CLASSES OF DISTRIBUTIONS APPLICABLE IN REPLACEMENT WITH RENEWAL THEORY IMPLICATIONS

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1. Introduction and summary

Age and block replacement policies are commonly used to diminish in-service failures. Unfortunately, for some items (say, those with decreasing failure rate), use of these policies may actually *increase* the number of in-service failures.

In this paper we determine the largest classes of life distributions for which age and block replacement diminishes, either stochastically or in expected value, the number of failures in service. We obtain bounds on survival probability, moment inequalities, and renewal quantity inequalities for distributions in these classes. We show that under certain reliability operations on components in a given class of life distributions (such as formation of systems, addition of life lengths, and mixtures of distributions), life distributions are obtained which remain within the class.

We consider items which perform a function that is to be continued over an indefinite period of time. To make this possible, an item which fails while in service is immediately replaced by a new item of the same kind.

Sometimes the interruption caused by an in-service failure is costly compared with the item replacement cost. If it is possible to make a "planned" replacement of an unfailed item, thus avoiding the high cost associated with a failure replacement, then planned replacements provide a practical means for avoiding reliance upon aged or worn items.

It has long been realized that for units with certain kinds of life distributions, planned replacements actually increase the frequency of failures. A goal of this paper is to identify the life distributions for which planned replacements are, or are not, beneficial.

We assume that the life lengths of all items to be placed in service are independent and have a common distribution F. Without further mention, we assume that $F(z) = 0$ for $z < 0$, and we denote the *survival function* by $\bar{F} \equiv 1 - F$.

The two planned replacement policies most commonly employed are age and block replacement. Under an *age replacement policy*, a unit is replaced upon failure or upon reaching a specified age T, whichever comes first. Under a *block replacement policy*, a replacement is made whenever a failure occurs and additionally at specified times $T, 2T, 3T, \cdots$. Age replacement results in fewer planned replacements, since replacements are planned according to a unit's age. On the other hand, block replacement can be scheduled in advance, and perhaps coordinated with the replacement of associated units.

Barlow and Proschan [3] have compared these replacement policies with respect to the number of failures, the number of planned replacements, and the total number of replacements by any time t. They reference several other authors who have studied these replacement policies.

Barlow and Proschan assume that the distribution F has an increasing failure rate. A distribution F is said to have an *Increasing Failure Rate* (IFR) if $\log \bar{F}$ is concave, that is, if for all $x > 0$, $\bar{F}(x + t)/\bar{F}(t)$ is decreasing in t such that $t \geqq 0$ and $\bar{F}(t) > 0$. If F has a density, this is equivalent to the condition that for some version f of the density, the failure rate $r(t) = f(t)/\bar{F}(t)$ is increasing in t for which $\bar{F}(t) > 0$.

We find two other classes of distributions important in the comparison of replacement policies.

A distribution F (or survival function $\bar{F}$) is said to be

(i) *New Better than Used* (NBU) if

$$\bar{F}(x + y) \leqq \bar{F}(x)\bar{F}(y) \tag{1.1}$$

for all $x, y \geqq 0$,

(ii) *New Better than Used in Expectation* (NBUE) if the mean μ of F is finite and

$$\mu \geqq \int_0^\infty \bar{F}(t + x)\, dx/\bar{F}(t) \tag{1.2}$$

for all $t \geqq 0$ such that $\bar{F}(t) > 0$. Notice that equation (1.1) can be interpreted as saying that the chance $\bar{F}(x)$ that a new unit will survive to age x is greater than the chance $\bar{F}(x + y)/\bar{F}(y)$ that an unfailed unit of age y will survive an additional time x. On the other hand, (1.2) says only that the expected life length of a new unit is greater than the expected remaining life of a used but unfailed unit.

Another class of distributions known to be important in reliability consists of the distributions with an *Increasing Failure Rate Average* (IFRA). A distribution F is said to be IFRA if $-[\log \bar{F}(t)]/t$ is increasing in $t > 0$, or equivalently, $[\bar{F}(s)]^{1/s} \geqq [\bar{F}(t)]^{1/t}$ for $0 < s < t$. Notice that if t is a multiple of s, this inequality is also satisfied when F is NBU.

The chain of implications

$$\text{IFR} \Rightarrow \text{IFRA} \Rightarrow \text{NBU} \Rightarrow \text{NBUE} \tag{1.3}$$

is readily established. To prove the last implication, one needs the fact that NBU distributions have finite means (proved in Section 4).

Each of the above classes of distributions has a companion class defined by reversing the inequality of the definition. Thus, we define *Decreasing Failure Rate* (DFR), *Decreasing Failure Rate Average* (DFRA), *New Worse than Used* (NWU) and *New Worse than Used in Expectation* (NWUE). The corresponding chain of implications

$$\text{DFR} \Rightarrow \text{DFRA} \Rightarrow \text{NWU} \tag{1.4}$$

holds. If F is NWU and if additionally it has a finite mean, then F is NWUE.

The classes NBU, NWU, NBUE and NWUE have received little attention in the literature in spite of their intuitive appeal. Their importance in replacement policy evaluation is demonstrated in Section 2, and their role in renewal theory is discussed in Section 3. Some basic inequalities for the classes are presented in Section 4, and the preservation of these classes under reliability operations is the subject of Section 5.

2. Replacement policy comparisons

In this section we obtain some comparisons for age and block replacement policies assuming that the underlying life distribution F belongs to an appropriate class. For these comparisons, we need the following notation:

$N(t)$ = number of failures (renewals) in $[0, t]$ for an ordinary renewal process, with no planned replacements. This quantity records the number of failures in $[0, t]$ if replacements are made only upon failure.

$N_A(t, T)$ = number of failures in $[0, t]$ under an age replacement policy with replacement age T.

$N_B(t, T)$ = number of failures in $[0, t]$ under a block replacement policy with replacement interval T.

These quantities do not record planned replacements, but only replacements due to an in-service failure. However, in-service failures are not recorded if they should happen to coincide with a planned replacement.

We caution the reader on one point: contrary to what is sometimes assumed in renewal theory (for example, by Feller, [8], p. 346), these quantities do *not* automatically count the origin as a renewal point (point of failure).

Y_1 = time of first failure when no planned replacements are made = $\inf\{t: N(t) \geqq 1\}$.

Y_i = length of time between $(i-1)$st and ith failure when no planned replacements are made = $\inf\{t: N(t) \geqq i\} - \inf\{t: N(t) \geqq i-1\}, i = 2, 3, \cdots$. Similarly define

$$Y_{i,A}(T), \qquad Y_{i,B}(T), \qquad i = 1, 2, \cdots \tag{2.1}$$

for the processes $N_A(t, T)$ and $N_B(t, T)$, respectively.

In the following theorems and lemma, we give the proof of the first of two parallel results. The second result in brackets in each case can be proved in the same way but with all inequalities reversed. (The symbol $\underset{\text{st}}{\leqq}$ will be used for "stochastically less than.")

THEOREM 2.1. $Y_i \underset{\text{st}}{\leqq} Y_{i,A}(T)$ $[Y_i \underset{\text{st}}{\geqq} Y_{i,A}(T)]$ *for all* $T > 0$, $i = 1, 2, \cdots \Leftrightarrow F$ *is* NBU [NWU].

PROOF. Note that Y_i and $Y_{i,A}(T)$ have distributions independent of i. Clearly,

$$P\{Y_1 > t\} = \bar{F}(t), \qquad P\{Y_{1,A}(T) > t\} = [\bar{F}(T)]^j \bar{F}(t - jT) \quad \text{for } jT \leqq t < (j+1)T, \quad j = 0, 1, \cdots. \tag{2.2}$$

If F is NBU, then $\bar{F}(t) \leqq [\bar{F}(T)]^j \bar{F}(t - jT)$ by a repeated application of the definition.

If $P\{Y_i > t\} \leqq P\{Y_{i,A}(T) > t\}$ for all t, T, take $T = \max(x, y)$, $t - T = \min(x, y)$ to obtain $\bar{F}(x + y) \leqq \bar{F}(x)\bar{F}(y)$. *Q.E.D.*

COROLLARY 2.1. $N(t) \underset{\text{st}}{\geqq} N_A(t, T)$ $[N(t) \underset{\text{st}}{\leqq} N_A(t, T)]$ *for all* $t, T > 0 \Leftrightarrow F$ is NBU [NWU].

