

## CIRCULAR CHROMATIC NUMBER AND GRAPH MINORS

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**Abstract.** This paper proves that for any integer  $n \geq 4$  and any rational number  $r$ ,  $2 \leq r \leq n - 2$ , there exists a graph  $G$  which has circular chromatic number  $r$  and which does not contain  $K_n$  as a minor.

### 1. INTRODUCTION

The circular chromatic number of a graph (also known as the “star chromatic number”) is a natural generalization of the chromatic number of a graph. For a pair of integers  $p \geq q$ , a  $(p, q)$ -colouring of a graph  $G$  is a mapping  $c$  from  $V(G)$  to the set  $\{0, 1, \dots, p - 1\}$  such that for any adjacent vertices  $x, y$  of  $G$ ,  $q \leq |c(x) - c(y)| \leq p - q$ . The *circular chromatic number*  $\chi_c(G)$  of a graph  $G$  is the infimum of the ratios  $p/q$  for which there exists a  $(p, q)$ -colouring of  $G$ .

It was shown by Vince [13] (cf. also [2] for a combinatorial proof) that for finite graphs  $G$ , the infimum in the definition above is always attained, and hence can be replaced by minimum.

Note that a  $(p, 1)$ -colouring of a graph is just a  $p$ -colouring of  $G$ . Therefore we have  $\chi_c(G) \leq \chi(G)$ . On the other hand, it was proved in [13] that for any graph  $G$  we have  $\chi(G) - 1 < \chi_c(G)$ . Hence if we know the circular chromatic number of a graph  $G$ , then we can obtain its chromatic number by taking the ceiling of  $\chi_c(G)$ , i.e.,  $\chi(G) = \lceil \chi_c(G) \rceil$ . However, two graphs of the same chromatic number may have different circular chromatic numbers. In this sense,  $\chi_c(G)$  is a refinement of the parameter  $\chi(G)$ , and  $\chi(G)$  is an approximation of  $\chi_c(G)$ .

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Since the infimum in the definition of  $\chi_c(G)$  can be replaced by minimum for finite graphs, we know that  $\chi_c(G)$  is a rational number for any finite graph  $G$ . On the other hand, for any rational  $p/q \geq 2$ , there is a finite graph  $G$  such that  $\chi_c(G) = p/q$ . Suppose  $p/q \geq 2$  and that  $(p, q) = 1$ . Let  $G_p^q$  be the graph with vertex set  $\{0, 1, \dots, p-1\}$ , in which  $ij$  is an edge if and only if  $q \leq |i-j| \leq p-q$ . Then it was shown in [13] that  $\chi_c(G_p^q) = p/q$ .

Given a property  $P$  of graphs, it is usually an interesting and difficult problem to determine whether or not there exists a graph  $G$  which has the property  $P$  and whose circular chromatic number is equal to a given rational number  $r$ . One such problem was discussed in [15]. For an integer  $g$ , let  $P(g)$  be the property of having girth at least  $g$ . It was shown in [15] that for any integer  $g$  and for any rational number  $r \geq 2$ , there exists a graph  $G$  which has property  $P(g)$  and which has circular chromatic number  $r$ . (For a constructive proof, see [20].) This result is a generalization of the result of Erdős concerning the existence of graphs with arbitrarily large girth and arbitrarily large chromatic number. Another such problem was discussed in [7, 18]. Let  $P$  be the property of being a planar graph. In [7, 18], the authors asked the problem that for which rational numbers  $r$  there exists a planar graph  $G$  which has circular chromatic number  $r$ . It follows from the Four Colour Theorem, that the number  $r$  is at most 4. It was shown in [7] that for any rational  $r$  between 2 and 3, there is a planar graph  $G$  with circular chromatic number  $r$ , and it was shown in [18] that for any rational number  $r$  between 3 and 4, there is a planar graph  $G$  with circular chromatic number  $r$ . Therefore a rational  $r$  is the chromatic number of a planar graph if and only if  $r = 1$  or  $2 \leq r \leq 4$ .

A graph  $H$  is called a *minor* of a graph  $G$  if  $H$  is isomorphic to a graph which is obtained from a subgraph of  $G$  by contracting some edges. We say a graph  $G$  is  *$H$ -minor free* if  $H$  is not a minor of  $G$ . As a generalization of the Four Colour Problem, Hadwiger conjectured that any graph  $G$  with chromatic number at least  $n$  contains  $K_n$  as a minor. Hadwiger's conjecture remains to be one of the major open problems in mathematics. The case  $n = 5$  of this conjecture is equivalent to the Four Colour Theorem, of which the only existing proofs rely on computer [1, 10]. The case  $n = 6$  was settled in [11], where the proof relies on the Four Colour Theorem and is quite complicated.

In this paper, we ask for which rational numbers  $r$  there exists a graph  $G$  which does not contain  $K_n$  as a minor and which has circular chromatic number  $r$ . If Hadwiger's conjecture is true, then for any rational number  $r > n - 1$ , any graph with circular chromatic number  $r$  does contain  $K_n$  as a minor. Therefore we shall only consider those rationals  $r \leq n - 1$ . Our main result is the following theorem:

**Theorem 1.1.** *Suppose  $n \geq 4$  is an integer and  $r$  is a rational. If  $2 \leq r \leq n - 2$ , then there is a  $K_n$ -minor free graph which has circular chromatic number  $r$ .*

In case that  $r$  is between  $n - 2$  and  $n - 1$ , it remains an open question whether or not there exists a  $K_n$ -minor free graph with circular chromatic number  $r$ . (The case  $r \geq n$  concerns Hadwiger's conjecture, as explained above.) However, we have some partial results. First of all, it was shown in [18] that for any rational number between 3 and 4, there is a planar graph with circular chromatic number  $r$ . As planar graphs are  $K_5$ -minor free, we have the following result:

**Theorem 1.2.** *For any rational number  $r$  between 3 and 4, there exists a  $K_5$ -minor free graph  $G$  with circular chromatic number  $r$ .*

If  $n = 4$ , then  $K_4$ -minor free graphs have a relative simple structure. The circular chromatic number of  $K_4$ -minor free graphs has been studied in [5, 3, 8]. P. Hell and the author [5] proved a somehow surprising result stating that for a  $K_4$ -minor graph  $G$ , either  $\chi_c(G) = 3$  or  $\chi_c(G) \leq 8/3$ .

For  $n \geq 6$ , we shall prove the following theorem in this paper:

**Theorem 1.3.** *Suppose  $n \geq 6$  and  $n - 2 \leq r \leq n - 1$ . If the Farey sequence of  $r$  has length at most 2 and if  $\alpha_2 = 2$  (see Section 2 for the definitions of Farey sequence and  $\alpha_i$ ), then there is a graph  $G$  which is  $K_n$ -minor free and which has circular chromatic number  $r$ .*

## 2. THE PROOF OF THEOREM 1.1

In this section, we shall prove Theorem 1.1. If  $n = 4$ , then the result is trivial. If  $n = 5$  or 6, then the result follows from theorems of [7, 18], where it was proved that any rational number between 2 and 4 is the circular chromatic number of a planar graph, since planar graphs are known to be  $K_5$ -minor free. Thus we assume that  $n \geq 7$  is a fixed integer,  $r = p/q$  is a fixed rational number (where  $p$  and  $q$  are integers with  $(p, q) = 1$ ), and that  $4 \leq r \leq n - 2$ . (The case that  $m \leq 4$  follows from the results in [7, 18] about planar graphs.) We shall construct a  $K_n$ -minor free graph, denoted by  $Z(p, q)$ , such that  $\chi_c(Z(p, q)) = p/q$ .

