# CIRCUMFERENTIAL STRESS WAVES IN A NON-LINEAR CYLINDRICAL ANNULUS IN A NEW CLASS OF ELASTIC MATERIALS

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[Received 4 September 2013. Revise 12 December 2013. Accepted 9 January 2014]

**Summary**

We study the propagation of circumferential stress waves in an annular cylindrical region comprised of a new class of isotropic homogeneous elastic bodies. The class of elastic bodies under consideration does not belong to either the classical Cauchy elastic or Green elastic bodies. Unlike the case of classical elasticity theory that reduces to the classical linearized elasticity theory when the displacement gradients are sufficiently small, it is possible that in the new class the linearized strain is related non-linearly to the stress. This possibility leads to a non-linear system of equations that admits the propagation of circumferential shear stress waves in the body.

## 1. Introduction

Recently, Kannan *et al.* (**1**) studied the propagation of stress waves in a slab of a new class of elastic bodies that does not belong to either Cauchy elastic or Green elastic bodies. The problem that they considered, unlike wave propagation in a classical non-linear elastic body, is governed by a system of two coupled partial differential equations. This situation is a consequence of the constitutive expression for the body being a prescription for the strain in terms of a possibly non-invertible function of the stress which implies that one cannot substitute for the stress in terms of the linearized strain, which is the symmetric part of the displacement gradient, and hence arrive at a single partial differential equation for the displacement. While in the case of the special model that we are considering, the two governing equations can be manipulated to reduce to one equation, the result is a higher order partial differential equation for which an additional initial condition would be necessary for the problem to be well-posed, and it is not clear what this additional initial condition ought to be. In view of this, it is in general best to solve the two equations as a system of lower order equations. We show that the system of equations allows for the propagation of shear stress waves.

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**Q. Jl Mech. Appl. Math, Vol. 67. No. 2 © The Author, 2014. Published by Oxford University Press;**

**Advance Access publication 21 March 2014. doi:10.1093/qjmam/hbu003**

Rajagopal (**2**, **3**) introduced a new class of elastic bodies that includes bodies that are neither Cauchy elastic nor Green elastic.[1] However, the new class introduced by Rajagopal (**2**) includes Cauchy elastic and Green elastic bodies as a special subclass. It also includes special subclasses wherein the Cauchy–Green tensor $\boldsymbol{B}$ is a function of the stress. Rajagopal and Srinivasa (**7**, **8**) have provided a thermodynamic basis for the new class of elastic bodies introduced by Rajagopal (**2**).

Several studies have been carried out recently within the context of the new class of elastic bodies introduced by Rajagopal (**2**). Simple homogeneous deformations within the new class of elastic bodies have been studied by Rajagopal (**3**), and Bustamante and Rajagopal (**9**) have studied simple inhomogeneous solutions. In a series of papers, Freed and co-workers have used implicit constitutive theories to describe the response of soft matter such as biological tissues (**10–12**) and the book on soft matter by Freed (**13**) provides a detailed discussion of the mechanics and thermo-mechanics of implicit constitutive theories for elastic bodies. Criscione and Rajagopal (**14**) have studied bodies wherein the linearized strain is related to the Cauchy stress in a non-linear manner and they use such a model to describe the experimental work of Penn (**15**) on rubber.

There is a considerable amount of experimental literature on titanium alloys, Gum metal alloys and other alloys, wherein the relationship between the linearized strain and the Cauchy stress is non-linear. Such a relationship can never be captured within the context of the Cauchy theory of elasticity as for such small strains Cauchy elasticity theory would collapse to the linearized theory of elasticity. On the other hand, when we linearize the new class of constitutive equations proposed by Rajagopal (**2**, **3**) under the assumption that the displacement gradient is appropriately small, we can obtain an expression for the linearized strain as a non-linear function of the stress. Rajagopal (**16**) has shown recently that the experimental data of Saito *et al.* (**17**) that clearly implies a non-linear relationship between the linearized strain and the stress, can be described very well within the context of the new class of models. The paper of Saito *et al.* (**17**) has been followed by numerous other experimental papers on titanium alloys and Gum metals which clearly indicate that even in the small strain regime the relationship between the strain and the stress is non-linear (see for example (**18–21**)) lending credence to the class of models that are studied in this article.

