CIRCULAR POLYA DISTRIBUTIONS OF ORDER k

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Two circular Polya distributions of order k are derived by means of generalized urn models and by compounding, respectively, the type I and type II circular binomial distributions of order k of Makri and Philippou (1994) with the beta distribution. It is noted that the above two distributions include, as special cases, new circular hypergeometric, negative hypergeometric, and discrete uniform distributions of the same order and type. The means of the new distributions are obtained and two asymptotic results are established relating them to the above-mentioned circular binomial distributions of order k.

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1. Introduction. In five pioneering papers, Philippou and Muwafi [18], Philippou et al. [17], Philippou [14], and Philippou et al. [15, 16] introduced the study of univariate and multivariate distributions of order k. Since then, the subject matter received a lot of attention from many researchers. For comprehensive reviews at the time of publication, we refer to Johnson et al. [7, 8].

Makri and Philippou [11] introduced two circular binomial distributions of order k, as the distribution of nonoverlapping and possibly overlapping success runs of length k in n Bernoulli trials arranged on a circle. They also derived two exact formulas for the reliability of a consecutive-k-out-of-$n : F$ system whose components are ordered circularly. The distribution of circular nonoverlapping or possibly overlapping success runs of length k was also studied by Chryssaphinou et al. [5], Koutras et al. [9], Charalambides [4], and Koutras et al. [10].

In the present paper, we introduce two circular Polya distributions of order k, type I and II, say $P_{k,\mathrm{I}}^c(\cdot)$ and $P_{k,\mathrm{II}}^c(\cdot)$, respectively (see Definitions 2.4 and 3.2), as the distribution of nonoverlapping ($N_{n,k,s}^c$) and possibly overlapping ($M_{n,k,s}^c$) success runs of length k in a generalized sampling scheme (see Theorems 2.1 and 3.1). We also derive $P_{k,\mathrm{I}}^c(\cdot)$ and $P_{k,\mathrm{II}}^c(\cdot)$ by compounding, respectively, the circular binomial distribution of order k, type I ($B_{k,\mathrm{I}}^c(\cdot)$) and type II ($B_{k,\mathrm{II}}^c(\cdot)$) of Makri and Philippou [11] with the beta distribution (see Remarks 2.5 and 3.3). We introduce as special cases of $P_{k,\mathrm{I}}^c(\cdot)$ and $P_{k,\mathrm{II}}^c(\cdot)$, respectively, new circular hypergeometric, negative hypergeometric and Bose-Einstein uniform distributions of the same order and type, and we recover $B_{k,\mathrm{I}}^c(\cdot)$ and $B_{k,\mathrm{II}}^c(\cdot)$. Moreover, we obtain the factorial moments of $P_{k,\mathrm{I}}^c(\cdot)$ and the mean of $P_{k,\mathrm{II}}^c(\cdot)$

(see Propositions 2.6 and 3.4) and we relate them asymptotically to $B_{k,\mathrm{I}}^c(\cdot)$ and $B_{k,\mathrm{II}}^c(\cdot)$, respectively (see Propositions 2.7 and 3.5). Potential applications are also indicated.

2. Circular Polya distribution of order k, type I. In this section, we derive the circular Polya distribution of order k, type I, by means of a generalized urn model and by compounding the circular binomial distribution of order k, type I, of Makri and Philippou [11]. The factorial moment of this new distribution is also obtained.

THEOREM 2.1. *Consider an urn containing $c_0 + c_1$ $(= c)$ balls of which c_0 bear the letter S (success) and c_1 bear the letter F (failure). A ball is drawn at random and then it is replaced together with s balls bearing the same letter. We repeat this procedure n (≥ 1) times and we assume that the outcomes are bent into a circle. Let $N_{n,k,s}^c$ be a random variable denoting the number of nonoverlapping success runs of length k (≥ 1). Then, for $x = 0,1,\dots,[n/k]$,*

$$
\begin{aligned}
&P\big(N_{n,k,s}^c = x\big)\\
&= \sum_{i=1}^{k} i \sum_{1} \binom{\Sigma_{j=1}^k x_j + x}{x_1,\dots,x_k,x} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j + 1]}}{(c/s)^{[n]}}\\
&\quad + k \sum_{i=1}^{k} \sum_{1} \frac{x}{\Sigma_{j=1}^k x_j + 1} \binom{\Sigma_{j=1}^k x_j + x}{x_1,\dots,x_k,x} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j + 1]}}{(c/s)^{[n]}}\\
&\quad + \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}} \delta_{x,[n/k]},
\end{aligned}
$$

