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CIRCLE PACKINGS IN DIFFERENT GEOMETRIES

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1. Circle packings. In the last few years there have been many papers which have explored the connection between circle packings in the Euclidean plane and complex analysis (see, for example, [3], [4] and [5]), and in [1] we considered packings in the hyperbolic plane. In this paper, we shall show how the three classical geometries of constant curvature (namely, spherical geometry $\boldsymbol{C}_\infty$, Euclidean geometry $\boldsymbol{C}$, and hyperbolic geometry viewed as the unit disc $\varDelta$ in $\boldsymbol{C}$) control the possible circle packings in that geometry. The idea is that the curvature of any one of these spaces determines its trigonometry, and that this, in turn, exerts a strong influence on any circle packing that the space supports.

Throughout this paper, S will denote any one of these spaces. By a *circle packing of S* we mean a collection $\{D_\alpha\}$ of closed, non-overlapping discs in S with the properties

(i) the discs D_α do not accumulate in S, and

(ii) the closure of each component of $S-(\bigcup D_\alpha)$ is a compact subset of S bounded by exactly three circular arcs, each arc lying on the boundary of some D_α.

Roughly speaking, in any circle packing, each disc is tangent to several others, and the regions between the discs are circular triangles. Note that (i) implies that any circle packing of $\boldsymbol{C}_\infty$ can only contain a finite number of circles, while (ii) implies that any circle packing of $\boldsymbol{C}$, or of $\varDelta$, must contain infinitely many circles: this is a fundamental distinction.

Given a circle packing of S, the *flower* of a circle C in the packing is the configuration consisting of C together with all of the circles tangent to it. The circle C is the *centre* of the flower, the circles tangent to C are the *petals* of C, and the *degree* of C is the number of petals of C. In this paper, our interest centres largely (but not entirely) on the *degree k circle packings*, that is, on circle packings in which each circle has exactly k petals. As an illustration of how the curvature of S influences its circle packings, we prove the following existence and uniqueness theorem for degree k packings.

THEOREM 1. (i) *There exists a degree k circle packing of $\boldsymbol{C}_\infty$ if and only if $k=2$, 3, 4 or 5.*

(ii) *There exists a degree k circle packing of $\boldsymbol{C}$ if and only if $k=6$.*

(iii) *There exists a degree k circle packing of $\varDelta$ if and only if $k\geq 7$.*

Further, in each case, a degree k circle packing of S is unique up to a conformal automorphism of S.

This rather special result illustrates our basic idea, for we can give an intuitive explanation of Theorem 1 as follows. Consider a degree k circle packing of S, choose any circle C_0 in the packing, and define a sequence of circles C_n inductively by the condition that C_{n+1} is any one of the smallest petals of C_n. In order that the sequence C_n does not accumulate in S, the radii of the C_n cannot converge to zero too quickly. It follows that the larger values of k must correspond to those geometries in which the length of the circumference of a circle is a relatively large function of the radius (the case of negative curvature) and conversely.

Of course, circle packings with variable degree are of greater interest than constant degree packings and we shall also prove:

THEOREM 2. (i) *Suppose that S supports a circle packing in which each circle has degree at most* 5; *then S is the complex sphere.*

(ii) *Suppose that S supports a circle packing in which each circle has degree at most* 6; *then S is either the complex plane or the complex sphere.*

This result is concerned with upper bounds on the degree; in the other direction, we prove:

THEOREM 3. *Suppose that S supports a circle packing in which each circle has degree at least seven. Then S is the hyperbolic plane.*

We remark that any circle packing of S gives rise to a (possibly infinite) triangulation of S which is obtained by joining the centres of mutually tangent circles by a geodesic segment. It is often easier to discuss the triangulation, or its graph, rather than the circle packing, and we shall pass freely between a circle packing, its triangulation, and the corresponding graph, without much comment.

