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Circulation of matter and evolution of the internal magnetic field in neutron stars

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ABSTRACT

Neutron stars may be in hydrostatic equilibrium only for very particular magnetic configurations with conservative magnetohydrodynamic forces. We consider the magnetohydrodynamic evolution of a neutron star with a non-zero external dipole field component. The ohmic dissipation should slightly change this equilibrium distribution of the currents, producing a slow circulation of matter which tends to maintain the magnetic configuration close to an equilibrium one. The evolution of a strong magnetic field ($B \simeq 10^{12}$–10^{13} G) under the influence of both ohmic dissipation and circulation is analysed in detail. The resulting field decay turns out to be crucially dependent on the thermal history of the neutron star. The time-scale of decay in the core may be rather short, particularly for old neutron stars with low surface temperatures T_s. For instance, this time-scale may be of the order of $\sim 10^6$–10^7 yr for $T_s \sim 5 \times 10^5$–10^6 K. The circulation accompanying field decay in the core is, in some aspects, like the classical meridional circulation in rotating stars. For example, the 'magnetic' circulation can partially mix the core matter. The velocity of circulation decreases with the age of the neutron star as a result of the decay of the magnetic field.

Key words: MHD – stars: interiors – stars: magnetic fields – stars: neutron.

1 INTRODUCTION

A large fraction of observed radio pulsars have very strong magnetic fields $B \geq 10^{12}$ G. The origin of these fields has been a subject of debate for many years. There are two essentially different points of view regarding the origin of neutron-star magnetic fields: one holds that these strong fields are inherited from their progenitors, and amplified during the collapse to the neutron star, and the other contends that these fields are generated by thermomagnetic effects after the neutron star is born (see e.g. Blandford, Applegate & Hernquist 1983; Urpin, Levshakov & Yakovlev 1986). It is important to realize that the later evolution of the field and its influence on thermal and magnetohydrodynamic processes within the star are likely to depend on its origin. The thermally generated magnetic field is expected to be confined to the crustal region, but its long-term evolution will be determined by the properties of both the crust and the liquid core (Urpin & Van Riper 1993). If the magnetic field is a fossil remnant of that of the progenitor star, then the magnetic flux is expected to thread the interior regions, where the density ρ may be above the nuclear density, $\rho_n = 2.8 \times 10^{14}$ g cm^{-3}. Undoubtedly, these strong magnetic fields not only have an influence on the transport processes or thermal balance of the superdense core matter (Haensel, Urpin & Yakovlev 1990; Urpin & Shalybkov 1992) but can also affect a number of important magnetohydrodynamic phenomena in neutron-star interiors. The present paper considers one such phenomenon initiated by a strong magnetic field passing through the interior regions: namely, we argue that in neutron-star cores with normal (non-superfluid or non-superconducting) npe-matter there is a large-scale circulation of matter which accompanies a quasi-static change in the field strength.

The reasons for such a circulation are as follows. It was argued by Chandrasekhar & Prendergast (1956) that a star with an arbitrary magnetic field cannot be in hydrostatic equilibrium. Equilibrium is possible only for a rather particular magnetic configuration with a conservative magnetohydrodynamic force. We have considered one such configuration with a non-zero dipole component outside the star. Dissipative processes can, however, change the magnetic configuration in the course of evolution. The ohmic dissipation of currents changes the distribution of these currents within the star. Changes in the currents and the magnetic field will slightly change the magnetohydrodynamic equilibrium through a set of quasi-static configurations. These changes produce a slow circulatory flow of matter which maintains the field configuration through a series of self-similar stages. The dissipation and circulation together result in a decay of the configuration, which is accompanied by a decrease in the magnitude of the currents alone, without any change in their spatial distribution. This circulation is very similar to the standard meridional circulation in stars (see e.g. Schwarzschild 1958). The only difference is in the causes of the circulatory flows: in the classical case the circulation tends to balance a departure from thermal equilibrium, whereas in the 'magnetic' case it compensates deviations produced by ohmic dissipation from a hydrostatic equilibrium magnetic configuration.

The paper is organized as follows. In Section 2 we consider hydrostatic equilibrium in magnetized neutron stars. Following the early work of Chandrasekhar & Prendergast (1956), we generalize their analysis to the case of a degenerate star with a non-uniform density. The general equations governing the hydromagnetic equilibrium configurations, and the solution of these equations with a non-zero dipole component outside the star, are obtained. Section 3 briefly discusses the electrical resistivity and conductivity of a superdense plasma in a strong magnetic field. Here we deal with a neutron-star model in which there is normal npe-matter in the core; other possibilities will be considered elsewhere. Section 4 analyses the decay of equilibrium hydromagnetic configurations due to ohmic dissipation. In Section 5 we argue for the existence of a large-scale circulatory flow in neutron-star interiors and calculate the velocity and configuration of the circulation. Section 6 discusses the results obtained.

