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Classical adiabatic angles and quantal adiabatic phase

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Abstract. A semiclassical connection is established between quantal and classical properties of a system whose Hamiltonian is slowly cycled by varying its parameters round a circuit. The quantal property is a geometrical phase shift γ_n associated with an eigenstate with quantum numbers $n=\{n_l\}$; the classical property is a shift $\Delta\theta_l(I)$ in the lth angle variable for motion round a phase-space torus with actions $I=\{I_l\}$; the connection is $\Delta\theta_l=-\partial\gamma/\partial n_l$. Two applications are worked out in detail: the generalised harmonic oscillator, with quadratic Hamiltonian whose parameters are the coefficients of q^2, qp and p^2; and the rotated rotator, consisting of a particle sliding freely round a non-circular hoop slowly turned round once in its own plane.

1. Introduction

Consider a quantal or classical system with N freedoms, whose Hamiltonian $H(q,p;X(t))$ depends on a set of slowly changing parameters $X=\{X_\mu\}$ as well as dynamical variables or operators $q=\{q_j\}$, $p=\{p_j\}$ $(i\leqslant j\leqslant N)$. The evolution of the system is governed by an adiabatic theorem. In the quantal case (Messiah 1962), this states that a system originally in an eigenstate, labelled by one or more parameters $n=\{n_i\}$, willl remain in the same eigenstate $|n;X(t)\rangle$, with energy $E_n(X(t))$, as the X vary. In the classical case (Dirac 1925), the theorem states that an orbit initially on an N-dimensional phase-space torus with actions $I=\{I_j\}$ (Arnold 1978) will continue to explore the tori with the same values of I (adiabatic invariants), in spite of the changing Hamiltonian corresponding to $X(t)$, provided such tori continue to exist (for example if the system remains integrable for all parameters X).

These well known adiabatic theorems fail to describe an important feature of the evolution, which manifests itself if the Hamiltonian returns to its original form after a (long) time T, i.e. $X(T)=X(0)$. We shall describe such changes as *taking the system round a circuit* C in the space of parameters X.

Quantally, the feature is a *geometrical phase factor* $\exp(\mathrm{i}\gamma_n(C))$ accumulated round C by a system in the nth state: if the state is initially $|\Psi(0)\rangle$, then the state at T is

$$|\Psi(T)\rangle=\exp(\mathrm{i}\gamma_n(C))\exp\left(-\frac{\mathrm{i}}{\hbar}\int_0^T \mathrm{d}t\,E_n(X(t))\right)|\Psi(0)\rangle. \tag{1}$$

(The second factor contains the familiar dynamical phase, and is present even if the parameters remain constant, and the third factor $|\Psi(0)\rangle$ is an expression of the adiabatic theorem.) A discussion of $\gamma_n(C)$, as well as general formulae and illustrative examples, is given by Berry (1984a) and Simon (1983), commenting on an early version of that

paper, explains how the geometrical phase embodies the anholonomy (non-integrable connection) of Hermitian line bundles.

Classically, the feature that the adiabatic theorem does not describe is *shifts* $\Delta\theta(I; C)$ in the *angles* $\theta = \{\theta_j\}$ conjugate to the actions I, in addition to those expected on the basis of the instantaneous frequencies $\omega = \{\omega_j(I; X)\}$: if the initial angles are $\theta(0)$, then after the circuit C the position of the system on its torus I is given by

$$\theta(T) = \theta(0) + \int_0^T \mathrm{d}t\, \omega(I; X(t)) + \Delta\theta(I; C). \tag{2}$$

The existence of the $\Delta\theta$ as a general feature of slowly cycled integrable systems was discovered by Hannay (1984), and I will refer to these angle shifts as 'classical adiabatic angles', or as 'Hannay's angles'.

My purpose here is to show by a semiclassical argument that Hannay's angles $\Delta\theta$ are indeed classical analogues of the quantal adiabatic phase $\gamma_n(C)$, and to establish the precise relation between these quantities. This analysis will be presented in § 3. As a preliminary, § 2 will contain an alternative to Hannay's (1985) derivation of his angles. Sections 4 and 5 give a discussion of two families of one-dimensional systems (both suggested by Hannay) for which classical and quantum mechanics can be worked out in detail, thus confirming the correctness of the general theory of § 3. The first system is the 'generalised harmonic oscillator', consisting of a quadratic Hamiltonian for which the coefficients of q^2, qp and p^2 are slowly varied; the second is the 'rotated rotator', consisting of a particle sliding freely round a non-circular hoop which is slowly rotated in its own plane. Two appendices give instructive 'elementary' derivations of Hannay's angles for these systems, based on asymptotic analyses of the corresponding Newtonian equations.

