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CLASS $\\textrm{VII}_0$ SURFACES WITH b_2 CURVES — Source link

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CLASS VII_0 SURFACES WITH b_2 CURVES

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Abstract. We give an affirmative answer to the following conjecture of Ma. Kato: Let S be a compact complex surface in Kodaira's class VII_0 which contains a strictly positive number of rational curves being exactly equal to the second Betti number of S. Then S admits a global spherical shell.

Introduction. Kodaira's class VII_0, which consists of minimal compact complex surfaces S having $b_1(S) = 1$, is not completely understood so far. In fact, only the case when $b_2(S) = 0$ is completely classified, by the work of Kodaira [13], Inoue [11], Bogomolov [1], Li-Yau-Zheng [14] and Teleman [18].

For $b_2(S) > 0$ a construction method and thus a large subclass of surfaces have been introduced by Kato [12]. These are exactly the minimal surfaces with $b_2 > 0$ containing *global spherical shells* (see the next section for the definition). One can show that a surface S of class VII_0 has at most $b_2(S)$ rational curves on it and that if moreover S admits a global spherical shell, then there are exactly $b_2(S)$ rational curves on S. Kato conjectured that the converse should be true as well. Important progress towards this conjecture was made by Nakamura [15], [16], who showed that if S has $b_2(S)$ rational curves, then their configuration is that of the curves of a surface with global spherical shells and S is a deformation of a blown-up Hopf surface.

This paper is devoted to the proof of Kato's conjecture:

MAIN THEOREM. *If S is a surface of class VII_0 with $b_2(S) > 0$ and with $b_2(S)$ rational curves, then S admits global spherical shells.*

At present all known surfaces S of class VII_0 with $b_2(S) > 0$ contain global spherical shells. In fact, by making additional assumptions on S like the existence of a homologically trivial divisor [8] or of two cycles of rational curves [15] or of a holomorphic vector field [6], [7], it was shown that S contains global spherical shells. The present paper makes a further step in this direction.

The paper is organized as follows. Section 1 is preparatory. We recall some facts on surfaces with global spherical shells and on surfaces S of class VII_0 with $b_2(S) > 0$ rational curves. Such surfaces were called *special* by Nakamura [16]. We also prove a fact we shall

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need later, namely that if S is special, then the canonical bundle of a suitable finite ramified covering of S is numerically divisorial.

Using the knowledge of the configuration of rational curves on a special surface, we prove in Section 2 the existence of a logarithmic 1-form twisted by a flat line bundle. Passing to a finite ramified covering we get a global twisted holomorphic vector field. The twisting is again by some flat line bundle. This induces a true holomorphic vector field on the universal covering $\tilde{S}$ of S.

In Section 3 we prove that this holomorphic vector field is completely integrable and that the universal covering of the complement of the curves of S is isomorphic to $\boldsymbol{H} \times \boldsymbol{C}$, where $\boldsymbol{H}$ denotes the complex half plane. Section 4 is devoted to the computation of the action of the fundamental group on $\boldsymbol{H} \times \boldsymbol{C}$. This allows us to recover in Section 5 the contracting rigid germ of holomorphic map which gave birth to our surface S. Using the work of Favre [9] which classifies such germs, we are able to conclude.

1. Preliminaries. We start by recalling some definitions and known facts. A compact complex surface S is said to belong to Kodaira's class VII if $b_1(S) = 1$ and to class VII_0 if it is moreover minimal. Surfaces of class VII_0 with $b_2 = 0$ have been completely classified, see for instance [18]. In this paper we deal with the case $b_2 > 0$. It is then well-known that the Kodaira dimension of S is negative and that the algebraic dimension vanishes. In particular, S has finitely many irreducible curves.

At present the only known surfaces of class VII_0 with $b_2 > 0$ contain *global spherical shells* (GSS). A global spherical shell is a neighborhood V of $S^3 \subset \boldsymbol{C}^2 \setminus \{0\}$ which is holomorphically embedded in the surface S such that $S \setminus V$ is connected. All surfaces with GSS may be constructed by a procedure due to Kato [12]. As a consequence, they all have exactly $b_2(S)$ rational curves; some of them admit an elliptic curve as well. Kato also made the following

CONJECTURE. *If S is a class VII_0 surface with $b_2 > 0$ rational curves, then S admits a* GSS.

Following Nakamura [16], we shall call a class VII_0 surface *special*, if it has $b_2 > 0$ rational curves. Since special surfaces admitting homologically trivial divisors or with all rational curves organized in one or two cycles have been shown to admit GSS, [8], [15], [3], [12], we concentrate our attention on the remaining ones. We shall call them *special surfaces of intermediate type*.

In [16], Nakamura proved that the configuration of the rational curves of a special surface of intermediate type is the same as that of a surface with GSS with the same b_2. In particular the dual graph of such a configuration is connected and contains a cycle to which some trees are attached. Nakamura also showed that these surfaces are deformations of blown-up primary Hopf surfaces, in particular that their fundamental group is isomorphic to $\boldsymbol{Z}$. Furthermore, he proposed a line of attack to Kato's conjecture. However, one of his conjectures proved to be not correct, see [19].

NOTATION. We denote by D the maximal reduced divisor of a special surface S, by $M(S)$ the intersection matrix of the rational curves of S and set $k(S) := \sqrt{|\det M(S)|} + 1$. (It is well-known that $\sqrt{|\det M(S)|}$ is the index of the subgroup generated by classes of curves in $H^2(S, \boldsymbol{Z})$ and thus it is an integer). Moreover we denote by $\tilde{S}$ the universal cover of S and by $\tilde{D}$ the preimage of D in $\tilde{S}$. Then $\tilde{D}$ is the universal cover of D.

In this section we show that after passing to a ramified covering and a resolution of singularities, we may suppose that the anticanonical bundle of our surface is numerically divisorial, i.e., there exists a flat line bundle L on S and a divisor D_{-K} such that $K_S^{-1} \simeq L \otimes \mathcal{O}(D_{-K})$.

Since under our assumptions $\pi_1(S) \simeq \boldsymbol{Z}$, it is easy to see that the flat line bundles are parametrized by $\boldsymbol{C}^* \simeq \mathrm{Hom}(\pi_1(S), \boldsymbol{C}^*) \simeq \mathrm{Pic}^0(S)$. We shall often write L^λ for the line bundle corresponding to the complex number $\lambda \in \boldsymbol{C}^*$.

LEMMA 1.1. *Let S be a special surface of intermediate type. Then there exist a positive integer m, a flat line bundle L and an effective divisor D_m such that*

$$(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)\,.$$

PROOF. Since the cohomology classes associated to the rational curves of S generate $H^2(S, \boldsymbol{Q})$, there always exist $m \in \boldsymbol{N}^*$, $L \in \mathrm{Pic}^0(S)$ and a divisor D_m such that

$$(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)\,.$$

We have only to check that $D_m \geq 0$. Let $D_m = D_+ - D_-$ with $D_+, D_- \geq 0$ and $D_+.D_- \geq 0$. The adjunction formula implies that $D_m.C \leq 0$ for any irreducible curve C on S. Hence

$$0 \geq D_m.D_- = (D_+ - D_-).D_- \geq -D_-^2 \geq 0\,.$$

But then $D_- = 0$, since S does not admit homologically trivial divisors. □

DEFINITION 1.2. The smallest possible $m \in \boldsymbol{N}^*$ for which a decomposition

$$(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)$$

as in Lemma 1.1 exists, will be called the *index* of the surface S and denoted by $m(S)$. When $m(S) = 1$, we denote $D_{-K} = D_1$ and call this the *numerically anticanonical divisor* of S.

Notice that D_m is unique when S is of intermediate type. The following proposition will enable us to reduce the proof of the Main Theorem to the case of special surfaces of index 1. We have formulated it for simplicity for special surfaces of intermediate type, but the general case can be proved similarly.

PROPOSITION 1.3. *Let S be a special surface of intermediate type with index $m := m(S) > 1$. Then there exists a diagram*

$$\begin{tikzcd}
 & S' \arrow[dr, "\pi'"] & \\
T \arrow[ur, "c"] \arrow[d, "\rho"'] & & Z' \\
Z \arrow[dr, "\pi"'] & & T' \arrow[u, "\rho'"'] \arrow[dl, "c'"] \\
 & S &
\end{tikzcd}$$

where

(i) (Z, π, S) *is a m-fold cyclic ramified covering space of* S, *branched over* D_m,

(ii) (T, ρ, Z) *is the minimal desingularization of* Z,

(iii) (T, c, S') *is the contraction of the (possible) exceptional curves of the first kind*,

(iv) S' *is a special surface with action of the group of* m*-th roots of unity* $\boldsymbol{U}_m$, *with index* $m(S') = 1$,

(v) Z' *is the quotient space of* S' *by* $\boldsymbol{U}_m$,

(vi) (T', ρ', Z') *is the minimal desingularization of* Z', *and*

(vii) (T', c', S) *is the contraction of the (possible) exceptional curves of the first kind such that the restriction over* $S \setminus D$ *is commutative, i.e.*,

$$\theta := \pi \circ \rho \circ c^{-1} = c' \circ {\rho'}^{-1} \circ \pi' : S' \setminus D' \to S \setminus D\,.$$

We have that $\theta : S' \setminus D' \to S \setminus D$ *is a m-fold non-ramified covering. Moreover*, S' *has a* GSS *if and only if* S *has a* GSS.

PROOF. We have

$$(K_S \otimes L)^{\otimes m} = \mathcal{O}(-D_m)\,.$$

Let X be the total space of the line bundle $K^{-1} \otimes L^{-1}$. We choose an open trivialisation covering $\mathcal{U} = (U_i)$ for K and L with local coordinates (z_1^i, z_2^i), defining cocycles (k_{ij}) and (g_{ij}) of K and L such that $D_m \cap U_i = \{f_i = 0\}$ and

$$k_{ij}^m g_{ij}^m \frac{f_i}{f_j} = 1\,.$$

If ζ_i is the fiber variable of X over U_i, the equations $\zeta_i^m = f_i(z)$ fit together and define an analytic subspace $Z \subset X$. It is easy to see that $\boldsymbol{U}_m$ acts holomorphically and effectively on Z and that $Z/\boldsymbol{U}_m = S$. Let $\pi : Z \to S$ be the projection on S. The ramified covering (Z, π, S) is branched exactly over $\mathrm{supp}(D_m)$. The local meromorphic 2-forms

$$\omega_i = \frac{dz_1^i \wedge dz_2^i}{\zeta_i}$$

yield a twisted meromorphic 2-form ω on X, hence on Z, for

$$\omega_i = \frac{dz_1^i \wedge dz_2^i}{\zeta_i} = \frac{k_{ij}^{-1} dz_1^j \wedge dz_2^j}{(k_{ij} g_{ij})^{-1} \zeta_j} = g_{ij} \frac{dz_1^j \wedge dz_2^j}{\zeta_j} = g_{ij} \omega_j\,.$$

Now, let (T, ρ, Z) be the minimal desingularization of Z. This includes of course the normalization of Z. Notice that the normalization is connected by the minimality of m. Set $H = \rho^\star L$. Then $\tau = \rho^\star\omega$ is a twisted meromorphic 2-form on T, which does not vanish and has a non trivial polar divisor E and $K_T \otimes H = \mathcal{O}(-E)$. Finally, in order to obtain S', we contract all exceptional curves of the first kind. It is clear that the index S' is one. As before, $\boldsymbol{U}_m$ acts holomorphically on S' and $S' \setminus D'$ is a covering manifold of $S \setminus D$. The quotient $Z' = S'/\boldsymbol{U}_m$ is a normal surface. The desingularization (T', ρ', Z') yields S after contraction of the exceptional curves of the first kind. Since S has no non-constant meromorphic functions, the same holds for S'. Applying then the classification of Kodaira, S' is a K3 surface, a torus or a surface of class VII_0. The first two cases have a trivial canonical bundle, and hence S' belongs to class VII_0. Moreover $b_2(S') > 0$, because it contains a cohomologically non trivial divisor.

