CHAPTER 30

CIRCULATION KINEMATICS IN NONLINEAR LABORATORY WAVES

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ABSTRACT

A weakly nonlinear solution is presented for the two-dimensional wave kinematics forced by a generic wavemaker of variable-draft. The solution is valid for both piston and hinged wavemakers of variable-draft that may be double articulated. The second-order propagating waves generated by a planar wave board are composed of two components; viz., a Stokes second-order wave and a second-harmonic wave forced by the wavemaker which travels at a different speed. A previously neglected time-independent solution that is required to satisfy a kinematic boundary condition on the wavemaker as well as a mixed boundary condition on the free surface is included for the first time. A component of the time-independent solution is found to accurately estimate the mean return current (correct to second-order) in a closed wave flume. This mean return current is usually estimated from kinematic considerations by a conservation of mass principle

INTRODUCTION

Flick and Guza (1980) investigated the motion of a hinged wavemaker that is hinged either on or below the channel bottom using a Stokes expansion. They studied the relationship between the second-harmonic (secondary) waves forced by the wavemaker and the Stokes waves by computing the coefficients for the propagating eigenmode numerically. Their solution, like that of Daugaard (1972), neglects the interactions of the first-order evanescent eigenmodes at the free-surface boundary near the wavemaker because these evanescent eigenmodes do not contribute to the propagating waves. Furthermore, their solution as well as that of Madsen (1971) and Daugaard (1972) is not exact because they neglect the time-independent, second-order solutions which are required to satisfy exactly the boundary conditions at the wavemaker and at the free surface.

Massel (1981) attempted to extend the work of Flick and Guza (1980) by including a time-independent solution but only for the kinematic boundary condition at the wavemaker.

A closed-form solution is presented that is correct to second-order (except for the singularities at the irregular points) for the fluid motion forced by a sinusoidally moving generic wavemaker of variable draft. The generic wavemaker motion is doubly articulated and includes both piston and hinged wavemakers. The previously

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neglected time-independent solutions required to satisfy both the nonlinear free surface and wavemaker boundary conditions are compared with the Eulerian mean horizontal momentum per unit area. The mean return current required to satisfy conservation of mass in closed wave flumes is estimated reasonably well by the time-independent, second-order solution for a broad class of planar wavemakers.

NONLINEAR WAVEMAKER THEORY

For convenience, all physical variables (denoted by superscript asterisks, *) will be made dimensionless by the following: $(x,z,h,d,b,\Delta,L) = k^*(x^*,z^*,h^*,d^*,b^*,\Delta^*,L^*)$; $(t,T) = \sqrt{g^*k^*}\,(t^*,T^*)$; $(H,\eta,S,\xi,\chi) = (H^*,\eta^*,S^*,\xi^*,\chi^*)/a^*$; $(u,w) = (u^*,w^*)/(a^*\sqrt{g^*k^*})$; $\Phi = \Phi^*/(a^*\sqrt{g^*/k^*})$; $B = B^*/(a^*g^*)$; and $p = p^*/(\rho^*a^*g^*)$ where a^* = amplitude of the first-harmonic wave component; $k^*(= 2\pi/L^*)$ = the wave number; L^* = wave length; g^* = gravitational constant; ρ^* = fluid mass density; and T^* = wave period = the period of the wavemaker oscillation.

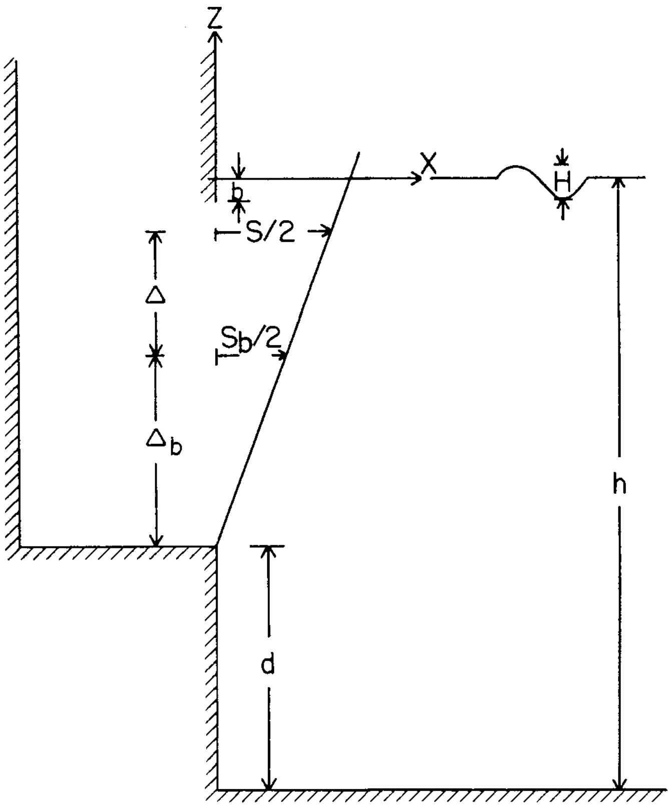

Fig. 1. Definition sketch for generic wavemaker

A generic wavemaker is shown in Fig. 1 which generates two-dimensional, irrotational motion of an inviscid, incompressible fluid in a semi-infinite channel of constant, still water depth, h. The fluid motion may be obtained from a scalar velocity potential $\Phi(x,z,t)$ by

$$[u,w] = -\vec{\nabla}\Phi \tag{1}$$

in which the two-dimensional gradient operator is $\vec{\nabla}(\cdot) = [\partial/\partial x, \partial/\partial z]$.

