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Circulant Interval Valued Fuzzy Matrices

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Abstract. Circulant matrix is a square matrix whose rows are obtained by cyclically rotating by its first row. In this paper, we define some operations on circulant interval valued fuzzy matrices (CIVFMs). Some elementary operators on circulant interval valued fuzzy matrices (CIVFMs) are presented here. The idea of reflexive, symmetric, transitive, determinant and adjoint of circulant interval valued fuzzy matrices (CIVFMs) are also defined.

Keywords. Circulant matrices, circulant interval valued fuzzy matrices determinant, fuzzy matrices, interval valued fuzzy matrices.

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1. Introduction

The concept of fuzzy set was introduced by Zadeh [17] in 1965. Fuzzy Matrix (FM) is a very important topic in Fuzzy Algebra. In FM, the elements belongs to the interval [0,1]. When the elements of FM are subintervals of the unit interval [0,1], then the FM is known as interval - valued Fuzzy Matrix(IVFM).

Thomason [16] defined fuzzy Matrices for the first time in 1977 and discussed about the convergence of the powers of fuzzy matrix. Several authors presented number of results on the convergence of power sequence of fuzzy matrices [3, 6]. Ragab and Emam [12] presented some properties on determinant and adjoint of square fuzzy matrix. Ragab and Emam [13] introduced some properties of the min-max composition of fuzzy matrix. Kim [5] investigated some important results on determinant of a square fuzzy matrices.

Pal [9,19] introduced the concept of interval-valued fuzzy matrices with interval-valued fuzzy rows and columns. Shyamal and Pal [14,15,18] introduced two new operators and applications of fuzzy matrices. Bhowmik and Pal [2,20] introduced the concept of circulant triangular fuzzy number matrices (TFNMs) and some result on TNFMs. Hemasinha, Pal and Bezdek [4] studies the max-min iterates of fuzzy circulant matrices.

In this paper, the concept of circulant interval -valued fuzzy matrices (CIVFMs) are defined with some of its properties. The determinant, some binary operations on circulant interval -valued fuzzy matrix are defined and some important theorems are proved with examples.

2. Circulant interval valued fuzzy matrices

Definition 2.1. [14] An m×n matrix $A = [a_{ij}]$ whose components are in the unit interval $[0,1]$ is called a fuzzy matrix.

Definition 2.2. The determinant $|A|$ of an $m\times n$ fuzzy matrix A is defined as follows;
$|A| = \sum_{\sigma\in S_n} a_{1_{\sigma(1)}} a_{2_{\sigma(2)}} \ldots\ldots a_{n_{\sigma(n)}}$ where S_n denotes the symmetric group of all permutations of the indices $\{1,2,\ldots.n\}$.

Definition 2.3. [9] An interval-valued fuzzy matrix(IVFM) of order m×n is defined as $A = (a_{ij})_{m\times n}$ where $a_{ij} = [a_{ijL}, a_{ijU}]$ is the ij^{th} element of A represents the membership value. All the elements of an IVFM are intervals and all the intervals are the subintervals of the interval $[0,1]$.

Definition 2.4. [12] The interval-valued fuzzy determinant (IVFD) of an IVFM A of order n×n is denoted by $|A|$ or $det(A)$ and we defined as

$$|A| = \sum_{\sigma\in S_n} a_{1_{\sigma(1)}} a_{2_{\sigma(2)}} \ldots\ldots\ldots a_{n_{\sigma(n)}} = \sum_{\sigma\in S_n} \prod_{i=1}^{n} a_{i_{\sigma(i)}} .$$

where $a_{i_{\sigma(i)}} = [a_{i_{\sigma(i)L}}, a_{i_{\sigma(i)U}}]$ and S_n denotes the symmetric group of all permutations of the indices $\{1, 2, \ldots\ldots n\}$.

Definition 2.5. [2] An Interval Valued Fuzzy Matrix (IVFM) A is said to be circulant interval valued fuzzy matrix if all the elements of A can be determined completely by its first row.

The first row of A is $[[a_{1L}, a_{1U}] \quad [a_{2L}, a_{2U}] \ldots\ldots\ldots\ldots \quad [a_{nL}, a_{nU}]]$

Then any element $a_{ij} = [a_{ijL}, a_{ijU}]$ of A can be determined (throughout the element of the first row) as $a_{ij} = a_{1(n-i+j+1)}$ with $a_{1(n+k)} = a_{1k}$

A circulant IVFM is of the form

$$A = \begin{bmatrix} [a_{1L}, a_{1U}] & [a_{2L}, a_{2U}] \ldots\ldots & \ldots\ldots [a_{nL}, a_{nU}] \\ [a_{nL}, a_{nU}] & [a_{1L}, a_{1U}] \ldots\ldots & [a_{(n-1)L}, a_{(n-1)U}] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & & \\ [a_{3L}, a_{3U}] & [a_{4L}, a_{4U}] \ldots\ldots\ldots & [a_{2L}, a_{2U}] \\ [a_{2L}, a_{2U}] & [a_{3L}, a_{3U}] \ldots\ldots\ldots & [a_{1L}, a_{1U}] \end{bmatrix}.$$

Remark 2.1. An IVFM A is circulant if and only if
$[a_{ijL}, a_{ijU}] = [a_{(k\oplus i)(k\oplus j)L}, a_{(k\oplus i)(k\oplus j)U}]$, for every $i, j, k \in \{1,2,3, \ldots\ldots n\}$, where $\oplus$ is sum modulo n. Here all the diagonal elements are equal.