The new class of elastic bodies introduced by Rajagopal (**2**) contains within it bodies that are strain limiting. Such constitutive relations have far reaching implications with regard to problems in fracture mechanics, especially the fracture of brittle elastic solids. Within the classical theory of linearized elasticity, at the tip of a crack, the strain grows like $O(1/\sqrt{r})$, where $r$ is the radial distance from the crack tip, for a large class of problems. Such a growth of the strain violates the approximation under which the linearized theory of elasticity is developed. The approximation within the context of the new class of materials could possibly imply that the linearized strain does not grow in a manner that contradicts the assumption that it be small. That this is indeed the case has been verified by an asymptotic analysis carried out by Rajagopal and Walton (**22**) within the context of anti-plane strain problems. Numerical calculations also seem to support bounded stresses at the edge of a V-notch (**23**) and at a hole in a slab (**24**). Bulicek *et al.*, unpublished data, have obtained rigorous mathematical results concerning existence of solutions to anti-plane stress problems for bodies with cracks.

In this article, we are interested in finding the propagation of circumferential shear waves in an annular cylindrical region in a body wherein the linearized strain depends non-linearly on the stress. The input to the problem under consideration is the stress applied at the boundary of the annular

[1] The class of Green elastic bodies is a subclass of Cauchy elastic bodies. However, recently Carroll (**4**) has shown that Cauchy elastic bodies that are not Green elastic could be an infinite source of energy. Green (**5**, **6**) himself had recognized that Cauchy elastic bodies that are not Green elastic could lead to perpetual motion machines.

region. We also determine the manner in which the displacement evolves with time. We find that unlike the case of classical linearized elasticity the speed of propagation depends on the amplitude of the stress. We also follow the motion after the wave undergoes reflection at the other boundary of the annulus. The nature of these solutions is discussed later.

Two kinds of input are considered in this work, a sinusoidal input and a triangular input with periodicity. We find that the wave motion in the interior gets distorted as time progresses, in keeping with the earlier results obtained by Kannan *et al.* (**1**) in the case of a slab.

## 2. Kinematics and the governing equations

Let $B$ denote the abstract body and let $\kappa_R(B)$ and $\kappa_t(B)$ denote the reference configuration and the configuration at time $t$, respectively. By the motion of a body, we mean a sufficiently smooth mapping $\boldsymbol{\chi}$, such that

$$\boldsymbol{\chi} : \kappa_R(B) \times \mathcal{R} \to \boldsymbol{\mathcal{E}}, \tag{2.1}$$

with

$$\boldsymbol{x} = \boldsymbol{\chi}(\boldsymbol{X}, t). \tag{2.2}$$

where $\boldsymbol{X} \in \kappa_R(B)$ and $\boldsymbol{x} \in \kappa_t(B)$ and $\boldsymbol{\mathcal{E}}$ denotes a three dimensional Euclidean space. The deformation gradient $\boldsymbol{F}$ and the Cauchy–Green tensor $\boldsymbol{B}$ are defined, respectively, through

$$\boldsymbol{F} = \frac{\partial \boldsymbol{x}}{\partial \boldsymbol{X}}, \ \boldsymbol{B} = \boldsymbol{F}\boldsymbol{F}^{\boldsymbol{T}} \tag{2.3}$$

where the superscript $T$ denotes the transpose. We are interested in the propagation of circumferential waves in an elastic material which is a subclass of implicit constitutive relations defined through

$$f(\boldsymbol{T}, \boldsymbol{B}) = \boldsymbol{0}, \tag{2.4}$$

where $\boldsymbol{T}$ denotes the Cauchy stress. We are interested in the particular sub-class wherein

$$\boldsymbol{B} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{T}^2, \tag{2.5}$$

where the $\alpha_i$, $i = 0, 1, 2$ depend on the density $\rho$ and the invariants $\operatorname{tr} \boldsymbol{T}$, $\operatorname{tr} \boldsymbol{T}^{\boldsymbol{2}}$ and $\operatorname{tr} \boldsymbol{T}^{\boldsymbol{3}}$. We first observe that the class (2.5) is not necessarily obtainable by inverting the classical Cauchy elastic representation

$$\boldsymbol{T} = \beta_0 \boldsymbol{I} + \beta_1 \boldsymbol{B} + \beta_2 \boldsymbol{B}^{\boldsymbol{2}}, \tag{2.6}$$

where the $\beta_i$, $i = 0, 1, 2$ depend on the density and the invariants $\operatorname{tr} \boldsymbol{B}$, $\operatorname{tr} \boldsymbol{B}^{\boldsymbol{2}}$ and $\operatorname{tr} \boldsymbol{B}^{\boldsymbol{3}}$. We are particularly interested in models wherein (2.6) is not necessarily invertible.[2] When we linearize

[2] Truesdell and Moon (**25**) have investigated when (2.6) can be inverted. They were not interested in the development of constitutive theories wherein the Cauchy–Green tensor is a function of the stress. They were interested in the invertibility of isotropic functions.

