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Circulants, displacements and decompositions of matrices — Source link

Israel Gohberg, Vadim Olshevsky

Institutions: Tel Aviv University

Published on: 01 Sep 1992 - Integral Equations and Operator Theory (Birkhäuser-Verlag)

Topics: Toeplitz matrix and Circulant matrix

CIRCULANTS, DISPLACEMENTS AND DECOMPOSITIONS OF MATRICES [1]

I.Gohberg and **V.Olshevsky**

In this paper are suggested new formulas for representation of matrices and their inverses in the form of sums of products of factor circulants, which are based on the analysis of the factor cyclic displacement of matrices. The results in applications to Toeplitz matrices generalized the Gohberg-Semencul, Ben-Artzi-Shalom and Heinig-Rost formulas and are useful for complexity analysis.

0. INTRODUCTION

About a decade ago T.Kailath and his coauthors [KKM] [FMKL] (see also [K]) proposed a concept of classification of matrices, which turned out to be an effective tool for the investigation of the complexity of matrix inversion. We start with a brief review of their results. Let $\mathbf{C}^{n\times n}$ stand for the algebra of all $n\times n$ matrices with complex entries. Following [KKM] introduce in $\mathbf{C}^{n\times n}$ the linear operator, which transforms each matrix A in its *displacement*

$$\nabla_0(A) = A - Z_0 A Z_0^T,$$

where

$$Z_0 = \begin{bmatrix} 0 & \cdots & \cdots & \cdots & 0 \\ 1 & 0 & & & \vdots \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is the lower shift matrix and the sign ^{T} means transpose. The number $\alpha = \mathrm{rank}\nabla_0(A)$ is referred to as *displacement rank* of the matrix A. It was shown in [KKM] that any matrix $A\in\mathbf{C}^{n\times n}$ is uniquely determined by its displacement. Moreover, the equality

$$\nabla_0(A) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T \qquad (\mathbf{f}_m, \mathbf{g}_m^T \in \mathbf{C}^n)$$

[1] Integral Equations and operator theory, 1992, **15**, 853 - 863.

holds if and only if

$$A = \sum_{m=1}^{\alpha} L(\mathbf{f}_m) \cdot U(\mathbf{g}_m),$$

where

$$L(\mathbf{r}) = \begin{bmatrix} r_0 & 0 & \cdots & \cdots & 0 \\ r_1 & r_0 & & & \vdots \\ \vdots & r_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ r_{n-1} & \cdots & \cdots & r_1 & r_0 \end{bmatrix}, \qquad U(\mathbf{s}) = \begin{bmatrix} s_0 & s_1 & \cdots & \cdots & s_{n-1} \\ 0 & s_0 & s_1 & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & s_1 \\ 0 & \cdots & \cdots & 0 & s_0 \end{bmatrix}$$

are the lower triangular Toeplitz matrix with the first column $\mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix}$ and the upper triangular Toeplitz matrix with the first row $\mathbf{s}^T = \begin{bmatrix} s_0 & s_1 & \cdots & s_{n-1} \end{bmatrix}$.

It is not difficult to realize that the displacement rank of any Toeplitz matrix $A = (a_{i-j})_{i,j=0}^{n-1}$ does not exceed 2. Various applications give rise to matrices that, while not Toeplitz themselves, are closely related to Toeplitz matrices. For example, the inverses of the Toeplitz matrices, or product of two Toeplitz matrices are "close" to the Toeplitz matrices in the sense that their displacement rank is comparatively small [FMKL]. The displacement rank may be considered as a measure of the above mentioned closeness.

The "displacement" approach gave a new explanation of the nature of the Gohberg-Semencul formula. In [GS] (see also [GF]) it was shown that if $A = (a_{i-j})_{i,j=0}^{n-1}$ is a Toeplitz matrix and the equations

$$A\mathbf{x} = \mathbf{e}_0 \qquad A\mathbf{y} = \mathbf{e}_{n-1} \tag{0.1}$$

with $\mathbf{e}_0 = \begin{bmatrix} 1 & 0 & \cdots & 0 \end{bmatrix}^T$, $\mathbf{e}_{n-1} = \begin{bmatrix} 0 & \cdots & 0 & 1 \end{bmatrix}^T$ have solutions $\mathbf{x} = (x_i)_{i=0}^{n-1}$, $\mathbf{y} = (y_i)_{i=0}^{n-1}$ and $x_0 \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{x_0}\left(\begin{bmatrix} x_0 & 0 & \cdots & \cdots & 0 \\ x_1 & x_0 & & & \vdots \\ \vdots & x_1 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & 0 \\ x_{n-1} & \cdots & \cdots & x_1 & x_0 \end{bmatrix} \cdot \begin{bmatrix} y_{n-1} & y_{n-2} & \cdots & \cdots & y_0 \\ 0 & y_{n-1} & y_{n-2} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & y_{n-2} \\ 0 & \cdots & \cdots & 0 & y_{n-1} \end{bmatrix} - \right.$$

$$\left. - \begin{bmatrix} 0 & 0 & \cdots & \cdots & 0 \\ y_0 & 0 & & & \vdots \\ \vdots & y_0 & \ddots & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ y_{n-2} & \cdots & \cdots & y_0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 0 & x_{n-1} & \cdots & \cdots & x_1 \\ 0 & 0 & x_{n-1} & & \vdots \\ \vdots & & \ddots & \ddots & \vdots \\ \vdots & & & \ddots & x_{n-1} \\ 0 & \cdots & \cdots & \cdots & 0 \end{bmatrix} \right). \tag{0.2}$$

Obviously, formula (0.2) means that the displacement rank for the inverse of a Toeplitz matrix is less than or equal to 2.