Remark 2.2. For a circulant IVFM A we notice that $[a_{inL}, a_{inU}] = [a_{(i\oplus 1)1L}, a_{(i\oplus 1)1U}]$ and $[a_{njL}, a_{njU}] = [a_{1(j\oplus 1)L}, a_{1(j\oplus 1)U}]$, for every $i, j \in \{1,2, \ldots\ldots\ldots n\}$.

Remark 2.3. For a circulant IVFM A we notice that
$\left[a_{(i\oplus(n-1))j\,L,}\, a_{(i\oplus(n-1))j\,U}\right] = \left[a_{i(j\oplus 1)L},\ a_{i(j\oplus 1)}\ a_{i(j\oplus 1)U}\right]$ for every $i,\ j \in \{1, 2, \ldots\ldots\ldots\ldots n\}$.

Theorem 2.1. An $n \times n$ IVFM A is circulant if and only if $AC_n = C_nA$, where C_n is the permutation matrix of IVFM.

$$C_n = \begin{bmatrix} [0,0] & [0,0] \ldots\ldots\ldots & [0,0] & [1,1] \\ [1,1] & [0,0] \ldots\ldots\ldots & [0,0] & [0,0] \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ [0,0] & [0,0] \ldots\ldots\ldots & [0,0] & [0,0] \\ [0,0] & [0,0] \ldots\ldots\ldots & [1,1] & [0,0] \end{bmatrix}. \qquad .$$

Proof: Let A be an IVFM and $X = AC_n$, then $x_{ij} = \left[x_{ijL,}\, x_{ijU}\right] = \sum_{k=1}^{n} a_{ik}c_{kj}$
In the first row, only c_{1n} is [1,1] and all the other elements are [0,0].Therefore we get $x_{ij} = a_{1(j\oplus 1)}$.
Let $Y = C_nA$, then $y_{ij} = \left[y_{ijL,}\, y_{ijU}\right] = \sum_{k=1}^{n} c_{ik}a_{kj} = a_{(i\oplus(n-1))j}$.
By remark (2.3), $x_{ij} = y_{ij}$ for all $i, j \in \{1,2, \ldots\ldots\ldots\ldots n\}$.
Hence $C_n = C_nA$. Therefore we get A is circulant IVFM.
The converse is straightforward.

Example 2.1. Let A and C are two circulant IVFM of order 3×3, where

$$A = \begin{bmatrix} [.3,\ .6] & [.7,\ .8] & [.4,\ .8] \\ [.4,\ .8] & [.3,\ .6] & [.7,\ .8] \\ [.7,\ .8] & [.4,\ .8] & [.3,\ .6] \end{bmatrix} \text{ and } C = \begin{bmatrix} [0,0] & [0,0] & [1,1] \\ [1,1] & [0,0] & [0,0] \\ [0,0] & [1,1] & [0,0] \end{bmatrix}.$$

Then $CA = \begin{bmatrix} [.7,\ .8] & [.4,\ .8] & [.3,\ .6] \\ [.3,\ .6] & [.7,\ .8] & [.4,\ .8] \\ [.4,\ .8] & [.3,\ .6] & [.7,\ .8] \end{bmatrix}$ and $AC = \begin{bmatrix} [.7,\ .8] & [.4,\ .8] & [.3,\ .6] \\ [.3,\ .6] & [.7,\ .8] & [.4,\ .8] \\ [.4,\ .8] & [.3,\ .6] & [.7,\ .8] \end{bmatrix}$.

Therefore $CA = AC$.

Theorem 2.2. For the circulant IVFM A and B, $i)$ $A + B$ is a circulant IVFM. $ii)$ A' is a circulant IVFM, $iii)$ AB is also a circulant IVFM. In particular A^k is also a circulant IVFM.

Proof: i) Proof is straight forward
ii) Since A is circulant IVFM, then A commetes with C_n.
Therefore we get $AC_n = C_nA$. Transpossing both sides, we get $C_n'A' = A'C_n'$,
Premultiply by C_n, we get $C_nC_n'A' = C_nA'C_n' \Rightarrow A' = C_nA'C_n'$
Postmultiply by C_n, we get $A'C_n = C_nA'C_n'C_n = C_nA'$. Hence $A'C_n = C_nA'$.
By theorem 2.1, we have A' is circulant IVFM.
iii) Since A and B are circulant IVFM, each of A and B commutes with C_n.
Hence AB commutes with C_n. By remark 2.3 and theorem 2.1 we get AB is circulant IVFM.
Similarly A^k is circulant IVFM.
iv) Since A and $A'$are circulant IVFM. By remark 2.3 AA' commutes with C_n.

