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Circular inclined punch problem with two corners to contact with a half plane with a surface crack

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Abstract

A circular rigid punch with two corners coming into contact with an elastic half plane is presented to initiate a surface crack on the boundary of the half plane. The punch is usually inclined by an angle owing to the existence of the crack and the frictional force on the contact region. The problem is solved by three steps; the first step is to map the half plane with a surface vertical crack into a unit circle by a rational mapping function; the second step is to transform the problem into a Riemann-Hilbert problem; the third step is to solve the R-H equation. Then stress components and stress intensity factors of the crack are calculated. The inclination of the punch is decided by the condition that the resultant moment on the contact region about the center of the punch is zero.

1 Introduction

The punch problem is an important branch not only in the fields of mechanical and civil engineering, but in the field of applied mathematics. Though many problems have been solved[1,2,7,9], there still exist many cases that have not been considered or are difficult to be solved by analytical methods.

The punch problem is often connected with fracture mechanics since infinite tensile stress occurs at the end of the punch to initiate a crack or a pre-existed

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crack in the matrix is developed by the action of the punch. It is also important for solving a fretting problem. When a rigid punch on a half plane with a crack is concerned, some difficulties will be met. Though the problem is possible to be calculated by some numerical methods[6], the modelling of the contact interface between the punch and the half plane and the infinite property of the half plane as well as the singularity at the tip of the crack will result in some inconveniences. Therefore it is important to present an analytical or a semi-analytical method to solve this type of the problem efficiently.

A frictional flat-ended punch problem with a crack at one end of the punch has been solved by the complex variable method[5,8]. A circular inclined punch problem is progressively considered in the present paper. The half plane with a vertical crack is mapped into a unit circle by a rational mapping function so that the general solution of the problem can be obtained explicitly by solving the Riemann-Hilbert equation, which is formed according to the boundary conditions of the problem. The whole problem is separated into two cases to simplify the situation: one is the circular punch acted by an eccentric load to keep the punch vertical; the other is the flat-ended punch inclined for an angle by the moment on the contact region. The superposition of the moments of the two cases represents the resultant moment of the whole problem. Since the resultant moment about the center of the punch must vanish when the load is acted at the center of the punch, this property can be employed to determine the inclination of the punch The stress distributions and stress intensity factors of the crack can then be obtained by the superposition of the corresponding results in each case under different circumstances, including different frictional coefficients on the contact region, crack lengths, the distances from the crack to the punch, the radius of the curvature of the punch and the Poisson's ratios of the half plane. If the radius of the curvature of the punch tends to infinite, the related results tend to that of the corresponding inclined flat-ended punch problem[5,8].

2 The method of analysis

The problem is shown in Fig.1, in which the punch is assumed to be acted by a vertical concentrated force P at the center of the punch. Coulomb's frictional force is supposed to exist on the contact region. A vertical surface crack is located at or away from the right end of the punch. The punch is usually inclined for an angle owing to the existence of the crack and the frictional force. The inclined angle of the punch is taken as a positive value when the punch is inclined in a clockwise direction

The problem can be separated into two cases A and B according to the character of the boundary conditions of the problem. Case A is due to the circular punch acted by an eccentric load to keep the punch vertical by the moment on the contact region; case B is due to the flat-ended punch inclined with an angle by the moment on the contact region.

The loading conditions for each case can then be presented as follows:
For case A,

$$p_x = p_y = 0 \qquad \text{on } L = L_1 + L_2 \tag{1a}$$

$$p_x = \mu p_y, \int p_y \, ds = P \qquad \text{on } M \tag{1b}$$

$$V_A = x^2 / 2R \qquad \text{on } M \tag{1c}$$

For case B,

$$p_x = p_y = 0 \qquad \text{on } L = L_1 + L_2 \tag{2a}$$

$$p_x = \mu p_y, \int p_y \, ds = 0 \qquad \text{on } M \tag{2b}$$

$$V_A = -\varepsilon x \qquad \text{on } M \tag{2c}$$

where $L_1 = ABCD'D$, $L_2 = EA$, $M = DE$ in Fig.1; μ represents the Coulomb's frictional coefficient on M; p_x and p_y represent the components of traction in x and y directions on the surface of the half plane, respectively; V_A and V_B represent the displacements on the interface due to R and ε, respectively; and R represents the radius of the curvature of the punch.

In order to solve the problem analytically, the following rational mapping function is used to map the half plane with a vertical crack into a unit circle[3]:

$$z = \omega(\zeta) = \frac{E_0}{1-\zeta} + \sum_{k=1}^{24} \frac{E_k}{\zeta_k - \zeta} + E_c \tag{3}$$

where E_0, E_k and ζ_k are known constants, and E_c is decided by the distance from the crack to the punch.