PROOF. If F is NBU, then since $Y_1, Y_2, Y_3, \cdots$ are independent and $Y_{1,A}, Y_{2,A}, \cdots$ are independent, it follows from Theorem 2.1 that

$$\begin{aligned} P\{N(t) \geqq n\} &= P\{Y_1 + \cdots + Y_n \leqq t\} \\ &\geqq P\{Y_{1,A}(T) + \cdots + Y_{n,A}(T) \leqq t\} = P\{N_A(t, T) \geqq n\}. \end{aligned} \tag{2.3}$$

If $N(t) \underset{\text{st}}{\geqq} N_A(t, T)$ then

$$P\{Y_1 > t\} = P\{N(t) = 0\} \leqq P\{N_A(t, T) = 0\} = P\{Y_{1,A}(T) > t\}, \tag{2.4}$$

and hence F is NBU by Theorem 2.1. *Q.E.D.*

THEOREM 2.2. $\mu = EY_i \leqq EY_{i,A}(T)$ $[\mu \geqq EY_{i,A}(T)]$ *for all* $T > 0$, $i = 1, 2, \cdots \Leftrightarrow F$ *is* NBUE [NWUE].

PROOF. It is sufficient to prove the theorem for $i = 1$. We compute

$$\begin{aligned} EY_{1,A}(T) &= \int_0^\infty P\{Y_{1,A}(T) > t\}\, dt \\ &= \int_0^T \bar{F}(t)\, dt + \bar{F}(T) \int_T^{2T} \bar{F}(t - T)\, dt + \bar{F}^2(T) \int_{2T}^{3T} \bar{F}(t - 2T)\, dt + \cdots \\ &= \int_0^T \bar{F}(t)\, dt / F(T). \end{aligned} \tag{2.5}$$

(See [11].) But $\int_0^T \bar{F}(t)\, dt/F(T) \geqq \mu$ for all $T \Leftrightarrow$

$$\bar{F}(T) \int_0^\infty \bar{F}(t)\, dt = \int_0^\infty \bar{F}(t)\, dt - F(T) \int_0^\infty \bar{F}(t)\, dt \geqq \int_T^\infty \bar{F}(t)\, dt \tag{2.6}$$

for all T, that is, $\Leftrightarrow F$ is NBUE. *Q.E.D.*

Because the process $N_B(t, T)$ has more dependencies than the process $N_A(t, T)$, results for block replacement are not quite as easily obtained as for age replacement. We require the following lemma which is of some independent interest.

LEMMA 2.1. *Let planned replacements occur at fixed time points* $0 < t_1 < t_2 < \cdots$ *under Policy* 1, *and at these and the additional point* $t_0 > 0$ *under Policy* 2. *Let* $N_i(t)$ *be the number of failures in* $[0, t]$ *under Policy* i, $i = 1, 2$. *Then* $N_1(t) \underset{\text{st}}{\geqq} N_2(t)$ $[N_1(t) \underset{\text{st}}{\leqq} N_2(t)]$ *for all* $t > 0$ *and all* $t_0, t_1, t_2, \cdots \Leftrightarrow F$ *is* NBU [NWU].

PROOF. Suppose first that F is NBU. For $t < t_0$, $N_1(t)$ and $N_2(t)$ have the same distribution. Next assume that $t_0 \leqq t \leqq t_k$ where t_k is the smallest $t_j > t_0$; take $t_k = \infty$ if $t_0 > t_j$ for all $j > 0$. Let Z be the age of the unit in operation at time t_0^-; the distribution of Z does not depend upon the policy. Let τ_i denote the interval between t_0 and the time of first failure subsequent to t_0^- under policy i, $i = 1, 2$. Since F is NBU,

$$P\{\tau_1 > t | Z\} \leqq P\{\tau_2 > t | Z\} \quad \text{for all} \quad t \geqq 0. \tag{2.7}$$

Let U_i be the number of failures in $[t_0, t]$ under policy i, $i = 1, 2$. If $X_1, X_2, \cdots$ are independent and have distribution F,

$$\begin{aligned} P\{U_1 \geqq n | Z\} &= P\{t_0 + \tau_1 + X_1 + \cdots + X_{n-1} \leqq t | Z\} \\ &\geqq P\{t_0 + \tau_2 + X_1 + \cdots + X_{n-1} \leqq t | Z\} = P\{U_2 \geqq n | Z\} \end{aligned} \tag{2.8}$$

for $n = 0, 1, 2, \cdots$. By unconditioning on Z, we conclude that $P\{U_1 \geqq n\} \geqq P\{U_2 \geqq n\}$, $n = 0, 1, 2, \cdots$. Thus $N_1(t) \underset{\text{st}}{\geqq} N_2(t)$.

Finally, assume that $t > t_k$. Let $N_i(t_k, t)$ denote the number of failures in $(t_k, t]$ under Policy i, $i = 1, 2$. Then $N_i(t_k) + N_i(t_k, t) = N_i(t)$, with $N_i(t_k)$ and $N_i(t_k, t)$ independent, $i = 1, 2$. Since $N_1(t_k) \underset{\text{st}}{\geqq} N_2(t_k)$ and $N_1(t_k, t) \underset{\text{st}}{\equiv} N_2(t_k, t)$, we conclude that $N_1(t) \underset{\text{st}}{\geqq} N_2(t)$.

Now suppose $N_1(t) \underset{\text{st}}{\geqq} N_2(t)$ for all t and all $t_0, t_1, \cdots$. Choose $0 < t_0 < t_1$. Then

$$\bar{F}(t_1) = P\{N_1(t_1) = 0\} \leqq P\{N_2(t_1) = 0\} = \bar{F}(t_0)\bar{F}(t_1 - t_0). \tag{2.9}$$

Since $0 < t_0 < t_1$ are arbitrary, F is NBU. *Q.E.D.*

THEOREM 2.3. $Y_i \underset{\text{st}}{\leqq} Y_{i,B}(T)$ $[Y_i \underset{\text{st}}{\geqq} Y_{i,B}(T)]$ *for all* $T > 0$, $i = 1, 2, \cdots \Leftrightarrow$ F *is* NBU [NWU].

PROOF. For given $S_{i-1,B} = Y_{1,B}(T) + \cdots + Y_{i-1,B}(T)$, let k be the smallest integer for which $kT \geqq S_{i-1,B}$. Then $P\{Y_{i,B}(T) > t | S_{i-1,B}\} = P\{Y_1^* > t\}$, where Y_1^* is the time to first failure when planned replacements are made at $kT - S_{i-1,B} = \delta, \delta + T, \delta + 2T, \cdots$.

If F is NBU, apply Lemma 2.1 to compare successive pairs of a sequence of replacement policies, in which the ith policy calls for planned replacement at time points $\{0, \delta, T + \delta, 2T + \delta, \cdots, iT + \delta\}$. This comparison yields

$$\bar{F}(t) = P\{Y_i > t\} = P\{Y_1 > t\} \leqq P\{Y_1^* > t\} \qquad \text{for all} \quad t \geqq 0, \tag{2.10}$$

and $Y_i \underset{\text{st}}{\leqq} Y_{i,B}(T)$ follows upon unconditioning.

The converse follows as in Theorem 2.1 since $Y_{1,B}(T)$ and $Y_{1,A}(T)$ have the same distribution. *Q.E.D.*

THEOREM 2.4. *$N(t) \underset{\text{st}}{\geqq} N_B(t, T)$ $[N(t) \underset{\text{st}}{\leqq} N_B(t, T)]$ for all $t \geqq 0$, $T > 0 \Leftrightarrow$ F is* NBU [NWU].

PROOF. If F is NBU, apply Lemma 2.1 to compare successive pairs of a sequence of replacement policies in which the jth policy calls for planned replacement at time points $0, T, \cdots, (j-1)T$. The converse follows as in the converse of Corollary 2.1 with $i = 1$. *Q.E.D.*

We remark that $N(t) \underset{\text{st}}{\geqq} N_B(t, T)$ is equivalent to $Y_1 + \cdots + Y_n \underset{\text{st}}{\leqq} Y_{1,B}(T) + \cdots + Y_{n,B}(T)$, $n = 1, 2, \cdots$. This result is not immediate from Theorem 2.3 because the $Y_{i,B}(T)$ are not independent.

THEOREM 2.5. *$\mu = EY_i \leqq EY_{i,B}(T)$ $[EY_i \geqq EY_{i,B}(T)]$ for all $T > 0$, $i = 1, 2, \cdots \Leftrightarrow F$ is* NBUE. [NWUE].