In a recent paper G.Ammar and P.Gader, following one of the concluding remarks of [KKM], considered and analyzed the *cyclic displacement*

$$\nabla_1(A) = A - Z_1 A Z_1^T$$

of a matrix A, where

$$Z_1 = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix}$$

is the cyclic lower shift matrix. Before stating their results [AG] [G], let us introduce the necessary notations. By $\mathrm{Circ}_\varphi(\mathbf{r})$ we shall denote the *φ-circulant* with the first column $\mathbf{r} = \begin{bmatrix} r_0 & r_1 & \cdots & r_{n-1} \end{bmatrix}^T$, i.e. the matrix of the form

$$\mathrm{Circ}_\varphi(\mathbf{r}) = \begin{bmatrix} r_0 & \varphi r_{n-1} & \cdots & \cdots & \varphi r_1 \\ r_1 & r_0 & \varphi r_{n-1} & & \vdots \\ \vdots & r_1 & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & \varphi r_{n-1} \\ r_{n-1} & \cdots & \cdots & r_1 & r_0 \end{bmatrix}.$$

When convenient, we shall not specify the number φ and refer such a matrix as a *factor circulant*. The matrix $\mathrm{Circ}_1(\mathbf{r})$ is called a *circulant*, and the matrix $\mathrm{Circ}_{-1}(\mathbf{r})$ is said to be *skew-circulant*. In [AG] [G] it was shown that if the equality

$$\nabla_1(A) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T \qquad (\mathbf{f}_m, \mathbf{g}_m^T \in \mathbf{C}^n)$$

holds, then

$$A = \mathrm{Circ}_{lr} + \sum_{m=1}^{\alpha} L(\mathbf{f}_m) \cdot \mathrm{Circ}_1(\mathbf{g}_m)^T, \tag{0.3}$$

where Circ_{lr} is the circulant with the same last row as that of A. The formulas involving the factor circulants instead of the Toeplitz matrices are more attractive from the computation point of view. By using the formula (0.3) G.Ammar and P.Gader reduced the complexity of the inversion of positive definite Toeplitz matrices. For the inverses of positive definite Toeplitz matrices G.Ammar and P.Gader obtained also a formula, involving only circulants and skew-circulants.

In this paper we continue to study the inversion of matrices with the help of the cyclic displacement in a more general situation and more systematically. We show how any square matrix can be decomposed in a sum of products of factor circulants. It allows to obtain new formulas for inversion of Toeplitz matrices, which are factor circulant analogs of well known formulas, involving lower and upper triangular Toeplitz matrices. These formulas are useful for the analysis of complexity of computations with Toeplitz matrices. This topic will be considered in a separate publication. On the other hand, this approach allows also to simplify the proofs of Gohberg-Semencul, Ben-Artzi-Shalom and Heinig-Rost formulas for inversion of Toeplitz matrices.

1. FACTOR CIRCULANT DECOMPOSITIONS OF MATRICES

The φ*-cyclic displacement* of a matrix $A \in \mathbf{C}^{n\times n}$ is defined as

$$\nabla_\varphi(A) = A - Z_\varphi A Z^T_{\frac{1}{\varphi}}, \tag{1.1}$$

where

$$Z_\varphi = \begin{bmatrix} 0 & \cdots & \cdots & 0 & \varphi \\ 1 & 0 & & & 0 \\ 0 & 1 & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & 1 & 0 \end{bmatrix} \qquad (\varphi \neq 0)$$

is the φ-cyclic lower shift matrix. The number $\alpha = \mathrm{rank}\nabla_1(A)$ is referred to as φ*-cyclic displacement rank* of the matrix A. Obviously, the difference between φ-cyclic and ψ-cyclic displacement ranks ($\varphi \neq \psi$) of any matrix A does not exceed 2. The transformation $\nabla_\varphi(\cdot)$, which maps any matrix A to its φ-cyclic displacement $\nabla_\varphi(A)$ is a linear operator, defined on the algebra $\mathbf{C}^{n\times n}$. Since the operator $\nabla_\varphi(\cdot)$ has a nontrivial kernel (it contains, for example, the identity matrix I), therefore, its image can not coincide with the whole $\mathbf{C}^{n\times n}$. The full description of the kernel and image of the φ-cyclic displacement operator is given by the following theorem.

THEOREM 1.1 *Let $\nabla_\varphi(\cdot)$ be the linear operator in $\mathbf{C}^{n\times n}$ defined by* (1.1). *Then the following statements hold:*

(i). *The equality $\nabla_\varphi(A) = 0$ holds if and only if A is φ-circulant.*

(ii). *If the equation*

$$\nabla_\varphi(X) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T, \tag{1.2}$$

where $\mathbf{f}_m, \mathbf{g}_m$ $(m = 1, 2, ..., \alpha)$ are given vectors, is solvable with respect to $X \in \mathbf{C}^{n\times n}$, then

$$\sum_{m=1}^{\alpha} \mathrm{Circ}_\varphi(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T = 0. \tag{1.3}$$

(iii). *If 2α vectors $\mathbf{f}_m$ and $\mathbf{g}_m$ $(m = 1, 2, ..., \alpha)$ satisfy the condition* (1.3), *then the equation* (1.2) *has the solution*

$$X = \mathrm{Circ}_{lr} + \frac{\varphi}{\varphi - \psi} \sum_{m=1}^{\alpha} \mathrm{Circ}_\psi(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T. \tag{1.4}$$

Here ψ $(\neq \varphi)$ is an arbitrary complex number and Circ_{lr} is any φ-circulant. The last rows of the matrices X and Circ_{lr} are the same.

