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Classical and advanced multilayered plate elements based upon PVD and RMVT. Part 1: Derivation of finite element matrices

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SUMMARY

This paper deals with the formulation of finite plate elements for an accurate description of stress and strain fields in multilayered, thick plates subjected to static loadings in the linear, elastic cases. The so-called zig-zag form and interlaminar continuity are addressed in the considered formulations. Two variational statements, the *principle of virtual displacements* (*PVD*) and the *Reissner mixed variational theorem* (*RMVT*) are employed to derive finite element matrices. Transverse stress assumptions are made in the framework of RMVT and the resulting finite elements describe *a priori* interlaminar continuous transverse shear and normal stresses. Both modellings which preserve the number of variables independent of the number of layers (equivalent single-layer models, ESLM) and layer-wise models (LWM) in which the same variables are independent in each layer, have been treated. The order N of the expansions assumed for both displacement and transverse stress fields in the plate thickness direction z as well as the number of element nodes N_n have been taken as free parameters of the considered formulations. By varying N, N_n, variable treatment (LW or ESL) as well as variational statements (PVD and RMVT), a large number of newly finite elements have been presented. Finite elements that are based on PVD and RMVT have been called *classical* and *advanced*, respectively.

In order to write the matrices related to the considered plate elements in a concise form and to implement them in a computer code (see Part 2), extensive indicial notations have been set out. As a result, all the finite element matrices have been built from only five arrays that were called fundamental nuclei (four are related to RMVT applications and one to PVD cases). These arrays have 3×3 dimensions and are therefore constituted of only nine terms each. The different formulations are then obtained by expanding the indices that were introduced for the N-order expansion, for the number of nodes N_n and for the constitutive layers N_l. Compliances and/or stiffness are accumulated from layer to multilayered level according to the corresponding variable treatment (ESLM or LWM). The numerical evaluations and assessment for the presented plate elements have been provided in the companion paper (Part 2), where it has been concluded that it is convenient to refer to RMVT as a variational tool to formulate multilayered plate elements that are able to give a quasi-three-dimensional description of stress/strain fields in multilayered thick structures. Copyright © 2002 John Wiley & Sons, Ltd.

KEY WORDS: finite element; plates; multilayers; classical and mixed formulation; composite materials

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1. INTRODUCTION

Multilayered structures are increasingly used in aerospace, ships, automotive vehicles, advanced optical mirrors and semiconductor technologies. Examples of multilayered, anisotropic structures are sandwich constructions, composite structures made of orthotropic laminae, layered structures made of different isotropic layers (such as those employed for thermal protection) as well as intelligent structures embedding piezo-layers. In most of the applications, these structures mostly appear as flat (plates) or curved panels (shells). In this paper, attention has been restricted to flat geometries, although most of the presented derivations and techniques could be extended to shell cases. Examples of multilayered plates are given in Figure 1.

The analysis of multilayered, anisotropic structures is difficult when compared to one-layered ones made of traditional isotropic materials. A number of complicating effects arise when their mechanical behaviour as well as failure mechanisms have to be correctly understood. Interesting discussions on these effects have been reported by Pagano [1]. Some of these complicating effects have clearly been shown by early [2–4] and recent [5, 6] three-dimensional, elasticity solutions. Unfortunately, these elasticity solutions are only available in a very few cases, which are mainly restricted to sample geometries, loadings and boundary conditions as well as orthotropic behaviour of constitutive layers.