(2.1)

where the summation $\sum_1$ is taken over all nonnegative integers $x_1,\dots,x_k$ such that $\sum_{j=1}^k j x_j = n - i - kx$ and δ is the Kronecker delta function.

First, we will establish two preliminary lemmas. The first one gives a formula for $P(N_{n,k,s}^c = x)$, $x = 0,1,\dots,[n/k]$.

LEMMA 2.2. *Let $N_{n,k,s}^c$, $\sum_1$, and δ as in Theorem 2.1. Then, for $x = 0,1,\dots, [n/k]$,*

$$
\begin{aligned}
&P\big(N_{n,k,s}^c = x\big)\\
&= \sum_{r=0}^{x} \sum_{i=1}^{k} i \sum_{1} \binom{\Sigma_{j=1}^k x_j + x - r}{x_1,\dots,x_k,x-r} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}\\
&\quad + \zeta(1,x) \sum_{r=0}^{x-1} \sum_{i=1}^{k-1} (k-i)
\end{aligned}
$$

$$\times \sum_{1} \binom{\Sigma_{j=1}^{k} x_j + x - r - 1}{x_1, \ldots, x_k, x - r - 1} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^{k} x_j]} (c_1/s)^{[\Sigma_{j=1}^{k} x_j + 1]}}{(c/s)^{[n]}}$$

$$+ \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}} \delta_{x,[n/k]},$$

$$(2.2)$$

where $\zeta(\cdot, \cdot)$ is the zeta function defined by $\zeta(\upsilon, u) = 1$ if $u \geq \upsilon$ and 0 otherwise.

PROOF. A typical element of the event $(N_{n,k,s}^{c} = x)$ is either a circular arrangement

$$\underbrace{\overbrace{SS\cdots S}^{k} \cdots \overbrace{SS\cdots S}^{k}}_{r} \overbrace{SS\cdots S}^{\alpha} F \alpha_1 \alpha_2 \cdots \alpha_{x_1 + \cdots + x_k + x - r} \overbrace{SS\cdots S}^{\beta} \quad (0 \leq x \leq [n/k])$$

$$(2.3)$$

such that x_j of the α's are $e_j = \underbrace{SS\cdots S}_{j-1} F$ $(1 \leq j \leq k)$, $x - r$ of the α's are $\tilde{e}_k = \underbrace{SS\cdots S}_{k}$, and

$$\sum_{j=1}^{k} j x_j = n - 1 - kx - (\alpha + \beta), \quad 0 \leq r \leq x, \ 0 \leq \alpha, \beta \leq k-1, \ \alpha + \beta \leq k-1, \quad (2.4)$$

or a circular arrangement

$$\underbrace{\overbrace{SS\cdots S}^{k} \cdots \overbrace{SS\cdots S}^{k}}_{r} \overbrace{SS\cdots S}^{\alpha} F \alpha_1 \alpha_2 \cdots \alpha_{x_1 + \cdots + x_k + x - r - 1} \overbrace{SS\cdots S}^{\beta} \quad (2.5)$$

$(1 \leq x \leq [n/k])$ such that x_j $(1 \leq j \leq k)$ and $x - r - 1$ of the α's are as above, and

$$\sum_{j=1}^{k} j x_j = n - 1 - k(x-1) - (\alpha + \beta),$$

$$x \geq 1, \quad 0 \leq r \leq x-1, \quad 1 \leq \alpha, \beta \leq k-1, \quad k \leq \alpha + \beta \leq 2k-2,$$

$$(2.6)$$

or a circular arrangement $\underbrace{SS\cdots S}_{n}$ $(x = [n/k])$.

