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Circle Packings and Polyhedral Surfaces*

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Abstract. Given a triangulated surface, a euclidean or hyperbolic polyhedral surface can be constructed by assigning radii to the vertices of the triangulation. We develop necessary and sufficient conditions for the existence of such a polyhedral surface having specified characteristics.

0. Introduction and Definitions

Suppose S is a compact bordered or closed surface with a triangulation T. A *radius function* R assigns a positive extended real number $R(v)$ to each vertex v of T. Thurston [23] described the following method for using a finite radius function to construct a euclidean polyhedral surface. For each face in T, consider three mutually tangent circles in the plane whose radii are $R(v_1)$, $R(v_2)$, and $R(v_3)$, where v_1, v_2, and v_3 are the vertices of the face. A *triangle of centers* is formed by connecting the centers of the circles by lines to produce a triangle. The triangles formed in this way are then pasted together along corresponding edges in a natural way to form a polyhedral surface $E(R)$ such that there is a homeomorphism from S to $E(R)$ which maps each face in S onto the corresponding triangle in $E(R)$. T can then be regarded as a triangulation either of S or of $E(R)$, and each face in $E(R)$ is a euclidean triangle.

The canonical metric on $E(R)$ has constant Gauss curvature zero except for possible isolated cone-type singularities at the interior vertices, and this metric can be transferred back to S via the homeomorphism. The *euclidean curvature* induced by R at any interior vertex v of T is denoted $\kappa_R^{\text{euc}}(v)$ and is defined as 2π

* The results in this paper are included in the author's doctoral dissertation [12].

minus the sum of the angles at v in each face in $E(R)$ that has v as a vertex. The metric on $E(R)$ has no singularity at an interior vertex v where $\kappa_R^{\text{euc}}(v) = 0$.

Any radius function R can also be used to determine a hyperbolic polyhedral surface $H(R)$ by following the same procedure using circles and geodesics in the hyperbolic plane to produce hyperbolic triangles. In this case the radius function need not be finite; a circle of infinite hyperbolic radius is interpreted as a horocycle under the disk model of the hyperbolic plane. The *hyperbolic curvature* induced by R at any interior vertex v of T is denoted $\kappa_R^{\text{hyp}}(v)$ and is again defined to be 2π minus the sum of the angles at v in each face of $H(R)$ that has v as a vertex. The metric on $H(R)$ has Gauss curvature -1 everywhere except for isolated singularities at the interior vertices v where $\kappa_R^{\text{hyp}}(v) \neq 0$. See [17] for further details.

Thurston [23] proved the following results, which we refer to as Thurston's theorems. Given a closed surface with triangulation T, if the Euler characteristic χ is negative, then there is a unique radius function R on T such that $\kappa_R^{\text{hyp}} \equiv 0$. Similarly, if $\chi = 0$, then there is a finite radius function R such that $\kappa_R^{\text{euc}} \equiv 0$; in this case R is unique up to constant multiples. These theorems show that for any triangulation of a closed surface S of Euler characteristic $\chi \leq 0$, there is a metric of constant Gauss curvature on S such that the triangulation can be represented using geodesics for edges [10]. This result is also true for a sphere as a consequence of the Andreev–Thurston theorem; see Corollary 2.4.

For a surface with border, it seems natural to ask the following question, which we call the *nonsingular hyperbolic* (or euclidean) *boundary-value problem*. Assign a radius $r(v)$ to each vertex in the boundary of the surface. Is there a radius function R that agrees with the prescribed radii on boundary vertices, and which induces $\kappa_R^{\text{hyp}} \equiv 0$ (or $\kappa_R^{\text{euc}} \equiv 0$) on the interior vertices? Carter and Rodin [9] used a Perron family technique to prove that, in the euclidean case, there is always such a radius function on an orientable surface of genus 0 or 1 with a border. Beardon and Stephenson [4] showed that the hyperbolic problem always has a solution R if the surface is simply connected with a border. Minda and Rodin [17] and Doyle [11] proved that the hyperbolic problem is solvable for any orientable surface with a border, and they showed how this case can lead to the Poincaré metric on the surface.