2. Hannay's angles

The evolution of angle variables, which by (2) determine the classical adiabatic angles $\Delta\theta$, can be determined by making a canonical transformation to action-angle variables. This is achieved (Landau and Lifshitz 1976) in terms of a generating function $S^{(\alpha)}(q, I; X(t))$, according to the scheme

$$\{q, p\} \leftarrow S^{(\alpha)}(q, I; X(t)) \rightarrow \{\theta, I\}, \tag{3a}$$

$$p^{(\alpha)} = \partial S^{(\alpha)}/\partial q, \qquad \theta^{(\alpha)} = \partial S^{(\alpha)}/\partial I. \tag{3b, c}$$

In these formulae, the superscript α labels the branches of S, a function whose unavoidable multivaluedness reflects the fact that, for a given torus I, q does not uniquely determine p (figure 1).

The new Hamiltonian $\bar{H}$ differs from the old Hamiltonian H in value as well as functional form, because the canonical transformation is time dependent through the slowly changing parameters $X(t)$. In fact

$$\bar{H}(\theta, I, t) = \mathscr{H}(I; X(t)) + (\mathrm{d}X/\mathrm{d}t)(\partial/\partial X) S^{(\alpha)}(q, I; X(t)), \tag{4}$$

where

$$\mathscr{H}(I; X(t)) \equiv H(q(\theta, I; X(t)), p(\theta, I; X(t)); X(t)) \tag{5}$$

is the (angle-independent) 'action' Hamiltonian corresponding to constant X. At

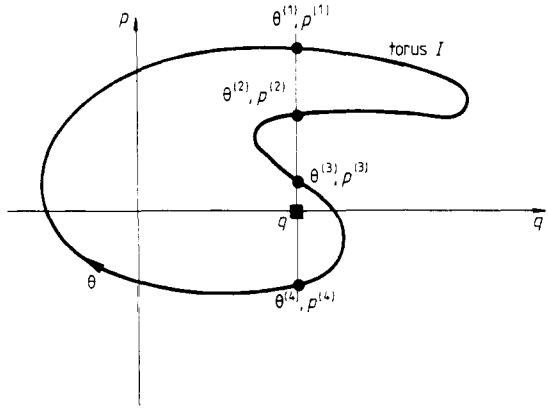

Figure 1. Torus with action $I=(1/2\pi)\oint p\,\mathrm{d}q$ for a system with one freedom, illustrating multivaluedness of mappings from q to p and q to θ.

any time t, the branch α and the value of q occurring in (4) are uniquely defined by θ and I.

To obtain an explicit form for $\bar{H}$, we define the single-valued function

$$\mathscr{S}(\theta, I; X) \equiv S^{(\alpha)}(q(\theta, I; X), I; X), \qquad (0 \leqslant \theta < 2\pi) \tag{6}$$

so that

$$\frac{\partial S^{(\alpha)}}{\partial X} = \frac{\partial \mathscr{S}}{\partial X} - \frac{\partial S^{(\alpha)}}{\partial q}\frac{\partial q}{\partial X} = \frac{\partial \mathscr{S}}{\partial X} - p^{\alpha}\frac{\partial q}{\partial X}. \tag{7}$$

Thus the new Hamiltonian becomes

$$\bar{H}(\theta, I, t) = \mathscr{H}(I; X(t)) + \frac{\mathrm{d}X(t)}{\mathrm{d}t}\left(\frac{\partial \mathscr{S}}{\partial X}(\theta, I; X(t)) - p(\theta, I; X(t))\frac{\partial q}{\partial X}(\theta, I; X(t))\right). \tag{8}$$

This is globally single-valued, because q and p are periodic functions of θ, and the increment of $\mathscr{S}$ round a circuit is

$$\mathscr{S}(\theta+2\pi, I; X) - \mathscr{S}(\theta, I; X) = \oint p\,\mathrm{d}q = 2\pi I, \tag{9}$$

which does not depend on X.

Hamilton's equation for the angles is

$$\mathrm{d}\theta/\mathrm{d}t = \partial\bar{H}/\partial I. \tag{10}$$

When applied to (8), the first term gives that part of the evolution which would occur even if the parameters remained constant, arising from the frequencies

$$\omega(I; X) = \partial\mathscr{H}(I; X)/\partial I. \tag{11}$$

What we are seeking, however, is the angle shift defined by (2), and this arises from the second term in (8):

$$\Delta\theta = \int_0^T \mathrm{d}t\frac{\mathrm{d}X}{\mathrm{d}t}\frac{\partial}{\partial I}\left(\frac{\partial \mathscr{S}}{\partial X} - p\frac{\partial q}{\partial X}\right). \tag{12}$$

As it stands, this integral is difficult to evaluate because the integrand depends on time implicitly through the changes in θ and I as well as explicitly through the variations of X. It is natural at this point to invoke the adiabatic technique (Arnold 1978) of averaging over the implicit (fast) variations by integrating over the torus at each time t. When applied to the Hamilton equation conjugate to (10), this procedure shows that the actions I remain constant in spite of the (slow) variations in X—which is of course the familiar adiabatic theorem. When applied to (12), it gives

$$\Delta\theta(I;C)=\oint \mathrm{d}X\frac{\partial}{\partial I}\frac{1}{(2\pi)^N}\oint \mathrm{d}\theta\left\{\frac{\partial\mathscr{S}}{\partial X}(\theta,I;X)-p(\theta,I;X)\frac{\partial q}{\partial X}(\theta,I;X)\right) \tag{13}$$

where

$$\oint \mathrm{d}\theta=\prod_{j=1}^{N}\int_0^{2\pi}\mathrm{d}\theta_j. \tag{14}$$