PROOF. Suppose F is NBUE. Let $S_{i-1,B}$ and δ be as in the proof of Theorem 2.3. Then

$$\begin{aligned} E[Y_{i,B}(T) | S_{i-1,B}] = & \int_0^\delta \bar{F}(t)\,dt + \bar{F}(\delta) \int_\delta^{\delta+T} \bar{F}(t-\delta)\,dt \\ & + \bar{F}(\delta)\bar{F}(T) \int_{\delta+T}^{\delta+2T} \bar{F}(t-\delta-T)\,dt + \cdots \\ & + \bar{F}(\delta)[\bar{F}(T)]^{j-1} \int_{\delta+jT}^{\delta+(j+1)T} \bar{F}(t-\delta-jT)\,dt + \cdots \\ = & \int_0^\delta \bar{F}(t)\,dt + \bar{F}(\delta) \int_0^T \bar{F}(t)\,dt/F(T). \end{aligned} \tag{2.11}$$

But for all $T, \delta > 0$,

$$\int_0^\delta \bar{F}(t)\,dt/F(\delta) \geqq \mu \qquad \text{and} \qquad \int_0^T \bar{F}(t)\,dt/F(T) \geqq \mu. \tag{2.12}$$

Thus

$$\int_0^\delta \bar{F}(t)\,dt + \bar{F}(\delta) \int_0^T \bar{F}(t)\,dt/F(T) \geqq \mu F(\delta) + \mu \bar{F}(\delta) = \mu. \tag{2.13}$$

It follows that $E[Y_{i,B}(T) | S_{i-1,B}] \geqq \mu$.

The converse follows as in Theorem 2.2 since $Y_{1,B}(T)$ and $Y_{1,A}(T)$ are identically distributed. *Q.E.D.*

The preceding results compare policies with and without planned replacements. We consider now the comparison of age replacement policies with differing replacement age T.

THEOREM 2.6. *$N_A(t, T)$ is stochastically increasing [decreasing] in $T > 0$ for each fixed $t \Leftrightarrow F$ is* IFR [DFR].

PROOF. Suppose F is IFR. For fixed $T > 0$, $\{N_A(t, T), t \geqq 0\}$ is a renewal process with underlying distribution

$$(2.14)\quad S_T(x) = 1 - [\bar{F}(T)]^n \bar{F}(x - nT), \quad nT \leqq x < (n + 1)T, n = 0, 1, \cdots.$$

Barlow and Proschan ([4], p. 61) show that $S_T(x)$ is increasing in $T > 0$ for fixed $x \geqq 0$. Hence the nth convolution $S_T^{(n)}(x) = P\{N_A(t, T) \geqq n\}$ is increasing in $T > 0$ for fixed $t \geqq 0$.

Barlow and Proschan ([4], p. 61) also show that $P\{N_A(t, T) = 0\}$ decreasing in $T > 0$ for all $t \geqq 0$ implies F is IFR, and this completes the proof. *Q.E.D.*

It is possible, under weaker conditions, to compare two age replacement policies for which the planned replacement age of one policy is a multiple of that of the other policy. This comparison is in fact a generalization of Corollary 2.1, obtained by setting $k > T/t$ below.

THEOREM 2.7. *$N_A(t, kT) \underset{st}{\geqq} N_A(t, T)$ $[N_A(t, kT) \underset{st}{\leqq} N_A(t, T)]$ for all $t \geqq 0$, $T > 0, k = 1, 2, \cdots \Leftrightarrow F$ is* NBU [NWU].

PROOF. Suppose first that F is NBU. With the notation introduced in the proof of Theorem 2.6, we have, for $nT \leqq x < (n + 1)T$,

$$(2.15)\quad \bar{S}_T(x) - \bar{S}_{kT}(x) = [\bar{F}(T)]^n \bar{F}(x - nT) - [\bar{F}(kT)]^{[n/k]} \bar{F}(x - [n/k]kT).$$

Since F is NBU,

$$(2.16)\quad [\bar{F}(kT)]^{[n/k]} \leqq [\bar{F}(T)]^{[n/k]k},$$

and

$$(2.17)\quad \bar{F}(x - [n/k]kT) \leqq [\bar{F}(T)]^{n-k[n/k]} \bar{F}(x - nT).$$

Thus $\bar{S}_T(x) \geqq \bar{S}_{kT}(x)$. Hence

$$(2.18)\quad P\{N_A(t, kT) \geqq n\} = S_{kT}^{(n)}(t) \geqq S_T^{(n)}(t) = P\{N_A(t, T) \geqq n\}$$

for $t \geqq 0, T > 0$, that is, $N_A(t, kT) \underset{st}{\geqq} N_A(t, T)$.

Next suppose $N_A(t, 2T) \underset{st}{\geqq} N_A(t, T)$ for all $t, T > 0$. Take $T = \max(x, y)$ and $t - T = \min(x, y)$. Then $t \leqq 2T$, so that

$$(2.19)\quad \bar{F}(x)\bar{F}(y) = \bar{F}(T)\bar{F}(t - T) = P\{N_A(t, T) = 0\} \geqq P\{N_A(t, 2T) = 0\} = \bar{F}(t) = \bar{F}(x + y).$$

THEOREM 2.8. *$EY_{i,A}(T)$ is decreasing [increasing] in $T > 0$, $i = 1, 2, \cdots \Leftrightarrow$*

$$(2.20)\quad \int_0^T \bar{F}(t)\, dt/F(T) \text{ is decreasing [increasing] in } T \text{ such that } F(T) > 0.$$

PROOF. We have seen in the proof of Theorem 2.2 that

$$(2.21)\quad EY_{i,A}(T) = \int_0^T \bar{F}(t)\, dt/F(T). \qquad Q.E.D.$$

The condition that $\int_0^T \bar{F}(t)\,dt/F(T)$ is decreasing in T has to our knowledge not been encountered previously in reliability theory. Its meaning or significance, apart from that given by Theorem 2.8, is not presently clear. One can easily show that F IFR $\Rightarrow$ (2.20) $\Rightarrow$ F NBUE, and that the conditions are distinct.

In view of Theorem 2.6, and because the condition there that F is IFR means $\bar{F}(x + t)/\bar{F}(t)$ is decreasing in t, one might have expected in place of (2.20) to have encountered in Theorem 2.8 the condition that

$$\int_0^\infty \bar{F}(x + t)\,dx/\bar{F}(t) = \int_t^\infty \bar{F}(x)\,dx/\bar{F}(t) \tag{2.22}$$

is decreasing in $t \geqq 0$ such that $\bar{F}(t) > 0$. A distribution which satisfies (2.22) is said to have a *Decreasing Mean Residual Life* (DMRL). It is of some interest that (2.20) and (2.22) are not related. If $\bar{F}(x) = e^{-x}$ for $0 \leqq x < 1$ and $\bar{F}(x) = e^{-2x}$ for $x \geqq 1$, then F satisfies (2.20) (and also F is IFRA) but it is not DMRL because of the discontinuity at 1 within its interval of support. On the other hand, if $\bar{F}(x) = e^{-x}$ for $0 \leqq x < 1$, $\bar{F}(x) = e^{-1}$ for $1 \leqq x < 2$, and $\bar{F}(x) = 0$ for $x \geqq 2$, then F is DMRL, but F does not satisfy (2.20) because its support is not an interval (neither is F IFRA, although F is NBUE as a consequence of its being DMRL).

Results parallel to Theorems 2.6 and 2.8 for block replacement are unknown. A theorem identical with Theorem 2.7 except with block in place of age replacement is easily obtained using Lemma 2.1.

3. Renewal theory inequalities

We obtain here several results which hold for renewal processes when times between failures have a distribution that is NBU or NBUE. The first of these, like Theorems 2.1 and 2.2, provides a characterization of the NBU class. A similar characterization of the NBUE class is also obtained. The renewal theory implications obtained in Propositions 3.1 through 3.9 and Example 3.1 below are summarized in Figure 1. Following this, moment inequalities are obtained for the renewal quantity; in each case the Poisson process yields a bound.

In cases where there is a parallel result with inequalities reversed, we give only one proof, as in Section 2.

Again, we caution the reader. $N(t)$ does *not* count the origin as a renewal point (unless the initial item placed in service has life length zero).

For any two random variables U and V, dependent or not, we write $U * V$ to represent a random variable with a distribution that is the convolution of the distributions of U and V. For any distribution function F, $F^{(n)}$ denotes the nth convolution of F, and $F^{(0)}$ is degenerate at 0.

PROPOSITION 3.1. *$N(s) * N(t) \underset{\text{st}}{\leqq} N(s + t)$ $[N(s) * N(t) \underset{\text{st}}{\geqq} N(s + t)]$ for all s, $t \underset{\text{st}}{\geqq} 0 \Leftrightarrow F$ is* NBU [NWU].

