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Circuit Waveforms For Periodic Waves

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Circuit Waveforms for Periodic Waves

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FELLOW IEEE

Summary: A method using steady-state transforms is described which indicates how the sum function of a Fourier series may be obtained. The method can be applied to problems arising from circuits containing concentrated circuit parameters, but is not quite so useful for those having distributed parameters. Tables of Fourier series and their sum functions are presented, and several examples of the application of the method in circuit problems are given in this paper.

IN THE CIRCUIT theory of linear networks the use of nonsinusoidal periodic waves usually leads to a Fourier series representation of voltages and cur-

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rents. The series is useful in determining the amplitude and phase of the harmonic voltages and currents, but it is not very useful in determining the waveforms of the voltages and currents. The sum function of the Fourier series might be defined as the repetitive waveform in the time domain, or, alternately, the closed-form expression of the sum of the Fourier series. If the sum function is known, the required waveform may be reproduced with ease. The sum function may be obtained in various ways,[1-9] although none of the methods seem to be well known, probably because of their difficulty. The purpose of this paper is to recover the waveform from the Fourier series, and this will be done by presenting another method of summing Fourier series and giving some examples of how the method may be used in circuit theory. The method consists of changing the Fourier series into a steady-state transform,[10,11] and then evaluating the transform by residues[10] or by using a table of steady-state transforms.[6,12] It is believed that this method using the transform tables will provide a comparatively simple way of obtaining the required sum function of the Fourier series.

Theory

The direct steady-state transform[8] is

$$S[f(t)]=\int_0^T e^{-pt}f(t)dt=F(p) \tag{1}$$

where $f(t)$ is given in the time interval $O<t<T$ and p is a complex variable. The inverse transform is

$$S^{-1}[F(p)]=\frac{1}{2\pi j}\int_W \frac{e^{pt}F(p)dp}{1-e^{-pT}}=f(t) \tag{2}$$

where W is the contour in the complex p plane shown in Fig. 1. The evaluation of equation 2 is presented in some detail in the Appendix. If equation 2 is evaluated at the poles $p=p_n=(j2\pi n/T)$ within W where n is an integer ($n=0, \pm 1, \pm 2, \ldots$), the resulting Fourier series is

$$(t)=\frac{1}{T}\sum_{n=-\infty}^{\infty} F(j2\pi n/T)e^{j(2\pi nt/T)} \tag{3}$$

or

$$f(t)=\frac{1}{T}F(O)+\frac{2}{T}\sum_{n=1}^{\infty} Re[F(j2\pi n/T)] \cos(2\pi nt/T)-\frac{2}{T}\sum_{n=1}^{\infty} Im[F(j2\pi n/T)] \sin(2\pi nt/T) \tag{4}$$

If $F(p)$ has one or more poles at $p_n=(j2\pi n/T)$, then equations 3 and 4 have to be modified for the multiple poles and this may be done by evaluating equation 2 directly. The function $F(p)$ may be obtained from the coefficients of the series of equations 3 or 4 by replacing n by $(-jpT/2\pi)$. The function $f(t)$ then may be evaluated as a sum function[10] by evaluating the integral of equation 2 or by the use of the steady-state transforms.[6,12] Fortunately, in linear circuit theory the $F(p)$ obtained from a given Fourier series will not have poles at $p_n=(j2\pi n/T)$ and so the procedure mentioned above for multiple poles need not be considered.

As an example showing how the sum function may be determined from a Fourier series consider

$$f(t)=\sum_{n=1}^{\infty}\frac{(-1)^n(\cos nt+n\sin nt)}{n^2+1} \tag{5}$$

Comparing equation 5 with equation 4 the following quantities are found:

$$T=2\pi$$

$$\frac{1}{\pi}Re[F(j2\pi n/T)]=\frac{(-1)^n}{n^2+1}$$

$$-\frac{1}{\pi}Im[F(j2\pi n/T)]=\frac{(-1)^n n}{n^2+1}$$

Then

$$F(j2\pi n/T)=\frac{(-1)^n(1-n)j\pi}{n^2+1} \tag{6}$$

and put $p=(jz\pi 2n/T)=jn$. The $(-1)^n$ term should be replaced by $(e^{-j\pi})^n$ whenever alternating signs are encountered.

$$F(p)=\frac{(e^{-j\pi})^{-j\pi}(1-p)\pi}{-p^2+1}+\frac{\pi e^{-p\pi}}{p+1} \tag{7}$$

The quantity $[F(O)/T]=(\pi/T)=(1/2)$ and let

$$f_1(t)=\frac{1}{2}+\sum_{n=1}^{\infty}\frac{(-1)^n(\cos nt+n\sin nt)}{n^2+1} \tag{8}$$

where

$$f(t)=f_1(t)-(1/2) \tag{9}$$

Now $F(p)$ and $f_1(t)$ are a pair of steady-state transforms, and if $f_1(t)$ is evaluated as the sum function[10] of the Fourier series by the use of equation 2 or by the use of the steady-state transform tables[6,12] the result is

$$f_1(t)=\begin{cases}\dfrac{\pi e^{-(t+\pi)}}{1-e^{-2\pi}} & 0<t<\pi\\[2ex] \dfrac{\pi e^{-(t-\pi)}}{1-e^{-2\pi}} & \pi<t<2\pi\end{cases} \tag{10}$$

or from equations 9 and 10,

$$f(t)=\begin{cases}\dfrac{\pi e^{-(t+\pi)}}{1-e^{-2\pi}}-\dfrac{1}{2}, & 0<t<\pi\\[2ex] \dfrac{\pi e^{-(t-\pi)}}{1-e^{-2\pi}}-\dfrac{1}{2} & \pi<t<2\pi\end{cases} \tag{11}$$

Equation 11 then is the sum function of the fourier series of equation 5. The waveform of equation 5 as obtained in equation 11 is shown in Fig. 2.