2. The sphere. There are three Platonic solids with triangular faces, namely the tetrahedron, the octahedron and the icosahedron. We view these as being embedded in $\boldsymbol{R}^3$ with their vertices on the unit sphere, and in each case, there is a unique value of r such that the spherical caps of radius r centred at the vertices provide a degree k circle packing of $\boldsymbol{C}_\infty$ (k is 3 for the tetrahedron, 4 for the octahedron and 5 for the icosahedron). In addition, we can place three equal circles symmetrically about the equator of the unit sphere to obtain a degree two circle packing of $\boldsymbol{C}_\infty$.

Conversely, given any degree k circle packing of $\boldsymbol{C}_\infty$, we can apply Euler's formula to its graph (as we do when we seek all Platonic solids) and immediately find that k is 2, 3, 4 or 5. This completes the proof of the existence part of Theorem 1 (i).

The uniqueness of these packings follows from the uniqueness results in [1]. Applying a Möbius map, we may consider one circle as the unit circle, with all other circles lying inside the unit disc Δ. This yields an Andreev configuration for the packing with the first circle removed, and this is known to be unique up to an automorphism of Δ.

3. The Euclidean plane. In this section, we shall prove Theorem 1, (ii) and Theorem 3, and first we prove Theorem 3.

Consider a flower in $\boldsymbol{C}$ with k petals, where $k \geq 3$, and suppose that the centre circle C_0 has radius r_0, and that the petals C_j have radii $r_j, j=1, \cdots, k$. A computation shows that if all of the petals have the same radius, then this common radius is $\lambda_k r_0$, where

$$\lambda_k = \frac{\sin(\pi/k)}{1-\sin(\pi/k)} .$$

Of course, $\lambda_k > 1$ for $k \leq 5$, $\lambda_k = 1$ for $k=6$, and $\lambda_k < 1$ for $k \geq 7$. We prove:

LEMMA 4. *In the situation described above,*

$$\min(r_1, \ldots, r_k) \leq \lambda_k r_0 ,$$

and λ_k is best possible.

REMARK. This may be viewed as a companion to the Rodin-Sullivan Ring Lemma, [3], which asserts that for some μ_k,

$$\min(r_1, \ldots, r_k) \geq \mu_k r_0 : \tag{3.1}$$

see [2] for the best choice of μ_k.

PROOF OF LEMMA 4. Let

$$\lambda = \min(r_1/r_0, \ldots, r_k/r_0) .$$

Now consider the three mutually tangent circles, C_0, C_1 and C_2 of radii r_0, r_1 and r_2, respectively, construct the triangle with vertices at the centres of the circles, and let θ be the angle of this triangle at the centre of C_0: thus

$$\cos\theta = \frac{(r_0+r_1)^2 + (r_0+r_2)^2 - (r_1+r_2)^2}{2(r_0+r_1)(r_0+r_2)} .$$

Now θ decreases as we decrease r_1 and r_2; hence $\cos\theta$ increases and so, on replacing r_1 and r_2 by λr_0, we obtain

$$\cos\theta \leq 1 - \frac{2\lambda^2}{(1+\lambda)^2} = g(\lambda) ,$$

say. The same argument holds for any pair of consecutive petals so, if θ_j is the angle associated with r_j and r_{j+1}, then we have $\cos\theta_j \leq g(\lambda)$. We deduce that

$$2\pi = \sum_{j=1}^{k} \theta_j \geq k \cos^{-1}[g(\lambda)] ,$$

and hence that

$$g(\lambda) \geq \cos(2\pi/k) = g(\lambda_k) .$$

As g is a decreasing function, we obtain $\lambda \leq \lambda_k$ as required.