2 HYDROSTATIC EQUILIBRIUM IN MAGNETIZED DEGENERATE STARS

It was pointed out by Chandrasekhar & Prendergast (1956) that a stellar magnetic field can be in hydrostatic equilibrium only for some particular magnetic configurations because of the non-conservative character of the Lorentz force. In this section we consider magnetic configurations that satisfy hydromagnetic equilibrium in liquid degenerate stars. The equation governing the magnetic field $\boldsymbol{B}$ of such a configuration is

$$-\frac{\nabla p}{\rho}+\nabla\psi+\frac{1}{4\pi\rho}(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=0, \tag{1}$$

where p is the pressure, ρ the density and ψ the gravitational potential. Calculating the curl of equation (1) and taking into account the fact that in degenerate stars $p\approx p(\rho)$, we obtain

$$\nabla\times\left[\frac{1}{\rho}\boldsymbol{B}\times(\nabla\times\boldsymbol{B})\right]=0. \tag{2}$$

Strictly speaking, $\nabla p\times\nabla\rho\neq 0$ because of small thermal corrections to the pressure of degenerate particles. If, however, the temperature T is not very high we can neglect these small corrections. For instance, for magnetic fields typical of neutron stars, the thermal corrections are negligible in equation (1) if $T<3\times10^7$ K. It follows from equation (2) that the equilibrium magnetic field must require the condition

$$(\nabla\times\boldsymbol{B})\times\boldsymbol{B}=\rho\nabla\eta, \tag{3}$$

where η is an arbitrary function. Only in this special circumstance can the pressure and gravitational force balance the magnetic force. If magnetohydrodynamic equilibrium does not apply, the magnetic force will drive a large-scale flow which, in its turn, will change the magnetic configuration. According to equation (3), hydrostatic equilibrium is possible for a much wider class of configurations than was considered by Easson (1976), who argued that only force-free configurations satisfy equilibrium equations. Easson's conclusion was based on an analysis of separate hydrostatic equilibria for neutron and electron–proton components, coupled with a chemical equilibrium between the charged particles and neutrons due to weak interactions. As was argued by Goldreich & Reisenegger (1992), however, the rate of weak interactions is very slow, and at a low temperature they cannot wipe out perturbations from chemical equilibrium. In that case, equilibrium configurations should satisfy equation (3).

The paper by Chandrasekhar & Prendergast (1956) treats the case of a uniform-density star ($\rho=$ constant), but, since the density can change by a large factor within the core of a neutron star/white dwarf, we will generalize their results to non-uniform-density stars. We restrict ourselves to axially symmetric configurations. The magnetic field, quite generally, can be expressed as the sum of toroidal and poloidal parts in the form

$$\boldsymbol{B}=\boldsymbol{e}_\varphi\Theta r\sin\theta+\nabla\times(\boldsymbol{e}_\varphi\Lambda r\sin\theta), \tag{4}$$

with Θ and Λ being any two functions that do not depend on the φ-coordinate. In what follows we will use both spherical (r, θ, φ) and cylindrical (s, z, φ) coordinates. For axial symmetry, the φ-component of the Lorentz force $(\nabla\times\boldsymbol{B})\times\boldsymbol{B}/4\pi$ should vanish (see equation 3). This condition implies

$$\frac{\partial(s^2\Theta, s^2\Lambda)}{\partial(s,z)}\equiv\frac{\partial}{\partial s}(s^2\Theta)\frac{\partial}{\partial z}(s^2\Lambda)-\frac{\partial}{\partial z}(s^2\Theta)\frac{\partial}{\partial s}(s^2\Lambda)=0, \tag{5}$$

where we use the standard notation for the Jacobian. In an equilibrium magnetic configuration, the toroidal and poloidal components must therefore be related by

$$s^2\Theta=F(s^2\Lambda), \tag{6}$$

with F being an arbitrary function of the specified argument. The φ-component of equation (2) gives

$$\frac{\partial}{\partial z}\left[Q\frac{\partial}{\partial s}(s^2\Lambda)+\frac{\Theta}{\rho}\frac{\partial}{\partial s}(s^2\Theta)\right] -\frac{\partial}{\partial s}\left[s^2\frac{\Theta}{\rho}\frac{\partial\Theta}{\partial z}-Qs^2\frac{\partial\Lambda}{\partial z}\right]=0, \tag{7}$$

where $Q=\Delta_5\Lambda/\rho$. Following Chandrasekhar & Prendergast (1956), we introduce the operator Δ_5 given by

$$\Delta_5\equiv\frac{\partial^2}{\partial s^2}+\frac{3}{s}\frac{\partial}{\partial s}+\frac{\partial^2}{\partial z^2}\equiv\frac{\partial^2}{\partial r^2}+\frac{4}{r}\frac{\partial}{\partial r} +\frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}+\frac{4}{r}\operatorname{ctg}\theta\frac{\partial}{\partial\theta} \tag{8}$$

in the cylindrical (the first equality) and spherical (the second equality) coordinates. Simplifying equation (7), one obtains

$$\frac{\partial(Q, s^2\Lambda)}{\partial(s,z)} = \frac{\partial(s^2\Theta, \Theta/\rho)}{\partial(s,z)}. \tag{9}$$