At first glance it would appear as if any Fourier series whose general term could be expressed in terms of n could be summed by this method, but this is not true. If the series cannot be written in the form of equation 3 except for a finite number of terms, then the sum function of the fourier series can not be found by this method. An example of a series that can not be summed by this method is

$$f(t)=\sum_{n=1}^{\infty}\frac{\sin nt}{n^2+1} \tag{12}$$

If equation 12 is written in the form of equation 3,

$$f(t)=\sum_{n=1}^{\infty}\frac{e^{jnt}}{2j(n^2+1)}+\sum_{n=1}^{\infty}\frac{(-e^{-jnt})}{2j(n^2+1)}$$

$$=\sum_{n=1}^{\infty}\frac{e^{jnt}}{2j(n^2+1)}+\sum_{n=-1}^{-\infty}\frac{(-e^{-jnt})}{2j(n^2+1)} \tag{13}$$

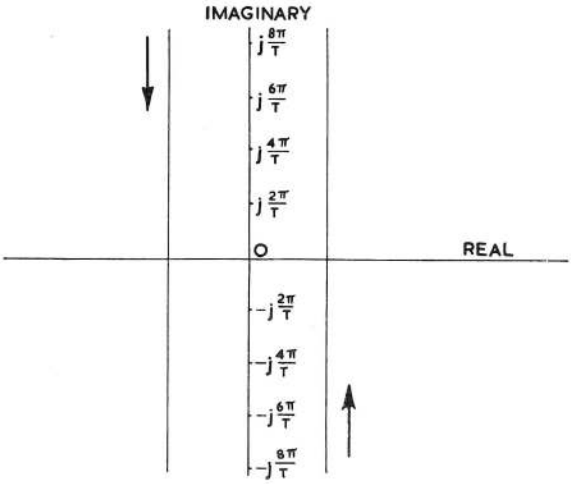

Fig. 1. The contour of the inverse transform

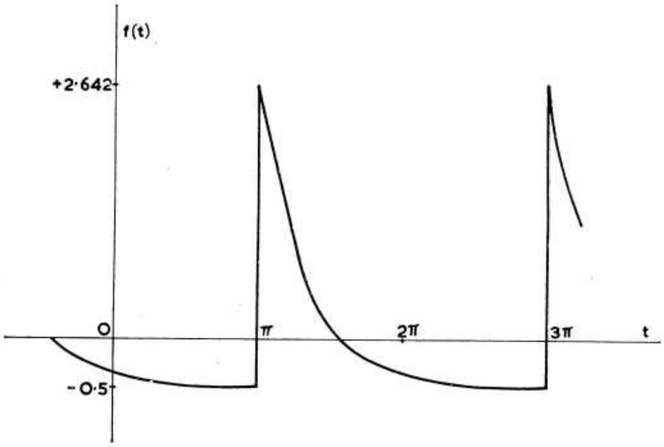

Fig. 2. The waveform of the example of a Fourier series

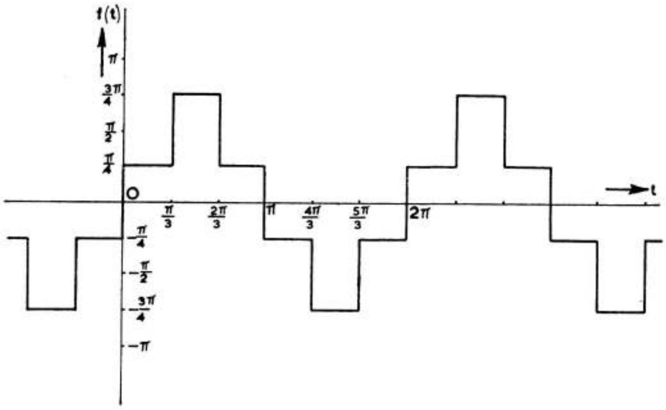

Fig. 3. The waveform of the example involving an essential singularity

From the first part of equation 13

$$\frac{1}{T}F(j2\pi n/T)=\frac{1}{2j(n^2+1)} \tag{14}$$

while from the second part

$$\frac{1}{T}F(j2\pi n/T)=-\frac{1}{2j(n^2+1)} \tag{15}$$

Thus, there is no one analytic function $F(p)$ in closed form that can be assigned to the series of equation 12 and so the sum function in closed form cannot be found by using this method. An infinite series for $F(p)$ can be obtained, but $F(p)$ is not useful in finding the sum function in closed form. The constant term is missing in equation 13, but this could be supplied as was done in equations 5, 8, and 9 of the first example.

Another difficulty that may arise is that which occurs often in series coming from problems involving distributed parameters such as those of transmission lines and wave guides. In almost all cases the problems arising from linear circuit theory with concentrated parameters such as resistance, inductance, and capacitance involve an $F(p)$ which has a finite number of single or multiple poles as the singularities in the complex p plane. The linear circuit theory involving distributed parameters, on the other hand, involves an $F(p)$ which may have one or more essential singularities or branch points in the p plane.[13] This $F(p)$, when evaluated as a sum function, may lead to another infinite series or to an integral which cannot be evaluated in terms of known functions. The infinite series may or may not be a Fourier series, but it may have the advantage that it converges faster than the original series, and thus, an approximate sum of the series may be obtained more quickly. An example of this type of Fourier series is

$$f(t)=\sum_{n=1}^{\infty}\frac{\sin(2n-1)t}{(2n-1)\cos[(2n-1)\pi/3]} \tag{16}$$

Before proceeding it is helpful to put $m=2n-1$ where the summation in equation 16 is taken over the odd values of m or

$$f(t)=\sum_{m=1,3,5,7,\ldots}^{\infty}\frac{\sin mt}{m\cos(m\pi/3)} \tag{17}$$

Then by comparison with equation 4

$$T=2\pi$$

$$\frac{1}{\pi}Re[F(jm)]=0$$

$$-\frac{1}{\pi}Im[F(jm)]=$$

$$\begin{cases}\dfrac{1}{m\cos\left(\dfrac{\pi m}{3}\right)} & \text{for } m=1,3,5,7\ldots\\ 0 & \text{for } m=2,4,6,8\ldots\end{cases} \tag{18}$$

To obtain the required $F(p)$ it is necessary to multiply the imaginary part of $F(jm)$ in equation 18 by a function of m which takes on the value of unity when m is odd, and zero when m is even. One such function is

