducible T–invariant effective divisors of B.

Lemma A.2 (see [17, Section 3.4]) *Any holomorphic line bundle $E\to B$ is isomorphic to $\mathcal{O}\big(\sum n_iD_i\big)$ for some $n_i\in\mathbb{Z}$.*

Example The description of $\mathcal{O}(-k)\to\mathbb{P}^m$ in Example A.1 is $\mathcal{O}(-kD_{m+1})$.

Lemma A.3 *Let $E=\mathcal{O}\big(\sum n_iD_i\big)\xrightarrow{\pi}B$, for $n_i\in\mathbb{Z}$. A fan for E is given by the edges*

$$e_1=(b_1,-n_1),\ \dots,\ e_r=(b_r,-n_r),\ e_{r+1}=(0,\dots,0,1)\in\mathbb{Z}^{n+1},$$

with the following cones: whenever the $\mathbb{R}_{\geq 0}$–span of a subset $b_{j_1}, \dots, b_{j_k}$ of the b_i is a cone for B, then the $\mathbb{R}_{\geq 0}$–span of $e_{j_1}, \dots, e_{j_k}$ is a cone for E and the $\mathbb{R}_{\geq 0}$–span of $e_{j_1}, \dots, e_{j_k}, e_{r+1}$ is a cone for E. That subset corresponds to the subvariety $V_J = (x_{j_1} = \cdots = x_{j_k} = 0) \subset B$, and it gives rise to two subvarieties in E, respectively $\pi^{-1}(V_J) \subset E$ and $V_J \subset B \hookrightarrow E$.

Proof In general, an effective divisor D can be described by $f_\alpha = 0$ on U_α, where U_α are open affines covering B, f_α are nonzero rational functions, and f_α / f_β are nowhere zero regular functions on $U_\alpha \cap U_\beta$. Then $\mathcal{O}(D)$ is the $\mathcal{O}_B$–subsheaf of the sheaf of rational functions $\mathcal{M}_B$ on B generated by $1/f_\alpha$ on U_α. Viewed as a line bundle, this has transition function $g_{\beta\alpha} = (f_\beta \circ \varphi_{\beta\alpha})/f_\alpha \colon U_\alpha \times \mathbb{C} \to U_\beta \times \mathbb{C}$, where $x' = \varphi_{\beta\alpha}(x)$ is the change of coordinates $U_\alpha \to U_\beta$. Indeed, comparing inside $\mathcal{M}_B$ we have $z \cdot (1/f_\alpha(x)) = g_{\beta\alpha} z \cdot (1/f_\beta(\varphi_{\beta\alpha}(x)))$.

First, consider the simple case $\mathcal{O}(D_1)$. Take $f_1 = 1$ and the other $f_i = x_1$, where $U_i = (x_i \neq 0) \subset B$. So $\mathcal{O}(D_1) = \mathcal{O}_B \subset \mathcal{M}_B$ on U_1, and $\mathcal{O}(D_1) = (1/x_1)\mathcal{O}_B \subset \mathcal{M}_B$ on the other U_i. Let us check g_{21}; the other cases are similar. Denote by G the group determined by the fan for E described in the claim, and let G_B denote the analogous group for the fan of B. There is an element $(t_1, \dots, t_r) \in G_B$ which, via multiplication, identifies

$$\begin{gathered} \varphi_{21} \colon U_1 \to U_2, \\ (1, x_2, \dots, x_r) \mapsto (x_1', 1, x_2', \dots, x_r') = (t_1 \cdot 1, t_2 \cdot x_2, \dots, t_r \cdot x_r). \end{gathered}$$

In particular, $t_1 = x_1'$. Now G is the kernel of a homomorphism $(\mathbb{C}^*)^{r+1} \to (\mathbb{C}^*)^{n+1}$ with last entry $t_1^{-1} t_{r+1}$ (the powers are the last entries of e_1, e_{r+1}). So $t_{r+1} = t_1 = x_1'$. So for the fiber coordinate $x_{r+1} \mapsto t_{r+1} x_{r+1} = x_1' x_{r+1}$. So $g_{21} = x_1' = f_2(x')/f_1(x)$.

Now consider the general case $\mathcal{O}\big(\sum n_i D_i\big)$. Take $f_i(x) = \prod_{j \neq i} x_j^{n_j}$. Now we need

$$g_{21} = \frac{f_2(x')}{f_1(x)} = \frac{\prod_{j \neq 2} (x_j')^{n_j}}{\prod_{j \neq 1} x_j^{n_j}} = \frac{(x_1')^{n_1}}{x_2^{n_2}} \prod_{j \geq 3} \left(\frac{x_j'}{x_j} \right)^{n_j}.$$

This time $t_1^{-n_1} \cdots t_r^{-n_r} t_{r+1} = 1$, where $t_1 = x_1'$, $t_2 = 1/x_2$ and $t_i = x_j'/x_j$. Thus, $x_{r+1} \mapsto t_{r+1} x_{r+1} = g_{21} x_{r+1}$ as required. □

A2 Toric symplectic manifolds and moment polytopes

Let (X, ω_X) be a closed real $2n$–dimensional symplectic manifold together with an effective Hamiltonian action of the n–torus $U(1)^n$. This action determines a *moment map* $\mu_X \colon X \to \mathbb{R}^n$, which is determined up to an additive constant (we tacitly identify $\mathbb{R}^n$ with the dual of the Lie algebra of $U(1)^n$). The image $\Delta = \mu_X(X) \subset \mathbb{R}^n$ is a convex polytope, called the *moment polytope*. By Delzant's theorem Δ determines, up to isomorphism, (X, ω_X) together with the action. More precisely, two toric manifolds with moment polytopes Δ_1, Δ_2 are equivariantly symplectomorphic if and only if $\Delta_2 = A\Delta_1 + \text{constant}$, where $A \in \mathrm{SL}(n, \mathbb{Z})$.

The moment polytope has the form

$$\Delta = \{y \in \mathbb{R}^n : \langle y, e_i \rangle \geq \lambda_i \text{ for } i = 1, \dots, r\}, \tag{35}$$

where $\lambda_i \in \mathbb{R}$ are parameters and $e_i \in \mathbb{Z}^n$ are the primitive inward-pointing normal vectors to the facets of Δ (the codimension 1 faces). For Delzant's theorem to hold, we always assume that at each vertex p of Δ there are exactly n edges of Δ meeting at p; that the edges are rational, that is they are of the form $p + \mathbb{R}_{\geq 0} v_i$ for $v_i \in \mathbb{Z}^n$; and that these $v_1, \dots, v_n$ are a $\mathbb{Z}$–basis for $\mathbb{Z}^n$.

From the polytope, one can construct a fan as follows. A face F is determined by a subset I_F of indices i for which the inequality $\langle y, e_i \rangle \geq \lambda_i$ is an equality. The data $(e_1, \dots, e_r)$ and $(I_F : F \text{ is a face of } \Delta)$ defines the fan, taking cones σ_{I_F} to be the $\mathbb{R}_{\geq 0}$–span of the $(e_i : i \in I_F)$.

By construction, the polytope Δ is combinatorially dual to the fan; in particular, the facets are orthogonal to the edges $e_1, \dots, e_r$ of the fan for X. The fan determines the complex structure on X, but of course does not encode the λ_i. In particular, the location of Δ in $\mathbb{R}^n$ depends on the choice of additive constant in μ_X, which in turn depends on the choice of symplectic form on X, and this is not encoded in the fan.

The λ_i are related to the symplectic form ω_X on X by the cohomological condition

$$[\omega_X] = -\sum \lambda_i \ \mathrm{PD}[D_i] \in H^2(X; \mathbb{Z}),$$

where the $\mathrm{PD}[D_i]$ are the Poincaré duals of the divisors $D_i = (x_i = 0) \subset X$ corresponding to the vanishing of one of the homogeneous coordinates x_i (which correspond to the edges e_i of the fan). This does not usually determine the λ_i, since the $[D_i]$ can be linearly dependent, but a refinement [23, Appendix A.2.1] of the above formula determines the Kähler form ω_X in terms of the λ_i as

$$\omega_X\big|_{\mu_X^{-1}(\mathrm{int}(\Delta))} = \frac{\sqrt{-1}}{2\pi} \partial\bar{\partial}\Big(\sum \lambda_i \log[\langle \mu_X(\cdot), e_i \rangle - \lambda_i] + \langle \mu_X(\cdot), \sum e_i \rangle\Big).$$

Remark A.4 The 2π ensures that for $\mathbb{P}^m$ one obtains the normalized Fubini–Study form, $\int_{[\mathbb{P}^1]} \omega_{FS} = 1$. The 2π is missing in [23, Appendix 2.1(1.3) and 2.3(4.5)], but should be there for [23, Appendix 2.1(1.6)] to hold, as can be checked for $\mathbb{P}^1$. Therefore we differ from Cho and Oh [10, Theorem 3.2] by the rescaling $\omega_P = 2\pi\omega_X$.

A3 The moment map

The moment map μ_X is determined by the diagram

$$\begin{array}{ccccc} \mathbb{C}^r - Z & \xrightarrow{\text{inclusion}} & \mathbb{C}^r & \xrightarrow{\mu_{\mathbb{C}^r}} & \mathbb{R}^r \\ \Big\downarrow{\scriptstyle\text{quotient}} & & & & \Big\uparrow{\scriptstyle\beta^t} \\ X = (\mathbb{C}^r - Z)/G & \xrightarrow[f_G]{\cong} & f^{-1}(0)/G_{\mathbb{R}} & \xrightarrow[\mu_X]{} & \mathbb{R}^n \end{array}$$

where, summarizing [23, Appendix 1],

(1) $r =$ number of edges e_i in the fan for X, and $n = \dim_{\mathbb{C}} X$.

(2) $\mu_{\mathbb{C}^r}(x) = \frac{1}{2}(|x_1|^2, \dots, |x_r|^2) + (\lambda_1, \dots, \lambda_r)$, the moment map for $U(1)^r$ acting by multiplication on $\mathbb{C}^r$ (using the convention $z \mapsto e^{i\theta} z$ for $\theta \in \mathbb{R}/2\pi\mathbb{Z}$, $z \in \mathbb{C}$).

(3) $Z \subset \mathbb{C}^r$ is the union of the vanishing sets $(x_i = 0 : i \in I)$ for those multi-indices I for which the $(e_i : i \in I)$ do *not* span a cone of the fan.

(4) $\beta\colon \mathbb{R}^r \to \mathbb{R}^n$ is the matrix whose columns are the edges $e_i \in \mathbb{R}^n$ of the fan. The matrix β has full rank, so the transpose matrix β^t is injective.

(5) Let $\mathfrak{g}_{\mathbb{C}} = \ker(\beta\colon \mathbb{C}^r \to \mathbb{C}^n) = \operatorname{Lie} G$ and $\mathfrak{g}_{\mathbb{R}} = \ker(\beta\colon \mathbb{R}^r \to \mathbb{R}^n) = \operatorname{Lie} G_{\mathbb{R}}$.

(6) $G \subset (\mathbb{C}^*)^r$ and $G_{\mathbb{R}} \subset U(1)^r$ are the images of $\mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{g}_{\mathbb{R}}$ respectively, via $\alpha \mapsto (e^{i\alpha_1}, \dots, e^{i\alpha_r})$.

(7) G and $G_{\mathbb{R}}$ act on $\mathbb{C}^r$ by multiplication; and $G = \ker(\operatorname{Exp}(\beta)\colon (\mathbb{C}^*)^r \to (\mathbb{C}^*)^n)$, where $\operatorname{Exp}(\beta)$ is β conjugated by the maps $\mathbb{C} \to \mathbb{C}/2\pi\mathbb{Z} \to \mathbb{C}^*$ given by $w \mapsto e^{iw}$.