Fix r, x_j $(1 \leq j \leq k)$, α, and β. Then, the number of the circular arrangements of type (2.3) is

$$\binom{\Sigma_{j=1}^{k} x_j + x - r}{x_1, \ldots, x_k, x - r}$$

$$(2.7)$$

and each one of them has probability

$$\frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}. \tag{2.8}$$

Furthermore, the number of the circular arrangements of type (2.5) is

$$\binom{\Sigma_{j=1}^k x_j + x - r - 1}{x_1, \ldots, x_k, x - r - 1} \tag{2.9}$$

and each one of them has probability

$$\frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}. \tag{2.10}$$

Finally, we observe that

$$P\left(\underbrace{SS\cdots S}_{n}\right) = \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}}. \tag{2.11}$$

But r, x_j $(1 \le j \le k)$, α, and β may vary subject to (2.4) and (2.6) for the elements of type (2.3) and (2.5), respectively. Denote by $\sum_1'$ and $\sum_1''$ the summation over all nonnegative integers $x_1, \ldots, x_k$ satisfying (2.4) and (2.6), respectively. Then, for $x = 0, 1, \ldots, [n/k]$ and $\sum_1$ as in the theorem, we have

$$P\left(N_{n,k,s}^c = x\right)$$

$$= \sum_{r=0}^{x} \sum_{\alpha=0}^{k-1} \sum_{\substack{\beta=0 \\ \alpha+\beta \le k-1}}^{k-1} {\sum_1}' \binom{\Sigma_{j=1}^k x_j + x - r}{x_1, \ldots, x_k, x - r}$$

$$\times \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}$$

$$+ \zeta(1,x) \sum_{r=0}^{x-1} \sum_{\alpha=1}^{k-1} \sum_{\substack{\beta=1 \\ k \le \alpha+\beta \le 2k-2}}^{k-1} {\sum_1}'' \binom{\Sigma_{j=1}^k x_j + x - r - 1}{x_1, \ldots, x_k, x - r - 1}$$

$$\times \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}$$

$$+ \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}} \delta_{x,[n/k]}$$

$$= \sum_{r=0}^{x} \sum_{\alpha+\beta=0}^{k-1} (\alpha+\beta+1) {\sum_1}' \binom{\Sigma_{j=1}^k x_j + x - r}{x_1,\ldots,x_k,x-r}$$

$$\times \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]} (c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}$$

$$+ \zeta(1,x) \sum_{r=0}^{x-1} \sum_{y=1}^{k-2} (2k-\alpha-\beta-1) {\sum_1}'' \binom{\Sigma_{j=1}^k x_j + x - r - 1}{x_1,\ldots,x_k,x-r-1}$$

$$\times \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]} (c_1/s)^{\left[\Sigma_{j=1}^k x_j+1\right]}}{(c/s)^{[n]}}$$

$$+ \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}} \delta_{x,[n/k]}$$

$$= \sum_{r=0}^{x} \sum_{i=1}^{k} i \sum_1 \binom{\Sigma_{j=1}^k x_j + x - r}{x_1,\ldots,x_k,x-r} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]} (c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}$$

$$+ \zeta(1,x) \sum_{r=0}^{x-1} \sum_{i=1}^{k-1} (k-i) \sum_1 \binom{\Sigma_{j=1}^k x_j + x - r - 1}{x_1,\ldots,x_k,x-r-1}$$

$$\times \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]} (c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}}$$

$$+ \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}} \delta_{x,[n/k]}, \tag{2.12}$$

which establishes the lemma. $\qquad\square$

The next lemma (see Makri and Philippou [11]) provides a formula for executing the summation with respect to r appearing in Lemma 2.2.