The velocity potential is a solution to

$$\nabla^2\Phi = 0 \qquad ; x > \varepsilon\chi(z,t) \ , \ -h < z < \varepsilon\eta(x,t) \tag{2a}$$

with boundary conditions (Phillips, 1977)

$$\partial\Phi/\partial z = 0 \qquad ; x > \varepsilon\chi(-h,t) \ , \ z = -h \tag{2b}$$

$$\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z} - [\varepsilon \frac{\partial}{\partial t} - \frac{1}{2}\varepsilon^2 \vec{\nabla}\Phi \ \vec{\nabla}]|\vec{\nabla}\Phi|^2 + \frac{dB}{dt} = 0; x > \varepsilon\chi(\eta,t), z = \varepsilon\eta(x,t) \tag{2c}$$

$$\partial\Phi/\partial x + \partial\chi/\partial t - \varepsilon\partial\Phi/\partial z \ \partial\chi/\partial z = 0 \quad ; x = \varepsilon\chi(z,t) \ ; -h < z < \varepsilon\eta(t) \tag{2d}$$

where $B(t)$ = the Bernoulli constant and the parameter $\varepsilon = a^*k^* << 1$. In addition, a radiation condition is required at infinity as $x \to +\infty$ in order to insure that propagating waves be only right progressing or that evanescent eigenmodes be bounded. The instantaneous wavemaker displacement from its mean position, $\chi(z,t)$, is given by

$$\chi(z,t) = \xi(z)[U(z+h-d)-U(z+b)]\sin\omega_o t = \xi(z)\Delta U \sin\omega_o t \tag{3}$$

where $U(\cdot)$ = the Heaviside step function. The amplitude of the wavemaker displacement, $\xi(z)$, for a double-articulated piston or hinged wavemaker of variable draft is given by the following equation for a straight line:

$$\xi(z) = [(S/2)/(\Delta/h)][M(1+z/h)+B'] \tag{4}$$

where $M = (1-S_b/S)$; and $B' = [\Delta/h-M(d/h+\Delta_b/h+\Delta/h)]$; in which $S/2$ = the dimensionless wavemaker stroke measured at an arbitrary elevation above the wave flume bottom at $z = -h+d+\Delta_b+\Delta$. A piston wavemaker is represented by $S_b = S$; and a wavemaker of full-depth draft is represented by $b = d = 0$ and $\Delta = h$. The dimensionless free-surface $\eta(x,t)$ and total pressure $p(x,z,t)$ are

$$\eta(x,t) = \partial\Phi/\partial t - \frac{1}{2}\varepsilon|\vec{\nabla}\Phi|^2 + B(t) \quad ; x > \varepsilon\chi(\eta,t) \ , \ z = \varepsilon\eta(x,t) \tag{5}$$

$$p(x,z,t) = \partial\Phi/\partial t - \frac{1}{2}\varepsilon|\vec{\nabla}\Phi|^2 - z/\varepsilon + B(t) \ ; x > \varepsilon\chi(z,t) \ , \ -h < z < \varepsilon\eta(x,t) \tag{6}$$

Equations (2c & d) & (5) may be expanded in a Maclauren series by

$$\sum_{n=0}^{\infty} \frac{(\varepsilon\eta)^n}{n!}\frac{\partial^n}{\partial z^n}[\frac{\partial^2\Phi}{\partial t^2} + \frac{\partial\Phi}{\partial z} - (\varepsilon\frac{\partial}{\partial t} - \frac{1}{2}\varepsilon^2\vec{\nabla}\Phi\cdot\vec{\nabla})|\vec{\nabla}\Phi|^2 + \frac{dB}{dt}] = 0 \ ; x > 0 \ , z = 0 \tag{7a}$$

$$\eta - \sum_{n=0}^{\infty} \frac{(\varepsilon\eta)^n}{n!}\frac{\partial^n}{\partial z^n}[\frac{\partial\Phi}{\partial t} - \frac{1}{2}\varepsilon|\vec{\nabla}\Phi|^2 + B] = 0 \quad ; x > 0 \ , \ z = 0 \tag{7b}$$

$$\sum_{n=0}^{\infty} \frac{(\varepsilon\chi)^n}{n!} \frac{\partial^n}{\partial x^n} \left[\frac{\partial\Phi}{\partial x} + \frac{\partial\chi}{\partial t} - \varepsilon\frac{\partial\Phi}{\partial z}\frac{\partial\chi}{\partial z}\right] = 0 \quad ; x = 0 ; -h < z < 0 \qquad (7c)$$

In addition, the functions Φ, η, B, p, and ω may also be expanded in the small parameter, ε, by the following:

$$\Phi(x,z,t) = \sum_{n=0}^{\infty} \varepsilon^n \, {}_{n+1}\Phi(x,z,t) \qquad (8a)$$

$$\eta(x,t) = \sum_{n=0}^{\infty} \varepsilon^n \, {}_{n+1}\eta(x,t) \qquad (8b)$$

$$B(t) = \sum_{n=0}^{\infty} \varepsilon^n \, {}_{n+1}B(t) \qquad (8c)$$

$$p(x,z,t) = p_s(z) + \sum_{n=0}^{\infty} \varepsilon^n \, {}_{n+1}P(x,z,t) \qquad (8d)$$

$$\omega t = \tau = \left(\sum_{n=0}^{\infty} \varepsilon^n \omega_n\right)t \qquad (8e)$$

in which $p_s(z) = z/\varepsilon$ = the dimensionless hydrostatic pressure.

Substituting Eqs. (7) & (8) into Eqs. (2)-(6) and collecting terms of the same order in ε results in a set of linear boundary value problems which may be solved in successive order.

Linear Solution

The linear boundary value problem for first-order (ε^o) is

$$\nabla^2 \, {}_1\Phi = 0 \quad ; x > 0 , -h < z < 0 \qquad (9a)$$

$$\partial_1\Phi/\partial z = 0 \quad ; x > 0 , z = -h \qquad (9b)$$

$$\mathcal{L}\{{}_1\Phi\} + \omega_o \, \partial_1 B/\partial\tau = 0 \quad ; x > 0 , z = 0 \qquad (9c)$$

$$\partial_1\Phi/\partial x = -\omega_o \partial\chi/\partial\tau \quad ; x = 0 , -h < z < 0 \qquad (9d)$$

in which the linear, free-surface operator, $\mathcal{L}\{\cdot\}$, is defined by

$$\mathcal{L}\{\cdot\} = (\omega_o^2 \, \partial^2/\partial\tau^2 + \partial/\partial z)\{\cdot\} \qquad (10)$$

The solution to Eqs. (9) must also satisfy a radiation condition at infinity as $x \to +\infty$ that will admit only right progressing waves or bounded evanescent eigenmodes.