(8) $f\colon \mathbb{C}^r \to \mathbb{R}^{r-n}$ is the moment map for the $G_{\mathbb{R}}$–action on $\mathbb{C}^r$. Pick an identification $\ker\beta \cong \mathbb{R}^{r-n}$; then $f(x) = \kappa^t \mu_{\mathbb{C}^r}(x)$, where $\kappa\colon \mathbb{R}^{r-n} \to \mathbb{R}^r$ is the inclusion of $\ker\beta$.

(9) $f_G\colon G \cdot x \mapsto ((G \cdot x) \cap f^{-1}(0))/G_{\mathbb{R}}$ and $f_G^{-1}\colon G_{\mathbb{R}} \cdot x \mapsto G \cdot x$.

(10) $\mu_X(x) = (\beta^t)^{-1}_{\text{left}} \cdot \mu_{\mathbb{C}^r}(x)$ on $f^{-1}(0)/G_{\mathbb{R}}$ using the left inverse, $(\beta^t)^{-1}_{\text{left}}\beta^t = \mathrm{id}$.

The moment polytope Δ can be recovered from μ_X by $\mu_X(X) = \Delta$; the T–fixed points of X are in bijection with the vertices of Δ via μ_X; the T–orbits are in bijection with the faces of Δ via μ_X.

Since $(\beta)^{-1}_{\mathrm{left}} e_i$ is the i^{th} standard vector, when $x \in f^{-1}(0)$ the formula for ω_X simplifies:

$$\begin{aligned}
&\langle \mu_X(x), e_i\rangle - \lambda_i = \langle (\beta^t)^{-1}_{\mathrm{left}} \mu_{\mathbb{C}^r}(x), e_i\rangle - \lambda_i = \langle \mu_{\mathbb{C}^r}(x), (\beta)^{-1}_{\mathrm{left}} e_i\rangle - \lambda_i = \tfrac{1}{2}|x_i|^2, \\
&\Big\langle \mu_X(x), \sum e_i \Big\rangle = \Big\langle \mu_{\mathbb{C}^r}(x), \sum (\beta)^{-1}_{\mathrm{left}} e_i \Big\rangle = \sum \tfrac{1}{2}|x_i|^2 + \sum \lambda_i, \\
&\omega_X|_{f^{-1}(0)\cap(x_i\neq 0)} = \frac{\sqrt{-1}}{2\pi}\, \partial\bar{\partial}\Big(\sum \lambda_i \log \tfrac{1}{2}|x_i|^2 + \sum \tfrac{1}{2}|x_i|^2\Big).
\end{aligned}$$

A4 The polytope of a Fano variety

A polytope $\Delta \subset \mathbb{R}^n$ is called *reflexive* if its vertices lie in $\mathbb{Z}^n$, the only $\mathbb{Z}^n$–point lying in the interior of Δ is 0, and the $\lambda_i = -1$. These were studied by Batyrev (see a discussion in [12, Section 3.5]); in particular, a closed toric variety X is Fano if and only if X admits a polytope Δ which is reflexive. Recall that *Fano* means that the anticanonical bundle $\Lambda^{\mathrm{top}}_{\mathbb{C}} TB$ is ample. For a reflexive Δ, the associated Kähler form ω_Δ lies in the class $[\omega_\Delta] = \sum -\lambda_i\, \mathrm{PD}[D_i] = c_1(TX)$ since $\lambda_i = -1$ (as $c_1(TX) = \sum \mathrm{PD}[D_i]$ holds in general). So monotone negative line bundles $E \to B$ over toric B always arise as follows:

1. (1) B is a Fano variety with an integral Kähler form ω_Δ coming from a reflexive Δ;
2. (2) by rescaling ω_Δ we obtain a primitive integral Kähler form ω_B;
3. (3) $c_1(TB) = [\omega_\Delta] = \lambda_B[\omega_B]$, where $\lambda_B \in \mathbb{Z}$ is called the *index* of the Fano variety;
4. (4) up to isomorphism there is only one negative line bundle $E = E_k$ with $c_1(TE) = -k[\omega_B]$, for $k \in \mathbb{Z}_{>0}$;
5. (5) E_k is monotone if and only if $1 \le k \le \lambda_B - 1$ (consequently we need $\lambda_B =$ Fano index ≥ 2).

Example A smooth complete intersection $B \subset \mathbb{P}^{n+s}$ defined by equations of degrees $d_1 \ge \cdots \ge d_s > 1$ is Fano if and only if $1 + n + s - (d_1 + \cdots + d_s) \ge 1$, and this difference is the Fano index. One can replace $\mathbb{P}^{n+s}$ by weighted projective space $\mathbb{P}(a_0, \dots, a_m)$, then for $\dim_{\mathbb{C}} B \ge 3$ the index is $\sum a_j - \sum d_i \ge 1$; see Kollar [27, page 245]. There are only finitely many Fano toric varieties of dimension n up to isomorphism since there are only finitely many reflexive polytopes up to unimodular transformation. The example shows there are many of index > 1.

A5 Using nonreflexive polytopes

For $\mathbb{P}^2$ the reflexive polytope has vertices $(-1,-1)$, $(2,-1)$, $(-1,2)$, barycenter $(0,0)$ and symplectic form $\omega = (1+2)\omega_{\mathbb{P}^2}$. But often one prefers to use the nonreflexive polytope $(1/(1+2))(\Delta-(-1,-1))$, which has vertices $(0,0)$, $(1,0)$, $(0,1)$, barycenter $(1/(1+2), 1/(1+2))$ and symplectic form $\omega_{\mathbb{P}^2}$. Here, $1+2$ plays the role of $\lambda_{\mathbb{P}^m} = 1+m$.

The next lemma explains how, from a reflexive polytope Δ for B, one obtains a polytope Δ_B which induces $[\omega_B] = (1/\lambda_B)[\omega_\Delta]$ and has 0 as a vertex. Using Δ_B we can construct the polytope Δ_E of negative line bundles in Section A6 (whereas for Δ one has $[\omega_\Delta] = c_1(TB)$, so it would be unclear how to get $E = \mathcal{O}\big(\sum n_i D_i\big)$ with $c_1(E) = -(k/\lambda_B)[\omega_\Delta]$, since k/λ_B is fractional).

Lemma A.5 *Let v be a vertex of Δ. Then $\Delta_B = (1/\lambda_B)(\Delta - v)$ is a polytope with vertices in $\mathbb{Z}^n$, parameters $\lambda_i^B \le 0$ in $\mathbb{Z}$, and associated symplectic form cohomologous to ω_B, so $[\omega_B] = \sum -\lambda_i^B\,\mathrm{PD}[D_i]$. Moreover, the barycenter y_{bar} of Δ_B satisfies $\langle y_{\mathrm{bar}}, b_i\rangle - \lambda_i^B = 1/\lambda_B$.*

Proof Let b_i be the edges of the fan for B (so the inward primitive normals of Δ). Adding $-v$ to Δ changes $\lambda_i = -1$ to $\lambda_i + \langle -v, b_i\rangle \in \mathbb{Z}$, which changes $[\omega]$ by $\sum \langle v, b_i\rangle\,\mathrm{PD}[D_i]$. But in general, $\sum b_i[D_i] = 0$ are n relations satisfied by the divisor classes $[D_i]$ (combining the lemma on page 61 and the corollary on page 64 in Fulton [17]). So $[\omega]$ does not change.

The translated polytope $\Delta - v$ still has normals b_i and has $0 = v - v \in \partial\Delta$, so from the equations $\langle 0, b_i\rangle \ge \lambda_i^B$ we obtain $\lambda_i^B \le 0$.

Applying $A \in \mathrm{GL}(n,\mathbb{Z})$ to a polytope changes b_i to $A'b_i$, where $A' = (A^{-1})^T \in \mathrm{GL}(n,\mathbb{Z})$. So the λ_i don't change. The fan changes by applying A', but that keeps the toric variety unchanged. The moment map μ_B becomes $A\mu_B$, so the symplectic form associated to the polytope does not change.

Consider an edge from v to v' in Δ. For some A as above, $A(v'-v) = (a, 0, \dots, 0)$, for some $a \in \mathbb{Z}$. By Guillemin [23, Theorem 2.10], the Euclidean volume of a polytope is the symplectic volume of the toric manifold. Thus the Euclidean length of an edge in $A(\Delta - v)$ is the symplectic area of the 2–sphere in B corresponding to that edge. Let u denote the sphere for the edge joining 0 to $A(v'-v)$. Then $a = \int u^*[\omega] = \lambda_B \int u^*[\omega_B]$. But $\int u^*[\omega_B] \in \mathbb{Z}$ since ω_B is an integral form. So a is divisible by λ_B. So also $v' - v \in \mathbb{Z}^n$ has entries divisible by λ_B. Applying this argument to any v shows that differences of vertices lying on edges of Δ are divisible by λ_B. Thus the vertices of $\Delta - v$ are divisible by λ_B, so $(1/\lambda_B)(\Delta - v)$ is an integral polytope. Since each face of this polytope lies on a hyperplane given by equations $\langle y, b_i\rangle = \lambda_i^B$ for a certain subset of the indices i, we also conclude that $\lambda_i^B = (1/\lambda_B)(\lambda_i + \langle -v, b_i\rangle)$ are integers. In particular, since also μ_B gets rescaled by $1/\lambda_B$, the associated symplectic form gets rescaled by $1/\lambda_B$ so it now lies in the cohomology class $[\omega_B]$.

For the barycenter, by construction $\langle y, b_i\rangle - \lambda_i$ is invariant under translating Δ (and y), and for Δ the barycenter 0 gives the value 1. Rescaling the polytope by $1/\lambda_B$ rescales this 1 by $1/\lambda_B$. □

A6 The moment polytope of toric negative line bundles

Lemma A.6 *The moment polytope Δ_E for $E = \mathcal{O}\big(\sum n_i D_i\big) \to B$ inducing the symplectic form $[\omega_E] = [\pi^*\omega_B]$ is*

$$\begin{aligned}\Delta_E &= \{y \in \mathbb{R}^{n+1} : \langle y, e_i\rangle \geq \lambda_i^E\}\\ &= \{(Y, y_{n+1}) \in \mathbb{R}^n \times \mathbb{R} : y_{n+1} \geq 0,\ \ \langle Y, b_i\rangle \geq \lambda_i^B + y_{n+1} n_i \ \text{ for } i = 1, \dots, r\}.\end{aligned}$$

Namely, $\lambda_{r+1}^E = 0$, $\lambda_i^E = \lambda_i^B$ and $e_{r+1} = (0, \dots, 0, 1)$, $e_i = (b_i, -n_i)$; see Lemma A.3. In particular, $\omega_E\big|_B = \omega_B$. Geometrically, $\Delta_E \subset \mathbb{R}^{n+1}$ lies in the upper half-space $y_{n+1} \geq 0$, its facet along $y_{n+1} = 0$ is $\Delta_B \hookrightarrow \mathbb{R}^{n+1}$, it has the same vertices as Δ_B, and the other facets lie in the hyperplanes in $\mathbb{R}^{n+1} \cap (y_{n+1} \geq 0)$ which are normal to the $e_i = (b_i, -n_i)$ and pass through the facets of Δ_B (with the exception of e_{r+1}, which is normal to the facet Δ_B).