(iv). *Under the conditions of the assertion* (iii) *the solution* X *of the equation* (1.2) *may also be written in the form*

$$X = \mathrm{Circ}_{lc} + \frac{\psi}{\psi - \varphi} \sum_{m=1}^{\alpha} \mathrm{Circ}_{\varphi}(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\psi}}(\mathbf{g}_m)^T. \tag{1.5}$$

Here $\psi(\neq \varphi)$ *is an arbitrary complex number and* Circ_{lc} *is any* φ*-circulant. The last columns of the matrices* X *and* Circ_{lc} *are the same.*

The formulas (1.3), (1.4) and (1.5) are generalizations of the formulas of G.Ammar and P.Gader [AG] [G]. In the latter papers the formulas (1.3) and (1.4) were obtained for the extremal case $\varphi = 1$, $\psi = 0$. In this case $\mathrm{Circ}_{\psi}(\mathbf{f}_m)$ are replaced by lower triangular Toeplitz matrices. The expression similar to (1.5) was obtained in [AG] [G] for the extremal case $\varphi = 1$, $\psi = \infty$, and there $\mathrm{Circ}_{\frac{1}{\psi}}(\mathbf{g}_m)^T$ are replaced by upper triangular Toeplitz matrices.

PROOF. Let matrix $A = (a_{ij})_{i,j=0}^{n-1}$ satisfies the condition $A = Z_{\varphi} A Z_{\frac{1}{\varphi}}^T$. From this equality it follows that $a_{ij} = a_{i-1,j-1}$ if $i, j \neq 0$, $a_{ij} = \frac{1}{\varphi} a_{i-1,n-1}$ if $j = 0$ and $a_{ij} = \varphi a_{n-1,j-1}$ if $i = 0$. These relations describe, obviously, a φ-circulant.

Let $\nabla_{\varphi}(X) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T$. Then, taking into account the equalities $Z_{\varphi}^n = \varphi I$ and $Z_{\varphi}^{-1} = Z_{\frac{1}{\varphi}}^T$, we have

$$0 = \sum_{k=0}^{n-1} Z_{\varphi}^k (X - Z_{\varphi} X Z_{\varphi}^{-1}) Z_{\varphi}^{-k} =$$

$$= \sum_{k=0}^{n-1} \sum_{m=1}^{\alpha} Z_{\varphi}^k \mathbf{f}_m \cdot (Z_{\frac{1}{\varphi}}^k \mathbf{g}_m)^T = \sum_{m=1}^{\alpha} \mathrm{Circ}_{\varphi}(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T.$$

The last equality follows from the general identity $U \cdot V^T = \sum_{k=0}^{n-1} \mathbf{u}_k \cdot \mathbf{v}_k^T$, where $\mathbf{u}_k$ and $\mathbf{v}_k$ are the k-th columns of the matrices U and V, correspondingly. The assertion (ii) is proved.

Now we shall prove the assertion (iii). Suppose that vectors $\mathbf{f}_m$, $\mathbf{g}_m$ $(m = 1, 2, ..., \alpha)$ satisfy the condition (1.3) and compute the φ-cyclic displacement of the matrix X defined by (1.4). The matrices $\mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T$ $(m = 1, 2, ..., \alpha)$ and $Z_{\frac{1}{\varphi}}^T$ are φ-circulants and hence are commuting. From here follows :

$$\nabla_{\varphi}(X) = \nabla_{\varphi}(\mathrm{Circ}_{lr}) + \frac{\varphi}{\varphi - \psi} \sum_{m=1}^{\alpha} \nabla_{\varphi}(\mathrm{Circ}_{\psi}(\mathbf{f}_m)) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T. \tag{1.6}$$

It is easy to see that φ-cyclic displacement for ψ-circulant $\mathrm{Circ}_{\psi}(\mathbf{r})$ with the first column $\mathbf{r} = \begin{bmatrix} r_0 \\ r_1 \\ \vdots \\ r_{n-1} \end{bmatrix}$ has the following simple form

$$\nabla_{\varphi}(\mathrm{Circ}_{\psi}(\mathbf{r})) = \left[\begin{array}{c|ccc} 0 & (\psi - \varphi) r_{n-1} & \cdots & (\psi - \varphi) r_1 \\ \hline (1 - \frac{\psi}{\varphi}) r_1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (1 - \frac{\psi}{\varphi}) r_{n-1} & 0 & \cdots & 0 \end{array} \right] =$$

$$= \frac{\varphi - \psi}{\varphi}\mathbf{r} \cdot \mathbf{e}_0^T - \frac{\varphi - \psi}{\varphi}\mathbf{e}_0 \cdot \tilde{\mathbf{r}}^T,$$

where $\tilde{\mathbf{r}}^T = [r_0 \ \varphi r_{n-1} \ \dots \ \varphi r_1]$ is the first row of the φ-circulant $\mathrm{Circ}_\varphi(\mathbf{r})$. Calculating in this way the φ-cyclic displacement for each matrix $\mathrm{Circ}_\psi(\mathbf{f}_m)$ in the right hand side of (1.6) and taking into account that $\nabla_\varphi(\mathrm{Circ}_{lr}) = 0$ in view of (i), we have

$$\nabla_\varphi(X) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{e}_0^T \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T - \sum_{m=1}^{\alpha} \mathbf{e}_0 \cdot \tilde{\mathbf{f}}_m^T \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T. \tag{1.7}$$

Here $\tilde{\mathbf{f}}_m^T$ are the first rows of the matrices $\mathrm{Circ}_\varphi(\mathbf{f}_m)$ $(m = 1, 2, ..., \alpha)$. Therefore, in view of (1.3) the sum of the last α terms in (1.7) is equal to the zero matrix. Furthermore, $\mathbf{e}_0^T \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T = \mathbf{g}_m^T$ $(m = 1, 2, ..., \alpha)$, and hence the matrix X defined by (1.4) satisfies the equation (1.2). Finally, the last row of the matrix $\mathrm{Circ}_\psi(\mathbf{f}_m)$ does not depend upon the number ψ, therefore in view of (1.3) the last rows of the matrices X and Circ_{lr} coincide. The assertion (iii) is now completely proved.