The first-order, free-surface elevation, ${}_1\eta(x,\tau)$, and the dynamic pressure ${}_1P(x,z,\tau)$ may be determined from

$${}_1\eta(x,\tau) = \omega_o \, \partial_1\Phi/\partial\tau \quad ; x > 0 , z = 0 \qquad (11)$$

$${}_1P(x,z,\tau) = \omega_o \, \partial_1\Phi/\partial\tau \quad ; x > 0 , -h < z < 0 \qquad (12)$$

A simple-harmonic solution to the linear problem requires that ${}_1B$ be identically zero in Eq. (9c). The linear solution which satisfies the radiation condition as $x \to +\infty$ is well-known and may be expressed by the following eigenfunction expansion (Hudspeth and Chen, 1981):

$$_1\Phi(x,z,\tau) = -a_1\phi_1(z)\sin(x-\tau) - \cos\tau \sum_{m=2} a_m\phi_m(z)\exp(-\alpha_m x) \quad (13)$$

in which the orthonormal eigenfunctions, $\phi_m(z)$ in the interval of orthogonality $[-h<z<0]$ are given by $\phi_m(z) = \cos[\alpha_m(z+h)]/n_m$; where the normalizing constants, n_m, are computed from $n_m^2 = [2\alpha_m h+\sin 2\alpha_m h]/4\alpha_m$; provided that $\omega_o^2 h+\alpha_m h \tan\alpha_m h = 0$ where $\alpha_1 = +i$.

The dimensionless coefficients, a_m, in Eq. (13) are given by

$$a_1 = \frac{\omega_o(S/2\Delta)}{n_1} D_1(h) \; ; \; a_m = \frac{\omega_o(S/2\Delta)\, D_m(\alpha_m h)}{\alpha_m^3\, n_m} \; ; \; m \geq 2 \quad (14)$$

$$D_1(h) = h[M(1-b/h)+B']\sinh[h(1-b/h)]-h[M(d/h)+B']\sinh[h(d/h)\cdot U(d/h)]$$

$$- M\{\cosh[h(1-b/h)]-\cosh[h(d/h)\cdot U(d/h)]\} \quad (15a)$$

$$D_m(\alpha_m h) = -(\alpha_m h)[M(1-b/h)+B']\sin[(\alpha_m h)(1-b/h)]$$

$$+(\alpha_m h)[M(d/h)+B']\sin[(\alpha_m h)(d/h)\cdot U(d/h)]-M\{\cos[(\alpha_m h)(1-b/h)]$$

$$-\cos[(\alpha_m h)(d/h)\cdot U(d/h)]\} \; ; \; m \geq 2 \quad (15b)$$

where M and B' are defined by Eqs. (4) and U(•) = Heaviside step function which is required for negative-draft wavemakers ($d < 0$).

Second-Order Solution

The boundary value problem for second-order (ε^1) is

$$\nabla^2\,{}_2\Phi = 0 \quad ; \; x > 0 \;,\; -h < z < 0 \quad (16a)$$

$$\partial\,{}_2\Phi/\partial z = 0 \quad ; \; x > 0 \;,\; z = -h \quad (16b)$$

$$\mathcal{L}\{{}_2\Phi\} + \omega_o\, \partial\,{}_2B/\partial\tau = -2\omega_o\omega_1\, \partial^2\,{}_1\Phi/\partial\tau^2 + \omega_o \frac{\partial}{\partial\tau}|\vec{\nabla}\,{}_1\Phi|^2$$

$$- {}_1\eta \frac{\partial}{\partial z}(\omega_o^2\, \partial^2\,{}_1\Phi/\partial\tau^2 + \partial\,{}_1\Phi/\partial z) \quad ; \; x > 0 \;,\; z = 0 \quad (16c)$$

$$\partial\,{}_2\Phi/\partial x = -\omega_1\partial\chi/\partial\tau+\partial\,{}_1\Phi/\partial z\; \partial\chi/\partial z-(\partial^2\,{}_1\Phi/\partial x^2)\chi \; ; \; x=0 \;,-h<z<0 \quad (16d)$$

The solution to Eqs. (16) must also satisfy a radiation condition at infinity as $x\to+\infty$ that will admit only right progressing waves or bounded eigenmodes. Because Eq. (16d) is an inhomogeneous Neumann condition, any constant times x may also be used for any time-independent solution.

The Bernoulli constant is ${}_2B = (a_1/2n_1)^2$ and $\partial\,{}_2B/\partial\tau = 0$ in Eq. (16c). The first term in the right hand side of Eq. (16c) must vanish since $\partial^2\,{}_1\Phi/\partial\tau^2$ is a homogeneous solution of the linear operator on the left hand side of Eq. (16c) so that $\omega_1 = 0$.

It is customary in boundary value problems with inhomogeneous boundary conditions on orthogonal boundaries such as those given by Eqs. (16c & d) to linearly decompose the solution into complementary homogeneous and inhomogeneous solutions. Accordingly, the solution to Eqs. (16) may be expressed as the linear sum of four scalar velocity potentials given by ${}_2\Phi = {}_2\Phi^s + {}_2\Phi^e + {}_2\Phi^f + \Psi$ in which ${}_2\Phi^s$ is a second-order Stokes wave potential; ${}_2\Phi^e$ is a near-field evanescent interaction potential; ${}_2\Phi^f$ is a wavemaker-forced potential; and Ψ is a time-independent potential needed to satisfy Eqs. (16c & d) exactly. This linear decomposition of ${}_2\Phi$ reduces Eqs. (16c & d) to

$$\mathcal{L}\{{}_2\Phi^s+{}_2\Phi^e+{}_2\Phi^f+\Psi\} = a_1^2 f_1(\phi_1)\sin 2(x-\tau)$$

$$- a_1 \sin(x-2\tau)\sum_{m=2} a_m \exp(-\alpha_m x) f_2(\phi_1,\phi_m)$$

$$+ a_1 \cos(x-2\tau)\sum_{m=2} a_m \exp(-\alpha_m x) f_3(\phi_1,\phi_m)$$

$$- \sin 2\tau \sum_{m=2}\sum_{n=2} a_m a_n \exp[-(\alpha_m+\alpha_n)x] f_4(\phi_m,\phi_n)$$

$$- a_1 \cos x \sum_{m=2} a_m \exp(-\alpha_m x) f_5(\phi_1,\phi_m) \; ; \; x>0 \; , \; z=0 \tag{17a}$$

$$\frac{\partial}{\partial x}\{{}_2\Phi^s+{}_2\Phi^e+{}_2\Phi^f+\Psi\} = +\frac{a_1}{2} W_1(\phi_1,\xi,z)[1-\cos 2\tau]$$

$$+ \frac{\sin 2\tau}{2}\sum_{m=2} a_m \alpha_m W_2(\phi_m,\xi,z) \; ; \; x=0, \; -h<z<0 \tag{17b}$$

in which the nonlinear, free surface interaction terms f_1, f_2, f_3, f_4, and f_5, and the nonlinear, wavemaker interaction terms W_1 and W_2 represent nonlinear interactions involving first-order quantities that are defined in Appendix I.