We shall now show that if S has a GSS, then S' will also have this property. We may choose a GSS, $S^3 \subset V \subset \boldsymbol{C}^2 \setminus \{0\}$, which after embedding in S cuts only one curve C of our surface S. For a suitable choice of global coordinates (z_1, z_2) on V, the intersection $C \cap V$ is given by the equation $z_1 = 0$. Moreover $K_S|_V$ and $L|_V$ are trivial on V. Let ζ be the fiber coordinate of X over V. Suppose $D_m \cap V = n(C \cap V)$. The pull-back $\pi^{-1}(S^3)$ of the sphere S^3 to Z is given by the equations

$$|z_1|^2 + |z_2|^2 = 1, \quad z_1^n = \zeta^m.$$

Let $d :=$ g.c.d.(n, m) and $n' = n/d$, $m' = m/d$. Then there are d irreducible components around $\pi^{-1}(S^3)$ in Z, which will become disjoint after normalization. We denote by $\Sigma_1, \dots, \Sigma_d$ the corresponding components of $\pi^{-1}(S^3)$. Let us choose Σ_1 with equations

$$|z_1|^2 + |z_2|^2 = 1, \quad z_1^{n'} = \zeta^{m'}.$$

This component is normalized by the map

$$(t, z_2) \mapsto (t^{m'}, z_2, t^{n'}).$$

The pull-back Σ_1' of Σ_1 to T will be given in these coordinates by the equation

$$|t|^{2m'} + |z_2|^2 = 1.$$

We check now that $T \setminus \Sigma_1'$ is connected. Let $P, Q \in T \setminus \Sigma_1'$ two points in a neighborhood of $\Sigma_1' \cap \rho^{-1}(\pi^{-1}(D))$ which find themselves on different sides of Σ_1'. Their projections $\pi(\rho(P)), \pi(\rho(Q))$ on S may be connected by a path avoiding S^3 and D. We may lift this path to a path γ in Z which connects $\rho(P)$ to some $Q' \in \pi^{-1}(\pi(\rho(Q)))$. In order to ensure that Q' and $\rho(Q)$ coincide, it is enough to let the initial path in S turn the needed number of times around the components of D. By further lifting γ to T, we obtain the desired connectedness. For $m' > 1$,

$$\Sigma_1' = \{(t, z) \in \boldsymbol{C}^2 \mid |t|^{2m'} + |z_2|^2 = 1\}$$

is no longer a sphere, but remains the border of a bounded Stein domain of $\boldsymbol{C}^2$. Noticing that Σ_1' may be approximated by a strictly pseudoconvex hypersurface (for example, of equation

$\varepsilon|t|^2 + |t|^{2m'} + |z_2|^2 = 1$) it is possible to use the same arguments as in [2] in order to get a contracting holomorphic germ and hence a GSS on S'.

In particular, when S has a GSS, S' has exactly $b_2(S') = -K_{S'}^2$ rational curves. But since for a special surface S, the dual graph of the curves is the same as for some surface with a GSS, and since the intersection matrix of the curves of S' depends only on this graph, we see that our S' also has $b_2(S') = -K_{S'}^2$ rational curves. Thus S' is special under the weaker condition that S is special (of intermediate type). Finally, it is not difficult to show (cf. [4]) that quotients of GSS surfaces by the actions of finite cyclic groups of automorphisms remain GSS surfaces. Thus if S' has a GSS, then S will also have one. □

2. Existence of a twisted logarithmic 1-form. In this section we shall consider a special surface S of intermediate type and prove that it always admits a twisted logarithmic 1-form. Let $D = D_{\max} = \sum_{i=1}^n C_i$ be the maximal reduced divisor of S.

LEMMA 2.1. *If S has index $m(S) = 1$, then the numerically anticanonical divisor satisfies $D_{-K} > D$. In particular, for every flat line bundle L on S, $H^0(S, K_S \otimes L) = 0$.*

PROOF. Suppose that D_{-K} does not contain all the curves of S. Since by [16] the maximal divisor D is connected, there exists an irreducible curve C which is not contained in D_{-K} but such that $\mathrm{supp}(D_{-K}) \cap C \neq \emptyset$. Then $K_S.C = -D_{-K}.C < 0$, which gives a contradiction to the adjunction formula. Thus $D_{-K} \geq D$. But the equality $D_{-K} = D$ would imply $D^2 = -b_2(S)$, which may happen only on Inoue-Hirzebruch surfaces by [16]. But Inoue-Hirzebruch surfaces are not of intermediate type, so in our case we have $D_{-K} > D$. □

LEMMA 2.2. *For every $L \in \mathrm{Pic}^0(S)$ we have $\Gamma(S, \Omega^1 \otimes L) = 0$.*

PROOF. We may suppose $L \neq 0$. Using the exact sequence of sheaves

$$0 \to d\mathcal{O}(L) \to \Omega^1 \otimes L \to \Omega^2 \otimes L \to 0\,,$$

we get

$$\Gamma(\Omega^1 \otimes L) = \Gamma(d\mathcal{O}(L))\,.$$

Take now $\omega \in \Gamma(d\mathcal{O}(L))$ and denote by $\tilde{\omega}$ its pull-back on the universal covering $\tilde{S}$ of S. Then $g^*\tilde{\omega} = \lambda\tilde{\omega}$, where g is a generator of $\pi_1(S) \simeq \boldsymbol{Z}$ and λ is the twisting factor which corresponds to $L \in \mathrm{Pic}^0(S) \simeq \boldsymbol{C}^*$. Let f be a primitive of $\tilde{\omega}$ and $c \in \boldsymbol{C}$ such that $f \circ g = \lambda f + c$. Replacing f by $h := f + c/(\lambda - 1)$, we get $h \circ g = \lambda h$, which means that h induces a section in $\Gamma(S, \mathcal{O}(L))$. By our assumption on S, h has then to be the zero section and thus $\omega = 0$. □

LEMMA 2.3. *A non-trivial twisted logarithmic 1-form on S is always closed and has poles along each curve of S.*

PROOF. Let $0 \neq \omega \in \Gamma(S, \Omega^1(\log D) \otimes L)$ for a flat line bundle L. Then $d\omega \in \Gamma(S, \Omega^2(D) \otimes L)$. If $d\omega \neq 0$, then its associated divisor Γ satisfies $0 \leq -\Gamma \leq D$, which contradicts Lemma 2.1. Thus $d\omega = 0$. By Lemma 2.2, the pole divisor D_∞ of ω is non-trivial. Then D_∞ must contain the cycle of rational curves of S, otherwise one could write ω

as a non-trivial logarithmic 1-form in a neighborhood V of D_∞, and since the dual graph of D_∞ would be contractible, ω would be holomorphic on V by [17].

So let now C_1 be an irreducible component of D_∞, and suppose there exists a rational curve C_2 not contained in D_∞ such that $\emptyset \neq C_1 \cap C_2 = \{p\}$. Choose coordinates (z_1, z_2) locally around p such that $C_i = \{z_i = 0\}$, $i = 1, 2$, and write

$$\omega = \alpha_1 \frac{dz_1}{z_1} + \alpha_2 dz_2$$

in these coordinates. Since

$$0 = d\omega = \frac{\partial \alpha_1}{\partial z_2} dz_2 \wedge \frac{dz_1}{z_1} + \frac{\partial \alpha_2}{\partial z_1} dz_1 \wedge dz_2 \,,$$

we see that α_1 must have the form

$$\alpha_1(z_1, z_2) = \beta(z_1) + z_1 z_2 \gamma(z_1, z_2)\,.$$

If $\beta(0) \neq 0$, then the restriction of ω to C_2 would have, as its only pole, a simple pole in p, which is impossible. Therefore $\beta(0) = 0$ and

$$\omega = \left(\frac{\beta}{z_1} + z_2\gamma\right) dz_1 + \alpha_2 dz_2$$

has no pole along C_1, which gives a contradiction. Thus $D_\infty = D$. $\square$

LEMMA 2.4. *Two logarithmic* 1*-forms* ω_1, ω_2 *on* S *twisted by flat line bundles* L_1, L_2 *are necessarily linearly dependent.*

PROOF. If ω_1, ω_2 were not linearly dependent, then their exterior product $\omega_1 \wedge \omega_2 \in \Gamma(S, \Omega^2(D) \otimes L_1 \otimes L_2)$ would be non-identically zero. This contradicts Lemma 2.1. $\square$

LEMMA 2.5. *If* L *is a flat line bundle on* S *and* $\Gamma(S, \Omega^1(\log D) \otimes L^{-1}) = 0$, *then the morphism*

$$H^2(S, \boldsymbol{C}_S(L)) \to H^2(D, \boldsymbol{C}_D(L))$$

induced by restriction is bijective.