Taking into account that $s^2\Lambda$ and $s^2\Theta$ can be expressed in terms of each other in accordance with relationship (6), we can define a function $G(s^2\Lambda)$ such that

$$\frac{\partial(s^2\Lambda, G/s^2\rho)}{\partial(s,z)} = \frac{\partial(s^2\Theta, \Theta/\rho)}{\partial(s,z)}. \tag{10}$$

It is easy to show that the function G, satisfying this equality, is governed by the equation

$$2G(s^2\Lambda) = \frac{\mathrm{d}}{\mathrm{d}(s^2\Lambda)}(s^4\Theta^2). \tag{11}$$

Now, combining equations (9) and (10), we finally obtain

$$\Delta_5\Lambda + \frac{G(s^2\Lambda)}{s^2} = \rho\phi(s^2\Lambda), \tag{12}$$

where ϕ is an arbitrary function of the argument. Equations (11) and (12) yield the most general axially symmetric magnetic configuration possible for hydrostatic equilibrium in stars. Equation (12) differs from the analogous equation obtained by Chandrasekhar & Prendergast (1956) for a star with a uniform density only by the factor ρ on the right-hand side. The most general force-free fields are obtained by setting $\phi = 0$.

The present paper deals with pure poloidal configurations for which $G = 0$. Other possibilities will be considered elsewhere. Particular cases of poloidal configurations for stars with $\rho =$ constant have been investigated by Ferraro (1954). The case of particular interest for our purposes is the equilibrium configuration with the dipole field outside the star. The internal field in this case can be presented in the form (4) with $\Theta = 0$ and $\Lambda = \Lambda(r)$. Since the magnetic force is very small compared to the gravitational force within the core, we can neglect deviations from sphericity of the density distribution in equation (12). Evidently, Λ is a function of r alone only if $\phi =$ constant $\equiv A$. Hence the equation governing this configuration is

$$\frac{\partial^2\Lambda}{\partial r^2} + \frac{4}{r}\frac{\partial\Lambda}{\partial r} = A\rho. \tag{13}$$

Continuity of $\boldsymbol{B}$ at the stellar surface $r = R$ implies that the function Λ has to satisfy the boundary condition

$$R\frac{\partial\Lambda}{\partial r} + 3\Lambda = 0. \tag{14}$$

The general solution of equation (13), which has no singularity at $r = 0$ and satisfies the boundary condition (14), is

$$\Lambda = A\int_R^r \frac{\mathrm{d}r_1}{r_1^4}\int_0^{r_1}\rho(r_2)r_2^4\,\mathrm{d}r_2 - \frac{A}{3R^3}\int_0^R \rho(r_1)r_1^4\,\mathrm{d}r_1. \tag{15}$$

For a uniform-density distribution $\rho = \rho_0 =$ constant we have

$$\Lambda = \frac{A\rho_0}{2}\left(\frac{r^2}{5} - \frac{R^2}{3}\right), \tag{16}$$

in accordance with the result of Ferraro (1954). If the density profile within the star can be approximated by the second-order polynomial $\rho = \rho_0(1 - r^2/R^2)$, where ρ_0 is the central density, then

$$\Lambda = \frac{A\rho_0}{2}\left[\frac{r^2}{5}\left(1 - \frac{5}{4}\frac{r^2}{R^2}\right) - \frac{R^2}{6}\right]. \tag{17}$$

The parameter A characterizes the magnetic field strength in the configuration. It is more convenient, however, to characterize the field not by the value of A but by the surface values of the magnetic field at the pole, B_s, which are

$$B_\mathrm{s} = -\frac{2}{15}A\rho_0R^2, \qquad B_\mathrm{s} = -\frac{4}{105}A\rho_0R^2 \tag{18}$$

for the configurations (16) and (17), respectively. Using equations (16), (17) and (4), one can obtain the distribution of the magnetic field within the star for the case of uniform density,

$$B_\mathrm{r} = \frac{15}{2}B_\mathrm{s}\cos\theta\left(\frac{1}{3} - \frac{r^2}{5R^2}\right),$$
$$B_\theta = \frac{15}{2}B_\mathrm{s}\sin\theta\left(\frac{2r^2}{5R^2} - \frac{1}{3}\right), \tag{19}$$

and for the density profile $\rho = \rho_0(1 - r^2/R^2)$,

$$B_\mathrm{r} = \frac{21}{4}B_\mathrm{s}\left[\frac{5}{6} - \frac{r^2}{R^2}\left(1 - \frac{5}{14}\frac{r^2}{R^2}\right)\right]\cos\theta,$$
$$B_\theta = \frac{21}{2}B_\mathrm{s}\left[\frac{r^2}{R^2}\left(1 - \frac{15}{28}\frac{r^2}{R^2}\right) - \frac{5}{12}\right]\sin\theta. \tag{20}$$

For non-uniform density the field concentration in the central regions is higher than for $\rho =$ constant. For instance, for non-uniform density B_r at the surface is ≈ 4 times weaker than in the centre, whereas at $\rho =$ constant this ratio is equal to 2.5.