Equation (13) has the form of a line integral over a single-valued function in parameter space. The first term vanishes because $\partial\mathscr{S}/\partial X$ is a gradient. The second term can be transformed by Stokes' theorem into an integral over any surface A, in parameter space, whose boundary is C. In the language of differential forms (Arnold 1978):

$$\Delta\theta(I;C)=-\frac{\partial}{\partial I}\int_{\partial A=C} W \tag{15}$$

where the *angle 2-form* is expressed in terms of W, given by

$$W=\frac{1}{(2\pi)^N}\oint \mathrm{d}\theta\,\mathrm{d}p\wedge\mathrm{d}q. \tag{16}$$

Of course the forms in this expression are parameter-space forms, not the more familiar phase-space ones. A more explicit expression, for the case where $\{X\mu\}$ can be written as a three-dimensional vector $\boldsymbol{X}$, is

$$\Delta\theta_l(I_j;C)=-\frac{\partial}{\partial I_l}\iint_{\partial A=C}\mathrm{d}\boldsymbol{A}\cdot\boldsymbol{W}(I;\boldsymbol{X}) \tag{17}$$

where $\mathrm{d}\boldsymbol{A}$ denotes the element of area in parameter space and

$$\boldsymbol{W}(I;\boldsymbol{X})=\frac{1}{(2\pi)^N}\oint \mathrm{d}\theta\,\nabla_{\boldsymbol{X}}p_j(\theta,I;\boldsymbol{X})\wedge\nabla_{\boldsymbol{X}}q_j(\theta,I;\boldsymbol{X}). \tag{18}$$

The formulae (15)-(18) for the classical adiabatic angles constitute one version of the expressions obtained by Hannay (1985). It is not difficult to show that Hannay's angles are invariant under parameter-dependent and action-dependent deformations of the (arbitrary) origin from which the angles θ are measured, i.e. under

$$\theta\to\theta+\beta(I;X), \tag{19}$$

provided β is single-valued across the area A in parameter space. This classical invariance corresponds to the invariance of the quantal phase factor under parameter-dependent changes in the phases of the eigenvectors $|n;X\rangle$ (see the appendix of Berry 1984a).

3. Angles and phase: semiclassical theory

The quantal geometrical phase $\gamma_n(C)$ defined by equation (1) will be written in the form

$$\gamma_n(C) = -\iint_{\partial A = C} \mathrm{d}\boldsymbol{A} \cdot \boldsymbol{V}(n; \boldsymbol{X}) \tag{20}$$

where

$$\boldsymbol{V}(n; \boldsymbol{X}) \equiv \operatorname{Im} \nabla_{\boldsymbol{X}} \wedge \langle n; \boldsymbol{X} | \nabla_{\boldsymbol{X}} | n; \boldsymbol{X} \rangle, \tag{21}$$

corresponding to equation (7) of Berry (1984a) and analogous to (17) and (18) of the preceding section. In position representation we define the wavefunction ψ_n by

$$\psi_n(q; \boldsymbol{X}) \equiv \langle q \,|\, n; \boldsymbol{X} \rangle, \tag{22}$$

so that the phase 2-form becomes

$$\boldsymbol{V}(n; \boldsymbol{X}) = \operatorname{Im} \nabla_{\boldsymbol{X}} \wedge \int \mathrm{d}q\, \psi_n^*(q; \boldsymbol{X}) \nabla_{\boldsymbol{X}} \psi_n(q; \boldsymbol{X}) \tag{23}$$

where

$$\int \mathrm{d}q \equiv \prod_{j=1}^{N} \int_{-\infty}^{\infty} \mathrm{d}q_j. \tag{24}$$

Semiclassically, ψ_n is associated with a torus whose actions are quantised by the corrected Bohr–Sommerfeld rule (Keller 1958)

$$I_j = (n_j + \sigma_j)\hbar \tag{25}$$

where the σ_j are N constants whose values are unimportant in the present context. The wavefunction is obtained from the torus by projection from phase space to q space, according to the method of Maslov (see Maslov and Fedoriuk (1981) and simplified presentations by Percival (1977) and Berry (1983)):

$$\psi_n(q; \boldsymbol{X}) = \sum a_\alpha(q, I; \boldsymbol{X}) \exp[\mathrm{i}\hbar^{-1} S^{(\alpha)}(q, I; \boldsymbol{X})], \tag{26}$$

where $S^{(\alpha)}$ is the classical generating function (equation (3)), the summation over α corresponds to all branches $p^{(\alpha)}$ contributing at q (figure 1), and the amplitude is given in terms of the projection Jacobian by

$$a_\alpha^2 = \frac{1}{(2\pi)^N} \frac{\mathrm{d}\theta^{(\alpha)}}{\mathrm{d}q} = \frac{1}{(2\pi)^N} \det\left(\frac{\mathrm{d}\theta_i}{\mathrm{d}q_j}\right) \tag{27}$$

(this quantity may be positive or negative, corresponding to $\pi/2$ phase shifts across turning points).