### 2.1 The Construction

The construction method is a modification of the method used in [10], where a sparse subgraph of  $G_p^q$  which has the same circular chromatic number as  $G_p^q$  is constructed.

If  $q = 1$ , then  $r = p$  is an integer. It is easy to see that in this case we may let  $Z(p, q) = K_p$ . Thus we assume that  $m < r < m + 1$  for some integer  $4 \leq m \leq n - 3$ .

Since  $(p, q) = 1$ , there exist unique integers  $p', q'$  such that  $p' < p, q' < q$  and  $pq' - qp' = 1$ . It is straightforward to verify that  $p'/q' < p/q$  and that  $p'/q'$  is the largest fraction with the property that  $p'/q' < p/q$  and  $p' \leq p$ . Similarly, we let  $p'', q''$  be positive integers such that  $p'' < p', q'' < q'$  and  $p'q'' - p''q' = 1$ . Then  $p''/q''$  is the largest fraction with the property that  $p''/q'' < p'/q'$  and that  $p'' \leq p'$ . Repeating this process of finding smaller and smaller fractions, we must reach the fraction  $m/1$  in a finite number of steps. Thus to any rational  $p/q$  between  $m$  and  $m + 1$ , there corresponds a unique sequence of fractions

$$\frac{m}{1} = \frac{p_0}{q_0} < \frac{p_1}{q_1} < \frac{p_2}{q_2} < \dots < \frac{p_k}{q_k} = \frac{p}{q}.$$

We call the sequence  $(p_i/q_i : i = 0, 1, \dots, k)$  the *Farey sequence* of  $p/q$ . The number  $k$  is called the *length* of the Farey sequence of  $p/q$ .

For convenience, we let  $p_{-1} = -1$  and  $q_{-1} = 0$ . Then for  $i = 1, 2, \dots, n$ , we have  $p_i q_{i-1} - p_{i-1} q_i = 1$  and  $p_{i-1} q_{i-2} - p_{i-2} q_{i-1} = 1$ . It follows that, for  $1 \leq i \leq k$ , we have  $p_{i-1}(q_i + q_{i-2}) = q_{i-1}(p_i + p_{i-2})$ . As  $p_{i-1}, q_{i-1}$  are co-prime,

$$\alpha_i = \frac{p_i + p_{i-2}}{p_{i-1}} = \frac{q_i + q_{i-2}}{q_{i-1}}$$

is an integer, which is greater than 1, and hence is at least 2. We call  $(\alpha_1, \alpha_2, \dots, \alpha_k)$  the *alpha sequence* of  $p/q$ , which is obviously uniquely determined by  $p/q$ . The process of deducing the alpha sequence from the rational  $p/q$  can also be reversed. In other words, each sequence  $(\alpha_1, \alpha_2, \dots, \alpha_k)$  with  $\alpha_i \geq 2$  determines a rational  $p/q$  between  $m$  and  $m + 1$ . Indeed, given the alpha sequence  $(\alpha_1, \alpha_2, \dots, \alpha_k)$ , the fractions  $p_i/q_i$  can be easily determined by solving the difference equations

$$(*) \quad p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}$$

with the initial condition that  $(p_{-1}, q_{-1}) = (-1, 0)$  and  $(p_0, q_0) = (m, 1)$ .

Having determined the alpha sequence of the rational  $p/q$ , we can start constructing the  $K_n$ -minor graph  $G$  which has circular chromatic number  $p/q$ . We shall indeed construct a sequence of graph  $G_i$ , for  $i = 1, 2, \dots, k$ , such that  $\chi_c(G_i) = p_i/q_i$ , and that each of  $G_i$  is  $K_{m+3}$ -minor free (and hence  $K_n$ -minor free, as  $n \geq m + 3$ ).

Before constructing the graphs  $G_i$ , we shall recursively construct ordered graphs  $F_i, H_i$  (the vertices of  $F_i$  and  $H_i$  are linearly ordered). Let  $f_i = |F_i|$

and  $h_i = |H_i|$ ; the vertices of  $F_i$  will usually be named  $(x_{i,1}, x_{i,2}, \dots, x_{i,f_i})$ , and considered in this order, and the vertices of  $H_i$  will usually be named  $(y_{i,1}, y_{i,2}, \dots, y_{i,h_i})$ , and considered in this order. (Sometimes we shall use simpler indices to denote the vertices, when no confusion occurs.)

For an edge  $e = (x, y)$  of an ordered graph, we define the *order length* of  $e$ , denoted by  $\ell(e)$ , to be the positive difference between the positions of  $x$  and  $y$ .

**Definition 2.1.** *Suppose  $X$  and  $Y$  are disjoint ordered graphs whose vertex orderings are  $(x_1, x_2, \dots, x_s)$  and  $(y_1, y_2, \dots, y_t)$ , respectively. When we say hook*

Type 1 hook                      Type 2 hook

Figure 1.

$X$  to  $Y$  with type 1 hook, we are adding the following edges between  $X$  and  $Y$ :

$$x_1y_t, x_sy_1, x_sy_2, x_sy_3, \dots, x_sy_{m-1}.$$

When we say hook  $X$  to  $Y$  with type 2 hook, we are adding the following edges between  $X$  and  $Y$ :

$$x_1y_t, x_1y_{t-1}, \dots, x_1y_{t-m+2}, x_sy_1.$$

In case the graph  $X$  (resp.  $Y$ ) is a singleton, then in the definition of the hooks, we set  $x_1 = x_2 = \dots = x_s$  (resp.  $y_1 = y_2 = \dots = y_t$ ). The edge  $x_1y_t$  of either type of hooks will be called a long edge of that hook.

Figure 1 above depicts the two types of hooks.

For an integer  $t$ , we let  $Q_t$  be the  $(m - 1)$ st power of the path of length  $t - 1$ , i.e.,  $Q_t$  has vertex set  $\{v_1, v_2, \dots, v_t\}$  in which two vertices  $v_i$  and  $v_j$  are adjacent if  $|i - j| \leq m - 1$ . The graph  $Q_t$  is considered as an ordered graph with the order  $(v_1, v_2, \dots, v_t)$  of the vertices.

First of all, we let  $F_1$  be a singleton, let  $H_1 = Q_{m\alpha_1}$ , and let  $F_2 = Q_{m(\alpha_1-1)}$ .

For  $i \geq 1$ , to construct the graph  $H_{i+1}$ , we take  $\alpha_{i+1}$  copies of  $F_i$ , denoted by  $F_i^1, F_i^2, \dots, F_i^{\alpha_{i+1}}$ , and  $\alpha_{i+1} - 1$  copies of  $H_i$ , denoted by  $H_i^1, H_i^2, \dots, H_i^{\alpha_{i+1}-1}$  and hook them together as follows:

- If  $i$  is odd, then for  $j = 1, 2, \dots, \alpha_{i+1} - 1$ , we hook  $F_i^j$  to  $H_i^j$  with type 1 hook, and hook  $F_i^{j+1}$  to  $H_i^j$  with type 2 hook;
- If  $i$  is even, then for  $j = 1, 2, \dots, \alpha_{i+1} - 1$ , we hook  $H_i^j$  to  $F_i^j$  with type 2 hook, and hook  $H_i^j$  to  $F_i^{j+1}$  with type 1 hook.

The resulting graph is  $H_{i+1}$ .