$\frac{1}{2}(1-2^{-jm\pi})$ and then for all m's

$$-\frac{1}{\pi}Im[F(jm)]=\frac{1-e^{-jm\pi}}{2m\cos\dfrac{m\pi}{3}} \tag{19}$$

Now, from equations 18 and 19,

$$F(jm)=0-\frac{j\pi(1-e^{-jm\pi})}{2m\cos\dfrac{m\pi}{3}} \tag{20}$$

and putting $p=jm$

$$F(p)=\frac{(1-e^{-p\pi})}{2p\cosh(p\pi/3)} \tag{21}$$

From equation 20,

$$(1/T)F(0)=(\pi/4) \tag{22}$$

Now let

$$f_1(t)=(\pi/4)+\sum_{m=1,3,5,7,\ldots}^{\infty}\frac{\sin mt}{m\cos\dfrac{m\pi}{3}} \tag{23}$$

where

$$f(t)=f_1(t)-(\pi/4)$$

The function $F(p)$ is to be changed into a sum function using equation 2 by evaluating the integrand by residues[10] at the poles of $F(p)$. The following series are then obtained:

$0<t<\pi$

$$_1(t)=(\pi/2)+\sum_{K=1,3,5,7,\ldots}^{\infty}(1/K)\times[(-1)^{(K+1)/2}\cos(3/2)Kt+\sin(3/2)Kt] \tag{24}$$

$\pi<t<2\pi$

$$f_1(t)=\sum_{K=1,3,5,7,\ldots}^{\infty}(1/K)\times[(-1)^{(K+1)/2}\cos(3/2)Kt-\sin(3/2)Kt]$$

From equations 23 and 24,

$0<t<\pi$

$$f(t)=(\pi/4)+\sum_{K=1,3,5,7,\ldots}^{\infty}(1/K)\times[(-1)^{(K+1)/2}\cos(3/2)Kt+\sin(3/2)Kt] \tag{25}$$

$\pi<t<2\pi$

$$f(t)=-(\pi/4)+\sum_{K=1,3,5,7,\ldots}^{\infty}(1/K)\times[(-1)^{(K+1)/2}\cos(3/2)Kt-\sin(3/2)Kt]$$

The process of finding the sum function has led to two Fourier series, one of which is to be used in the first half-period, and the second of which is to be used in the second half-period. In this case a closed-form expression of the sum function is not obtained although other methods[13] may lead to a closed form. In the example whose results are given in equation 25 the time function $f(t)$ may be recognized as the superposition of two square waves and this leads to the waveform shown in Fig. 3. The original Fourier series of equation 17 may also be written in the following form:

$$(ft)=2\sum_{m=1,3,5,7,\ldots}^{\infty}\frac{\sin mt}{m}-\sum_{m=1,3,5,7,\ldots}^{\infty}\frac{\sin 3mt}{m} \tag{26}$$

This also is the difference of two square waves and again leads to the waveform of Fig. 3.

Another question which arises in this solution is the choice of the function $1/2\,(1-e^{-jm\pi})$ used in equation 19. Another possible choice is $1/4(1-e^{-jm\pi})^2$ which leads to

$$F_1(p)=\frac{(1-e^{-p\pi})^2}{4p\cosh(p\pi/3)} \tag{27}$$

where

$(1/T)F_1(0)=0$

If this quite different value of $F_1(p)$ is used in place of the $F(p)$ of equation 2 and evaluated at the poles of $F_1(p)$ to obtain the sum function, it will be found that equation 25 again results. This remarkable result that the same $f(t)$ results from two quite different $F(p)$'s may be explained in the following manner. From equations 21 and 27 let

$$F_2(p)=F(p)-F_1(p)$$

$$=\frac{\pi(1-e^{-p\pi})}{2p\cosh(p\pi/3)}-\frac{\pi(1-e^{-p\pi})^2}{4p\cosh(p\pi/3)}$$

$$=\frac{\pi(1-e^{-p2\pi})}{4p}+\frac{\pi(1-e^{-p2\pi})[1-\cosh(p\pi/3)]}{4p\cosh\left(\frac{p\pi}{3}\right)} \tag{28}$$

When $F_2(p)$ from equation 28 is substituted in equation 2 the first term of equation 28 yields $(\pi/4)$ which is the same $(\pi/4)$ as that of equations 22 and 23, while the second term of equation 28 yields zero. The fact that certain direct transforms that might be called null transforms yield zero when evaluated in the inverse transform has been discussed previously.[10,12] Therefore, both $F(p)$ and $F_1(p)$ should yield the same $f(t)$.

It appears as if this method will be very useful for Fourier series arising from the problems of circuits containing concentrated circuit parameters, but not quite as useful for those of circuits containing distributed parameters. In the result of equation 25, Fourier series were obtained because the poles of $F(p)$ lay on the imaginary axis. In general, the poles of $F(p)$ will lie anywhere in the p plane and the series will contain exponential terms as well as the trigonometric terms and thus will not be Fourier series.

Tables of Fourier Series

From the foregoing theory it appears then that for a given function of a complex variable, $F(p)$, the corresponding time function, $f(t)$, may be obtained either as a Fourier series or the sum function of the Fourier series. Table I presents a short table of basic $F(p)$'s with their corresponding sum functions, but much more complete tables are available in the literature.[12-15]

Table I. Steady-State Transforms

Direct Transform, F(p)	Sum Function, f(t)
1. $\frac{e^{-pkT}}{p}$, $0\le K\le 1$	$0,\ 0<t<KT$; $1,\ kT<t<T$
2. $\frac{e^{-pkT}}{p+\alpha}$, $0\le K\le 1$	$\frac{e^{-\alpha(t-KT)}}{1-e^{-\alpha T}},\ KT<t<(KT+T)$
3. $\frac{(\alpha_1 p+\alpha_0)e^{-pKT}}{(p+\alpha)(p+\beta)}$, $0\le K\le 1$	$\frac{(\alpha_0-\alpha_1\alpha)e^{-\alpha(t-KT)}}{(\beta-\alpha)(1-e^{-\alpha T})}+\frac{(\alpha_0-\alpha_1\beta)e^{-\beta(t-KT)}}{(\alpha-\beta)(1-e^{-\beta T})},\ KT<t<(KT+T)$
4. $\frac{(\alpha_1 p+\alpha_0)e^{-pKT}}{p(p+\alpha)}$, $0\le K\le 1$	$-\frac{(\alpha_0-\alpha_1\alpha)e^{-\alpha(t-KT+T)}}{\alpha(1-e^{-\alpha T})},\ 0<t<KT$; $\frac{\alpha_0}{\alpha}-\frac{(\alpha_0-\alpha_1\alpha)e^{-\alpha(t-KT)}}{\alpha(1-e^{-\alpha T})},\ KT<t<T$
5. $\frac{(\alpha_1 p+\alpha_0)e^{-pKT}}{p^2}$, $0\le K\le 1$	$0,\ 0<t<KT$; $\alpha_1+\alpha_0(t-KT),\ KT<t<T$
6. $\frac{(\alpha_1 p+\alpha_0)e^{-pKT}}{(p+\alpha)^2}$, $0\le K\le 1$	$\frac{[(\alpha_0-\alpha_1\alpha)(t-KT)+\alpha_1]e^{-\alpha(t-KT)}}{(1-e^{-\alpha T})}+\frac{(\alpha_0-\alpha_1\alpha)Te^{-\alpha(t-KT+T)}}{(1-e^{-\alpha T})^2},\ KT<t<(KT+T)$