We can now give:

THE PROOF OF THEOREM 3. Consider a circle packing of S as given in Theorem 3 and suppose first that S is the complex plane. As described in the introduction, we create a sequence of circles C_n, each C_{n+1} being one of the smallest petals of C_n. By Lemma 4, the radius of C_n, say ρ_n, is at most $(\lambda_7)^n r_0$, so, as $\lambda_7 < 1$,

$$\sum_{n=1}^{\infty} \rho_n \leq r_0 \sum_{n=1}^{\infty} (\lambda_7)^n < +\infty ,$$

and the circles C_n must therefore accumulate in $\mathbf{C}$. As this cannot happen, S is not $\mathbf{C}$. Finally, we show that S cannot be the sphere. Now any circle packing of the sphere can be stereographically projected into the plane without changing the degree of the circles, and the argument above shows that each circle C in $\mathbf{C}$ has a petal which has a strictly smaller Euclidean radius than C. As this implies that there are infinitely many circles present, S is not the sphere so it must be the hyperbolic plane.

REMARK. The circle packing of $\mathbf{C}$ consisting of circles of radius $1/2$ with centres at $m+in$ (m, n integers), and circles of radius $(\sqrt{2}-1)/2$ with centres at $(m+in)/2$ (m, n odd integers), has infinitely many circles of degree eight.

We now give:

THE PROOF OF THEOREM 1, (ii). First, the regular hexagonal packing of circles of equal size is a degree 6 circle packing of $\mathbf{C}$. We must show that for any degree k circle packing of $\mathbf{C}$, $k=6$, and as an immediate consequence of Theorem 3 is that $k \leq 6$, it remains to show that $k \geq 6$ for such a packing.

Consider now a (necessarily infinite) degree k circle packing of $\mathbf{C}$, or of Δ, and suppose that $k \leq 5$: we propose to reach a contradiction and so show that no such packings exist. This will

(1) complete the proof of the existence part of Theorem 1 (ii);

(2) prove Theorem 2 (i),

and finally (for use later),

(3) show that $k \geq 6$ for any degree k circle packing of Δ.

We proceed now to a contradiction, so let S be either of the spaces $\mathbf{C}$ or Δ. The given packing of S gives rise to an infinite triangulation, and hence to a graph G on S, and we begin by constructing a Jordan curve Γ in G. We denote the interior of Γ by Σ, so G induces a triangulation T of $\Sigma \cup \Gamma$. Clearly, we may construct Γ so that

(i) Γ has at least 21 edges, and

(ii) each triangle in T meets Γ in a connected set (this means that Σ has no narrow neck spanned by just one triangle).

Suppose that Γ contains n vertices v_j, and n edges e_j (so $n \geq 21$), and let k_j be the number of triangles having v_j as a vertex. Suppose also that T contains t triangles, and

n_0 interior vertices w_j, and let a_j be the number of triangles having w_j as a vertex. First, Euler's formula applied to T yields

$$t-\left(\frac{3t-n}{2}+n\right)+(n_0+n)=1\,. \tag{3.2}$$

Next, each w_j has a_j triangles meeting there and, by assumption, $a_j \leq 5$: thus

$$3t-\sum_j k_j=\sum_{w_j} a_j \leq 5n_0 \tag{3.3}$$

and, eliminating n_0 from (3.2) and (3.3),

$$5n+t\leq 2\sum_j k_j+10\,. \tag{3.4}$$

We must now estimate the sum $2\sum k_j$ in two different ways. Obviously, $k_j \leq 4$ (as otherwise, v_j would be a vertex inside Γ) so certainly, $\sum k_j \leq 4n$: this, however, is not sufficient for our needs. Suppose for the moment that, say, $k_2=4$. Then the edges e_1 and e_2 which meet at v_2 must be two consecutive edges of one triangle outside of T (as otherwise, in the graph G, v_j would have valency exceeding 5). It follows that if $k_j=k_{j+1}=4$, then three consecutive edges of Γ bound the same triangle and so $n=3$, contrary to (i). We deduce that