The graph  $F_{i+2}$  is constructed in the same way as the graph  $H_{i+1}$ , but with one less copy of  $F_i$  and  $H_i$ , i.e.,  $F_{i+2}$  is constructed from  $\alpha_{i+1} - 1$  copies of  $F_i$  and  $\alpha_{i+1} - 2$  copies of  $H_i$ .

The graphs  $H_{i+1}$  and  $F_{i+2}$  are regarded as ordered graphs. The order of the vertices is: the vertices of  $F_i^1$  in order, followed by the vertices of  $H_i^1$  in order, followed by the vertices of  $F_i^2$  in order, etc.

Finally when  $i$  is even, we let  $G_i$  be the graph obtained by hooking  $H_i$  to  $F_i$  with type 1 hook; when  $i$  is odd, we let  $G_i$  be the graph obtained by hooking  $F_i$  to  $H_i$  with type 1 hook. We shall regard  $G_i$  as an ordered graph as well, where the order of the vertices is: those of  $F_i$  in order, followed by those of  $H_i$  in order.

We note that the mapping  $f : V(H_i) \rightarrow V(H_i)$  defined as  $f(y_{i,j}) = y_{i,h_i-j+1}$  is an automorphism of  $H_i$ , and that the mapping  $g : V(F_i) \rightarrow V(F_i)$  defined as  $f(x_{i,j}) = x_{i,f_i-j+1}$  is an automorphism of  $F_i$ . Therefore, in the construction of  $G_i$ , when we hook  $F_i$  to  $H_i$  or  $H_i$  to  $F_i$ , it makes no difference which of the two types of hooks is used.

This finishes the construction of the graphs  $G_i$ . The graph  $Z(p, q)$  is equal to  $G_n$ .

## 2.2 The Minors of $G_i$

In these two sections we shall prove that the graphs  $G_i$  are  $K_{m+3}$ -minor free, and that  $\chi_c(G_i) = \chi_c(G'_i) = p_i/q_i$ .

To prove that each of the graphs  $G_i$  is  $K_{m+3}$ -minor free, we shall need the following lemmas, which are quite obvious.

**Lemma 2.1.** *Suppose a graph  $G$  contains  $K_k$  as a minor, and that  $x$  is a vertex of  $G$  of degree at most  $k - 2$ . Then there is a neighbour  $y$  of  $x$  such that the graph  $G|_{xy}$ , which is obtained from  $G$  by contracting the edge  $xy$ , also contains  $K_k$  as a minor.*

Let  $G$  be a graph. A *decomposition* of  $G$  by means of a subgraph  $H$  is an expression of  $G$  in the form

$$G = G_1 + G_2, \quad G_1 \cap G_2 = H.$$

In case  $H$  is a complete graph, then the expression above is called a *simplex decomposition* of  $G$ .

**Lemma 2.2.** *Suppose  $G = G_1 + G_2$  is a simplex decomposition of  $G$ . If both  $G_1$  and  $G_2$  are  $K_k$ -minor free, then  $G$  is  $K_k$ -minor free.*

Suppose  $X$  is an ordered graph with vertices  $\{x_1, x_2, \dots, x_t\}$  in that order, and  $t \geq m$ . Let  $\bar{X}$  be the graph obtained from  $X$  by adding two vertices  $u, v$  and the following edges:

$$x_1x_t, ux_1, ux_2, \dots, ux_{m-1}, ux_t, vx_1, vx_t, vx_{t-1}, \dots, vx_{t-m+2}, uv.$$

Let  $\tilde{X}$  be the graph obtained from  $X$  by adding the edge  $x_1x_t$ .

**Theorem 2.1.** *For any  $i \geq 1$ , if  $i$  is odd, then the graphs  $\overline{H}_i$  and  $\tilde{F}_i$  are  $K_{m+3}$ -minor free; if  $i$  is even then the graphs  $\tilde{H}_i$  and  $\overline{F}_i$  are  $K_{m+3}$ -minor free.*

*Proof.* We shall prove it by induction on  $i$ . First we consider the case that  $i = 1$ . It is easy to see (or prove by induction) that the  $j$ th power of a path is  $K_{j+2}$ -minor free. Hence for any integer  $t$ , the graph  $Q_t$  is  $K_{m+1}$ -minor free. Thus  $H_1 = Q_{m\alpha_1}$  is  $K_{m+1}$ -minor free, and hence  $(\overline{H}_1)$  is  $K_{m+3}$ -minor free, as  $(\overline{H}_1)$  is obtained from  $H_1$  by adding two vertices. Similarly,  $\overline{F}_2$  is  $K_{m+3}$ -minor free. We note that  $\tilde{F}_1$  is obviously  $K_{m+3}$ -minor free.

Assume that Theorem 2.1 is true for  $i \leq k - 1$ . Assume first that  $k$  is even. We consider the graph  $\tilde{H}_k$ . The graph  $H_k$  is obtained from  $\alpha_k$  copies of  $F_{k-1}$  and  $\alpha_k - 1$  copies of  $H_{k-1}$ . Suppose the vertices of  $F_{k-1}$  are  $x_1, x_2, \dots, x_s$  and that the vertices of  $H_{k-1}$  are  $y_1, y_2, \dots, y_t$ . We shall denote by  $x_1^j, x_2^j, \dots, x_s^j$  the vertices of the  $j$ th copy of  $F_{k-1}$ , and denote by  $y_1^j, y_2^j, \dots, y_t^j$  the vertices of the  $j$ th copy of  $H_{k-1}$ . Recall that the  $j$ th copy of  $F_{k-1}$  is hooked to the  $j$ th copy of  $H_i$  by type 1 hook, and the  $(j + 1)$ th copy of  $F_{k-1}$  is hooked to the  $j$ th copy of  $H_{k-1}$  by type 2 hook.

We shall add to the graph  $\tilde{H}_k$  the following edges:

- for  $j = 1, 2, \dots, \alpha_k$ , add the edge  $x_1^j x_s^j$ ;
- for  $j = 1, 2, \dots, \alpha_k - 1$ , add the edges  $x_s^j y_t^j, x_1^{j+1} y_1^j, x_s^j x_1^{j+1}, y_1^j y_t^j$ .

We shall prove that after adding these edges to  $\tilde{H}_k$ , the resulting graph, denoted by  $M_k$ , is still  $K_{m+3}$ -minor free.

It is straightforward to verify that for  $j = 1, 2, \dots, \alpha_k$ , each of the sets  $\{x_1^j, x_s^j\}$  is a cutset of the graph  $M_k$  which induces a complete subgraph of  $M_k$ , and also each of the sets  $\{x_s^j, y_1^j, y_t^j, x_1^{j+1}\}$  is a cutset of the graph  $M_k$  which induces a complete subgraph of  $M_k$ . By repeatedly applying Lemma 2.2 and the induction hypothesis, we conclude that  $M_k$  contains  $K_{m+3}$  as a minor if and only if the subgraph of  $M_k$  induced by the set

$$\left\{ x_1^1, x_s^1, y_1^1, y_t^1, x_1^2, x_s^2, y_1^2, y_t^2, \dots, y_1^{\alpha_k-1}, y_t^{\alpha_k-1}, x_1^{\alpha_k}, x_s^{\alpha_k} \right\}$$

contains  $K_{m+3}$  as a minor. However, this subgraph is easily seen to be  $K_7$ -minor free (hence  $K_{m+3}$ -minor free). Indeed, if the vertex  $x_1^1$  is deleted from this subgraph, then the resulting subgraph is a subgraph of the 3rd power of a path, and hence is  $K_5$ -minor free.