PROOF. We may suppose L to be non-trivial, since otherwise the conclusion holds by our hypotheses on S. The long exact cohomology sequence of the diagram

$$\begin{tikzcd}
0 \arrow[r] & \boldsymbol{C}_D(L) \arrow[r] & \mathcal{O}_D(L) \arrow[r, "d"] & \bigoplus_{i=1}^n \Omega_{C_i} \otimes L \arrow[r] & 0 \\
0 \arrow[r] & \boldsymbol{C}_S(L) \arrow[r] \arrow[u, "\mathrm{restr}"'] & \mathcal{O}_S(L) \arrow[r, "d"] \arrow[u, "\mathrm{restr}"'] & d\mathcal{O}_S(L) \arrow[r] \arrow[u, "\mathrm{restr}"'] & 0
\end{tikzcd} \tag{1}$$

gives

$$\begin{tikzcd}
H^1(\mathcal{O}_D(L)) \arrow[r] & H^1(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L) \arrow[r] & H^2(\boldsymbol{C}_D(L)) \arrow[r] & H^2(\mathcal{O}_D(L)) \\
H^1(\mathcal{O}_S(L)) \arrow[r] \arrow[u] & H^1(d\mathcal{O}_S(L)) \arrow[r] \arrow[u] & H^2(\boldsymbol{C}_S(L)) \arrow[r] \arrow[u] & H^2(\mathcal{O}_S(L))\,. \arrow[u]
\end{tikzcd}$$

Using the identities

$$H^2(\mathcal{O}_S(L)) = 0\,,\quad H^0(\mathcal{O}_S(L)) = 0\,,\quad H^2(\mathcal{O}_D(L)) = 0\,,\quad H^0(\mathcal{O}_D(L)) = 0$$

and the theorem of Riemann-Roch, we get $H^1(\mathcal{O}_S(L)) = 0$ and $H^1(\mathcal{O}_D(L)) = 0$. Thus our task comes to showing the bijectivity of the morphism

$$H^1(d\mathcal{O}_S(L)) \to H^1\Big(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L\Big).$$

On the other hand, the commutative triangle

$$\begin{tikzcd} & \bigoplus_{i=1}^n \Omega_{C_i} \otimes L & \\ d\mathcal{O}_S(L) \arrow[ur, "\mathrm{restr}"] \arrow[rr] & & \Omega^1(L)\,, \arrow[ul, "\mathrm{restr}"'] \end{tikzcd}$$

the long exact cohomology sequence associated to

$$0 \to d\mathcal{O}_S(L) \to \Omega^1_S(L) \to \Omega^2_S(L) \to 0 \tag{2}$$

and the fact that $H^0(\Omega^2_S(L)) = 0$, allow us to get the commutative triangle

$$\begin{tikzcd} & H^1\big(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L\big) & \\ H^1(d\mathcal{O}_S(L)) \arrow[ur] \arrow[rr] & & H^1(\Omega^1(L))\,, \arrow[ul] \end{tikzcd}$$

where the horizontal arrow is an isomorphism. Thus, we have only to prove that the morphism

$$H^1(\Omega^1(L)) \to H^1\Big(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L\Big) \tag{3}$$

is bijective.

We examine the dimensions first.

$$\dim H^1\Big(\bigoplus_{i=1}^n \Omega_{C_i} \otimes L\Big) = \sum_{i=1}^n \dim H^1(\Omega_{C_i} \otimes L) = n$$

and

$$\begin{aligned} \dim H^1(\Omega^1(L)) &= -\chi(\Omega^1(L)) + h^0(\Omega^1(L)) + h^2(\Omega^1(L)) \\ &= -\chi(\Omega^1(L)) = -\chi(\Omega^1) = b_2\,, \end{aligned}$$

since we know that $h^0(\Omega^1(L)) = h^0(\Omega^1(L^{-1})) = 0$. By assumption, $b_2 = n$, so it is enough to show only the surjectivity of the morphism (3). In order to do so, let us compute the kernel

$\mathcal{N}$ of the surjective morphism

$$\Omega^1_S \to \bigoplus_{i=1}^n \Omega_{C_i} .$$

Locally around a point $C_1 \cap C_2$, where $C_i = \{z_i = 0\}$, this morphism is given by

$$f_1 dz_1 + f_2 dz_2 \mapsto (f_1(z_1, 0)dz_1, f_2(0, z_2)dz_2) .$$

Thus a section of the kernel must have the form $z_2 g_1 dz_1 + z_1 g_2 dz_2$, where g_1, g_2 are holomorphic functions. Therefore we get the duality

$$\mathcal{N} \otimes \Omega^1_S(\log D) \xrightarrow{\wedge} \Omega^2_S ,$$

mapping

$$\left(z_2 g_1 dz_1 + z_1 g_2 dz_2, \frac{h_1}{z_1} dz_1 + \frac{h_2}{z_2} dz_2\right) \longmapsto (g_1 h_2 - g_2 h_1) dz_1 \wedge dz_2 ,$$

proving that

$$\mathcal{N} \simeq \Omega^1_S(\log D)^\vee \otimes \Omega^2_S .$$

In order to finish the proof we consider now the long exact cohomology sequence of

$$0 \to \mathcal{N} \otimes L \to \Omega^1_S \otimes L \to \bigoplus_{i=1}^n \Omega_{C_i} \otimes L \to 0 ,$$

and use the fact that $H^2(\mathcal{N} \otimes L) = H^0(\Omega^1_S(\log D) \otimes L^{-1})^* = 0$ which is ensured by hypothesis. □

THEOREM 2.6. *Let S be a special surface of intermediate type and $k := k(S)$. Then there exists a choice of a generator of $\pi_1(S) \simeq \mathbf{Z}$ such that S admits a closed logarithmic 1-form twisted by the flat line bundle L^k.*

PROOF. By [16] there exists a surface S' with a GSS such that the dual graph of the maximal reduced divisor D' of S' coincides with that of D. By [5] and [9, Thm. 1.2.24], we may choose the generator of $\pi_1(S') \simeq \pi_1(D')$ such that $\Gamma(S', \Omega^1(\log D') \otimes L^k) \neq 0$. We further fix the generator of $\pi_1(S) \simeq \pi_1(D) \simeq \mathbf{Z} \simeq \pi_1(S') \simeq \pi_1(D')$ to be the same as above. We may now suppose that $\Gamma(S, \Omega^1(\log D) \otimes L^{1/k}) = 0$, otherwise we change the generator of $\pi_1(S)$. We shall identify sections of sheaves on S twisted by L^k with sections of the pullback-sheaves on the universal cover $\tilde{S}$ of S which respect the representation $\rho : \pi_1(S) \to \mathbf{C}^*$ defining L^k. We start with the exact sequence

$$0 \to d\mathcal{O}(L^k) \to d\mathcal{O}(\log D) \otimes L^k \to \bigoplus_{i=1}^n \mathcal{O}_{\tilde{C}_i} \otimes L^k \to 0 , \tag{4}$$

where $\tilde{C}_i$ are the normalisations of the curves $C_i \subset S$ and with a certain element

$$a \in \Gamma\Big(S, \bigoplus_{i=1}^{n} \mathcal{O}_{\tilde{C}_i} \otimes L^k\Big) = \Gamma\Big(\tilde{S}, \bigoplus_{i=-\infty}^{\infty} \mathcal{O}_{\tilde{C}_i}\Big)^{\rho}$$

to be defined below. Seen as an element in

$$\Gamma\Big(\tilde{S}, \bigoplus_{i=-\infty}^{\infty} \mathcal{O}_{\tilde{C}_i}\Big) \simeq \bigoplus_{i=-\infty}^{\infty} \Gamma(\tilde{S}, \mathcal{O}_{\tilde{C}_i})\,,$$

a becomes a vector in $\boldsymbol{C}^{\boldsymbol{Z}}$ with the property $a_{i+n} = ka_i$ for all $i \in \boldsymbol{Z}$.

Choose a non-trivial element

$$\omega' \in \Gamma(S', \Omega^1(\log D') \otimes L^k)$$

and put

$$a_i := \int_{\gamma_i} \omega'\,,$$

where γ_i is a small path around $C'_i \subset \tilde{D}'$. We denote again by ω' the pull-back of ω' to the universal cover $\tilde{S}'$ of S'. The Camacho-Sad formula for the foliation defined by ω' gives

$$C_j^2 = -\sum_{\substack{i \neq j \\ C_i \cap C_j \neq \emptyset}} \frac{a_i}{a_j}\,,$$

see [7]. Moreover we have seen in [7] that ω' may be chosen such that $a \in \boldsymbol{Z}[1/k]^{\boldsymbol{Z}}$. For such a choice let further U_j be small neighborhoods of the curves C_j on $\tilde{S}$ and consider divisors D_j in these neighborhoods of the form

$$D_j := k^{\nu}\Big(a_j C_j + \sum_{\substack{i \neq j \\ C_i \cap C_j \neq \emptyset}} a_i C_i\Big)\,,$$

for some $\nu \in \boldsymbol{N}$ which is sufficiently large. Since $D_j.C_j = 0$, D_j is the zero divisor of some holomorphic function $f_j \in \mathcal{O}(U_j)$. Put then

$$\omega_j := k^{-\nu}\frac{df_j}{f_j} \in \Gamma(U_j, d\mathcal{O}(\log \tilde{D}))\,.$$

One may choose the functions $f_j \in \mathcal{O}(U_j)$ such that

$$g^*\omega_{j+n} = k\omega_j$$

holds for all $j \in \boldsymbol{Z}$; here $g \in \pi_1(S)$ denotes the "positive" generator of $\pi_1(S)$. There exist local coordinate functions z_j, z_i which define the curves C_j, C_i such that around $C_i \cap C_j$ we have

$$\omega_j = a_j\frac{dz_j}{z_j} + a_i\frac{dz_i}{z_i}\,,$$

and around points of C_j away from any other compact curve

$$\omega_j = a_j \frac{dz_j}{z_j}.$$

We look now at the situation on $U_i \cap U_j$. Here $\omega_j - \omega_i$ is a closed holomorphic 1-form and thus exact if $U_i \cap U_j$ is simply connected, which we shall always suppose. Indeed, $\omega_j - \omega_i$ has the form

$$a_j \frac{dg_j}{g_j} + a_i \frac{dg_i}{g_i},$$

with $g_i, g_j \in \mathcal{O}^*(U_i \cap U_j)$. Let $\mathcal{U}$ be the covering of $\tilde{S}$ which consists of the open sets U_i and of $\tilde{S} \setminus \tilde{D}$. We set $\omega = 0$ on $\tilde{S} \setminus \tilde{D}$. Then we get a cocycle $(\omega_j - \omega_i)_{j,i} \in H^1(\mathcal{U}, d\mathcal{O}(L^k))$ which represents the image of a through the canonical connecting homomorphism associated to the sequence (4). We follow this image further through the isomorphism

$$H^1(d\mathcal{O}(L^k)) \xrightarrow{\simeq} H^2(\boldsymbol{C}(L^k)),$$

which comes from the short exact sequence

$$0 \to \boldsymbol{C}(L^k) \to \mathcal{O}(L^k) \to d\mathcal{O}(L^k) \to 0.$$

In order to do this, we pass to a finer covering $\mathcal{V} = (V_\nu)_\nu$ which has the property that the sets $V_{\mu\nu} := V_\mu \cap V_\nu$ are simply connected. Let γ be the refinement map and set $\omega_\nu := \omega_{\gamma(\nu)}|_{V_\nu}$ for $V_\nu \subset U_{\gamma(\nu)}$. Since $\omega_\nu - \omega_\mu$ is exact on $V_{\mu\nu}$, we may choose primitives $f_{\nu\mu}$. This is done such that

$$f_{\nu\mu} = 0 \quad \text{if } \gamma(\mu) = \gamma(\nu) \quad \text{and}$$
$$f_{\nu\mu} = f_{\gamma(\nu)\gamma(\mu)} \quad \text{if } \gamma(\mu) \neq \gamma(\nu) \text{ and } \gamma(\mu), \gamma(\nu) \in \mathbf{Z}.$$

The cocycle

$$((f_{\mu\nu} - f_{\lambda\nu} + f_{\lambda\mu})|_{V_{\lambda\mu\nu}})_{\lambda\mu\nu}$$

is the desired element in $H^2(\mathcal{V}, \boldsymbol{C}(L^k))$. But we remark now that the trace of this cocycle on $\tilde{D}$ is zero and thus its image in $H^2(D \cap \mathcal{V}, \boldsymbol{C}(L^k))$ by the restriction morphism will be zero as well. One sees this by checking the different possibilities for λ, μ, ν, such that $V_{\lambda\mu\nu} \cap \tilde{D} \neq \emptyset$:

(i) when $\gamma(\lambda) = \gamma(\mu) = \gamma(\nu)$, all primitives are zero,

(ii) when $\gamma(\lambda) = \gamma(\mu) \neq \gamma(\nu)$, one gets again $f_{\mu\nu} - f_{\lambda\nu} + f_{\lambda\mu}|_{V_{\lambda\mu\nu}} = 0$.