$$k_j+k_{j+1}\leq 4+3=7\,,$$

and so we obtain our first estimate, namely

$$2\sum_j k_j \leq 7n\,.$$

Combining this with (3.4) we obtain

$$t\leq 10+2n\,. \tag{3.5}$$

Next, we obtain a second estimate. The assumptions on Γ imply that there are

$$m_j=\max\{k_j-2, 0\}$$

triangles with v_j as a vertex and not having an edge in common with Γ. As no such triangle appears in this set for different values of j, we find that

$$\sum_j (k_j-2)\leq \sum_j m_j \leq t-n\,,$$

$t-n$ being the total number of triangles not having an edge on Γ. This leads to our second estimate, namely,

$$\sum_j k_j \leq t+n\,,$$

and this, with (3.4) yields

$$3n \leq t+10 . \tag{3.6}$$

Finally, elimination of t from (3.5) and (3.6) yields $n \leq 20$, a contradiction. We have now completed the proofs of (1), (2) and (3) above.

The uniqueness of the packing in Theorem 1, (ii) follows directly from Appendix 1 in [4].

4. The hyperbolic plane. We begin by constructing, for each $k \geq 7$, a degree k circle packing of the hyperbolic plane. First, construct an equilateral (hyperbolic) triangle T with each angle equal to $2\pi/k$: this is possible if and only if $k \geq 7$. The group Γ generated by the reflections across the sides of T is discrete, and T is a fundamental region for Γ. This means that the Γ-images of T tesselate Δ and clearly, this gives rise to an infinite triangulation of Δ in which each edge has the same hyperbolic length, say $2R_k$, and in which each vertex has degree k. The desired circle packing is now obtained by constructing circles of radius R_k at each vertex of each Γ-image of T.

As the value of R_k plays an important role in what follows, we shall find an explicit expression for it. By bisecting T and considering the triangle with sides R_k, r (say), and $2R_k$ (these being opposite angles of π/k, $2\pi/k$ and $\pi/2$, respectively), we obtain

$$\frac{\sinh(R_k)}{\sinh(2R_k)} = \sin(\pi/k) ,$$

or, equivalently,

$$2\sin(\pi/k)\cosh(R_k) = 1 . \tag{4.1}$$

This definition holds for $k \geq 7$, but it is convenient to extend it to $k=6$, so $R_6=0$. To complete the proof of the existence part of Theorem 1, we must prove that $k \geq 7$ for any degree k circle packing of Δ. Note that from (3) in §3, we know that $k \geq 6$, so R_k is defined for the range of k we are considering.

We shall need to use the function

$$\Phi : (a, b) \mapsto \frac{\sinh a}{\sinh(a+b)}, \qquad a>0, b>0,$$

which arises naturally in hyperbolic trigonometry. Observe that

(i) for fixed b, the map $a \mapsto \Phi(a, b)$ is increasing;
(ii) for fixed a, the map $b \mapsto \Phi(a, b)$ is decreasing;
(iii) $\Phi(0+, b)=0$, $\Phi(+\infty, b)=e^{-b}$;
(iv) the map $a \mapsto \Phi(a, a) = 2(\cosh a)^{-1}$ is decreasing.

Now take any λ in $(0, 1/2]$. The properties (i) and (iii) guarantee that we can define a function

$$f\colon (0, \log \lambda^{-1}) \to (0, +\infty)$$

by the formula

$$\Phi(f(x), x) = \lambda ,$$

where we have supressed the dependence of f on λ in our notation. To obtain an explicit formula for f, we can express the defining equation in terms of exponentials and re-arrange to obtain the formula

$$\exp[2f(x)] = \frac{1 - \lambda e^{-x}}{1 - \lambda e^{x}} .$$

This shows that f is a strictly increasing map of $(0, \log \lambda^{-1})$ onto $(0, +\infty)$, and also that f can be extended to an analytic function in some neighbourhood of the origin with $f(0) = 0$.