When the wavefunction (26) is substituted into (23), products of contributions from different branches α give rapid oscillations and cancel semiclassically on integrating over q, leaving

$$\boldsymbol{V}(n; \boldsymbol{X}) = \frac{1}{\hbar} \nabla_{\boldsymbol{X}} \wedge \int \mathrm{d}q \frac{1}{(2\pi)^N} \sum_\alpha \frac{\mathrm{d}\theta^{(\alpha)}}{\mathrm{d}q} \nabla_{\boldsymbol{X}} S^{(\alpha)}(q, I; \boldsymbol{X}). \tag{28}$$

Transformation of the variables of integration from q to θ, and use of the formulae

(6) and (7) give

$$V(n;X)=\frac{1}{\hbar}\nabla_X\wedge\frac{1}{(2\pi)^N}\oint \mathrm{d}\theta(\nabla_X\mathscr{S}-p\nabla_X q) \tag{29}$$

$$=\frac{-1}{\hbar(2\pi)^N}\oint \mathrm{d}\theta\,\nabla_X p_j(\theta,I;X)\wedge\nabla_X q_j(\theta,I;X)$$

$$=-\frac{1}{\hbar}\mathbf{W}(I;X), \tag{30}$$

thus relating the phase 2-form to the angle 2-form (18).

Finally, this relation, together with (17) and (20), immediately gives the connection between Hannay's angles and the geometrical phase:

$$\Delta\theta_l(I;C)=-\hbar\frac{\partial}{\partial I_l}\gamma_n(C)=-\frac{\partial\gamma_n(C)}{\partial n_l}, \tag{31}$$

where the association (25) enables the quantum numbers n to be considered as continuous variables.

4. Example: generalised harmonic oscillator

The classical Hamiltonian for this system with one freedom is

$$H=\tfrac{1}{2}(X(t)q^2+2Y(t)qp+Z(t)p^2). \tag{32}$$

The parameters are X, Y, Z; when these are held fixed, H describes oscillatory motion round elliptic contours in the phase plane (figure 2), provided

$$XZ>Y^2, \tag{33}$$

and we assume henceforth that this remains the case as X, Y, Z vary. For given energy E the area of the contour is $2\pi E/(XZ-Y^2)^{1/2}$, and this is $2\pi I$ by definition, so the action Hamiltonian (5) is

$$\mathscr{H}(I;X)=I(XZ-Y^2)^{1/2} \tag{34}$$

giving the action-independent frequency (11) as

$$\omega=(XZ-Y^2)^{1/2}. \tag{35}$$

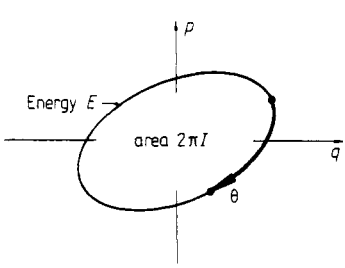

Figure 2. Elliptic phase-plane contour for a generalised harmonic oscillator with Hamiltonian (32).

It will be shown in appendix 1 that when the parameters vary the Hamiltonian (32) describes an oscillator with parametrically forced frequency, whose classical motion, including Hannay's angle, can be determined by the WKB method commonly employed in quantum mechanics. Here the angle 2-form will be calculated from (16), and requires the solution of Hamilton's equations for fixed X, Y, Z in the form $q(\theta, I; X)$ and $p(\theta, I; X)$. Choosing the origin of angle at the positive extreme of the q motion (figure 2) the solutions are

$$q = \left(\frac{2ZI}{\omega}\right)^{1/2} \cos\theta, \qquad p = -\left(\frac{2ZI}{\omega}\right)^{1/2}\left(\frac{Y}{Z}\cos\theta + \frac{\omega}{Z}\sin\theta\right) \tag{36}$$

as can easily be verified. Equation (16) now gives

$$\begin{aligned} W &= -2I\frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}\theta \cos^2\theta \, \mathrm{d}\left(\frac{Y}{Z}\left(\frac{Z}{\omega}\right)^{1/2}\right) \wedge \mathrm{d}\left(\left(\frac{Z}{\omega}\right)^{1/2}\right) \\ &= \frac{I}{2}\mathrm{d}\left(\frac{Z}{(XZ-Y^2)^{1/2}}\right) \wedge \mathrm{d}\left(\frac{Y}{Z}\right). \end{aligned} \tag{37}$$