The second-order Stokes wave potential, ${}_2\Phi^s$, must satisfy exactly Eqs. (16a & b); a radiation condition at infinity as $x \to +\infty$ requiring only right progressing waves, as well as the inhomogeneous part of the nonlinear free-surface condition in Eq. (17a) given by

$$\mathcal{L}\{{}_2\Phi^s\} - a_1^2 f_1(\phi_1)\sin 2(x-\tau) = 0 \qquad ; \; x > 0 \; , \; z = 0 \tag{18}$$

The well-known Stokes (1847) wave potential is simply ${}_2\Phi^s = -(3\omega_o/8)\mathrm{cosech}^4 h \cosh 2(z+h) \sin 2(x-\tau)$.

The near-field evanescent wave potential, ${}_2\Phi^e$, must satisfy Eqs. (16a & b), a radiation condition at infinity as $x \to +\infty$ requiring only bounded evanescent eigenmodes, as well as that part of the inhomogeneous free-surface boundary condition given by

$$\mathcal{L}\{{}_2\Phi^e\} + a_1 \sin(x-2\tau)\sum_{m=2} a_m \exp(-\alpha_m x) f_2(\phi_1,\phi_m)$$

$$- a_1 \cos(x-2\tau)\sum_{m=2} a_m \exp(-\alpha_m x) f_3(\phi_1,\phi_m)$$

$$+ \sin 2\tau \sum_{m=2}\sum_{n=2} a_m a_n \exp[-(\alpha_m+\alpha_n)x] f_4(\phi_m,\phi_n) = 0 \; ; \; x > 0 \; , \; z = 0 \tag{19}$$

A solution for the near-field evanescent potential that satisfies a radiation condition as $x \to +\infty$ is assumed to be given by

$$
\begin{aligned}
{}_2\Phi^e(x,z,\tau) &= a_1 \cos(x-2\tau) \sum_{m=2} a_m \exp(-\alpha_m x)[A_m \phi_1(z)\phi_m(z) + B_m \phi_1'(z)\phi_m'(z)] \\
&- a_1 \sin(x-2\tau) \sum_{m=2} a_m \exp(-\alpha_m x)[A_m \phi_1'(z)\phi_m'(z) - B_m \phi_1(z)\phi_m(z)] \\
&- \sin 2\tau \sum_{m=2}\sum_{n=2} a_m a_n \exp[-(\alpha_m+\alpha_n)x] C_{mn} [\phi_m(z)\phi_n(z) - \phi_m'(z)\phi_n'(z)]
\end{aligned}
\tag{20a}
$$

where

$$\phi_1'(z) = \sinh(z+h)/n_1; \qquad \phi_m'(z) = \sin[\alpha_m(z+h)]/n_m; \qquad m > 2 \tag{20b}$$

Substituting Eqs. (20) into Eq. (19) and equating coefficients of like functions gives

$$A_m = \frac{\alpha_m^2 \omega_o^3 \{12 + 2(\alpha_m^2 - 1)/\omega_o^4 + \operatorname{cosec}^2 \alpha_m h - \operatorname{cosech}^2 h\}}{[4\omega_o^4 + \alpha_m^2 - 1]^2 + (2\alpha_m)^2} \tag{21a}$$

$$B_m = \frac{\alpha_m \omega_o^3}{2} \frac{\{[4\omega_o^4 + \alpha_m^2 - 1][\operatorname{cosech}^2 h - \operatorname{cosec}^2 \alpha_m h - 4] + 8(\alpha_m/\omega_o^2)^2\}}{[4\omega_o^4 + \alpha_m^2 - 1]^2 + (2\alpha_m)^2} \tag{21b}$$

$$C_{mn} = \alpha_m \alpha_n \omega_o^3 \frac{[1 + (\alpha_n \alpha_m/\omega_o^4) + \frac{1}{4}(\operatorname{cosec}^2 \alpha_m h + \operatorname{cosec}^2 \alpha_n h)]}{(2\omega_o^2)^2 + (\alpha_m - \alpha_n)^2} \tag{21c}$$

The wavemaker-forced potential, ${}_2\Phi^f$, must satisfy Eqs. (16a & b); a homogeneous form of the linear, free-surface operator defined by Eq. (10); a radiation condition at infinity as $x \to +\infty$ requiring only right progressing waves and bounded eigenmodes; as well as that part of the inhomogeneous wavemaker boundary condition given by

$$
\begin{aligned}
&\frac{\partial}{\partial x}\{{}_2\Phi^f\} + \frac{\partial}{\partial x}\{{}_2\Phi^s + {}_2\Phi^e\} + \frac{a_1}{2}\cos 2\tau \, W_1(\phi_1,\xi,z) \\
&\quad - \frac{\sin 2\tau}{2} \sum_{m=2} a_m \alpha_m W_2(\phi_m,\xi,z) = 0 \; ; \; x = 0, \; -h < z < 0
\end{aligned}
\tag{22}
$$

A solution for the wavemaker-forced potential that satisfies a radiation condition at $x \to +\infty$ is assumed to be given by

$$
\begin{aligned}
{}_2\Phi^f(x,z,\tau) &= \{E_1 \cos(\beta_1 x - 2\tau) + F_1 \sin(\beta_1 x - 2\tau)\} Q_1(z) \\
&- \sum_{j=2} \exp(-\beta_j x)\{E_j \sin 2\tau + F_j \cos 2\tau\} Q_j(z)
\end{aligned}
\tag{23}
$$

in which the orthonormal eigenfunctions, $Q_j(z)$, in the interval of orthogonality $[-h < z < 0]$ are $Q_j(z) = \cos\beta_j(z+h)/N_j$; where the normalizing constants are $N_j^2 = (2\beta_j h + \sin 2\beta_j h)/(4\beta_j)$; provided that $4\omega_o^2 h + \beta_j h \tan\beta_j h = 0$ and that $\beta_1 = i\beta_1$.