As an example of the use of Table I let

$$F(p)=\frac{e^{-pKT}}{p+\alpha}\quad 0\le K\le 1 \tag{29}$$

which is item 2 of the table. This $F(p)$ has one pole at $p=-\alpha$ and by use of equation 4 and Table I,

$$f(t)=\frac{1}{\alpha T}+\frac{2}{T}\sum_{n=1}^{\infty}Re\left[\frac{e^{-j2\pi nK}}{\alpha+(j2\pi n/T)}\right]\cos(2\pi nt/T)-\frac{2}{T}\sum_{n=1}^{\infty}Im\left[\frac{e^{-j2\pi nK}}{\alpha+(j2\pi n/T)}\right]\sin(2\pi nt/T)$$

$$=\frac{e^{-\alpha(t-KT)}}{1-e^{-\alpha T}},\quad KT<t<(K+1)T \tag{30}$$

In equation 30 let $x=(2\pi t/T)$, $x_0=2^{\pi}K$, and $a=(\alpha T/2\pi)$, then

$$\sum_{n=1}^{\infty}\frac{a\cos n(x-x_o)+n\sin n(x-x_o)}{n^2+a^2}=\frac{\pi e^{-a(x-x_0)}}{1-e^{-2\pi a}}-\frac{1}{2a},\quad x_o<x<x_o+2\pi \tag{31}$$

Results similar to equation 31 may be obtained for the other transforms of Table I, and these along with equation 31 are shown in Table II.

A second example is that of item 1 of Table I where

$$F(p)=(e^{-pKT}/p),\quad 0\le K\le 1 \tag{32}$$

This $F(p)$ has a single pole at $p=0$ so equation 4 may not be used but we obtain the Fourier series from equation 2:

$$f(t)=\frac{1}{2\pi j}\int_W\frac{e^{p(t-KT)}dp}{p(1-e^{-pT})}$$

$$=\frac{d}{dp}\left[\frac{pe^{p(t-KT)}}{1-e^{-pT}}\right]_{p=0}+\frac{1}{T}\sum_{n=-\infty}^{\infty}{}'\frac{e^{[j2\pi n(t-KT)/T]}}{(j2\pi n/T)}$$

$$=\frac{t}{T}-K+\frac{1}{2}+\frac{1}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin[2\pi n(t-KT)/T]$$

$$=\begin{cases}0, & 0<t<KT\\ 1, & KT<t<T\end{cases} \tag{33}$$

The prime over the summation sign indicates that the $n=0$ term is omitted. With $x=(2\pi t/T)$ and $x_0=2\pi K$, equation 33 becomes

$$\sum_{n=1}^{\infty}\frac{1}{n}\sin n(x-x_o)=\frac{\pi-(x-x_c)}{2},\quad x_o<x<(x_o+2\pi) \tag{34}$$

Table II. Sum Functions of Basic Fourier Series

	Fourier Series	Sum Functions, $x_0<x<(x_0+2\pi)$
1.	$\sum_{n=1}^{\infty}\frac{1}{n}\sin n(x-x_0)$	$\frac{\pi-(x-x_0)}{2}$
2.	$\sum_{n=1}^{\infty}\frac{a\cos n(x-x_0)+n\sin n(x-x_0)}{n^2+a^2}$	$\frac{\pi e^{-a(x-x_0)}}{1-e^{-2\pi a}}-\frac{1}{2a}$
3.	$\sum_{n=1}^{\infty}\frac{[n^2(a_1a+a_1b-a_0)+aba_0]\cos n(x-x_0)}{(n^2+a^2)(n^2+b^2)}$	$\frac{\pi(a_0-a_1a)e^{-a(x-x_0)}}{(b-a)(1-e^{-2\pi a})}$
	$+\sum_{n=1}^{\infty}\frac{[a_1n^3+(a_0a+a_0b-a_1ab)n]\sin n(x-x_0)}{(n^2+a^2)(n^2+b^2)}$	$+\frac{\pi(a_0-a_1b)e^{-b(x-x_0)}}{(a-b)(1-e^{-2\pi a})}-\frac{a_0}{2ab}$
4.	$\sum_{n=1}^{\infty}\frac{(a-a_0)\cos n(x-x_0)}{n^2+a^2}$	$\frac{\pi(a_1a-a_0)e^{-a(x-x_0)}}{a(1-e^{-2\pi a})}$
	$+\sum_{n=1}^{\infty}\frac{[a_1n^3+a_0an]\sin n(x-x_0)}{n^2(n^2+a^2)}$	$+\frac{a_0-a_1a}{2a^2}+\frac{a_0}{2a}[\pi-(x-x_0)]$
5.	$\sum_{n=1}^{\infty}\left[\frac{a_1\sin n(x-x_0)}{n}-\frac{a_0\cos n(x-x_0)}{n^2}\right]$	$\frac{a_1}{2}[\pi-(x-x_0)]$
		$-\frac{a_0}{4}\left\{[\pi-(x-x_0)]^2-\frac{\pi^2}{3}\right\}$
6.	$\sum_{n=1}^{\infty}\frac{[n^2(2aa_1-a_0)+a^2a_0]\cos n(x-x_0)}{(n^2+a^2)^2}$	$\frac{\pi[(a_0-a_1a)(x-x_0)+a_1]e^{-a(x-x_0)}}{(1-e^{-2\pi a})}$
	$+\sum_{n=1}^{\infty}\frac{[a_1n^3+(2aa_0-a^2a_1)n]\sin n(x-x_0)}{(n^2+a^2)^2}$	$+\frac{2\pi^2(a_0-a_1a)e^{-a(x-x_0+2\pi)}}{(1-e^{-2\pi a})^2}-\frac{a_0}{2a^2}$

