## CIRCULAR INTERLACING WITH RECIPROCAL POLYNOMIALS

Abstract. The purpose of this paper is to show that all zeros of the reciprocal polynomial

$$P_m(z) = \sum_{k=0}^m A_k z^k \quad (z \in \mathbb{C})$$

of degree  $m \ge 2$  with real coefficients  $A_k \in \mathbb{R}$  (i.e.  $A_m \ne 0$  and  $A_k = A_{m-k}$  for all  $k = 0, \dots, \left\lfloor \frac{m}{2} \right\rfloor$ ) are on the unit circle, if there is a  $B \in \mathbb{R}$  such that  $A_m B \ge 0, |A_m| \ge |B|$  and

$$|A_m + B| \geqslant \sum_{k=1}^{m-1} |A_k + B - A_m|$$

holds.

If the inequality is strict then the zeros of  $P_m$  have the form  $e^{\pm u_j}$   $(j = 1, \dots, \lfloor \frac{m}{2} \rfloor)$  where

$$\frac{2(j-1)\pi}{m} < u_j < \frac{2j\pi}{m} \quad (j=1,\ldots,\left\lfloor\frac{m}{2}\right\rfloor)$$

and they are simple (for odd *m*, in addition to these zeros,  $-1 = e^{-i\pi}$  is a zero too).

This implies that the polynomial  $P_m$  (with  $A_m > 0$ ) and  $z^{2m} - 1$  satisfy the circular interlacing condition.

If in the inequality (for the coefficients) equality holds, then double zeros may arise, we discuss how this can happen.

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