PROOF. Suppose first that F is NBU. Then the result follows from Lemma 2.1 with $t_0 = s$, $t_1 > s + t$. Next, suppose that $N(s) * N(t) \leqq N(s + t)$. Then

$P\{N(s+t)=0\} \leqq P\{N(s)*N(t)=0\} = P\{N(s)=0\}P\{N(t)=0\}$, which is the condition that F is NBU. *Q.E.D.*

Denote the distribution of time between u and the next following renewal by F_u; it is often called the distribution of residual life at time u. In order to write F_u in terms of F, it is convenient to use the standard notation $M(t) = EN(t)$.

PROPOSITION 3.2. *F is* NBU [NWU] $\Rightarrow \bar{F}_u(t) \leqq \bar{F}(t)$ $[\bar{F}_u(t) \geqq \bar{F}(t)]$ *for all* $t, u \geqq 0$.

PROOF. If F is NBU, then

$$(3.1) \qquad \begin{aligned} \bar{F}_u(t) &= \bar{F}(t+u) + \int_0^u \bar{F}(t+u-z)\, dM(z) \\ &\leqq \bar{F}(t)\bar{F}(u) + \int_0^u \bar{F}(t)\bar{F}(u-z)\, dM(z) \\ &= \bar{F}(t)\bar{F}_u(0) \leqq \bar{F}(t). \end{aligned}$$

Q.E.D.

The process $\{N_u(t), t \geqq 0\}$, where $N_u(t) = N(t+u) - N(u)$, is a modified renewal process in which the distribution of time to first renewal is F_u, and the distribution of time between successive renewals is F.

PROPOSITION 3.3. *For each* $u \geqq 0$, $\bar{F}_u(t) \leqq \bar{F}(t)$ $[\bar{F}_u(t) \geqq \bar{F}(t)]$ *for all* $t \geqq 0$ $\Leftrightarrow N(t) \underset{\text{st}}{\leqq} N_u(t)$ $[N(t) \underset{\text{st}}{\geqq} N_u(t)]$ *for all* $t \geqq 0$.

PROOF. Suppose first that $\bar{F}_u(t) \leqq \bar{F}(t)$ for all $t, u \geqq 0$. Then for $n > 0$,

$$(3.2) \qquad \begin{aligned} P\{N(t) \geqq n\} &= \int_0^t F^{(n-1)}(t-x)\, dF(x) \\ &\leqq \int_0^t F^{(n-1)}(t-x)\, dF_u(x) = P\{N_u(t) \geqq n\}. \end{aligned}$$

Next, suppose that $N(t) \underset{\text{st}}{\leqq} N_u(t)$ for all $t \geqq 0$. Then $P\{N(t)=0\} \geqq P\{N_u(t) = 0\}$, that is, $\bar{F}_u(t) \leqq \bar{F}(t)$. *Q.E.D.*

We have shown in Proposition 3.2 that F is NBU implies $\bar{F}_u(t) \leqq \bar{F}(t)$ for all $t, u \geqq 0$, but the truth of the converse has not been determined. The following proposition provides a weaker conclusion.

PROPOSITION 3.4. $\bar{F}_u(t) \leqq \bar{F}(t)$ *for all* $t, u \geqq 0 \Rightarrow F$ is NBUE.

PROOF. It is well known (see, for example, Feller (1968), p. 355) that $\lim_{u\to\infty} \bar{F}_u(t) = \mu^{-1}\int_t^\infty \bar{F}(z)\, dt$ if F has a finite mean μ, and otherwise $\lim_{u\to\infty} \bar{F}_u(t) \equiv 1$. Since $\bar{F}_u(t) \leqq \bar{F}(t)$, we can choose t for which $\bar{F}(t) < 1$ to conclude that $\mu < \infty$. Then $\lim_{u\to\infty} \bar{F}_u(t) = \mu^{-1}\int_t^\infty \bar{F}(z)\, dz \leqq \bar{F}(t)$ is the desired result. *Q.E.D.*

The parallel result, that $\bar{F}_u(t) \geqq \bar{F}(t)$ for all $t, u \geqq 0 \Rightarrow F$ is NWUE, is true only with the additional assumption that $\mu < \infty$.

The converse of Proposition 3.4 is false, as we shall later show in Example 3.1.

The class of NBUE distributions can also be characterized in terms of a renewal quantity stochastic ordering. Let $\hat{N}(t)$ = number of failures in $[0, t]$

for a stationary renewal process. The definition of this modified renewal process requires $\mu < \infty$. The distribution of time to first renewal has density $\bar{F}(t)/\mu$, $t > 0$, and the distribution of subsequent interrenewal times is F.

PROPOSITION 3.5. *F is* NBUE [NWUE] iff $N(t) \underset{\text{st}}{\leqq} \hat{N}(t)$ $[N(t) \underset{\text{st}}{\geqq} \hat{N}(t)]$ *for all* $t > 0$.

This result is in fact a special case of Proposition 3.3, obtained by letting $u \to \infty$. Alternatively, it can be directly proved using the argument of Proposition 3.3 by taking $\hat{N}$ in place of N_u and $\mu^{-1} \int_0^\infty \bar{F}(t + x)\, dx$ in place of $\bar{F}_u(t)$. That F is NBUE implies $N(t) \underset{\text{st}}{\leqq} \hat{N}(t)$ was obtained by Barlow and Proschan ([3], Theorem 4.1).

Let us turn now to some results concerning the renewal function $M(t) = EN(t)$.

PROPOSITION 3.6. $\bar{F}_u(t) \leqq \bar{F}(t)$ $[\bar{F}_u(t) \geqq \bar{F}(t)]$ *for all* $u, t > 0 \Rightarrow M(s + t) \geqq M(s) + M(t)$ $[M(s + t) \leqq M(s) + M(t)]$ *for all* $s, t > 0$.

PROOF. $M(s + t) - M(t) = \sum_{k=0}^\infty (F_t * F^{(k)})(s) \geqq \sum_{k=0}^\infty (F * F^{(k)})(s) = M(s)$.

PROPOSITION 3.7. $M(s + t) \geqq M(s) + M(t) \Rightarrow M(t) \leqq t/\mu$.

PROOF. Since $\lim_{s\to\infty} [M(s)/s] = 1/\mu$, this result is trivial. *Q.E.D.*

Again the parallel result, that $M(s + t) \leqq M(s) + M(t) \Rightarrow M(t) \geqq t/\mu$, requires the additional assumption that $\mu < \infty$.

PROPOSITION 3.8. *F is* NBUE [NWUE] $\Rightarrow M(t) \leqq t/\mu$ $[M(t) \geqq t/\mu]$.

This result was obtained by Barlow and Proschan [3], and is immediate from Proposition 3.5 upon taking expectations. It is extended below in Proposition 3.10 and 3.11.

PROPOSITION 3.9. *F is* NBU $\Rightarrow$ Var $N(t) \leqq M(t)$.

PROOF. The argument given by Barlow and Proschan (1964, Theorem 4.2) applies, as their hypothesis that F is IFR is unnecessarily strong.

A weaker upper bound for Var $N(t)$ has been obtained by Esary, Marshall and Proschan ([7], Section 6) which holds for general interrenewal time distributions F. They show that

$$\text{Var } N(t) \leqq M(t) + [M(t)]^2. \tag{3.3}$$

The following example is of interest, as it provides a counterexample to converses of several preceding propositions.

EXAMPLE 3.1. Suppose that F places mass $\frac{1}{2}$ at 1 and mass $\frac{1}{2}$ at 3. Then F has mean $\mu = 2$ and $\int_0^\infty \bar{F}(t + x)\, dx/\bar{F}(t) \leqq 2$ for $t \geqq 0$, so that F is NBUE.

Observe that for this example, $M(1+) = \frac{1}{2}$ and $M(2+) = 1(\frac{1}{2})^2 + 2(\frac{1}{2})^2 = \frac{3}{4}$ so that $M(2+) < M(1+) + M(1+)$. Hence F is NBUE $\not\Rightarrow M(s + t) \geqq M(s) + M(t)$ for all $s, t \geqq 0$. Consequently it cannot be that F is NBUE $\Rightarrow \bar{F}_u(t) \leqq \bar{F}(t)$, because by Proposition 3.6, this would be an even stronger conclusion than $M(s + t) \geqq M(s) + M(t)$.

Of course for this F, we still have from Proposition 3.8 that $M(t) \leqq t/\mu$, so that the converse of Proposition 3.7 is false.

Figure 1 summarizes the results of Propositions 3.1 to 3.8 and Example 3.1. Also indicated are some dotted implications which have not yet been proved or disproved.