The graphs  $F_{k+1}$  and  $H_k$  have the same structure. The same argument shows that  $\tilde{F}_{k+1}$  is still  $K_{m+3}$ -minor free.



Assume next that  $k$  is odd. We consider the graph  $\overline{H}_k$ . The graph  $H_k$  is obtained from  $\alpha_k$  copies of  $F_{k-1}$  and  $\alpha_k - 1$  copies of  $H_{k-1}$ . Suppose the vertices of  $F_{k-1}$  are  $x_1, x_2, \dots, x_s$  and that the vertices of  $H_{k-1}$  are  $y_1, y_2, \dots, y_t$ . We shall denote by  $x_1^j, x_2^j, \dots, x_s^j$  the vertices of the  $j$ th copy of  $F_{k-1}$ , and denote by  $y_1^j, y_2^j, \dots, y_t^j$  the vertices of the  $j$ th copy of  $H_{k-1}$ . Recall that the  $j$ th copy of  $H_{k-1}$  is hooked to the  $j$ th copy of  $F_i$  by type 2 hook, and the  $j$ th copy of  $H_{k-1}$  is hooked to the  $(j + 1)$ th copy of  $F_{k-1}$  by type 1 hook.

We shall add to the graph  $\overline{H}_k$  the following edges:

- for  $j = 1, 2, \dots, \alpha_k$ , add the edge  $x_1^j x_s^j$ ;
- for  $j = 1, 2, \dots, \alpha_k - 1$ , add the edges  $x_1^j y_1^j, x_s^{j+1} y_t^j, y_1^j y_t^j$ ;
- for  $j = 1, 2, \dots, \alpha_k - 2$ , add the edge  $y_s^j y_1^{j+1}$ ;
- and finally, add the edges  $ux_s^1, uy_1^1, vy_t^{\alpha_k - 1}, vx_1^{\alpha_k}$ .

We shall prove that after adding these edges to  $\overline{H}_k$ , the resulting graph, denoted by  $M_k$ , is still  $K_{m+3}$ -minor free.

It is straightforward to verify that

- for  $j = 1, 2, \dots, \alpha_k - 1$ , each of the sets  $\{y_1^j, y_t^j\}$  is a cutset of the graph  $M_k$  which induces a complete subgraph of  $M_k$ ;
- for  $j = 2, 3, \dots, \alpha_k - 1$ , each of the sets  $\{y_t^{j-1}, x_1^j, x_s^j, y_1^j\}$  is a cutset of the graph  $M_k$  which induces a complete subgraph of  $M_k$ ;
- each of the two sets  $\{u, x_1^1, x_s^1, y_1^1\}, \{v, x_1^{\alpha_k}, x_s^{\alpha_k}, y_t^{\alpha_k - 1}\}$  is a cutset of the graph  $M_k$  which induces a complete subgraph.

By repeatedly applying Lemma 2.2 and the induction hypothesis, we conclude that  $M_k$  contains  $K_{m+3}$  as a minor if and only if the subgraph of  $M_k$  induced by the set

$$\left\{ u, x_1^1, x_s^1, y_1^1, y_t^1, x_1^2, x_s^2, y_1^2, y_t^2, \dots, y_1^{\alpha_k - 1}, y_t^{\alpha_k - 1}, x_1^{\alpha_k}, x_s^{\alpha_k}, v \right\}$$

contains  $K_{m+3}$  as a minor. However, this subgraph is easily seen to be  $K_7$ -minor free (hence  $K_{m+3}$ -minor free). Indeed, if the vertices  $u, v$  are deleted from this subgraph, then the resulting subgraph is a subgraph of the 3rd power of a path, and hence is  $K_5$ -minor free. Similarly, we can show that  $\overline{F}_{k+1}$  is  $K_{m+3}$ -minor free. This completes the proof of Theorem 2.1. ■

Note that  $G_i$  is a subgraph of  $H_{i+1}$ . Hence  $G_i$  is  $K_{m+3}$ -minor free for  $1 \leq i \leq n - 1$ . The same argument can prove that  $G_n$  is  $K_{m+3}$ -minor free.

**Corollary 2.1.** *For any  $1 \leq i \leq n$ , the graph  $G_i$  is  $K_{m+3}$ -minor free.*

### 2.3 The Circular Chromatic Number of $G_i$

It remains to show that for each  $i$  the graph  $G_i$  has circular chromatic number  $p_i/q_i$ . We shall use the same idea of the proof presented in [19]. However, the graphs in consideration are not exactly the same, and hence the proof is also technically different. In the argument below, we shall omit some of the details which are straightforward and are contained in [19].

Let  $g_i$  be the number of vertices of  $G_i$ . Straightforward calculation shows that  $g_i$  satisfies the same difference equation as  $p_i$ , and also  $g_i$  has the same initial value as  $p_i$ . Therefore  $|G_i| = p_i$  for  $i = 1, 2, \dots, k$ .

Now we shall prove that the circular chromatic number of  $G_i$  is at most  $p_i/q_i$ . Before proving this, we need some preliminary results about the relation between the Farey sequence and the alpha sequence. We observed before that the Farey sequence is uniquely determined by the alpha sequence. The numbers  $p_i$  and  $q_i$  are obtained by solving the following difference equations:

$$(*) \quad p_i = \alpha_i p_{i-1} - p_{i-2}, \quad q_i = \alpha_i q_{i-1} - q_{i-2}$$

with the initial condition that  $(p_{-1}, q_{-1}) = (-1, 0)$  and  $(p_0, q_0) = (m, 1)$ .

By repeatedly applying the equation (\*), we may express  $p_i$  (respectively,  $q_i$ ) in terms of  $p_j$  and  $p_{j-1}$  (respectively,  $q_j$  and  $q_{j-1}$ ) for any  $0 \leq j \leq i - 2$ . Lemma 2.3 below, which can be proved easily by induction, gives the explicit expressions. For  $1 \leq r \leq s \leq n$ , we let

$$\Lambda_{r,s} = \det \begin{pmatrix} \alpha_r & 1 & 0 & \cdots & 0 & 0 \\ 1 & \alpha_{r+1} & 1 & \cdots & 0 & 0 \\ 0 & 1 & \alpha_{r+2} & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & \alpha_{s-1} & 1 \\ 0 & 0 & 0 & \cdots & 1 & \alpha_s \end{pmatrix}.$$

**Lemma 2.3.** For  $0 \leq j \leq i - 2$ , we have

$$(**) \quad p_i = p_j \Lambda_{j+1,i} - p_{j-1} \Lambda_{j+2,i}, \quad q_i = q_j \Lambda_{j+1,i} - q_{j-1} \Lambda_{j+2,i}.$$

By letting  $j = 0$  in (\*\*), and by using the initial condition, we have

$$(***) \quad p_i = m \Lambda_{1,i} + \Lambda_{2,i}, \quad q_i = \Lambda_{1,i}.$$

**Lemma 2.4.** For  $0 \leq j \leq i - 2$ ,  $p_j q_i = p_i q_j - \Lambda_{j+2,i}$ .

This is proved by induction on  $j$ , and detailed calculations can be found in [19].

**Lemma 2.5.** For any  $2 < t < i$ ,  $\Lambda_{t,i} < \Lambda_{t-1,i}$ .

This is easily proved by induction, by noting that  $\alpha_j \geq 2$ .