Notice that at least two elements among $\gamma(\lambda), \gamma(\mu), \gamma(\nu)$ must be equal. Now Lemma 2.5 shows that the class of our cocycle in $H^2(S, \boldsymbol{C}(L^k))$ vanishes. Hence there exists a non-trivial element ω in $\Gamma(S, d\mathcal{O}(\log D) \otimes L^k)$ which maps onto $a \in \Gamma(S, \bigoplus_{i=1}^n \mathcal{O}_{\tilde{C}_i} \otimes L^k)$. □

COROLLARY 2.7. *The only singularities of the foliation associated to* $\omega \in \Gamma(S, \Omega^1(\log D) \otimes L^k)$ *are simple and are located at the intersection points of the rational curves of* S. *The rational curves are invariant with respect to this foliation and their Camacho-Sad indices are rational numbers.*

PROOF. Let $\mathcal{F}$ denote the foliation associated to ω. Lemma 2.3 implies that every rational curve of S is invariant for $\mathcal{F}$. This shows already that the b_2 intersection points of the rational curves are singularities of the foliation. On the other side if Z denotes the complex subspace of the singularities of $\mathcal{F}$, we can compute its length out of the following short exact sequence induced by ω

$$0 \to \mathcal{O}(-D) \otimes L^{1/k} \to \Omega^1 \to L^k \otimes \mathcal{J}_Z \otimes K_S(D) \to 0\,.$$

We get $b_2 = c_2(\Omega^1) = \mathrm{length}(Z) - D.K_S - D^2$. But it is easy to check using the adjunction formula that in our case $D.K_S + D^2 = 0$, so $\mathcal{F}$ has exactly b_2 singularities counted with multiplicities. Thus the intersection points of the rational curves are the only singularities of the foliation and they are simple.

Next we prove the rationality of the Camacho-Sad indices of the curves with respect to $\mathcal{F}$. If ω is obtained as in the main part of the proof of Theorem 2.6, then the associated Camacho-Sad indices are a_i/a_j, i.e., the same as those associated to ω' on S', and thus rational. But if $\omega \in \Gamma(S, \Omega^1(\log D) \otimes L^{1/k})$, where the orientation on $\pi_1(S)$ and $\pi_1(S')$ is chosen to be the same, we have to reconsider the Camacho-Sad equations of the components C_i of D. Let C_1, C_2 be two such components which intersect at p, and (z_1, z_2) local coordinate functions around p such that $C_i = \{z_i = 0\}$ for $i = 1, 2$. We may write ω around p as

$$\omega = g_1 \frac{dz_1}{z_1} + g_2 \frac{dz_2}{z_2}\,,$$

where g_1, g_2 are holomorphic functions in z_1, z_2.

Consider now small paths γ_i turning around C_i, contained say in the local curve $\{z_{3-i} = c_{3-i}\}$, for two constants $c_1, c_2 \in \boldsymbol{C}$. Since ω is closed, the integrals

$$\int_{\gamma_1} \omega = 2\pi i \mathrm{Res}_{z_1=0}(\omega|_{z_2=c_2}) = 2\pi i g_1(0, c_2)\,,$$
$$\int_{\gamma_2} \omega = 2\pi i \mathrm{Res}_{z_2=0}(\omega|_{z_1=c_1}) = 2\pi i g_2(c_1, 0)$$

are independent of c_1, c_2. Moreover, since ω has true poles of order one along C_1 and C_2, both integrals are non-zero; in particular, $g_1(0,0) \neq 0 \neq g_2(0,0)$ and p is a simple singularity for $\mathcal{F}$. Now the foliation $\mathcal{F}$ is defined locally around p by the kernel of the form $z_2 g_1 dz_1 + z_1 g_2 dz_2$ and the Camacho-Sad index of $\mathcal{F}$ with respect to C_2 is by definition

$$\begin{aligned}\mathrm{CS}(\mathcal{F}, C_2, p) :&= \mathrm{Res}_{z_1=0}\left(\frac{\partial}{\partial z_2}\left(-\frac{z_2 g_1}{z_1 g_2}\right)\Big|_{C_2}\right) = \mathrm{Res}_{z_1=0}\left(-\frac{g_1(z_1,0)}{z_1 g_2(z_1,0)}\right)\\ &= -\frac{g_1(0,0)}{g_2(0,0)} = -\int_{\gamma_2}\omega \Big/ \int_{\gamma_1}\omega\end{aligned}$$

and similarly for C_1. As a consequence, all Camacho-Sad indices of $\mathcal{F}$ along the curves of S are non-zero.

Now recall that the maximal divisor D of S consists of a cycle $\sum_{i=1}^m C_i$ of rational curves, to which a non zero number of trees of rational curves is attached. It is easy to see that the Camacho-Sad indices associated to the curves of the trees are all rational. Indeed, if $B_{i,1}, \ldots, B_{i,m_i}$ is the tree with root C_i, and if $-B_{i,j}^2 =: b_{i,j}$, we get:

$$\mathrm{CS}(\mathcal{F}, B_{i,m_i}, B_{i,m_i} \cap B_{i,m_i-1}) = -b_{i,m_i}$$

for the top,

$$\mathrm{CS}(\mathcal{F}, B_{i,m_i-1}, B_{i,m_i} \cap B_{i,m_i-1}) = -\frac{1}{b_{i,m_i}}\,,$$

$$\mathrm{CS}(\mathcal{F}, B_{i,m_i-1}, B_{i,m_i-1} \cap B_{i,m_i-2}) = -b_{i,m_i-1} + \frac{1}{b_{i,m_i}}\,,$$

and so on. Setting as in [7], $b_i := -\mathrm{CS}(\mathcal{F}, C_i, C_i \cap B_{i,1})$, $d_i := -C_i^2 - b_i$ and $\alpha_i := -\mathrm{CS}(\mathcal{F}, C_i, C_{i-1} \cap C_i)$, $i = 1, \dots, m$, we get the equations

$$\alpha_i + \frac{1}{\alpha_{i+1}} = d_i\,, \quad \text{for} \quad 1 \le i \le m-1$$

and

$$\alpha_m + \frac{1}{\alpha_1} = d_m\,.$$

Each α_i is the solution of a quadratic equation with rational coefficients, and since we know already that a rational solution exists by working with ω' on S', the other solution has to be rational as well. □

COROLLARY 2.8. *A special surface of intermediate type of index* 1 *possesses a non-trivial holomorphic vector field θ twisted by some flat line bundle and its vanishing divisor D_θ contains all the curves of D except perhaps certain summits of the trees. In particular, θ vanishes on all the curves of the cycle of D and θ has no isolated zeroes.*

PROOF. Let S be a special surface of intermediate type, and let D_{-K} be the (unique) numerical anticanonical divisor on S. This means that there exists a torsion factor $\kappa \in \boldsymbol{C}^*$ such that

$$\mathcal{O}(D_{-K}) = K_S^{-1} \otimes L^\kappa\,.$$

Let now $0 \neq \omega \in \Gamma(S, \Omega^1(\log D) \otimes L^k)$ and Z the subspace of the intersection points of the curves of S. By Corollary 2.7 this is exactly the space of singularities of the associated foliation to ω. Thus ω induces an exact sequence

$$0 \to \mathcal{O}(-D) \otimes L^{1/k} \to \Omega^1 \to L^{\kappa k} \otimes \mathcal{J}_Z \otimes \mathcal{O}(D - D_{-K}) \to 0\,,$$

and by duality we get a non-trivial section

$$\theta \in \Gamma(S, \Theta_S \otimes L^{k\kappa})$$

vanishing precisely on $D_\theta := D_{-K} - D$. We know by Lemma 2.1 that $D_\theta > 0$. Let now $C_1 \not\subset \mathrm{supp}(D_\theta)$ such that C_1 intersects an irreducible curve $C_2 \subset D_\theta$. Since θ defines the same foliation as ω the curve C_1 is θ-invariant. On the other hand, since $C_1 \cap C_2$ is a singularity of this foliation and θ vanishes on C_2, the vanishing order of the restriction of θ to C_1 is at least two. But then since C_1 is rational, this order is exactly two and C_1 cannot intersect another curve of D. Thus C_1 is the summit of a tree of rational curves. This implies our statement. □

3. The universal covering of the complement of the curves. From now on we shall consider a special surface S of intermediate type which admits a numerical anticanonical divisor. We have seen that in this case S possesses a non-trivial twisted logarithmic 1-form ω with twisting factor $k = k(S)$ and a non-trivial twisted holomorphic vector field θ with twisting factor, say $\lambda \in \boldsymbol{C}^*$. The case $\lambda = 1$, i.e., θ is a holomorphic vector field on S, was considered and completely understood in [6] and [7]. In this section we prove that in the general case, as in the case $\lambda = 1$, the universal covering of $S \setminus D$ is isomorphic to $\boldsymbol{H} \times \boldsymbol{C}$, where $\boldsymbol{H}$ denotes the complex half-plane.

Let U be a small open neighborhood of D such that D is a deformation retract of U. We have

$$\pi_1(U) = \pi_1(D) = \pi_1(S) = \boldsymbol{Z},$$

and we denote as before by g a generator of this group. There is a fundamental domain U_0 for the action of $\boldsymbol{Z}$ in the inverse image $\tilde{U}$ of U in the universal cover $\tilde{S}$ of S, such that the border of U_0 in $\tilde{U}$ cuts $\tilde{D}$ in a component C_0 and in its translated $g(C_0)$ along a circle S^1. Set $Y_0 := \bigcup_{\nu \geq 0} g^\nu(U_0)$. We keep the notation ω for the logarithmic 1-form one gets on $\tilde{S}$.