Proof The facets are normal to the edges e_i of the fan for E because moment polytopes are combinatorially dual to fans. The T–invariant divisors in E are $\pi^{-1} D_i = (x_i = 0)$ for $i \leq r$, and $B = (x_{r+1} = 0)$. As locally finite cycles, these are Poincaré dual to $\pi^* \operatorname{PD}_B[D_i]$ and $\pi^* c_1(E)$ respectively. Recall $[\omega_B] = -\sum \lambda_i^B \operatorname{PD}_B[D_i]$, therefore

$$[\omega_E] = -\sum\big(\lambda_i^E \operatorname{PD}_E[\pi^{-1} D_i] + \lambda_{r+1}^E[B]\big) = -\sum \lambda_i^B \pi^* \operatorname{PD}_B[D_i] = \pi^*[\omega_B].$$

That $\omega_E\big|_B = \omega_B$ can be seen from the explicit formula for ω at the end of Section A3, using the fact that $\lambda_{r+1}^E = 0$ (which ensures the $\log \frac{1}{2}|x_{r+1}|^2$ term does not appear), restricting to the subspace $(dx_{r+1} = 0) \subset TE$ and letting $x_{r+1} \to 0$. □

Lemma A.7 *For any negative line bundle E over toric B, the form ω_E agrees with the form ω of Section 4A, with Hermitian norm $(1/\sqrt{2\pi})|w|$ for the fiber at a point x satisfying $\mu_E(x) = (\dots, \frac{1}{2}|w|^2)$.*

Sketch of proof The proof requires studying the construction in [23, Appendix A.2.1] of the Kähler form ω_X on a toric X. The ω_X is determined from its restriction to the fixed point set X_r of the involution on X induced by complex conjugation on $\mathbb{C}^r$. On X_r, the Kähler metric becomes $(1/2\pi)\sum(dx_i)^{\otimes 2}$ (see [23, Appendix A.2.2(2.5)] using Remark A.4). So for $X = E$ the fiber component is $(1/2\pi)(dw)^{\otimes 2}$. Since this recovers the standard metric $(d\rho)^{\otimes 2}$ on the real ray $\mathbb{R}_{>0} \subset \mathbb{C}$ in the fiber, we deduce that the Hermitian norm for the fiber is $\rho = (1/\sqrt{2\pi})|w|$. □

A7 The Landau–Ginzburg superpotential

Definition (preliminary version) The superpotential $W\colon (\mathbb{C}^*)^n \to \mathbb{C}$ for a toric variety X, with $\dim_{\mathbb{C}} X = n$, is the Laurent polynomial $W(z_1,\dots,z_n) = \sum_{i=1}^r e^{\lambda_i} z^{e_i}$ defined on the domain $(\mathbb{C}^*)^n \cap \{|e^{\lambda_i} z^{e_i}| < 1 \ \forall i\} = \mathrm{Log}^{-1}(\mathrm{int}(\Delta))$ described by

$$\begin{tikzcd}
X \setminus \cup D_i = (\mathbb{C}^*)^r/G \arrow[r, "\mathrm{Exp}(\mu)"] \arrow[d, "\text{inclusion}"'] & (\mathbb{C}^*)^n \supset \mathrm{Log}^{-1}(\mathrm{int}(\Delta)) \arrow[r, "W"] \arrow[dr, "\mathrm{Log}"] & \mathbb{C} \\
X = (\mathbb{C}^r - Z)/G \arrow[rr, "\mu_X"'] & & \Delta \subset \mathbb{R}^n
\end{tikzcd}$$

where $\mathrm{Log}(z) = (-\log|z_1|,\dots,-\log|z_n|)$, $\mathrm{Exp}(\mu)(x) = (e^{-\mu_{X,1}(x)},\dots,e^{-\mu_{X,n}(x)})$, involving the components $\mu_{X,j}(x) \in \mathbb{R}$ of $\mu_X(x) \in \mathbb{R}^n$.

Explanation (Auroux [3, Proposition 4.2]) The domain of W is actually the moduli space M of gauge equivalence classes of special Lagrangian submanifolds L inside the torus $T = X \setminus \cup D_i \cong (\mathbb{C}^*)^n$ equipped with a flat unitary connection on the trivial complex line bundle over L. Then W is a weighted count of $\mu = 2$ holomorphic discs bounding L with a boundary marked point constraint through a generic point of L, and the weight in the count is $e^{-\omega[u]} \cdot \mathrm{holonomy}(u|_{\partial\mathbb{D}})$.

A Lagrangian $L \subset X \setminus \bigcup D_i$ is called *special* if some imaginary part

$$\mathrm{Im}(e^{-i\,\mathrm{constant}}\Omega\big|_L) = 0,$$

where $\Omega = d\log x_1 \wedge \cdots \wedge d\log x_n$ is a nonvanishing holomorphic n–form on $X \setminus \cup D_i$ (indeed, it is a section of the canonical bundle K_X with poles along D_i). Such L have the form $S^1(r_1) \times \cdots \times S^1(r_n) \subset (\mathbb{C}^*)^n$, where the r_i denote the radii.

The biholomorphism $M \cong \mathrm{Log}^{-1}(\mathrm{int}(\Delta))$ is given by

$$z_j = e^{-\mu_{X,j}(L)} \cdot \mathrm{holonomy}([S^1(r_j)]),$$

where $\mu_{X,j}$ is constant on L since L is a T–orbit, and where $[S^1(r_j)] \in H_1(L)$ determines the holonomy for the connection. Therefore the equations $|e^{\lambda_i} z^{e_i}| \le 1$ correspond via $y = \mu_X(x) = \mathrm{Log}(z)$ to the equations $\lambda_i - \sum_{j=1}^n y_j e_{i,j} \le 0$ defining Δ.

After this biholomorphic identification, by Cho and Oh [10] and Auroux [3, Proposition 4.3] $W = \sum e^{\lambda_i} z^{e_i}$, where e_i is the primitive inward-pointing normal to the facet of Δ defined by the equation $\{y \in \mathbb{R}^n : \langle y, e_i \rangle = \lambda_i\} \cap \Delta$.

Remark No factors of 2π arise in our z_j, Log, Exp and W due to Remark A.4.

Example For $\mathbb{P}^m$ we have $W = z_1 + \cdots + z_m + e^{-1} z_1^{-1} \cdots z_m^{-1}$.

Example A.8 If the special Lagrangian L with connection ∇ in the explanation above is monotone, and λ_L, λ_X are the monotonicity constants for L, X (recall that $2\lambda_L \lambda_X = 1$), then

$$W(L, \nabla) = \#(\text{discs})\, e^{-2\lambda_L} = \#(\text{discs})\, e^{-1/\lambda_X},$$

where #(discs) is the weighted count of Maslov 2 discs bounding L as in the explanation, the weights being the holonomies around the boundary of the discs.

One now actually wants to deform the Floer theory for the special Lagrangians. This is explained in Auroux [4, Section 4.1]: it can be done either by allowing nonunitary connections on L, or by deforming L by a non-Hamiltonian Lagrangian isotopy (and using a unitary connection), or by formally deforming the Fukaya category by a cocycle $b_L \in CF^1(L, L)$ by the machinery of Fukaya, Oh, Ohta and Ono [15] (this corresponds to the connection $\nabla = d + b_L$). We will allow nonunitary connections, which is an idea that goes back to Cho [9], and is explained also in Fukaya, Oh, Ohta and Ono [16, Sections 4 and 12] and in Auroux [3, Remark 3.5]. So we work over the Novikov ring Λ with $\mathbb{K} = \mathbb{C}$, the e^{-1} above is replaced by t, and we work with $\mathcal{W}(E)$, the wrapped category with local systems (see Ritter and Smith [37]).

Definition A.9 (corrected version) The superpotential $W\colon (\Lambda \setminus \{0\})^n \to \Lambda$ (with $\mathbb{K} = \mathbb{C}$) for a toric variety X with $\dim_{\mathbb{C}} X = n$ is

$$W(z_1, \dots, z_n) = \sum_{i=1}^{r} t^{-\lambda_i} z^{e_i}.$$

Example For $\mathbb{P}^m$ we have $W = z_1 + \ldots + z_m + t z_1^{-1} \cdots z_m^{-1}$.

The condition that a point $z = (z_1, \dots, z_r)$ land inside the polytope is now the condition $\mathrm{val}_t(t^{-\lambda_i} z^{e_i}) > 0$ for $i = 1, \dots, r$, where val_t is the valuation for the t–filtration, whose value on a Laurent series is the lowest exponent of t arising in the series. Indeed, if $z_i \in t^{y_i} \mathbb{C}^* +$(higher order), then that condition becomes $\langle y, e_i \rangle > \lambda_i$, the equations defining interior(Δ). We recover the point of the polytope over which the toric fiber L lies and the holonomy around each generating circle of $\pi_1(L)$ by

$$\Lambda^n \ni z \mapsto (\mathrm{val}_t(z), t^{-\mathrm{val}_t(z)} z) \in \mathrm{interior}(\Delta) \times (\Lambda_0^\times)^n \subset \mathbb{R}^n \times H^1(L, \Lambda_0^\times),$$

where $\Lambda_0^\times$ is the multiplicative group of units in the subring $\Lambda_0 \subset \Lambda$ of series with $\mathrm{val}_t \geq 0$. We say z *lands at* $y \in \mathbb{R}^n$ if $y = \mathrm{val}_t(z) = (\mathrm{val}_t(z_1), \dots, \mathrm{val}_t(z_n))$.

Example A.10 For $E = \mathcal{O}\big(\sum n_i D_i\big) \to B$,

$$W(z_1, \dots, z_{n+1}) = \sum_{i=1}^{r} t^{-\lambda_i^B} z^{(b_i, -n_i)} + z_{n+1}$$

by Lemma A.6. Since $[\omega_B] = \sum -\lambda_i^B \,\mathrm{PD}[D_i]$, the bundle $E = \mathcal{O}\big(k \sum \lambda_i^B D_i\big)$ has $c_1(E) = -k[\omega_B]$ and

$$\begin{aligned} W_E(z_1, \dots, z_n, z_{n+1}) &= \sum_{i=1}^{r} z^{b_i} (t z_{n+1}^k)^{-\lambda_i^B} + z_{n+1} \\ &= z_{n+1} + W_B(z)\big|_{(t \text{ replaced by } t z_{n+1}^k)}. \end{aligned}$$

Theorem A.11 *For a monotone negative line bundle $E \to B$ over a toric B, the critical values $W(p)$ of the superpotential W are a subset of the eigenvalues of $c_1(TE)$: $QH^*(E) \to QH^{*+2}(E)$ acting by quantum cup-product. The critical points p of W correspond to Lagrangian toric fibers L_p of the moment map, together with a choice of holonomy data, such that*

$$HF^*(L_p, L_p) \cong HW^*(L_p, L_p) \neq 0.$$

Thus the L_p are nondisplaceable Lagrangians, and the existence of an L_p forces $SH^*(E) \neq 0$.

Proof This is proved by mimicking Auroux [3, Section 6], except that in the non-compact setup we use locally finite cycles to represent cohomology classes. The monotonicity condition on E ensures that B and E are Fano toric varieties. The fact that Floer cohomology can be defined independently of whether L is monotone or not is due to Cho and Oh [10, Section 7]: the m_1–obstruction class vanishes when (L, ∇) arises as a critical point of W and the toric Fano assumption then ensures that Lagrangian Floer cohomology exists and is nontrivial (see also Auroux [3, Proposition 6.9, Lemma 6.10]). That $HF^*(L_p, L_p) \cong HW^*(L_p, L_p)$ follows because the torus L_p is compact. The final claim follows because if $SH^*(E) = 0$, then $HW^*(L, L) = 0$ since it is a module over $SH^*(E)$ (see [34]). □

Example A.12 For $E = \mathcal{O}_{\mathbb{P}^m}(-k)$,

$$W = z_1 + \cdots + z_m + t^k z_1^{-1} \cdots z_m^{-1} z_{m+1}^k + z_{m+1}.$$

The critical points of W are $z = (w, \ldots, w, -kw)$, with critical values $W(z) = (1+m-k)w$, for any solution w of $w^{1+m-k} = (-k)^k t^k$. The w and $W(z)$ are eigenvalues respectively of $\pi^*[\omega_{\mathbb{P}^m}]$ and $c_1(TE) = (1+m-k)\pi^*[\omega_{\mathbb{P}^m}]$. Here $1 \le k \le m$ is required for E to be monotone.