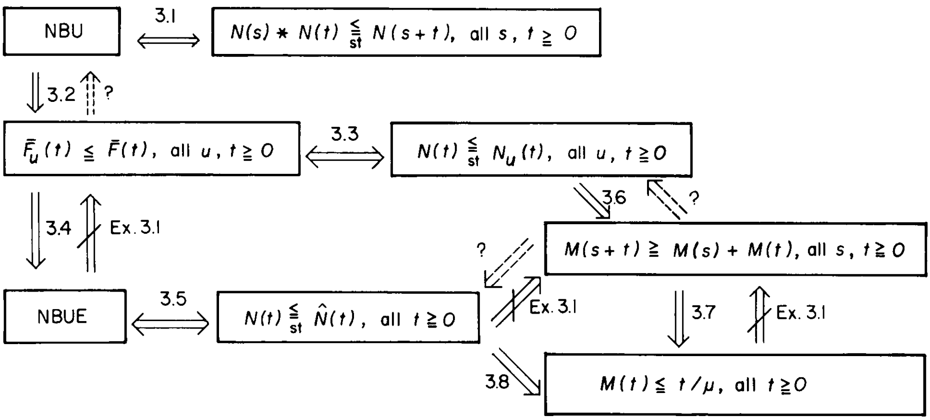

FIGURE 1
Summary of the results of Propositions 3.1 to 3.8 and Example 3.1.

Let us now consider a generalized renewal process in which the interrenewal times have not necessarily identical distributions. We denote the distribution of time to first renewal by F_1, and the distribution of time from the jth to $(j + 1)$st renewal by $F_{j+1}, j = 1, 2, \cdots$. For convenience we often write $F^{[j]}$ in place of $F_1 * \cdots * F_j$. Denote a generalized renewal process by $\{N_0(t), t \geqq 0\}$, and let $M_0(t) = EN_0(t)$. Then M_0 is given by

$$M_0(t) = \sum_{j=1}^{\infty} F^{[j]}(t), \; t \geqq 0. \tag{3.4}$$

PROPOSITION 3.10. *If M_0 is a generalized renewal function in which each F_i has common mean μ and is* NBUE [NWUE], *then*

$$M_0(t) \leqq \frac{t}{\mu} \quad \left[M_0(t) \geqq \frac{t}{\mu}\right] \quad \text{for all} \quad t \geqq 0. \tag{3.5}$$

PROOF. First, note that by Proposition 3.8, the result holds when the F_i are identical. We shall proceed to prove the result by induction.

Suppose that the result holds whenever $F_k \equiv F_{k+1} \equiv \cdots$ (that is, only the first k distributions can differ), and consider a renewal process where $F_{k+1} \equiv F_{k+2} \equiv \cdots$. Then by the induction hypothesis the result holds when interrenewal time distributions are $F_2, F_3, \cdots$, since only the first k distributions can differ. Thus,

$$\begin{aligned} M_0(t) &= \int_0^t \left[1 + \sum_{j=1}^{\infty} F_2 * \cdots * F_{j+1}(t - x)\right] dF_1(x) \\ &\leqq \int_0^t \left[1 + \frac{t - x}{\mu}\right] dF_1(x) = \frac{t}{\mu} + F_1(t) - \frac{1}{\mu}\int_0^t \bar{F}_1(x)\, dx \leqq \frac{t}{\mu} \end{aligned} \tag{3.6}$$

since F_1 is NBUE. The result follows by taking the limit as $k \to \infty$. *Q.E.D.*

The result of Proposition 3.10 can be viewed as a comparison with the Poisson process in which each of the interrenewal times is exponential with mean μ. For this process $M_0(t) \equiv t/\mu$. Other moment comparisons with the Poisson process were made by Barlow and Proschan [3] that hold also for generalized renewal processes.

PROPOSITION 3.11. *If N_0 is a generalized renewal process in which each interrenewal time distribution F_i has a common mean μ and is NBUE, then*

$$E\binom{N_0(t) + n}{m} \leqq \sum_{j=0}^{\infty} \binom{j + n}{m} e^{-t/\mu} \left(\frac{t}{\mu}\right)^j \frac{1}{j!}, \tag{3.7}$$

$$EN_0^n(t) \leqq \sum_{j=0}^{\infty} j^n e^{-t/\mu} \left(\frac{t}{\mu}\right)^j \frac{1}{j!}, \tag{3.8}$$

for $m, n = 0, 1, 2, \cdots$, and $0 \leqq t < \infty$. The reverse inequalities hold if F_i is NWUE.

PROOF OF (3.7). First, assume that $n = 0$, $m = 0$; then both sides equal one. Assume that (3.7) holds for $n = 0$, and $m = 0, 1, \cdots, k - 1$. Then, with $G(t) = 1 - e^{-t/\mu}$,

$$\begin{aligned} E\binom{N(t)}{k} \tag{3.9} &= \sum_{i=k}^{\infty} \binom{i}{k} [F^{[i]}(t) - F^{[i+1]}(t)] = \sum_{i=k}^{\infty} \binom{i-1}{k-1} F^{[i]}(t) \\ &= \sum_{\ell=0}^{\infty} \binom{k+\ell-2}{k-2} \sum_{i=1}^{\infty} F^{[k+\ell-1+i]}(t) \\ &= \sum_{\ell=0}^{\infty} \binom{k+\ell-2}{k-2} \sum_{i=1}^{\infty} \int F_{k+\ell} * \cdots * F_{k+\ell-1+i}(t - x)\, dF^{[k+\ell-1]}(x) \\ &\leqq \sum_{\ell=0}^{\infty} \binom{k+\ell-2}{k-2} \sum_{i=1}^{\infty} \int G^{(i)}(t - x)\, dF^{[k+\ell-1]}(x) \\ &= \sum_{\ell=0}^{\infty} \binom{k+\ell-2}{k-2} \sum_{i=1}^{\infty} \int F^{[k+\ell-1]}(t - x)\, dG^{(i)}(x) \\ &\leqq \sum_{i=1}^{\infty} \int \sum_{\ell=0}^{\infty} \binom{k+\ell-2}{k-2} G^{(k+\ell-1)}(t - x)\, dG^{(0)}(x) \\ &= \sum_{i=k}^{\infty} \binom{i-1}{k-1} G^{(i)}(t) = \sum_{j=0}^{\infty} \binom{j}{k} e^{-t/\mu} \left(\frac{t}{\mu}\right)^j \frac{1}{j!}. \end{aligned}$$

The first inequality above follows from Proposition 3.10; the second inequality follows from the inductive hypothesis.

To establish (3.7) for $n = 1, 2, \cdots$, use induction and the identity

$$\binom{i+n}{m} = \binom{i+n-1}{m-1} + \binom{i+n-2}{m-1} + \cdots + \binom{m-1}{m-1} \tag{3.10}$$

for $i + n > m$.

4. Moment inequalities

In this section we obtain some inequalities for distributions that are NBU, NWU, NBUE, or NWUE. We use the notation

$$\mu_r = \int x^r \, dF(x), \qquad \lambda_r = \mu_r/\Gamma(r+1), \qquad r \geqq 0. \tag{4.1}$$

Barlow, Marshall, and Proschan [2] have shown that if F is IFR, then λ_{r+s}/λ_r is decreasing in $r \geqq 0$ for fixed $s \geqq 0$. A somewhat weaker condition, that $\lambda_r^{1/r}$ is decreasing in $r > 0$, holds when F is IFRA [7]. These results can be expressed somewhat differently as:

(i) if $\log \bar{F}(t)$ is concave in $t \geqq 0$, then $\log \lambda_r$ is concave in $r \geqq 0$;

(ii) if $-\log \bar{F}(t)$ is starshaped in $t \geqq 0$, then $-\log \lambda_r$ is starshaped in $r \geqq 0$. (ϕ is *starshaped* on $[0, \infty)$ if $\phi(ax) \leqq a\phi(x)$ for all $x \geqq 0$, $0 \leqq a \leqq 1$.)

The condition that F is NBU is just the condition that $-\log \bar{F}(t)$ is superadditive in $t \geqq 0$; consequently it is natural to conjecture that

(iii) if $-\log \bar{F}(t)$ is superadditive in $t \geqq 0$, then $-\log \lambda_r$ is superadditive in $r \geqq 0$. The truth of this conjecture is established in the following theorem.

THEOREM 4.1. *If F is* NBU [NWU], *then*

$$\lambda_{r+s} \leqq \lambda_r \lambda_s \qquad [\lambda_{r+s} \geqq \lambda_r \lambda_s] \tag{4.2}$$

for all $r, s \geqq 0$.