LEMMA 3.1. *There is a normalization of ω such that the representation*

$$\rho : \pi_1(\tilde{S} \setminus \tilde{D}) \to \boldsymbol{C},$$

which maps

$$\gamma \mapsto \int_\gamma \omega,$$

has as image $2\pi i\boldsymbol{Z}[1/k] \subset \boldsymbol{C}$. Furthermore, one can choose this normalization such that $\rho(\pi_1(Y_0 \setminus \tilde{D})) = 2\pi i\boldsymbol{Z}$.

PROOF. Since $\tilde{S}$ is simply connected, the group $\pi_1(\tilde{S} \setminus \tilde{D})$ is generated by small paths around the irreducible components of $\tilde{D}$. Thus keeping the notations of the previous section, we see that $\rho(\pi_1(\tilde{S} \setminus \tilde{D}))$ is a $\boldsymbol{Z}[1/k]$-module generated by

$$2\pi i a_0 = \int_{\gamma_0} \omega, \dots, 2\pi i a_{n-1} = \int_{\gamma_{n-1}} \omega,$$

where $\gamma_0, \dots, \gamma_{n-1}$ are small paths around the curves $C_0, \dots, C_{n-1}$ in U_0. Using Corollary 2.7 and the way we computed the Camacho-Sad indices, we see that we can normalize the form ω such that the numbers $a_0, \dots, a_{n-1}$ are non-zero integers with g.c.d. equal to 1; so $\rho(\pi_1(\tilde{S} \setminus \tilde{D}))$ is free of rank 1 as $\boldsymbol{Z}[1/k]$-module. In a similar way, the group $\rho(\pi_1(Y_0 \setminus \tilde{D}))$ is generated as a $\boldsymbol{Z}$-module by small paths around the irreducibles components of $\tilde{D}$ which meet U_0, and we get the announced result. □

In what follows, we suppose ω to be normalized such that $\rho(\pi_1(Y_0 \setminus \tilde{D})) = 2\pi i\boldsymbol{Z}$. This is the same as to say that $a_0, \dots, a_{n-1}$ are non-zero integers with g.c.d. equal to 1. Let A be a fundamental domain for the action of $\boldsymbol{Z}$ on $\tilde{S}$ and $X_0 := \bigcup_{j \geq 0} g^j(A)$. Translating by g, we may suppose that $\tilde{D} \cap X_0 \subset Y_0$. We remark that after such a translation we have

$$\rho(\pi_1((Y_0 \cup X_0) \setminus \tilde{D})) = \rho(\pi_1(Y_0 \setminus \tilde{D})) = 2\pi i\boldsymbol{Z}.$$

Fix $z_0 \in U_0$. We define a holomorphic function f on $(Y_0 \cup X_0) \setminus \tilde{D}$ by

$$f(z) = \exp\left(\int_{z_0}^{z} \omega + \frac{1}{k-1} \int_{z_0}^{g(z_0)} \omega \right).$$

One verifies easily that f is well-defined and that

$$(\dagger) \qquad f(g(z)) = f^k(z)$$

for $z \in (Y_0 \cup X_0) \setminus \tilde{D}$.

Let C be the smooth part in $\tilde{D}$ of an irreducible component of $\tilde{D} \cap Y_0$. Since ω is a closed logarithmic 1-form, an easy computation shows that one can extend f meromorphically across C such that C belongs to the zero or to the polar set of the extension of f. By replacing ω with $-\omega$ if necessary, one finds at least one preimage on $\tilde{S}$ of a component of the cycle of D in the zero-set of f. This means that the connected components of the polar set of f are exeptional divisors in Y_0. Thus the polar set is empty, see [17], and f vanishes on $\tilde{D} \cap Y_0$. In particular, we see that the integers $a_0, \ldots, a_{n-1}$ are positive.

LEMMA 3.2. $|f(z)| < 1$ *for any* $z \in Y_0 \cup X_0$.

PROOF. Remark first that one can extend the function $|f|$ to the whole of $\tilde{S}$. The extended function is still denoted by $|f|$. Then the statement of the Lemma can be rephrased by saying that the image of $|f|$ is the interval $[0, 1[$. Suppose now that this is not so. Then $|f|$ has as image $[0, \infty[$, since $|f| \circ g = |f|^k$. We consider the real hypersurface $\tilde{H} := |f|^{-1}(1)$, which is $\pi_1(S)$-invariant and thus descends to a compact real hypersurface H on S. Obviously, H is a (compact) leaf of the foliation defined by $\Re e\, \omega$. Remark that the morphism $\pi_1(H) \to \pi_1(S)$ is non-trivial. Otherwise the connected components of $\tilde{H}$ would be compact and their intersection with $f^{-1}(1)$ would give compact analytic curves in the complement of $\tilde{D}$, which is absurd. By passing to a finite unramified covering of S, we may even suppose that $\pi_1(H) \to \pi_1(S)$ is surjective.

Next we prove that ω must have non-vanishing periods on $\tilde{H}$. If not, consider a $\pi_1(S)$-invariant neighborhood $\tilde{V}$ of $\tilde{H}$ which is the preimage of a neighborhood V of H in S and on which ω has no periods; thus $\omega|_{\tilde{V}}$ is exact. We define the following holomorphic function on $\tilde{V}$:

$$h(z) := \int_{z_0}^{z} \omega + \frac{1}{k-1} \int_{z_0}^{g(z_0)} \omega\,,$$

where $z_0 \in \tilde{V}$ is a fixed base point, $z \in \tilde{V}$ and integration is done along paths in $\tilde{V}$. We have $h(g(z)) = kh(z)$ for all $z \in \tilde{V}$. The function h does not take the value 0, because the set $\{h = 0\}$ would then be a $\pi_1(S)$-invariant analytic curve giving rise to a compact curve in the complement of D in S, which is absurd. Hence one may consider the 1-form $d(\log|h|)$, which is closed and descends to a closed non-twisted form on V. This form obviously defines the same foliation as $\Re e\, \omega$. But this means that this foliation has trivial holonomy. This implies that the leaves near H are also compact. In particular, they are contained in V. The inverse images of these leaves are leaves of the foliation defined by $|f|$ on $\tilde{S}$. On the other hand, they are completely contained in $\tilde{V}$ and intersect all the sets $g^\nu(\tilde{S} \setminus X_0)$, $\nu \in \boldsymbol{Z}$. But the relation $f(g(z)) = f^k(z)$ and the choice of V imply that

$$\bigcap_{\nu \in \boldsymbol{Z}} |f|(g^\nu(\tilde{V} \setminus X_0)) = \{1\},$$

giving a contradiction. Thus $\omega|_{\tilde{H}}$ has non-vanishing periods.

The next point is to show that f may be extended holomorphically to

$$U_1 := |f|^{-1}(]1, \infty[).$$

In order to do this we consider the sets

$$U_\alpha := |f|^{-1}(]\alpha, \infty[)$$

for $\alpha \geq 1$ and

$$M := \{\alpha > 1 | f \text{ admits a holomorphic extension to } U_\alpha\}.$$

Then M is non-empty, since $U_\alpha \subset X_0$ for $\alpha > \max_{z \in \partial X_0} |f(z)|$. Furthermore, the set M is closed in $]1, \infty[$, since $U_{\alpha_0} = \bigcup_{\alpha > \alpha_0} U_\alpha$ for each α_0. It remains to check that M is open as well. Let $\alpha \in M$. There exists some $\nu \in \boldsymbol{N}$ such that $U_\alpha \subset g^{-\nu}(X_0)$. On $g^{-\nu}(X_0)$ one can extend f^{k^ν} holomorphically. Take now a finite open covering $V_1, \ldots, V_\mu$ of $(\partial U_\alpha) \setminus X_0$ such that each V_i intersects U_α and that on V_i a k^ν-th root of f^{k^ν} is defined. Consider next on each V_i that k^ν-th root of f^{k^ν} which coincides with f on $V_i \cap U_\alpha$. This gives an extension of f to

$$U_\alpha \cup X_0 \cup \bigcup_{i=1}^{\mu} V_i.$$

Remark that this set will contain some U_β with $\beta < \alpha$.

We finish the proof of the Lemma by considering a path $\gamma \subset U_1$ such that $\int_\gamma \omega \neq 0$, which is possible, since ω has non-vanishing periods on H. But for ν sufficiently large

$$\int_{g^{-\nu} \circ \gamma} \omega = k^{-\nu} \int_\gamma \omega,$$

cannot be a multiple of $2\pi i$, which is incompatible with the definition of f on the whole of U_1. □

PROPOSITION 3.3. *The holomorphic vector field $\tilde{\theta} \in \Gamma(\tilde{S}, \tilde{\Theta}_{\tilde{S}})$ induced by θ is completely integrable on $\tilde{S}$.*

PROOF. We consider the integrability of $\tilde{\theta}$ along a leaf F of the foliation. It suffices to show that one can find a local integration radius which is uniform for all F.

We set $A_0 := \overline{X_0 \cup Y_0} \setminus g(X_0 \cup Y_0)$. The set A_0 intersects $\tilde{D}$ along a curve C_0 and its translated by g, C_n. There is a constant $c \in]0, 1[$ such that $F \subset |f|^{-1}(c)$. Using the relation $|f| \circ g = |f|^k$ again and Lemma 3.2, we see that $\lim_{\nu \to \infty} |f| \circ g^\nu = 0$ uniformly on compact sets. Hence there exists a $\nu_0 \in \boldsymbol{N}$ such that $F \cap \bigcup_{\nu > \nu_0} g^\nu(A_0) = \emptyset$. On the other side there exists a $\nu_1 \in \boldsymbol{N}$ such that $F \cap \bigcup_{\nu \geq \nu_1} g^{-\nu}(A_0) \subset \tilde{U}$. By passing to a suitable translation, we may assume that $\nu_1 = 0$. We may further assume that around $\tilde{U} \cap \partial A_0$ we have coordinate functions (z_0, z_1) such that $C_0 = \{z_0 = 0\}$,

$$\omega = a_0 \frac{dz_0}{z_0}$$

with $a_0 \in \boldsymbol{N}^*$ and

$$\tilde{\theta} = \alpha(z_0, z_1) z_0^s \frac{\partial}{\partial z_1}$$

with $s \in \boldsymbol{N}^*$ and α a nowhere vanishing holomorphic function. Here we have $f(z_0, z_1) = z_0^{a_0}$. Thus

$$(z_0, z_1) \in F \cap \partial\Big(\bigcup_{\nu \geq 0} g^\nu(A_0)\Big)$$

implies that $|z_0|^{a_0} = c$. But since $F \cap \bigcup_{\nu \geq 0} g^\nu(A_0)$ is contained in the compact set $\bigcup_{\nu=0}^{\nu_0} g^\nu(A_0)$, the integration radius of $\tilde{\theta}$ at points of

$$F \cap \bigcup_{\nu \geq 0} g^\nu(A_0)$$

is at least as large as the integration radius of $\tilde{\theta}$ at

$$F \cap \partial\Big(\bigcup_{\nu=0}^{\nu_0} g^\nu(A_0)\Big) = F \cap \partial\Big(\bigcup_{\nu \geq 0} g^\nu(A_0)\Big),$$

and this is the minimal integration radius of $\alpha(z_0, z_1) z_0^s \partial/\partial z_1$ at points (z_0, z_1) with $|z_0|^{a_0} = c$. Looking now at $\tilde{\theta}$ on

$$F \cap \partial\Big(\bigcup_{\nu \geq -r} g^\nu(A_0)\Big)$$

for $r \in \boldsymbol{N}$, we see, by applying g_*^r, that the integration radius at these points will be at least as large as the minimal integration radius of $\lambda^r \alpha(z_0, z_1) z_0^s \partial/\partial z_1$ at points $(z_0, z_1) \in F \cap \partial(\bigcup_{\nu \geq 0} g^\nu(A_0))$ with $|z_0|^{a_0} = c^{k^r}$. But the sequence of velocities

$$|\lambda|^r c^{sk^r/a_0} \sup |\alpha|$$

is obviously bounded and thus there is an uniform integration radius for $\tilde{\theta}$ on F. $\square$