The assertion (iv) can be proved with the same arguments.Ξ

According to the proposition (i) of the theorem 1.1, every matrix A is determined by its φ-cyclic displacement up to a φ-circulant. Therefore, an arbitrary matrix is uniquely determined by its φ-cyclic displacement and any one of its rows or columns. For example, on the basis of the information on the last row of A, one can use the formula (1.4), or, knowing the last column of A, we can reconstruct A from its φ-cyclic displacement with the help of (1.5).

2. INVERSION OF MATRICES

It is easy to observe that for invertible matrix $A \in \mathbf{C}^{n\times n}$ there exists a simple interrelation between the φ-cyclic displacement of the inverse matrix A^{-1} and the φ-cyclic displacement of A. This connection is given by

$$\nabla_\varphi(A) = -A \cdot \nabla_\varphi(A^{-1}) \cdot Z_\varphi A Z_{\frac{1}{\varphi}}^T. \tag{2.1}$$

The immediate conclusion from (2.1) is that φ-cyclic displacement rank is inherited under matrix inversion. The equation (2.1) also allows to make use of the φ-cyclic displacement technique for the calculation of the inverse matrix. If the φ-cyclic displacement of $A \in \mathbf{C}^{n\times n}$ is given as the outer sum

$$\nabla_\varphi(A) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T, \tag{2.2}$$

then, according to (2.1), one can receive the analogous representation for $\nabla_\varphi(A^{-1})$ by solving 2α matrix equations, involving the matrix A and the vectors of outer sum (2.2). To apply the formula (1.4) it remains only to solve one more equation, providing us with the information on the last row of A^{-1}.

According to this scheme, set $\hat{\mathbf{g}}_m^T = \mathbf{g}_m^T \cdot Z_\varphi$ $(m = 1, 2, ..., \alpha)$ and let the vectors $\mathbf{u}_m$ and $\hat{\mathbf{v}}_m$ be the solutions of the following equations

$$A\mathbf{u}_m = \mathbf{f}_m \quad (m = 1, 2, ..., \alpha), \tag{2.3}$$

and

$$\hat{\mathbf{v}}_m^T A = \hat{\mathbf{g}}_m^T \quad (m = 1, 2, ..., \alpha). \tag{2.4}$$

Furthermore, in view of (2.1), we have

$$\nabla_{\varphi}(A^{-1}) = -\sum_{m=1}^{\alpha} \mathbf{u}_m \cdot \mathbf{v}_m^T, \tag{2.5}$$

where $\mathbf{v}_m^T = \hat{\mathbf{v}}_m^T \cdot Z_{\frac{1}{\varphi}}^T$. Solving the equation

$$\mathbf{y}^T A = \mathbf{e}_{n-1}^T \tag{2.6}$$

we get the last row of A^{-1}. Note that in our consideration matrix A was supposed to be invertible from the very beginning. As we shall see in the proof of the the next theorem, the solvability of equations (2.4) and (2.6) yields the invertibility of A, and, therefore, the procedure of reconstructing matrix A^{-1} from (2.2) will be successful in this case.

THEOREM 2.1 *Let $A \in \mathbf{C}^{n \times n}$ and its φ-cyclic displacement is given as the outer sum $\nabla_{\varphi}(A) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T$. If the equations* (2.3), (2.4) *and* (2.6) *have the solutions $\mathbf{u}_m$, $\hat{\mathbf{v}}_m$ and $\mathbf{y}$, correspondingly, then A is invertible and*

$$A^{-1} = \mathrm{Circ}_{lr} - \frac{\varphi}{\varphi - \psi} \sum_{m=1}^{\alpha} \mathrm{Circ}_{\psi}(\mathbf{u}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{v}_m)^T, \tag{2.7}$$

where ψ $(\neq \varphi)$ is an arbitrary complex number, $\mathbf{v}_m = Z_{\frac{1}{\varphi}} \hat{\mathbf{v}}_m$ $(m = 1, 2, ..., \alpha)$ and Circ_{lr} is the φ-circulant with the last row $\mathbf{y}^T$.

PROOF. In view of the arguments preceding the formulation of the theorem, we only need to show that the solvability of the corresponding equations yields the invertibility of A. Thus, suppose that we succeeded in solving the equations (2.4) and (2.6). If vector $\mathbf{r} \in \mathbf{C}^n$ satisfies the equality $A\mathbf{r} = \mathbf{0}$, then (2.2) and (2.4) imply that $(A - Z_{\varphi} A Z_{\frac{1}{\varphi}}^T) Z_{\varphi} \mathbf{r} = \mathbf{0}$, i.e. $A Z_{\varphi} \mathbf{r} = \mathbf{0}$. With the same arguments we also get the equalities $A Z_{\varphi}^k \mathbf{r} = \mathbf{0}$ $(k = 0, 1, ..., n-1)$. Premultiplying the last identities by the solution $\mathbf{y}^T$ of the equation (2.6), we conclude that all coordinates of the vector $\mathbf{r}$ are identically equal to zero, and hence matrix A is invertible.Ξ

3. INVERSION OF TOEPLITZ MATRICES

In this section we consider the Toeplitz matrices, which have a low φ-cyclic displacement rank. Below we make use of the fact that an arbitrary Toeplitz matrix A has a property of *persymmetry* (the symmetry with respect to the reverse diagonal), or, what is the same that A satisfies the condition

$$A = J A^T J, \text{ where } J = \begin{bmatrix} 0 & \cdots & \cdots & 0 & 1 \\ \vdots & & \iddots & 1 & 0 \\ \vdots & \iddots & \iddots & \iddots & \vdots \\ 0 & 1 & \iddots & & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \end{bmatrix} \tag{3.1}$$

is the reverse identity matrix. It is not difficult to see that this property is invariant under matrix inversion.