3 CONDUCTIVITY AND RESISTIVITY OF A NEUTRON-STAR CORE

The equilibrium configuration (19) should evolve under the influence of dissipative processes. This section briefly discusses the conductive properties of neutron-star cores with normal npe-matter (without neutron superfluidity and proton superconductivity). In the present paper we will neglect the influence of the solid crust on the evolution of the magnetic field. Strictly speaking, this is not entirely correct because the crustal conductivity may reach high values (see e.g. Itoh & Kohyama 1993) and the time-scale of field diffusion through the crust is very long (Sang & Chanmugam 1990; Urpin & Van Riper 1993). This time-scale can significantly exceed the decay time-scale of internal magnetic configurations with currents perpendicular to the magnetic field (Haensel et al. 1990). It is therefore quite possible that due to the high crustal conductivity the external field will not decay substantially in the course of evolution, being maintained by the crustal currents, in spite of the comparatively rapid reduction of the internal field in the core. However, since the properties of the neutron-star crust are not well established, one cannot exclude the situation in which the solidity of the crust is not very important; this may be especially relevant for strong magnetic fields. Depending upon the crustal yield strength and the magnetic stresses, the crust can 'flow' under magnetic stresses like a normal fluid (Ruderman 1972). Which behaviour is more relevant in real conditions (i.e. whether the crust really behaves like a solid or 'flows' under the stresses) depends on very delicate properties of astrophysical lattices, such as the shear modulus, the yield strength, etc. Unfortunately, the theoretical estimates give a rather uncertain value for the critical magnetic stresses that can break the crust. For instance, according to Ruderman (1972) the critical value B_{cr} of a magnetic field that can be compensated by lattice stresses is not very high and lies in the range 10^{12}–10^{13} G. Thus under the magnetic stresses produced by the field $B > B_{cr}$ a lattice will flow, and hence the fluid approximation is valid essentially down to the stellar surface. The conductive properties of such a fluid 'crust' are evidently not of particular importance, and in this case the field evolution is mainly determined by the evolution in the core, which is in turn determined by the conductivity of the core and its hydrodynamics. The present paper deals with the evolution of such 'strong' magnetic fields that break the crust. A forthcoming paper will consider the opposite case of 'weak' magnetic fields and strong crustal solidity.

The plasma of neutron-star cores is usually regarded as a mixture of strongly degenerate fermions. At densities $\rho \sim \rho_n$ the main constituents of such matter are neutrons with an admixture, typically several per cent, of protons and electrons. The conductivity of core matter has been a subject of study in several papers. The paper by Baym, Pethick & Pines (1969) examined the relaxation times of charge carriers with respect to p–e, n–p and n–e interactions, and obtained an expression for the conductivity at $B=0$. It was, however, argued by Haensel et al. (1990) that magnetic fields with strength $\sim 10^{12}$–10^{13} G can magnetize the plasma of neutron-star cores, resulting in strong anisotropy of the transport processes. The component of the electrical resistivity tensor perpendicular to $\boldsymbol{B}$ may be much larger than the component parallel to $\boldsymbol{B}$, and so perpendicular currents should decay much more rapidly. Recently, a more detailed study of the conductive properties of core matter with different chemical compositions and in the presence of a magnetic field has been carried out by Yakovlev & Shalybkov (1991), who obtained simple analytical expressions for components of the resistivity and conductivity tensors.

In the presence of a magnetic field the charge flow is described by the conductivity and resistivity tensors $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{R}}$. In a coordinate frame in which the z-axis is directed along the magnetic field $\boldsymbol{B}$, these tensors are

$$\hat{\boldsymbol{\sigma}} = \begin{pmatrix} \sigma_\perp & \sigma_\wedge & 0 \\ -\sigma_\wedge & \sigma_\perp & 0 \\ 0 & 0 & \sigma_\parallel \end{pmatrix}, \qquad \hat{\boldsymbol{R}} = \begin{pmatrix} R_\perp & R_\wedge & 0 \\ -R_\wedge & R_\perp & 0 \\ 0 & 0 & R_\parallel \end{pmatrix}. \tag{21}$$

Here $\sigma_\parallel$ and $R_\parallel$ are the conductivity and resistivity components along $\boldsymbol{B}$, $\sigma_\perp$ and $R_\perp$ are the components perpendicular to $\boldsymbol{B}$, and $\sigma_\wedge$ and $R_\wedge$ are the so-called Hall components which appear due to the Hall effect. The resistivity components $R_\parallel$ and $R_\perp$ determine the dissipation of the parallel ($\boldsymbol{j}_\parallel$) and perpendicular ($\boldsymbol{j}_\perp$) components of electric currents: the rate of Joule heating per unit volume is $\dot{Q} = R_\parallel j_\parallel^2 + R_\perp j_\perp^2$. The magnetic fields in neutron-star interiors are probably non-quantizing, and hence $R_\parallel = \sigma_\parallel^{-1}$ is equal to the resistivity at $B=0$.