(2.5) by appealing to

$$\max_{X \in \kappa_R(B),\ t \in \mathcal{R}} \|\nabla \boldsymbol{u}\| = O(\delta), \quad \delta \ll 1, \tag{2.7}$$

where $\boldsymbol{u}$ is the displacement, we obtain

$$\boldsymbol{\epsilon} = \alpha_0 \boldsymbol{I} + \alpha_1 \boldsymbol{T} + \alpha_2 \boldsymbol{T}^2, \tag{2.8}$$

where

$$\boldsymbol{\epsilon} := \frac{1}{2}\left[\left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right) + \left(\frac{\partial \boldsymbol{u}}{\partial \boldsymbol{X}}\right)^T\right] \tag{2.9}$$

is the linearized strain. It may be worth noting that, for small deformations, it does not matter whether the gradient is computed with respect to the current or reference configuration. We shall require that the linearized strain is zero when the stress is zero and hence

$$\alpha_0(\rho, 0, 0, 0) = 0. \tag{2.10}$$

Let us consider the specific sub-class of models defined by (2.8), namely the class

$$\boldsymbol{\epsilon} = \beta(\operatorname{tr} \boldsymbol{T})\boldsymbol{I} + \frac{\alpha}{2}(1 + \frac{\gamma}{2}(\operatorname{tr} \boldsymbol{T}^2))^n \boldsymbol{T}, \tag{2.11}$$

where $\alpha \geqslant 0$, $\beta \leqslant 0$, $\gamma \geqslant 0$ and $n$ are constants.

### 2.1 *Governing equations*

In this article we study the propagation of circumferential stress waves in the cylindrical annular region

$$D = \{(R, \Theta, Z) \mid R_i \leqslant R \leqslant R_0,\ 0 \leqslant \Theta \leqslant 2\pi,\ -\infty \leqslant Z \leqslant \infty\}. \tag{2.12}$$

We shall consider a stress of the form

$$\boldsymbol{T} = T(r, t)(\boldsymbol{e_r} \otimes \boldsymbol{e_\theta} + \boldsymbol{e_\theta} \otimes \boldsymbol{e_r}) \tag{2.13}$$

and a displacement of the form

$$\boldsymbol{u} = f(r, t)\boldsymbol{e_\theta}, \tag{2.14}$$

where $\boldsymbol{e_r}, \boldsymbol{e_\theta}, \boldsymbol{e_z}$ are the unit vectors in the $r$, $\theta$ and $z$ directions, respectively. The assumptions (2.13) and (2.14) are more restrictive than the usual semi-inverse solution procedure wherein a specific form of the motion or the velocity is assumed. Here, we assume a specific structure for both the stress field and the motion. Solutions with such structure may not exist. However as the following analysis shows, we are able to find solutions of the form that is sought. Since the general governing equations are non-linear, it is possible that more general solutions exist. We shall find it convenient to use the following non-dimensionalization to study the problem

$$\bar{r} = \frac{r}{R_0},\ \bar{t} = \frac{t}{t_0} = \frac{t}{R_0\sqrt{\alpha\rho}}, \tag{2.15}$$

and

$$\bar{f} = \frac{f}{R_0},\ \bar{T} = \frac{T}{\left(1/\sqrt{\gamma}\right)}. \tag{2.16}$$

It follows from the constitutive relation (2.11), (2.15) and (2.16) that

$$\frac{\partial\bar{f}}{\partial\bar{r}} - \frac{\bar{f}}{\bar{r}} = \frac{\alpha}{\sqrt{\gamma}}\left[1 + \bar{T}^2\right]^n \bar{T} \tag{2.17}$$

and (2.15), (2.16) and the balance of linear momentum imply

$$\frac{\partial\bar{T}}{\partial\bar{r}} + \frac{2\bar{T}}{\bar{r}} = \frac{\sqrt{\gamma}}{\alpha}\frac{\partial^2\bar{f}}{\partial\bar{t}^2}. \tag{2.18}$$