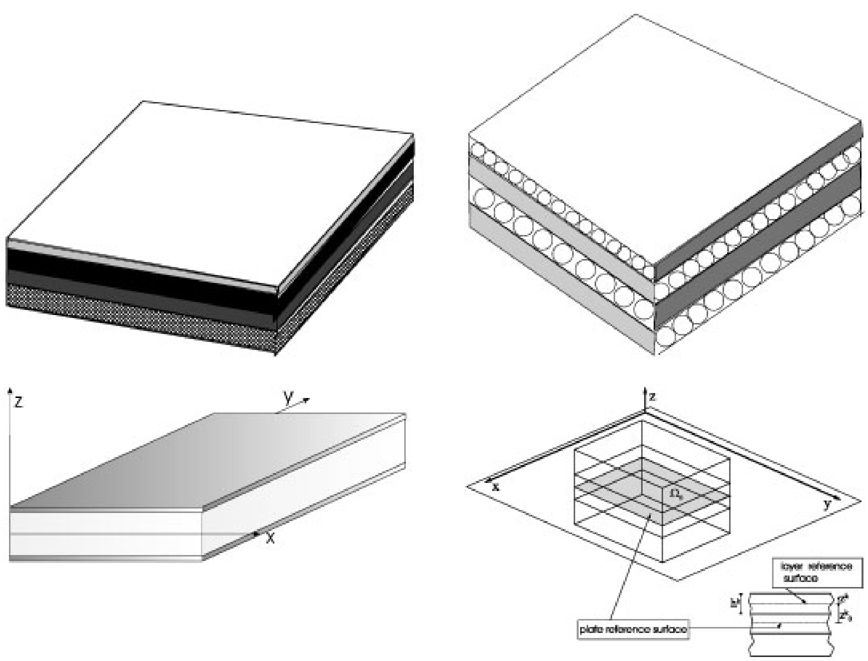

Figure 1. Examples of multilayered structures and plate geometries and notations. Plates (upper part) are made of layers of different materials (left) and by unidirectional fibres (right) and sandwich flat panel (lower, left part).

As far as two-dimensional modelling is concerned, the subject to which this paper is devoted, layered structures also require special attention. This is due to the intrinsic discontinuity of the thermomechanical properties at each layer-interface to which high shear and normal transverse deformability is associated. An accurate description of the stress and strain fields of these structures requires theories that are able to describe the so-called zig-zag (ZZ) form of displacement fields in the thickness z-direction as well as interlaminar continuous (IC) transverse shear and normal stresses (see [7, 2] as examples). Transverse and in-plane anisotropy of multilayered structures make it difficult to find closed-form solutions when these structures are subjected to the usual static and dynamical loadings of the environment to which these structures are exposed when in use. The use of approximated solutions is necessary in these cases. It can therefore be concluded that the use of both refined two-dimensional theories and computational methods become mandatory to solve practical problems related to multilayered structures.

A large number of refined theories and computational strategies have been proposed and implemented over the last four decades. Among the implemented computational strategies, the iterative techniques based on *a posteriori* evaluations developed by Noor and co-authors [8–10], the recent differential quadrature technique proposed by Malik [11], Malik and Bert [12] and recently applied by Teo and Liew [13], and the interesting boundary element formulation proposed by Davì [14] and recently applied by Davì [15], Davì and Milazzo [16] and Milazzo [17] are herein mentioned. Excellent overview papers are available on the topics of computational methods for multilayered structures analyses (see Section 2).

Among the several available computational methods, the finite element method (FEM) has played, and continues to play, a significant role. Most of the commercial codes that are used in small and large companies as well as in research centres are, in fact, finite element oriented. The subject of the present work consists of multilayered finite elements that are able to furnish an accurate description of strain/stress fields in multilayer flat structure analysis. Reissner's mixed variational theorem (RMVT) is used to derive what have been called *advanced*[‡] multilayered finite elements. As a main property, RMVT permits one to assume two independent fields for displacement and transverse stress variables. The resulting advanced finite elements therefore describe *a priori* interlaminar continuous transverse shear and normal stress fields. *Classical* finite elements with only displacement variables are formulated on the basis of the principle of virtual work (PVD) for comparison purposes. The number of both the order N of expansion in z and the number of nodes N_n of the elements are taken as free parameters of the considered RMVT and PVD formulations. As a result, apart from the new finite elements based on RMVT, a number of new classical FE based on PVD are proposed in this work.

In order to lower the number of equations related to the several presented finite elements as much as possible, the indicial notation already used in the first author's papers [18–21] has herein been extended to finite element applications. As a fundamental property, such an indicial notation has led to the writing of all the finite element matrices in terms of a few arrays, which are called *fundamental nuclei*, the dimension of which is 3×3. These fundamental nuclei are herein written at a layer level; such a choice has permitted the authors

[‡]The use of the word 'advanced' has been preferred by the authors, over a few others, such as 'refined', 'mixed' or 'higher order'.

to treat both modellings which preserve the number of variables independent of the number of layers (equivalent single-layer models, ESLM) and layer-wise models (LWM) in which the same variables are independent in each layer, at the same time. The variational statements and continuity requirements for stresses and displacements as well as non-homogeneous boundary conditions at each interface, for displacement and/or transverse stress variables, are used to derive matrices from layers to multilayers and from elements to structure levels.