PROOF. If F is NBU, then $\bar{F}(x+y) \leqq \bar{F}(x)\bar{F}(y)$, and so

$$\frac{x^{r-1}}{\Gamma(r)} \frac{y^{s-1}}{\Gamma(s)} \bar{F}(x+y) \leqq \frac{x^{r-1}\bar{F}(x)}{\Gamma(r)} \frac{y^{s-1}\bar{F}(y)}{\Gamma(s)}, \qquad x, y \geqq 0. \tag{4.3}$$

It follows that

$$\int_0^\infty \int_0^\infty \frac{x^{r-1}}{\Gamma(r)} \frac{x^{s-1}}{\Gamma(s)} \bar{F}(x+y) \, dx dy \tag{4.4}$$

$$\leqq \int_0^\infty \frac{x^{r-1}\bar{F}(x)}{\Gamma(r)} dx \int_0^\infty \frac{y^{s-1}\bar{F}(y)}{\Gamma(s)} dy = \lambda_r \lambda_s.$$

The left member of this inequality is

$$(4.5)\quad \int_0^\infty \int_0^\infty \frac{x^{r-1}}{\Gamma(r)} \frac{y^{s-1}}{\Gamma(s)} \int_{x+y}^\infty dF(z)\, dxdy = \int_0^\infty dF(z) \int_0^z \frac{x^{r-x}}{\Gamma(r)}\, dx \int_0^{z-1} \frac{y^{s-1}}{\Gamma(s)}\, dy$$

$$= \int_0^\infty dF(z) \int_0^z \frac{x^{r-1}(z-x)^s}{\Gamma(r)\Gamma(s+1)}\, dx$$

$$= \int_0^\infty dF(z) \frac{z^{r+s}}{\Gamma(r+s+1)} = \lambda_{r+s}.$$

In case F is NWU, a proof is obtained by reversing the above inequalities.

THEOREM 4.2. *If F is* NBUE [NWUE], *then*

$$(4.6)\quad \lambda_{r+1} \leqq \lambda_r \lambda_1 \qquad [\lambda_{r+1} \geqq \lambda_r \lambda_1]$$

for all $r > 0$.

PROOF. If F is NBUE, then $\int_t^\infty \bar{F}(x)\, dx \leqq \mu_1 \bar{F}(t)$ so that

$$(4.7)\quad \int_0^\infty \frac{t^{r-1}}{\Gamma(r)} \int_t^\infty \bar{F}(x)\, dxdt \leqq \mu_1 \int_0^\infty \frac{t^{r-1}}{\Gamma(r)} \bar{F}(t)\, dt = \lambda_1 \lambda_r.$$

By interchanging the order of integration, we easily compute that the left side of this inequality is λ_{r+1}. If F is NWUE, the proof is modified by reversing inequalities. *Q.E.D.*

Notice that with $r = 1$ in Theorem 4.2 we obtain that if F is NBUE[NWUE] then the *coefficient* of *variation* $\sigma/\mu \leqq 1$ $[\sigma/\mu > 1]$, where σ is the standard deviation of F.

We have in the previous theorems ignored an interesting question: are the moments finite?

PROPOSITION 4.1. *If F is* NBU, *then $\mu_r < \infty$ for all $r > 0$.*

PROOF. Choose $t < \infty$ such that $\bar{F}(t) < 1$. Since $\bar{F}$ is monotone and NBU, it follows that if x and k satisfy $kt \leqq x < (k+1)t$, then

$$(4.8)\quad \bar{F}(x) \leqq \bar{F}(kt) \leqq [\bar{F}(t)]^k \leqq [\bar{F}(t)]^{(x/t)-1}$$

Thus

$$(4.9)\quad \mu_r = r \int_0^\infty x^{r-1} \bar{F}(x)\, dx$$

$$\leqq r \int_0^t x^{r-1}\, dx + r \int_t^\infty x^{r-1} [\bar{F}(t)]^{(x/t)-1}\, dx < \infty. \ Q.E.D.$$

If F is NWU, μ_r need not be finite for any $r > 0$. In fact, Barlow, Marshall and Proschan [2] have pointed out following their Theorem 6.2 that a DFR distribution (which, of course, is NWU) may have infinite moments of all positive orders.

We already know, as an integral part of the definition, that NBUE and NWUE distributions have a finite mean.

PROPOSITION 4.2. *If F is* NBUE, *then $\mu_r < \infty$ for all $r > 0$.*

PROOF. This is a consequence of $\mu_1 < \infty$ and Theorem 4.2.

Barlow and Marshall [1] have shown that if F is IFR and F has mean μ, then $F(t) \leqq 1 - e^{-t/\mu}$ for all $t < \mu$. A somewhat weaker bound can be obtained if F is known only to be NBUE.

THEOREM 4.3. *If F is* NBUE *and μ is the mean of F, then $F(t) \leqq t/\mu$ for all $t \leqq \mu$.*

PROOF. If F is NBUE, then $\mu F(t) \leqq \int_0^t \bar{F}(x)\,dx$. Trivially, $\int_0^t \bar{F}(x)\,dx \leqq t$. *Q.E.D.*

The inequality of Theorem 4.3 is sharp in the sense that equality can be attained. Moreover, the bound cannot be improved even with the stronger condition that F is NBU. To see this, we exhibit an NBU distribution G which attains equality: let

$$\log \bar{G}(x) = -k\Delta, \qquad kt \leqq x < (k+1)t, \qquad k = 0, 1, \cdots. \tag{4.10}$$

With Δ chosen to satisfy

$$\mu = \int_0^\infty \bar{G}(x)\,dx = \sum_{k=0}^\infty te^{-k\Delta} = t/(1 - e^{-\Delta}), \tag{4.11}$$

we have

$$\bar{G}(t) = e^{-\Delta} = 1 - \frac{t}{\mu}. \tag{4.12}$$

Theorem 4.3 provides a lower bound on $\bar{F}(t)$ for $t \leqq \mu$; the upper bound for $t \geqq \mu$ provided by Markov's inequality can be improved under the assumption that F is NBUE. One way to do this for large t is to combine the result of Theorem 4.2 with the Markov inequality $\bar{F}(t) \leqq \mu_r/t^r$. However, this does not provide a sharp bound. Sharp upper bounds for $\bar{F}(t)$, $t > \mu$, under the conditions that F is NBU or NBUE are not known.

Bounds for NBU survival functions $\bar{F}$ can be obtained in terms of a percentile.

THEOREM 4.4. *If F is* NBU *and $\bar{F}(t) = \alpha$, then*

$$\begin{aligned} &\bar{F}(x) \geqq \alpha^{1/k}, \qquad \frac{t}{k+1} < x \leqq \frac{t}{k}, \qquad k = 1, 2, \cdots, \\ &\bar{F}(x) \leqq \alpha^k, \qquad kt \leqq x < (k+1)t, \qquad k = 0, 1, \cdots. \end{aligned} \tag{4.13}$$

PROOF. For $t/(k+1) < x \leqq t/k, \bar{F}(x) \geqq \bar{F}(t/k) \geqq [\bar{F}(t)]^{1/k}$; this establishes the lower bound. For $kt \leqq x < (k+1)t$, $\bar{F}(x) \leqq \bar{F}(kt) \leqq [\bar{F}(t)]^k$, which is the upper bound. *Q.E.D.*

The upper bound of Theorem 4.4 is itself an NBU survival function, which, of course, attains equality and shows that the inequality is sharp. On the other hand, the lower bound is *not* an NBU survival function, and sharpness is not so trivially established.

Let $\bar{G}_k(x) = \alpha^{(j-1)/k}$ for $[(j-1)/(k+1)]t < x \leqq [j/(k+1)]t, j = 1, 2, \cdots, k+1$, and let $\bar{G}_k(x) = 0$ for $x > t$. This survival function is NBU and attains equality for x in the interval $t/(k+1) < x \leqq t/k$.

THEOREM 4.5. *If F is* NWU *and $\bar{F}(t) = \alpha$, then*

$$\begin{aligned} &\bar{F}(x) \leqq \alpha^{1/(k+1)}, \qquad \frac{t}{k+1} \leqq x < \frac{t}{k}, \qquad k = 1, 2, \cdots, \\ &\bar{F}(x) \geqq \alpha^{k+1}, \qquad kt \leqq x < (k+1)t, \qquad k = 0, 1, \cdots. \end{aligned} \tag{4.14}$$

PROOF. For $t/(k+1) \leqq x \leqq t/k$, $\bar{F}(x) \leqq \bar{F}[t/(k+1)] \leqq [\bar{F}(t)]^{1/(k+1)}$; this establishes the upper bound. For $kt \leqq x < (k+1)t$, $\bar{F}(x) \geqq \bar{F}((k+1)t) \geqq [\bar{F}(t)]^{k+1}$, which is the lower bound. *Q.E.D.*

The lower bound of Theorem 4.5 is itself an NWU survival function, so that the inequality is sharp in the sense that equality can be attained. The upper bound is *not* an NWU survival function, but it is still true that equality can be attained, although not by the same distribution for each x.