While one can eliminate $\bar{f}$ and obtain an equation for $\bar{T}$, from (2.17) and (2.18), this will increase the order of the equation. We shall solve the system, (2.17) and (2.18), simultaneously. We shall consider the following initial conditions

$$\bar{f}(\bar{r}, 0) = \frac{\partial\bar{f}}{\partial t}(\bar{r}, 0) = 0, \tag{2.19}$$

and

$$\bar{T}(\bar{r}, 0) = 0, \tag{2.20}$$

that is, the annulus is stress-free initially, and the displacement and the velocity in the circumferential direction are also zero initially. For the sake of illustration, we shall assume that $R_i = (0.2)R_0$. Thus, $\bar{R_i} = 0.2$, and $\bar{R_0} = 1$. We shall assume two different boundary conditions:

(a) Sine wave input at the inner boundary:

$$\bar{T}(0.2, \bar{t}) = 0.5\ \sin(2\pi\bar{t}/0.4),\ \bar{f}(1, \bar{t}) = 0.$$

(b) Triangular wave input at the inner boundary:

$$\bar{T}(0.2, \bar{t}) = \max(1 - |5\bar{t} - 1|, 0),\ \bar{f}(1, \bar{t}) = 0.$$

## 3. Results

Numerical solutions to the governing equations (2.17) and (2.18) with appropriate boundary and initial conditions are obtained, using Comsol. The dimensionless strain is computed to an accuracy

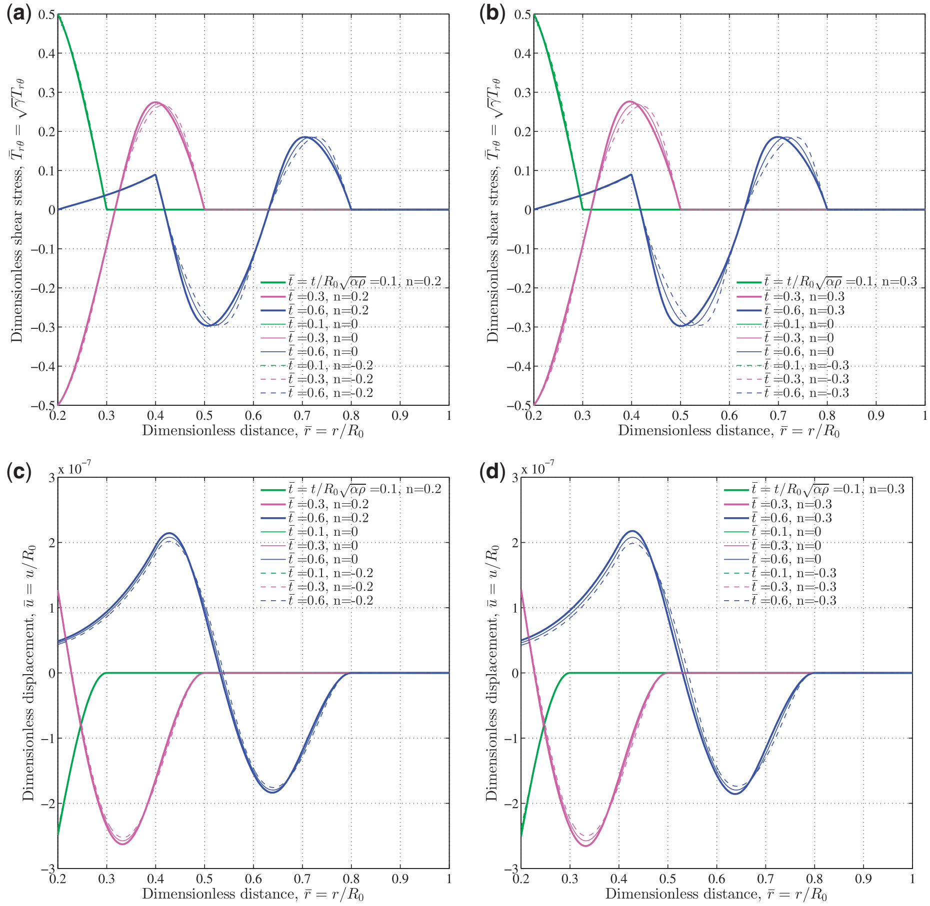


**Fig. 1** Propagation of shear stress wave and its corresponding displacement field for the sinusoidally varying boundary condition

of 0.0001% of its value. For each of the two types of boundary conditions (that is, sinusoidal and triangular input), five boundary value problems are solved and the numerical results for the shear stress wave and its corresponding displacement field are plotted in Figs. 1–4.