A little reduction produces the symmetrical form

$$W = -\tfrac{1}{4}I(XZ-Y^2)^{-3/2}(X\,\mathrm{d}Y \wedge \mathrm{d}Z + Y\,\mathrm{d}Z \wedge \mathrm{d}X + Z\,\mathrm{d}X \wedge \mathrm{d}Y). \tag{38}$$

If X, Y, Z are regarded as Cartesian components of a vector $\boldsymbol{X}$, this can be written

$$\boldsymbol{W}(I; \boldsymbol{X}) = -\tfrac{1}{4}I\boldsymbol{X}(XZ-Y^2)^{-3/2}. \tag{39}$$

Quantally, (32) corresponds to the Hamiltonian operator

$$\hat{H} = \tfrac{1}{2}[X\hat{q}^2 + Y(\hat{q}\hat{p} + \hat{p}\hat{q}) + Z\hat{p}^2]. \tag{40}$$

When the parameters are constant, this gives rise to the Schrödinger equation satisfied by the wavefunction (22), namely

$$-\frac{Z\hbar^2}{2}\frac{\mathrm{d}^2\psi_n}{\mathrm{d}q^2} - \mathrm{i}\hbar Yq\frac{\mathrm{d}\psi_n}{\mathrm{d}q} + \left(\frac{Xq^2}{2} - \mathrm{i}\hbar\frac{Y}{2}\right)\psi_n = E_n\psi_n. \tag{41}$$

As is easily verified, the normalised solution can be written as

$$\psi_n(q, \boldsymbol{X}) = \frac{(XZ-Y^2)^{1/8}}{Z^{1/4}\hbar^{1/4}}\chi_n\left(\frac{q(XZ-Y^2)^{1/4}}{Z^{1/2}\hbar^{1/2}}\right)\exp\left(\frac{-\mathrm{i}Yq^2}{2Z\hbar}\right), \tag{42}$$

where $\chi_n(\xi)$ are the real, normalised, Hermite functions satisfying

$$\frac{\mathrm{d}^2\chi_n(\xi)}{\mathrm{d}\xi^2} + (2n+1-\xi^2)\chi_n(\xi) = 0, \tag{43}$$

and the energies are exactly given by the semiclassical formula

$$E_n = (n+\tfrac{1}{2})\hbar\omega = (n+\tfrac{1}{2})\hbar(XZ-Y^2)^{1/2}. \tag{44}$$

This wavefunction must now be substituted into the exact formula (23) for the phase 2-form, giving

$$\boldsymbol{V} = -\nabla_x \wedge \frac{(XZ-Y^2)^{1/4}}{Z^{1/2}\hbar^{1/2}}\int_{-\infty}^{\infty} \mathrm{d}q\, \chi_n^2(\xi)\frac{q^2}{\hbar}\nabla_{\boldsymbol{X}}\left(\frac{Y}{2Z}\right). \tag{45}$$

Transforming the integration variable from q to ξ and using the standard result

$$\int_{-\infty}^{\infty} \mathrm{d}\xi\, \xi^2 \chi_n^2(\xi) = n + \tfrac{1}{2} \tag{46}$$

leads to

$$\boldsymbol{V} = -\frac{(n+\frac{1}{2})}{2} \nabla_{\boldsymbol{X}} \left(\frac{Z}{(XZ - Y^2)^{1/2}} \right) \wedge \nabla_{\boldsymbol{X}} \left(\frac{Y}{Z} \right). \tag{47}$$

The semiclassical quantisation rule $(n+\frac{1}{2}) = I/\hbar$ (cf (25)) is exact for this case, and comparison of (47) with the angle 2-form as given by (37) confirms the truth of the central semiclassical relation (30).

5. Example: rotated rotator

A particle of unit mass slides freely round a non-circular hoop, which is slowly turned through one complete rotation, in its own plane, about a centre 0 (figure 3). The rotation can be described by a single parameter X, namely the orientation angle of the hoop, which slowly varies from 0 to 2π. The particle motion in the plane with coordinates $\boldsymbol{q} = (x, y)$ can be described by the Hamiltonian

$$H(\boldsymbol{q}, \boldsymbol{p}; X(t) = \tfrac{1}{2}p_x^2 + \tfrac{1}{2}p_y^2 + V(x \cos X(t) + y \sin X(t), -x \sin X(t) + y \cos X(t)), \tag{48}$$

where V is a confining potential which is zero in a narrow strip centred on the hoop and very large elsewhere.

For fixed X, the particle executes periodic motion, whose constant speed relative to the hoop is the magnitude p of the momentum $\boldsymbol{p}$. The action is

$$I = \tfrac{1}{2}p\mathscr{L} \tag{49}$$

where $\mathscr{L}$ is the length of the hoop, and the angle may be taken as

$$\theta = 2\pi s/\mathscr{L} \tag{50}$$

where s is arc length measured relative to a material point A on the hoop (figure 3). (For this problem with two freedoms there is of course a second pair of action-angle variables, corresponding to transverse vibrations of the particle, but this motion is considered here to have zero amplitude.)