Let $\bar{H}_k(x) = \alpha^{(j+1)/(k+1)}$ for $jt/k \leqq x < [(j+1)/k]t$, $j = 0, 1, \cdots$. This survival function is NWU and attains equality for x in the interval $t/(k+1) \leqq x < t/k$.

THEOREM 4.6. *If F is* NBUE [NWUE] *and μ is the mean of F, then*

$$\mu e^{-t/\mu} \geqq \int_t^\infty \bar{F}(x)\,dx \qquad \left[\mu e^{-t/\mu} \leqq \int_t^\infty \bar{F}(x)\,dx\right] \qquad \text{for all} \quad t \geqq 0. \tag{4.15}$$

PROOF. Let F_1 be the distribution with density $f_1(x) = \bar{F}(x)/\mu$, $x \geqq 0$. If F is NBUE, then

$$\bar{F}_1(z) = \int_z^\infty \bar{F}(x)/\mu\,dx \leqq \mu\bar{F}(z)/\mu = \mu f_1(z), \tag{4.16}$$

or equivalently, $r_1(z) \equiv f_1(z)/\bar{F}_1(z) \geqq 1/u$. Thus

$$\begin{aligned} \frac{1}{\mu}\int_t^\infty \bar{F}(x)\,dx = \bar{F}_1(t) &\equiv \exp\left\{-\int_0^t r_1(z)\,dz\right\} \\ &\leqq \exp\left\{-\int_0^t \frac{1}{\mu}\,dz\right\} = e^{-t/\mu}. \quad Q.E.D. \end{aligned} \tag{4.17}$$

The bound $\mu e^{-t/\mu}$ of Theorem 4.6 may be viewed as $\int_t^\infty \bar{G}(x)\,dx$, where $\bar{G}(x) = e^{-x/\mu}$, $x \geqq 0$. Since F and G have the same mean, we obtain immediately the corollary:

COROLLARY 4.1. *If F is* NBUE *and μ is the mean of F, then*

$$\mu(1 - e^{-t/\mu}) \leqq \int_0^t \bar{F}(x)\,dx \qquad \text{for all} \quad t \geqq 0. \tag{4.18}$$

The inequality is reversed if F is NWUE.

Marshall and Proschan [10] have shown that the inequalities of Theorem 4.6 are preserved under the formation of parallel systems, and the inequalities of Corollary 4.1 are preserved under the formation of series systems. Moreover, all these inequalities are preserved under convolutions.

5. Class preservation under reliability operations

In this section we determine which of the classes of life distributions, NBU, NWU, NBUE, and NWUE, are preserved under the formation of (1) coherent systems, (2) convolutions, and (3) mixtures. These operations occur quite naturally in reliability models.

5.1. *Coherent systems.* Let $x_i = 1$ if the ith component in a system functions, 0 otherwise, $i = 1, \cdots, n$. Let $\phi(x) = 1$ if the system functions, 0 otherwise. The function ϕ is called the *structure function* of the system. A system is coherent if (a) its structure function ϕ is increasing and (b) ϕ is not identically 0 and not identically 1. See Birnbaum, Esary, and Saunders [5] and Barlow and Proschan [4].

THEOREM 5.1. *If each component of a coherent system of independent components has an* NBU *life distribution, then the system has an* NBU *life distribution.*

The proof is presented in Esary, Marshall, and Proschan [6].

NBUE *not preserved.* To show that the NBUE class need not be preserved under the formation of coherent systems, consider a series system of two independent components each having the life distribution F of Example 3.1. We have verified that F is NBUE, but not NBU. However, the mean life of a new system is

$$\nu = \int_0^3 \bar{F}^2(x)\,dx = \tfrac{3}{2}, \tag{5.1}$$

whereas the mean remaining life of a system of age one is two. Thus system life is *not* NBUE.

REMARK. We saw in the remark following Theorem 4.2 that an NBUE distribution has coefficient of variation $\leqq 1$. A simple counter example shows that a coherent system of independent components each having coefficient of variation $\leqq 1$ may itself have coefficient of variation > 1: form a series system of two independent components each having the distribution which places mass $\frac{1}{2}$ at 0 and 1.

NWU, NWUE *not preserved.* To show that neither the NWU nor the NWUE classes are preserved under the formation of coherent systems, consider a parallel system of two independent components each having life distribution $1 - e^{-t}$, which is both NWU and NWUE. Then system life distribution is given by

$$F(t) = (1 - e^{-t})^2, \tag{5.2}$$

so that system failure rate is

$$r(t) = 1 - \frac{1}{2e^t - 1}, \tag{5.3}$$

a strictly increasing function. Thus system life is neither NWU nor NWUE.

5.2. *Convolutions.* Both the NBU and the NBUE classes are preserved under convolution, as shown in the next two theorems.

THEOREM 5.2. *Let F_1, F_2 be* NBU *distributions. Then the convolution*

$$F(t) = \int_0^t F_1(t - x)\, dF_2(x) \tag{5.4}$$

is NBU.

PROOF.

$$\bar{F}(x + y) = \int_0^x \bar{F}_2(x + y - z)\, dF_1(z) + \int_0^\infty \bar{F}_2(y - z)\, d_z F_1(x + z). \tag{5.5}$$

But

$$\begin{aligned}\int_0^x \bar{F}_2(x + y - z)\, dF_1(z) &\leqq \bar{F}_2(y) \int_0^x \bar{F}_2(x - z)\, dF_1(z) \\ &= \bar{F}_2(y)[\bar{F}(x) - \bar{F}_1(x)],\end{aligned} \tag{5.6}$$

and, integrating by parts,

$$\begin{aligned}&\int_0^\infty \bar{F}_2(y - z)\, d_z F_1(x + z) \\ &\qquad = \bar{F}_2(y)\bar{F}_1(x) + \int_0^\infty \bar{F}_1(x + z)[-\, d_z F_2(y - z)] \\ &\qquad \leqq \bar{F}_2(y)\bar{F}_1(x) + \bar{F}_1(x) \int_0^\infty \bar{F}_1(z)[-\, d_z F_2(y - z)] \\ &\qquad = \bar{F}_2(y)\bar{F}_1(x) + \bar{F}_1(x)[\bar{F}(y) - \bar{F}_2(y)].\end{aligned} \tag{5.7}$$

Thus

$$\begin{aligned}\bar{F}(x + y) &\leqq \bar{F}_2(y)\bar{F}(x) + \bar{F}_1(x)\bar{F}(y) - \bar{F}_1(x)\bar{F}_2(y) \\ &= \bar{F}(x)\bar{F}(y) - [\bar{F}(x) - \bar{F}_1(x)][\bar{F}(y) - \bar{F}_2(y)] \leqq \bar{F}(x)\bar{F}(y).\end{aligned} \tag{5.8}$$

THEOREM 5.3. *Let F_1, F_2 be* NBUE. *Then the convolution $F(t) = \int_0^t F_1(t - x)\, dF_2(x)$ is* NBUE.

PROOF.

$$\begin{aligned}&\int_0^\infty \bar{F}(t + x)\, dx \\ &\qquad = \int_0^\infty \int_0^t \bar{F}_1(t + x - u)\, dF_2(u)\, dx + \int_0^\infty \int_t^\infty \bar{F}_1(t + x - u)\, dF_2(u)\, dx.\end{aligned} \tag{5.9}$$

But for $u \leqq t$, $\int_0^\infty \bar{F}_1(t - u + x)\, dx \leqq \mu_1 \bar{F}_1(t - u)$ by (1.2), where μ_i is the mean of F_i, $i = 1, 2$. Thus

$$\begin{aligned}\int_0^\infty \int_0^t \bar{F}_1(t + x - u)\, dF_2(u)\, dx &\leqq \mu_1 \int_0^t \bar{F}_1(t - u)\, dF_2(u) \\ &= \mu_1[\bar{F}(t) - \bar{F}_2(t)].\end{aligned} \tag{5.10}$$

For $u > t$, $\int_0^\infty \bar{F}_1(t + x - u)\, dx = u - t + \mu_1$, so that

$$\int_0^\infty \int_t^\infty \bar{F}_1(t + x - u)\, dF_2(u)\, dx = \int_t^\infty (u - t + \mu_1)\, dF_2(u). \tag{5.11}$$