This paper has been organized as follows. Section 2 outlines the necessary requirements (herein referred to as C_z^0-requirements) that should be taken into account for an accurate description of multilayered structures. Relevant contributions based on different approaches are briefly outlined. RMVT is introduced as a possible tool to completely meet the C_z^0-requirements. Available, relevant finite element implementations are also overviewed in this section. Section 3 quotes the preliminaries that are used in the subsequent sections. Geometries and Hooke's law in classical and mixed forms are given along with strain displacement relations and typical finite element descriptions. Section 4 briefly recalls the employed variational statements. RMVT and PVD are introduced along with their use in the framework of finite element applications. The used indicial notations are also explained in this section. Sections 5 and 6 describe the two-dimensional assumptions that one made on the displacements and transverse stresses. Section 7 derives the first 3×3 *fundamental nuclei* related to classical PVD applications. Section 8 derives the further four 3×3 *fundamental nuclei* related to RMVT applications. Section 9 discusses the possible treatment of stress variables for RMVT-based finite elements. Section 10 gives a summary of the derived multilayered finite elements along with concluding remarks.

Further derivations have been outlined in the form of appendices as follows. Appendix A reports an example that shows how the introduced indicial notations work. A well-known finite element based on PVD has been considered. Appendix B gives an example of loading vectors related to the finite element formulations that have been treated. Appendix C identifies the compliance/stiffness terms that require specific, numerical sub-integration schemes.

A companion paper (Part 2) has been written to provide numerical evaluations related to some of the herein proposed finite elements.

2. MODELLINGS AND FE IMPLEMENTATIONS

This section gives some insight into the peculiarities of two-dimensional modellings of multilayered plates (Section 2.1). Analytical developments are also considered in Section 2.1, while available, related finite element implementations are briefly discussed in Section 2.2.

The literature overview is not complete. A more exhaustive discussion on the several contributions that have been made in the recent past has been covered by recent state-of-the-art articles. Among these, one can mention the papers by Librescu and Reddy [22], Kapania and Raciti [23], Noor and Burton [9], Reddy and Robbins [24], Noor *et al.* [25], and the books by Librescu [26] and Reddy [27].

2.1. Two-dimensional modellings of multilayered structures

2.1.1. High transverse deformability. As far as two-dimensional modelling is concerned, the main task of multilayered constructions is related to the possibility of exhibiting different mechanical–physical properties in the thickness plate direction. These are *transversely anisotropic* (TA) structures. In addition, anisotropic multilayered structures often show both higher transverse shear and normal flexibility, with respect to in-plane deformability, than traditional isotropic one-layered ones. These are *transversely high deformable* (THD) structures. For instance, laminated structures made of advanced composite materials presently used in aerospace structures could exhibit high values of Young's moduli orthotropic ratio ($E_L/E_T = E_L/E_z = 5$–40, where L denotes the fibre directions while T and z are two-directions orthogonal to L) and low transverse shear moduli ratio ($G_{LT}/E_T \approx G_{TT}/E_T = 1/10$–$1/200$), leading to higher transverse shear and normal stress deformability than in isotropic cases.

As a direct consequence of both TA and THD, well-known thin-plate theories [28–30] that were originally developed for traditional isotropic one-layer structures could be inadequate to predict the response of multilayered structures. Extension of thin-plate theories to multilayered structures are often denoted as *classical lamination theories* (CLT); see Jones [31] as an example. Transverse shear as well as normal strains are, in fact, postulated to be negligible with respect to the other strains in CLT plate analyses.

Improvements of thin-plate theories should be made according to the well-known *Koiter's recommendation* (KR) [32]: a refinement of Kirchhoff thin-plate theory is indeed meaningless, in general, unless the effects of transverse shear and normal stresses are taken into account at the same time. A great deal of contributions have been presented in the literature in which thin-plate and improved theories, already known for isotropic one-layered structures, have been extended to multilayered structures. Extensions of Reissner [33] and/or Mindlin [34] refined-type models, which violate KR and include only transverse shear strains, to layered structures are known as the *shear deformation theory* (SDT) (*or first-order SDT*, $FSDT$); see Yang *et al.* [35], Whitney [7] and the recent book by Reddy [27]. KR can be retained by including both transverse shear and normal strains, as done by Hildebrand *et al.* [36]. Examples of applications of these types of models to laminated structures can be found in the works by Sun and Whitney [37] and Lo *et al.* [38]. These are all known as *higher-order theories* (HOT).