In this paper, we consider the following problem, which we call the hyperbolic or euclidean *boundary-value curvature problem*. This is a slightly generalized version of a problem posed in [17]. Let T be a triangulation of a compact bordered or closed surface S. Let $\mathscr{B}$ be the set of border vertices, if any, plus any selected interior vertices (called *punctured vertices*). We refer to $\mathscr{B}$ as the set of *boundary vertices*. Let $r\colon \mathscr{B} \to (0, \infty]$ be a function that prescribes radii on the boundary vertices. Let $\mathscr{I}$ be the set of nonboundary vertices. Let $\kappa\colon \mathscr{I} \to (-\infty, 2\pi]$ be a function that prescribes curvatures to be concentrated at the nonboundary vertices. Is there a radius function R which agrees with the prescribed radius $r(v)$ on each boundary vertex, and which induces the prescribed hyperbolic curvature $\kappa(v)$ at each vertex in $\mathscr{I}$? Such a radius function is called a *solution* to the problem.

For any subset $\mathscr{V}$ of $\mathscr{I}$, let $F(\mathscr{V})$ denote the number of faces having at least one vertex in $\mathscr{V}$, and let $F'(\mathscr{V})$ denote the number of faces having all three vertices in $\mathscr{V}$. In Theorem 1 we show that the hyperbolic boundary-value curvature problem has a unique solution if and only if, for any nonempty subset $\mathscr{V}$ of $\mathscr{I}$, the sum of the curvatures assigned to vertices in $\mathscr{V}$ is greater than $2\pi|\mathscr{V}| - \pi F(\mathscr{V})$.

Theorem 2 examines the nonsingular hyperbolic boundary-value problem, which is obtained by simply prescribing $\kappa \equiv 0$. We find that this problem has a solution if and only if $2|\mathscr{I}| < F(\mathscr{I})$. This condition is satisfied if and only if the surface has a border or has an Euler characteristic that is less than the number of punctured vertices.

In Theorem 3 we examine the euclidean boundary-value curvature problem. Provided $\mathscr{B}$ is nonempty, κ is strictly positive, and r is finite, this problem is solvable if and only if

$$2\pi|\mathscr{V}| - \pi F(\mathscr{V}) < \sum_{v \in \mathscr{V}} \kappa(v) < 2\pi|\mathscr{V}| - \pi F'(\mathscr{V})$$

for each subset $\mathscr{V}$ of $\mathscr{I}$. Our results can be applied to prove Thurston's theorems; see Corollaries 2.2 and 3.1.

We find it useful to adapt Colin de Verdière's concept of a coherent system of angles [10] to the present situation. A *system of angles* on T assigns an angle value $\theta(v, f) \in [0, \pi)$ to each ordered pair (v, f) where v is a vertex of the face f. We say $\theta(v, f)$ is the angle at v in f. Such a system is *hyperbolic coherent* for κ if

(i) the sum of the three angles in each face in T is strictly less than π, and
(ii) 2π minus the sum of the angles at any nonboundary vertex v is equal to the assigned curvature $\kappa(v)$.

Euclidean coherent systems of angles are defined similarly; in this case the assigned values are strictly positive and the sum of the angles in each face is exactly π. Colin de Verdière showed that a nonsingular boundary-value problem is solvable iff there exists a coherent system of angles; we shall find that the same is true for our more general case.

We prove our theorems by means of convergent algorithms which can be used for finding solutions to the boundary-value curvature problems. These algorithms use a hyperbolic or euclidean *relaxation operator* $\mathscr{R}$ which depends on the functions r and κ. Given a radius function G, $\mathscr{R}(G)$ is the radius function on T defined as follows. For each vertex $v \in \mathscr{B}$, $\mathscr{R}(G)(v) = r(v)$. For each vertex $v \in \mathscr{I}$, $\mathscr{R}(G)(v)$ is the unique radius, if any, such that the curvature induced at v would equal $\kappa(v)$ if the other radii remained unchanged. In other words, if v_0 is a nonboundary vertex, then the radius function

$$G'(v) = \begin{cases} G(v) & \text{if} \quad v \neq v_0, \\ \mathscr{R}(G)(v_0) & \text{if} \quad v = v_0 \end{cases}$$

induces the prescribed curvature at v_0. It is easily seen by Lemma 0.1, which follows, that the hyperbolic relaxation operator $\mathscr{R}$ is defined on all radius functions as long as, for each nonboundary vertex v in T, $2\pi - \pi F(\{v\}) < \kappa(v) \leq 2\pi$; in the euclidean case, strict inequality must hold.