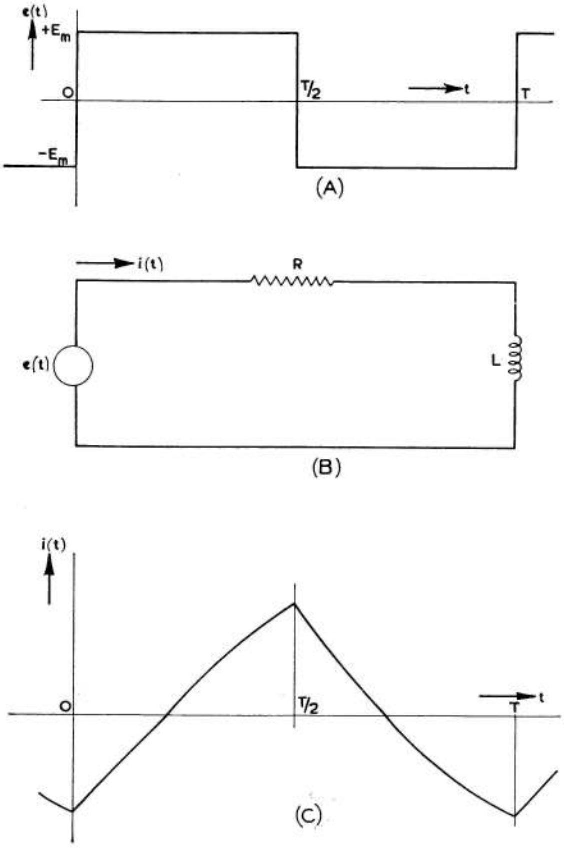

Fig. 4. First example of a circuit problem

A—Applied wave of voltage
B—Circuit diagram
C—Steady-state current with $(RT/L)=1.0$

which is item 1 of Table II. The result, equation 34, may also be obtained from equation 31 by letting a approach zero. From item 2 of Table II it is possible to obtain all of the other items of the table. Additional and more involved series and their sum functions may be obtained by using more involved steady-state transforms than those of Table I, or by obtaining them from combinations or limiting cases of the series given in Table II.

Simpler series may be obtained from those of Table II by various manipulations. As one example put $x_0=-\pi$ in item 2 of Table II. Then

$$\sum_{n=1}^{\infty}(-1)^n\left[\frac{a\cos nx+n\sin nx}{n^2+a^2}\right]=\frac{\pi e^{-ax}}{2\sin ha\pi}-\frac{1}{2a},\quad -\pi<x<\pi \tag{35}$$

In equation 35 $x=-x'$ then drop primes.

$$\sum_{n=1}^{\infty}(-1)^n\left[\frac{a\cos nx-n\sin nx}{n^2+a^2}\right]=\frac{\pi e^{+ax}}{2\sinh a\pi}-\frac{1}{2a},\quad -\pi<x<\pi \tag{36}$$

From equations 35 and 36:

$$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2+a^2}=\frac{\pi\cosh ax}{2a\sinh a\pi}-\frac{1}{2a^2},\quad -\pi<x<\pi \tag{37}$$

$$\sum_{n=1}^{\infty}(-1)^n\frac{n\sin nx}{n^2+a^2}=-\frac{\pi\sinh ax}{2\sinh a\pi},\quad -\pi<x<\pi \tag{38}$$

Equations 37 and 38 are presented as items 1 and 2 of Table III. Similarly, if $x_0=0$ in item 2 of Table II, items 3 and 4 of Table III are the result. In equation 37 let $a\rightarrow 0$ and the result is item 6 of Table III. In a similar manner, letting $a\rightarrow 0$ in items 1, 3, and 4 of Table III produces items 5, 7, and 8 of the same table. Items 5 and 7 may be recognized as the well-known saw-tooth waveforms, while items 6 and 8 have a parabolic wave shape. If item 1 is subtracted from item 3, item 9 results. Then if, in item 9, $a\rightarrow 0$, item 11 is produced and this is the familiar square wave. In the same way items 10 and 12 may be obtained from items 2 and 4.

Additional derived Fourier series similar to those of Table III may be obtained by methods such as those already given and others. Two examples of more complicated series are derived in the Appendix and a table of approximately 250 such Fourier series is available.[16,17] There are also available several shorter tables.[3,18,19]

The first application will be that of the square wave voltage of Fig. 4(A) applied to the series RL circuit of Fig. 4(B), and the purpose is to obtain the waveform of the current.

From item 11 of Table III

$$e(t)=\frac{4E_m}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega t}{(2n-1)} \tag{39}$$

is the Fourier series of the applied voltage where E_m is the maximum voltage of the square wave, $\omega=(2\pi/T)$ is the angular frequency of the wave, and t is the time. From Fig. 4(B) $Z=R+j\omega L=\sqrt{R^2+\omega^2L^2}\underline{/\arctan(\omega L/R)}$ and the Fourier series for the steady-state current is

$$i(t)=\frac{4E_m}{\pi}\sum_{n=1}^{\infty}\sin\frac{\{(2n-1)\omega t-\arctan[(2n-1)\omega L/R]\}}{(2n-1)\sqrt{R^2+(2n-1)^2\omega^2L^2}}$$

$$=\frac{4E_mR}{\pi}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega t}{(2n-1)[R^2+(2n-1)^2\omega^2L^2]}-\frac{4E_m\omega L}{\pi}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\omega t}{[R^2+(2n-1)^2\omega^2L^2]} \tag{40}$$

By using partial fractions the current expression can be written:

$$i(t)=\frac{4E_m}{\pi R}\sum_{n=1}^{\infty}\frac{\sin(2n-1)\omega t}{(2n-1)}-\frac{4E_m}{\pi R}\sum_{n=1}^{\infty}\frac{(2n-1)\sin(2n-1)\omega t}{(2n-1)^2+a^2}-\frac{4E_m}{\pi\omega L}\sum_{n=1}^{\infty}\frac{\cos(2n-1)\omega t}{(2n-1)^2+a^2} \tag{41}$$

where $a=(R/\omega L)$. Now by the use of items 11, 9, and 10 of Table III, the sum function of the steady-state current is for $0<t<(T/2)$

$$i(t)=\frac{4E_m}{\pi R}\left(\frac{\pi}{4}\right)-\frac{4E_m}{\pi R}\left(\frac{\pi}{4\sinh a\pi}\right)\times[\sinh a\omega t-\sinh a(\omega t-\pi)]-\frac{4E_m}{\pi\omega L}\left(\frac{\pi}{4a\sinh a\pi}\right)\times[\cosh a(\omega t-\pi)-\cosh a\omega t]$$

$$=\frac{E_m}{R}\left[1-\frac{2e^{-(Rt/L)}}{1+e^{-(RT/2L)}}\right] \tag{42}$$

In a similar manner for $(T/2)<t<T$

$$i(t)=-\frac{E_m}{R}\left\{1-\frac{2e^{-[R(t-T/2)/L]}}{1+e^{(RT/2L)}}\right\} \tag{43}$$

The current waveform for $(RT/L)=1.0$ is shown in Fig. 4(C).