The coefficients E_j and F_j are

$$E_j = -\beta_j^{-1}\Big\{\sum_{m=2} a_m[a_1(A_m+\alpha_m B_m)\langle\phi_1\phi_m,Q_j\rangle_z + \sum_{n=2} a_n(\alpha_m+\alpha_n)C_{mn}\langle\phi_m\phi_n,Q_j\rangle_z]$$

$$+ \sum_{m=2} a_m[a_1(B_m-\alpha_m A_m)\langle\phi_1'\phi_m',Q_j\rangle_z - \sum_{n=2} a_n(\alpha_m+\alpha_n)C_{mn}\langle\phi_m'\phi_n',Q_j\rangle_z]$$

$$- \sum_{m=2}\frac{a_m\alpha_m}{2}\langle W_2,Q_j\rangle_z\Big\} \qquad ; j > 1 \qquad (24a)$$

$$F_j = \beta_j^{-1}\Big\{\frac{3\omega_o}{4\sinh^4 h}\langle\cosh 2(z+h),Q_j\rangle_z + a_1\sum_{m=2} a_m[(A_m\alpha_m-B_m)\langle\phi_1\phi_m,Q_j\rangle_z +$$

$$+ (A_m+\alpha_m B_m)\langle\phi_1'\phi_m',Q_j\rangle_z] - \frac{a_1}{2}\langle W_1,Q_j\rangle_z\Big\} \qquad ; j > 1 \qquad (24b)$$

where the inner product terms $\langle\cdot,\cdot\rangle_z$ in Eqs. (24) are summarized in Appendix II.

An interesting feature of the second-order problem which has not previously been given much detailed attention is the time-independent potential, $\Psi(x,z)$, which must satisfy the following:

$$\nabla^2\Psi = 0 \qquad ; x > 0 , -h < z < 0 \qquad (25a)$$

$$\mathcal{L}\{\Psi\} = \frac{\partial\Psi}{\partial z} = -a_1 \cos x \sum_{m=2} a_m \exp(-\alpha_m x) f_5(\phi_1,\phi_m) \qquad ; x > 0 ; z = 0 \qquad (25b)$$

$$\frac{\partial\Psi}{\partial z} = 0 \qquad ; x > 0 ; z = -h \qquad (25c)$$

$$\frac{\partial\Psi}{\partial x} = \frac{a_1}{2} W_1(\phi_1,\xi,z) \qquad ; x = 0 ; -h < z < 0 \qquad (25d)$$

Because the time-independent solution is not a progressive wave, the radiation condition at infinity as $x \to +\infty$ is relaxed to admit bounded, time-independent velocities.

Similarly, Ψ may be decomposed into two linearly independent potentials according to $\Psi = \Psi^{fs} + \Psi^{wm}$.

The free surface potential, Ψ^{fs}, must satisfy Eqs. (25a & c); a boundedness condition as $x \to +\infty$; in addition to the inhomogeneous free surface boundary condition given by Eq. (25b). A solution which is bounded is given by

$$\Psi^{fs}(x,z) = a_1 \cos x \sum_{m=2} a_m \exp(-\alpha_m x)[b_m\phi_1(z)\phi_m(z) + c_m\phi_1'(z)\phi_m'(z)]$$

$$- a_1 \sin x \sum_{m=2} a_m \exp(-\alpha_m x)[b_m\phi_1'(z)\phi_m'(z) - c_m\phi_1(z)\phi_m(z)]$$

$$(26a)$$

where

$$b_m = -\frac{\omega_o^3 \alpha_m^2 \, [\mathrm{cosech}^2 h + \mathrm{cosec}^2 \alpha_m h]}{(\alpha_m^2 + 1)^2} \tag{26b}$$

$$c_m = -\frac{\omega_o^3 \alpha_m (\alpha_m^2 - 1) [\mathrm{cosech}^2 h + \mathrm{cosec}^2 \alpha_m h]}{2(\alpha_m^2 + 1)^2} \tag{26c}$$

The wavemaker potential, Ψ^{wm}, must satisfy Eqs. (25a & c); a boundedness condition on the velocity as $x \to +\infty$; in addition to a homogeneous free surface boundary condition given by

$$\mathcal{L}\{\Psi^{wm}\} = \frac{\partial \Psi^{wm}}{\partial z} = 0 \qquad ; x \geq 0 ; z = 0 \tag{27}$$

and an inhomogeneous wavemaker boundary condition given by

$$\frac{\partial \Psi^{wm}}{\partial x} - \frac{a_1}{2} W_1(\phi_1, \xi, z) + \frac{\partial \Psi^{fs}}{\partial x} = 0 \qquad ; x = 0 ; -h \leq z \leq 0 \tag{28}$$

A solution for Ψ^{wm} is given by the following eigenfunction expansion:

$$\Psi^{wm}(x,z) = \sum_{j=0} d_j \psi_j(z) [\exp(-\mu_j x) + \delta_{jo}(x-1)] \tag{29}$$

where the orthonormal eigenfunctions, $\psi_j(z)$, in the interval of orthogonality $[-h \leq z \leq 0]$ are given by

$$\psi_j(z) = \cos \mu_j (z+h) / [h/(2-\delta_{jo})]^{1/2} \qquad ; j \geq 0 \tag{30}$$

provided that the eigenvalues, μ_j, are given by $\mu_j = j\pi/h$.
The coefficients d_j are

$$d_j = -(\mu_j - \delta_{jo})^{-1} a_1 \Big\{ \tfrac{1}{2} \langle W_1, \psi_j \rangle_z + \sum_{m=2} a_m [b_m (\alpha_m \langle \phi_1 \phi_m, \psi_j \rangle_z + \langle \phi_1' \phi_m', \psi_j \rangle_z)$$
$$+ c_m (\alpha_m \langle \phi_1' \phi_m', \psi_j \rangle_z - \langle \phi_1 \phi_m, \psi_j \rangle_z)] \Big\} \qquad ; j \geq 0 \tag{31}$$

where the inner product terms $\langle \cdot , \cdot \rangle_z$ are summarized in Appendix II.