Let $w = u - t$. Then

$$(5.12)\quad \int_t^\infty (u - t + \mu_1)\, dF_2(u) = \mu_1\bar{F}_2(t) + \int_0^\infty w\, dF_2(w + t)$$
$$= \mu_1\bar{F}_2(t) + \int_0^\infty \bar{F}_2(w + t)\, dw$$
$$\leqq \mu_1\bar{F}_2(t) + \mu_2\bar{F}_2(t).$$

Thus

$$(5.13)\quad \int_0^\infty \bar{F}(t + x)\, dx \leqq \mu_1\bar{F}(t) + \mu_2\bar{F}_2(t) \leqq (\mu_1 + \mu_2)\bar{F}(t).$$

REMARK. If F_1 and F_2 each have coefficient of variation $\leqq a$. then their convolution $F_0(t) = \int_0^\infty F_1(t - x)\, dF_2(x)$ has coefficient of variation $\leqq a$. To prove this, let F_i have mean μ_i, variance σ_i^2, $i = 0, 1, 2$. Then

$$(5.14)\quad a^2\mu_0^2 - \sigma_0^2 = a^2(\mu_1 + \mu_2)^2 - (\sigma_1^2 + \sigma_2^2)$$
$$= (a\mu_1^2 - \sigma^2) + (a_2\mu_2^2 - \sigma^2) + 2a^2\mu_1\mu_2 \geqq 0.$$

NWU, NWUE *not preserved.* To show that neither the NWU nor the NWUE classes are preserved under convolution, let $F(t) = 1 - e^{-t}$, which is both NWU and NWUE. Then $F^{(2)}(t) = 1 - (1 + t)e^{-t}$, a gamma distribution of order two, which has strictly increasing failure rate, and thus is neither NWU nor NWUE.

5.3. *Mixtures.*

DEFINITION. *Let* $\mathscr{F} = \{F_\alpha : \alpha \in \mathscr{A}\}$ *be a family of probability distributions and G a probability distribution. Then*

$$(5.15)\quad F(t) = \int F_\alpha(t)\, dG(\alpha)$$

is a mixture of probability distributions *from* $\mathscr{F}$.

As we shall see below, none of the classes, NBU, NBUE, NWU, NWUE, is preserved under mixtures. However, we can demonstrate preservation of a subclass of the NWU and of the NWUE classes under mixtures.

THEOREM 5.4. *Suppose F is the mixture of* F_α, $\alpha \in \mathscr{A}$, *with each* F_α NWU [NWUE] *and no two distinct* F_α, $F_{\alpha'}$ *crossing on* $(0, \infty)$. *Then F is* NWU [NWUE].

PROOF. NWU *case.* By the Chebyshev inequality for similarly ordered functions (Hardy, Littlewood, and Pólya [9], Theorem 43),

$$(5.16)\quad \bar{F}(s)\bar{F}(t) \equiv \int \bar{F}_\alpha(s)\, dG(\alpha) \int \bar{F}_\alpha(t)\, dG(\alpha) \leqq \int \bar{F}_\alpha(s)\bar{F}_\alpha(t)\, dG(\alpha).$$

By the NWU property

$$(5.17)\quad \int \bar{F}_\alpha(s)\bar{F}_\alpha(t)\, dG(\alpha) \leqq \int \bar{F}_\alpha(s + t)\, dG(\alpha) \equiv \bar{F}(s + t).$$

Thus F is NWU.

NWUE *case.* As above,

$$\mu\bar{F}(t) \equiv \int \mu_\alpha \, dG(\alpha) \int \bar{F}_\alpha(t) \, dG(\alpha) \leqq \int \mu_\alpha \bar{F}_\alpha(t) \, dG(\alpha). \tag{5.18}$$

Using the NWUE defining property (1.2) first and the Fubini Theorem next, we have

$$\begin{aligned} \int \mu_\alpha \bar{F}_\alpha(t) \, dG(\alpha) &\leqq \int\int \bar{F}_\alpha(t + x) \, dx dG(\alpha) \\ &= \int\int \bar{F}_\alpha(t + x) \, dG(\alpha) \, dx = \int \bar{F}(t + x) \, dx. \end{aligned} \tag{5.19}$$

Thus by (1.2), F is NWUE. *Q.E.D.*

REMARK. We may readily show that if F is a mixture of F_α, $\alpha \in \mathscr{A}$, with each F_α having $CV \geqq a$, then F has $CV \geqq a$, $a \geqq 0$.

NBU, NBUE *classes not preserved.* To see that neither the NBU nor the NBUE class is preserved under mixtures, note that a mixture of nonidentical exponential distributions has a strictly decreasing failure rate, and thus cannot be NBU nor NBUE.

NWU *class not preserved.* To see that the NWU class is not preserved under mixtures, consider the following example. Let $\bar{F}_\delta(x) = e^{-k\delta}$ for $(k - 1)\delta < x \leqq k\delta$, $k = 1, 2, \cdots$. Then it is easy to verify that F is NWU.

Next we show that the mixture $F = \frac{1}{2}F_\delta + \frac{1}{2}F_\gamma$ does not satisfy the NWU property

$$D \equiv \bar{F}(x + y) - \bar{F}(x)\bar{F}(y) \geqq 0 \quad \text{for} \quad 0 < y < \delta < x < \gamma < 2\delta < x + y < 2\gamma \tag{5.20}$$

(for example, take $y = 3$, $\delta = 4$, $x = 6$, $\gamma = 7$). Then

$$\begin{aligned} &\bar{F}_\delta(x) = e^{-2\delta}, \quad \bar{F}_\delta(y) = e^{-\delta}, \quad \bar{F}_\delta(x + y) = e^{-3\delta}; \\ &\bar{F}_\gamma(x) = e^{-\gamma}, \quad \bar{F}_\gamma(y) = e^{-\gamma}, \quad \bar{F}_\gamma(x + y) = e^{-2\gamma}. \end{aligned} \tag{5.21}$$

We compute

$$\begin{aligned} D &= \tfrac{1}{2}(e^{-3\delta} + e^{-2\gamma}) - \tfrac{1}{4}(e^{-2\delta} + e^{-\gamma})(e^{-\delta} + e^{-\gamma}) \\ &= \tfrac{1}{4}(e^{-\gamma} - e^{-\delta})(e^{-\gamma} - e^{-2\delta}) < 0. \end{aligned} \tag{5.22}$$

We summarize the preservation results of this section in Table I.

TABLE I

PRESERVATION OF LIFE DISTRIBUTION CLASSES UNDER RELIABILITY OPERATIONS

	Formation of Coherent Systems	Convolutions	Arbitrary Mixtures	Mixtures of Distributions That Do Not Cross
NBU	preserved	preserved	not preserved	not preserved
NBUE	not preserved	preserved	not preserved	not preserved
NWU	not preserved	not preserved	not preserved	preserved
NWUE	not preserved	not preserved	?	preserved

REFERENCES

[1] R. E. BARLOW and A. W. MARSHALL, "Bounds for distributions with monotone hazard rate," *Ann. Math. Statist.*, Vol. 35 (1964), pp. 1234–1257.

[2] R. E. BARLOW, A. W. MARSHALL and F. PROSCHAN, "Properties of probability distributions with monotone hazard rate," *Ann. Math. Statist.*, Vol. 34 (1963), pp. 375–389.

[3] R. E. BARLOW and F. PROSCHAN, " Comparison of replacement policies and renewal theory implications," *Ann. Math. Statist.*, Vol. 35 (1964), pp. 577–589.

[4] ———, *Mathematical Theory of Reliability*, New York, Wiley, 1965.

[5] Z. W. BIRNBAUM, J. D. ESARY and S. C. SAUNDERS, "Multi-component systems and structures and their reliability," *Technometrics*, Vol. 3 (1961), pp. 55–77.

[6] J. D. ESARY, A. W. MARSHALL and F. PROSCHAN, "Some reliability applications of the hazard transform," *SIAM J. Appl. Math.*, Vol. 18 (1970), pp. 849–860.

[7] ———, "Shock models and wear processes," to appear in *Ann. Math. Statist.*

[8] W. FELLER, *An Introduction to Probability Theory and Its Applications*, Vol. 2, New York, Wiley, 1966.

[9] G. H. HARDY, J. E. LITTLEWOOD and G. PÓLYA, *Inequalities*, Cambridge, Cambridge University Press, 1952.

[10] A. W. MARSHALL and F. PROSCHAN, "Mean life of series and parallel systems," *J. Appl. Probability*, Vol. 7 (1970), pp. 165–174.

[11] G. WEISS, "On the theory of replacement of machinery with a random failure time," *Naval Res. Logist. Quart.*, Vol. 3 (1956), pp. 279–293.