Hence, $A\,A'$ is circulant IVFM.

Theorem 2.3. If A and B are circulant IVFM then $AB = BA$.

Proof: Let $AB = X$ and $x_{ij} = [x_{ijL,}\, x_{ijU}]$, for $i, j \in \{1,2,\ldots\ldots\ldots\ldots n\}$,

Let $BA = Y$ and $y_{ij} = [y_{ijL,}\, y_{ijU}]$, for $i, j \in \{1,2,\ldots\ldots\ldots\ldots n\}$

Then both X and Y are circulant, by theorem 2.2(iii) and their first rows are

$[[x_{1L},\ x_{1U}] \quad [x_{2L},\ x_{2U}]\ldots\ldots\ldots\ldots\ [x_{nL}, x_{nU}]]$ and

$[y_{2L},\ y_{2U}]\ldots\ldots\ldots\ldots\ [y_{nL}, y_{nU}]]$ respectively.

Then K^{th} element of the first row of X is,

$$x_k = [x_{kL},\ x_{kU}] = \sum_{p=1}^{k} [a_{pL},\ a_{pU}][b_{(k-p+1)L},\ b_{(k-p+1)U}] + \sum_{p=k+1}^{n} [a_{pL},\ a_{pU}][b_{(n-p+k+1)L},\ b_{(n-p+k+1)U}]$$

$$= [a_{1L},\ a_{1U}][b_{kL},\ b_{kU}] + [a_{2L},\ a_{2U}][b_{(k-1)L},\ b_{(k-1)U}] + \ldots\ldots\ldots\ldots + [a_{(k-1)L},\ a_{(k-1)U}][b_{2L},\ b_{2U}] + [a_{kL},\ a_{kU}][b_{1L},\ b_{1U}] + [a_{(k+1)L},\ a_{(k+)1U}][b_{nL},\ b_{nU}] + \ldots\ldots\ldots [a_{nL},\ a_{nU}][b_{(k+1)L},\ b_{(k+1)U}]$$

K^{th} element of the first row of Y is,

$$y_k = [y_{kL},\ y_{kU}] = \sum_{p=1}^{k} [b_{pL},\ b_{pU}][a_{(k-p+1)L},\ a_{(k-p+1)U}] + \sum_{p=k+1}^{n} [b_{pL},\ b_{pU}][a_{(n-p+k+1)L},\ a_{(n-p+k+1)U}]$$

$$= [b_{1L},\ b_{1U}][a_{kL},\ a_{kU}] + [b_{2L},\ b_{2U}][a_{(k-1)L},\ a_{(k-1)U}] + \ldots\ldots\ldots\ldots + [b_{(k-1)L},\ b_{(k-1)U}][a_{2L},\ a_{2U}] + [b_{kL},\ b_{kU}][a_{1L},\ a_{1U}] + [b_{(k+1)L},\ b_{(k+)1U}][a_{nL},\ a_{nU}] + \ldots\ldots\ldots [b_{nL},\ b_{nU}][a_{(k+1)L},\ a_{(k+1)U}]$$

Hence, we get $x_k = y_k$. ie., $x_{ij} = y_{ij}$. ie., $X = Y$. Therefore we have $AB = BA$.

Theorem 2.4. If a circulant IVFM A is circulant, then DA is symmetric, where D is a permutation matrix of unit circulant IVFM.

$$D = \begin{bmatrix} [0,0] & [0,0]\ldots\ldots\ldots & [0,0]\ [1,1] \\ [0,0] & [0,0]\ldots\ldots\ldots & [1,1]\ [0,0] \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots\ldots\ldots\ldots & \ldots\ldots\ldots \\ [0,0] & [1,1]\ldots\ldots\ldots & [0,0]\ [0,0] \\ [1,1] & [0,0]\ldots\ldots\ldots & [0,0]\ [0,0] \end{bmatrix}$$

Proof: Let $M = DA$, then $m_{ij} = [m_{ijL,}\, m_{ijU}] = \sum_{k=1}^{n} d_{ik}a_{kj}$

$= \sum_{k=1}^{n} [d_{ikL},\ d_{ikU}][a_{kjL},\ a_{kjU}]$ for all $i, j \in 1, 2, \ldots\ldots\ldots\ldots n\}$

Now, D is permutation matrix of unit circulant IVFM and only the elements $d_{1n,}\ d_{2(n-1),}\ d_{3(n-2),}\ldots\ldots\ d_{n1,}$ are $[1, 1]$ and all other elements are $[0, 0]$.

Then $m_{ij} = [a_{(n-i+1)jL},\ a_{(n-i+1)jU}]$.Since A is circulant, we get

$$m_{ij} = a_{(n-i+1)j} = a_{((n-i+1)\oplus k)(k+j)}, \quad \text{for all } i, j, k = 1,2,\ldots\ldots\ldots\ldots n.$$

When $k = i$, $m_{ij} = a_{(n-i+1)j} = a_{((n-i+1)\oplus i))(i+j)} = a_{(n\oplus 1)(i\oplus j)} = a_{1(i\oplus j)}$ and

$$m_{ji} = a_{(n-j+1)i} = a_{((n-j+1)\oplus k))(k+i)}, \text{ for all } i, j, k = 1,2,\ldots\ldots\ldots\ldots n.$$

When $k = j$, $m_{ji} = a_{(n-j+1)i} = a_{((n-j+1)\oplus j))(j+i)} = a_{(n\oplus 1)(i\oplus j)} = a_{1(i\oplus j)}$
Therefore, we get $m_{ij} = m_{ji}$. Hence M is symmetric.