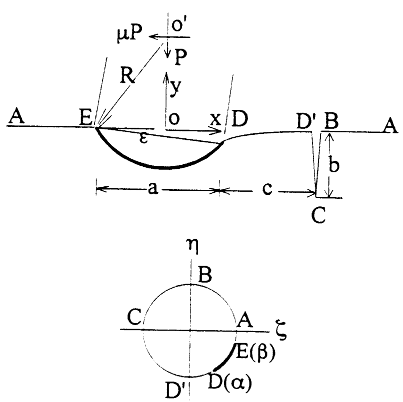

Figure 1: The punch and the associated unit circle.

Each case can be transformed into a Riemann-Hilbert problem as follows[5]:

$$\phi^+(\sigma) - \phi^-(\sigma) = i\int(p_x + ip_y)ds \equiv f_L \qquad \text{on } L = L_1 + L_2 \tag{4a}$$

$$\phi^+(\sigma) + \frac{1}{g}\phi^-(\sigma) = f_M \qquad \text{on } M \tag{4b}$$

where

$$f_M = \frac{4(1-i\mu)GiV + (1+i\mu)(1+\kappa)R(\sigma)}{(\kappa+1) - i\mu(\kappa-1)}$$

$$\frac{1}{g} = \frac{(\kappa+1) + i\mu(\kappa-1)}{(\kappa+1) - i\mu(\kappa-1)}, \qquad R(\zeta) = \phi(\zeta) + \frac{1-i\mu}{1+i\mu}\overline{\phi\left(\frac{1}{\bar{\zeta}}\right)}$$

$f_L = 0$ on L_1 and $P(1-i\mu)$ on L_2, $\kappa = 3-4\nu$ for plane strain state and $(3-\nu)/(1+\nu)$ for plane stress state, $V = x^2/2R$ for case A and $-\varepsilon x$ for case B, and ν is the Poisson's ratio of the half plane.

The general solution of the complex stress function for each case can be expressed as follows:

For case A,

$$\phi_A(\zeta) = H_A(\zeta) + \frac{1+i\mu}{2}J_A(\zeta) + Q_A(\zeta)\chi(\zeta) \tag{5}$$

where

$$H_A(\zeta) = P(1-i\mu)\frac{\chi(\zeta)}{2\pi i}\int_\beta^1 \frac{d\sigma}{\chi(\sigma)(\sigma-\zeta)} + \frac{Gi(1-i\mu)}{R(\kappa+1)}\frac{\chi(\zeta)}{2\pi i}\oint_M \frac{\{\omega(\sigma)\}^2}{\chi(\sigma)(\sigma-\zeta)}d\sigma$$

$$J_A(\zeta) = -\sum_{k=1}^{24}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{\overline{A_k}B_k}{\zeta_k - \zeta} + \frac{1-i\mu}{1+i\mu}\sum_{k=1}^{24}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{A_k\overline{B_k}\zeta_k'^2}{\zeta_k' - \zeta}$$

$$Q_A(\zeta)\chi(\zeta) = -\sum_{k=1}^{24}\frac{\chi(\zeta)\overline{A_k}B_k}{\chi(\zeta_k)(\zeta_k - \zeta)}$$

and

$$\chi(\zeta) = (\zeta-\alpha)^m(\zeta-\beta)^{1-m}, \quad m = 0.5 - i\ln g/2\pi$$

A_k and $\overline{A_k}$ are determined by solving 48 linear simultaneous equations from $A_k = \phi_A'(\zeta_k')$, (k=1,2,...,24) and $B_k = E_k/\overline{\omega'(\zeta_k')}$, $\zeta_k' = 1/\overline{\zeta_k}$, G is the shear modulus of the half plane.

For case B

$$\phi_B(\zeta) = H_B(\zeta) + \frac{1+i\mu}{2}J_B(\zeta) + Q_B(\zeta)\chi(\zeta) \tag{6}$$

where

$$H_B(\zeta) = \frac{2Gi(1-i\mu)}{\kappa+1}\oint_M \frac{-\varepsilon\omega(\sigma)}{\chi(\sigma)(\sigma-\zeta)}d\sigma$$

$$J_B(\zeta) = -\sum_{k=1}^{24}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k)}\right]\frac{\overline{C_k}B_k}{\zeta_k - \zeta} + \frac{1-i\mu}{1+i\mu}\sum_{k=1}^{24}\left[1 - \frac{\chi(\zeta)}{\chi(\zeta_k')}\right]\frac{C_k\overline{B_k}\zeta_k'^2}{\zeta_k' - \zeta}$$

$$Q_B(\zeta)\chi(\zeta) = -\sum_{k=1}^{24} \frac{\chi(\zeta)\overline{C_k}B_k}{\chi(\zeta_k)(\zeta_k - \zeta)}$$

C_k and $\overline{C_k}$ are determined by solving 48 linear simultaneous equations for real and imaginary parts of $C_k = \phi_B'(\zeta_k')$ (k=1,2,...,24).