We will discover that, if there is a solution to the boundary-value curvature problem, then $\mathscr{R}$ is defined and $\mathscr{R}^k(G)$ converges pointwise to the solution as $k \to \infty$, provided that, in the euclidean case, there is at least one boundary vertex. Conversely, if $\mathscr{R}^k(G)$ converges to a radius function L, then L solves the problem, provided that L is finite in the euclidean case. $\mathscr{R}$ is very similar to the operators defined in [9] and originally suggested by Thurston [23].

We make frequent use of the following fairly obvious results; the reader may refer to [4], [12], and [23] for proofs. Except as noted, these results apply to both the hyperbolic and euclidean cases.

Lemma 0.1. *Consider the triangle of centers of circles of radii a, b, and r, where a and b are fixed. Let $\alpha(r)$, $\beta(r)$, and $\rho(r)$ be the angles that correspond to a, b, and r, respectively. Then*

(i) *$\rho(r)$ is a strictly decreasing continuous function with $\lim_{r \to 0^+} \rho(r) = \pi$ and $\lim_{r \to \infty} \rho(r) = 0$.*
(ii) *$\alpha(r)$ and $\beta(r)$ are strictly increasing continuous functions with $\lim_{r \to 0^+} \alpha(r) = \lim_{r \to 0^+} \beta(r) = 0$. (Exception: In the hyperbolic case, if $a = \infty$, then $\alpha \equiv 0$; if $b = \infty$, then $\beta \equiv 0$.)*

Lemma 0.2. *$\mathscr{R}$ is a continuous function that preserves inequalities in the sense that if A and B are two radius functions such that $A \le B$ (that is, $A(v) \le B(v)$ for each vertex v in T), then $\mathscr{R}(A) \le \mathscr{R}(B)$.*

Lemma 0.3. *Let A be a radius function, and assume that the relaxation operator $\mathscr{R}$ is defined. The following are equivalent, and the corresponding statements with the inequalities reversed are also equivalent:*

(a) *$\mathscr{R}(A) \le A$.*
(b) *For each integer $k \ge 0$, $\mathscr{R}^{k+1}(A) \le \mathscr{R}^k(A)$.*
(c) *For each nonboundary vertex v, the curvature induced by A concentrated at v is greater than or equal to $\kappa(v)$.*
(d) *For each integer $k \ge 0$ and for each nonboundary vertex v, the curvature induced by $\mathscr{R}^k(A)$ concentrated at v is greater than or equal to $\kappa(v)$.*

Lemma 0.4. *Let A be a radius function on T. Suppose that the sequence $\{\mathscr{R}^k(A)\}$ converges pointwise to a radius function L and that, in the euclidean case, L is finite. Then L is a solution to the problem.*

1. The Hyperbolic Boundary-Value Curvature Problem

Theorem 1. *Consider any hyperbolic boundary-value curvature problem. Let $\mathscr{R}$ be the hyperbolic relaxation operator for r and κ. The following are equivalent:*

(a) *The problem has a unique solution.*
(b) *There exists a hyperbolic coherent system of angles for κ on T.*

(c) *Given a nonempty $\mathscr{V}$ of $\mathscr{I}$, the prescribed curvatures satisfy*

$$2\pi|\mathscr{V}| - \pi F(\mathscr{V}) < \sum_{v \in \mathscr{V}} \kappa(v).$$

(d) *There exists a radius function G such that $\mathscr{R}^k(G)$ converges pointwise to a radius function as $k \to \infty$.*

(e) *Given any radius function G, $\mathscr{R}^k(G)$ converges pointwise to the solution to the problem as $k \to \infty$.*

The following lemmas will be useful in proving Theorem 1. Lemma 1.1 is due to Doyle [11]. An analytic proof based upon the Hyperbolic Cosine Rule is given by Xiangyang Liu in [17], and a proof based on hyperbolic and euclidean geometries is given in [12].