For the particluar subclass of constitutive relations considered in this article, only the strain measure is linearized and is given by (2.11). Therefore, (2.7) requires that the gradient of displacement be very small, small enough that the square of the norm of the displacement gradient can be neglected in terms of the norm of the displacement gradient. In the constitutive equation (2.17), the parameter $\alpha/\sqrt{\gamma}$ is set to $10^{-5}$, exponent $n$ is chosen between $-0.3$ to 0.3, and $\bar{T}$, the dimensionless circumferential shear stress, is limited by the amplitude of the wave input which is of order 1. This implies that the dimensionless strain in the equation, the left-hand side of (2.17), is limited to a value of order $10^{-6}$ approximately. However, it is not necessary that displacement gradients have to be that small for

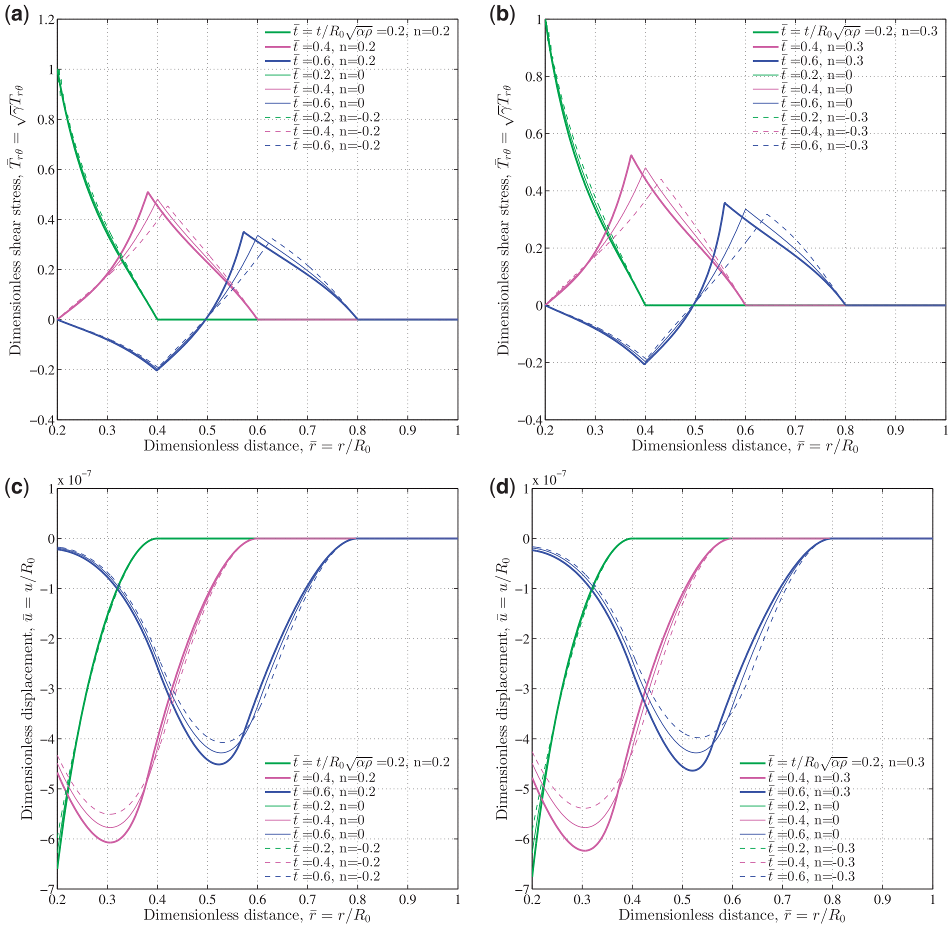


**Fig. 2** Propagation of shear stress wave and its corresponding displacement field for the triangularly varying boundary condition

the linearized strain to be used. Even a linearized strain of, say, 0.01, is such that its square can be neglected with regard to the value of strain (see Rajagopal (**16**) for an extended discussion of this issue).