Corollary A.13 *The critical values of W are homogeneous in t of order $t^{1/\lambda_E} = T$.*

Proof Since E is monotone, $QH^*(E)$ is $\mathbb{Z}$–graded using grading $|t| = 2\lambda_E$ (see Section 2A). This grading of t ensures that the quantum cup product is grading preserving on $QH^*(E)$. Since $c_1(TE)$ lies in degree 2, the eigenvalues must lie in degree 2. Finally, $|t^{1/\lambda_E}| = 2$. □

Remark A.14 $\mathrm{Crit}(W)$ only detects the eigenvalues of the action of $c_1(TE)$ on $SH^*(E)$ by Theorem 4.19, so it forgets the zero eigenvalues of the action on $QH^*(E)$.

A8 Critical points of W_X arise in λ_X–families

Lemma A.15 *For (X, ω_X) a monotone toric manifold with integral $\omega_X \in H^2(X, \mathbb{Z})$,*

$$W_X(\xi z) = \xi W_X(z) \quad \textit{whenever} \quad \xi^{\lambda_X} = 1.$$

It also holds for Fano toric manifolds (X, ω_F) for the F–twisted superpotential (Section 5F).

Proof After relabeling the indices, we can assume $W_X = \sum t^{-\lambda_i} z^{e_i}$ has the form

$$W_X = z_1 + \cdots + z_n + d_{n+1}(z) + \cdots + d_r(z),$$

where $d_j(z)$ are monomials in the free variables $z_i^{\pm 1}$. This is proved as follows. Consider a top-dimensional cone for X, say $\mathrm{span}_{\mathbb{R}_{\ge 0}}\{e_1, \ldots, e_n\}$ (relabel indices if necessary). This corresponds to a chart for X. Making an $\mathrm{SL}(n, \mathbb{Z})$ transformation so that the $e_1, \ldots, e_n$ become the standard basis of $\mathbb{R}^n$, and translating the moment polytope so that $\lambda_1, \ldots, \lambda_n$ become zero, will not affect X (up to an equivariant symplectomorphism). It follows that W_X has the above form, and the functions d_j express the linear dependence relations amongst edges. Explicitly, $d_j(z) = t^{-\lambda_j} z_1^{a_1} \cdots z_n^{a_n}$ means $e_j = a_1 e_1 + \cdots + a_n e_n$.

Example For $\mathbb{P}^2$ we have $W_X(z) = z_1 + z_2 + t z_1^{-1} z_2^{-1}$ so $d_3(z) = t z_1^{-1} z_2^{-1}$, which expresses the fact that $e_3 = (-1,-1) = -(1,0) - (0,1) = -e_1 - e_2$.

Now observe how W_X changes under the action, as

$$\begin{aligned} W_X(\xi z) &= \xi z_1 + \cdots + \xi z_n + d_{n+1}(\xi z) + \cdots + d_r(\xi z) \\ &= \xi z_1 + \cdots + \xi z_n + \xi^{\langle e_{n+1},(1,\dots,1)\rangle} d_{n+1}(z) + \cdots + \xi^{\langle e_r,(1,\dots,1)\rangle} d_r(z) \end{aligned}$$

since, in the above notation, $d_j(\xi z) = \xi^{a_1+\cdots+a_n} z^{e_j}$. Thus $W_X(\xi z) = \xi W_X(z)$ follows if we can show that $a_1 + \cdots + a_n$ is congruent to 1 modulo λ_X.

The $\mathbb{Z}$–linear relation amongst edges, $e_j - a_1 e_1 - \cdots - a_n e_n = 0$, corresponds to a class $\gamma \in H_2(X)$ determined by the intersection products $\gamma \cdot D_i = -a_i$ for $i \le n$ and $\gamma \cdot e_j = 1$ (see Section 3A). Since $c_1(TX) = \sum \mathrm{PD}[D_i]$, it follows that $c_1(TX)(\gamma) = 1 - a_1 - \cdots - a_n$. By monotonicity, $c_1(TX) = \lambda_X \omega_X$, so $1 - a_1 - \cdots - a_n$ is an integer multiple of λ_X, since ω_X is integral.

Since z^{e_i} gets rescaled by ξ via $z \mapsto \xi z$, the proof also holds for the F–twisted superpotential (note that we use $c_1(TX) = \lambda_X \omega_X$ at the end of the proof, not the nonmonotone ω_F). □

Corollary A.16 *For (X, ω_X) as above, the critical points of W_X arise in λ_X–families: if $p \in \mathrm{Crit}(W_X)$ then $\xi p \in \mathrm{Crit}(W_X)$, with $W_X(\xi p) = \xi W_X(p)$ whenever $\xi^{\lambda_X} = 1$.*

Proof This follows since $d(W_X(\xi z)) = d(\xi W_X(z)) = \xi d(W_X(z))$. □

A9 The barycenter of the moment polytope for monotone toric manifolds

Fukaya, Oh, Ohta and Ono showed in [16, Theorem 7.11] that for closed monotone toric manifolds B, the points $p \in \mathrm{Crit}(W)$ always land at the barycenter of Δ, and the corresponding L_p is the unique monotone Lagrangian torus fiber of the moment map. This can be proved as follows.

Lemma A.17 *When B is a closed monotone toric manifold, $p \in \mathrm{Crit}(W)$ always lands at the barycenter of Δ and L_p is the unique monotone Lagrangian torus fiber of μ_B.*

Proof Take the reflexive polytope Δ for B (Section A4), so $W = t\big(\sum z^{e_i}\big)$. Now simply calculate the critical points $z \in (\mathbb{C}^*)^n$ of the map $\sum z^{e_i} \colon (\mathbb{C}^*)^n \to \mathbb{C}$. So $y = \mathrm{val}_t(z) = 0$, which is the barycenter. By Delzant's theorem, any other choice of polytope for B is $A\Delta + \text{constant}$, for $A \in \mathrm{SL}(n, \mathbb{Z})$, but such transformations preserve the barycenter.

To prove monotonicity of L, we need $\omega[u] = (1/2\lambda_B)\mu[u]$ for discs $u \colon (\mathbb{D}, \partial\mathbb{D}) \to (B, L)$ (recall that $2\lambda_L \lambda_B = 1$). By the long exact sequence

$$\pi_2(L) = 0 \to \pi_2(B) \to \pi_2(B, L) \to \pi_1(L) \to 0 = \pi_1(B),$$

using that toric B are simply connected and that a torus L has $\pi_2(L) = 0$, a basis for such discs are the standard $\mu = 2$ discs u_i in the homogeneous coordinate x_i (keeping the other x_j fixed). A calculation [10, Theorem 8.1] shows that $\omega[u_i] = \langle y, e_i \rangle - \lambda_i$ (without 2π due to Remark A.4). For the barycenter y the latter equals $1/\lambda_B$, so $\omega[u_i] = (1/2\lambda_B)\mu[u_i]$ as required. □

Theorem A.18 *For any admissible toric manifold M (Definition 1.4, for example a monotone toric negative line bundle $E \to B$), using the toric monotone form $\omega_\Delta = \sum \mathrm{PD}[D_i] = c_1(TM)$ so that the $\lambda_i = -1$ define the polytope Δ for M, the critical points of the superpotential all lie over $y = 0 \in \Delta$. More generally, for any polytope Δ_M for M inducing a monotone toric symplectic form ω_M (so $c_1(TM) = \lambda_M[\omega_M]$), the critical points of the superpotential lie over the "barycenter" of Δ_M, which is defined as the unique point $y \in \Delta_M$ satisfying*

$$\langle y, e_i \rangle - \lambda_i = \frac{1}{\lambda_M}. \tag{36}$$

Moreover, the corresponding Lagrangians L_p are monotone.

Proof Follows by the same proof as in Lemma A.17, using the fact that $\Delta_M = (1/\lambda_M)(A\Delta + \text{constant})$.

Sanity check Let's run the calculation of the critical points of W, as described above Example A.10. By Corollary A.13 we expect critical points $z = t^y \cdot c$ with $c \in (\mathbb{C}^*)^n$ and critical value $W \in T \cdot \mathbb{C}^*$ (recall that $T = t^{1/\lambda_M}$). Indeed, we evaluate

$$W = \sum t^{-\lambda_i} z^{e_i} = \sum t^{-\lambda_i + \langle y, e_i \rangle} c^{e_i} = t^{1/\lambda_M} \sum c^{e_i} \in T \cdot \mathbb{C}^*, \tag{37}$$

where in the last equality we used the barycenter equation (36). □

Corollary A.19 *The critical points of W_E all give rise (with various holonomy data) to the unique monotone Lagrangian torus L in the sphere bundle $SE \subset E$ of radius $1/\sqrt{\pi\lambda_E}$, which projects to the monotone Lagrangian torus in B lying over the barycenter of Δ_B.*

Proof By Theorem A.18, the last entry of $\mu_E(L)$ is $\mathrm{val}_t(z_{r+1}) = \langle y, e_{r+1} \rangle = 1/\lambda_E$ (using the fact that $\lambda^E_{r+1} = 0$ and $e_{r+1} = (0, \dots, 0, 1)$). By Lemma A.7, the Hermitian norm of points in L is $1/\sqrt{\pi\lambda_E}$. We now verify that the projection to B is as claimed. Since for critical p of W_E the Lagrangian L_p lies over the barycenter of Δ_E, we have $\omega_E[u_i^E] = 1/\lambda_E$ for the standard discs u_i^E bounding L_p (see the proof of Lemma A.17). The projection $\pi\colon E \to B$ forgets the last homogeneous coordinate x_{r+1}, so $\pi(u_i^E) = u_i^B$ are the standard discs in B bounding $\pi(L_p)$ for

$i = 1, \dots, r$. Since x_{r+1} is constant on u_i^E for those i, it follows that $\omega_E|_{u_i^E} = (1 + k\pi r^2)\pi^*\omega_B$, where r is the radius of the sphere bundle SE where L_p lies. Thus,

$$(1 + k\pi r^2)\omega_B[u_i^B] = \omega_E[u_i^E] = \frac{1}{\lambda_E}.$$

By the proof of Lemma A.17, $\pi(L_p)$ lies over the barycenter of Δ_B precisely if $\omega_B[u_i^B] = 1/\lambda_B$ for $i = 1, \dots, r$. This holds precisely if $\lambda_B = (1 + k\pi r^2)\lambda_E$, which, using $\lambda_E = \lambda_B - k$, is equivalent to $r = 1/\sqrt{\pi\lambda_E}$ as claimed. □

Appendix B: Matrix perturbations and Grassmannians

B1 Set-up of the eigenvalue problem

(Needed in Section 6F.) Suppose that a family of matrices

$$A(x) \in \mathrm{End}(\mathbb{C}^n)$$

is given, depending holomorphically on a complex parameter x near $x = 0$, and the generalized eigenspace decomposition of $\mathbb{C}^n$ for $A(x)$ needs to be compared with that of $A(0)$.

When the $A(x)$ are normal matrices, this problem is rather well-behaved [26, Chapter 2, Theorem 1.10]: the eigenvalues and the projections onto the eigenspaces are all holomorphic in x. However, in our situation the $A(x)$ are not normal ($A(0)$ will typically not be diagonalizable), and the outcome is significantly more complicated. In this case, the eigenvalues $\lambda_j(x)$ and the *eigenprojections* $P_j(x)$ (the projection onto the generalized eigenspace $\ker(A(x) - \lambda_j(x))^n$) are only holomorphic when x is constrained to lie in a simply connected domain D which does not contain *exceptional points*, that is, points $x = x_0$ where the characteristic polynomial $\chi_{A(x_0)}$ has repeated roots. The examples in Kato [26, Chapter 2, Examples 1.1 and 1.12] show that near x_0, the $\lambda_j(x)$ and $P_j(x)$ can have branch points, and the $P_j(x)$ typically have poles at x_0 (the $\lambda_j(x)$ are always continuous at x_0). Even when x_0 is not a branch point of $\lambda_j(x)$, then even though $\lambda_j(x)$ is holomorphic on a whole disc around x_0 and $P_j(x)$ is single-valued on a punctured disc around x_0, nevertheless $P_j(x)$ can have a pole at x_0!