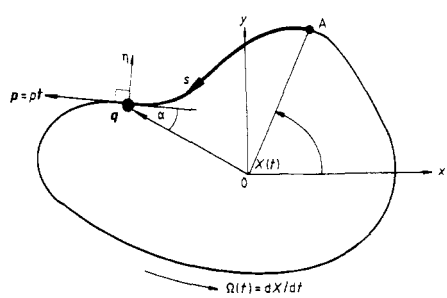

Figure 3. Coordinates and notation for the rotated rotator; the hoop has length $\mathscr{L}$ and area $\mathscr{A}$.

The classical angle shift for a complete turn can be written as a line integral, using (13) with the first term omitted because it gives zero when integrated

$$\Delta\theta = \frac{-1}{2\pi}\frac{\partial}{\partial I}\int_0^{2\pi} \mathrm{d}X \int_0^{2\pi} \mathrm{d}\theta\, \boldsymbol{p}(\theta, I; X)\frac{\partial}{\partial X}\boldsymbol{q}(\theta, I; X). \tag{51}$$

Changing variables from θ to s, and writing the momentum using (49) as

$$\boldsymbol{p} = (2\pi I/\mathscr{L})\boldsymbol{t}, \tag{52}$$

where $\boldsymbol{t}$ is the hoop's unit tangent vector at the point $\boldsymbol{q}$, this becomes

$$\Delta\theta = \frac{-2\pi}{\mathscr{L}^2}\int_0^{2\pi} \mathrm{d}X \int_0^{\mathscr{L}} \mathrm{d}s\, \boldsymbol{t}\cdot\frac{\partial \boldsymbol{q}}{\partial X}. \tag{53}$$

Elementary geometry gives

$$\boldsymbol{t}\cdot\partial q/\partial X = q(s)\sin\alpha(s) \tag{54}$$

where q is the radius of the hoop at s and α the angle between the radius and the tangent; thus $\boldsymbol{t}\cdot\partial\boldsymbol{q}/\partial X$ is independent of the orientation X, and

$$\begin{aligned}\Delta\theta &= \frac{-(2\pi)^2}{\mathscr{L}^2}\int_0^{\mathscr{L}} \mathrm{d}s\, q(s)\sin\alpha(s)\\ &= -8\pi^2\mathscr{A}/\mathscr{L}^2\end{aligned} \tag{55}$$

where $\mathscr{A}$ is the area of the hoop. A more conventional derivation of this result is given in appendix 2.

Writing Hannay's angle in the form

$$\Delta\theta = -2\pi + 2\pi(1 - 4\pi\mathscr{A}/\mathscr{L}^2), \tag{56}$$

we see that the first term gives the expected phase slippage resulting from the fact that the origin A, from which the angle is measured, has made a complete rotation. The non-trivial aspect of the anholonomy is embodied in the second term. By the isoperimetric inequality this term is never negative, so that for small deviations from circularity the particle appears to have advanced further round the hoop than would be predicted by an argument that neglected anholonomy. Hannay's hoop is thus a detector of complete rotations and, more generally, of *absolute angular displacements*, closely analogous to the ring gyroscopes (Forder 1984) employed to detect angular velocities, and to the Sagnac effect (Post 1967).

Quantally, the rotator eigenstates for fixed X are

$$\psi_n(\boldsymbol{q}; X) = (a(\boldsymbol{q}; X)/\mathscr{L}^{1/2})\exp[2\pi i n s(\boldsymbol{q}; X)/\mathscr{L}] \tag{57}$$

where the amplitude a which confines particles to the hoop can be expressed in terms of a perpendicular coordinate η (figure 3) by

$$a^2 = \delta(\eta(\boldsymbol{q}; X)). \tag{58}$$

The adiabatic phase γ_n, accumulated during one rotation of the hoop, can be calculated as a circuit integral (cf (20) and (21) and Berry 1984a) from

$$\gamma_n = -\mathrm{Im}\int_0^{2\pi} \mathrm{d}X \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y\, \psi_n^*(\boldsymbol{q}; X)\frac{\partial}{\partial X}\psi_n(\boldsymbol{q}; X). \tag{59}$$

Substituting (57) and changing to s, η coordinates gives

$$\gamma_n = \frac{-2\pi n}{\mathscr{L}^2} \int_0^{2\pi} \mathrm{d}X \int_0^{\mathscr{L}} \mathrm{d}s \frac{\partial s}{\partial X}(\boldsymbol{q}; X). \tag{60}$$

Now, inspection of figure 3 shows that

$$\partial s / \partial X = -q \sin \alpha \tag{61}$$

so that

$$\gamma_n = -8\pi^2 n \mathscr{A} / \mathscr{L}^2. \tag{62}$$

But $n = I/\hbar$, so that when compared with the angle shift (55) this result gives another confirmation of the central semiclassical relation (31).

6. Discussion

At the heart of this paper lies the relation (31) between the geometric adiabatic phase shift in quantum mechanics and Hannay's adiabatic angles in classical mechanics. The relation holds at the level of semiclassical approximation, and its applicability is restricted to systems whose classical motion is integrable (for fixed parameters) and whose quantal stationary states are associated with phase-space tori. How realistic is a treatment dependent upon this restriction?