Table III. Derived Fourier Series

	Series	Sum Functions
1.	$\sum_{n=1}^{\infty}(-1)^n\frac{n\sin nx}{n^2+a^2}$	$-\frac{\pi\sinh ax}{2\sinh a\pi}$, $-\pi<x<\pi$
2.	$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2+a^2}$	$\frac{\pi\cosh ax}{2a\sinh a\pi}-\frac{1}{2a^2}$, $-\pi<x<\pi$
3.	$\sum_{n=1}^{\infty}\frac{n\sin nx}{n^2+a^2}$	$-\frac{\pi\sinh a(x-\pi)}{2\sinh a\pi}$, $0<x<2\pi$
4.	$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+a^2}$	$\frac{\pi\cosh a(x-\pi)}{2a\sinh a\pi}$, $0<x<2\pi$
5.	$\sum_{n=1}^{\infty}(-1)^n\frac{\sin nx}{n}$	$-\frac{x}{2}$, $-\pi<x<\pi$
6.	$\sum_{n=1}^{\infty}(-1)^n\frac{\cos nx}{n^2}$	$\frac{1}{4}\left(x^2-\frac{\pi^2}{3}\right)$, $-\pi<x<\pi$
7.	$\sum_{n=1}^{\infty}\frac{\sin nx}{n}$	$\frac{\pi-x}{2}$, $0<x<2\pi$
8.	$\sum_{n=1}^{\infty}\frac{\cos nx}{n^2}$	$\frac{1}{4}\left[(\pi-x)^2-\frac{\pi^2}{3}\right]$, $0<x<2\pi$
9.	$\sum_{n=1}^{\infty}\frac{(2n-1)\sin(2n-1)x}{(2n-1)^2+a^2}$	$\frac{\pi}{4\sinh a\pi}[\sinh ax-\sinh a(x-\pi)]$, $0<x<\pi$
		$-\frac{\pi}{4\sinh a\pi}[\sinh a(x-\pi)-\sinh a(x-2\pi)]$, $\pi<x<2\pi$
10.	$\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2+a^2}$	$\frac{\pi}{4a\sinh a\pi}[\cosh a(x-\pi)-\cosh ax]$, $0<x<\pi$
		$-\frac{\pi}{4a\sinh a\pi}[\cosh a(x-2\pi)-\cosh a(x-\pi)]$, $\pi<x<2\pi$
11.	$\sum_{n=1}^{\infty}\frac{\sin(2n-1)x}{(2n-1)}$	$\pi/4$, $0<x<\pi$
		$-\pi/4$, $\pi<x<2\pi$
12.	$\sum_{n=1}^{\infty}\frac{\cos(2n-1)x}{(2n-1)^2}$	$\frac{\pi}{4}\left(\frac{\pi}{2}-x\right)$, $0<x<\pi$
		$-\frac{\pi}{4}\left(\frac{3\pi}{2}-x\right)$, $\pi<x<2\pi$

The second application will be that of the sine-loop voltage of Fig. 5(A) applied to the filter circuit of Fig. 5(B), and the object is to obtain the output voltage waveform. The Fourier series for the input voltage is

$$e_i(t)=\frac{2E_m}{\pi}\left[1-2\sum_{n=1}^{\infty}\frac{\cos(n2\pi t/T)}{4n^2-1}\right] \tag{44}$$

The Fourier series for the steady-state output voltage is

$$e_o(t)=\frac{2E_m}{\pi}\left\{1-2\sum_{n=1}^{\infty}\frac{\cos[(n2\pi t/T)-\arctan(nb/1-n^2a^2)]}{(4n^2-1)\sqrt{(1-n^2a^2)^2+n^2b^2}}\right\} \tag{45}$$

where $a=(2\pi\sqrt{LC}/T)$, $b=(2\pi L/TR)$. Take care that the correct sign of the square root in equation 45 is employed.

Now equation 45 can be written

$$e_o(t)=\frac{2E_m}{\pi}+\frac{4E_m}{\pi}\sum_{n=1}^{\infty}\frac{(n^2a^2-1)\cos(n2\pi t/T)}{(4n^2-1)[(n^2a^2-1)^2+n^2b^2]}-\frac{4E_m}{\pi}\sum_{n=1}^{\infty}\frac{nb\sin(n2\pi t/T)}{(4n^2-1)[(n^2a^2-1)^2+n^2b^2]}$$

$$=\frac{2E_m}{\pi}+\frac{E_m}{\pi a^2}\sum_{n=1}^{\infty}\frac{[n^2-(1/a^2)]\cos(n2\pi t/T)}{[n^2-(1/4)](n^2+A^2)(n^2+B^2)}-\frac{E_mb}{\pi a^4}\sum_{n=1}^{\infty}\frac{n\sin(n2\pi t/T)}{[n^2-(1/4)](n^2+A^2)(n^2+B^2)} \tag{46}$$

where

$$A=(1/2a^2)(b+\sqrt{b^2-4a^2})$$

$$B=(1/2a^2)(b-\sqrt{b^2-4a^2})$$

Then, by the use of equations 49, 51, 52, and 53 of the Appendix, the steady-state output voltage of equation 46 becomes for $0<t<T$:

$$e_o(t)=-\frac{E_m}{2a^2}\left\{2K_1\sin(\pi t/T)-\frac{K_2\cosh A[(2\pi t/T)-\pi]}{A\sinh A\pi}-\frac{K_3\cosh B[(2\pi t/T)-\pi]}{B\sinh B\pi}\right\}-\frac{E_m b}{2a^4}\left\{K_4\cos(\pi t/T)-\frac{K_5\sinh A[(2\pi t/T)-\pi]}{\sinh A\pi}-\frac{K_6\sinh B[(2\pi t/T)-\pi]}{\sinh B\pi}\right\} \quad (47)$$