Example B.1 $\left(\begin{smallmatrix} x & 1 \\ 0 & 0 \end{smallmatrix}\right)$ has eigenvalues $\lambda_1(x) = x$, $\lambda_2(x) = 0$, eigenvectors $v_1(x) = (1, 0)$, $v_2(x) = (1, -x)$, but eigenprojections $P_1(x) = \left(\begin{smallmatrix} 1 & x^{-1} \\ 0 & 0 \end{smallmatrix}\right)$ and $P_2(x) = \left(\begin{smallmatrix} 0 & -x^{-1} \\ 0 & 1 \end{smallmatrix}\right)$ which blow up at the exceptional point $x = 0$.

Although one can [26, Chapter 2, Section 4.2] construct a basis of holomorphic generalized eigenvectors $v_j(x)$ for $P_j(x)\cdot\mathbb{C}^n$ in a domain D, as above, they cannot in general be constructed near exceptional x_0. We believe that the correct setting, to fix these convergence issues, is to instead consider how the eigenspaces and generalized eigenspaces vary in the relevant Grassmannian. It seems this is missing in the literature, so we develop it here.

In our notation, $x=0$ will be a possibly exceptional point of $A(x)$ and we write

$$\mathbb{D}=\{x\in\mathbb{C}:|x|\le\varepsilon\},\quad \mathbb{D}^\times=\{x\in\mathbb{C}:0<|x|\le\varepsilon\},$$

to mean a small disc and punctured disc, respectively, around $x=0$, containing no other exceptional points. The purpose of this appendix is two-fold:

(1) Suppose that λ is a *semisimple eigenvalue* of $A(0)$, meaning that the generalized eigenspace for λ coincides with the eigenspace for λ (for example, this holds for diagonalizable $A(0)$). Then the $\lambda_j(x)$ with $\lambda_j(0)=\lambda$ are all holomorphic on $\mathbb{D}$. If, moreover, their derivatives $\lambda_j'(0)$ are all distinct, then the eigenprojections $P_j(x)$ are also holomorphic on $\mathbb{D}$ and there is a linearly independent collection of holomorphic eigenvectors $v_j(x)$ on $\mathbb{D}$ converging to a basis of the λ–eigenspace of $A(0)$. In particular, the $E_j(x)=\mathbb{C}v_j(x)=P_j(x)\cdot\mathbb{C}^n$ vary holomorphically in projective space $\mathbb{P}(\mathbb{C}^n)$ and their sum converges to a decomposition of the λ–eigenspace of $A(0)$ into 1–dimensional summands.

(2) More generally, suppose that $A(0)$ has a generalized sub-eigenspace $GE_i(0)\subset\ker(A(0)-\lambda)^n$, corresponding to a $k\times k$ Jordan block of $A(0)$ for λ. Then we want to show that the eigenvalues $\lambda_j(x)$ arise in families indexed by $j\in J_i=\{j_1,\dots,j_k\}$ such that $E_{j_1}(x)\oplus\cdots\oplus E_{j_k}(x)$ converges continuously to $GE_i(0)$ in the Grassmannian $\mathrm{Gr}_k(\mathbb{C}^n)$ of k–dimensional vector subspaces of $\mathbb{C}^n$ as $x\to 0$, where $E_j(x)=P_j(x)\cdot\mathbb{C}^n$. Moreover, each $E_j(x)$, for $j\in J_i$, converges in $\mathbb{P}(\mathbb{C}^n)$ to the 1–dimensional sub-eigenspace $E_i(0)\subset GE_i(0)$.

In particular, this implies that if a map

$$\mathcal{OC}_j(x)\in\mathrm{End}(\mathbb{C}^n)$$

depends continuously on x and is nonzero only in the summand $E_j(x)=P_j(x)\cdot\mathbb{C}^n$ of $\mathbb{C}^n=\oplus E_i(x)$ for $x\ne 0$, then this also holds at $x=0$: $\mathrm{Image}(\mathcal{OC}_j(0))\subset E_j(0)$.

Moreover, if $(A(x)-\lambda_j(x))\cdot\mathcal{OC}_j(x)=0$ for $x\ne 0$, then the same holds for $x=0$. That is, if the image of $\mathcal{OC}_j(x)$ consists only of eigenvectors (rather than *generalized* eigenvectors) for $x\ne 0$ then the same holds for $x=0$.

B2 The case when λ is a semisimple eigenvalue of $A(0)$

We recall from Kato [26, Chapter 2] some basic properties of matrices $A(x)$ depending holomorphically on $x \in \mathbb{C}$. We will assume that $x=0$ is an exceptional point (that is, $\chi_{A(0)}$ has a repeated root λ), since otherwise $\lambda_j(x)$ and $P_j(x)$ extend holomorphically over $x=0$ and there is nothing to prove. The eigenvalues $\lambda_1(x),\dots,\lambda_n(x)$ of $A(x)$ are branches of analytic functions on a punctured disc around $x=0$, but they are continuous even at $x=0$.

The $\lambda_j(x)$ are holomorphic on simply connected domains avoiding exceptional points, therefore they define multi-valued analytic functions near $x=0$. Analytic continuation, as x travels around 0, will permute the functions $\lambda_j(x)$ and so we can reindex them so that the permutation is described by the disjoint cycle decomposition

$$(\lambda_1(x),\dots,\lambda_p(x))\cdot(\lambda_{p+1}(x),\dots,\lambda_{p+p'}(x))\cdots,$$

where the periods of the cycles are $p, p', \dots$. The $\lambda_j(x)$ can then be expanded in Puiseux series around $x=0$ (which is a branch point of the same period as the relevant cycle); namely, for $j=1,\dots,p$,

$$\lambda_j(x)=\lambda+c_1\xi_p^j x^{1/p}+c_2\xi_p^{2j}x^{2/p}+\cdots,$$

where $\xi_p=e^{2\pi i/p}$ and $c_1,c_2,\dots\in\mathbb{C}$ are constants. We recall that the $\lambda_j(x)$ are continuous everywhere in x, and their value at 0 is some eigenvalue λ of $A(0)$.

The eigenprojections

$$P_j(x)=\frac{1}{2\pi i}\int_{\Gamma_j} R(z,x)\,dz$$

are defined by integrating the resultant $R(z,x)=(A(x)-z)^{-1}$ around a circle Γ_j surrounding $\lambda_j(x)$ but no other $\lambda_i(x)$. The $P_j(x)$ have branch point $x=0$ of period p just like the $\lambda_j(x)$, so $P_1(x)+\cdots+P_p(x)$ will be single-valued on a punctured disc around 0. However, $P_1(x)+\cdots+P_p(x)$ may have a pole at $x=0$, as was the case in Example B.1.

However, supposing $\lambda_1(x),\dots,\lambda_r(x)$ are all the eigenvalues which converge to λ at $x=0$, the *total projection*

$$P_\lambda^{\mathrm{tot}}(x)=P_1(x)+\cdots+P_r(x)$$

is single-valued and holomorphic even at $x=0$. In particular, $P_\lambda^{\mathrm{tot}}(0)$ is the eigenprojection onto the generalized eigenspace of $A(0)$ for λ. In Example B.1, $P_0^{\mathrm{tot}}(x)=P_1(x)+P_2(x)=\mathrm{id}$.

Suppose now that λ is a semisimple eigenvalue of $A(0)$, that is, the Jordan blocks of $A(0)$ for λ all have size 1×1. We work with indices j such that $\lambda_j(0) = \lambda$. Semisimplicity of λ ensures that the holomorphic function $(A(x)-\lambda)P_\lambda^{\mathrm{tot}}(x)$ vanishes at $x = 0$, so

$$\widetilde{A}(x) = \tfrac{1}{x}(A(x)-\lambda)P_\lambda^{\mathrm{tot}}(x)$$

is well-defined at $x = 0$ and is holomorphic in x. Let $\widetilde{\lambda}_j(x)$ denote the eigenvalues of $\widetilde{A}(x)$ restricted to $\mathrm{Image}(P_\lambda^{\mathrm{tot}}(x))$. The reduction process [26, Chapter 2, Section 2.3] shows that the $\widetilde{\lambda}_j(x)$ are related to the $\lambda_j(x)$ for $A(x)$ by

$$\lambda_j(x) = \lambda + \widetilde{\lambda}_j(0) \cdot x + c_j x^{1+1/p_j} + \cdots,$$

where p_j is the period of the cycle for $\widetilde{\lambda}_j(x)$. In particular, it follows that $\lambda_j(x)$ is not just continuous at $x = 0$, but also differentiable at $x = 0$ with derivative $\lambda_j'(0) = \widetilde{\lambda}_j(0)$. This breaks up the collection $\lambda_1(x), \dots, \lambda_r(x)$ into smaller collections depending on the value of the derivative $\lambda_j'(0) = \widetilde{\lambda}$, which is an eigenvalue of $\widetilde{A}(0)$. The projection $\widetilde{P}_j(x)$ for $\widetilde{A}(x)$ in fact coincides with the projection $P_j(x)$, therefore if $\lambda_1(x), \dots, \lambda_s(x)$ are the eigenvalues of $A(x)$ with derivative $\widetilde{\lambda}$ at $x = 0$, then

$$P_1(x) + \cdots + P_s(x) = \widetilde{P}_1(x) + \cdots + \widetilde{P}_s(x) = \widetilde{P}_{\widetilde{\lambda}}^{\mathrm{tot}}(x)$$

is single-valued and holomorphic even at $x = 0$ since it is the total projection for $\widetilde{A}(x)$ for $\widetilde{\lambda}$. If all $\lambda_j'(0)$ are distinct (for those j with $\lambda_j(0) = \lambda$), then $s = 1$ above, and so all $P_j(x)$ are holomorphic even at $x = 0$. So by [26, Chapter 2, Section 4.2] one can then construct holomorphic eigenvectors $v_j(x)$ for each $P_j(x)$ on a disc around $x = 0$. Thus we have obtained:

Corollary B.2 *If $A(x) \in \mathrm{End}(\mathbb{C}^n)$ depends holomorphically on $x \in \mathbb{C}$ for small x, and λ is a semisimple eigenvalue of $A(0)$, then the eigenvalues $\lambda_j(x)$ with $\lambda_j(0) = \lambda$ are differentiable at $x = 0$. If the $\lambda_j'(0)$ are all distinct, then the eigenprojections $P_j(x)$ are single-valued near $x = 0$ and holomorphic even at $x = 0$, and there is a linearly independent collection of holomorphic eigenvectors $v_j(x)$ on a disc around 0 converging to a basis of the λ–eigenspace of $A(0)$. In particular, the spaces $E_j(x) = \mathbb{C}v_j(x) = P_j(x) \cdot \mathbb{C}^n$ vary holomorphically in projective space $\mathbb{P}(\mathbb{C}^n)$ and their sum converges to a decomposition of the λ–eigenspace of $A(0)$ into 1–dimensional summands.*

B3 When λ is a nonsemisimple eigenvalue of $A(0)$

(Used in Sections 1E and 6G.) When λ is not semisimple, the classical literature on matrix perturbation theory does not appear to address the issue of convergence of eigenspaces and generalized eigenspaces at exceptional points.