Theorem 2.5. For a Circulant Interval -Valued Fuzzy Matrix A, we have $adj\ A$ is circulant.
Proof. We have to prove co-factor of the elements $a_{i(j\oplus 1)}$ and $a_{(i\oplus(\mathrm{n}-1))j}$ are same.
By remark 2.3, we have $a_{i(j\oplus 1)} = a_{(i\oplus(\mathrm{n}-1))j}$. So the minor of $a_{i(j\oplus 1)}$ and $a_{(i\oplus(\mathrm{n}-1))j}$ are will be same.
co-factor of $a_{i(j\oplus 1)} = (-1)^{i\oplus(j+1)} \sum_{\sigma\in S_n} \prod_{k=1,k\neq i,k\neq j\oplus 1}^{n} [a_{k\sigma(k)L}, a_{k\sigma(k)U}]$
co-factor of
$a_{(i\oplus(\mathrm{n}-1))j} = (-1)^{(1\oplus(n-1))+j} \sum_{\sigma\in S_n} \prod_{k=1,k\neq j,k\neq(1\oplus(n-1))}^{n} [a_{k\sigma(k)L}, a_{k\sigma(k)U}]$
Now, the sign of $(-1)^{i\oplus(j+1)}$ = the sign of $(-1)^{(i\oplus(n-1))+j}$ (since n is fixed)
So, the co-factor of $a_{i(j\oplus 1)}$ and $a_{(i\oplus(\mathrm{n}-1))j}$ are same.
Hence $adj\ A$ is circulant interval -valued fuzzy matrix.

3. Operators on circulant IVFM

In this section, some operators, viz . ∧,∨ are defined and explained with numerical examples.

Definition 3.1. Let $A = [a_{ij}]_{n\times n} = [a_{ijL}, a_{ijU}]$ and $B = [b_{ij}]_{n\times n} = [b_{ijL}, b_{ijU}]$ are two circulant IVFM, then
$A \vee B = [a_{ij}] \vee [b_{ij}] = [a_{ijL}, a_{ijU}] \vee [b_{ijL}, b_{ijU}] = [a_{ijL} \vee b_{ijL}, a_{ijU} \vee b_{ijU}]$.

Theorem 3.1. If A and B are two circulant IVFM then A∨B is also circulant IVFM.
Proof: Proof is straight forward.

Definition 3.2. The ∧ operation is similar to ∨ operation.
Let $A = [a_{ij}]_{n\times n} = [a_{ijL}, a_{ijU}]$ and B=$[b_{ij}]_{n\times n} = [b_{ijL}, b_{ijU}]$ are two CIVFM,then
A∧B = $[a_{ij}] \wedge [b_{ij}] = [a_{ijL}, a_{ijU}] \wedge [b_{ijL}, b_{ijU}] = [a_{ijL} \wedge b_{ijL}, a_{ijU} \wedge b_{ijU}]$.

Theorem 3.2. If A and B are two CIVFM, then A∧B is also a CIVFM.
Proof: Proof is straight forward.

Example 3.1. Let A = $\begin{bmatrix} [.2, .4] & [.5, .8] & [.7, .9] \\ [.7, .9] & [.2, .4] & [.5, .8] \\ [.5, .8] & [.7, .9] & [.2, .4] \end{bmatrix}$ and

$$B = \begin{bmatrix} [.3, .5] & [.6, .7] & [.1, .8] \\ [.1, .8] & [.3, .5] & [.6, .7] \\ [.6, .7] & [.1, .8] & [.3, .5] \end{bmatrix}$$

Then, $A \vee B = \begin{bmatrix} [.3, .5] & [.6, .8] & [.7, .9] \\ [.7, .9] & [.3, .5] & [.6, .8] \\ [.6, .8] & [.7, .9] & [.3, .9] \end{bmatrix}$ and

A∧B = $\begin{bmatrix} [.2, .4] & [.5, .7] & [.1, .8] \\ [.1, .8] & [.2, .4] & [.5, .7] \\ [.5, .7] & [.1, .8] & [.2, .4] \end{bmatrix}$. Therefore A∨B and A∧B are CIVFM

Definition 3.3. Let A=$[a_{ij}]_{n\times n}$ = $[a_{ijL}, a_{ijU}]$ and B=$[b_{ij}]_{n\times n}$ = $[b_{ijL}, b_{ijU}]$ are two CIVFM.