2.1.2. Zig-zag effects and interlaminar continuity: C_z^0-requirements. In addition to the discussed refinements known for one-layer plates made of isotropic materials, the layered construction introduces further complicating effects. Transverse discontinuous mechanical properties cause, in fact, displacement fields $\mathbf{u} = (u_1, u_2, u_3)$ (bold letters denote arrays, while subscripts 1, 2, 3, denote the components in the x, y, z, directions, respectively) in the thickness direction which can exhibit rapid changes and different slopes in correspondence to each layer interface (see Figures 1 and 2). This is known as the *zig-zag* (ZZ) form of displacement field in the thickness shell direction. Although in-plane stresses $\boldsymbol{\sigma}_p = (\sigma_{11}, \sigma_{22}, \sigma_{12})$ can in general be discontinuous, equilibrium reasons, i.e. the Cauchy theorem, demand continuous transverse stresses $\boldsymbol{\sigma}_n = (\sigma_{13}, \sigma_{23}, \sigma_{33})$ at each layer interface (see Figure 3). This is often referred to in the literature as *interlaminar continuity* (IC) of transverse shear and normal stresses. Figure 2 shows, from a qualitative point of view, what could be the scenario of displacement $\mathbf{u}$ and transverse stress $\boldsymbol{\sigma}_n$ distributions in a multilayered structure in exact solutions and/or experiments. Stresses at the interfaces are displayed in Figure 3. In-plane components, which can be discontinuous, are also depicted for comparison. Figures 2 and 3 show that both displacement and transverse stresses, due to compatibility and equilibrium reasons, respectively, are C^0-continuous functions in the thickness z direction. $\mathbf{u}$ and $\boldsymbol{\sigma}_n$ have, in the most general case,

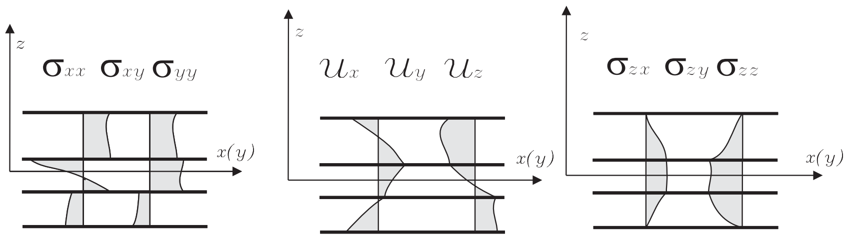

Figure 2. C_z^0-requirements. Displacement and stress (in-plane and transverse) fields in the thickness plate direction. Three-layered plate.