Lemma 1.1. *Let a, b, and c be positive extended real numbers such that a is finite. For $t > 0$, consider the hyperbolic triangle of centers of mutually tangent circles of hyperbolic radii ta, tb, and tc, and let $\theta(t)$ be the angle corresponding to ta. Then $\theta(t)$ is a strictly decreasing function of t.*

Lemma 1.2. *If a hyperbolic boundary-value curvature problem has a solution, then the solution is unique.*

Proof. Suppose A and B are solutions. Certainly A and B agree on boundary vertices, and $A(v) = B(v) = \infty$ at any vertex $v \in \mathscr{I}$ where $\kappa(v) = 2\pi$. Now suppose $A(v) > B(v)$ at some vertex $v \in \mathscr{I}$ where $\kappa(v) < 2\pi$, and choose v_0 to maximize $A(v)/B(v)$ among such vertices. Let $c = A(v_0)/B(v_0)$. Then Lemmas 0.1 and 1.1 imply $\kappa_B^{\text{hyp}}(v_0) < \kappa_{cB}^{\text{hyp}}(v_0) \leq \kappa_A^{\text{hyp}}(v_0)$, which violates the assumption that $\kappa_B^{\text{hyp}}(v_0) = \kappa_A^{\text{hyp}}(v_0) = \kappa(v_0)$. Thus the solution is unique. □

We are now ready to prove Theorem 1.

Proof of Theorem 1. (a) ⇒ (b) The solution to the boundary-value curvature problem generates a hyperbolic coherent system of angles.

(b) ⇒ (c) Given a subset $\mathscr{V}$ of $\mathscr{I}$, certainly

$$\begin{aligned}\Sigma(\text{angles at vertices in } \mathscr{V}) &\leq \Sigma(\text{angles of faces that have at least one vertex in } \mathscr{V})\\ &< \pi F(\mathscr{V}).\end{aligned}$$

We obtain the desired inequality by subtracting the first and last expressions from $2\pi|\mathscr{V}|$.

(c) ⇒ (d) $\mathscr{R}$ is defined because, for any vertex $v \in \mathscr{I}$, $\kappa(v) \leq 2\pi$ by assumption, and the inequality in (c) yields $2\pi - \pi F(\{v\}) < \kappa(v)$.

Let $G = P$, where $P(v) = \infty$ at each vertex v; certainly $\mathscr{R}(P) \leq P$. By Lemma 0.3, $\{\mathscr{R}^k(P)\}$ is a nonincreasing sequence which is bounded below at each vertex by zero. Thus $\mathscr{R}^k(P)$ converges pointwise to a function L such that $L(v) \geq 0$ for each vertex v.

Now we must show that $L(v) \neq 0$. Let $\mathscr{V}$ be the set of vertices v where $L(v) = 0$. Suppose that $\mathscr{V}$ is nonempty. We will show that the inequality in (c) is violated.

As in [16], we classify each angle at any vertex $v \in \mathscr{V}$ as type α, β, or γ if the face that contains the angle has exactly one, two, or three vertices in $\mathscr{V}$, respectively. Now consider the angles induced by $\mathscr{R}^k(P)$. It follows from Lemma 0.1 that any angle of type α approaches π as $k \to \infty$. If a face has two type β angles, the third angle approaches zero by Lemma 0.1, and the area of the face approaches zero. Since the area of a hyperbolic triangle is π minus the sum of its angles, this means that the sum of the two type β angles approaches π as $k \to \infty$. For a face with three type γ angles, the area of the face approaches 0, so the sum of the three angles in the face approaches π.

Thus, as $k \to \infty$, the sum of the angles at vertices in $\mathscr{V}$ approaches $\pi F(\mathscr{V})$. Hence $\sum_{v \in \mathscr{V}} \kappa_k^{\text{hyp}}(v)$ approaches $2\pi|\mathscr{V}| - \pi F(\mathscr{V})$, where $\kappa_k^{\text{hyp}}(v)$ denotes the hyperbolic curvature induced by $\mathscr{R}^k(P)$ at the vertex v. However, for all $v \in \mathscr{V}$ and for all k, $\kappa(v) \leq \kappa_k^{\text{hyp}}(v)$ by Lemma 0.3. Therefore,

$$\sum_{v \in \mathscr{V}} \kappa(v) \leq 2\pi|\mathscr{V}| - \pi F(\mathscr{V});$$

this violates condition (c).