Then A⊕B = $[a_{ijL} \oplus b_{ijL} - a_{ijL}.b_{ijL},\ a_{ijU} \oplus b_{ijU} - a_{ijU}.b_{ijU}]$ for all i,j

A⊙B = $[a_{ijL}.b_{ijL},\ a_{ijU}.b_{ijU}]$ for all i,j

A@B = $\left[\frac{1}{2}(a_{ijL} + b_{ijL}),\ \frac{1}{2}(a_{ijU} + b_{ijU})\right]$ for all i,j.

Definition 3.4. The **complement of CIVFM** A =$[a_{ij}]_{n\times n}$ = $[a_{ijL}, a_{ijU}]$ is defined as $A^C = [1\text{-}a_{ij}]_{n\times n}$.

Definition 3.5. An CIVFM is called **self complement** $(A^C)^C = A$.

Theorem 3.3. If A be a CIVFM, then $(A^C)^C = A$.

Proof: Let $B = A^C$. Then $b_{ij} = 1\text{-}a_{ij} = [1 - a_{ijU}, 1 - a_{ijL}]$

If $D = B^C = (A^C)^C$, then $d_{ij} = 1 - b_{ij} = [1 - b_{ijL}, 1 - b_{ijU}]$

$= [1 - (1 - a_{ijL}), 1 - (1 - a_{ijU})]$

$= [a_{ijL}, a_{ijU}] = A$. Therefore $(A^C)^C = A$.

Theorem 3.4. If a CIVFM A =$[a_{ij}]_{n\times n}$ is self complement then $a_{ijL} + a_{ijU} = 1$ for all i, j.

Proof: By the definition of complement

$$a_{ij}^c = [1,1] - [a_{ijL},\ a_{ijU}] = [1 - a_{ijU},\ 1 - a_{ijL}] = [a_{ijL},\ a_{ijU}].$$

Since A is self complement , then A^C= A. Hence $a_{ijL} + a_{ijU} = 1$, for all i, j.

Theorem 3.5. (De Morgan's laws) Let A =$[a_{ij}]_{n\times n}$ = $[a_{ijL}, a_{ijU}]$ and B =$[b_{ij}]_{n\times n}$ = $[b_{ijL}, b_{ijU}]$ are two CIVFM, then $i)$(A ∨ B$)^C = A^C \wedge B^C$, $ii)$(A ∧ B$)^C = A^C \vee B^C$.

Proof: i). Let $P =$ A ∨ B, then $p_{ij} = [a_{ij} \vee b_{ij}] = [a_{ijL} \vee b_{ijL},\ a_{ijU} \vee b_{ijU}]$

Let $Q = P^C$, then $Q_{ij} = p_{ij} = 1\text{-}a_{ij} \vee b_{ij} = [1,1] - [a_{ijL} \vee b_{ijL},\ a_{ijU} \vee b_{ijU}]$

$= [1 - a_{ijU} \vee b_{ijU},\ 1 - a_{ijL} \vee b_{ijL}]$

Let $R = A^C \wedge B^C$, then $R_{ij} = (1 - a_{ij}) \wedge (1 - b_{ij})$

$= ([1,1] - [a_{ijL}, a_{ijU}]) \wedge ([1,1] - b_{ijL}, b_{ijU})$

$= [1 - a_{ijU} \vee b_{ijU},\ 1 - a_{ijL} \vee b_{ijL}] = Q_{ij}$.

Hence (A ∨ B$)^C = A^C \wedge B^C$.

ii). Proof is similar to (i).

Example 3.2. Let A and B are two CIVFM of order 3×3, where

$$A = \begin{bmatrix} [.2,\ .4] & [.5,\ .8] & [.7,\ .9] \\ [.7,\ .9] & [.2,\ .4] & [.5,\ .8] \\ [.5,\ .8] & [.7,\ .9] & [.2,\ .4] \end{bmatrix} \text{ and } B = \begin{bmatrix} [.3,\ .5] & [.6,\ .7] & [.1,\ .8] \\ [.1,\ .8] & [.3,\ .5] & [.6,\ .7] \\ [.6,\ .7] & [.1,\ .8] & [.3,\ .5] \end{bmatrix}$$

$$(\mathrm{A} \vee \mathrm{B})^C = \begin{bmatrix} [.5,\ .7] & [.2,\ .4] & [.1,\ .3] \\ [.1,\ .3] & [.5,\ .7] & [.2,\ .4] \\ [.2,\ .4] & [.1,\ .3] & [.5,\ .7] \end{bmatrix} \qquad (3.1)$$

$$A^C = \begin{bmatrix} [.6,\ .8] & [.2,\ .5] & [.1,\ .3] \\ [.1,\ .3] & [.6,\ .8] & [.2,\ .5] \\ [.2,\ .5] & [.1,\ .3] & [.6,\ .8] \end{bmatrix} \text{ and } B^C = \begin{bmatrix} [.5,\ .7] & [.3,\ .4] & [.2,\ .9] \\ [.2,\ .9] & [.5,\ .7] & [.3,\ .4] \\ [.3,\ .4] & [.2,\ .9] & [.5,\ .7] \end{bmatrix}$$

$$A^C \wedge B^C = \begin{bmatrix} [.5,\ .7] & [.2,\ .4] & [.1,\ .3] \\ [.1,\ .3] & [.5,\ .7] & [.2,\ .4] \\ [.2,\ .4] & [.1,\ .3] & [.5,\ .7] \end{bmatrix} \qquad (3.2)$$

From (3.1) and (3.2) we get $(\mathrm{A} \vee \mathrm{B})^C = A^C \wedge B^C$.