CIRCULATION KINEMATICS

It is of interest to compare the second-order (ε), time-independent solution forced by the weakly nonlinear boundary conditions at both the free surface and the wavemaker boundaries given by Eqs. (17) with the mean horizontal momentum per unit area. The time- and depth-averaged dimensionless mean horizontal momentum per unit area is defined by (Phillips, 1977)

$$M_E = \Big\langle \int_{-h}^{\eta} U_E \, dz \Big\rangle_{2\pi} \tag{32}$$

where the temporal averaging operator $\langle\cdot\rangle_{2\pi} = (2\pi)^{-1} \int_0^{2\pi} (\cdot)d\tau$ and U_E is an Eulerian horizontal velocity component.

The horizontal component of the dimensionless Eulerian velocity may be determined approximately from

$$M_E = U_\Psi + U_\Phi + O(\varepsilon^2) \tag{33}$$

The dimensionless horizontal component, U_Ψ, that is forced by f_5 and W_1 in Eqs. (17) may be estimated from the time-independent velocity potential according to $U_\Psi = U_{\Psi,\infty}(d_o) + U_{\Psi,e}(a_m)$ where

$$U_{\Psi,\infty}(d_o) = -\varepsilon(2\omega_o)^{-1}\{\frac{n_1}{D_1(h)\phi_1'(o)}\}\{h[M(1-b/h)+B']\cosh b(\omega_o^2-\tanh b)$$
$$-h[M(d/h)+B']\sinh d \operatorname{sech} h$$
$$+\omega_o^6 \sum_{m=2} \frac{\phi_m(o)D_m(\alpha_m h)}{n_m\alpha_m^2(\alpha_m^2+1)} [\operatorname{cosech}^2 h+\operatorname{cosech}^2\alpha_m h]\} \tag{34a}$$

$$U_{\Psi,e}(a_m) = \varepsilon\frac{\omega_o^5 n_1}{D_1(h)\phi_1'(o)} \cos x \sum_{m=2} \frac{\phi_m(o)D_m(\alpha_m h)}{n_m\alpha_m^3(\alpha_m^2+1)} \exp-\alpha_m x$$
$$[\operatorname{cosech}^2 h+\operatorname{cosech}^2\alpha_m h][\alpha_m-\tan x] \tag{34b}$$

Similarly, the dimensionless horizontal component, U_Φ, may be estimated from the first-order eigenmodes by a Maclauren series expansion about the still water level according to

$$U_\Phi = -\varepsilon\omega_o\langle(\partial_1\Phi/\partial x)(\partial_1\Phi/\partial\tau)\rangle_{2\pi} = U_{\Phi,\infty}(\omega_o)+U_{\Phi,e}(a_m) \;;\; z=0 \tag{35a}$$

The dimensionless, far-field component, $U_{\Phi,\infty}(\omega_o)$, is given by

$$U_{\Phi,\infty}(\omega_o) = \varepsilon(2\omega_o)^{-1} \tag{35b}$$

which is the well-known Eulerian description of the dimensionless Stokes drift (Longuet-Higgins, 1953). The dimensionless evanescent component, $U_{\Phi,e}(a_m)$, is given by

$$U_{\Phi,e}(a_m) = -(\varepsilon/2)\cos x \sum_{m=2} a_m\phi_m(o)\exp - \alpha_m x \;[\alpha_m - \tan x] \tag{35c}$$

In the far-field ($x > 3h$, say), the evanescent components of the mean horizontal momentum per unit area, $U_{\Psi,e}(a_m) + U_{\Phi,e}(a_m)$, are negligible. This implies that far away from the local wavemaker effects, the mean horizontal momentum per unit area is approximately

$$M_E \sim \varepsilon(2\omega_o)^{-1}\{1 - [\frac{n_1}{D_1(h)\phi_1'(o)}][h(M(1-b/h)+B')\cosh b(\omega_o^2-\tanh b)$$
$$- h(M(d/h)+B')\sinh d \operatorname{sech} h$$
$$+\omega_o^6 \sum_{m=2} \frac{\phi_m(o)D_m(\alpha_m h)}{n_m\alpha_m^2(\alpha_m^2+1)} (\operatorname{cosech}^2 h + \operatorname{cosech}^2\alpha_m h)]\} \tag{36}$$

which implies that for wavemakers intersecting the free surface (i.e., b=o), the leading-order coefficient, d_o, approximately estimates the uniform return current required to insure a zero net mass flux in a closed wave flume. This may be observed by plotting the dimensionless, time-independent Eulerian velocities.

The dimensionless horizontal component of the second-order (ε), time-independent fluid motion is

$$U_\Psi(x,z) = -\varepsilon \frac{\partial \Psi^{wm}}{\partial x} - \varepsilon \frac{\partial \Psi^{fs}}{\partial x}$$

$$= \varepsilon \sum_{j=0} d_j(\mu_j \psi_j(z) \exp -\mu_j x - \delta_{jo})$$

$$- \varepsilon \frac{\omega_o^3}{4} \frac{n_1}{D_1(h)} [1+2h \text{ cosech } 2h] \cos x \sum_{m=2} \frac{D_m(\alpha_m h)}{n_m} \frac{\exp -\alpha_m x}{\alpha_m^2(\alpha_m^2+1)}$$

$$[\text{cosech}^2 h + \text{cosec}^2 \alpha_m h][\phi_1(z)\phi_m(z)(1+\alpha_m \tan x)$$

$$+ \phi_1'(z)\phi_m'(z)(\alpha_m - \tan x)] \tag{37a}$$

and the dimensionless vertical component is

$$V_\Psi(x,z) = -\varepsilon \frac{\partial \Psi^{wm}}{\partial z} - \varepsilon \frac{\partial \Psi^{fs}}{\partial z}$$

$$= \varepsilon \sum_{j=1} d_j \mu_j \psi_j'(z) \exp -\mu_j x$$

$$- \varepsilon \frac{\omega_o^3}{4} \frac{n_1}{D_1(h)} [1+2h \text{ cosech } 2h] \cos x \sum_{m=2} \frac{D_m(\alpha_m h)}{n_m} \frac{\exp -\alpha_m x}{\alpha_m^2(\alpha_m^2+1)}$$

$$[\text{cosech}^2 h + \text{cosec}^2 \alpha_m h][\phi_1(z)\phi_m'(z)(1+\alpha_m \tan x)$$

$$- \phi_1'(z)\phi_m(z)(\alpha_m - \tan x)] \tag{37b}$$

where $\psi_j'(z) = \sin \mu_j(z+h)/\sqrt{h}$.