(d) $\Rightarrow$ (e) The hyperbolic boundary-value curvature problem has a solution R by Lemma 0.4. Let G be any radius function. For each vertex v, $R(v) = \infty$ iff ($\kappa(v) = 2\pi$ or $r(v) = \infty$) iff $\mathscr{R}(G(v)) = \infty$, so we can choose $\varepsilon \in (0, 1)$ such that $\varepsilon R \leq \mathscr{R}(G)$. Let $N = \varepsilon R$. By Lemma 1.1 $\kappa_N^{\text{hyp}}(v) \leq \kappa(v)$ at all vertices $v \in \mathscr{I}$. Thus by Lemma 0.3, $\mathscr{R}^{k+1}(N) \geq \mathscr{R}^k(N)$ for all $k \geq 0$. Furthermore, $N \leq R$, so $\mathscr{R}^k(N) \leq \mathscr{R}^k(R) = R$.

Thus, at any vertex v where $R(v)$ is finite, $\{\mathscr{R}^k(N)(v)\}$ is nondecreasing and bounded. At other vertices, $\mathscr{R}^k(N)(v) = \infty$ for all k. Therefore, $\mathscr{R}^k(N)$ converges pointwise to a radius function L; $L = R$ by Lemmas 0.4 and 1.2. A similar argument shows that $\mathscr{R}^k(P)$ decreases to R, where $P(v) = \infty$ at each vertex v.

Now $N \leq \mathscr{R}(G) \leq P$, so, by Lemma 0.2, $\mathscr{R}^k(N) \leq \mathscr{R}^{k+1}(G) \leq \mathscr{R}^k(P)$ for all k. Since $\{\mathscr{R}^k(N)\}$ and $\{\mathscr{R}^k(P)\}$ both approach R as $k \to \infty$, so does $\{\mathscr{R}^k(G)\}$.

(e) $\Rightarrow$ (a) This is obvious. □

Any hyperbolic boundary-value curvature problem can be modified by changing the assigned radii and/or puncturing some of the vertices in $\mathscr{I}$ and assigning radii instead of curvatures to them. If the original problem has a solution R, then the new problem will also have a solution, since R generates a coherent system of angles.

The proof of Theorem 1 shows that if no solution exists, then either $\mathscr{R}$ is undefined or there is at least one vertex v in $\mathscr{I}$ such that, for any radius function G, $\mathscr{R}^k(G)(v) \to 0$ as $k \to \infty$. In fact, as long as $\mathscr{R}$ is defined, the sequence $\{\mathscr{R}^k(G)\}$ converges pointwse to a limit L which does not depend on G. L is the solution if the problem is solvable; otherwise, $L(v) = 0$ for at least one vertex $v \in \mathscr{I}$. This assertion is proved in [12].

2. The Nonsingular Hyperbolic Boundary-Value Problem

Theorem 2. *Consider a hyperbolic boundary-value curvature problem in which the assigned curvature function κ is identically zero. Assume $\mathscr{I}$ is nonempty. Then the following are equivalent:*

(a) *The problem has a solution.*
(b) $2|\mathscr{I}| < F(\mathscr{I})$.
(c) *The Euler characteristic χ is strictly less than the number of punctured vertices, or the surface has a border.*

Proof. (a)$\Leftrightarrow$(b) By Theorem 1 we know that (a)$\Rightarrow$(b); we aim to show that if $2|\mathscr{I}| < F(\mathscr{I})$, then condition (c) of Theorem 1 is satisfied, i.e., $2|\mathscr{V}| - F(\mathscr{V}) < 0$ for any nonempty subset $\mathscr{V}$ of $\mathscr{I}$. We assume without loss of generality that $\mathscr{V}$ is connected.

