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CIRCULANT, NEGACYCLIC AND SEMICIRCULANT MATRICES WITH THE MODIFIED PELL, JACOBSTHAL AND JACOBSTHAL-LUCAS NUMBERS

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Abstract

In this paper, we give some properties of the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers. We then define the circulant, negacyclic and semicirculant matrices with these numbers and investigate the norms, eigenvalues and determinants of these matrices.

Keywords: Modified Pell numbers, Jacobsthal numbers, Jacobsthal -Lucas numbers, Circulant Matrix.

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1. Introduction

In this section, we introduce the main notation used throughout the paper, and briefly review some of the work on circulant, negacyclic and semicirculant matrices.

Let $x \in \mathbb{C}^n$, $x = (x_0, x_1, \dots, x_{n-1})^T$. The $n \times n$ circulant matrix $C(x) = c_{ij}$ is given by $c_{ij} = x_{j-i(\mod n)}$. That is, a circulant matrix of order n is a square matrix of the form:

$$C(x) = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & \dots & x_{n-1} \\ x_{n-1} & x_0 & x_1 & x_2 & & x_{n-2} \\ x_{n-2} & x_{n-1} & x_0 & x_1 & \ddots & \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & & x_1 \\ x_1 & \dots & & x_{n-1} & & x_0 \end{bmatrix}$$

The elements of each row of $C(x)$ are identical to those of the previous row, but are moved one position to the right and wrapped around. The whole circulant matrix is

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evidently determined by the first row (or column). A skew circulant matrix is a circulant by a change in sign to all elements below the main diagonal. Skew circulant matrices have also been called negacyclic matrices or (-1) – factor matrices. Hence, a $n \times n$ negacyclic matrix $N(x)$ is given by

$$N(x) = \begin{bmatrix} x_0 & x_1 & x_2 & x_3 & \dots & x_{n-1} \\ -x_{n-1} & x_0 & x_1 & x_2 & & x_{n-2} \\ -x_{n-2} & -x_{n-1} & x_0 & x_1 & \ddots & \\ \vdots & & \ddots & \ddots & & \vdots \\ & & & & & x_1 \\ -x_1 & \dots & & -x_{n-1} & & x_0 \end{bmatrix}$$

where $x = (x_0, x_1, \dots, x_{n-1})^T$, $x \in \mathbb{C}^n$. Note that $C(x)$ and $N(x)$ are special types of Toeplitz matrix.

Let $x \in \mathbb{C}^n$, $x = (x_1, x_2, \dots, x_n)^T$. The $n \times n$ semicirculant matrix $S(x) = (s_{ij})$ is given by

$$s_{ij} = \begin{cases} x_{j-i+1} & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

We will have occasion to use the $n \times n$ Fourier matrix $F = (f_{ij})$ given by

$$f_{ij} = \frac{1}{\sqrt{n}} \omega^{(i-1)(j-1)},$$

where $\omega = e^{2\pi i/n}$ is the n^{th} primitive root of unity. The unitary matrix $G = (g_{pq})$ defined by

$$g_{pq} = \frac{1}{\sqrt{n}} \omega^{p\left(\frac{2q+1}{2}\right)}$$

is related to the Fourier matrix by the equality

$$G = \operatorname{diag}\left(1,\ \omega^{1/2},\ \dots,\ \omega^{(n-1)/2}\right) F.$$

Circulant and negacyclic matrices are an especially tractable class of matrices since their inverses, conjugate transposes, products, and sums are also, respectively, circulant, negacyclic matrices, and hence both straightforward to construct and normal [1].

1.1. Theorem. [7] *Let $C(x)$ be a general $n \times n$ circulant matrix. Then*

$$C(x) = F^* \operatorname{diag}\left(\lambda_0(x),\ \lambda_1(x), \dots, \lambda_{n-1}(x)\right) F, \tag{1.1}$$

where $\lambda_j(x) = \sum_{k=0}^{n-1} x_k \omega^{-jk}$, $j = 0, 1, \dots, n-1$ and ω is the n^{th} primitive root of unity.

1.2. Theorem. [7] *Let $N(x)$ be an $n \times n$ negacyclic matrix. Then*

$$N(x) = G \operatorname{diag}\left(\mu_0(x),\ \mu_1(x), \dots,\ \mu_{n-1}(x)\right) G^*, \tag{1.2}$$

where $\mu_j(x) = \sum_{k=0}^{n-1} x_k \omega^{(2j+1)k/2}$, $j = 0, 1, \dots, n-1$.