**Lemma 2.6.** Let  $C_i = \{q_i, q_{i+1}, \dots, p_i - q_i\}$ . If  $0 \leq j \leq i - 1$ , then

$$p_j q_i \pmod{p_i} \notin C_i, \quad (p_j + 1)q_i \pmod{p_i} \notin C_i,$$

but

$$(p_j - 1)q_i \pmod{p_i} \in C_i.$$

*Proof.* Consider first the case that  $0 \leq j \leq i - 2$ . By Lemma 2.4,

$$p_j q_i = p_i q_j - \Lambda_{j+2,i},$$

and by Lemma 2.5, (\*\*\*) and the definition of  $\Lambda_{r,s}$ ,

$$2 \leq \alpha_i = \Lambda_{i,i} \leq \Lambda_{j+2,i} < \Lambda_{1,i} = q_i.$$

Thus,

$$\begin{aligned} p_i - q_i &< p_j q_i \pmod{p_i} \leq p_i - 2, \\ 0 &< (p_j + 1)q_i \pmod{p_i} \leq q_i - 2, \\ p_i - 2q_i &< (p_i - 1)q_i \pmod{p_i} \leq p_i - q_i - 2, \end{aligned}$$

giving the required exclusions and inclusions.

Next consider the case  $j = i - 1$ . Since the definition of the Farey sequence gives  $p_i q_{i-1} - p_{i-1} q_i = 1$ , the conclusion is trivially true. ■

**Lemma 2.7.** For each  $i$ ,  $\chi(G_i) \leq p_i/q_i$ .

*Proof.* We consider the graph  $G_i$  as an ordered graph, where the order of the vertices is: the vertices of  $F_i$  in order, followed by the vertices of  $H_i$  in order. Suppose the vertices of  $G_i$  are  $(v_1, v_2, \dots, v_{p_i})$  in order. Let  $c(v_j) = jq_i \pmod{p_i}$ . We shall show that  $c$  is a  $(p_i, q_i)$ -colouring of  $G_i$ , i.e., for every edge  $e = xy$  of  $G_i$ ,  $|c(x) - c(y)| \in C_i$ , where  $C_i$  is the set defined as in Lemma 2.6. Recall that the order length  $\ell(e)$  of an edge  $e = xy$  is the positive difference of the positions of  $x$  and  $y$  in  $G_i$  (as an ordered graph). It follows from the definition of the colouring  $c$  that  $|c(x) - c(y)| = \ell(e)q_i \pmod{p_i}$ . Therefore it suffices to show that for any edge  $e$  of  $G_i$ , we have

$$\ell(e)q_i \pmod{p_i} \in C_i.$$

Let  $L_j = \{1, 2, \dots, m - 1\} \cup \{p_t - 1 : 0 \leq t \leq j - 1\}$ . It is not difficult to show by induction on  $j$  that for any edge  $e$  of  $H_j$  (resp.,  $F_j$ ), we have

$\ell(e) \in L_j$ . Indeed, when  $H_j$  is constructed from copies of  $F_{j-1}$  and  $H_{j-1}$ , the edges of  $H_j$  are either those carried over from the copies of  $F_{j-1}$  and  $H_{j-1}$ , or the hooking edges. For those edges carried over from the copies of  $F_{j-1}$  and  $H_{j-1}$ , their order lengths remain unchanged. For those hooking edges, the long edge has order length  $p_{j-1} - 1$ , and the other edges have order length at most  $m - 1$ .

Since  $G_i$  as an ordered graph is isomorphic to the subgraph of  $H_{i+1}$  induced by the union of the first copy of  $F_i$  and the first copy of  $H_i$ , we conclude that for any edge  $e$  of  $G_i$ , either  $1 \leq \ell(e) \leq m - 1$  or  $\ell(e) = p_j - 1$  for some  $j \leq i$ . If  $1 \leq \ell(e) \leq m - 1$ , then obviously

$$\ell(e)q_i \pmod{p_i} \in C_i,$$

as  $p_i > mq_i$ . If  $\ell(e) = p_i - 1$ , then  $\ell(e)q_i \pmod{p_i} = p_i - q_i \in C_i$ . If  $\ell(e) = p_j - 1$  for some  $j \leq i - 1$ , then it follows from Lemma 2.6 that  $\ell(e)q_i \pmod{p_i} \in C_i$ . ■

Next we shall prove that for each  $i$ ,  $\chi_c(G_i) = p_i/q_i$ . By Lemma 2.7, it suffices to show that  $\chi_c(G_i) \geq p_i/q_i$ . We shall prove it by induction on  $i$ . First we need a few lemmas.

Lemma 2.8 below was proved in [4] and also implicitly used in [13, 14].

Given a  $(k, d)$ -colouring  $c$  of a graph  $G$ , we define a directed graph  $D_c(G)$  on the vertex set of  $G$  by putting a directed edge from  $x$  to  $y$  if and only if  $(x, y)$  is an edge of  $G$  and that  $c(x) - c(y) = d \pmod{k}$ .

**Lemma 2.8.** *For any graph  $G$ ,  $\chi_c(G) = k/d$  if and only if  $G$  is  $(k, d)$ -colourable, and that for any  $(k, d)$ -colouring  $c$  of  $G$ , the directed graph  $D_c(G)$  contains a directed cycle.*

A simple calculation shows that the length of the directed cycle in  $D_c(G)$  is a multiple of  $k$ , and hence is at least  $k$ .

**Corollary 2.2.** *For any graph  $G$ , if  $\chi_c(G) = k/d$  where  $(k, d) = 1$ , then  $G$  has a cycle of length at least  $k$ . In particular,  $k \leq |V(G)|$ .*

Suppose that  $\chi_c(G_i) = p_i/q_i$ , and that  $\Delta$  is an  $(p_i, q_i)$ -colouring of  $G_i$ . It follows from Lemma 2.8 that there is a directed cycle of  $D_\Delta(G_i)$  of length at least  $p_i$ . Since  $|G_i| = p_i$ , we conclude that there is a Hamiltonian cycle, say  $Q = (c_1, c_2, \dots, c_{p_i}, c_1)$ , of  $G_i$  such that  $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$ .

We say a Hamiltonian cycle  $Q = (c_1, c_2, \dots, c_t, c_1)$  of the graph  $G$  is a *good Hamiltonian cycle* (with respect to  $p/q$ ) if for any edge  $c_k c_\ell$  of  $G$  we have  $k - \ell \neq p_i, p_i + 1$  for any  $0 \leq i < n$ . In particular,  $k - \ell \neq m, m + 1$  for any edge  $c_k c_\ell$  of  $G$ . Similarly, a Hamiltonian path  $P = (c_1, c_2, \dots, c_t)$  of a graph  $G$  is

a good Hamiltonian path if for any edge  $c_k c_\ell$  of  $G$ , we have  $k - \ell \neq p_{i'}, p_{i'} + 1$  for any  $i' < n$ .

Now we shall show that if  $\chi_c(G_i) = p_i/q_i$ , then the Hamiltonian cycle induced by any  $(p_i, q_i)$ -colouring of  $G_i$  is a good Hamiltonian cycle.