The kernel $\ker \rho$ defines a covering

$$\pi : X' \to \tilde{S} \setminus \tilde{D}.$$

One checks immediately that the $\mathbf{Z}$-action induced by g on $\pi_1(\tilde{S} \setminus \tilde{D})$ stabilizes $\ker \rho$ and thus induces a $\mathbf{Z}$-action on X'. We denote again by g a lift of g on X'. Thus we get an action of the semi-direct product $\mathbf{Z} \ltimes \mathbf{Z}[1/k]$ on X' whose quotient is $S \setminus D$. Let $\omega' = \pi^* \omega$ and $\phi : X' \to \boldsymbol{C}$ be a primitive of ω' on X' such that $\exp(\phi)$ and $f \circ \pi$ coincide on a connected component of the π-preimage of $(Y_0 \cup X_0) \setminus \tilde{D}$. Since $\phi \circ g = k\phi$, the image of ϕ is invariant under the action of the multiplicative group $\{k^\nu | \nu \in \mathbf{Z}\}$; this image is also invariant under the action of the additive group $2\pi i \boldsymbol{Z}[1/k]$. Since f takes its values in the unit disk by Lemma 3.2, we see now that $\phi(X')$ must coincide with the left half plane $\boldsymbol{H}_l := \{w \in \boldsymbol{C} \,|\, \Re e\, w < 0\}$. The function $\phi : X' \to \boldsymbol{H}_l$ is a surjective holomorphic submersion since θ has no zeroes on $S \setminus D$. The connected components of its fibers are leaves of the inverse image foliation induced by $\mathcal{F}$. For the proof of the next Proposition the reader is referred to [7, Prop. 2.2]. Note that the Camacho-Sad indices of our foliation here, are positive integers, just as in [7].

PROPOSITION 3.4. *The fibers of ϕ are connected.*

COROLLLARY 3.5. *The foliation defined by θ on $\tilde{S} \setminus \tilde{D}$ has no closed leaf.*

PROOF. If F were a closed leaf on $\tilde{S} \setminus \tilde{D}$, its preimage $\pi^{-1}(F)$ in X' would also be closed. But since $\rho|_{\pi_1(F)}$ is trivial, the group $\boldsymbol{Z}[1/k]$ operates on X' by permuting components of $\pi^{-1}(F)$. Using the previous Proposition, the non-discreteness of the $2\pi i\boldsymbol{Z}[1/k]$-orbits in $\boldsymbol{H}_l$ and the fact that ϕ is a submersion, we get a contradiction. □

LEMMA 3.6. *The fibers of ϕ are isomorphic to $\boldsymbol{C}$.*

PROOF. Since $\tilde{\theta}$ is completely integrable, there is a holomorphic $\boldsymbol{C}$-action on $\tilde{S} \setminus \tilde{D}$. The fixed point set of a non-trivial element of $\boldsymbol{C}$ is a closed analytic subset of $\tilde{S} \setminus \tilde{D}$ which is a union of leaves of the foliation. By the previous Corollary, such a fixed point set cannot have dimension 1. Hence it is either empty or the whole space $\tilde{S} \setminus \tilde{D}$. We must exclude the second case. In this situation the $\boldsymbol{C}$-action factorizes to a $\boldsymbol{C}^*$-action. Thus all fibers of ϕ are isomorphic to $\boldsymbol{C}^*$. Moreover the $\boldsymbol{C}^*$-action lifts to X' making $\phi : X' \to \boldsymbol{H}_l$ into a principal $\boldsymbol{C}^*$-bundle. But such a bundle over $\boldsymbol{H}_l$ is always trivial. We may therefore see ϕ as the first factor projection

$$X' \simeq \boldsymbol{H}_l \times \boldsymbol{C}^* \to \boldsymbol{H}_l\,.$$

If (w, z) denote coordinate functions on $\boldsymbol{H}_l \times \boldsymbol{C}^*$, then the pull-back of the vector field has the form $\alpha(w)z\partial/\partial z$ for some $\alpha \in \mathcal{O}^*(\boldsymbol{H}_l)$. We now consider generators f_g, f_γ of the groups $\boldsymbol{Z}$ and $\boldsymbol{Z}[1/k]$ acting on $\boldsymbol{H}_l \times \boldsymbol{C}^*$.

By passing to a double covering of S, we may assume that f_g acts on the $\boldsymbol{C}^*$-fibers by homotheties. Suppose that the same holds for f_γ. Then we have

$$f_g(w, z) = (kw, \beta(w)z)\,,$$

$$f_\gamma(w, z) = (w + 2\pi i, \gamma(w)z)$$

for some $\beta, \gamma \in \mathcal{O}^*(\boldsymbol{H}_l)$. The compatibility with the pulled-back vector field implies that

$$\alpha(kw) = \lambda\alpha(w)\,, \quad \alpha(w + 2\pi i) = \alpha(w)$$

for all $w \in \boldsymbol{H}_l$. The second relation implies that α is the composition of a function u on the punctured unit disc with the exponential. Then the first relation translates into

$$u(\zeta^k) = \lambda u(\zeta) \quad \text{for all } \ \zeta \in \Delta^\star\,;$$

(by Δ we denoted the unit disk in $\boldsymbol{C}$). By comparing the Laurent expansion of this equality at 0, one sees that u is constant and $\lambda = 1$. Hence we obtain an effective $\boldsymbol{C}^*$-action on S, which is excluded by [10].

When f_γ is composed with an inversion a similar argument applies, working with α^2 instead of α for instance. Thus the $\boldsymbol{C}$-action is effective and the fibers of ϕ are isomorphic to $\boldsymbol{C}$. □

Using now the lift of the $\boldsymbol{C}$-action on X', we get a $\boldsymbol{C}$-principal bundle structure on X' over $\boldsymbol{H}_l$. Again, such a bundle is holomorphically trivial. In conclusion we have proven the following

THEOREM 3.7. *The universal covering of $S \setminus D$ is isomorphic to $\boldsymbol{H}_l \times \boldsymbol{C}$.*

4. The action of the fundamental group. We consider a system of holomorphic coordinates (w, z) in $\boldsymbol{H}_l \times \boldsymbol{C} \simeq \widetilde{S \setminus D}$. The integrable vector field induced here by θ has no zeros and is therefore constant on each fiber of the projection of $\boldsymbol{H}_l \times \boldsymbol{C}$ on $\boldsymbol{H}_l$. Consequently, this vector field is of the form $\alpha(w)\partial/\partial z$ on $\boldsymbol{H}_l \times \boldsymbol{C}$, where $\alpha \in \mathcal{O}^\star(\boldsymbol{H}_l)$. Conjugating by the automorphism

$$(w, z) \mapsto (w, \alpha^{-1} \cdot z)\,,$$

one gets $\alpha \equiv 1$.

Let $\gamma \in \pi_1(\tilde{S} \setminus \tilde{D})$ be a closed path in $\tilde{S} \setminus \tilde{D}$ with $\rho(\gamma) = 2\pi i$. Denoting by g_γ the automorphism of $\boldsymbol{H}_l \times \boldsymbol{C} \simeq \widetilde{S \setminus D}$ corresponding to γ, we have:

$$\begin{aligned} g_\gamma(w, z) &= (w + 2\pi i, z + f_\gamma(w))\,,\\ g(w, z) &= (kw, \lambda z + f_g(w))\,. \end{aligned}$$

The case $\lambda = 1$ was treated in [7]. From now on we shall therefore assume that $\lambda \in \boldsymbol{C} \setminus \{0, 1\}$.

The automorphism g_γ generates an action of $\boldsymbol{Z}$ on $\boldsymbol{H}_l \times \boldsymbol{C}$, which induces a holomorphic $\boldsymbol{C}$-principal bundle

$$\boldsymbol{H}_l \times \boldsymbol{C}/\boldsymbol{Z} \to \boldsymbol{H}_l/\boldsymbol{Z} \simeq \Delta^\star\,.$$

The triviality of this bundle proves the existence of a holomorphic function $h : \boldsymbol{H}_l \to \boldsymbol{C}$ such that

$$h(w + 2\pi i) - h(w) = f_\gamma(w)\,,$$

and conjugation by $(w, z) \mapsto (w, z + h(w))$ gives us the new form

$$g_\gamma(w, z) = (w + 2\pi i, z)\,.$$

In what follows we suppose therefore that $f_\gamma \equiv 0$. We have

$$g \circ g_\gamma \circ g^{-1} = g_\gamma^k\,,$$

which gives a group isomorphism

$$\langle g_\gamma, g\rangle \simeq \boldsymbol{Z} \ltimes \boldsymbol{Z}\left[\frac{1}{k}\right] \simeq \pi_1(S \setminus D)$$

on the one hand, and the $2\pi i$-periodicity of the function f_g on the other hand.