Example B.3 Let

$$A(x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & x & 1 \\ 0 & 0 & 0 \end{pmatrix}.$$

The eigenvalues are $\lambda_1(x) = 0$, $\lambda_2(x) = x$, $\lambda_3(x) = 0$, and $x = 0$ is an exceptional value (the lower-right 2×2 block is the matrix from Example B.1). The eigenvectors $v_1(x) = e_1$, $v_2(x) = e_2$, $v_3(x) = e_2 - xe_3$ converge to $v_1 = e_1$, $v_2 = v_3 = e_2$. But in fact the spans of the bases for each Jordan block converge in the relevant Grassmannian: $\mathbb{C}v_1(x) = \mathbb{C}e_1 \to \mathbb{C}e_1$ in $\mathbb{P}(\mathbb{C}^n)$, and $\mathbb{C}v_1(x) + \mathbb{C}v_2(x) = \mathbb{C}e_2 + \mathbb{C}e_3 \to \mathbb{C}e_2 + \mathbb{C}e_3$ in $\mathrm{Gr}_2(\mathbb{C}^n)$. It is a "singular" linear combination $(1/x)(v_2(x) - v_3(x)) = e_3$ of $v_2(x)$ and $v_3(x)$ that will converge to the generalized eigenvector e_3 of $A(0)$.

Recall that for a linear map $A \in \mathrm{End}(\mathbb{C}^n)$,

$$\mathbb{C}^n = \oplus\, GE_\lambda(A)$$

is a direct sum of the generalized eigenspaces $GE_\lambda(A) = \ker(A-\lambda)^n$ as λ varies over the eigenvalues of A. The $GE_\lambda(A)$ can be further decomposed into vector subspaces

$$GE_\lambda(A) = J_{\lambda,1}(A) \oplus \cdots \oplus J_{\lambda,r}(A)$$

corresponding to the Jordan blocks of the Jordan canonical form of A, namely each $J_{\lambda,j}(A)$ intersects the eigenspace $E_\lambda(A) = \ker(A-\lambda)$ in a one-dimensional subspace $E_j(A) = \mathbb{C} \cdot v_j$, and there is a basis $v_j, v_{j,2}, \dots, v_{j,k}$ for the k–dimensional subspace $J_{\lambda,j}(A)$ such that

$$(A-\lambda)^m v_{j,m} = 0 \quad \text{and} \quad (A-\lambda)^{m-1} v_{j,m} \neq 0 \in E_j(A) = \mathbb{C} \cdot v_j.$$

We call such a generalized eigenvector $v_{j,m}$ an *m–gevec over v_j*.

Let $A(x) \in \mathrm{End}(\mathbb{C}^n)$ depend holomorphically on $x \in \mathbb{C}$ for small x. Recall that the eigenvalues $\lambda_j(x)$ are continuous everywhere, and are branches of multivalued analytic functions on $\mathbb{D}^\times$. The eigenprojections $P_j(x)$ are branches of analytic matrix-valued functions on $\mathbb{D}^\times$.

We will henceforth assume that $A(x)$ is *not permanently degenerate* on $\mathbb{D}^\times$, meaning the characteristic polynomial $\chi_{A(x)}$ has distinct roots for at least some $x \in \mathbb{D}^\times$. This implies that the eigenvalues $\lambda_j(x)$ are not pairwise identical analytic functions on any simply connected subset of $\mathbb{D}^\times$. Although this assumption is presumably not essential for our argument, it will always hold in our applications, and it does simplify the discussion.

Since the analytic functions $\lambda_j(x)$ are pairwise distinct, they are pointwise distinct except at finitely many points x (the exceptional points). So, making ε smaller if necessary, we can assume that the values $\lambda_j(x)$ are all distinct at each $x \in \mathbb{D}^\times$. Hence $A(x)$ is diagonalizable for $x \in \mathbb{D}^\times$. In particular, each $P_j(x)$ is then the projection onto a 1–dimensional eigenspace $E_j(x) = P_j(x) \cdot \mathbb{C}^n$ of $A(x)$, for $x \in \mathbb{D}^\times$. By [26, Chapter 2, Section 4.2] the $P_j(x)$ therefore give rise to eigenvectors $v_j(x)$ which are branches of analytic vector-valued functions on $\mathbb{D}^\times$. Therefore

$$\mathbb{D}^\times \setminus \mathbb{R}_{<0} \to \mathbb{P}(\mathbb{C}^n), \quad x \mapsto \mathbb{C} v_j(x),$$

is a holomorphic map into projective space, where we made a cut along the negative real axis to deal with the branching issue (as x travels around 0, the $P_j(x)$ will cycle according to how the $\lambda_j(x)$ cycle, and so $\mathbb{C} v_j(x) = E_j(x) = P_j(x) \cdot \mathbb{C}^n$ will cycle through different eigenspaces). We claim the above map extends continuously over $x = 0$.

Lemma B.4 *Let $v_1, v_2, \dots, v_r$ be a basis of eigenvectors of $A(0)$. Then each $v_j(x)$ converges continuously in $\mathbb{P}(\mathbb{C}^n)$, with $P_j(x) \cdot \mathbb{C}^n = \mathbb{C} v_j(x) \to \mathbb{C} v_i$ as $x \to 0$ in $\mathbb{D}^\times \setminus \mathbb{R}_{<0}$ for some unique $i \in \{1, \dots, r\}$. Conversely, each v_i arises as such a limit: $\mathbb{C} v_j(x) \to \mathbb{C} v_i$ for some j.*

Thus, for each v_i there is a nonempty collection of $j \in J_i$ for which

$$E_j(x) = P_j(x) \cdot \mathbb{C}^n = \mathbb{C} v_j(x) \to \mathbb{C} v_i = E_i(0)$$

in $\mathbb{P}(\mathbb{C}^n)$ as $x \to 0$. If $j \in J_i$ then also $k \in J_i$ for any $\lambda_k(x)$ arising in the cycle for $\lambda_j(x)$. That is, the various branches of $v_j(x)$ all converge in $\mathbb{P}(\mathbb{C}^n)$ to the same v_i.

Proof Suppose $\lambda_j(x) \to \lambda$ as $x \to 0$. Observe that the operator $A(0) - \lambda$ is bounded away from zero on $S \setminus \bigcup C_i$, where S is the unit sphere and the C_i are disjoint neighborhoods of $S^1 \cdot v_i / \|v_i\|$ (here S^1 is multiplication by phases $e^{i\theta}$). Indeed, $\bigcup C_i$ is a neighborhood of the zero set $\bigcup S^1 \cdot v_i / \|v_i\|$ of $A(0) - \lambda$ (the eigenvectors). By continuity, also $A(x) - \lambda_j(x)$ on $S \setminus \bigcup C_i$ is bounded away from zero for small x. Hence $v_j(x)$ is trapped inside some $\mathbb{C} C_i$ for all small enough x, since $A(x) - \lambda_j(x)$ vanishes on $v_j(x)$. It follows that $\mathbb{C} v_j(x)$ converges to $\mathbb{C} v_i$ in $\mathbb{P}(\mathbb{C}^n)$ as $x \to 0$.

To show that all v_i arise as limits, we first make a small perturbation of $A(x)$. Call $A_\varepsilon(x) = A(x) + \varepsilon_1 \,\mathrm{id}_1 + \cdots + \varepsilon_r \,\mathrm{id}_r$, where id_i is the projection operator for the summand $J_{\lambda,i}(A(0))$ in the above generalized eigenspace decomposition of $A(0)$. Thus $A_\varepsilon(0)$ differs from $A(0)$ by having added ε_i to each diagonal entry of the i^{th} Jordan block of $A(0)$. For small $\varepsilon_i > 0$ this has the effect of separating the repeated eigenvalue λ according to Jordan blocks. Repeating the argument above, for $A_\varepsilon(x)$ in place of $A(x)$, shows that each eigenvector $v_{j,\varepsilon}(x)$ of $A_\varepsilon(x)$ is trapped inside some $\mathbb{C}C_i$ for all small x, ε. Running the argument above, but with $A(x)$, $A_\varepsilon(x)$ in place of $A(0)$, $A(x)$, and keeping x fixed but letting $\varepsilon \to 0$, shows that $v_{j,\varepsilon}(x)$ is trapped in some $\mathbb{C}C_{k,x}$ for small ε, where the $C_{k,x}$ are neighborhoods of $S^1 \cdot v_k(x)/\|v_k(x)\|$. Combining these two facts shows that for those indices i, k, we have $\mathbb{C}v_k(x) \to \mathbb{C}v_i$ in $\mathbb{P}(\mathbb{C}^n)$ as $x \to 0$. □

Theorem B.5 *Let v_1 be an eigenvector of $A(0)$, and let $v_{1,m}$ be m–gevecs over v_1, so $v_1, v_{1,2}, \dots, v_{1,k}$ is a basis of a Jordan summand $J_1(A(0))$ as described above. By Gram–Schmidt we may assume $v_1, v_{1,2}, \dots, v_{1,k}$ are orthonormal. Suppose $v_1(x), v_2(x), \dots$ are the eigenvectors of $A(x)$ which converge to v_1 in $\mathbb{P}(\mathbb{C}^n)$ (see Lemma B.4). Then the Gram–Schmidt process applied to $v_1(x), v_2(x), \dots$ will produce orthonormal vectors $w_1(x), w_2(x), \dots$ such that $\mathbb{C}w_j(x) \to \mathbb{C}v_{1,j}$ in $\mathbb{P}(\mathbb{C}^n)$, and $\mathbb{C}w_1(x) + \dots + \mathbb{C}w_k(x) \to J_1(A(0))$ in $\mathrm{Gr}_k(\mathbb{C}^n)$.*

Proof Let $w_1(x) = v_1(x)/\|v_1(x)\| = v_1(x)$, and recall $w_2(x) = W_2(x)/\|W_2(x)\|$, where

$$W_2(x) = v_2(x) - \langle v_2(x), w_1(x)\rangle w_1(x).$$

Then

$$(A(x) - \lambda_2(x))w_2(x) = \frac{1}{\|W_2(x)\|}(-\langle v_2(x), w_1(x)\rangle(\lambda_1(x) - \lambda_2(x)))\, w_1(x).$$

Observe that $\langle v_2(x), w_1(x)\rangle$ is nonzero for small x since $\mathbb{C}v_2(x) \to \mathbb{C}v_1$, $\mathbb{C}w_1(x) = \mathbb{C}v_1(x) \to \mathbb{C}v_1$ and $\langle v_1, v_1\rangle \neq 0$. Since the unit sphere $S \subset \mathbb{C}^n$ is compact, a subsequence $w_2(\delta_n)$ will converge as $\delta_n \to 0$. The limit w_2 will be a unit vector orthogonal to v_1 since $w_2(\delta_n)$ is orthogonal to $w_1(\delta_n)$ and $\mathbb{C}w_1(\delta_n) \to \mathbb{C}v_1$, and will satisfy $(A(0) - \lambda)w_2 \in \mathbb{C}v_1$ since $(A(x) - \lambda_2(x))w_2(x) \in \mathbb{C}w_1(x) \to \mathbb{C}v_1$. So w_2 is either an eigenvector of $A(0)$ orthogonal to v_1 or a 2–gevec over v_1 for $A(0)$ orthogonal to v_1. The first case can be ruled out if the eigenvectors of $A(0)$ are never orthogonal to each other (which is a generic condition). Even if this is not the case for the standard inner product on $\mathbb{C}^n$, we can always perturb the inner product so that this condition holds. It follows a posteriori that also for the standard inner product the limit w_2 was not an eigenvector.