LEMMA 2.3. *Let*

$$\binom{x_1 + \cdots + x_k}{x_1,\ldots,x_k} = \frac{(x_1 + \cdots + x_k)!}{x_1! \cdots x_k!} \tag{2.13}$$

denote the multinomial coefficient. Then, for $x = 0,1,\ldots,$

$$\sum_{r=0}^{x} \binom{x_1 + \cdots + x_k + r}{x_1,\ldots,x_k,r} = \frac{x+1}{x_1 + \cdots + x_k + 1} \binom{x_1 + \cdots + x_k + x + 1}{x_1,\ldots,x_k,x+1}. \tag{2.14}$$

PROOF OF THEOREM 2.1. By means of Lemma 2.3, for $x = 0, 1, \ldots,$ we get

$$\zeta(1,x) \sum_{r=0}^{x-1} \binom{x_1 + \cdots + x_k + x - r - 1}{x_1, \ldots, x_k, x - r - 1} = \frac{x}{x_1 + \cdots + x_k + 1} \binom{x_1 + \cdots + x_k + x}{x_1, \ldots, x_k, x}$$

$$\sum_{r=0}^{x} \binom{x_1 + \cdots + x_k + x - r}{x_1, \ldots, x_k, x - r}$$

$$= \binom{x_1 + \cdots + x_k + x}{x_1, \ldots, x_k, x} + \zeta(1,x) \sum_{r=1}^{x} \binom{x_1 + \cdots + x_k + x - r}{x_1, \ldots, x_k, x - r}$$

$$= \binom{x_1 + \cdots + x_k + x}{x_1, \ldots, x_k, x} + \frac{x}{x_1 + \cdots + x_k + 1} \binom{x_1 + \cdots + x_k + x}{x_1, \ldots, x_k, x}.$$
$$(2.15)$$

Introducing these expressions into the formula of Lemma 2.2, we get, for $x = 0, 1, \ldots, [n/k]$,

$$P\left(N_{n,k,s}^c = x\right)$$

$$= \sum_{i=1}^{k} i \sum_{1} \binom{\Sigma_{j=1}^k x_j + x}{x_1, \ldots, x_k, x} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j + 1]}}{(c/s)^{[n]}}$$

$$+ \sum_{i=1}^{k} i \sum_{1} \frac{x}{\Sigma_{j=1}^k x_j + 1} \binom{\Sigma_{j=1}^k x_j + x}{x_1, \ldots, x_k, x} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j + 1]}}{(c/s)^{[n]}}$$

$$+ \sum_{i=1}^{k} (k-i) \sum_{1} \frac{x}{\Sigma_{j=1}^k x_j + 1} \binom{\Sigma_{j=1}^k x_j + x}{x_1, \ldots, x_k, x}$$

$$\times \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j + 1]}}{(c/s)^{[n]}}$$

$$+ \frac{(c_0/s)^{[n]}}{(c/s)^{[n]}} \delta_{x,[n/k]},$$
$$(2.16)$$

from which the theorem follows.   □

DEFINITION 2.4. A random variable X is said to have the circular Polya distribution of order k, type I, with parameters n, s, c, and c_0 (s integer and n, c, and c_0 positive integers), to be denoted by $P_{k,\mathrm{I}}^c(n;s;c,c_0)$ if for $x = 0, 1, \ldots, [n/k]$, $P(X = x)$ is given by (2.1).

REMARK 2.5. It may be noted that if X and P are two random variables such that $(X \mid P = p)$ is distributed as $B_{k,\mathrm{I}}^c(n,p)$ and P is distributed as $B(\alpha, \beta)$

(the beta distribution with positive real parameters α and β), then, for $x = 0, 1, \ldots, [n/k]$,

$$
\begin{aligned}
&P(X = x) \\
&= \sum_{i=1}^{k} i \sum_{1} \binom{\Sigma_{j=1}^{k} x_j + x}{x_1, \ldots, x_k, x} \frac{B(\alpha + n - 1 - \sum_{j=1}^{k} x_j, \beta + \sum_{j=1}^{k} x_j + 1)}{B(\alpha, \beta)} \\
&\quad + k \sum_{i=1}^{k} \sum_{1} \frac{x}{\Sigma_{j=1}^{k} x_j + 1} \binom{\Sigma_{j=1}^{k} x_j + x}{x_1, \ldots, x_k, x} \\
&\qquad\qquad\qquad \times \frac{B(\alpha + n - 1 - \sum_{j=1}^{k} x_j, \beta + \sum_{j=1}^{k} x_j + 1)}{B(\alpha, \beta)} \\
&\quad + \frac{B(\alpha + n, \beta)}{B(\alpha, \beta)} \delta_{x, [n/k]},
\end{aligned}
\tag{2.17}
$$

where $\sum_1$ and δ are as in Theorem 2.1. Relation (2.17) reduces to (2.1) if $\alpha = c_0/s$ and $\beta = c_1/s$ $(s \neq 0)$, which indicates that (2.17) may be considered as another form of $P_{k,\mathrm{I}}^c(n; s; c, c_0)$.