Consider the subcomplex $S'_{\mathscr{V}}$ that consists of all faces that have at least one vertex in $\mathscr{V}$. If $S'_{\mathscr{V}}$ is closed, then it must be that $S'_{\mathscr{V}} = S$ and S is closed, so $F(\mathscr{V}) = F(\mathscr{I})$. Then

$$2|\mathscr{V}| - F(\mathscr{V}) = 2|\mathscr{V}| - F(\mathscr{I}) \leq 2|\mathscr{I}| - F(\mathscr{I});$$

this quantity is negative by assumption (b).

We are left with the case where $S'_{\mathscr{V}}$ has a border; in the ensuing discussion, "border" refers to the border of the subcomplex. Let $S_{\mathscr{V}}$ be the subcomplex that results from "separating" any locations in $S'_{\mathscr{V}}$ where a border vertex is shared by more than two border edges, so that the number of border edges equals the number of border vertices; see Fig. 1. Let n be the number of contours in the border of $S_{\mathscr{V}}$ plus the number of vertices in the interior of $S_{\mathscr{V}}$ that are not in $\mathscr{V}$. Let m be the number of edges in the border of $S_{\mathscr{V}}$.

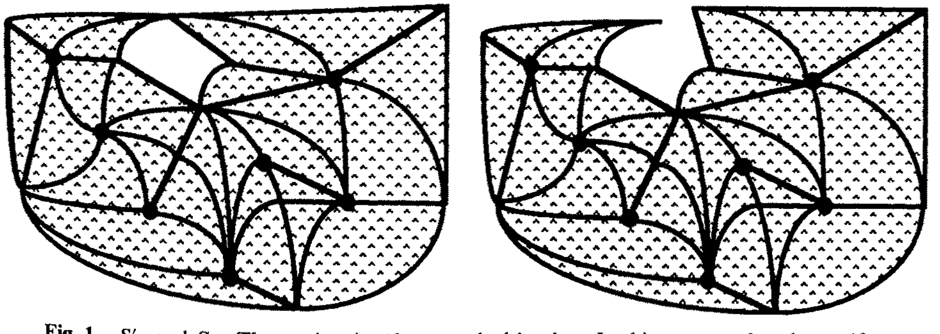

Fig. 1. $S'_{\mathscr{V}}$ and $S_{\mathscr{V}}$. The vertices in $\mathscr{V}$ are marked by dots. In this case $n = 3$ and $m = 10$.

Now construct a triangulated closed surface as follows. Attach a disk to each contour in the border of $S_{\mathscr{V}}$, and triangulate the disk by placing a vertex in its interior and constructing edges between this vertex and each vertex in the contour. The resulting surface has $|\mathscr{V}| + n + m$ vertices and $F(\mathscr{V}) + m$ faces. Let χ' be the Euler characteristic of this surface.

Now for any triangulated closed surface with F faces, V vertices, E edges, and Euler characteristic χ', we know that $V - E + F = \chi'$ and $3F = 2E$, so $2V - F = 2\chi'$. In the present case this implies $2|\mathscr{V}| - F(\mathscr{V}) = 2\chi' - 2n - m$. Since $S_{\mathscr{V}}$ has a border, we know $n \geq 1$ and $m \geq 3$; since $\chi' \leq 2$, this tells us that $2|\mathscr{V}| - F(\mathscr{V}) < 0$.

(b)$\Leftrightarrow$(c) Apply the above argument to the case $\mathscr{V} = \mathscr{I}$. If the subcomplex $S_{\mathscr{I}}$ has a border, then both (b) and (c) are true. Otherwise, $m = 0$, so $2|\mathscr{I}| - F(\mathscr{I}) = 2(\chi - n)$. However, in this case n is the number of punctured vertices, so (b)$\Leftrightarrow$(c) is obvious. □

The next three corollaries follow immediately from the fact that the Euler characteristic χ of an orientable closed surface of genus g is $2 - 2g$, and, for a nonorientable closed surface, $\chi = 2 - g$. Note that Corollary 2.3 resolves a question posed in [17] regarding the thrice-punctured sphere.