The geometric significance of the function Φ is that if a circle C_1 of radius a is tangent to a circle C_2 of radius b, then C_1 subtends and angle 2θ at the centre of C_2, where

$$\sin \theta = \Phi(a, b) .$$

In particular, if a flower contains a central circle C of radius r, and n petals C_j of equal size, then each petal has radius $f(r)$, where f is given by

$$\Phi(f(x), x) = \sin(\pi/n) .$$

We shall now use the dynamics of the iterates of f to investigate the geometry of circle packings. With this in mind, observe that if $\lambda < 1/2$, then f has exactly two fixed points, namely 0 and x_λ, where

$$2\lambda \cosh x_\lambda = 1 :$$

in particular, if $\lambda = \sin(\pi/k)$, the fixed points of f are 0 and R_k. We shall show that the fact that R_k is a fixed point of f corresponds to the existence of the regular degree k circle packing of Δ with each circle having radius R_k. The fact that R_k is an *unstable* fixed point of f corresponds to the fact that the regular packing is *rigid*, and, by further analysis, unique.

For $0 < \lambda < 1/2$, the additional relevant features of f are as follows:

(v) $f(x) < x$ on $(0, x_\lambda)$, and

(vi) $f(x) > x$ on $(x_\lambda, \log \lambda^{-1})$.

For example, $f(x) < x$ is equivalent to

$$\Phi(x_\lambda, x_\lambda) = \lambda = \Phi(f(x), x) < \Phi(x, x)$$

which, in turn, is equivalent to $x < x_\lambda$. These facts show that if f^n denotes the *n-th* iterate of f, then

(vii) $f^n \to 0$ on $(0, x_\lambda)$, and

(viii) if x is in $(x_\lambda, \log \lambda^{-1})$, then for some n,

$$f^n(x) > \log \lambda^{-1}$$

and so $f^{n+1}(x)$ is not defined.

If $\lambda = 1/2$, then f fixes zero only, $f(x) > x$ on $(0, \log 2)$, and (viii) holds with $x_\lambda = 0$.

We come now to the stability argument, and the essence of this is contained in the following Lemma (which is a type of hyperbolic Ring Lemma).

LEMMA 5. *Suppose that $\lambda = \sin(\pi/k)$, where $k \geq 6$, and consider a flower in Δ with central circle C_0 and petals $C_1, \ldots, C_k$, where each C_j has radius r_j. Then $r_0 < \log \lambda^{-1}$, so $f(r_0)$ is defined, and*

$$\min\{r_1, \ldots, r_k\} \leq f(r_0) \leq \max\{r_1, \ldots, r_k\} .$$

With this, the rest of the proof of Theorem 1 is easy. Suppose first that $k \geq 7$. Let C_0 be a circle in a degree k circle packing of Δ, and construct a sequence C_n of circles, each C_{n+1} being one of the smallest petals of C_n. Let ρ_n be the radius of C_n, and suppose that $\rho_0 < R_k$. Then from Lemma 5, $\rho_n \leq f^n(\rho_0)$, and, as $\rho_0 < R_k$, we see that $\rho_n \to 0$ as $n \to \infty$. This means that for sufficiently large n,

$$\rho_{n+1} \leq (f'(0) + \varepsilon)\rho_n ,$$

and as $f'(0) < 1$, there is a number τ, $\tau < 1$, such that $\rho_{n+1} \leq \tau \rho_n$ for all sufficiently large n. The convergence argument (as given in the proof of Theorem 3) is now applicable, and this shows that the circles C_n must accumulate in Δ. As this cannot be so, we deduce that for $k \geq 7$, *every circle in every degree k circle packing of Δ has radius at least R_k.*

Now suppose that $k \geq 6$ and $\rho_0 > R_k$, and construct the C_n as above, except that now, C_{n+1} is one of the largest petals of C_n. Then $\rho_{n+1} \geq f(\rho_n)$, whence $\rho_n \geq f^n(\rho_0)$, and so, according to (viii), some ρ_n exceeds $\log \lambda^{-1}$. This contradicts Lemma 5, however, because in Lemma 5, C_0 is any petal in the packing, hence $\rho_n < \log \lambda^{-1}$ for all n. We deduce that for $k \geq 6$, *every circle in every degree k circle packing of Δ has radius at most R_k.*

When $k = 6$, $R_k = 0$ and this argument shows that every circle in the packing has radius zero: thus no such packings exist, and $k \geq 7$ for any degree k circle packing of Δ. When $k \geq 7$, the argument above shows that *every circle in the packing has radius equal to R_k*, and the uniqueness is now obvious.