To apply theorem 2.1 for Toeplitz matrix A we need the outer sum representation of $\nabla_\varphi(A)$. Using a special structure of A, it is easy to see that

$$\nabla_\varphi(A) = \left[\begin{array}{c|ccc} 0 & a_{-1}-\varphi a_{n-1} & \cdots & a_{-n+1}-\varphi a_1 \\ \hline a_1 - \frac{1}{\varphi}a_{-n+1} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ a_{n-1}-\frac{1}{\varphi}a_{-1} & 0 & \cdots & 0 \end{array}\right] =$$

$$= \mathbf{e}_0 \cdot \begin{bmatrix} -\beta \\ a_{-1}-\varphi a_{n-1} \\ \vdots \\ a_{-n+1}-\varphi a_1 \end{bmatrix}^T + \begin{bmatrix} \beta \\ a_1-\frac{1}{\varphi}a_{-n+1} \\ \vdots \\ a_{n-1}-\frac{1}{\varphi}a_{-1} \end{bmatrix} \cdot \mathbf{e}_0^T, \tag{3.2}$$

where β may be an arbitrary complex number. Thus, the φ-cyclic displacement rank of Toeplitz matrix is less than or equal to 2. Moreover, in view of the theorem 1.1, the special form (3.2) of the φ-cyclic displacement of Toeplitz matrix is reflected in the fact that for any $\varphi, \psi \in \mathbf{C}$ $(\varphi \neq \psi)$ an arbitrary Toeplitz matrix can be decomposed into a sum of φ-circulant and ψ-circulant. Furthermore, for A and representation (3.2) of its φ-cyclic displacement the corresponding equations (2.3) take the shape

$$A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{u} = \begin{bmatrix} \beta \\ a_1-\frac{1}{\varphi}a_{-n+1} \\ \vdots \\ a_{n-1}-\frac{1}{\varphi}a_{-1} \end{bmatrix}, \tag{3.3}$$

and equations (2.4) are of the form

$$\hat{\mathbf{v}}^T A = \begin{bmatrix} a_{-1}-\varphi a_{n-1} \\ \vdots \\ a_{-n+1}-\varphi a_1 \\ -\varphi\beta \end{bmatrix}^T, \quad \hat{\mathbf{z}}^T A = \varphi \mathbf{e}_{n-1}^T. \tag{3.4}$$

Set, as in theorem 2.1, $\mathbf{z} = Z_{\frac{1}{\varphi}}\hat{\mathbf{z}}$, $\mathbf{v} = Z_{\frac{1}{\varphi}}\hat{\mathbf{v}}$. Then transposing both equations in (3.4) and taking into account the identity (3.1), we can conclude that vectors $\mathbf{z}$ and $\mathbf{v}$ are connected with the solutions $\mathbf{x} = (x_i)_{i=0}^{n-1}$ and $\mathbf{u} = (u_i)_{i=0}^{n-1}$ of the equations (3.3) in the following way :

$$\mathbf{z} = \varphi Z_{\frac{1}{\varphi}} J\mathbf{x}, \quad \mathbf{v} = -\varphi Z_{\frac{1}{\varphi}} J\mathbf{u}. \tag{3.5}$$

Hence,

$$\nabla_\varphi(A^{-1}) = \begin{bmatrix} x_0 \\ x_1 \\ \vdots \\ x_{n-1} \end{bmatrix} \cdot \begin{bmatrix} u_0 & \varphi u_{n-1} & \cdots & \varphi u_1 \end{bmatrix} - \begin{bmatrix} u_0 \\ u_1 \\ \vdots \\ u_{n-1} \end{bmatrix} \cdot \begin{bmatrix} x_0 & \varphi x_{n-1} & \cdots & \varphi x_1 \end{bmatrix}. \tag{3.6}$$

THEOREM 3.1 *Let* $A = (a_{i-j})_{i,j=0}^{n-1}$ *be a Toeplitz matrix. If for some* $\varphi \neq 0$ *and* $\beta \in \mathbf{C}$ *the equations*

$$A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{u} = \begin{bmatrix} \beta \\ a_1 - \frac{1}{\varphi}a_{-n+1} \\ \vdots \\ a_{n-1} - \frac{1}{\varphi}a_{-1} \end{bmatrix} \tag{3.7}$$

have the solutions $\mathbf{x}$ *and* $\mathbf{u}$*, then* A *is invertible and*

$$A^{-1} = \frac{\varphi}{\varphi - \psi}(\mathrm{Circ}_\psi(\mathbf{x}) \cdot \mathrm{Circ}_\varphi(\mathbf{u}) - \mathrm{Circ}_\psi(\mathbf{u} - \frac{\varphi - \psi}{\varphi}\mathbf{e}_0) \cdot \mathrm{Circ}_\varphi(\mathbf{x})), \tag{3.8}$$

where ψ $(\neq \varphi)$ *is an arbitrary complex number.*

PROOF. Let the vectors $\mathbf{x}$ and $\mathbf{u}$ be the solutions of the equations (3.7), $\hat{\mathbf{z}}$ and $\hat{\mathbf{v}}$ be solutions of the equations (3.4) and $\mathbf{z} = Z_{\frac{1}{\varphi}}\hat{\mathbf{z}}$, $\mathbf{v} = Z_{\frac{1}{\varphi}}\hat{\mathbf{v}}$. According to the theorem 2.1

$$A^{-1} = \mathrm{Circ}_{lr} - \frac{\varphi}{\varphi - \psi}(\mathrm{Circ}_\psi(\mathbf{x}) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{v})^T + \mathrm{Circ}_\psi(\mathbf{u}) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{z})^T). \tag{3.9}$$