The anisotropy of $\hat{\boldsymbol{\sigma}}$ and $\hat{\boldsymbol{R}}$ is characterized by the magnetization parameters $a_e = \omega_e \tau_e$ and $a_p = \omega_p \tau_p$ of electrons and protons, respectively, where ω_α is a particle gyrofrequency and τ_α is an effective relaxation time. The electron relaxation time in nuclear matter is determined by the electromagnetic scattering on protons, and the proton relaxation time is mainly determined by strong interactions with neutrons. The influence of the magnetic field on transport by the particles α is pronounced if $a_\alpha \geq 1$. The appropriate values of the magnetization fields B_e and B_p, which correspond to $a_e = 1$ and $a_p = 1$, respectively, are

$$B_e = 8.4 \times 10^8 \, T_8^2 \left(\frac{\rho_n}{\rho} + 0.17 \right) \text{G},$$
$$B_p = 5.9 \times 10^{12} \, T_8^2 \, \frac{(\rho_n/\rho)}{(\rho_n/\rho) + 0.17} \, \text{G} \tag{22}$$

(see Yakovlev & Shalybkov 1991), where $T_8 = T/10^8$ K. The magnetization field B_e for electrons is rather weak: typical neutron-star magnetic fields may strongly magnetize the electron component in the core, even at high temperatures. In contrast, the field B_p is much stronger owing to the large proton mass and the short relaxation time. In real conditions, however, protons can be magnetized too, particularly at low temperatures. Generally, both electrons and protons can contribute to the transport of charge in npe-plasma, and therefore the resistivity is characterized by both the parameters a_e and a_p. The components of the resistivity tensor in this case are

$$R_\parallel = 6.06 \times 10^{-29} \, T_8^2 \left(\frac{\rho_n}{\rho} \right)^3 \left(1 + \frac{\rho}{2\rho_n} \right) s;$$
$$R_\perp = R_0 \left(1 + \frac{B^2}{B_0^2} \right), \qquad R_\wedge = R_0 \, \frac{B}{B_e}, \tag{23}$$

where

$$B_0 = (B_e B_p)^{1/2} = 7.1 \times 10^{10} \, T_8^2 \left(\frac{\rho_n}{\rho} \right)^{1/2} \text{G}. \tag{24}$$

At $B < B_0$ we have $R_\perp \approx R_\parallel$, and the direct influence of the magnetic field on the rate of dissipation is negligible. Even in this case, however, the field can lead indirectly to more rapid dissipation due to the Hall currents (see Jones 1988; Urpin & Shalybkov 1991). The Hall currents are, however, non-dissipative and do not contribute to the rate of Joule heating $\dot{Q}$ directly. Nevertheless, since the Hall parameter B/B_e may be rather large even at $B < B_0$, the Hall drift can rapidly change the configuration of the currents, producing strong non-uniformities in the field, and thus increase the dissipation rate. The strong magnetic field $B > B_0$ can essentially change the character of dissipative processes. The transverse resistivity $R_\perp$ is much higher than the parallel one at $B \gg B_0$. Thus enhancement of $R_\perp$ is clearly associated with the magnetization of charge carriers. For $B > B_0$ we find that the stronger magnetic field, the higher the dissipation rate, since $R_\perp \propto B^2$. Moreover, the magnetic field changes the dependence of dissipative processes on temperature. In a weak field, $B < B_0$, the Joule heating $Q \propto R_\parallel \propto T^2$ and hence the rate of field decay decreases at low temperatures. In a strong field, $B > B_0$, we have $\dot{Q} \propto R_\parallel / B_0^2 T^4 \propto T^{-2}$ and the rate of dissipation increases if the neutron star cools down.

4 OHMIC DECAY OF THE INTERNAL FIELD

The magnetic field in the liquid interior of neutron stars is governed by the standard induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\frac{c^2}{4\pi} \nabla \times [\hat{R} \cdot (\nabla \times \boldsymbol{B})] + \nabla \times (\boldsymbol{V} \times \boldsymbol{B}), \qquad (25)$$

where $\boldsymbol{V}$ is the velocity of the hydrodynamic flow in the core. To follow the evolution of the internal magnetic field, one has to take into account the effect of hydrodynamic flows, which can be induced as follows. As mentioned above, hydrostatic equilibrium in the liquid interiors is possible only for particular magnetic configurations. For instance, the equilibrium configuration producing the exterior dipole field is given by equation (20) or (21) within thc star. Clearly, a magnetic equilibrium configuration cannot be maintained without changes in the course of evolution, during which dissipative processes alter the distribution of currents inside the star. Small deviations from the equilibrium distribution must result in a hydrodynamic flow, which advects the magnetic field in such a way as to compensate the small unbalanced forces, and restores a new hydrostatic equilibrium. The evolution of the field in the neutron-star core is therefore determined by two processes, one of which (ohmic dissipation) tends to destroy the equilibrium configuration, and one of which (hydrodynamic flow) acts to reduce deviations and return the field distribution to equilibrium. Since the time-scale of hydrodynamic processes is much shorter than that of dissipative processes, deviations from the equilibrium configuration have to be very small. In the resulting configuration governed by the above two processes, the distribution of the field within the star should be close to that given by equation (20) (if $\rho = \text{constant}$). The ohmic dissipation can manifest itself only by a decrease of the factor B_p. The equation determining the magnitude of the field may be easily obtained by multiplying equation (25) by $\boldsymbol{B}/8\pi$ and integrating it over the stellar volume. Thus we have