4. Determinant of circulant interval –valued fuzzy matrix

Definition 4.1. The determinant of a CIVFM of order $n \times n$ is defined by $|A|$ and is defined as $|A| = \sum_{\sigma \in S_n} sgn\sigma \prod_{i=1}^{n} a_{i\sigma(i)}, where\ a_{i\sigma(i)} = [a_{i\sigma(i)L},\ \ a_{i\sigma(i)U}]$ is the IVFM and S_n denotes the symmetric group of all permutation of the indices $\{1,2, \dots \dots \dots .. n\}$ and $\sigma = 1\ or - 1$ according as the permutation

$\sigma = \begin{pmatrix} 1 & 2 \dots \dots \dots \dots \dots \dots n \\ \sigma(1) & \sigma(2) \dots \dots \dots \dots \dots \sigma(n) \end{pmatrix}$ is even or odd respectively.

The computation of $det(A)$ involves several product of IVFM. Since A is circulant IVFM, the value of $a_{ij} = [a_{ijL},\ \ a_{ijU}] = [a_{1(n-1+j+1)L},\ \ a_{1(n-1+j+1)U}]$, with $[a_{1(n+k)L},\ \ a_{1(n+K)U}] = [a_{1kL},\ \ a_{1kU}]$.

Theorem 4.1. If A be a CIVFM, then $A\ |adj|\ A$ is weakly reflexive.

Proof: Let $A = [a_{ij}]_{n \times n} = [a_{ijL},\ a_{ijU}]$ be a circulant IVFM with $[a_{1lL},\ a_{1lU}] \geq [a_{1iL}\ ,\ a_{1iU}]$

Let $C = A\ adj\ A$. Then C is circulant , since A and $adj\ A$ are circulant .

Now, we have $[C_{11L},\ \ C_{11U}] = \sum_{k=1}^{n} ([a_{1kL},\ a_{1kU}] . |[A_{1kL}, A_{1kU}]|)$

Here $|[A_{1lL},\ A_{1lU}]| = \sum_{\sigma \in S_{n1nl}} sgn\sigma \prod [a_{t\sigma(t)L},\ \ a_{t\sigma(t)U}]$

$= [a_{2\pi(2)L},\ \ a_{2\pi(2)U}] . [a_{3\pi(3)L},\ \ a_{3\pi(3)U}] \dots \dots \dots . [a_{n\pi(n)L},\ \ a_{n\pi(n)U}]$ for some $\pi \in s_n$

Since A is circulant, we get $[a_{1lL},\ a_{1lU}] = [a_{2(l \oplus 1)L},\ a_{2(l \oplus 1)U}]$ $= [a_{3(l \oplus 2)L},\ a_{3(l \oplus 2)U}] = \dots \dots \dots = [a_{n(l \oplus (n-1))L},\ a_{n(l \oplus (n-1))U}]$

Suppose $\pi \in S_{n1\ nl}$ be defined as

$\pi = \begin{pmatrix} 2 & 3 & 4 \dots \dots \dots \dots .. n \\ (l \oplus 1) & (l \oplus 2) & (l \oplus 3) \dots \dots \dots (l \oplus (n-1)) \end{pmatrix}$

Then $|[A_{1lL},\ A_{1lU}]| = [a_{2(l \oplus 1)L},\ a_{2(l \oplus 1)U}] . [a_{3(l \oplus 2)L},\ a_{3(l \oplus 2)U}] \dots \dots \dots [a_{n(l \oplus (n-1))L}, a_{n(l \oplus (n-1))U}] = [a_{1lL},\ a_{1lU}]$

Therefore, $[C_{11L},\ \ C_{11U}] \geq [a_{1lL},\ a_{1lU}]$. But $[a_{1lL},\ \ a_{1lU}] \geq [a_{1iL},\ a_{1iU}]$. Then $[C_{11L},\ C_{11U}] \geq [C_{1iL},\ C_{1iU}]$. Since, C is circulant, the elements of its diagonal are all equal.

Hence, $[C_{iiL}, C_{iiU}] \geq [C_{ijL}, C_{ijU}]$. Therefore $C = A\, adj\, A$ is weakly reflexive.

Theorem 4.2. If $A = [a_{ij}]_{n\times n} = [a_{ijL},\ a_{ijU}]$ be a $n \times n$ CIVFM, then CIVFM is transitive.