For the case of one freedom, all bound systems are integrable, with orbits on one-dimensional tori (closed energy contours) in the phase plane. It is therefore not unrealistic to consider a family of such systems, and the example of the generalised harmonic oscillator provides an illustration. Nevertheless, caution should be exercised when considering systems possessing more than one torus with the same action, because then barrier penetration effects may cause discordance between the quantal and classical adiabatic theorems, as discussed by Berry (1984b).

For more freedoms, integrability is exceptional, because of the occurrence of chaotic motion (Lichtenberg and Lieberman (1983)). Even in quasi-integrable systems, where most of the phase space is occupied by tori, variation of even one parameter X will, generically, cause the system to pass through many resonant zones where the tori are destroyed. Therefore our arguments (like most theories of semiclassical wavefunctions) apply only to very special systems when $N > 1$. In spite of this, some exceptional families of integrable systems are interesting, as illustrated by the example of the rotated rotator. More generally, the argument for that case could be adapted to apply to slow rigid rotation of any multidimensional integrable system—such as an elliptic or cubical cavity containing particles—or even any non-integrable system with a stable closed orbit.

In the case where classical motion is chaotic, it is not clear what is the classical analogue of the quantal adiabatic phase, or of the phase 2-form. In view of the fact that the latter quantity for state $|n\rangle$ has singularities (in parameter space) where $|n\rangle$ degenerates with the state above or below (Berry 1984a), and also because degeneracies presumably get denser as $\hbar \to 0$, the classical analogue of the phase 2-form at parameters X might give a measure of the average density of degeneracies near X.

Acknowledgments

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Appendix 1. Newtonian asymptotics of generalised harmonic oscillator

The time-dependent Hamiltonian (32) yields equations of motion for q and p. Eliminating p gives the Newtonian equation for the coordinate $q(t)$ as

$$\ddot{q}-(\dot{Z}/Z)\dot{q}+[XZ-Y^2+(\dot{Z}Y-\dot{Y}Z)/Z]q=0 \tag{A1.1}$$

where dots denote time derivatives. Defining a (small) adiabatic parameter ε and a 'slow time' variable τ by

$$X\equiv X(\tau)\text{ etc},\qquad \tau=\varepsilon t, \tag{A1.2}$$

and eliminating the term in $\dot{q}$ by introducing the new coordinate $Q(\tau)$ defined by

$$q(t)\equiv[Z(\tau)]^{1/2}Q(\tau), \tag{A1.3}$$

gives

$$Q''+\varepsilon^{-2}\{XZ-Y^2+\varepsilon(Z'Y-Y'Z)/Z+\varepsilon^2[\tfrac{1}{2}(Z'/Z)'-\tfrac{1}{4}(Z'/Z)^2]\}Q=0 \tag{A1.4}$$

where primes denote derivatives with respect to τ.

This equation decribes a parametrically driven oscillator whose variable frequency is given by adding to (35) some terms arising from the time-dependence of H. Because of the restriction (33) and the assumed smallness of ε the quantity in curly brackets in (A1.4) never vanishes, so that the motion is always oscillatory and never exponential. Therefore the most elementary form of WKB asymptotics (see e.g. Fröman and Fröman 1965) may be employed, without the complications that would arise from turning points, to determine $Q(\tau)$ for small ε. The leading-order behaviour is

$$Q(\tau)\approx[1+\mathrm{O}(\varepsilon)]\cos[\theta(\tau)](XZ-Y^2)^{-1/2} \tag{A1.5}$$

where

$$\theta(\tau)=\theta(0)+\frac{1}{\varepsilon}\int_0^\tau \mathrm{d}\tau'(XZ-Y^2)^{1/2}+\tfrac{1}{2}\int_0^\tau \mathrm{d}\tau'\frac{(Z'Y-Y'Z)}{Z(XZ-Y^2)^{1/2}}+\mathrm{O}(\varepsilon), \tag{A1.6}$$

$\theta(0)$ being the initial phase. Of course these terms correspond precisely to those in equation (2), enabling Hannay's angle to be identified as

$$\Delta\theta=\tfrac{1}{2}\int_0^T \mathrm{d}t\frac{(\dot{Z}Y-\dot{Y}Z)}{Z(XZ-Y^2)^{1/2}}. \tag{A1.7}$$

Transforming to a line integral in parameter space and thence, by Stokes' theorem, to a surface integral, gives

$$\Delta\theta=\int_{\partial A=C}\frac{1}{2}\left[\mathrm{d}\left(\frac{Y}{Z(XZ-Y^2)^{1/2}}\right)\wedge\mathrm{d}Z-\mathrm{d}\left(\frac{1}{(XZ-Y^2)^{1/2}}\right)\wedge\mathrm{d}Y\right] \tag{A1.8}$$

which reduces to

$$\Delta\theta = \tfrac{1}{4}\int_{\partial A=C} \frac{1}{(XZ-Y^2)^{3/2}}(X\,\mathrm{d}Y\wedge\mathrm{d}Z + Y\mathrm{d}Z\wedge\mathrm{d}X + Z\,\mathrm{d}X\wedge\mathrm{d}Y). \tag{A1.9}$$

This is precisely the angle shift given by the previously obtained (38) and (15).