The magnitude of the velocity $[U_\Psi^2(x,z) + V_\Psi^2(x,z)]^{1/2}$ is illustrated in Figs. 2 & 3 for both a piston and hinged wavemaker. Figures 2 & 3 demonstrate that the mean return current is estimated reasonably well by the leading coefficient, d_o.

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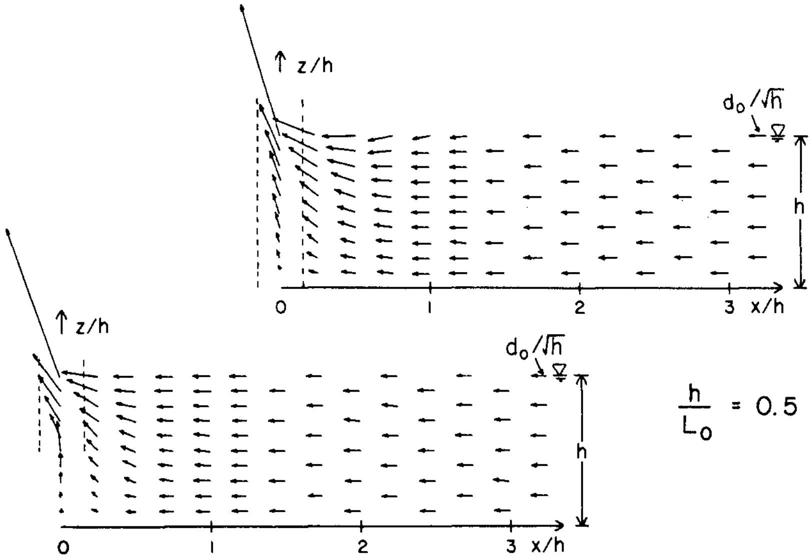

Fig. 2. Magnitude of dimensionless time-independent velocity (piston)

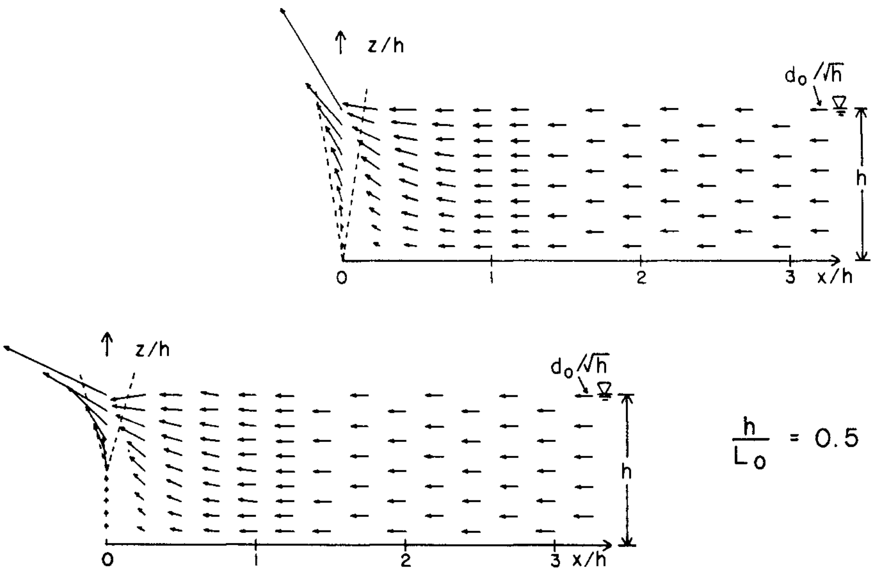

Fig. 3. Magnitude of dimensionless time-independent velocity (hinged)

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APPENDIX I: NONLINEAR INTERACTION COEFFICIENTS

The nonlinear, inhomogeneous free surface interaction terms in Eq. (17a) are defined from first-order quantities by the following:

$$f_1(\phi_1) = 3\omega_o n_1^{-2}/2 \tag{I.1}$$

$$f_2(\phi_1,\phi_m) = 2\alpha_m \omega_o \phi_1(0)\phi_m(0) \tag{I.2}$$

$$f_3(\phi_1,\phi_m) = \frac{\omega_o^5}{2}\,\phi_1(0)\,\phi_m(0)[\mathrm{cosech}^2 h - \mathrm{cosec}^2\alpha_m h - 4] \tag{I.3}$$

$$f_4(\phi_m,\phi_n) = \omega_o^5\,\phi_n(0)\,\phi_m(0)\,[\alpha_n\alpha_m/\omega_o^4+1+\tfrac{1}{4}\,(\mathrm{cosec}^2\alpha_m h+\mathrm{cosec}^2\alpha_n h)] \tag{I.4}$$

$$f_5(\phi_1,\phi_m) = (\omega_o^5/2)\phi_1(0)\,\phi_m(0)[\mathrm{cosech}^2 h + \mathrm{cosec}^2\alpha_m h] \tag{I.5}$$