$$\frac{1}{8\pi} \frac{\mathrm{d}}{\mathrm{d}t} \int B^2 \,\mathrm{d}V = -\frac{c^2}{16\pi^2} \int R_\perp (\nabla \times \boldsymbol{B})^2 \,\mathrm{d}V. \qquad (26)$$

In this equation we take into account the fact that the field and the current are perpendicular for the dipole configuration, so the field decay is determined by the resistivity tensor component $R_\perp$ alone. The hydrodynamic flow does not contribute directly to the field decay. Substituting $\boldsymbol{B}$ from equation (20) and $R_\perp$ from equation (24) and assuming that $B > B_0$, we obtain

$$\frac{1}{B_\mathrm{p}^4} \frac{\mathrm{d}B_\mathrm{p}^2}{\mathrm{d}t} = -\frac{5.87 c^2 R_\parallel}{\pi R^2 B_0^2}. \qquad (27)$$

Because of neutron-star cooling, the quantities $R_\parallel$ and B_0 depend on age. It is convenient to extract these dependences, introducing

$$\tau_B = \frac{\pi R^2 T_8^2}{5.87 c^2 R_\parallel} = 1.40 \times 10^{12}\ \mathrm{yr}$$

and

$$B_\mathrm{c} = B_0 T_8^{-2} = 5.02 \times 10^{10}\ \mathrm{G}$$

(numbers are given for the Friedman & Pandharipande neutron-star model with $M = 1.4\ \mathrm{M}_\odot$). The quantities τ_B and B_c are the time-scale of the internal field decay and the characteristic field magnetizing the core plasma at $T_8 = 1$, respectively. Integrating equation (27), we obtain

$$B_\mathrm{p}(t) = B_\mathrm{p}(0) \left[1 + \frac{B_\mathrm{p}^2(0)}{B_\mathrm{c}^2} \frac{1}{\tau_B} \int_0^t \frac{\mathrm{d}t'}{T_8^2(t')} \right]^{-1/2}. \qquad (28)$$

Evidently, field decay is essentially non-exponential and is strongly dependent on the thermal history. The particular law of time dependence of $B_\mathrm{p}(t)$ is determined by the neutron-star cooling. For comparatively strong original magnetic fields $B_\mathrm{p}(0) \sim 10^{12}$ G, the decay time-scale may be several orders of magnitude shorter than τ_B, and this time-scale decreases with neutron-star cooling. This result is in agreement with the qualitative conclusion by Haensel et al. (1990), who examined the internal field decay without analysing hydrostatic equilibrium. It should be noted that after a relatively short initial phase the field decay, guided by equation (28), reaches a self-similar regime that is practically independent of the original field strength. If

$$\int_0^t \frac{\mathrm{d}t'}{T_8^2(t')} > \frac{B_\mathrm{c}^2}{B_\mathrm{p}^2(0)} \tau_B, \qquad (29)$$

then we have

$$B_\mathrm{p}(t) \approx B_\mathrm{c} \left[\frac{1}{\tau_B} \int_0^t \frac{\mathrm{d}t'}{T_8^2(t')} \right]^{-1/2}. \qquad (30)$$

The initial stage which precedes the self-similar regime lasts ~ 1–10 Myr depending on the original field strength. At the late evolutionary stage ($t > 10^7$ yr) the temperature is reduced to low values ($T < 10^5$ K) and the field can decay rather rapidly: the characteristic time-scale may be shorter than 10^6 yr. The order of magnitude of the integral in equation (30) for old neutron stars is $\sim t/T_8^2(t)$. One can therefore estimate B_p for these stars as

$$B_\mathrm{p}(t) \sim B_\mathrm{c} \left(\frac{\tau_B}{t} \right)^{1/2} T_8(t). \qquad (31)$$

For $t \sim 10^7$–10^8 yr this formula gives $B_\mathrm{p} \sim 10^9$ G.