The rth factorial moment of the circular Polya distribution of order k, type I, is given next.

PROPOSITION 2.6. *Let X be a random variable following the circular Polya distribution of order k, type I, with probability function as given by (2.17) and let*

$$
\mu_{(r)}(n) = \sum_{x=r}^{[n/k]} (x)_r P(X = x), \quad r = 0, 1, \ldots, [n/k],
\tag{2.18}
$$

be the rth factorial moment of X. Then,

$$
\begin{aligned}
\mu_{(r)}(n) &= n(r-1)! \sum_{i=0}^{m_r} \sum_{j=r}^{[(n-i)/k]} (-1)^i \binom{r}{i} \\
&\qquad\qquad \times \binom{n - kj + r - i - 1}{r - 1} \binom{j - 1}{r - 1} \frac{B(\alpha + jk + 1, \beta)}{B(\alpha, \beta)} \\
&\quad + r! \binom{[n/k]}{r} (1 - k\delta_{n, k[n/k]}) \frac{B(\alpha + n, \beta)}{B(\alpha, \beta)}, \quad r = 1, 2, \ldots, [n/k], \; n \geq k,
\end{aligned}
\tag{2.19}
$$

where $m_r = \min\{r, n - kr\}$.

PROOF. It follows by means of Charalambides [4, relation (3.10)] and Remark 2.5.

The circular Polya distribution of order k, type I, includes as special cases four distributions of order k, of which three are new. Each one arises when success runs of length k are not allowed to overlap by means of Theorem 2.1. Their moments may be easily obtained from Proposition 2.7.

CASE 1. The $P_{k,\mathrm{I}}^c(n;s;c,c_0)$, for $s = -1$, reduces to a new distribution, which we call circular hypergeometric distribution of order k, type I, with parameters n, c, and c_0, and denote it by $H_{k,\mathrm{I}}^c(n;c,c_0)$.

CASE 2. The $P_{k,\mathrm{I}}^c(n;s;c,c_0)$, for $s = +1$, reduces to a new distribution, which we call circular negative hypergeometric distribution of order k, type I, with parameters n, c, and c_0, and denote it by $NH_{k,\mathrm{I}}^c(n;c,c_0)$.

CASE 3. The $P_{k,\mathrm{I}}^c(n;s;c,c_0)$, for $c = c_0 = s$, reduces to a new distribution, which we call circular Bose-Einstein uniform distribution of order k, type I.

CASE 4. For $s = 0$, $P_{k,\mathrm{I}}^c(n;s;c,c_0)$ reduces to the circular binomial distribution of order k, type I, of Makri and Philippou [11], with $p = c_0/c$.

Next, we relate $P_{k,\mathrm{I}}^c(\cdot)$ asymptotically to the circular binomial distribution of the same order and type. □

PROPOSITION 2.7. *Let X_{c_0,c_1} and X be two random vectors distributed as $P_{k,\mathrm{I}}^c(n;s;c,c_0)$ and $B_{k,\mathrm{I}}^c(n,p)$, respectively, and assume that $c_0/c \to p$ $(0 < p < 1)$ as $c_0, c_1 \to \infty$, then*

$$P(X_{c_0,c_1} = x) \longrightarrow P(X = x), \quad x = 0,1,\dots,[n/k]. \tag{2.20}$$

PROOF. We observe that

$$\frac{(c_0/s)^{[n-1-\Sigma_{j=1}^k x_j]}(c_1/s)^{[\Sigma_{j=1}^k x_j+1]}}{(c/s)^{[n]}} \longrightarrow p^{n-1-\Sigma_{j=1}^k x_j}(1-p)^{\Sigma_{j=1}^k x_j+1},$$
$$(c_0/s)^{[n]}/(c/s)^{[n]} \longrightarrow p^n \quad \text{as } c_0,c_1 \longrightarrow \infty \tag{2.21}$$

from which the proof follows. □

3. Circular Polya distribution of order k, type II. In this section, we derive the circular Polya distribution of order k, type II, by means of a generalized urn model and by compounding the circular binomial distribution of order k, type II, of Makri and Philippou [11]. The mean of this new distribution is also obtained.