Proof: Let $C = A\, adj\, A$,

then $C_{ij} = [C_{ijL},\ C_{ijU}] = \sum_{k=1}^{n}([a_{ikL},\ a_{ikU}].\left|[A_{jkL},\ A_{jkU}]\right|)$
$= [a_{itL},\ a_{itU}].\left|A_{jtL},\ A_{jtU}\right|$

$(C_{ij})^2 = [C_{ijL},\ C_{ijU}]^2 = \sum_{k=1}^{n} [C_{isL},\ C_{isU}]\,[C_{sjL},\ C_{sjU}]$
$= \sum_{s=1}^{n}[(\sum_{p=1}^{n}[a_{ipL},\ a_{ipU}]\left|[A_{spL},\ A_{spU}]\right|)\ \ (\sum_{q=1}^{n}[a_{sqL},\ a_{sqU}]\left|[A_{jqL},\ A_{jqU}]\right|)]$
$= \sum_{s=1}^{n}(\ [a_{ihL},\ a_{ihU}]\,|[A_{shL},\ A_{shU}]|)\,([a_{skL},\ a_{skU}](\left|[A_{jkL},\ A_{jkU}]\right|)$
$\leq\ [a_{ihL},\ a_{ihU}]\left|[A_{jkL},\ A_{jkU}]\right| \leq\ [a_{itL},\ a_{itU}]\left|[A_{jtL},\ A_{jtU}]\right|.$

Hence, $(A\, adj\, A)^2 \leq A\, adj\, A$. Therefore, $A\, adj\, A$ is transitive.

Theorem 4.3. If $A = [a_{ij}]_{n\times n} = [a_{ijL}, a_{ijU}]$ be a circulant interval valued fuzzy matrix, then $A\, adj\, A$ is idempotent.

Proof: Let $C = A\, adj\ A$. By theorem (4.2), $[C_{ijL},\ C_{ijU}]^2 \leq [C_{ijL},\ C_{ijU}]$

But $[C_{ijL},\ C_{ijU}]^2 = \sum_{k=1}^{n}[C_{ikL},\ C_{ikU}]\,[C_{kjL},\ C_{kjU}] \geq [C_{iiL},\ C_{iiU}][C_{ijL},\ C_{ijU}]$
$= [C_{ijL},\ C_{ijU}]$

Therefore we get, $[C_{ijL},\ C_{ijU}]^2 = [C_{ijL},\ C_{ijU}]$. Hence, $A\, adj\ A$ is idempotent.

Example 4.1. Let $A = \begin{bmatrix} [.3,\ .5] & [.6,\ .8] & [.7,\ .9] \\ [.7,\ .9] & [.3,\ .5] & [.6,\ .8] \\ [.6,\ .8] & [.7,\ .9] & [.3,\ .5] \end{bmatrix}$ then

$$Adj\ A = \begin{bmatrix} [.6,\ .8] & [.7,\ .9] & [.6,\ .8] \\ [.6,\ .8] & [.6,\ .8] & [.7,\ .9] \\ [.7,\ .9] & [.6,\ .8] & [.6,\ .8] \end{bmatrix}$$

Let $C = A\ adj\ A = \begin{bmatrix} [.7,\ .9] & [.6,\ .8] & [.6,\ .8] \\ [.6,\ .8] & [.7,\ .9] & [.6,\ .8] \\ [.6,\ .8] & [.6,\ .8] & [.7,\ .9] \end{bmatrix}$.

Now, $[C_{iiL},\ C_{iiU}] \geq [C_{ijL},\ C_{ijU}]$. Hence, $A\, adj\, A$ is weakly reflexive.

$$(A\, adj\, A)^2 = \begin{bmatrix} [.7,\ .9] & [.6,\ .8] & [.6,\ .8] \\ [.6,\ .8] & [.7,\ .9] & [.6,\ .8] \\ [.6,\ .8] & [.6,\ .8] & [.7,\ .9] \end{bmatrix}$$

Hence $(A\, adj\, A)^2 \leq A\, adj\, A$. Therefore, $A\, adj\, A$ is transitive.

Here $[C_{ijL}, C_{ijU}]^2 = [C_{ijL}, C_{ijU}]$. Hence, $C = A\, adj\, A$ is idempotent.

Theorem 4.4. If $A = [a_{ij}]_{n\times n} = [a_{ijL},\ a_{ijU}]$ be a circulant interval valued fuzzy matrix, then determinant of A is the largest element in A.

Proof: Let $[a_{1mL},\ a_{1mU}] \geq [a_{1iL},\ a_{1iU}]$ i.e, a_{1m} is the largest element of A.
Then by definition of $|A|$ we have $|A| = \sum_{\sigma\in S_n} sgn\ \sigma \prod_{i=1}^{n} a_{i\sigma(i)}$

$= \sum_{\sigma \in S_n} sgn\, \sigma \prod_{i=1}^{n} [a_{i\sigma(i)L},\ a_{i\sigma(i)U}] \ = \ \prod_{i=1}^{n} [a_{i\pi(i)L},\ a_{i\pi(i)U}]$

for some $\pi \in s_n$

$= [a_{1\pi(1)L},\ a_{1\pi(1)U}].[a_{2\pi(2)L},\ a_{2\pi(2)U}] \dots\dots\dots\dots\dots.. [a_{n\pi(n)L},\ a_{n\pi(n)U}]$

Let $\pi(1) = 1$. Since A is circulan,we get

$[A_{1mL},\ A_{1mU}] = [a_{2(m \oplus 1)L},\ a_{2(m \oplus 1)U}] = [a_{3(m \oplus 2)L},\ a_{3(m \oplus 2)U}]$..........