Therefore w_2 is a 2–gevec over v_1 orthogonal to v_1. But there is only a 1–dimensional space of 2–gevec over v_1 orthogonal to v_1, namely $\mathbb{C}v_{1,2}$, so $\mathbb{C}w_2 = \mathbb{C}v_{1,2}$ (using the fact that the $v_{1,j}$ are orthonormal). By uniqueness, it follows that $\mathbb{C}w_2(x) \to \mathbb{C}v_{1,2}$ in $\mathbb{P}(\mathbb{C}^n)$: indeed, if by contradiction $\mathbb{C}w_2(\varepsilon_n) \to \mathbb{C}v_{1,2}$ were to fail for a sequence $\varepsilon_n \to 0$, then no subsequence of $w_2(\varepsilon_n)$ would be allowed to converge, contradicting compactness of S.

The argument now continues by induction. We show one more step of the induction, since the general inductive step is then obvious. The next step of Gram–Schmidt is $W_3(x) = v_3(x) - \langle v_3(x), w_2(x)\rangle w_2(x) - \langle v_3(x), w_1(x)\rangle w_1(x)$, and $w_3(x)$ is the normalization of $W_3(x)$. Then $(A(x)-\lambda_3(x))(A(x)-\lambda_2(x))w_3(x)$ will again be a nonzero multiple of $w_1(x)$ for small x. A subsequence $w_3(\delta_n)$ will converge to some w_3 which is orthogonal to $\mathbb{C}v_1$ and $\mathbb{C}v_{1,2}$ and satisfies $(A(0)-\lambda)^2 w_3 \in \mathbb{C}v_1$. Now w_3 cannot be an eigenvector since it is orthogonal to v_1 (arguing as before). So w_3 is either a 2–gevec or a 3–gevec over v_1. We need to exclude the first case. But in the first case, $\mathbb{C}w_3 = \mathbb{C}v_{1,2}$ (since w_3 is orthogonal to v_1) contradicts that w_3 is orthogonal to $v_{1,2}$. Therefore w_3 is a 3–gevec over v_1 orthogonal to $\mathbb{C}v_1 + \mathbb{C}v_{1,2}$. But the space of such 3–gevecs is 1–dimensional and equal to $\mathbb{C}v_{1,3}$. Thus $\mathbb{C}w_3 = \mathbb{C}v_{1,3}$, and so $\mathbb{C}w_3(x) \to \mathbb{C}v_{1,3}$ in $\mathbb{P}(\mathbb{C}^n)$ (again, arguing by contradiction: if a subsequence $w_3(\varepsilon_n)$ were to make $\mathbb{C}w_3(\varepsilon_n) \to \mathbb{C}v_{1,3}$ fail, then $w_3(\varepsilon_n)$ would have no convergent subsequence in S).

It follows that in $\mathrm{Gr}_k(\mathbb{C}^n)$,

$$\begin{aligned}\mathbb{C}v_1(x) + \cdots + \mathbb{C}v_k(x) = \mathbb{C}w_1(x) + \cdots + \mathbb{C}w_k(x)&\\ \to \mathbb{C}v_1 + \mathbb{C}v_{1,2} + \cdots + \mathbb{C}v_{1,k} = J_1(A(0)).&\end{aligned}$$

In particular, it follows a posteriori that $v_1(x), v_2(x), \ldots$ involved exactly k vectors. $\square$

Appendix C: The extended maximum principle

C1 Symplectic manifolds conical at infinity

Let (M,ω) be a symplectic manifold *conical at infinity*. As in [35], this means ω is allowed to be nonexact, but outside of a bounded domain $M_0 \subset M$ there is a symplectomorphism

$$(M \setminus M_0, \omega|_{M\setminus M_0}) \cong (\Sigma \times [1,\infty), d(R\alpha)),$$

where (Σ,α) is a contact manifold, and R is the coordinate on $[1,\infty)$.

We call $\Sigma \times [1,\infty)$ the *collar* of M. On the collar, $\omega = d\theta$ is exact with $\theta = R\alpha$. We denote by Y the Reeb vector field on Σ (so $\alpha(Y) = 1$ and $d\alpha(Y,\cdot) = 0$), and by $Z = R\partial_R$ the Liouville vector field on the collar (so $\omega(Z,\cdot) = \theta$).

Recall that an ω–compatible almost complex structure J is called of *contact type* if, for large R, it satisfies $JZ = Y$. This is equivalent to the condition

$$\theta \circ J = dR.$$

Remark C.1 The contact type condition $JZ = Y$ can be slightly generalized to $JZ = c(R)Y$, or equivalently $\theta \circ J = c(R)dR$, where $c\colon \mathbb{R} \to \mathbb{R}$ satisfies $c > 0$ and $c' \geq 0$.

C2 The class of Hamiltonians

Usually, to make Floer theory work in this context one would require the Hamiltonians $H\colon M \to \mathbb{R}$ to depend only on R for large R, ie $H = h(R)$. Then the coordinate $\rho = R \circ u\colon S \to \mathbb{R}$ will satisfy a maximum principle for Floer solutions $u\colon S \to M$. The maximum principle is originally due to Viterbo [41] for Floer trajectories (ie when S is a cylinder), and it holds also when S is a punctured Riemann surface (due to Seidel [40], see the final appendix of [34] for a detailed description). By a *Floer solution* $u(s,t)$ we mean a solution of

$$(du - X_H \otimes \beta)^{0,1} = 0,$$

where X_H is the Hamiltonian vector field (so $\omega(\cdot, X_H) = dH$), the 1–form $\beta = \beta_s\, ds + \beta_t\, dt$ on S satisfies $d\beta \leq 0$, and the $(0,1)$ part is taken with respect to J (see [34, Section D.1] for details). The Hamiltonian orbits that u converges to at the punctures of S (near which β is a constant positive multiple of dt) are called the *asymptotic conditions*.

The condition that the Hamiltonian depend only on R on the collar is quite restrictive: it implies that X_H is a multiple of the Reeb vector field, $X_H = h'(R)Y$. This is too restrictive in the case of noncompact toric varieties, as one would like to allow the Hamiltonians arising from the natural S^1–actions around the various toric divisors, mentioned in Section 1.

The aim of this appendix is to prove the following maximum principle.

Theorem C.2 (extended maximum principle) *Let $H\colon M \to \mathbb{R}$ have the form*

$$H(y,R) = f(y)R$$

for large R, where $(y,R) \in \Sigma \times (1,\infty)$ are the collar coordinates, and $f\colon \Sigma \to \mathbb{R}$ is such that

- *f is invariant under the Reeb flow (that is, $df(Y) = 0$ for the Reeb vector field Y),*
- *$f \geq 0$.* [this condition can be omitted if $d\beta = 0$]

Let J be an ω–compatible almost complex structure of contact type at infinity.

Then the R–coordinate of any Floer solution u is bounded a priori in terms of the R–coordinates of the asymptotic conditions of u.

C3 Observations about functions invariant under the Reeb flow

The condition that $f\colon \Sigma \to \mathbb{R}$ is invariant under the Reeb flow is equivalent to

$$df(Y) = 0,$$

where Y is the Reeb vector field. This is equivalent to the condition

$$dR(X_f) = 0 \tag{38}$$

since $Y = X_R$ and $dR(X_f) = \omega(X_f, X_R) = -df(Y) = 0$.

For $h = h(R)$ depending only on the radial coordinate, $X_h = h'(R)Y$, therefore

$$df(X_h) = 0. \tag{39}$$

The condition that f only depends on $y \in \Sigma$ and not on R on the collar $\Sigma \times (1, \infty)$ implies

$$\theta(X_f) = 0 \tag{40}$$

since $\theta(X_f) = \omega(Z, X_f) = df(Z) = R\partial_R f = 0$, where $Z = R\partial_R$ is the Liouville vector field.

Lemma C.3 *For $H = f(y)R$ on $(\Sigma \times (1, \infty), d(R\alpha))$, with $f\colon \Sigma \to \mathbb{R}$ invariant under the Reeb flow, the flow g_t of H has the properties*

1. *g_t preserves α (the contact form on Σ);*
2. *g_t preserves the R–coordinate;*
3. *g_t preserves f.*

This also holds if f is allowed to be time-dependent.

Proof By (40), $\alpha(X_f) = 0$. Since $\alpha(Y) = 1$ and $X = X_H = fY + RX_f$, we deduce

$$\alpha(X) = f.$$

By comparing $f\,dR + R\,df = dH = d(R\alpha)(\,\cdot\,, X)$ with $(dR \wedge \alpha + R\,d\alpha)(\,\cdot\,, X) = f\,dR + R\,d\alpha(\cdot, X)$, using $dR(X_f) = 0$ from (38), we obtain

$$d\alpha(\,\cdot\,, X) = df.$$

The Lie derivative of α along X can be computed by Cartan's formula,

$$\mathcal{L}_X\alpha = di_X\alpha + i_X d\alpha = df - df = 0.$$

Since $g_t^* \mathcal{L}_X \alpha = \partial_s\big|_{s=t} g_s^* \alpha$, it follows that $g_t^* \alpha$ is constantly α (using $g_0 = \mathrm{id}$). This proves (1). Claim (2) follows from $dR(X) = 0$. Claim (3) follows from $df(X) = 0$ (using $df(Y) = 0$ and $df(X_f) = \omega(X_f, X_f) = 0$).

For time-dependent f, the derivative $\mathcal{L}_X^t \alpha$ depends on the time-parameter t: it is defined by $\mathcal{L}_X^t \alpha = \partial_s\big|_{s=t} g_{s,t}^* \alpha$, where $\partial_s g_{s,t} = X_{f_s R}$ with initial point $g_{t,t} = \mathrm{id}$. Then Cartan's formula, and hence the above argument, still holds. $\square$

Lemma C.4 *If a diffeomorphism g_t preserves α and R for large R, then it also preserves θ, $d\theta$, Y, $Z = R\partial_R$, and it preserves the splitting $TM = \ker\alpha \oplus \mathbb{R}Y \oplus \mathbb{R}Z$ (for large R).*

Proof It preserves $\theta = R\alpha$, since g_t preserves R and α. So $g_t^* d\theta = dg_t^*\theta = d\theta$.

Since $\ker\alpha = \ker g_t^*\alpha$, we get $dg_t(\ker\alpha) \subset \ker\alpha$, and this has to be an equality by dimensions (dg_t is invertible). Since $Y \in T\Sigma$ is the unique vector field determined by $\alpha(Y) = 1$, $d\alpha(Y, T\Sigma) = 0$, to prove that $dg_t \cdot Y = Y$ we just need to check these conditions for $dg_t \cdot Y$. Now $1 = \alpha(Y) = (g_t^*\alpha)(Y) = \alpha(dg_t Y)$ gives the first condition. The second follows from

$$0 = d\alpha(Y, T\Sigma) = d(g_t^*\alpha)(Y, T\Sigma) = g_t^* d\alpha(Y, T\Sigma) = d\alpha(dg_t Y, T\Sigma),$$

using that $dg_t(T\Sigma) = T\Sigma$ since g_t preserves R.

Since $d\theta(Z, \cdot) = \theta$, we obtain

$$\begin{aligned} d\theta(Z, \cdot) = \theta = g_t^*\theta = \theta(dg_t \cdot) = d\theta(Z, dg_t \cdot) &= (g_t^* d\theta)(dg_t^{-1} Z, \cdot) \\ &= d\theta(dg_t^{-1} Z, \cdot). \end{aligned}$$

Therefore $dg_t Z = Z$. The claim follows. $\square$

Corollary C.5 *The Hamiltonian flow g_t for H as in Theorem C.2 satisfies $dg_t = \mathrm{id}$ on $\mathrm{span}(Z, Y) \subset T(\Sigma \times (1, \infty))$ for large R.*

In particular, if J is of contact type at infinity, then so is $g^* J = dg_t^{-1} \circ J \circ dg_t$.