**Lemma 2.9.** *Suppose  $\chi_c(G_i) = p_i/q_i$  and that  $\Delta$  is an  $(p_i, q_i)$ -colouring of  $G_i$ . Let  $Q = (c_1, c_2, \dots, c_{p_i}, c_1)$  be the Hamiltonian cycle of  $G_i$  such that  $\Delta(c_j) - \Delta(c_{j-1}) = q_i \pmod{p_i}$ . Then  $Q$  is a good Hamiltonian cycle of  $G_i$ .*

*Proof.* Assume to the contrary that there is an edge  $(c_k, c_\ell)$  of  $G_i$  such that  $|k - \ell| = p_t$  or  $p_t + 1$  for some  $t \leq i - 1$ . Then it follows from Lemma 2.6 that  $\Delta(c_k) - \Delta(c_\ell) = (k - \ell)q_i \pmod{p_i} \notin C_i$ , contrary to the assumption that  $\Delta$  is a  $(p_i, q_i)$ -colouring of  $G_i$ . ■

For any  $i \leq n$ , let  $X_i$  be the path of  $F_i$  given by the order of the vertices of  $F_i$ , and let  $Y_i$  be the path of  $H_i$  given by the order of the vertices of  $H_i$ . The proof of Lemma 2.7 shows that the union of  $X_i$  and  $Y_i$  is a good Hamiltonian cycle of  $G_i$ . We may consider  $X_i$  and  $Y_i$  as the canonical good Hamiltonian paths of  $F_i$  and  $H_i$ , respectively.

It is easy to see that any good Hamiltonian cycle of  $G_i$  must be the join of a good Hamiltonian path  $X'_i$  of  $F_i$  and a good Hamiltonian path  $Y'_i$  of  $H_i$ , because when  $i$  is odd, the first and the last vertices of  $F_i$  form a 2-vertex cut of  $G_i$ , and when  $i$  is even the first and the last vertices of  $H_i$  form a 2-vertex cut of  $G_i$ . Our next two lemmas show that for any such good Hamiltonian paths  $X'_i, Y'_i$ , the initial part and the terminal part of  $X'_i$  (resp.,  $Y'_i$ ) coincide with the corresponding part of  $X_i$  (resp.,  $Y_i$ ).

**Lemma 2.10.** *The graphs  $H_1$  and  $F_2$  have a unique good Hamiltonian path, up to an isomorphism.*

*Proof.* Each of the graphs  $H_1$  and  $F_2$  is of the form  $Q_t$  for some positive integer  $t$ . We shall simply prove that for any positive integer  $t$ , the graph  $Q_t$  has a unique good Hamiltonian path (with respect to any  $p/q$ ), up to an isomorphism. When  $t \leq m$ ,  $Q_t$  is a complete graph, and there is nothing to be proved. Assume now that  $t \geq m + 1$ . Suppose the vertices of  $Q_t$  are  $1, 2, \dots, t$ , where  $(x, y)$  is an edge if and only if  $|x - y| \leq m - 1$ . Let  $P = (x_1, x_2, \dots, x_t)$  be a good Hamiltonian path of  $Q_t$ . Then for any edge  $(x_i, x_j)$  of  $Q_t$ , we have  $|i - j| \neq m, m + 1$ . This, in particular, implies that for any  $i \leq t - m$ , the pair  $(x_i, x_{i+m})$  is not an edge of  $Q_t$ . In other words, for any  $i \leq t - m$ ,  $|x_i - x_{m+i}| \geq m$ .

We shall assume that  $x_1 < x_{m+1}$ , hence  $x_1 \leq x_{m+1} - m$  (the case that  $x_1 > x_{m+1}$  is parallel). Because  $x_2 \leq x_1 + m - 1$  and  $x_{m+2} \geq x_{m+1} - m + 1$  (as

$(x_1, x_2)$  and  $(x_{m+1}, x_{m+2})$  are edges of  $Q_t$ , we conclude that  $x_2 \leq x_{m+2} + m - 2$ . Since  $|x_2 - x_{m+2}| \geq m$ , we conclude that  $x_2 \leq x_{m+2} - m$ . Repeating this argument, we can prove that  $x_i \leq x_{i+m} - m$  for all  $i \leq t - m$ .

This implies that  $\{x_1, x_2, \dots, x_m\} = \{1, 2, \dots, m\}$ , for otherwise there would exist an  $x \leq m$  and an  $i \geq 1$  such that  $x_{i+m} = x$  and hence  $1 \leq x_i \leq x_{i+m} - m = x - m \leq 0$ , an obvious contradiction.

Suppose  $x_i = m + 1$ . Then  $i \geq m + 1$ , by the previous paragraph. Since  $x_{i-m} \leq x_i - m = 1$ , we conclude that  $x_{i-m} = 1$ . Now by induction, it is easy to prove that if  $x_i = m + j$  then  $x_{i-m} = j$ . This implies that  $x_{j+m} = x_j + m$  for all  $1 \leq j \leq t - m$ . Now we shall show that  $x_1 < x_2 < \dots < x_{t-m}$ . Otherwise,  $x_i > x_{i+1}$  for some  $i \leq t - m - 1$ . Then  $1 \leq x_{i+m+1} - x_i \leq m - 1$ , and hence  $x_i x_{i+m+1}$  is an edge of  $Q_t$ , contrary to the assumption that  $P$  is a good Hamiltonian path. Now it follows easily that  $x_i = i$  for all  $i \leq t$ . This completes the proof of Lemma 2.10. ■

**Lemma 2.11.** *Suppose  $X'_i$  and  $Y'_i$  are good Hamiltonian paths of  $F_i$  and  $H_i$  respectively, such that the union of  $X'_i$  and  $Y'_i$  is a good Hamiltonian cycle of  $G_i$ . Then the first and the last vertex of  $X'_i$  (resp.,  $Y'_i$ ) coincide with the first and last vertex of  $X_i$  (resp.,  $Y_i$ ). Moreover, if  $i$  is even (resp., odd), then the first and last  $m$  vertices of  $X'_i$  (resp.,  $Y'_i$ ) coincide with the first and the last  $m$  vertices of  $X_i$  (resp.,  $Y_i$ ), up to an isomorphism.*

*Proof.* The first half of this lemma is more or less trivial. We shall only prove that when  $i$  is even (resp., odd), the first and the last  $m$  vertices of  $X'_i$  (resp.,  $Y'_i$ ) are the same as that of  $X_i$  (resp.,  $Y_i$ ) and are of the same order as in  $X_i$  (resp.,  $Y_i$ ). We shall prove this by induction on  $i$ . When  $i = 1, 2$ , this follows from Lemma 2.10. Suppose the lemma is true for all  $j < i$ . We shall prove it for  $i$ . First consider the case that  $i$  is odd. We shall prove that the first and the last  $m$  vertices of  $Y'_i$  coincide with the first and the last  $m$  vertices of  $Y_i$ , up to an isomorphism.

The graph  $H_i$  is constructed from copies of  $F_{i-1}$  and  $H_{i-1}$ . The first vertex and the last vertex of any of the copies of  $H_{i-1}$  form a 2-vertex cut of  $H_i$ . Therefore the Hamiltonian path  $Y'_i$  must be the concatenation of good Hamiltonian paths of the copies of  $F_{i-1}$  and the copies of  $H_{i-1}$ . Let  $X'_{i-1}$  be the good Hamiltonian path of the first copy of  $F_{i-1}$ , and let  $Y'_{i-1}$  be the good Hamiltonian path of the first copy of  $H_{i-1}$ . It is easy to see that the union of  $X'_{i-1}$  and  $Y'_{i-1}$  must be a good Hamiltonian cycle of  $G_{i-1}$  (note that the union of the first copy of  $F_{i-1}$  and the first copy of  $H_{i-1}$  is a copy of  $G_{i-1}$ ). Therefore by the induction hypotheses, the first  $m$  vertices of  $X'_{i-1}$  coincide with the first  $m$  vertices of  $X_{i-1}$ . Now the first  $m$  vertices of  $X'_{i-1}$  are the

first  $m$  vertices of  $Y'_i$ , and the first  $m$  vertices of  $X_{i-1}$  are the first  $m$  vertices of  $Y_i$ . Thus we have proved that the first  $m$  vertices of  $Y'_i$  coincide with the first  $m$  vertices of  $Y_i$ . The same argument can be used to prove that the last  $m$  vertices of  $Y'_i$  coincide with the last  $m$  vertices of  $Y_i$ . The case  $i$  is even can be proved by the same method, and we omit the details. This completes the proof of Lemma 2.11. ■