THEOREM 3.1. *Consider an urn containing $c_0 + c_1$ $(= c)$ balls of which c_0 bear the letter S (success) and c_1 bear the letter F (failure). A ball is drawn at random and then it is replaced together with s balls bearing the same letter. We repeat this procedure n (≥ 1) times and we assume that the outcomes are bent into a circle. Let $M_{n,k,s}^c$ be a random variable denoting the number of overlapping*

success runs of length k (≥ 1). Then, $P(M^c_{n,k,s} = x) = (c_0/s)^{[n]}/(c/s)^{[n]}$ if $x = n$, 0 if $n-k+1 \leq x \leq n-1$, and for $0 \leq x \leq n-k$,

$$P(M^c_{n,k,s} = x) = \sum_{i=1}^{x+k} i \sum_2 \binom{x_1 + \cdots + x_n}{x_1, \ldots, x_n}$$
$$\times \frac{(c_0/s)^{[n-1-\sum_{j=1}^n x_j]} (c_1/s)^{[\sum_{j=1}^n x_j + 1]}}{(c/s)^{[n]}}, \tag{3.1}$$

where the summation $\sum_2$ are taken over all nonnegative integers $x_1, \ldots, x_k$ such that $\sum_{j=1}^n j x_j = n - i$ $(1 \leq i \leq x+k)$ and $\max\{0, i-k\} + \sum_{j=k+1}^n (j-k) x_j = x$ $(0 \leq x \leq n-k)$.

PROOF. We first observe that, for $n-k+1 \leq x \leq n-1$, $P(M^c_{n,k,s} = x) = P(\varnothing) = 0$ and $P(M^c_{n,k,s} = n) = P(\underbrace{SS \cdots S}_{n}) = (c_0/s)^{[n]}/(c/s)^{[n]}$. For $0 \leq x \leq n-k$, a typical element of the event $(M^c_{n,k,s} = x)$ is a circular arrangement

$$\underbrace{SS \cdots S}_{\alpha} F \alpha_1 \alpha_2 \cdots \alpha_{x_1 + \cdots + x_n} \underbrace{SS \cdots S}_{\beta} \tag{3.2}$$

such that x_j of the α's are $e_j = \underbrace{SS \cdots S}_{j-1} F$ $(1 \leq j \leq n)$ and

$$0 \leq \alpha \leq x+k-1, \quad 0 \leq \beta \leq x+k-1, \quad \alpha + \beta \leq x+k-1, \tag{3.3}$$

$$\sum_{j=1}^n j x_j = n-1-(\alpha+\beta), \quad \max\{0, \alpha+\beta-k+1\} + \sum_{j=k+1}^n (j-k) x_j = x. \tag{3.4}$$

Fix x_j $(1 \leq j \leq n)$, α, and β. Then, the number of the above arrangements is

$$\binom{x_1 + \cdots + x_n}{x_1, \ldots, x_n} \tag{3.5}$$

and each one of them has probability

$$\frac{(c_0/s)^{[n-1-\sum_{j=1}^n x_j]} (c_1/s)^{[\sum_{j=1}^n x_j + 1]}}{(c/s)^{[n]}}. \tag{3.6}$$