The nonlinear, inhomogeneous wavemaker interaction terms in Eq. (17b) are defined by the following:

$$W_1(\phi_1,\xi,z) = [\phi_1'(z)\partial\xi/\partial z+\phi_1(z)\xi]\cdot[U(z+h-d)-U(z+b)] \tag{I.6}$$

$$W_2(\phi_m,\xi,z) = [\phi_m'(z)\partial\xi/\partial z+\alpha_m\phi_m(z)\xi]\cdot[U(z+h-d)-U(z+b)] \tag{I.7}$$

where $\xi(z)$ is defined in Eq. (4); ϕ_m' are defined in Eq. (20b); and

$$\partial\xi/\partial z = M(S/2\Delta) \tag{I.8}$$

APPENDIX II: INNER PRODUCTS $\langle\cdot,\cdot\rangle_z$

The inner product terms used to compute the coefficients of the second-order potential are determined from

$$\langle\cdot,\cdot\rangle_z = \int_{-h}^{o} \{\cdot,\cdot\}dz \tag{II.1}$$

These inner products are:

$$\langle\phi_m\phi_n,\Lambda_j\rangle_z = \phi_m(o)\phi_n(o)\Lambda_j(o)\left\{\frac{2(\omega_o\lambda_j)^2+\Omega_j[2\omega_o^4+\alpha_m^2+\alpha_n^2-\lambda_j^2]}{(\alpha_m^2+\alpha_n^2-\lambda_j^2)^2-(2\alpha_m\alpha_n)^2}\right\} \tag{II.2}$$

$$\langle\phi_m'\phi_n',\Lambda_j\rangle_z = \phi_m(o)\phi_n(o)\Lambda_j(o)\,|\alpha_m|\,|\alpha_n|$$

$$\left\{\frac{\omega_o^2[\alpha_m^4+\alpha_n^4-\lambda_j^2(\alpha_m^2+\alpha_n^2)+\omega_o^2\Omega_j(\alpha_m^2+\alpha_n^2-\lambda_j^2)]-2(\alpha_m\alpha_n)^2[\omega_o^2-\Omega_j]}{\alpha_m^2\alpha_n^2[(\alpha_m^2+\alpha_n^2-\lambda_j^2)^2-(2\alpha_m\alpha_n)^2]}\right\} \tag{II.3}$$

$$\langle \cosh 2(z+h), Q_j \rangle_z = \frac{4\omega_o^6 \, Q_j(0)}{(\omega_o^4 - 1)\,(4 + \beta_j^2)} \tag{II.4}$$

$$\langle W_1, \Lambda_j \rangle_z = \frac{S}{2\Delta} M \langle \phi_1' \Delta U, \Lambda_j \rangle_z + \langle \phi_1 \xi \Delta U, \Lambda_j \rangle_z \tag{II.5}$$

$$\langle W_2, \Lambda_j \rangle_z = \frac{S}{2\Delta} M \langle \phi_m' \Delta U, \Lambda_j \rangle_z + \alpha_m \langle \phi_m \xi \Delta U, \Lambda_j \rangle_z \tag{II.6}$$

$$\langle \phi_m' \Delta U, \Lambda_j \rangle_z = \frac{|\alpha_m|}{\alpha_m^2} [\lambda_j^2 - \alpha_m^2]^{-1} \{\phi_m(0)\Lambda_j(0) \cos \alpha_m b \cos \lambda_j b \tag{II.7}$$

$$[(\omega_o^2 + \alpha_m \tan \alpha_m b)(\Omega_j + \lambda_j \tan \lambda_j b) + \alpha_m^2 (1 + \tan \alpha_m h \tan \alpha_m b)(1 + \tan \lambda_j h \tan \lambda_j b)]$$

$$- \phi_m[d \cdot U(d) - h] \Lambda_j (d \cdot U(d) - h) \alpha_m^2 [1 + (\lambda_j / \alpha_m) \tan(\lambda_j d \cdot U(d)) \tan(\alpha_m d \cdot U(d))]\}$$

$$\langle \phi_m \xi \Delta U, \Lambda_j \rangle_z = \frac{S}{2\Delta} M (\alpha_m^2 - \lambda_j^2)^{-2} \{\phi_m(0) \Lambda_j(0) \cos \alpha_m b \cos \lambda_j b$$

$$\times [(\alpha_m^2 + \lambda_j^2)(1 - (\omega_o^2/\alpha_m) \tan \alpha_m b)(1 + \tan \lambda_j h \tan \lambda_j b)$$

$$+ 2(\omega_o^2 + \alpha_m \tan \alpha_m b)(\Omega_j + \lambda_j \tan \lambda_j b)] -$$

$$- \phi_m (d \cdot U(d) - h) \Lambda_j (d \cdot U(d) - h) [\alpha_m^2 + \lambda_j^2 + 2\alpha_m \lambda_j \tan(\alpha_m d \cdot U(d)) \tan(\lambda_j d \cdot U(d))]\}$$

$$+ \frac{S}{2\Delta} [M(h-b) + B'h] [\alpha_m^2 - \lambda_j^2]^{-1} \phi_m(0) \Lambda_j(0) \cos \alpha_m b \cos \lambda_j b$$

$$\times [(1 - (\omega_o^2/\alpha_m) \tan \alpha_m b)(\Omega_j + \lambda_j \tan \lambda_j b)$$

$$- (\omega_o^2 + \alpha_m \tan \alpha_m b)(1 + \tan \lambda_j h \tan \lambda_j b)]$$

$$- \frac{S}{2\Delta} [Md \cdot U(d) + B'h] [\alpha_m^2 - \lambda_j^2]^{-1} \phi_m (d \cdot U(d) - h) \Lambda_j (d \cdot U(d) - h)$$

$$\times [\alpha_m \tan(\alpha_m d \cdot U(d)) - \lambda_j (\tan \lambda_j d \cdot U(d))] \tag{II.8}$$

where

$$\Omega_j = \lambda_j \tan \lambda_j h = \begin{cases} 4\omega_o^2 \;;\; \text{for } \Lambda_j(z) = Q_j(z) \text{ and } \lambda_j = \beta_j \\ 0 \;;\; \text{for } \Lambda_j(z) = \psi_j(z) \text{ and } \lambda_j = \mu_j \end{cases}$$

$$\begin{Bmatrix} \alpha_1 \\ \beta_1 \end{Bmatrix} = i \begin{Bmatrix} 1 \\ \beta_1 \end{Bmatrix}$$

and ΔU is defined in Eq. (3).

APPENDIX III: REFERENCES

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