Proof The first part follows by the previous two lemmas. For the second part, recall that the contact condition is $JZ = Y$ for large R. So $g_t^* J$ is of contact type by the first part. $\square$

C4 The "no escape" lemma

Lemma C.6 *The Hamiltonians $H\colon M \to \mathbb{R}$ of the form $H(y,R) = f(y)R$ for large R, where $(y,R) \in \Sigma \times (1,\infty)$, and with $f\colon \Sigma \to \mathbb{R}$ invariant under the Reeb flow, are precisely the class of Hamiltonians whose Hamiltonian vector field X satisfies*

(2) $dR(X) = 0$;

(3) $H = \theta(X)$.

Proof Recall $\theta(X) = \omega(Z,X) = dH(Z) = R\partial_R H$, where $Z = R\partial_R$ is the Liouville vector field. Thus $\theta(X) = H$ corresponds to $\partial_R \log H = \partial_R \log R$. Integrating yields $H = f(y)R$, for some $f\colon \Sigma \to \mathbb{R}$. Since

$$X = f(y)X_R + RX_f = f(y)Y + RX_f,$$

the condition $dR(X) = 0$ is equivalent to $dR(X_f) = 0$, or equivalently $df(Y) = 0$ by (38). □

Theorem C.2 now follows by the following "no escape lemma" applied to u restricted to $u^{-1}(V)$, taking R_0 greater than or equal to the maximal value that R attains on the asymptotics of u (and we pick R_0 generically so that ∂V intersects u transversely).

Lemma C.7 (no escape lemma) *Let $H\colon M \to \mathbb{R}$ be any Hamiltonian. Let $(V, d\theta)$ be the region $R \geq R_0$ in $\Sigma \times (1,\infty)$, so $\partial V = \{R = R_0\}$. Assume that*

(1) *J is of contact type along ∂V;*

(2) *$dR(X) = 0$ on ∂V;*

(3) *$H = \theta(X)$ on ∂V;*

(4) *$H \geq 0$ on V,* [this condition can be omitted if $d\beta = 0$]

where $X = X_H$. Let S be a compact Riemann surface with boundary and let β be a 1–form on S with $d\beta \leq 0$. Then any solution $u\colon S \to V$ of $(du - X \otimes \beta)^{0,1} = 0$ with $u(\partial S) \subset \partial V$ is either constant or maps entirely into ∂V.

Proof We run the same argument as in [34, Section D.5]. The energy of u is:

$$\begin{aligned} E(u) &= \frac{1}{2}\int_S \|du - X \otimes \beta\|^2 \mathrm{vol}_S \\ &= \int_S u^* d\theta - u^*(dH) \wedge \beta \\ &= \int_S d(u^*\theta - u^* H\beta) + u^* H\, d\beta \qquad \text{expanding } d(u^* H\beta), \end{aligned}$$

$$
\begin{aligned}
&\leq \int_S d(u^*\theta - u^* H\beta) && \text{as when } d\beta \neq 0, \text{we use } H \geq 0,\ d\beta \leq 0,\\
&= \int_{\partial S} u^*\theta - u^* H\beta && \text{by Stokes' theorem,}\\
&= \int_{\partial S} u^*\theta - u^*\theta(X)\beta && \text{using } H = \theta(X) \text{ on } \partial V,\\
&= \int_{\partial S} \theta(du - X\otimes\beta) && \\
&= \int_{\partial S} -\theta J(du - X\otimes\beta) j && \text{since } (du - X\otimes\beta)^{0,1} = 0,\\
&= \int_{\partial S} -dR(du - X\otimes\beta) j && \text{since } J \text{ is of contact type on } \partial V,\\
&= \int_{\partial S} -dR(du) j && \text{since } dR(X) = 0 \text{ on } \partial V,\\
&\leq 0 && \text{since } \partial V \text{ minimizes } R \text{ on } V.
\end{aligned}
$$

For the last line: if $\hat{n}$ is the outward normal along $\partial S \subset S$, then $j\hat{n}$ is the oriented tangent direction along ∂S, so $-dR(du)j(j\hat{n}) = d(R\circ u)\cdot\hat{n} \leq 0$. Indeed, if we assumed that J was of contact type on all of V, then the above argument is just Green's formula [14, Appendix C.2] for S,

$$\int_{\partial S} u^*\theta - u^*\theta(X)\beta = \int_S \Delta(R\circ u)\, ds\wedge dt = \int_{\partial S} \frac{\partial(R\circ u)}{\partial\hat{n}}\, dS \leq 0.$$

Since $E(u) \geq 0$ by definition, the above inequality $E(u) \leq 0$ forces $E(u) = 0$, and so $du = X\otimes\beta$. But $X \in \ker dR = T\partial V$, so the image of du lies in $T\partial V$, and so u lies in ∂V. □

Remark C.8 (when $d\beta = 0$) Condition (4) can be dropped when $d\beta = 0$. This applies to cylinders ($\mathbb{R}\times S^1, \beta = dt$), continuation cylinders (see Theorem C.11), and to TQFT operations [34] where the sum of the weights at the inputs equals the sum at the outputs, eg for products $HF^*(H)\otimes HF^*(H) \to HF^*(2H)$.

Lemma C.9 (generalizing contact type) *In Lemma C.7 we can replace the contact type condition by $\theta\circ J = c(R)dR$ for any positive function $c\colon \mathbb{R}\to(0,\infty)$. Indeed, since we only need the condition along ∂V, the condition $\theta\circ J = c\cdot dR$ for a positive constant $c = c(R_0)$ suffices.*

Proof This only affects the integrand in $E(u)$ at the end: instead of

$$-dR(du - X\otimes\beta)j,$$

we get

$$-c(R_0) \cdot dR(du - X \otimes \beta) j,$$

but $c(R_0) > 0$, so it does not affect the sign. $\square$

We remark that $J = J_z$ can also be domain-dependent; this does not affect the above proofs as long as J_z is of contact type for each z.

Remark C.10 (time-dependent Hamiltonians) To achieve nondegeneracy of the 1–periodic Hamiltonian orbits (a requirement in the proof of transversality for Floer moduli spaces), one must in fact always make a time-dependent perturbation of the Hamiltonian $H = H_t$ near the ends of the surface S [34, Appendix A.5 and A.6]. Therefore $H = H_z$ will also depend on s near the ends, where $z = s + it$. When we expand $d(u^* H\beta)$ in the proof of Lemma C.7, we obtain new terms $u^*(\partial_t H)\, dt \wedge \beta$ and $u^*(\partial_s H)\, ds \wedge \beta$.

In the case of Floer trajectories, one wants $H = H_t$ and $\beta = dt$ on the whole cylinder $\mathbb{R} \times S^1$, so both those new terms vanish.

For more general Floer solutions on a Riemann surface S, we only need the z–dependence near the ends [34, Appendix A.5] where we can ensure that $\beta = C\, dt$ for a constant $C > 0$. Thus, to ensure that we can drop the new term $u^* \partial_s H\, ds \wedge \beta$, we require

(5) $\partial_s H \leq 0$.

This condition can always be achieved. Indeed, on a negative end, $\mathbb{R} \times (-\infty, 0)$, take H_t to be a very small time-dependent perturbation of $H + \varepsilon R$ for small $\varepsilon > 0$, then we can find a homotopy $H_{s,t}$ from H_t to H with $\partial_s H_{s,t} \leq 0$. On a positive end, one considers $H - \varepsilon R$.

Theorem C.11 (continuation maps) *Suppose the data $(H, J) = (H_z, J_z)$ now depends on the domain coordinates $z = s + it$ of the cylinder $(\mathbb{R} \times S^1, \beta = dt)$, such that (H_z, J_z) becomes independent of s for large $|s|$. We assume that for large R,*

(1) *J is of contact type;*

(2) *$dR(X) = 0$;*

(3) *$H = \theta(X)$;*

(5) *$\partial_s H_z \leq 0$.*

Then Theorem C.2 still applies (via Lemma C.7).

Proof Since $\beta = dt$, (4) is not necessary but we need (5) by Remark C.10. $\square$

C5 Symplectic cohomology using Hamiltonians $H = f(y)R$

Recall that
$$SH^*(M) = \varinjlim HF^*(H, J),$$
taking the direct limit over continuation maps $HF^*(H, J) \to HF^*(\widetilde{H}, \widetilde{J})$ for Hamiltonians linear in R at infinity, as the slope of H is increased, where J and $\widetilde{J}$ are of contact type at infinity. These continuation maps are defined since for large R there is a homotopy H_s from $\widetilde{H}$ to H with $\partial_s H_s \le 0$ (condition (5) in Theorem C.11). We now show we can enlarge the class to H as in Theorem C.2.

Theorem C.12 *$SH^*(M) = \varinjlim HF^*(H_k, J_k)$ for any sequence of almost complex structures J_k of contact type at infinity, and any sequence of Hamiltonians H_k on M such that*

(1) *$H_k = f_k(y)R +$ constant for large R;*

(2) *the $f_k\colon \Sigma \to \mathbb{R}$ are invariant under the Reeb flow;*

(3) $\max f_k \le \min f_{k+1}$ *and* $\min f_k \to \infty$.

Proof Adding a constant to H plays no role in Floer theory, so we can ignore the constant in (1). Condition (5) in Theorem C.11 ensures that continuation maps can be defined for the larger class of Hamiltonians of the form described in Theorem C.2. The slope of $f(y)R$ is bounded by $\max_{y\in\Sigma} f(y)$, and a homotopy H_s from $\widetilde{H} = \widetilde{f}(y)R$ to $H = f(y)R$ with $\partial_s H_s \le 0$ exists provided that
$$\min_{y\in\Sigma} \widetilde{f}(y) \ge \max_{y\in\Sigma} f(y).$$
Any sequence $H_k = f_k(y)R$ with $\max f_k \le \min f_{k+1}$ and $\min f_k \to \infty$ will be cofinal, and so enlarging the class of Hamiltonians as above does not affect the resulting group $SH^*(M)$. $\square$

Corollary C.13 *Let $H_j\colon M \to \mathbb{R}$ be Hamiltonians of the form $H_j = m_j R$ for large R, whose slopes $m_j \to \infty$, and let J be of contact type at infinity. Suppose g_t is the flow for a Hamiltonian of the form $K = f(y)R$ for large R, with $f\colon \Sigma \to \mathbb{R}$ invariant under the Reeb flow. Then the pairs $(g^* H_j, g^* J)$ compute symplectic cohomology,*
$$SH^*(M) = \lim_{j\to\infty} HF^*(g^* H_j, g^* J).$$

Proof For large R,
$$g^* H = H \circ g_t - K \circ g_t = m_j R - f(g_t y)R = (m_j - f(y))R,$$

using the fact that g_t preserves R and f (Lemma C.3). Moreover, g^*J is of contact type by Corollary C.5. The claim follows by Theorem C.12. $\square$

As explained in Theorem 2.6, when there is a loop of Hamiltonian diffeomorphisms whose Hamiltonian has strictly positive slope at infinity, then $SH^*(M)$ is a quotient of $QH^*(M)$.

Theorem C.14 (by Ritter [35] and Theorem 2.6) *Given $g\colon S^1 \to \mathrm{Ham}(M,\omega)$ generated by a Hamiltonian of the form $K(y,R) = f(y)R$ for large R, with $f\colon \Sigma \to \mathbb{R}$ invariant under the Reeb flow, and satisfying* $\min f > 0$, *then the canonical map $c^*\colon QH^*(M) \to SH^*(M)$ induces an isomorphism of Λ–algebras*

$$SH^*(M) \cong QH^*(M)/(\text{generalized } 0\text{–eigenspace of } r_{\tilde{g}}) \cong QH^*(M)_{r_{\tilde{g}}(1)},$$

where the last expression is localization at $r_{\tilde{g}}(1)$ (Section 4B), and c^* is the localization map.

Here $SH^*(M)$ is restricted to contractible loops, see Section 2B. When M is simply connected, for instance a toric manifold M, this is all of $SH^*(M)$.

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