1.3. Theorem. [6, Section 3.1, Exercise 19, page 157] *Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$. Then, A is a normal matrix if and only if the eigenvalues of AA^* are $|\lambda_1|^2, |\lambda_2|^2, \dots, |\lambda_n|^2$, where A^* is the conjugate transpose of the matrix A.*

Let $A = (a_{ij})$ be an $n \times n$ matrix. The Euclidean (or Frobenius) norm of the matrix A is

$$\|A\|_E = \left(\sum_{i,j=1}^{n} |a_{ij}|^2 \right)^{1/2},$$

and the spectral norm of the matrix A is

$$\|A\|_2 = \left(\max_{1 \le i \le n} \lambda_i \left(A^* A \right) \right)^{1/2},$$

where $\lambda_i \left(A^* A \right)$ is the i^{th} eigenvalue of the matrix $A^* A$.

The maximum column sum matrix norm of the $n \times n$ matrix A is

$$\|A\|_1 = \max_{1 \le j \le n} \sum_{i=1}^{n} |a_{ij}|,$$

and the maximum row sum matrix norm of the $n \times n$ matrix A is

$$\|A\|_\infty = \max_{1 \le i \le n} \sum_{j=1}^{n} |a_{ij}|.$$

The Pell and Pell-Lucas sequences are defined by the recurrences

(1.3) $\quad P_{n+2} = 2P_{n+1} + P_n \ ; \ P_0 = 0, \ P_1 = 1$

and

(1.4) $\quad Q_{n+2} = 2Q_{n+1} + Q_n \ ; \ Q_0 = 2, \ Q_1 = 2,$

respectively.

Define the sequence $\{q_n\}$ for all integers $n \ge 0$ by the recurrence

(1.5) $\quad q_{n+2} = 2q_{n+1} + q_n \ ; \ q_0 = 1, \ q_1 = 1.$

Clearly, the characteristic equation of (1.3, 1.4, 1.5) is

(1.6) $\quad x^2 - 2x - 1 = 0.$

Horadam [4] has called the sequence $\{q_n\}$ the modified Pell sequence. It is closely related to the Pell $\{P_n\}$ and Pell-Lucas $\{Q_n\}$ sequences. In fact the relationship between q_n and Q_n is $Q_n = 2q_n$. Consequently, the known properties of $\{Q_n\}$ are easily transferable to $\{q_n\}$.

Binet's formula for the modified Pell numbers is

(1.7) $$q_n = \frac{\alpha^n + \beta^n}{\alpha + \beta},$$

where $\alpha, \ \beta$ are the roots of the equation (1.6).

In [4] the following properties of the modified Pell numbers are given:

(1.8) $$P_n = \frac{q_n + q_{n-1}}{2}$$

(1.9) $$\sum_{k=1}^{n} q_{k-1}^2 = \frac{q_{2n-1} - (-1)^n + 2}{4}$$

(1.10) $$\sum_{k=0}^{n-1} q_k = \frac{q_n + q_{n-1}}{2}.$$

Define the sequences $\{J_n\}$ and $\{j_n\}$ for all integers $n \ge 0$ by the recurrences

(1.11) $\quad J_{n+2} = J_{n+1} + 2J_n \ ; \ J_0 = 0, \ J_1 = 1$

and

(1.12) $\quad j_{n+2} = j_{n+1} + 2j_n \ ; \ j_0 = 2, \ j_1 = 1.$

These sequences are to called the Jacobsthal and Jacobsthal-Lucas sequences, respectively. The characteristic equation of (1.11) and (1.12) is

(1.13) $\quad x^2 - x - 2 = 0.$

Binet's formula for the Jacobsthal and Jacobsthal-Lucas sequences are

$$J_n = \frac{2^n - (-1)^n}{3}, \quad j_n = 2^n + (-1)^n .$$

Hordam [3, 5] gives some properties of J_n and j_n. Cassini's identities for the Jacobsthal and Jacobsthal-Lucas numbers are

$$J_{n+1}J_{n-1} - J_n^2 = (-1)^n \, 2^{n-1}$$

and

$$j_{n+1}j_{n-1} - j_n^2 = 9\,(-1)^{n-1}\, 2^{n-1}$$

respectively. The summation formulae for the Jacobsthal and Jacobsthal-Lucas numbers are given by:

(1.14) $$\sum_{i=1}^{n} J_i = \frac{1}{2}\left(J_{n+2} - 1\right),$$

(1.15) $$\sum_{i=1}^{n} j_i = \frac{1}{2}\left(j_{n+2} - 5\right),$$

(1.16) $$\sum_{i=1}^{n} J_i^2 = \frac{1}{9}\left(2^{n+2}J_n + n\right),$$

(1.17) $$\sum_{i=1}^{n} j_i^2 = \frac{1}{3}\left(2^{n+2}j_n + 3n - 8\right).$$

The values of the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers for $n = 0, 1, 2, \ldots$ are given in the following table.

n	0	1	2	3	4	5	6	7	8	9	$\cdots$
q_n	1	1	3	7	17	41	99	239	577	1393	$\cdots$
J_n	0	1	1	3	5	11	21	43	85	171	$\cdots$
j_n	2	1	5	7	17	31	65	127	257	511	$\cdots$

2. Main results

2.1. Theorem. *Let $C(q) = (c_{ij})$ be the $n \times n$ circulant matrix with the modified Pell numbers, that is $c_{ij} = q_{j-i \ (\bmod n)}$, where q_n is the n^{th} modified Pell number. Then the Euclidean norm of $C(q)$ is*

$$\|C(q)\|_E = \begin{cases} \sqrt{\frac{n}{4}\left(q_{2n-1} + 1\right)} & \text{if } n \text{ is even,} \\ \sqrt{\frac{n}{4}\left(q_{2n-1} + 3\right)} & \text{if } n \text{ is odd.} \end{cases}$$

Proof. From the definition of the Euclidean norm, we obtain

$$\|C(q)\|_E^2 = n\sum_{k=1}^{n} q_{k-1}^2.$$

Using (1.9), we have

$$\|C(q)\|_E^2 = n\frac{q_{2n-1} - (-1)^n + 2}{4}.$$

Therefore, the Euclidean norm of $C(q)$ is

$$\|C(q)\|_E = \begin{cases} \sqrt{\frac{n}{4}(q_{2n-1}+1)} & \text{if } n \text{ is even,} \\ \sqrt{\frac{n}{4}(q_{2n-1}+3)} & \text{if } n \text{ is odd.} \end{cases}$$

□

It should be noted that the Euclidean norms of circulant and negacyclic matrices are the same. So, the Euclidean norm of the negacyclic matrices with the modified Pell numbers is also

$$\|N(q)\|_E = \begin{cases} \sqrt{\frac{n}{4}(q_{2n-1}+1)} & \text{if } n \text{ is even,} \\ \sqrt{\frac{n}{4}(q_{2n-1}+3)} & \text{if } n \text{ is odd.} \end{cases}$$

2.2. Corollary. *The maximum column (or row) sum matrix norms of $C(q)$ and $N(q)$ are*

$$\|C(q)\|_1 = \|C(q)\|_\infty = P_n$$

and

$$\|N(q)\|_1 = \|N(q)\|_\infty = P_n.$$

2.3. Theorem. *Let $C(q)$ be an $n \times n$ circulant matrix with the modified Pell numbers. Then the eigenvalues of $C(q)$ are*

$$\lambda_j(C(q)) = \frac{(1+q_{n-1})\,\omega^{-j} - (1-q_n)}{\omega^{-2j} + 2\omega^{-j} - 1},$$

where q_n is the n^{th} modified Pell number, ω is the n^{th} primitive root of unity and $j = 0, 1, \ldots, n-1$.

Proof. Using Theorem 1.1 and Binet's formula for the modified Pell numbers, we have

$$\begin{aligned}
\lambda_j(C(q)) &= \sum_{k=0}^{n-1} q_k \omega^{-jk} = \sum_{k=0}^{n-1}\left(\frac{\alpha^k+\beta^k}{\alpha+\beta}\right)\omega^{-jk} \\
&= \frac{1}{\alpha+\beta}\left(\sum_{k=0}^{n-1}\left(\alpha\omega^{-j}\right)^k + \sum_{k=0}^{n-1}\left(\beta\omega^{-j}\right)^k\right) \\
&= \frac{1}{\alpha+\beta}\left(\frac{\left(\alpha\omega^{-j}\right)^n - 1}{\alpha\omega^{-j}-1} + \frac{\left(\beta\omega^{-j}\right)^n - 1}{\beta\omega^{-j}-1}\right) \\
&= \frac{(2-(\alpha^n+\beta^n)) - \left((\alpha+\beta) + \left(\alpha^{n-1}+\beta^{n-1}\right)\right)\omega^{-j}}{(\alpha+\beta)\left(-\omega^{-2j} - 2\omega^{-j} + 1\right)} \\
&= \frac{(1+q_{n-1})\,\omega^{-j} - (1-q_n)}{\omega^{-2j} + 2\omega^{-j} - 1}.
\end{aligned}$$

□

As applications of the above theorem, we obtain the following results.