where

$$K_1=\frac{4a^2(a^2-4)}{(a^2-4)^2+4b^2}$$

$$K_2=-\frac{4(a^2A^2+1)}{a^2(4A^2+1)(A^2-B^2)}$$

$$K_3=\frac{4(a^2B^2+1)}{a^2(4B^2+1)(A^2-B^2)}$$

$$K_4=\frac{16a^4}{(a^2-4)^2+4b^2}$$

$$K_5=\frac{4}{(4A^2+1)(A^2-B^2)}$$

$$K_6=-\frac{4}{(4B^2+1)(A^2-B^2)}$$

An algebraic manipulation of equation 47 produces the following, for $0<t<T$,

$$e_o(t)=4E_m[(4-a^2)^2+4b^2]^{-1}\times[(4-a^2)\sin(\pi t/T)-2b\cos(\pi t/T)]-\frac{4E_m}{a^2(A-B)}\left[\frac{e^{-(A2\pi t/T)}}{(4A^2+1)(1-e^{-2\pi A})}-\frac{e^{-(B2\pi t/T)}}{(4B^2+1)(1-e^{-2\pi B})}\right] \quad (48)$$

The output voltage waveform $e_0(t)$ for $a=2.0$ and $b=4.1$, as calculated from equation 48 is shown in Fig. 5(C). Much of the algebraic manipulation, such as the use of partial fractions, could be eliminated by the use of a more extensive table of series similar to the one mentioned previously[16] and to be employed in place of Table III.

Conclusions

The following conclusions have been reached by the author:

1. A method using steady-state transforms has been presented that will find the sum function of most Fourier series occurring when periodic nonsinusoidal waves are applied to linear circuits with concentrated parameters. This method, however, may not be useful in some circuits with distributed parameters.

2. The tables of Fourier series presented are useful in applications to circuit problems. A more complete table of such series would aid in lessening the amount of algebraic manipulation required in obtaining the simplest form of the sum function of a Fourier series.

Appendix. Examples of More Complicated Series

Suppose the sum function of

$$f(x)=\sum_{n=1}^{\infty}\frac{(\alpha_4n^4+\alpha_2n^2+\alpha_0)\cos nx}{(n^2+a^2)(n^2+b^2)(n^2+c^2)} \quad (49)$$

is to be found where the α's and a^2, b^2, c^2 are real constants. Use the method of partial fractions to resolve equation 49 into the sum of three series

$$f(x)=K_1\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+a^2}+K_2\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+b^2}+K_3\sum_{n=1}^{\infty}\frac{\cos nx}{n^2+c^2} \quad (50)$$

where

$$K_1=\frac{\alpha_0-\alpha_2a^2+\alpha_4a^4}{(b^2-a^2)(c^2-a^2)}$$

$$K_2=\frac{\alpha_0-\alpha_2b^2+\alpha_4b^4}{(a^2-b^2)(c^2-b^2)}$$

$$K_3=\frac{\alpha_0-\alpha_2c^2+\alpha_4c^4}{(a^2-c^2)(b^2-c^2)}$$

Then, by the use of item 4 of Table III, the sum function of equation 49 is for $0<x<2\pi$

$$f(x)=\frac{\pi}{2}\left[\frac{K_1\cosh a(x-\pi)}{a\sinh a\pi}+\frac{K_2\cosh b(x-\pi)}{b\sinh b\pi}+\frac{K_3\cosh c(x-\pi)}{c\sinh c\pi}\right]-\frac{\alpha_0}{2a^2b^2c^2} \quad (51)$$

A second example is

$$f(x)=\sum_{n=1}^{\infty}\frac{(\alpha_5n^5+\alpha_3n^3+\alpha_1n)\sin nx}{(n^2+a^2)(n^2+b^2)(n^2+c^2)} \quad (52)$$

The sum function of this series could be found in the same way as that of equation 49, but perhaps a quicker way is to take the derivative of equations 49 and 51 and with respect to x and then change the constants to those of equation 52. The result for $0<x<2\pi$ is

$$f(x)=-\frac{\pi}{2}\left[\frac{K_4\sinh a(x-\pi)}{\sinh a\pi}+\frac{K_5\sinh b(x-\pi)}{\sinh b\pi}+\frac{K_6\sinh c(x-\pi)}{\sinh c\pi}\right] \quad (53)$$

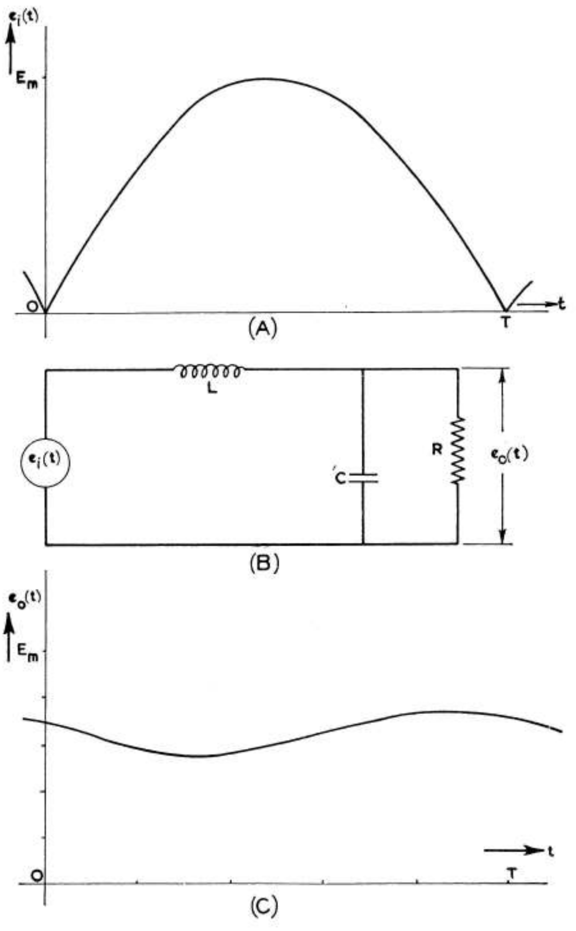

Fig. 5. Second example of a circuit problem

A—Applied sine-loop voltage
B—Circuit diagram
C—Steady-state output voltage with A=2.0 and B=4.1

where

$$K_4=\frac{\alpha_1-\alpha_3a^2+\alpha_5a^4}{(b^2-a^2)(c^2-a^2)}$$

$$K_5=\frac{\alpha_1-\alpha_3b^2+\alpha_5b^4}{(a^2-b^2)(c^2-b^2)}$$

$$K_6=\frac{\alpha_1-\alpha_3c^2+\alpha_5c^4}{(a^2-c^2)(b^2-c^2)}$$

Equations 52 and 53 can be found also by the method employing steady-state transforms as exemplified by the finding of the sum functions of equation 5.