Since there exists traction free boundary on the surface of the half plane, the other stress function can be expressed as follows[7]:

$$\psi_i(\zeta) = -\overline{\phi_i}(1/\zeta) - \overline{\omega}(1/\zeta)\phi_i'(\zeta)/\omega'(\zeta) \tag{7}$$

where i=A, B represent cases A and B, respectively.

3 The inclined angle of the punch

In order to find the inclined angle of the punch, the resultant moments R_{mA} for case A and R_{mB} for case B must be decided beforehand, which are calculated by

$$R_{mi} = \mathrm{Re}\left[\int_\alpha^\beta \psi_i(\sigma)\omega'(\sigma)d\sigma\right] - \mathrm{Re}\left[\omega(\sigma)\psi_i(\sigma)\right]_\alpha^\beta - \mathrm{Re}\left[\omega(\sigma)\overline{\omega}(\sigma)\frac{\phi_i'(\sigma)}{\omega(\sigma)}\right]_\alpha^\beta \tag{8}$$

where i=A, B represent cases A and B, respectively.

The non-dimensional resultant moment for each case is defined as

For case A,

$$M_{rA}^a = \frac{R_{mA}}{Pa} \qquad 0 \le b/a \le 1 \tag{9a}$$

$$M_{rA}^b = \frac{R_{mA}}{Pb} \qquad 0 \le a/b \le 1 \tag{9b}$$

For case B,

$$M_{rB}^a = \frac{(\kappa+1)R_{mB}}{G\varepsilon a^2} \qquad 0 \le b/a \le 1 \tag{10a}$$

$$M_{rB}^b = \frac{(\kappa+1)R_{mB}}{G\varepsilon b^2} \qquad 0 \le a/b \le 1 \tag{10b}$$

The inclined angle of the punch is determined by

$$R_{mA} + R_{mB} = 0 \tag{11}$$

i.e.

$$\varepsilon = -\frac{P(\kappa+1)}{Ga}\frac{M_{rA}^a}{M_{rB}^a} \qquad 0 \le b/a \le 1 \tag{12a}$$

$$\varepsilon = -\frac{P(\kappa+1)}{Gb}\frac{M_{rA}^b}{M_{rB}^b} \qquad 0 \le a/b \le 1 \tag{12b}$$

Whenever M_{rA} and M_{rB} are calculated, the inclined angle of the punch can be determined by (11).

Fig.2 shows ε with various μ and κ. For each κ, there exists a value of μ that makes ε vanish, which corresponds to the case that the punch does not

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incline. The inclined angle of the punch decreases from positive to negative value with the increase of μ for each κ, which means that the frictional force on the contact region changes the state of the inclination of the punch greatly. When the frictional force is relatively small, the punch is inclined in clockwise direction; on the other hand, if the frictional force is relatively large, the punch inclines in anti-clockwise direction. The larger the value of κ becomes, the greater the change of ε becomes with the increase of μ.

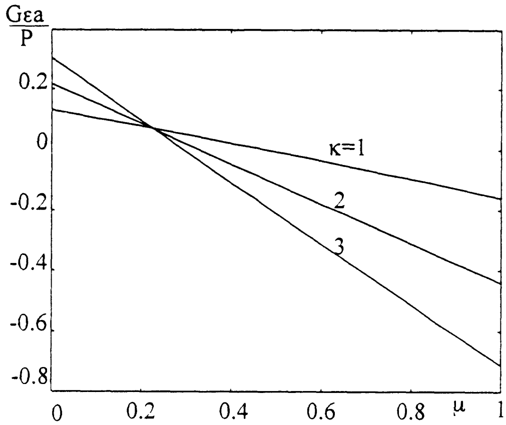

Figure 2: The inclined angle of the punch, b/a=0.5 and c/a=0.

4 The stress distributions

Fig.3 shows the stress distributions for $Ga^2 / PR = 1$, b/a=0.5, c/a=0, $\kappa = 2$ and $\mu = 0.5$. Cases A and B are superposed such that the resultant moment with respect to the origin vanishes on M. The inclined angle of the punch is determined by (12a) and $G\varepsilon a / P = -0.11371$ radian. $\tau_{xy} = \mu\sigma_{xy}$ holds true on M, which is not shown in Fig.3.