We now give:

PROOF OF LEMMA 5. First, we must show that $r_0 < \log \lambda^{-1}$. Place the centre of the circle C_0 at the origin, and let R be its Euclidean radius. If a circle C_1 touches C_0 and the unit circle, and subtends an angle $2\pi/k$ at the origin, then (from Euclidean geometry),

$$\lambda=\sin(\pi/k)=\frac{(1-R)/2}{R+(1-R)/2}=\frac{1-R}{1+R},$$

so

$$R=\frac{1-\lambda}{1+\lambda}=\mu,$$

say. If R exceeds μ then it is impossible for C_0 to have k petals, so C_0 has Euclidean radius at must μ. If follows that if C_0 has hyperbolic radius r_0, then

$$r_0<\log\left(\frac{1+\mu}{1-\mu}\right)=\log\lambda^{-1}$$

as required. Now let C_j have centre z_j, and, for convenience, put $C_{k+1}=C_1$. Consider the triangle with vertices z_0, z_j and z_{j+1}, and let the angle at z_0 be θ_j. Obviously,

$$\sum\theta_j=2\pi.$$

Now let

$$t=\min\{r_1,\ldots,r_k\}.$$

If we decrease r_j and r_{j+1} to t, then θ_j decreases to some angle, say α_j, where, by trigonometry,

$$\sinh t=\sin(\alpha_j/2)\sinh(r_0+t).$$

We deduce that for each j,

$$\Phi(t,r_0)=\sin(\alpha_j/2)\leq\sin(\theta_j/2).$$

Taking θ_j to be the smallest of the angles $\theta_1,\ldots,\theta_k$, we obtain

$$\Phi(t,r_0)\leq\sin(\pi/k)=\Phi(f(r_0),r_0),$$

whence $t\leq f(r_0)$ as required.

Now let

$$s=\max\{r_1,\ldots,r_k\}.$$

We let $2\phi_j$ be the angle subtended at z_0 by the circle C_j, so

$$\sum 2\phi_j\geq 2\pi.$$

From trigonometry,

$$\sin\phi_j=\Phi(r_j,r_0),$$

so we have

$$\sin^{-1}\Phi(f(r_0), r_0) = \pi/k \le \frac{1}{k}\sum_{j=1}^{k}\phi_j = \frac{1}{k}\sum_{j=1}^{k}\sin^{-1}\Phi(r_j, r_0)$$

$$\le \frac{1}{k}\sum_{j=1}^{k}\sin^{-1}\Phi(s, r_0) = \sin^{-1}\Phi(s, r_0)\,,$$

whence $f(r_0) \le s$, as required.

It only remains to prove Theorem 2, (ii) so suppose that there is some circle packing of Δ in which each circle has degree at most six. We define f by

$$\Phi(f(x), x) = \sin(\pi/6)\,,$$

and suppose that a circle C_0 with radius r_0 has q petals C_j with radii r_j, respectively, where $j = 1, \ldots, q$ and $q \le 6$. Then, as in the proof that $f(r_0) \le s$ above, we have

$$\Phi(f(r_0), r_0) = \sin(\pi/6) \le \sin(\pi/q) \le \Phi(s, r_0)\,,$$

where

$$s = \max\{r_1, \ldots, r_q\}\,.$$

We deduce that $s \ge f(r_0)$, and as $r_0 > R_6$ $(=0)$, we again see that there must be a sequence of circles in the packing whose radii tend to $+\infty$. For exactly the same reason as before, this cannot happen and the proof is complete.

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