Applying Lemmas 2.8, 2.9 and 2.11, and by the remark following the proof of Lemma 2.9, we have the following lemma:

**Lemma 2.12.** *Suppose  $\chi_c(G_i) = p_i/q_i$  for some  $i$ . Let  $\Delta$  be any  $(p_i, q_i)$ -colouring of  $G_i$ . Then the colours of the first vertex and the last vertex of  $F_i$  (resp.,  $H_i$ ) are uniquely determined by the colours of the first vertex and the last vertex of  $H_i$  (resp.,  $F_i$ ). Moreover, when  $i$  is even (resp., odd) then the colours of the first  $m$  vertices and the last  $m$  vertices of  $F_i$  (resp.,  $H_i$ ) are uniquely determined by the colours of the first vertex and the last vertex of  $H_i$  (resp.,  $F_i$ ).*

To prove that  $\chi_c(G_i) \geq p_i/q_i$  (and hence  $\chi_c(G_i) = p_i/q_i$ ), we need another gadget. If  $i \geq 2$  is even, let  $T_i$  be the graph obtained by hooking  $F_{i-1}$  to  $F_i$  by type 1 hook. If  $i \geq 2$  is odd, let  $T_i$  be the graph obtained by hooking  $F_i$  to  $F_{i-1}$  by type 1 hook.

**Theorem 2.2.** *For each  $i \geq 2$ ,  $\chi_c(G_i) = p_i/q_i$  and  $\chi_c(T_i) > p_{i-1}/q_{i-1}$ . Moreover,  $\chi_c(G_1) = p_1/q_1$ .*

*Proof.* First we prove that  $\chi_c(G_1) = p_1/q_1$ . By Lemma 2.7, it suffices to show that  $\chi_c(G_1) \geq p_1/q_1$ . It is easy to verify that  $\chi(G_1) = m + 1$ . Hence  $\chi_c(G_1) > m$ . Suppose  $\chi_c(G_1) = k/d > m$ , then  $k \leq |V(G_1)| = p_1$  by Corollary 2.2. Therefore  $k/d \geq p_1/q_1$ , because it follows from the construction of the Farey sequence that any fraction  $a/b$  strictly between  $m = p_0/q_0$  and  $p_1/q_1$  must have numerator  $a > p_1$ .

Next we show that  $\chi_c(T_2) > p_1/q_1$ . Again it is easy to verify that  $\chi(T_2) = m + 1$ . Suppose  $\chi_c(T_2) = k/d > m$ . As  $|V(T_2)| < p_1$  (because  $|V(F_2)| < |V(H_1)|$ ), we know that  $k < p_1$ . Therefore  $k/d > p_1/q_1$ , because by the construction of the Farey sequence, any fraction  $a/b$  strictly between  $m$  and  $p_1/q_1$  has numerator  $a > p_1$  (note that  $k/d \neq p_1/q_1$ ).

Now assume that  $i \geq 2$ ,  $\chi_c(T_i) > p_{i-1}/q_{i-1}$  and that  $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$ . We shall prove that  $\chi_c(G_i) = p_i/q_i$ .

Assume to the contrary that  $\chi_c(G_i) = k/d < p_i/q_i$ . Then  $k \leq p_i$  and hence  $k/d \leq p_{i-1}/q_{i-1}$ , because by the construction of the Farey sequence,

any fraction  $a/b$  strictly between  $p_{i-1}/q_{i-1}$  and  $p_i/q_i$  has numerator  $a > p_i$ . Since  $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$  and that  $G_{i-1}$  is a subgraph of  $G_i$ , it follows that  $\chi_c(G_i) = p_{i-1}/q_{i-1}$ .

Let  $\Delta$  be a  $(p_{i-1}, q_{i-1})$ -colouring of  $G_i$ . First we consider the case that  $i$  is odd. Then  $G_i$  is obtained by hooking  $F_i$  to  $H_i$ . Now  $H_i$  is constructed from  $\alpha_i$  copies of  $F_{i-1}$  and  $\alpha_i - 1$  copies of  $H_{i-1}$ . The first copy of  $H_{i-1}$  is hooked to the first copy of  $F_{i-1}$  by type 1 hooks and hooked to the second copy of  $F_{i-1}$  by type 2 hooks. The subgraph of  $H_i$  induced by the union of the first copy of  $F_{i-1}$  and the first copy of  $H_{i-1}$  is a copy of  $G_{i-1}$ . It is not difficult to verify that the subgraph of  $H_i$  induced by the first copy of  $H_{i-1}$  and the second copy of  $F_{i-1}$  is also isomorphic to  $G_{i-1}$ . (For this purpose, one only needs to observe that each of the graphs  $F_j$  and  $H_j$  has an automorphism which reverses the order of the vertices, i.e., the mapping  $h$  defined as  $h(x_{j,s}) = x_{j,f_j-s}$  is an automorphism of  $F_j$ , and  $h(y_{j,s}) = y_{j,h_j-s}$  is an automorphism of  $H_j$ .)

By using the induction hypotheses that  $\chi_c(G_{i-1}) = p_{i-1}/q_{i-1}$ , and by applying Lemma 2.12 to each of the two copies of  $G_{i-1}$ , we conclude that the first and last  $m$  vertices of the first copy of  $F_{i-1}$  are coloured the same way (under  $\Delta$ ) as the first and the last  $m$  vertices of the second copy of  $F_{i-1}$ .

Repeating the same argument, we can prove that the first and the last  $m$  vertices of the first copy of  $F_{i-1}$  are coloured the same way as the first and the last  $m$  vertices of the last copy of  $F_{i-1}$ . This implies that the restriction of  $\Delta$  to the union of  $F_i$  and the first copy of  $F_{i-1}$  of  $H_i$  is indeed a  $(p_{i-1}, q_{i-1})$ -colouring of  $T_i$  (recall that  $T_i$  is obtained by hooking  $F_i$  to  $F_{i-1}$  by type 1 hook). This is contrary to our assumption that  $\chi_c(T_i) > p_{i-1}/q_{i-1}$ .

Finally, assuming that  $i \geq 2$ ,  $\chi_i(G_i) = p_i/q_i$  and that  $\chi_c(T_i) > p_{i-1}/q_{i-1}$ , we shall prove that  $\chi_c(T_{i+1}) > p_i/q_i$ .

Assume to the contrary that  $\chi_c(T_{i+1}) = k/d \leq p_i/q_i$ . Since  $|F_{i+1}| < |H_i|$ , hence  $|T_{i+1}| < |G_i| = p_i$ . It follows from Corollary 2.2 that  $k < p_i$ . As  $p_{i-1}/q_{i-1}$  is the largest fraction satisfying the property that  $p_{i-1} < p_i$  and  $p_{i-1}/q_{i-1} \leq p_i/q_i$ , we conclude that  $\chi_c(T_{i+1}) \leq p_{i-1}/q_{i-1}$ .