$.\dots\dots\dots = [a_{m(m\oplus(n-1))L},\ a_{m(m\oplus(n-1))U}]$

Let the permutation π defined as $\pi = \begin{pmatrix} 1 & 2 \dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots n \\ m & m \oplus 1 \dots\dots\dots\dots\dots\dots m\oplus(n-1) \end{pmatrix}$

Therefore $|A| = [a_{1mL},\ a_{1mU}].[a_{2(m \oplus 1)L},\ a_{2(m \oplus 1)U}] =$

$[a_{3(m \oplus 2)L},\ a_{3(m \oplus 2)U}]$,...........$\dots\dots\dots = [a_{m(m\oplus(n-1))L},\ a_{m(m\oplus(n-1))U}]$

$|A| = [a_{1mL},\ a_{1mU}]$. Hence, $|A|$ is the largest element in A.

Example 4.4. Let $A = \begin{bmatrix} [.3,.6] & [.7,.9] & [.4,.3] \\ [.4,.3] & [.3,.6] & [.7,.9] \\ [.7,.9] & [.3,.6] & [.3,.6] \end{bmatrix}$ then $|A| = [.7,\ .9]$.

Hence, $|A|$ is the largest element in A.

5. Conclusion

In this paper, some properties of circulant interval valued fuzzy matrix (CIVFMs) are discussed with examples. De Morgan's laws are proved using elementary operators. The concept and some properties of determinant of circulant interval valued fuzzy matrix (CIVFMs) are also discussed.

REFERENCES

1. A.K.Adak, M.Bhowmik and M.Pal, Intuitionistic fuzzy block matrix and its some properties, *Annals of Pure and Applied Mathematics*, 1(1) (2012) 13-31.
2. M.Bhowmik, M.Pal and A.Pal Circulant triangular fuzzy number matrices, *Journal of physical Sciences*, 12 (2008) 141-154.
3. H.Hashimoto, Convergence of power of a fuzzy transitive matrix, *Fuzzy Sets and Systems,* 9 (1983) 153-160.
4. R.Hemasinha, N.R.Pal, and J.C.Bezdek**,** Iterates of fuzzy circulant matrices *Fuzzy Sets and Systems*, 60 (1993) 199-206.
5. J.B.Kim, Determinant theory for fuzzy and Boolean matrices, *Congressus Numerantium*, (1988) 273-276.
6. W.Kolodziejczyk, Convergence of powers of s-transitive fuzzy matrices, *Fuzzy Sets and Systems*, 26 (1988) 127-130.
7. S.Mondal and M.Pal, Intuitionistic fuzzy incline matrix and determinant, *Annals of Fuzzy Mathematics and Informatics*, 8(1) (2014) 19-32.
8. A.Nagoorgani and A.R.Manikandan, Det–norm on fuzzy matrices, *International Journal of Pure And Applied Mathematics*, 92(1) (2014) 1–12.
9. M..Pal, Interval-valued fuzzy matrices with fuzzy rows and columns, *Fuzzy Information and Engineering*, 7(3) (2015) 335-368.
10. R.Pradhan and M.Pal, Intuitionistic fuzzy linear transformations, *Annals of Pure and Applied Mathematics*, 1(1) (2012) 57-68.

11. V.Pushpalatha, α cuts of interval valued fuzzy matrices with interval valued fuzzy rows and columns, *IOSR Journal of Mathematics*, 13(3) (2017) 55-62.
12. M.Z.Ragab and E.C.Emam, The determinant and adjoint of a square fuzzy matrix, *Fuzzy Sets and Systems*, 61 (1994) 297-307.
13. M.Z.Ragab and E.G.Emam, On the min-max composition of fuzzy matrices, *Fuzzy Sets and Systems*, 75 (1995) 83-92.
14. A.K.Shyamal and M.Pal, Two new operations on fuzzy matrices, *Journal of Applied Mathematics and Computing*, 15 (2004) 91-107.
15. A.K.Shyamal and M.Pal, Distance between intuitionistic fuzzy matrices and its applications, *Journal of Natural and Physical Sciences*, 19 (2005) 39-58.
16. M.G.Thomason, Convergence of powers of a fuzzy matrix, *J. Math Anal. Appl*., 57 (1977) 476-480.
17. L.A.Zadeh, Fuzzy sets, *Information and Control*, 8 (1965) 338-353.
18. A.K.Shyamal and M.Pal, Triangular fuzzy matrices, *Iranian Journal of Fuzzy Systems,* 4 (1) (2007) 75-87.
19. M.Pal, Fuzzy matrices with fuzzy rows and fuzzy columns, *Journal of Intelligent and Fuzzy Systems*, 30(1) (2015) 561-573.
20. M.Bhowmik and M.Pal, Some results on intuitionistic fuzzy matrices and circulant intuitionistic fuzzy matrices, *International Journal of Mathematical Sciences,* 7(1-2) (2008) 81-96.