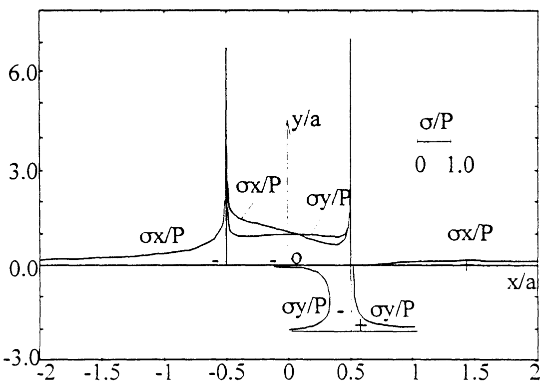

Figure 3: The stress distributions.

5 The stress intensity factors of the crack

The stress intensity factors of the crack are calculated by

$$K_{Ij} - iK_{IIj} = 2\sqrt{\pi} e^{-\frac{\delta}{2}i} \frac{\phi'_j(\sigma)}{\sqrt{\omega''(\sigma)}} \tag{13}$$

where $\sigma = -1$ is ζ on the unit circle corresponding to the tip of the crack, and $\delta = -\pi/2$ represents the angle between x-axis and the crack. j = A, B denote cases A and B , respectively.

The non-dimensional stress intensity factors for cases A and B are defined as

For case A,

$$F^a_{IA} + iF^a_{IIA} = \frac{\sqrt{a}(K_{IA} + iK_{IIA})}{P\sqrt{\pi}} \qquad 0 < b/a \le 1 \tag{14a}$$

$$F^b_{IA} + iF^b_{IIA} = \frac{\sqrt{b}(K_{IA} + iK_{IIA})}{P\sqrt{\pi}} \qquad 0 < a/b \le 1 \tag{14b}$$

For case B

$$F^a_{IB} + iF^a_{IIB} = \frac{(\kappa+1)(K_{IB} + iK_{IIB})}{G\varepsilon\sqrt{\pi a}} \qquad 0 < b/a \le 1 \tag{15a}$$

$$F^b_{IB} + iF^b_{IIB} = \frac{(\kappa+1)(K_{IB} + iK_{IIB})}{G\varepsilon\sqrt{\pi b}} \qquad 0 < a/b \le 1 \tag{15b}$$

After the stress intensity factors for cases A and B are calculated, with the known inclined angle of the punch, the non-dimensional stress intensity factors for the whole problem can be obtained by the superposition of the results in the two cases as follows:

$$F^a_I + iF^a_{II} = (F^a_{IA} + iF^a_{IIA}) + \frac{G\varepsilon a}{P(\kappa+1)}(F^a_{IB} + iF^a_{IIB}) \qquad 0 < b/a \le 1 \tag{16a}$$

$$F^b_I + iF^b_{II} = (F^b_{IA} + iF^b_{IIA}) + \frac{G\varepsilon b}{P(\kappa+1)}(F^b_{IB} + iF^b_{IIB}) \qquad 0 < a/b \le 1 \tag{16b}$$

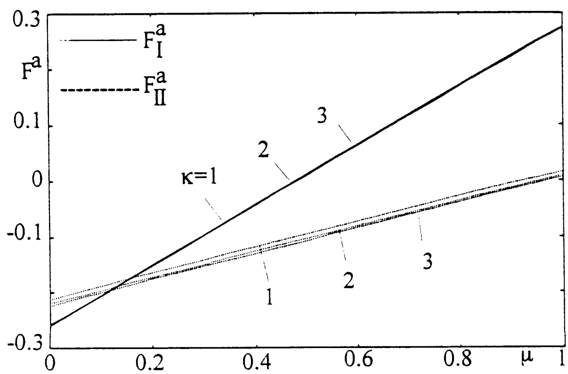

Figure 4: The stress intensity factors of the crack, $Ga^2/PR = 1$, b/a=0.5 and c/a=0.

The stress intensity factors of the crack are almost not influenced by the Poisson's ratio of the half plane, and both F_I^a and F_{II}^a increase with the increase of μ, as shown in Fig.4.

In the above analysis, $Ga^2 / PR = 1$ is adopted for the calculation. If $Ga^2 / PR = 0$, the results tend to that of the inclined flat-ended punch[5].

6 The conclusions

The inclined circular punch with two corners coming into contact with a half plane can be divided into two simple cases. The inclined angle of the punch can be decided by the resultant moment in each case. The radius of the circular punch R can be changed extensively. When the punch does not incline, i.e. case A, the vertical force P must be applied at the location of $e = R_{mA} / P$ away from the origin. When R tends to infinite, the results tend to that of the inclined flat-ended punch. Oblique crack and kinked crack can be solved by using the mapping function in[4].

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