As was already noted, the persymmetry condition (3.1) is invariant under matrix inversion, therefore the last row of A^{-1} is uniquely determined by its first column and, moreover, $\mathrm{Circ}_{lr} = \mathrm{Circ}_\varphi(\mathbf{x})$. Furthermore, the equalities (3.5) imply that $\mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{z})^T = \mathrm{Circ}_\varphi(\mathbf{x})$ and $\mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{v})^T = -\mathrm{Circ}_\varphi(\mathbf{u})$. Now, using these expressions, we conclude that (3.8) follows from (3.9).Ξ

The factor circulant representation (3.8) for matrix A^{-1} is based on the solutions of two matrix equations (3.7), where the right hand side of one equation depends on matrix A. In the following theorem we show, how to construct the factor circulant representation for A^{-1} using the solutions of the same equations (0.1), as in the standard Gohberg-Semencul formula.

THEOREM 3.2 *Let* $A = (a_{i-j})_{i,j=0}^{n-1}$ *be a Toeplitz matrix. If the vectors* $\mathbf{x} = (x_i)_{i=0}^{n-1}$ *and* $\mathbf{y}$ *satisfy the equations*

$$A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{y} = \mathbf{e}_{n-1} \tag{3.10}$$

and $x_0 \neq 0$*, then* A *is invertible and*

$$A^{-1} = \frac{1}{x_0(\varphi - \psi)}(\mathrm{Circ}_\psi(\mathbf{x}) \cdot \mathrm{Circ}_\varphi(Z_\varphi\mathbf{y}) - \mathrm{Circ}_\psi(Z_\psi\mathbf{y}) \cdot \mathrm{Circ}_\varphi(\mathbf{x})), \tag{3.11}$$

where φ *and* ψ $(\neq \varphi)$ *are any complex numbers.*

Formula (3.11) is a generalization of the formula of G.Ammar and P.Gader [AG], where they considered the case $\varphi = 1$, $\psi = -1$ and Hermitian Toeplitz matrix A. Note, if the matrix A is Hermitian, then the persymmetry of A^{-1} means that the solutions $\mathbf{x}$ and $\mathbf{y}$ of (3.10) are related by $\mathbf{y} = J\bar{\mathbf{x}}$. Thus, to determine in this case the parameters of factor circulants from the right hand side of (3.11) we have to solve only one equation.

According to theorem 3.2, if $x_0 \neq 0$, then the first and the last columns of A^{-1} are sufficient for restoring the whole matrix A^{-1}. In a general situation the inverse of an arbitrary Toeplitz matrix cannot be determined by any fixed pair of its columns. As was shown in [BAS] for this purpose three properly chosen columns of A^{-1} are always enough. Now let us formulate the factor circulant analog of this result.

Hereafter we shall assume that the coordinate vectors are indexed by the integers modulo n, such that $\mathbf{e}_n = \mathbf{e}_0$.

THEOREM 3.3 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If for an integer k $(0 \le k \le n-1)$ the equations*

$$A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{y} = \mathbf{e}_k, \quad A\mathbf{z} = \mathbf{e}_{k+1}$$

have solutions $\mathbf{x} = (x_i)_{i=0}^{n-1}$, $\mathbf{y}$ and $\mathbf{z}$, where $x_{n-1-k} \neq 0$, then A is invertible and

$$A^{-1} = \frac{1}{x_{n-1-k}(\varphi - \psi)}(\mathrm{Circ}_\psi(\mathbf{x}) \cdot \mathrm{Circ}_\varphi(Z_\varphi \mathbf{y} - \mathbf{z}) - \mathrm{Circ}_\psi(Z_\psi \mathbf{y} - \mathbf{z}) \cdot \mathrm{Circ}_\varphi(\mathbf{x})), \qquad (3.12)$$

where φ $(\neq 0)$ and ψ $(\neq \varphi)$ are an arbitrary complex numbers.

Theorem 3.2 is a particular case of the last theorem taking $k = n-1$ (i.e. $\mathbf{e}_{k+1} = \mathbf{e}_0$) and $\mathbf{z} = \mathbf{x}$. Moreover, taking $k = 0$ and $\mathbf{y} = \mathbf{x}$, (3.12) can be regarded as factor circulant analog of the Gohberg-Krupnik formula [GK] (see also [GF]).

Theorem 3.3 follows from the theorem 3.1, the next lemma, and the equality $y_{n-1} = x_{n-1-k}$, which holds in view of (3.1).

LEMMA 3.4 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If for some integer k $(0 \le k \le n-1)$ the equations*

$$A\mathbf{y} = \mathbf{e}_k \quad \text{and} \quad A\mathbf{z} = \mathbf{e}_{k+1}$$

have solutions $\mathbf{y} = (y_i)_{i=0}^{n-1}$ and $\mathbf{z} = (z_i)_{i=0}^{n-1}$, where $y_{n-1} \neq 0$, then the vector $\mathbf{u} = \frac{1}{\varphi y_{n-1}}(Z_\varphi \mathbf{y} - \mathbf{z})$ solve the second equation in (3.7) for some β.