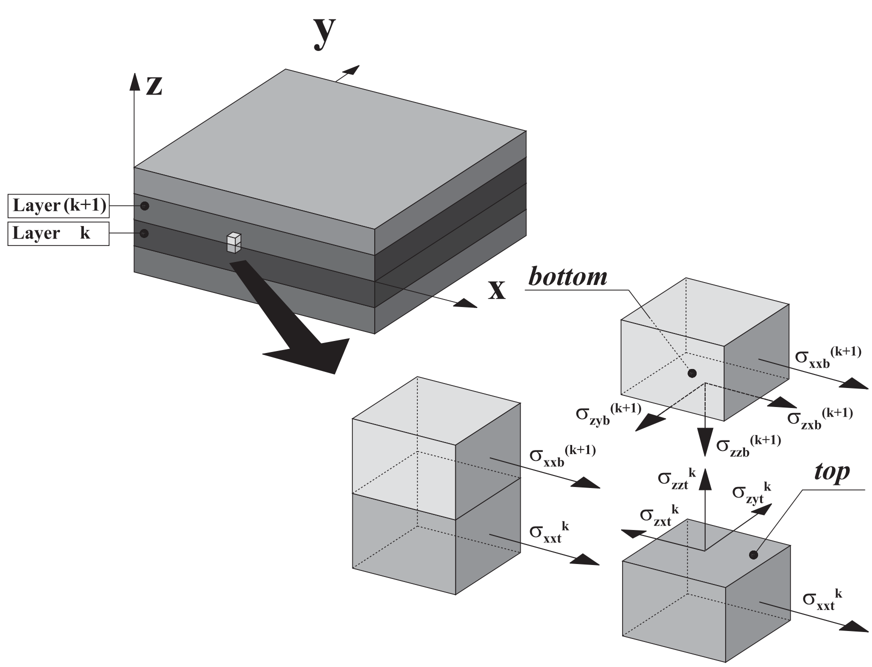

Figure 3. Details of the stress states at the interface between two consecutive layers.

discontinuous first derivatives with correspondence to each interface where the mechanical properties change. In References [18, 39], ZZ and IC were referred to as C_z^0-requirements. The fulfilment of C_z^0-requirements is a crucial point of two-dimensional modelling of multilayered structures.

The extension *sic et simpliciter* of CLT, FSDT and HOT to multilayered plates does not permit the fulfilment of the C_z^0-requirements, that is, ZZ and IC are not addressed by the mentioned CLT, SDT and HOT. An exception is given by Vlasov's [40] SDT-type theory which permits fulfilment of the homogeneous conditions for the transverse shear stresses in correspondence to the top and bottom shell/plate surfaces. Reddy [41, 42] and Reddy and

Phan [43] have shown that such simple inclusion could lead to significant improvement, with respect to SDT, in tracing the static and dynamic response of thick laminated structures.

The theories mentioned above all have the number of unknown variables that are independent of the number of constitutive layers N_l. Following Reddy [27], these types of theories are here grouped as *equivalent single-layer models* (*ESLM*). A possible, natural manner of including the ZZ effect could be implemented by applying CLT, FSDT or HOT at a layer level, that is, each layer is seen as an independent plate and compatibility of displacement is then imposed as a constraint. In these cases, *layer-wise models* (*LWM*) are obtained. Relevant examples of these types of theories are those found in the articles by Srinivas [44], who applied CLT in each layer, and Cho *et al.* [45], who implemented the HOT by Lo *et al.* [38] in each layer. Generalizations on these types of theories were given by Nosier *et al.* [46] and Reddy [27], who expressed the displacement variables in the thickness direction in terms of Lagrange polynomials (interface values were used as unknown variables), therefore permitting an easy linkage of compatibility conditions at each interface.

Literature has shown that LWM provides much better results than those related to ESLM-type analyses. The overviewed ESL or LW models, being formulated with only displacement unknowns, cannot describe *a priori* IC for the transverse shear and normal stresses, i.e. C_z^0-requirements are not completely fulfilled. Apart from the previously discussed contributions, special mention should be made of those works in which the description of both IC and ZZ effects is addressed. Among these, one should mention the pioneering paper by Lekhnitskii [47, 48] that was originally developed for beams, and which describes interlaminar continuous transverse shear stress as well as ZZ effects, and the plate/shell theories by Ambartsumian [49]. Lekhnitskii's theory was extended to plates by Ren [50] and Ren and Owen [51], while Ambartsumian's theory was first extended to unsymmetric plate cases by Whitney [7] and then to shell geometries by Rath and Das [52]. Hundreds of papers have been published that are based on the Ambartsumian–Whitney–Rath–Das theory (see the overview papers already mentioned at the beginning of this section). Most of the works based on these types of theories do not account for the interlaminar continuous transverse normal stress σ_{33} description, i.e. KR is discarded.

2.1.3. Use of RMVT. All the previously discussed theories are formulated on the basis of displacement variables. These types of theories are not designed to *a priori* describe interlaminar continuous transverse stresses. A post-processing procedure is required to recover $\boldsymbol{\sigma}_n$ stresses. Post-processing can be avoided if and only if stress assumptions are made. In-plane and transverse stresses can be assumed in the framework of mixed variational principles (see [41, 53]). *Reissner's mixed variational theorem consists of a mixed principle designed for multilayered structures*. RMVT, in fact, restricts the stress assumptions to transverse components. Murakami [54–56] was the first to apply RMVT to multilayered structures by assuming two independent fields for displacement and transverse stress variables. Toledano and Murakami [57, 58] showed that RMVT does not experience any particular difficulty when including transverse normal stresses in a plate theory.