When the parameter $n = 0$ in (2.11), it reduces to a model for a linearized, isotropic elastic body. The governing equations (2.17) and (2.18), when combined, represent a non-linear wave equation and on using $n = 0$, it reduces to the standard linear wave equation in cylindrical polar co-ordinates whose solution, for both the sinusoidal and triangular boundary condition, show an amplitude change in the wave which is unlike the behaviour of a standard linear wave in a slab where the amplitude remains unaffected.

Figures 1 and 2 show the radial propagation (outward) of the shear stress wave and its corresponding displacement in the cylindrical annulus for sinusoidal and triangular boundary

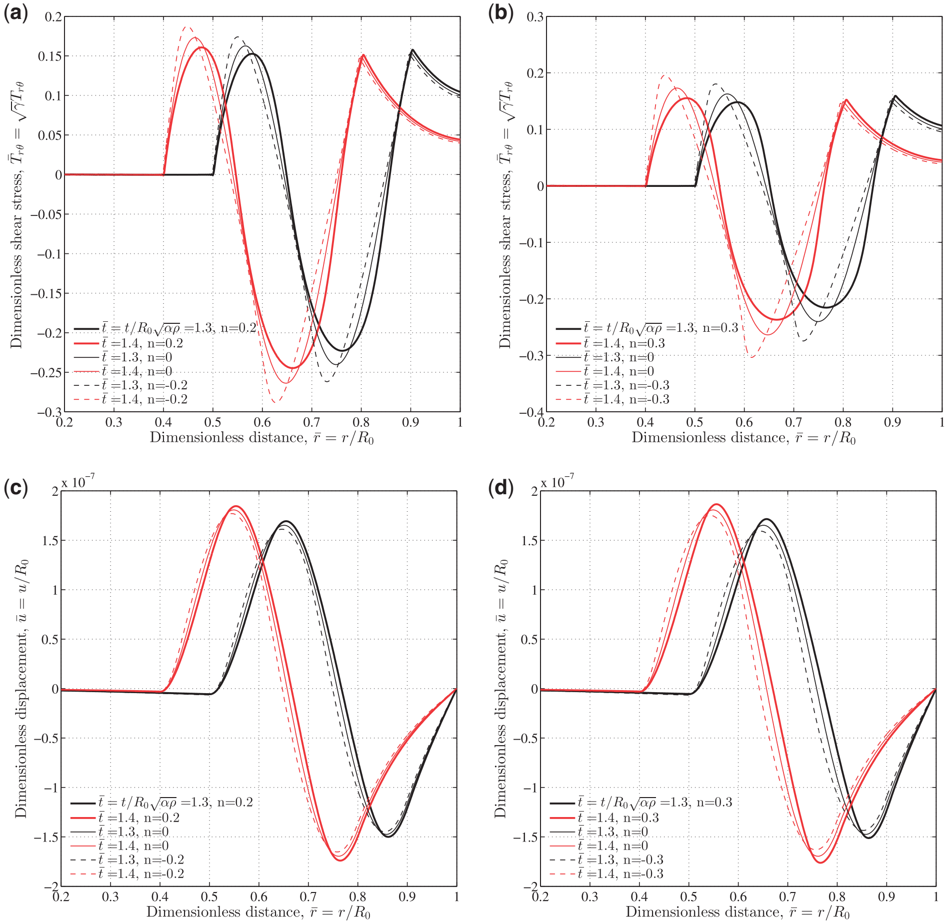


**Fig. 3** Interaction of shear stress wave and corresponding displacement field with the far-boundary and its reflection for the sinusoidally varying boundary condition

condition, respectively. At a given instant of time, the stress waves corresponding to non-zero values of $n$ show distortion, by which we mean the change in the shape of the waveforms only relative to the standard wave $n = 0$. This distortion of waveforms increases with the magnitude of $n$. At higher magnitude of stress, the distortion is significant and is such that the stress wave corresponding to $n < 0$ leads the standard wave and the stress wave corresponding to $n > 0$ trails the standard wave.