But α and β may vary subject to (3.3) and the nonnegative integers x_j $(1 \leq j \leq k)$ may vary subject to (3.4). Therefore, for $0 \leq x \leq n-k$ and $\sum_2$ as in the

theorem, we have

$$
\begin{aligned}
& P\left(M_{n,k,s}^c = x\right) \\
&= \sum_{\substack{\alpha=0 \\ \alpha+\beta \le x+k-1}}^{x+k-1} \sum_{\beta=0}^{x+k-1} \sum_{2} \binom{x_1+\cdots+x_n}{x_1,\ldots,x_n} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^n x_j]}(c_1/s)^{[\Sigma_{j=1}^n x_j+1]}}{(c/s)^{[n]}} \\
&= \sum_{i=1}^{x+k} i \sum_{2} \binom{x_1+\cdots+x_n}{x_1,\ldots,x_n} \frac{(c_0/s)^{[n-1-\Sigma_{j=1}^n x_j]}(c_1/s)^{[\Sigma_{j=1}^n x_j+1]}}{(c/s)^{[n]}}
\end{aligned}
\tag{3.7}
$$

which establishes the theorem. $\qquad\square$

DEFINITION 3.2. A random variable X is said to have the circular Polya distribution of order k, type II, with parameters n, s, c, and c_0 (s integer and n, c, and c_0 positive integers), to be denoted by $P_{k,\mathrm{II}}^c(n;s;c,c_0)$ if

$$
P(X = x) = \begin{cases}
\sum_{i=1}^{x+k} i \sum_{2} \binom{x_1+\cdots+x_n}{x_1,\ldots,x_n} \\
\times \dfrac{(c_0/s)^{[n-1-\Sigma_{j=1}^n x_j]}(c_1/s)^{[\Sigma_{j=1}^n x_j+1]}}{(c/s)^{[n]}}, & \text{if } 0 \le x \le n-k, \\[2ex]
0, & \text{if } n-k+1 \le x \le n-1, \\[1ex]
\dfrac{(c_0/s)^{[n]}}{(c/s)^{[n]}}, & \text{if } x = n,
\end{cases}
\tag{3.8}
$$

where the summation $\sum_2$ are taken over all nonnegative integers $x_1,\ldots,x_k$ such that $\sum_{j=1}^n jx_j = n-i$ ($1 \le i \le x+k$) and $\max\{0,i-k\}+\sum_{j=k+1}^n (j-k)x_j = x$ ($0 \le x \le n-k$).

REMARK 3.3. It may be noted that if X and P are two random variables such that $(X \mid P = p)$ is distributed as $B_{k,\mathrm{II}}^c(n,p)$ and P is distributed as $B(\alpha,\beta)$, then

$$
P(X = x) = \begin{cases}
\sum_{i=1}^{x+k} i \sum_{2} \binom{x_1+\cdots+x_n}{x_1,\ldots,x_n} \\
\times \dfrac{B(\alpha+n-1-\sum_{j=1}^n x_j, \beta+\sum_{j=1}^n x_j+1)}{B(\alpha,\beta)}, & \text{if } 0 \le x \le n-k, \\[2ex]
0, & \text{if } n-k+1 \le x \le n-1, \\[1ex]
\dfrac{B(\alpha+n,\beta)}{B(\alpha,\beta)}, & \text{if } x = n,
\end{cases}
\tag{3.9}
$$

where the summation $\sum_2$ is as in Theorem 3.1. Relation (3.9) reduces to (3.8) if $\alpha = c_0/s$ and $\beta = c_1/s$ ($s \ne 0$), which indicates that (3.9) may be considered as another form of $P_{k,\mathrm{II}}^c(n;s;c,c_0)$.

The mean of the circular Polya distribution of order k, type II, is given next.

PROPOSITION 3.4. *Let X be a random variable following the circular Polya distribution of order k, type II, with probability function as given by (3.9). Then,*

$$E(X) = \frac{nB(\alpha + k, \beta)}{B(\alpha, \beta)}.\tag{3.10}$$

PROOF. It follows by means of Proposition 2.7 of Makri and Philippou [11] and Remark 3.3.

The circular Polya distribution of order k, type II, includes as special cases four distributions of order k, of which three are new. Each one arises when success runs of length k are allowed to overlap by means of Theorem 3.1. Their means may be easily obtained from Proposition 3.5.

CASE 1. The $P_{k,\mathrm{II}}^c(n;s;c,c_0)$, for $s = -1$, reduces to a new distribution, which we call circular hypergeometric distribution of order k, type II, with parameters n, c, and c_0, and denote it by $H_{k,\mathrm{II}}^c(n;c,c_0)$.