PROOF. Set $\beta = a_0 + \frac{1}{\varphi y_{n-1}} \sum_{i=1}^{n-1} a_{-i} y_{i-1}$ if $k < n-1$, or $\beta = -\frac{1}{\varphi y_{n-1}} + a_0 + \frac{1}{\varphi y_{n-1}} \sum_{i=1}^{n-1} a_{-i} y_{i-1}$ in the case $k = n-1$. Then

$$A\mathbf{u} = \frac{1}{\varphi y_{n-1}} A \begin{bmatrix} \varphi y_{n-1} \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \frac{1}{\varphi y_{n-1}} A \begin{bmatrix} 0 \\ y_0 \\ \vdots \\ y_{n-2} \end{bmatrix} - \frac{1}{\varphi y_{n-1}} A\mathbf{z} =$$

$$= \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ a_{n-1} \end{bmatrix} + \frac{1}{\varphi y_{n-1}} \begin{bmatrix} \varphi y_{n-1}(\beta - a_0) \\ -y_{n-1} a_{-n+1} \\ \vdots \\ -y_{n-1} a_{-1} \end{bmatrix} + \frac{1}{\varphi y_{n-1}} \mathbf{e}_{k+1} - \frac{1}{\varphi y_{n-1}} \mathbf{e}_{k+1} = \begin{bmatrix} \beta \\ a_1 - \frac{1}{\varphi} a_{-n+1} \\ \vdots \\ a_{n-1} - \frac{1}{\varphi} a_{-1} \end{bmatrix}.$$

Ξ

Another representation of the inverse of the Toeplitz matrix as a sum of the products of lower and upper triangular Toeplitz matrices was obtained in [HR]. In this paper the parameters of these triangular Toeplitz matrices were determined by the solutions of another pair of the equations. The factor circulant analog of the Heinig-Rost formula appears in the following theorem.

THEOREM 3.5 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If for some $\gamma \in \mathbf{C}$ the equations*

$$A\mathbf{x} = \mathbf{e}_0 \quad \text{and} \quad A\mathbf{v} = \begin{bmatrix} \gamma \\ a_{-n+1} \\ \vdots \\ a_{-1} \end{bmatrix} \tag{3.13}$$

are solvable, then A is invertible and

$$A^{-1} = \frac{1}{\varphi - \psi}(\mathrm{Circ}_\psi(\mathbf{x}) \cdot \mathrm{Circ}_\varphi(\varphi\mathbf{e}_0 - \mathbf{v}) - \mathrm{Circ}_\psi(\psi\mathbf{e}_0 - \mathbf{v}) \cdot \mathrm{Circ}_\varphi(\mathbf{x})),$$

where φ $(\neq 0)$ and ψ $(\neq \varphi)$ are arbitrary complex numbers.

The proof of theorem 3.5 is straightforward consequence of theorem 3.1 and the following obvious lemma.

LEMMA 3.6 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If the second equation in* (3.13) *has a solution $\mathbf{v} \in \mathbf{C}^n$, then the vector $\mathbf{u} = \mathbf{e}_0 - \frac{1}{\varphi}\mathbf{v}$ fulfill the second equation in* (3.7) *with $\beta = a_0 - \frac{1}{\varphi}\gamma$.*

4. FORMULAS WITH TRIANGULAR TOEPLITZ MATRICES

With the help of the φ-cyclic displacement, the proof of the standard Gohberg-Semencul formula and some of its analogs can be simplified. As in the first section, let $\mathbf{f}_m$ and $\mathbf{g}_m$ $(m = 1, 2, ..., \alpha)$ be the 2α vectors which satisfy the condition

$$\sum_{m=1}^{\alpha} \mathrm{Circ}_\varphi(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T = 0.$$

In this case formulas (1.4) and (1.5) give explicit expressions for the solution X of the equation

$$\nabla_\varphi(X) = \sum_{m=1}^{\alpha} \mathbf{f}_m \cdot \mathbf{g}_m^T. \tag{4.1}$$

It is easy to observe that if the vector $\breve{\mathbf{g}}$ differs from the vector $\mathbf{g}$ only in the first coordinate, then the matrix $\mathrm{Circ}_{\frac{1}{\psi}}(\breve{\mathbf{g}})$ is a linear combination of $\mathrm{Circ}_{\frac{1}{\psi}}(\mathbf{g})$ and the identity matrix I. Therefore, in view of the assertion (i) of the theorem 1.1 the formula (1.5) for the solution of the equation (4.1) can be rewritten in the form

$$X = \mathrm{Circ}_{fc} + \frac{\psi}{\psi - \varphi} \sum_{m=1}^{\alpha} \mathrm{Circ}_\varphi(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\psi}}(\breve{\mathbf{g}}_m)^T, \tag{4.2}$$

where $\breve{\mathbf{g}}_m = R\mathbf{g}_m \quad (m = 1, 2, ..., \alpha)$ with $R = \mathrm{diag}(\frac{\varphi}{\psi}, 1, 1, ..., 1)$, and Circ_{fc} is an arbitrary φ-circulant. It is easy to see that in this case the first columns of the matrices X and Circ_{fc} are the same. Setting $\psi = 0$ in (1.4) and $\psi = \infty$ in (4.2) we obtain the following expressions for the solution of the equation (4.1) :

$$X = \mathrm{Circ}_{lr} + \sum_{m=1}^{\alpha} L(\mathbf{f}_m) \cdot \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T \tag{4.3}$$

and

$$X = \mathrm{Circ}_{fc} + \sum_{m=1}^{\alpha} \mathrm{Circ}_{\varphi}(\mathbf{f}_m) \cdot U(\tilde{\mathbf{g}}_m). \tag{4.4}$$

Here $\tilde{\mathbf{g}}_m = Q\mathbf{g}_m$ $(m = 1, 2, ..., \alpha)$, where $Q = \mathrm{diag}(0, 1, 1, ..., 1)$ is the canonical projector onto the last $n-1$ coordinates. As was already noted, for the case φ= 1 both formulas (4.3) and (4.4) are due to G.Ammar and P.Gader [AG].