A more comprehensive evaluation of LW and ESL theories was considered by Carrera [19, 20] where applications to the static analysis of plates were presented. Subsequent works extended the analysis to the dynamics case [21]. In References [19–21, 59–64], Carrera showed that RMVT leads to a quasi-three-dimensional description of the in-plane and out-of plane response. In particular, transverse stresses were determined *a priori* with excellent accuracy.

One should conclude that RMVT appears to be a natural tool to completely and *a priori* fulfil the C_z^0-requirements in both LW and ESL cases. An exhaustive review in RMVT has been recently proposed by Carrera in Reference [65].

2.2. Finite element implementations

Many finite elements have been proposed which were based on the approaches mentioned in the last section. Others based on some special finite element techniques (such as hybrid) have also been proposed. Some of these are briefly discussed in the next two subsections. A third subsection overviews available finite element implementation based on RMVT.

2.2.1. Implementation of refined two-dimensional theories. Early papers concerning laminated plate elements including transverse shear effects SDT have been developed by Pryor and Barker [66], Noor [67], Noor and Mathers [68], Panda and Natarayan [69], Reddy [70] and Kant and Kommineni [71]. Reduced, selective integration [72] and the assumed shear strain concept [73, 74] are known techniques to contrast shear locking and spurious modes associated to these implementations. Many refinements of SDT-type elements have been proposed (see the overview papers by Pandya and Kant [75], Reddy [76] and Barboni and Gaudenzi [77]).

Dozens of finite elements have been proposed that are based on the Ambartsumian–Whitney–Rath–Das-type theory. Among these, the recent works by Cho and Parmeter [78], Aitharaju and Averill [79], Idlbi *et al.* [80], Cho and Averill [81] and Polit and Touratier [82] can be mentioned.

Layer-wise finite elements have been discussed by Pinsky and Kim [83], Reddy [84], Robbin's and Reddy [85], Gaudenzi *et al.* [86], and more recently by Botello *et al.* [87].

The finite element models based on ESL or LW approaches have their own advantages/disadvantages in terms of solution accuracy and/or solution economy. However, these approaches can be combined to lead to the so-called 'multiple-method' or 'global/local technique': an LW description is used in those zones of the structures in which an accurate description is required while ESL is left for the remaining parts. Examples of these approaches can be found in Reddy [27]. Similar techniques, denoted as 'sub-laminate approaches', have recently been developed in the already mentioned Cho and Averill article [81], in the framework of zig-zag-type theories. The so-called 'hierarchy' finite elements for laminated plates were discussed by Babuska *et al.* [88] for similar reasons. The analytical derivations and numerical evaluations were restricted to laminated strips. Similar approaches, named 's-version', were used by Fish and Markolefas [89]. Finite elements based on asymptotic expansion of three-dimensional elasticity equations have been discussed by Turn *et al.* [90].

2.2.2. Mixed and hybrid FE. Discussions on mixed finite elements can be read in the interesting articles reported in the book by Atluri *et al.* [53]. Hybrid stress finite elements are based on a modified complementary energy statement in which equilibrating intra-element stresses and, independently, intra-element or element boundary displacements are interpolated in terms of stress parameter and nodal displacement, respectively. The stress parameters are then eliminated on an element level and a stiffness matrix is obtained. Four-node hybrid stress laminated plate elements, including transverse shear effects, have been developed by Mau and Pian [91] and Spilker *et al.* [92, 93]. Stress fields were defined for each layer (in the ESLM) or for the laminate (for the ESLM case) with interlayer traction continuity and upper/lower laminate traction-free conditions enforced exactly. More recently, three-dimensional hybrid stress elements have been proposed by Moriya [94] and Liou and Sun [95] and a partial hybrid stress element was developed by Jing and Liao [96]. Partial mixed finite elements have been proposed by Auricchio and Sacco [97] which were based on a re-elaboration of the classical Hellingher–Reissner mixed functional. These were directed to the building of improved FSDT-type finite elements.