2.4. Corollary. *Let $C(q)$ be an $n \times n$ circulant matrix with the modified Pell numbers. The spectral norm of $C(q)$ is*

$$\|C(q)\|_2 = P_n,$$

where P_n is the n^{th} Pell number.

Proof. Using Theorem 1.1 and Theorem 1.3, we have

$$\|C(q)\|_2 = \left(\max_{0 \le j \le n-1} \lambda j \left(C(q) C(q)^*\right)\right)^{1/2} = \left(\max_{0 \le j \le n-1} |\lambda j (C(q))|^2\right)^{1/2}.$$

If $j = 0$, then the eigenvalue is maximum. Therefore,

$$\|C(q)\|_2 = \left(|\lambda_0 (C(q))|^2\right)^{1/2} = |\lambda_0 (C(q))| = \frac{q_n + q_{n-1}}{2}.$$

From (1.8), we obtain $\|C(q)\|_2 = P_n$. $\square$

2.5. Corollary. *Let $C(q)$ be an $n \times n$ circulant matrix with the modified Pell numbers. Then*

$$\det(C(q)) = \frac{(1-q_n)^n - (1+q_{n-1})^n}{1 - 2q_n + (-1)^n},$$

where q_n is the n^{th} modified Pell number.

Proof. Using the fact that the determinant of a matrix is the product of the eigenvalues, we have

$$\begin{aligned}\det(C(q)) &= \prod_{k=0}^{n-1} \lambda_j (C(q)) \\ &= \prod_{k=0}^{n-1} \frac{(1-q_n) - (1+q_{n-1})\,\omega^{-j}}{(\alpha\omega^{-j} - 1)(\beta\omega^{-j} - 1)}.\end{aligned}$$

For all x and y we can write $\prod_{k=0}^{n-1} (x - y\omega_k) = x^n - y^n$. Then

$$\prod_{k=0}^{n-1} (1-q_n) - (1+q_{n-1})\,\omega^{-j} = (1-q_n)^n - (1+q_{n-1})^n$$

and

$$\prod_{k=0}^{n-1} \left(\alpha\omega^{-j} - 1\right)\left(\beta\omega^{-j} - 1\right) = (1-\alpha^n)(1-\beta^n) = 1 - 2q_n + (-1)^n,$$

where $Q_n = 2q_n$. Thus the proof is completed. $\square$

2.6. Theorem. *Let $N(q)$ be an $n \times n$ negacyclic matrix with the modified Pell numbers. Then the eigenvalues of $N(q)$ are*

$$\mu_j (N(q)) = \frac{(1-q_{n-1})\,\omega^{(2j+1)/2} - (q_n + 1)}{\omega^{2j+1} + 2\omega^{(2j+1)/2} - 1},$$

where ω is the n^{th} primitive root of unity and $j = 0, 1, \ldots, n-1$.

Proof. Using Theorem 1.2 and Binet's formula for the modified Pell numbers, we have

$$\begin{aligned}
\mu_j\left(N\left(q\right)\right) &= \sum_{k=0}^{n-1} q_k \omega^{(2j+1)k/2} \\
&= \sum_{k=0}^{n-1} \left(\frac{\alpha^k+\beta^k}{\alpha+\beta}\right)\omega^{(2j+1)k/2} \\
&= \frac{1}{\alpha+\beta}\left(\sum_{k=0}^{n-1}\left(\alpha\omega^{(2j+1)/2}\right)^k + \sum_{k=0}^{n-1}\left(\beta\omega^{(2j+1)/2}\right)^k\right) \\
&= \frac{1}{\alpha+\beta}\left(\frac{-\alpha^n-1}{\alpha\omega^{(2j+1)/2}-1} + \frac{-\beta^n-1}{\beta\omega^{(2j+1)/2}-1}\right) \\
&= \frac{\left(1-q_{n-1}\right)\omega^{(2j+1)/2}-\left(1+q_n\right)}{\omega^{2j+1}+2\omega^{(2j+1)/2}-1}.
\end{aligned}$$