Appendix 2: Rotated rotator in a rotating frame

In the frame of reference rotating with the angular velocity $\Omega = \mathrm{d}X/\mathrm{d}t$ of the hoop, the acceleration $\ddot{s}$ is determined by the centrifugal and Euler pseudo-forces. (The Coriolis force acts perpendicular to the motion and thus affects only the normal reaction of the hoop on the particle.) Thus

$$\ddot{s} = \boldsymbol{t}\cdot[-\boldsymbol{\Omega}\wedge(\boldsymbol{\Omega}\wedge\boldsymbol{q}) - \dot{\boldsymbol{\Omega}}\wedge\boldsymbol{q}] \tag{A2.10}$$

where the vector $\boldsymbol{\Omega}$ is normal to the hoop's plane, and $|\boldsymbol{\Omega}| = \Omega$. Referring to figure 3 we see that this can be written

$$\ddot{s}(t) = \Omega^2(t)q(s(t))\,\mathrm{d}q(s(t))/\mathrm{d}s - \dot{\Omega}(t)q(s(t))\sin[\alpha(s(t))]. \tag{A2.11}$$

Integrating twice we obtain

$$s(t) = s_0 + p_0 t + \int_0^t \mathrm{d}t'(t-t')\{\tfrac{1}{2}\Omega^2(t')\,\mathrm{d}q^2(s(t'))/\mathrm{d}s) \\ -\dot{\Omega}(t')q(s(t'))\sin[\alpha(s(t'))]\} \tag{A2.12}$$

as the equation implicitly determining $s(t)$, where s_0 and p_0 are the initial position and velocity.

Because Ω and $\dot{\Omega}$ are small, the particle makes many circuits while the hoop rotates a little, so that the s-dependent quantities in the square brackets can be replaced by their averages round the hoop, giving the following explicit formula for the position of the particle after the (long) time T in which the hoop turns once:

$$s(T) = s_0 + p_0 T + \int_0^T \mathrm{d}t(T-t)\left(\frac{\Omega^2(t)}{2\mathscr{L}}\int_0^{\mathscr{L}} \mathrm{d}s\frac{\mathrm{d}q(s)}{\mathrm{d}s} \right. \\ \left. -\frac{\dot{\Omega}(t)}{\mathscr{L}}\int_0^{\mathscr{L}} \mathrm{d}s\, q(s)\sin[\alpha(s)]\right). \tag{A2.13}$$

The first hoop integral (from the centrifugal force) vanishes. The entire anholomic effect thus arises, in this formulation, from the Euler force. Use of

$$\int_0^T \mathrm{d}t(T-t)\dot{\Omega}(t) = \int_0^T \mathrm{d}t\,\Omega(t) = 2\pi \tag{A2.14}$$

gives, finally,

$$S(T) = s_0 + p_0 T - \frac{4\pi\mathscr{A}}{\mathscr{L}}, \tag{A2.15}$$

in exact agreement with the previously calculated angle shift (55).

References

Arnol'd V I 1978 *Mathematical Methods of Classical Mechanics* (New York: Springer)
Berry M V 1983 in *Chaotic Behavior of Deterministic Systems* ed G Iooss, R H G Helleman and R Stora (Amsterdam: North-Holland) pp 171-271
—— 1984a *Proc. R. Soc.* A **392** 45-57
—— 1984b *J. Phys. A: Math. Gen.* **17** 1225-33
Dirac P A M 1925 *Proc. R. Soc.* **107** 725-34
Forder P W 1984 *J. Phys. A: Math. Gen.* **17** 1343-55
Fröman N and Fröman P O 1965 *JWKB Approximation; Contributions to the Theory* (Amsterdam: North-Holland)
Hannay J H 1985 *J. Phys. A: Math. Gen.* **18** in press
Keller J B 1958 *Ann. Phys., NY* **4** 180-8
Landau L D and Lifshitz E M 1976 *Mechanics* (*Course of Theoretical Physics*, vol 1) 3rd edn (Oxford: Pergamon)
Lichtenberg A J and Lieberman M A 1983 *Regular and Stochastic Motion* (New York: Springer)
Maslov V P and Fedoriuk M V 1981 *Semiclassical Approximation in Quantum Mechanics* (Dordrecht: Reidel)
Messiah A 1962 *Quantum Mechanics* (Amsterdam: North-Holland)
Percival I C 1977 *Adv. Chem. Phys.* **36** 1-61
Post E J 1967 *Rev. Mod. Phys.* **39** 475-93
Simon B 1983 *Phys. Rev. Lett.* **51** 2167-70