Factorizing by $\exp : \boldsymbol{H}_l \to \Delta^\star$, $w \mapsto e^w =: \zeta$, gives a Laurent series expansion

$$f_g(w) = \sum_{m\in\boldsymbol{Z}} a_m e^{mw} = \sum_{m\in\boldsymbol{Z}} a_m \zeta^m .$$

A conjugation by

$$(w, z) \mapsto (w, z + \beta(w)),$$

where β is a $2\pi i$-periodic function on $\boldsymbol{H}_l$, does not change the form of g_γ, but replaces f_g by

$$w \mapsto f_g(w) + \beta(kw) - \lambda\beta(w) .$$

Let

$$h(\zeta) := \sum_{m\in\boldsymbol{Z}} a_m \zeta^m, \quad h_+(\zeta) := \sum_{m>0} a_m \zeta^m .$$

The series $\sum_{l=0}^{\infty} \lambda^{-(l+1)} h_+(\zeta^{k^l})$ converges uniformly on compact sets in $\Delta^\star$. To see this, it is enough to write $h_+(\zeta) = \zeta(\zeta^{-1}h_+(\zeta))$ and to remark that $\zeta^{-1}h_+(\zeta)$ is holomorphic in 0. Let

$$\chi(\zeta) := \sum_{l=0}^{\infty} \lambda^{-(l+1)} h_+(\zeta^{k^l}) .$$

Then we have

$$\lambda\chi(\zeta) - \chi(\zeta^k) = h_+(\zeta) .$$

If we set $\beta(w) := \chi(e^w) + a_0/(\lambda - 1)$, then we get

$$f_g(w) + \beta(kw) - \lambda\beta(w) = \sum_{m<0} a_m e^{mw} .$$

We can therefore suppose that $f_g(w) = h(e^w)$, where $h \in \mathcal{O}(\boldsymbol{P}_1(\boldsymbol{C}) \setminus \{0\})$ and $h(\infty) = 0$.

One still has the possibility to conjugate with $(w, z) \mapsto (w, z + \chi(e^w))$, where $\chi \in \mathcal{O}(\boldsymbol{P}_1(\boldsymbol{C}) \setminus \{0\})$.

For a function

$$h(\zeta) = \sum_{m<0} a_m \zeta^m$$

and $l \in \boldsymbol{N}^\star$, we consider

$$h_l(\zeta) := \sum_{k^l|m,\ m<0} a_m \zeta^m$$

and

$$h_0 := h.$$

Each h_l is of the form $h_l(\zeta) = f_l(\zeta^{k^l})$, for a certain function f_l. Let χ be the formal series $-\sum_{l\geq 1} \lambda^{l-1} f_l$. Formally one has:

$$\begin{aligned} h(\zeta) + \chi(\zeta^k) - \lambda\chi(\zeta) &= (h(\zeta) - f_1(\zeta^k)) + \lambda(f_1(\zeta) - f_2(\zeta^k)) + \lambda^2(f_2(\zeta) - f_3(\zeta^k)) + \cdots \\ &= \sum_{l\geq 0} \lambda^{l-1}(f_l(\zeta) - f_{l+1}(\zeta^k)) \end{aligned}$$

and each term contains in its Laurent series expansion in ζ only terms $b_m\zeta^m$ with $k \nmid m$. For $0 < R < 1$, $l \geq 1$ and $|\zeta| \geq R$ one has:

$$|f_l(\zeta)| \leq \sum_{\substack{k^l|m\\ m<0}} |a_m| R^{\frac{m}{k^l}} \leq \sum_{\substack{k^l|m\\ m<0}} |a_m| R^{m+k^l-1} \leq R^{k^l-1} \sum_{m\leq 0} |a_m| R^m ,$$

which shows that the series defining χ is uniformly convergent on compact sets of $\boldsymbol{P}_1(\boldsymbol{C}) \setminus \{0\}$. One gets the following normal form for h:

$$h(\zeta) = \sum_{\substack{m<0\\ k\nmid m}} a_m \zeta^m .$$

Remark that if h is of this form, any modification of $h(\zeta)$ by conjugation gives

$$h(\zeta) + \chi(\zeta^k) - \lambda\chi(\zeta) ,$$

which is in normal form if and only if the function $\zeta \mapsto \chi(\zeta^k) - \lambda\chi(\zeta)$ is identically zero. To see it, write the power series expansions of these functions:

$$\chi(\zeta) = \sum_{m<0} b_m \zeta^m , \quad -\chi(\zeta^k) + \lambda\chi(\zeta) = \sum_{m<0} c_m \zeta^m .$$

If there existed a $c_r \neq 0$ for $r \in \mathbf{Z}_-$, then $k \nmid r$, $\lambda b_r = c_r$, $\lambda b_{kr} = c_{kr} + b_r = \lambda^{-1} c_r$, $\lambda^3 b_{k^2 r} = \lambda^2 c_{k^2 r} + \lambda^2 b_{kr} = c_r, \ldots$, and the series $\sum_{m\leq 0} b_m \zeta^m$ would not converge on $\boldsymbol{P}_1(\boldsymbol{C}) \setminus \{0\}$, which one verifies by putting $\zeta = |\lambda|$ or $\zeta = 1$. Hence we get a contradiction.

We are now in the following normalized situation:

PROPOSITION 4.1. *The action of the fundamental group* $\pi_1(S \setminus D) \simeq \mathbf{Z} \ltimes \mathbf{Z}[1/k]$ *on the universal cover* $\boldsymbol{H}_l \times \boldsymbol{C}$ *of* $S \setminus D$ *is generated by the two automorphisms*:

$$g_\gamma(w, z) = (w + 2\pi i, z),$$
$$g(w, z) = (kw, \lambda z + f_g(w))$$

with $f_g(w) = h \circ \exp(w) = H \circ \exp(-w)$, *where*

$$H(\zeta) = \sum_{\substack{m>0\\ k\nmid m}} A_m \zeta^m , \quad A_m = a_{-m} .$$

The elements $g_{l,n} := g^{-n} \circ g_\gamma^l \circ g^n$ for $n \in \boldsymbol{N}$, $l \in \mathbf{Z}$, form a subgroup Γ of $\pi_1(S \setminus D)$ isomorphic to $\mathbf{Z}[1/k]$. Explicitly, we have

$$g_{l,n}(w, z) = \left(w + 2\pi i l k^{-n}, z + \sum_{j=0}^{n-1} \lambda^{-j-1} (f_g(k^j w) - f_g(k^j w + 2\pi i l k^{j-n})) \right)$$

with $f_g(w) = H \circ \exp(-w)$, where H is of the form

$$H(\zeta) = \sum_{\substack{m>0\\ k\nmid m}} A_m \zeta^m .$$

We know that Γ acts properly discontinuously. Therefore H is non-trivial. One verifies easily that if $n \geq m$, $g_{p,n} \circ g_{q,m} = g_{p+qk^{n-m},n}$.

For $l = 1$, let

$$G_{n,j}(\zeta) = H(\zeta^{k^j}) - H(\zeta^{k^j}\exp(-2\pi i k^{j-n}))\,, \quad 0 \leq j < n\,,$$
$$F_n(\zeta) = \sum_{0 \leq j < n} \lambda^{-j-1} G_{n,j}(\zeta)\,.$$

With these and putting $\zeta = \exp(-w)$, we have

$$\sum_{j=0}^{n-1} \lambda^{-j-1}(f_g(k^j w) - f_g(k^j w + 2\pi i k^{j-n})) = \sum_{j=0}^{n-1} \lambda^{-j-1} G_{n,j}(\zeta) = F_n(\zeta)\,,$$

and

$$g_{1,n}(w, z) = \left(w + \frac{2\pi i}{k^n}, z + F_n(\zeta) \right).$$

The rest of this section is devoted to the proof of the following

THEOREM 4.2. *The function H is a non-constant polynomial.*

This theorem generalizes the similar statement of [7], whose proof we owe to Alexander Borichev. In what follows we use the notation:

$r\boldsymbol{T}$ (resp. $r\boldsymbol{D}$) denote the circle (resp. the open disc) of radius $r > 0$ and K the compact set $\{z \in \boldsymbol{C} \mid 3 \leq |z| \leq 3^k\}$. The different circles $r\boldsymbol{T}$ are equipped with the normalized Lebesgue measure $dm(\zeta)$, for which $\int_{r\boldsymbol{T}} dm(\zeta) = 1$. If f is holomorphic on $3\overline{\boldsymbol{D}}$, then $\mathrm{wind}(f)$ denotes the number of zeros of f in $3\boldsymbol{D}$.

The proofs of the following lemmas are left to the reader; see however [7] for the case $\lambda = 1$.

LEMMA 4.3. *Let f be a holomorphic function on the closed unit disc $3^k\overline{\boldsymbol{D}}$ such that for all $z \in K$ $|f(z)| \geq 1$. Then one has the relations*

$$\int_{3\boldsymbol{T}} \ln|f(\zeta)\zeta^{-\mathrm{wind}(f)}|\,dm(\zeta) = \int_{3^k\boldsymbol{T}} \ln|f(\zeta)\zeta^{-\mathrm{wind}(f)}|\,dm(\zeta)\,,$$
$$\int_{3\boldsymbol{T}} \ln|f(\zeta)|\,dm(\zeta) = \int_{3^k\boldsymbol{T}} \ln|f(\zeta)|dm(\zeta) - (k-1)\,\mathrm{wind}(f)\ln 3\,.$$

LEMMA 4.4. *For all $a, b \in \boldsymbol{C}$, such that $|a| > 2$ and $|b| < 1$, one has*

$$\ln|a+b| \geq \ln(|a|) - |b|\,.$$

LEMMA 4.5. *Let f be a holomorphic function defined in a neighborhood of the closed disc $3\overline{\boldsymbol{D}}$. If we set*

$$A = \int_{3\boldsymbol{T}} \ln^+|f(\zeta)|\,dm(\zeta)\,,$$

then there is a constant $C > 0$, independent of f, for which

$$\ln \int_{2\boldsymbol{T}} |f(\zeta)|\,dm(\zeta) \leq CA\,.$$

(*Here* $\ln^+ := \max(\ln, 0)$.)

LEMMA 4.6. *Let* $f = \sum_{s\geq 0} \hat{f}(s)z^s$ *be a holomorphic function in a neighborhood of the disc* $2\overline{\boldsymbol{D}}$. *Then, for all* $s \in \boldsymbol{N}$,

$$|\hat{f}(s)| = \left| \int_{2\boldsymbol{T}} f(\zeta)\zeta^{-s} dm(\zeta) \right| \leq 2^{-s} \int_{2\boldsymbol{T}} |f(\zeta)|\, dm(\zeta)\,.$$

LEMMA 4.7. *For all* $n \geq 0$ *we have*

$$\lambda F_{n+1}(\zeta) = G_{n+1,0}(\zeta) + F_n(\zeta^k)\,.$$

LEMMA 4.8. *For every* $\nu \geq 1$,

$$\widehat{F_n}(\nu k^{n-1}) = \lambda^{-n} A_\nu (1 - \exp(-2\pi i \nu/k))\,.$$

In particular,

$$|\widehat{F_n}(\nu k^{n-1})| \geq |\lambda|^{-n} \frac{|A_\nu|}{k}\,.$$

PROOF OF THEOREM 4.2. Since Γ acts properly discontinuously, the images $g_{1,n}(K)$ of the compact K tend to infinity. Since the first component of $g_{1,n}(w, z)$ converges, this implies that the sequence $(|F_n|)$ converges uniformly on K to $+\infty$.