Figures 3 and 4 represent the interaction and reflection of the shear wave and its displacement with the outer boundary for sinusoidal and triangular boundary condition respectively. On completion of wave interaction with the boundary, the direction of wave propagation is reversed, but the distortion of waves corresponding to non-zero values of $n$ continues.

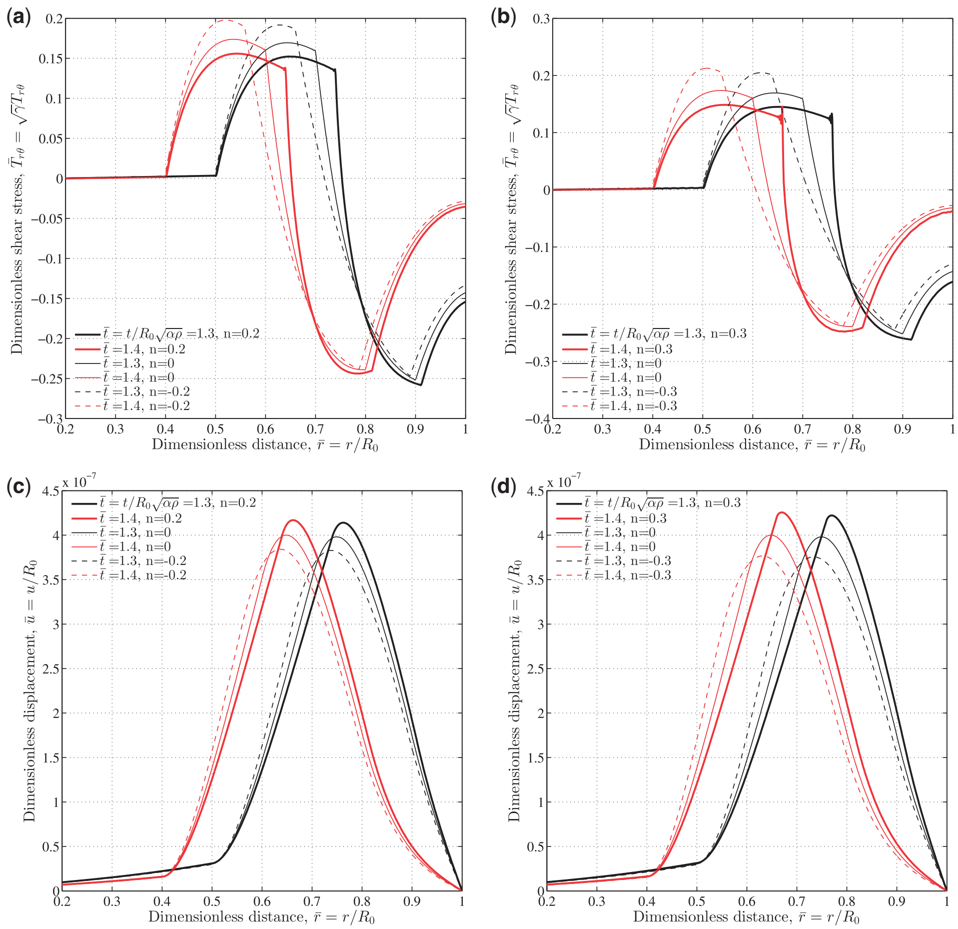

**Fig. 4** Interaction of shear stress wave and its corresponding displacement field with the far boundary and its reflection for the triangularly varying boundary condition

## 4. Conclusions

Non-linear stress-strain behaviour of materials even under small deformation, as observed in some metallic alloys of titanium (**17**), requires a constitutive equation that accounts for such non-linearities. Hooke's law cannot deal with the non-linear relationship between the linearized strain and the stress. However, a constitutive equation relating linearized strain to the Cauchy stress, such as the one given in (2.17), can account for non-linearlities under the condition of small deformation.

In general, whenever a distortion (shape change) of the wave is observed experimentally when the strains are small, one usually attributes the phenomenon to dissipation and overlooks the possibility that such changes are possible even within the context of elasticity theories. In this article, it is shown that shape change of a shear wave in a cylindrical annulus can occur in an elastic body undergoing small deformations, and is not a consequence of dissipation! Since the material is elastic, the propagation of waves is such that the total energy in the solid is always conserved.

## Acknowledgements

One of the authors, K.R.R. thanks the Office of Naval Research and the Army Research Office, for support of this work.

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