Through this section all solutions $X = (x_{ij})_{i=0}^{n-1}$ of the equation (4.1) will be assumed to have the property

$$x_{i0} = x_{n-1,n-1-i} \quad (i = 0, 1, ..., n-1), \tag{4.5}$$

or, in other words, the corresponding matrices Circ_{lr} in (4.3) and Circ_{fc} in (4.4) are the same. Representing

$$\mathrm{Circ}_{\varphi}(\mathbf{f}_m) = L(\mathbf{f}_m) + \varphi U(Z_0 J\mathbf{f}_m) \quad \text{and} \quad \mathrm{Circ}_{\frac{1}{\varphi}}(\mathbf{g}_m)^T = \frac{1}{\varphi} L(Z_{\varphi} J\mathbf{g}_m) + U(\tilde{\mathbf{g}}_m),$$

where J denotes as above the reverse identity matrix, and subtructing (4.3) from (4.4), we have

$$\frac{1}{\varphi} \sum_{m=1}^{\alpha} L(\mathbf{f}_m) \cdot L(Z_{\varphi} J\mathbf{g}_m) = \varphi \sum_{m=1}^{\alpha} U(Z_0 J\mathbf{f}_m) \cdot U(\tilde{\mathbf{g}}_m). \tag{4.6}$$

Since the sum in the left hand side of the last identity is the lower triangular matrix and the expression in the right hand side is the upper triangular matrix with zeros on the main diagonal, therefore sums in both sides of (4.6) are equal to zero matrix. Hence, (4.3) and (4.6) imply that formula

$$X = \mathrm{Circ}_{lr} + \sum_{m=1}^{\alpha} L(\mathbf{f}_m) \cdot U(\tilde{\mathbf{g}}_m) \tag{4.7}$$

with $\tilde{\mathbf{g}}_m = Q\mathbf{g}_m$, where Q is a canonical projector onto the last $n-1$ coordinates, gives us another representation for the solution of the equation (4.1).

If it is clear that the solution $X \in \mathbf{C}^{n \times n}$ of the equation (4.1) satisfies the condition (4.5), then the formula (4.7) holds, and it allows one to restore the whole matrix X from its φ-cyclic displacement and the last row. As was already mentioned, the Toeplitz matrices and also their inverses satisfy the stronger condition of persymmetry. Hence, for the inverse of the Toeplitz matrix the formula (4.7) and the representation (3.6) of its φ-cyclic displacement immediately imply the following theorem.

THEOREM 4.1 *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If for some $\varphi \neq 0$, $\beta \in \mathbf{C}$ the equations*

$$A\mathbf{x} = \mathbf{e}_0 \quad \text{and} \quad A\mathbf{u} = \begin{bmatrix} \beta \\ a_1 - \frac{1}{\varphi}a_{-n+1} \\ \vdots \\ a_{n-1} - \frac{1}{\varphi}a_{-1} \end{bmatrix}$$

are solvable, then A is invertible and

$$A^{-1} = \varphi(L(\mathbf{x}) \cdot U(Z_0J\mathbf{u} + \frac{1}{\varphi}\mathbf{e}_0) - L(\mathbf{u} - \mathbf{e}_0) \cdot U(Z_0J\mathbf{x})).$$

Theorem 4.1 and lemma 3.4 yields the following above mentioned result of Ben-Artzi and Shalom.

THEOREM 4.2 ([BAS]) *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If for some integer k $(0 \leq k \leq n-1)$ the equations*

$$A\mathbf{x} = \mathbf{e}_0, \quad A\mathbf{y} = \mathbf{e}_k \quad \text{and} \quad A\mathbf{z} = \mathbf{e}_{k+1}$$

have solutions $\mathbf{x} = (x_i)_{i=0}^{n-1}$, $\mathbf{y}$ and $\mathbf{z}$, where $x_{n-1-k} \neq o$, then A is invertible and

$$A^{-1} = \frac{1}{x_{n-1-k}}(L(\mathbf{x}) \cdot U(J\mathbf{y} - Z_0J\mathbf{z}) + L(\mathbf{z} - Z_0\mathbf{y}) \cdot U(Z_0J\mathbf{x})). \tag{4.8}$$

The well known Gohberg-Semencul [GS] and Gohberg-Krupnik [GK] (see also [GF]) formulas appear as natural extremal cases of (4.8). More precisely, the choice $k = n-1$ (it means that $\mathbf{e}_{k+1} = \mathbf{e}_0$) and $\mathbf{z} = \mathbf{x}$ corresponds to the Gohberg-Semencul formula (0.2), or taking $k = 0$ and $\mathbf{y} = \mathbf{x}$ we get the formula of Gohberg and Krupnik.

Theorem 4.1 and lemma 3.6 imply the following result of Heinig and Rost.

THEOREM 4.3 ([HR]) *Let $A = (a_{i-j})_{i,j=0}^{n-1}$ be a Toeplitz matrix. If the equations*

$$A\mathbf{x} = \mathbf{e}_0 \quad \text{and} \quad A\mathbf{v} = \begin{bmatrix} \gamma \\ a_{-n+1} \\ \vdots \\ a_{-1} \end{bmatrix} \quad (\gamma \in \mathbf{C})$$

are solvable, then A is invertible and

$$A^{-1} = L(\mathbf{x}) \cdot U(\mathbf{e}_0 - Z_0J\mathbf{v}) + L(\mathbf{v}) \cdot U(Z_0J\mathbf{x}).$$

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School of Mathematical Sciences
Raymond and Beverly Sackler
Faculty of Exact Sciences
Tel Aviv University
Ramat Aviv 69978. ISRAEL

MSC 1991 : Primary 15A09, Secondary 47B35

Submitted : February 9, 1992