Evaluation of the Inverse Steady-State Transform

The inverse steady-state transform, as given by equation 2 may be evaluated in two principal ways, the first of which leads to the Fourier series, and the second of which leads to the sum function, i.e., the closed-form of the sum of the Fourier series. These methods of evaluation may be found in a number of references[10,11,20,21] but are presented again here for convenience.

The Fourier series evaluation may be obtained in the following manner. The contour W as shown in Fig. 1 is assumed to contain only the poles $p=p_n=(j2\pi n/T)$, where n is an integer ($n=0, \pm 1, \pm 2, \ldots$). These are the poles of the function $1/(1-e^{-pT})$. When 2 is evaluated at the pole $p=p\acute{e}=(j2\pi n/T)$, the residue is

$$\underset{p=p_n}{\text{Residue at}}=\frac{1}{T}F(j2\pi n/T)e^{j(2\pi nt/T)} \tag{54}$$

and the sum of the residues is the Fourier series of equation 3. As an example consider item 2 of Table I

$$F(p)=\frac{e^{-pKT}}{p+\alpha},\ 0\leq K\leq 1 \tag{55}$$

For equation 55 the residue at p_n is

$$\underset{p=p_n}{\text{Residue at}}=\frac{1}{T}\left[\frac{e^{-j2\pi nK}}{(j2\pi n/T)+\alpha}\right]e^{j(2\pi nt/T)} \tag{56}$$

The Fourier series then is

$$S^{-1}[F(p)=f(t)$$
$$=\frac{1}{T}\sum_{n=-\infty}^{+\infty}\left[\frac{e^{-j2\pi nK}}{\alpha+(j2\pi n/T)}\right]e^{j(2\pi nt/T)} \tag{57}$$

which may also be written as the Fourier series of equation 30. The sum function evaluation consists in separating the integral of equation 2 into two integrals, the first of which is taken over the part W_1 of the contour W to the right of the imaginary axis as shown in Fig 1. The second integral is taken over the part W_2 of the contour W to the left of the imaginary axis of Fig. 1. For convenience the contour W_2 will be reversed in direction and called W_3 and the sign on the integral will then be negative. Then from equation 2

$$S^{-1}[F(p)]=\frac{1}{2\pi j}\int_{W_1}\frac{e^{pt}F(p)dp}{1-e^{-pT}}-\frac{1}{2\pi j}\int_{W_3}\frac{e^{pt}F(p)dp}{1-e^{-pT}} \tag{58}$$

The contour W_1 is the same as that used in the inverse Laplace transform and hence may be evaluated using L^{-1} as the symbol for the inverse Laplace transform as follows:

$$\frac{1}{2\pi j}\int_{W_1}\frac{e^{pt}F(p)dp}{1-e^{-pT}}=L^{-1}\left[\frac{F(p)}{1-e^{-pT}}\right]$$
$$=L^{-1}[F(p)]+L^{-1}[F(p)e^{-pT}]+L^{-1}[F(p)e^{-p2T}]+\ldots \tag{59}$$

For the interval $0<t<T$, equation 59 becomes

$$\frac{1}{2\pi j}\int_{W_1}\frac{e^{pt}F(p)}{1-e^{-pT}}dp=L^{-1}[F(p)] \tag{60}$$

since all of the other terms are zero.

The second integral of equation 58 must be evaluated at all of the poles of the integrand except the $p=p_n$ included within the contour W. If now $F(p)$ is assumed to have the poles $p=s_m$, where m is a positive integer, then the second integral of equation 58 becomes

$$\frac{1}{2\pi j}\int_{W_3}\frac{e^{pt}F(p)dp}{1-e^{-pT}}=\sum_m\left[\frac{e^{pt}F(p)(p-s_m)}{1-e^{-pT}}\right]_{p=s_m} \tag{61}$$

From equations 58, 60, and 61 and for $0<t<T$

$$S^{-1}[F(p)]=L^{-1}[F(p)]-\sum_m\left[\frac{e^{pt}F(p)(p-s_m)}{1-e^{-pT}}\right]_{p=s_m} \tag{62}$$

Consider again the example of equation 55. Here $m=1$, $s_1=-\alpha$, and using equation 29 for $0<t<T$

$$S^{-1}[F(p)]=f(t)=L^{-1}\left[\frac{e^{-pKT}}{p+\alpha}\right]-\left[\frac{e^{pt}e^{-pKT}}{1-e^{-pT}}\right]_{p=-\alpha}$$

Now for $0<t<KT$

$$f(t)=0-\frac{e^{-\alpha t}e^{\alpha KT}}{1-e^{\alpha T}}=\frac{e^{-\alpha(t-KT+T)}}{1-e^{-\alpha T}} \tag{63}$$

and for $KT<t<T$

$$f(t)=e^{-\alpha(t-KT)}-\frac{e^{-\alpha t}e^{\alpha KT}}{1-e^{\alpha T}}=\frac{e^{-\alpha(t-KT)}}{1-e^{-\alpha T}} \tag{64}$$

Actually, equation 64 may be used until the next discontinuous jump occurs at $t=(K+1)T$ and this is indicated in equation 30. Hence, equation 63 and equation 64 are the sum function of the Fourier series of equation 57.

A physical interpretation of equation 58 may prove helpful.[20] The integral over W_1 is the usual inverse Laplace transform of the response of a system, and hence is the sum of the steady-state plus the transient response and might be called the total response. The steady-state response would be the inverse steady-state transform on the left-hand side of equation 58. Thus, the integral over W_3 must represent the transient response. Then equation 58 might be written: (steady-state response) = (total response) − (transient response).

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