We consider two cases:

Case 1:  $\alpha_i = 2$ . In this case  $F_{i+1} = F_{i-1}$ , and hence  $T_{i+1} = T_i$ . By induction hypothesis,  $\chi_c(T_i) > p_{i-1}/q_{i-1}$ .

Case 2:  $\alpha_i > 2$ . In this case  $F_{i+1}$  consists of  $\alpha_i - 1$  copies of  $F_{i-1}$  and  $\alpha_i - 2$  copies of  $H_{i-1}$ . The union of any copy of  $F_{i-1}$  and the consecutive copy of  $H_{i-1}$  induces a copy of  $G_{i-1}$ . Therefore we must have  $\chi_c(T_{i+1}) = p_{i-1}/q_{i-1}$ . Using the same argument as before (cf. the proof of the fact that  $\chi_c(G_i) = p_i/q_i$ ), we conclude that for any  $(p_{i-1}, q_{i-1})$ -colouring  $\Delta$  of  $T_{i+1}$ , the restriction of  $\Delta$  to the union of  $F_i$  and the first copy of  $F_{i-1}$  in  $F_{i+1}$  is indeed a  $(p_{i-1}, q_{i-1})$ -colouring of  $T_i$ , contrary to our assumption that  $\chi_c(T_i) > p_{i-1}/q_{i-1}$ . This



completes the proof of Theorem 2.2. ■

### 3. RATIONAL NUMBERS BETWEEN $n - 2$ AND $n - 1$

Up to now, we have proved Theorem 1.1, which asserts the existence of a  $K_n$ -minor free graph of circular chromatic number  $r$  for any  $2 \leq r \leq n - 2$ . For rational  $r$  between  $n - 2$  and  $n - 1$ , it is generally unknown whether or not there exists a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$ . However, for  $n = 4$ , P. Hell and the author [5] have recently proved the surprising result that for any  $K_4$ -minor free graph  $G$ , one has either  $\chi_c(G) = 3$  or  $\chi_c(G) \leq 8/3$ . In other words, there is a gap among the rationals that are the circular chromatic numbers of  $K_4$ -minor free graphs. For  $n \geq 6$ , we do not know if such gaps exist. Even for  $n = 4$ , we do not have a complete answer to the question, “Which rationals are the circular chromatic number of a  $K_4$ -minor free graph?”

Assume that  $n \geq 6$ . We shall prove Theorem 1.3, which says that if  $r$  is a rational between  $n - 2$  and  $n - 1$  whose Farey sequence either has length 1, or has length 2 and  $\alpha_2 = 2$ , then there is a  $K_n$ -minor free graph  $G$  with  $\chi_c(G) = r$ .

*Proof of Theorem 1.3.* If the Farey sequence of  $r$  has length 1, then  $r = n - 2 + 1/d$  for some integer  $d \geq 1$ . Let  $t = (n - 2)d + 1$ . Let  $G$  be the graph with vertex set  $V = \{1, 2, \dots, t\}$  and edge set  $E = \{ij : 1 \leq |i - j| \leq n - 3\} \cup \{1t\}$ . It is easy to verify that  $\chi(G) = n - 1$ , and that  $c(j) = dj \pmod{t}$  is a  $(t, d)$ -colouring of  $G$ , and hence  $\chi_c(G) \leq t/d = r$ . On the other hand, any fraction  $p/q$  between  $n - 2$  and  $r$  must have  $p > t$ . Therefore  $\chi_c(G) = r$  by Corollary 2.2.

It is straightforward to verify that  $G$  is  $K_n$ -minor free.

Finally we assume that the Farey sequence of  $r$  has length 2, and that the alpha sequence is  $(\alpha_1, 2)$ . Suppose the Farey sequence of  $r$  is  $(p_0/q_0, p_1/q_1, p_2/q_2)$ . Then  $p_0 = n - 2, q_0 = 1, p_1 = (n - 2)\alpha_1 + 1, q_1 = \alpha_1, p_2 = (2\alpha_1 - 1)(n - 2) + 2$  and  $q_2 = 2\alpha_1 - 1$ .

We construct the graph  $G$  as follows:

First let  $H_1$  be the  $(n - 3)$ rd power of a path with  $n - 1$  vertices (or equivalently  $H_1 = K_{n-1} - e$ , where  $e$  is an edge of  $K_{n-1}$ ). Let  $F_1$  be the  $(n - 3)$ rd power of a path of length  $(\alpha_1 - 1)(n - 2)$ . Both  $H_1$  and  $F_1$  are considered as ordered graphs as before. Then we take two copies of  $F_1$  and one copy of  $H_1$  and hook  $H_1$  to the first copy of  $F_1$  by type 2 hook, and hook  $H_1$  to the second copy of  $F_1$  by type 1 hook. (Note that we should take  $m = n - 2$  in the definition of hooks.) Finally we add one more vertex  $u$ , and connect  $u$  to the first  $n - 3$  vertices of the first copy of  $F_1$ , and to the last vertex of the second copy of  $F_1$ .

Figure 2 below shows the graph  $G$ , in the case that  $n = 6$  and  $r = 14/3$  (hence the alpha sequence is  $(2, 2)$ ).

It is straightforward to verify that  $|F_1| + |H_1| = p_1$  and  $|G| = p_2$ . With the same argument as in Section 2.3, we can prove that  $\chi_c(G) = r$ , by first proving that for  $i = 1, 2, \dots, \alpha_2 - 1$ , the subgraph induced by the union of the first copy of  $F_1$  and the first copy of  $H_1$ , as well the subgraph induced by the union of the first copy of  $H_1$  and the second copy of  $F_1$  has circular chromatic number  $p_1/q_1$ . Moreover, for any  $(p_1, q_1)$ -colouring of  $G - u$ , the first copy of  $F_1$  and the second

Figure 2.

copy of  $F_1$  are coloured the same way. This means that such a colouring cannot be extended to a  $(p_1, q_1)$ -colouring of  $G$ , and hence  $\chi_c(G) > p_1/q_1$ . It is easy to produce a  $(p_2, q_2)$ -colouring of  $G$  (cf. Lemma 2.7), hence  $\chi_c(G) = p_2/q_2$  (because any fraction strictly between  $p_1/q_1$  and  $p_2/q_2$  has a numerator greater than  $p_2$ ). We shall omit the details, and refer readers to the proof of Lemma 2.7 in Section 2.3.

Now we shall show that  $G$  is  $K_n$ -minor free. Suppose the vertices of the  $j$ th copy of  $F_1$  is  $x_1^j, x_2^j, \dots, x_s^j$ , for  $j = 1, 2$ , and that the vertices of  $H_1$  is  $y_1, y_2, \dots, y_t$ . Let  $G'$  be the graph obtained from  $G$  by adding the following edges:

$$uy_1, x_1^1y_1, y_t x_s^2, y_1 y_t.$$

Then each of the following sets is a cutset of  $G'$  which induces a complete subgraph:

$$\{y_1, y_t\}, \{u, x_1^1, y_1\}, \{y_t, x_s^2\}.$$

By applying Lemma 2.2, it is now straightforward to show that  $G'$  does not contain  $K_n$  as a minor. Hence  $G$  is  $K_n$ -minor free. This completes the proof of Theorem 1.3. ■

**Remark.** Very recently, Liaw, Pan and Zhu [6] have proved that for  $n \geq 5$ , any rational number between  $n - 2$  and  $n - 1$  is the circular chromatic number of a  $K_n$ -minor free graph. Pan and Zhu [9] have proved that any rational number between 2 and  $8/3$  is the circular chromatic number of a  $K_4$ -minor free graph.

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