□

2.7. Corollary. *The spectral norm of $N\left(q\right)$ is*

$$\left\|N\left(q\right)\right\|_2 = \left(\max_{0\le j\le n-1}\left|\frac{\left(1-q_{n-1}\right)\omega^{(2j+1)/2}-\left(1+q_n\right)}{\omega^{2j+1}+2\omega^{(2j+1)/2}-1}\right|^2\right)^{1/2}.$$

2.8. Corollary. *Let $N\left(q\right)$ be an $n\times n$ negacyclic matrix with the modified Pell numbers. Then*

$$\det\left(N\left(q\right)\right) = \frac{\left(1+q_n\right)^n+\left(-1\right)^n\left(q_{n-1}-1\right)^n}{1+2q_n+\left(-1\right)^n}.$$

2.9. Theorem. *Let $C\left(J\right)=\left(c_{ij}\right)$, where $c_{ij}=J_{j-i(\mod n)}$, be the $n\times n$ circulant matrix with the Jacobsthal numbers. The Euclidean norm of $C\left(J\right)$ is*

$$\left\|C\left(J\right)\right\|_E = \sqrt{\frac{n}{9}\left(2^{n+1}J_{n-1}+n-1\right)},$$

where J_n is the n^{th} Jacobsthal number.

Proof. From the definition of the Euclidean norm, we obtain

$$\left\|C\left(J\right)\right\|_E^2 = n\sum_{k=0}^{n-1} J_k^2.$$

Using (1.16), we have

$$\left\|C\left(J\right)\right\|_E^2 = n\frac{2^{n+1}J_{n-1}+n-1}{9}.$$

□

2.10. Corollary. *The maximum column (or row) sum matrix norm of $C\left(J\right)$ is*

$$\left\|C\left(J\right)\right\|_1 = \left\|C\left(J\right)\right\|_\infty = \frac{1}{2}\left(J_{n+1}-1\right).$$

Also, the Euclidean norm and the maximum column (or row) sum matrix norm of negacyclic matrices with the Jacobsthal numbers are respectively

$$\left\|N\left(J\right)\right\|_E = \sqrt{\frac{n}{9}\left(2^{n+1}J_{n-1}+n-1\right)}$$

and

$$\left\|N\left(J\right)\right\|_1 = \left\|N\left(J\right)\right\|_\infty = \frac{1}{2}\left(J_{n+1}-1\right).$$

2.11. Theorem. *Let $C(j) = (c_{sk})$, where $c_{sk} = j_{k-s(\mod n)}$, be the $n \times n$ circulant matrix with the Jacobsthal-Lucas numbers. The Euclidean norm of $C(j)$ is*

$$\|C(j)\|_E = \sqrt{\frac{n}{3}\left(2^{n+1} j_{n-1} + 3n + 1\right)},$$

where j_n is the n^{th} Jacobsthal-Lucas number. □

2.12. Corollary. *The maximum column (or row) sum matrix norm of $C(j)$ is*

$$\|C(j)\|_1 = \|C(j)\|_\infty = \frac{1}{2}\left(j_{n+1} - 1\right).$$

Also, the Euclidean norm and the maximum column (or row) sum matrix norm of negacyclic matrices with the Jacobsthal-Lucas numbers are respectively

$$\|N(j)\|_E = \sqrt{\frac{n}{3}\left(2^{n+1} j_{n-1} + 3n + 1\right)}$$

and

$$\|N(j)\|_1 = \|N(J)\|_\infty = \frac{1}{2}\left(j_{n+1} - 1\right).$$

Now, we give the Euclidean norms of semicirculant matrices with the modified Pell, Jacobsthal and Jacobsthal-Lucas numbers.