Since H is $\mathcal{C}^1$ on K, there is a constant $c > 0$ independent of n for which

$$\beta_n := \sup_{\zeta\in K} |G_{n,0}(\zeta)| = \sup_{\zeta\in K} |H(\zeta) - H(\zeta \exp(-2\pi i k^{-n}))| \leq ck^{-n}\,.$$

Fix a positive integer N such that

$$(*) \qquad \text{for all } \zeta \in K,\ n \geq N\,, \quad |F_n(\zeta)| \geq 2$$

and

$$(**) \qquad \lambda^{-1} \sum_{n>N} \beta_n \leq 1\,.$$

Set $W := \mathrm{wind}(F_N)$ and $F(\zeta) := F_N(\zeta^k)$. Using $(*)$, we get $\mathrm{wind}(F) = kW$. We recall that $\lambda F_{N+1}(\zeta) = F(\zeta) + G_{N+1,0}(\zeta)$. Combine now Rouché's Theorem applied to F and $G_{N+1,0}$ with the inequalities $(*)$ and $(**)$ to obtain $\mathrm{wind}(F_{N+1}) = kW$. By induction, one shows that for $p \in \boldsymbol{N}$,

$$\mathrm{wind}(F_{N+p}) = k^p W\,.$$

The Lemmas 4.3 and 4.7 imply that

$$\begin{aligned}
\int_{3\boldsymbol{T}} \ln|F_N(\zeta)|\, dm(\zeta) &= \int_{3^k\boldsymbol{T}} \ln|F_N(\zeta)|\, dm(\zeta) - (k-1)W\ln 3 \\
&= \int_{3\boldsymbol{T}} \ln|F_N(\zeta^k)|\, dm(\zeta) - (k-1)W\ln 3 \\
&= \int_{3\boldsymbol{T}} \ln|\lambda F_{N+1}(\zeta) - G_{N+1,0}(\zeta)|\, dm(\zeta) - (k-1)W\ln 3\,.
\end{aligned}$$

Applying Lemma 4.4 and the inequalities $(*)$ and $(**)$, one gets

$$\begin{aligned}\int_{3\boldsymbol{T}} \ln|F_N(\zeta)|\,dm(\zeta) &\geq \int_{3\boldsymbol{T}} (\ln|\lambda F_{N+1}(\zeta)| - |G_{N+1,0}(\zeta)|)dm(\zeta) - (k-1)W\ln 3 \\ &\geq \int_{3\boldsymbol{T}} \ln|\lambda F_{N+1}(\zeta)|\,dm(\zeta) - \beta_{N+1} - (k-1)W\ln 3\,,\end{aligned}$$

and furthermore

$$\int_{3\boldsymbol{T}} \ln|\lambda F_{N+1}(\zeta)|\,dm(\zeta) \leq \int_{3\boldsymbol{T}} \ln|F_N(\zeta)|\,dm(\zeta) + \beta_{N+1} + (k-1)W\ln 3\,.$$

Induction gives the inequalities

$$(\dagger\dagger)\qquad \begin{aligned}\int_{3\boldsymbol{T}} \ln|F_{N+p}(\zeta)|\,dm(\zeta) &\leq \int_{3\boldsymbol{T}} \ln|F_N(\zeta)|\,dm(\zeta) + \sum_{N<s\leq N+p} \beta_s \\ &\quad + (k-1)W\ln 3 \sum_{0\leq s<p} k^s - p\ln|\lambda| \\ &\leq C + Wk^p\ln 3 - p\ln|\lambda|\,,\end{aligned}$$

for a certain constant $C > 0$ independent of $p \in \boldsymbol{N}$. Lemma 4.5 and the inequalities $(*)$ and $(\dagger\dagger)$ give the existence of a constant $C_1 > 0$ independent of p such that

$$\ln \int_{2\boldsymbol{T}} |F_{N+p}(\zeta)|\,dm(\zeta) \leq C_1 + C_1 k^p\,.$$

If H *is not a polynomial*, one can find an integer ν verifying

$$\nu > C_1 k^{1-N}/\ln 2\,, \quad \text{and} \quad A_\nu \neq 0\,.$$

But now, on the one hand Lemma 4.6 gives

$$\begin{aligned}\ln|\widehat{F_{N+p}}(\nu k^{N+p-1})| &\leq \ln\int_{2\boldsymbol{T}} |F_{N+p}(\zeta)|\,dm(\zeta) - \nu k^{N+p-1}\ln 2 \\ &\leq C_1 + C_1 k^p - \nu k^{N+p-1}\ln 2 \\ &= C_1 - (\nu k^{N-1}\ln 2 - C_1)k^p.\end{aligned}$$

On the other hand, by Lemma 4.8,

$$|\widehat{F_{N+p}}(\nu k^{N+p-1})| \geq |\lambda|^{-N-p}\frac{|A_\nu|}{k} > 0\,,$$

and therefore $\ln|\widehat{F_{N+p}}(\nu k^{N+p-1})| \geq C_2 - C_3 p$ for some constants C_2 and C_3 independent of p. This is a contradiction. □

5. The contracting germ. We have seen that the action of the group $\pi_1(S\setminus D)$ on $\boldsymbol{H}_l \times \boldsymbol{C}$ is generated by the two automorphisms

$$\begin{cases} g_\gamma(w,z) = (w + 2\pi i, z)\,, \\ g(w,z) \;= (kw, \lambda z + H(e^{-w}))\,,\end{cases}$$

where $H(\zeta) = \sum_{m=1}^{s} A_m\zeta^m$ is a polynomial in normal form, i.e., $A_m = 0$ for all $m > 0$ with $k|m$ and $A_s \neq 0$.

Let $l := [s/k] + 1$. We will conjugate our group by

$$\phi(w, z) = \left(w, z + \lambda^{-1} \sum_{m=1}^{l-1} A_m e^{-mw}\right).$$

This has no effect on g_γ, but

$$\phi \circ g \circ \phi^{-1}(w, z) = (kw, \lambda z + Q(e^{-w})),$$

where $Q(\zeta) = H(\zeta) - \sum_{m=1}^{l-1} A_m \zeta^m + \lambda^{-1} \sum_{m=1}^{l-1} A_m \zeta^{mk}$ is a polynomial of degree s with

$$\zeta^{min(k,l)} | Q(\zeta).$$

Iterating this procedure if necessary, we end up with a polynomial Q of degree s such that $\zeta^l | Q(\zeta)$. Let $Q(\zeta) := \sum_{m=l}^{s} b_m \zeta^m$ and $d :=$ g.c.d.$\{k,\ m \mid b_m \neq 0\}$.

We conjugate now with $\phi(w, z) = (dw, z)$:

$$\phi \circ g_\gamma \circ \phi^{-1}(w, z) = (w + 2\pi i d, z),$$

$$\phi \circ g \circ \phi^{-1}(w, z) = (kw, \lambda z + Q(e^{-w/d})).$$

One verifies directly that the group generated is the same as the group G' generated by

$$(w, z) \mapsto (w + 2\pi i, z),$$

$$(w, z) \mapsto (kw, \lambda z + Q(e^{-w/d})).$$

Let now

$$l' := [s/kd] + 1$$

and

$$Q'(\zeta) := \sum_{m=l}^{s} b_m \zeta^{m/d}.$$

Using the inequality $d[x/d] < [x] + 1, d \in \boldsymbol{N}^\star, x \in \boldsymbol{R}$ and the fact that the indices of the non-vanishing coefficients of Q' are divisible by d, one verifies easily that $\zeta^{l'} | Q'(\zeta)$. We conjugate now with $(w, z) \mapsto (w, e^{l'w} z)$ and the generators of G' become

$$(w, z) \mapsto (w + 2\pi i, z),$$

$$(w, z) \mapsto (kw, \lambda e^{l'(k-1)w} z + P(e^w)),$$

where P is the polynomial defined by

$$P(\xi) := \xi^{l'k} Q'(\xi^{-1}).$$

Remark that deg$P \leq l'(k-1)$ and that $P(0) = 0$. Let

$$P(\xi) = \sum_{m=1}^{l'(k-1)} c_m \xi^m.$$

We have g.c.d.$\{k, m \mid c_m \neq 0\} = 1$. This relation implies that the contracting germ

$$f : \Delta^* \times \boldsymbol{C} \to \Delta^* \times \boldsymbol{C},$$

$$f(\xi, z) := (\xi^k, \lambda \xi^{l'(k-1)} z + P(\xi))$$

is locally injective around $(0,0)$: If $f(\xi_1, z_1) = f(\xi_0, z_0)$, then $\xi_0^k = \xi_1^k$ and $\lambda \xi_0^{l'(k-1)} z_0 + P(\xi_0) = \lambda \xi_1^{l'(k-1)} z_1 + P(\xi_1)$. Put $\varepsilon := \xi_1/\xi_0$. One has $\varepsilon^k = 1$ and

$$z_1 = \varepsilon^{l'} \left[z_0 + \lambda^{-1} \xi_0^{-l'(k-1)} \sum_{m=1}^{l'(k-1)} c_m \xi_0^m (1 - \varepsilon^m) \right].$$

If $\varepsilon = 1$, one has $z_1 = z_0$. Otherwise, take m_0 the smallest index such that $c_{m_0} \neq 0$ and $\varepsilon^{m_0} \neq 1$. The existence of such an index is ensured by the relation

$$\text{g.c.d.}\{k, m \mid c_m \neq 0\} = 1 .$$

We write now

$$z_1 = \varepsilon^{l'} \left[z_0 + \lambda^{-1} \xi_0^{-l'(k-1)+m_0} \left(c_{m_0}(1 - \varepsilon^{m_0}) + \sum_{m=m_0+1}^{l'(k-1)} c_m \xi_0^{m-m_0} (1 - \varepsilon^m) \right) \right]$$

and we see that for z_0 and ξ_0 sufficiently small, z_1 stays away from 0. The local injectivity follows now directly.

By Proposition 1.2.8 of [9], we see that f is a defining germ for a minimal GSS surface S' whose maximal divisor we denote by D'. One can verify that the quotient of $\boldsymbol{H}_l \times \boldsymbol{C}$ by the action of G' is the same as the one of $\Delta^* \times \boldsymbol{C}$ by the equivalence relation $u_1 \sim u_2$: there exist $n_1, n_2 \in \boldsymbol{N}$ such that $f^{\circ n_1}(u_1) = f^{\circ n_2}(u_2)$. It follows by construction that $S \setminus D$ and $S' \setminus D'$ are isomorphic. Since the intersection matrices of D and of D' are negative-definite and neither D nor D' contain exceptional curves of the first kind, this isomorphism is extendable to an isomorphism of S onto S'. This completes the proof of the Main Theorem. □

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