Not only does the thermal evolution influence the field decay, but, in its turn, additional heating due to ohmic dissipation can slow down the cooling. The total magnetic energy of the poloidal field configuration (20) is

$$E_{\rm m} \approx 0.274 B_{\rm p}^2(t) R^3. \tag{32}$$

By making use of equations (27) and (28), the rate of energy production within the star, $\dot{\epsilon} = -\mathrm{d}E_{\rm m}/\mathrm{d}t$, can be written in the form

$$\dot{\epsilon} = 0.274 \frac{B_{\rm p}^2(0) R^3}{\tau_B} \left[\frac{B_{\rm p}(0)}{B_{\rm c}}\right]^2 \frac{1}{T_8^2} \left[1 + \frac{B_{\rm p}^2(0)}{B_{\rm c}^2} \frac{1}{\tau_B} \int_0^t \frac{\mathrm{d}t'}{T_0^2(t')}\right]^{-2}. \tag{33}$$

At the late evolutionary stage, when inequality (29) is fulfilled, $\dot{\epsilon}$ becomes independent of the original field strength:

$$\dot{\epsilon} \approx 0.274 R^3 B_{\rm c}^2 \frac{\tau_B}{T_8^2(t)} \left[\int_0^t \frac{\mathrm{d}t'}{T_8^2(t')}\right]^{-2}. \tag{34}$$

This additional heating may slow down the cooling. This effect is of particular importance for old neutron stars with surface temperatures $T_{\rm s} \leq 10^5$ K.

5 FIELD DECAY AND CIRCULATION

As mentioned in the previous section, the field decay violates hydrostatic equilibrium and induces a slow circulatory flow which tends to maintain the magnetic configuration close to equilibrium. Since the hydrodynamic time-scale is very short in comparison to the dissipative one, the departure from the equilibrium configuration is small. The reasons for this circulation are very similar to those for the classical Eddington–Sweet meridional circulation in stars. The classical circulation tends to compensate deviations from thermal balance due to stellar rotation. The circulation due to field decay in neutron stars should balance the departure from hydrostatic equilibrium that is continuously produced by dissipative processes. The circulatory flow, which maintains the magnetic configuration close to equilibrium, is governed by equation (25). We will consider the circulation accompanying the decay of the dipole configuration, with Λ given by equation (20). For the sake of simplicity we neglect the Hall component of the resistivity tensor, assuming that $B > B_{\rm p}$. Since we examine the axially symmetric configuration, equation (25) can be written in the form

$$\boldsymbol{V} \times \boldsymbol{B} = \boldsymbol{e}_\varphi \frac{\sin\theta}{r} \left\{ r^2 \dot{\Lambda} - \frac{c^2 R_\perp}{4\pi} \left[\frac{\partial^2}{\partial r^2}(r^2 \Lambda) - 2\Lambda\right]\right\}. \tag{35}$$

Clearly this equation determines only the component of the velocity ($\boldsymbol{V}_\perp$) perpendicular to the magnetic field. Multiplying equation (35) by $\boldsymbol{B}/B^2$, one obtains

$$\boldsymbol{V}_\perp = \frac{\boldsymbol{B} \times \boldsymbol{e}_\varphi}{B^2} \frac{\sin\theta}{r} \left\{ r^2 \dot{\Lambda} - \frac{c^2 R_\perp}{4\pi} \left[\frac{\partial^2}{\partial r^2}(r^2 \Lambda) - 2\Lambda\right]\right\}. \tag{36}$$

The total hydrodynamic velocity is then

$$\boldsymbol{V} = \boldsymbol{V}_\perp + q\boldsymbol{B}, \tag{37}$$

where q is a function of r, θ and t. This function can be calculated from the continuity equation, which in the case ρ = constant reads $\nabla \cdot \boldsymbol{V} = 0$. We therefore have

$$\nabla q \cdot \boldsymbol{B} = -\nabla \cdot \boldsymbol{V}_\perp. \tag{38}$$

It should be noted that equations (36)–(38) are not valid in the surface layers, where the circulation velocity should satisfy the standard hydrodynamic boundary conditions. Our self-similar solution describes the circulation only in comparatively deep layers where the flow becomes insensitive to the boundary conditions.

From equations (36)–(38) it is easy to estimate the order of magnitude of the circulation velocity,

$$V \sim \frac{r}{\tau_B} \frac{B_{\rm p}^2}{B_{\rm c}^2} T_8^{-2}. \tag{39}$$

Using the estimate (31) for the polar magnetic field, we obtain

$$V \sim \frac{r}{t}. \tag{40}$$

Naturally the circulation velocity tends to zero with age. The circulation should compensate the change of the magnetic configuration due to dissipation, but the latter becomes slower as the field becomes weaker. Since the velocity decreases with age approximately in accordance with an inverse linear law, the core matter probably does not turn over more than once in the course of the evolution.

The circulation is of particular interest in the dense central region. Using equation (17), one can represent $V_\perp$ in this region as a Taylor expansion of r. To the lowest order, equation (36) then becomes

$$\boldsymbol{V}_\perp = -V_0 \frac{r}{R} \sin\theta (\sin\theta \boldsymbol{e}_{\rm r} + \cos\theta \boldsymbol{e}_\theta),$$
$$V_0 = 0.548 \frac{R}{\tau_B} \frac{B_{\rm p}^2}{B_{\rm c}^2} T_8^{-2}. \tag{41}$$