CASE 2. The $P_{k,\mathrm{II}}^c(n;s;c,c_0)$, for $s = +1$, reduces to a new distribution, which we call circular negative hypergeometric distribution of order k, type II, with parameters n, c, and c_0, and denote it by $NH_{k,\mathrm{II}}^c(n;c,c_0)$.

CASE 3. The $P_{k,\mathrm{II}}^c(n;s;c,c_0)$, for $c = c_0 = s$, reduces to a new distribution, which we call circular Bose-Einstein uniform distribution of order k type II.

CASE 4. For $s = 0$, $P_{k,\mathrm{II}}^c(n;s;c,c_0)$ reduces to the circular binomial distribution of order k, type II, $B_{k,\mathrm{II}}^c(n,p)$, of Makri and Philippou [11], with $p = c_0/c$.

Finally, we relate $P_{k,\mathrm{II}}^c(\cdot)$ asymptotically to the circular binomial distribution of the same order and type. $\qquad\square$

PROPOSITION 3.5. *Let X_{c_0,c_1} and X be two random vectors distributed as $P_{k,\mathrm{I}}^c(n;s;c,c_0)$ and $B_{k,\mathrm{I}}^c(n,p)$, respectively, and assume that $c_0/c \to p$ $(0 < p < 1)$ as $c_0, c_1 \to \infty$, then*

$$P(X_{c_0,c_1} = x) \longrightarrow P(X = x), \quad x = 0,1,\dots,n.\tag{3.11}$$

PROOF. We observe that

$$\frac{(c_0/s)^{[n-1-\Sigma_{j=1}^n x_j]}(c_1/s)^{[\Sigma_{j=1}^n x_j+1]}}{(c/s)^{[n]}} \longrightarrow p^{n-1-\Sigma_{j=1}^n x_j}(1-p)^{\Sigma_{j=1}^n x_j+1},$$

$$(c_0/s)^{[n]}/(c/s)^{[n]} \longrightarrow p^n \quad \text{as } c_0, c_1 \longrightarrow \infty,\tag{3.12}$$

from which the proof follows. $\qquad\square$

The distribution of the number of success runs of length k in a circular sequence of n Bernoulli trials, with a constant success probability p, can have potential applications in several fields (see Koutras et al. [10]) as in quality control (modifying the Prairie et al.'s sampling process [19]) and in reliability

theory (introducing the circular-m-consecutive-k-out-of-n : F system, which fails whenever at least m nonoverlapping failure runs of length k occur, see Alevizos et al. [2], Chao et al. [3], Papastavridis and Koutras [13], and Makri and Philippou [12]). In these cases, when p is allowed to vary from trial to trial according to the beta distribution, the circular Polya distribution of order k arises.

Agin and Godbole [1] proposed a nonparametric test for randomness based on the number of success runs of length k in a linear sequence of n Bernoulli trials, for an alternative hypothesis of clustering. Furthermore, tests based on the number of success runs of length k in a linear sequence of trials in the sampling scheme described in Theorem 2.1 are valid even for finite populations (see Godbole [6]). An analogous test (which is expected to be more sensitive for the alternative hypothesis of circular clustering) can be established on the number $N_{n,k,s}^c$. Other potential applications are indicated by Theorems 2.1 and 3.1.
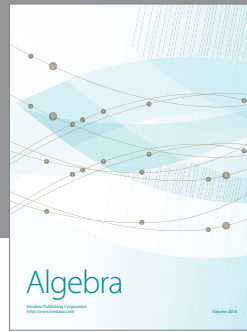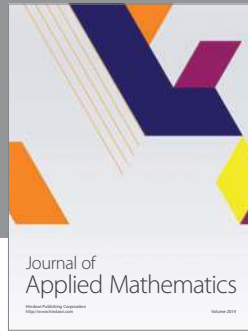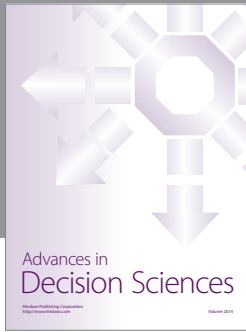
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