2.13. Theorem. *The Euclidean norm of an $n \times n$ semicirculant matrix $S(q)$ with the modified Pell numbers is*

$$\|S(q)\|_E = \begin{cases} \sqrt{\frac{1}{8}\left(q_{2n+2} - 4n - 3\right)} & \text{if } n \text{ even,} \\ \sqrt{\frac{1}{8}\left(q_{2n+2} - 4n - 5\right)} & \text{if } n \text{ odd.} \end{cases}$$

Proof. For the $n \times n$ semicirculant matrix $S(q) = (s_{ij})$ with the modified Pell numbers we have

$$s_{ij} = \begin{cases} q_{j-i+1} & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

The Euclidean norm of $S(q)$ is

$$\|S(q)\|_E^2 = \sum_{j=1}^{n} \sum_{i=1}^{j} \left(q_{j-i+1}\right)^2.$$

Using $Q_n = 2q_n$, we have

$$\|S(q)\|_E^2 = \frac{1}{4} \sum_{j=1}^{n} \sum_{i=1}^{j} \left(Q_{j-i+1}\right)^2.$$

From the properties of the Pell-Lucas numbers, we obtain

$$\|S(q)\|_E^2 = \frac{1}{4}\left(\frac{Q_{2n+2} - 2(-1)^{n+1} - 8 - 8n}{4}\right) = \frac{q_{2n+2} - (-1)^{n+1} - 4 - 4n}{8}.$$

Thus,

$$\|S(q)\|_E = \begin{cases} \sqrt{\frac{1}{8}\left(q_{2n+2} - 4n - 3\right)} & \text{if } n \text{ even,} \\ \sqrt{\frac{1}{8}\left(q_{2n+2} - 4n - 5\right)} & \text{if } n \text{ odd.} \end{cases}$$

□

2.14. Theorem. *The Euclidean norm of an $n \times n$ semicirculant matrix $S(J) = (s_{ij})$ with the Jacobsthal numbers is*

$$\|S(J)\|_E = \left(\frac{8}{81}\left(2^n j_{n+1} - 1\right) + \frac{n(n+1)}{18}\right)^{1/2},$$

where j_n is n^{th} Jacobsthal-Lucas number.

Proof. For the semicirculant matrix $S(J) = (s_{ij})$ with the Jacobsthal numbers we have

$$s_{ij} = \begin{cases} J_{j-i+1} & \text{if } i \leq j, \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of the Euclidean norm , we obtain

$$\|S(J)\|_E^2 = \sum_{j=1}^{n} \sum_{i=1}^{j} \left(J_{j-i+1}\right)^2 = \sum_{j=1}^{n} \left(\sum_{k=1}^{j} J_k^2\right).$$

Using (1.16), we have

$$\begin{aligned} \|S(J)\|_E^2 &= \sum_{j=1}^{n} \left(\frac{2^{j+2} J_j + j}{9}\right) \\ &= \frac{1}{27}\left(\sum_{j=1}^{n} 2^{2j+2} - 4\sum_{j=1}^{n} (-2)^j\right) + \frac{n}{9} \\ &= \frac{1}{27}\left(\frac{4}{3} 2^{n+1}\left(2^{n+1} + (-1)^{n+1}\right) - \frac{8}{3}\right) + \frac{n(n+1)}{18} \\ &= \frac{8}{81}\left(2^n j_{n+1} - 1\right) + \frac{n(n+1)}{18}. \end{aligned}$$

Thus, the proof is complete. □

2.15. Theorem. *The Euclidean norm of an $n \times n$ semicirculant matrix $S(j)$ with the Jacobsthal-Lucas numbers is*

$$\|S(j)\|_E = \left(\frac{2^{n+4} J_{n+1} + 3n^2 - 13n - 16}{6}\right)^{1/2},$$

where J_n is n^{th} Jacobsthal number.

Proof. For the semicirculant matrix $S(j) = (s_{ik})$ with the Jacobsthal-Lucas numbers we have

$$S(j) = \begin{cases} j_{k-i+1} & \text{if } i \leq k, \\ 0 & \text{otherwise.} \end{cases}$$

From the definition of the Euclidean norm, we have

$$\|S(j)\|_E^2 = \sum_{k=1}^{n} \sum_{i=1}^{k} \left(j_{k-i+1}\right)^2 = \sum_{k=1}^{n} \left(\sum_{t=1}^{k} j_t^2\right).$$

Using (1.17), we obtain

$$\begin{aligned}
\|S(j)\|_E^2 &= \sum_{k=1}^{n} \frac{1}{3}\left(2^{k+2} j_k + 3k - 8\right) \\
&= \frac{1}{3}\left(\sum_{k=1}^{n} 2^{2k+2} + 4\sum_{k=1}^{n} (-2)^k\right) + \frac{n(n+1)}{2} - \frac{8n}{3} \\
&= \frac{2^{n+4} J_{n+1} + 3n^2 - 13n - 16}{6}.
\end{aligned}$$

□

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