Substituting this expression into equation (38) and taking into account that in the central regions $\boldsymbol{B} \simeq (5/2) B_{\rm p}(\cos\theta \boldsymbol{e}_{\rm r} - \sin\theta \boldsymbol{e}_\theta)$, we obtain

$$\cos\theta \frac{\partial q}{\partial r} - \frac{\sin\theta}{r} \frac{\partial q}{\partial \theta} = \frac{4}{5} \frac{V_0}{R B_{\rm p}}. \tag{42}$$

This equation can be easily integrated to give

$$q = \frac{4}{5} \frac{V_0}{B_{\rm p}} \frac{r}{R} \sin\theta \int_\theta^{\pi/2} \frac{\mathrm{d}y}{\sin^2 y}. \tag{43}$$

Finally, we obtain the following expression for the circulation velocity in the central region of the neutron star:

$$\boldsymbol{V} = -\frac{V_0 r}{R} \sin\theta \left[\boldsymbol{e}_{\rm r} \left(\sin\theta - 2\cos\theta \int_\theta^{\pi/2} \frac{\mathrm{d}y}{\sin^2 y}\right) + \boldsymbol{e}_\theta \left(\cos\theta + 2\sin\theta \int_\theta^{\pi/2} \frac{\mathrm{d}y}{\sin^2 y}\right)\right]. \tag{44}$$

As mentioned above, in the less dense regions, at $r \sim R$, the configuration of the flow may be more complicated. It is seen from equation (44) that, in the deep layers, matter flows towards the centre of the star near the polar axis, while near the equatorial plane it flows towards the surface. Generally, the angle θ_0 at which the radial velocity changes sign depends on the radius, but in the central regions this angle is determined by the equation

$$\tan\theta_0 = 2\int_{\theta_0}^{\pi/2}\frac{dy}{\sin^2 y}, \tag{45}$$

and is $\sim 54°$. A sketch of the circulation pattern in the deep layers of the core is shown in Fig. 1.

It should be noted that, although the Hall currents do not influence the rate of field decay (see equation 26), they may be very important for a circulation at $B_p > B > B_e$. Certainly, in this case the Hall currents contribute appreciably in equation (25), and the circulation will mainly compensate changes of the magnetic configuration due to these currents.

6 CONCLUSION

We have examined the decay of strong magnetic fields and the accompanying circulation of matter in a neutron-star core in which the neutrons and protons are normal (i.e. non-superfluid and superconducting). In conclusion, we briefly summarize the main results of the paper.

Hydrostatic equilibrium in a strongly degenerate neutron-star core may be reached only for very particular magnetic configurations with conservative magnetohydrodynamic forces. One such configuration with a non-zero dipole component outside the star has been considered in detail. In the course of evolution this configuration should change under the influence of dissipative processes. However, as the distribution of currents changes, the configuration deviates from the initial hydrostatic equilibrium configuration. Since dissipation is very slow in comparison with hydrodynamic processes, the departure from the equilibrium configuration has to be very small. In practice this implies that the evolution of the magnetic field goes through a set of self-similar equilibrium configurations with the same spatial field distributions but with the currents that maintain the configurations decreasing in magnitude. Because the dissipation alone cannot provide evolution of this nature, decay of the field should be accompanied by the very slow circulation of neutron-star matter. Since the magnetic field lines are partly 'frozen in' in a highly conductive core plasma, this circulation advects the magnetic field, compensating the changes of the configuration due to dissipation. This circulation, together with the dissipation, results in quasi-static evolution of the magnetic field without changes in the spatial distribution of currents. To provide evolution of this nature in the deep layers of the core, matter flows towards the centre of the star near the polar axis, while it moves outwards near the equatorial plane. The velocity of circulation decreases with neutron-star age because of a decrease in the field strength (see equations 39–40). The core matter probably does not turn over more than once during the lifetime of the neutron star.

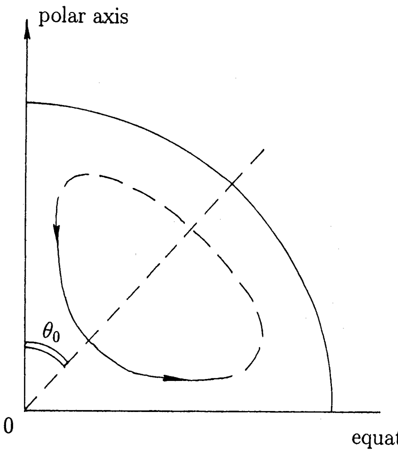

Figure 1. The circulation pattern for the dipole field decay. The dashed curve shows schematically the flow far from the centre, where equation (44) does not apply.

The circulation considered in the present paper can play an important role in the chemical equilibrium in the core. In analysing the circulation we have completely neglected beta-processes in the flowing matter. These processes can, however, change the chemical composition of the npe-plasma since the chemical equilibrium depends on the density, but the hydrodynamic flow causes the plasma to pass through core regions that have non-uniform densities. As pointed out by Pethick (1991), beta-decay can also lead to additional dissipation of the magnetic field due to a change of the chemical equilibrium in the moving plasma. This 'beta-decay' dissipation can accelerate field decay for some conditions, and can influence the velocity and configuration of the circulation. We plan to consider these questions in a forthcoming paper.

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