Corollary 2.1. *The nonsingular hyperbolic boundary-value problem on a closed surface of genus g with n punctured vertices has a solution iff $2g + n \geq 3$ (orientable case) or $g + n \geq 3$ (nonorientable case).*

Corollary 2.2 (Thurston's Theorem for $\chi < 0$). *Let T be a triangulation of a closed orientable surface with genus $g \geq 2$ or a closed nonorientable surface of genus $g \geq 3$. There exists a radius function on T that induces hyperbolic curvature zero at each vertex in T.*

Corollary 2.3. *Let T be the triangulation of any closed surface, and suppose that at least three vertices are punctured. The nonsingular hyperbolic boundary-value problem has a solution. Note that if the boundary radii are set to infinity, the Poincaré metric on the punctured surface is obtained.*

Corollary 2.4 (Existence Portion of the Andreev–Thurston Theorem). *A* circle packing *is a collection of closed disks whose interiors are disjoint. The* nerve *of a circle packing is a graph that has one vertex for each disk; two vertices are connected by an edge iff the corresponding closed disks are tangent. Let T be a triangulation of the sphere. Then there exists a circle packing on the sphere whose nerve is isomorphic to the graph formed by the vertices and edges in T* [16].

Proof. Choose any face in T. Puncture its three vertices and assign arbitrary radii to them. By Corollary 2.3, there is a radius function R that solves the nonsingular hyperbolic boundary-value problem. Construct the hyperbolic polyhedral surface $H(R)$ as discussed in the introduction. Now remove the chosen face from $H(R)$; what remains is a triangulated triangle in the hyperbolic plane. For each vertex v in $H(R)$, construct the disk of hyperbolic radius $R(v)$ centered at v; this yields a circle packing in the hyperbolic plane which has the desired nerve. This packing can be embedded in the complex plane and then transferred to the sphere by stereographic projection. □

3. The Euclidean Boundary-Value Curvature Problem

Our euclidean theorem requires at least one boundary vertex, but a closed surface is easily handled by puncturing one vertex and assigning radius 1. In this case the curvature concentrated at the punctured vertex is uniquely determined by the curvatures at the other vertices; this is easily seen by applying the Gauss–Bonnet formula, which simplifies to $\Sigma\ \kappa(v) = 2\pi\chi$ in the case of a closed surface of Euler characteristic χ with a metric of Gauss curvature 0 except for isolated singularities v with concentrated curvatures $\kappa(v)$.

Theorem 3. *Consider a euclidean boundary-value curvature problem in which $\mathscr{B}$ is nonempty, r is finite, and $\kappa(v) < 2\pi$ for all $v \in \mathscr{I}$. Let $\mathscr{R}$ denote the euclidean relaxation operator for r and κ. The following are equivalent:*

(a) *The problem has a unique solution.*
(b) *There exists a euclidean coherent system of angles for κ on T.*
(c) *Given a nonempty subset $\mathscr{V}$ of $\mathscr{I}$, the assigned curvatures satisfy*

$$2\pi|\mathscr{V}| - \pi F(\mathscr{V}) < \sum_{v \in \mathscr{V}} \kappa(v) < 2\pi|\mathscr{V}| - \pi F'(\mathscr{V}).$$

(d) *There exists a finite radius function that solves the* hyperbolic *boundary-value curvature problem determined by κ and r, and, for any nonempty subset $\mathscr{V}$ of $\mathscr{I}$,*

$$2\pi|\mathscr{V}| - \pi F'(\mathscr{V}) > \sum_{v \in \mathscr{V}} \kappa(v).$$

(e) *There exists a finite radius function G such that $\mathscr{R}^k(G)$ converges to a finite radius function as $k \to \infty$.*
(f) *If G is any finite radius function, then $\mathscr{R}^k(G)$ converges to the problem's solution as $k \to \infty$.*

The proof of Theorem 3 follows along the lines of the proof of Theorem 1, with the following exceptions. The uniqueness proof is somewhat different; see Lemma 3.2. Also, in order to prove (d) $\Rightarrow$ (e), we let G be the radius function that solves the hyperbolic problem; it turns out that $\{\mathscr{R}^k(G)\}$ is a nondecreasing, bounded sequence. To show that $\{\mathscr{R}^k(G)\}$ is bounded, let $\mathscr{V}$ be the set of nonboundary vertices such that $\mathscr{R}^k(G) \to \infty$. It is easily shown that condition (d) is violated if $\mathscr{V}$ is nonempty. The fact that $\{\mathscr{R}^k(G)\}$ is nondecreasing follows from this lemma:

Lemma 3.1. *Let a, b, and c be positive real numbers that satisfy the triangle inequality. Consider the hyperbolic and euclidean triangles with sides of lengths a, b, and c. Each angle of the euclidean triangle is larger than the corresponding angle of the hyperbolic triangle.*

Proof. Consider the hyperbolic triangle with sides ta, tb, and tc, and let $\theta(t)$ be the angle opposite the side tc. Let φ be the angle opposite c in the euclidean triangle. Now $\cos\theta(t)$ approaches $\cos\varphi$ as $t \to 0$, so $\theta(t)$ approaches φ. However, $\theta(t)$ is a strictly decreasing function of t by Lemma 1.1, so $\varphi > \theta(t)$ for all positive t. Thus $\varphi > \theta(1)$. □

Lemma 3.2. *If a euclidean boundary-value curvature problem has a solution, then the solution is unique, provided the set $\mathscr{B}$ is nonempty.*

Proof. Suppose A and B are two radius functions which solve the problem. Choose a vertex v_0 which maximizes $A(v)/B(v)$ over all vertices v in a connected component of $\mathscr{I}$; let $c = A(v_0)/B(v_0)$. Then at each vertex v which is adjacent to v_0, we must have $A(v)/B(v) = c$, for otherwise we would have $\kappa_A^{\text{euc}}(v_0) > \kappa_B^{\text{euc}}(v_0)$. Repeating this argument along chains of vertices in $\mathscr{I}$ shows that $A(v)/B(v) = c$ for all vertices v in and adjoining the connected component. Since A and B certainly must agree on boundary vertices, we conclude that $c = 1$. □

As in the hyperbolic case, any euclidean boundary-value curvature problem can be modified by changing the assigned radii and/or by puncturing some of the vertices in $\mathscr{I}$ and assigning radii instead of curvatures to them. If the original problem had a solution, then so will the new one.

We now show how Thurston's theorem for $\chi = 0$ follows from our results.

Corollary 3.1. *Suppose T is a triangulation of a torus or a Klein bottle. Then there is a radius function R on T such that $\kappa_R^{\text{euc}}(v) = 0$ for each vertex v in T. R is unique up to constant multiples.*

Proof. Let V, E, and F denote the number of vertices, edges, and faces in the triangulation, respectively. Now $V - E + F$ is the Euler characteristic zero, and $3F = 2E$, so $2V - F = 0$.

Now puncture one vertex v_0 in T, and prescribe radius $r(v_0) = 1$. Then $\mathscr{B} = \{v_0\}$, and $\mathscr{I}$ is the set of all remaining vertices in T. There is a solution to the nonsingular hyperbolic boundary-value problem on T by Theorem 2, so by Theorem 1 we know that, for any nonempty subset $\mathscr{V}$ of $\mathscr{I}$, we have

$$2|\mathscr{V}| - F(\mathscr{V}) < 0.$$

Now if $\mathscr{V}^c$ denotes the set of vertices not in $\mathscr{V}$, we know that

$$2|\mathscr{V}| - F(\mathscr{V}) + 2|\mathscr{V}^c| - F'(\mathscr{V}^c) = 2V - F = 0,$$

so

$$2|\mathscr{V}^c| - F'(\mathscr{V}^c) > 0.$$

However, v_0 was chosen arbitrarily, so this inequality actually holds for *any* nonempty proper subset $\mathscr{V}^c$ of the vertices in T. Hence condition (d) of Theorem 3 is satisfied, so there is a unique radius function R such that $\kappa_R^{\mathrm{euc}}(v) = 0$ for all $v \neq v_0$, and $R(v_0) = 1$. Now $\kappa_R^{\mathrm{euc}}(v_0) = 0$ by the Gauss–Bonnet formula. Clearly, any scalar multiple kR will also generate euclidean curvature zero at each vertex. □

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