2.2.3. Available plate elements based on RMVT. The first FE approach to multilayered structures by means of RMVT is due to Jing and Liao [96]. A partial hybrid formulation was presented; a self-equilibrated stress field was restricted to the three in-plane stresses. As usual in hybrid formulation, stress unknowns were eliminated at an element level in the implemented finite hexahedron element for each layer. The results were restricted to cross-ply plates and showed good accuracy with respect to the exact solution and improvements with respect to other refined analyses.

Application of RMVT to develop standard finite elements was proposed by Rao and Meyer-Piening [98]. The Toledano and Murakami [57, 58] theory was used. Stress unknowns were eliminated before introducing FE approximations by employing a technique that is equivalent to the so-called weak form of Hooke's law (WFHL), which was introduced in Reference [18], that is, only displacements were taken as nodal variables in the ESLM framework. Applications were quoted for laminate and sandwich plates and were related to eight-noded plate isoparametric elements.

The extension of the standard Reissner–Mindlin model to multilayered structures was discussed by Carrera [99]. The obtained finite plate elements (four, eight and nine nodes) were denoted with the acronym RMZC (Reissner–Mindlin zig-zag interlaminar continuity). The weak form of Hooke's law proposed in Reference [18] was used to eliminate transverse shear stress variables. The numerical efficiency of the RMZC FE model was tested for non-linear problems in subsequent works. Large deflection of post-buckling was analysed by Carrera and Kröplin [100]. Non-linear dynamics problems were solved by Carrera and Krause [101]. Applications to linear and non-linear multilayered plates embedding piezo-layers were given by Carrera [102]. Extensive application to sandwich plates was quoted by Carrera [59] while extension to shells has recently been provided by Brank and Carrera [74].

It can be concluded that no study is available in which a systematic application of RMVT to develop ESLM as well as LW advanced multilayered plate elements is made. This is the subject of the present paper, in which PVD applications are mainly developed for comparison purposes.

3. PRELIMINARY ASSUMPTIONS

3.1. Geometry and notations for multilayered plates

The geometry and co-ordinate system of the laminated plates of N_l layers have been shown in Figure 1. The integer k, which is extensively used as both subscripts or superscripts, denotes the layer number that starts from the plate bottom. x and y are the plate middle surface Ω^k co-ordinates. Ω_0 and Ω will be also used to denote the reference surface. Γ^k is the layer boundary on Ω^k. z and z_k are the plate and layer thickness co-ordinates; h and h_k denote plate and layer thickness, respectively. $\zeta_k = 2z_k/h_k$ is the non-dimensional local plate co-ordinate;

A_k will denote the k-layer thickness domain. Symbols not affected by k subscripts/superscripts refer to the whole plate.

3.2. *Hooke's law for orthotropic lamina in the material reference system*

The laminae are considered to be homogeneous and to operate in the linear elastic range. By employing stiffness coefficients, Hooke's law for the anisotropic k-lamina is written in the form $\sigma_i = C_{ij}\varepsilon_j$, where the sub-indices i and j, ranging from 1 to 6, stand for the index couples $11, 22, 33, 13, 23$ and 12, respectively. The material is assumed to be orthotropic, as specified, by $C_{14} = C_{24} = C_{34} = C_{64} = C_{15} = C_{25} = C_{35} = C_{65} = 0$. This implies that σ_{13}^k and σ_{23}^k only depend on ε_{13}^k and ε_{23}^k. In matrix form,

$$\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{12} \\ \sigma_{13} \\ \sigma_{23} \\ \sigma_{33} \end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & 0 & 0 & 0 & C_{13} \\ C_{12} & C_{22} & 0 & 0 & 0 & C_{23} \\ 0 & 0 & C_{66} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{55} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ C_{13} & C_{23} & 0 & 0 & 0 & C_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{12} \\ \varepsilon_{13} \\ \varepsilon_{23} \\ \varepsilon_{33} \end{bmatrix} \tag{1}$$

3.2.1. *Hooke' law for orthotropic lamina in the plate reference system.*

Multilayered plates are often composed of layers made up with different orientation. It is therefore of interest to write the previous Hooke's law from the material axis 1, 2, 3 into the reference (or problem) Cartesian system x, y, z.

$$\boldsymbol{\sigma}_m = [\sigma_{11} \;\; \sigma_{22} \;\; \sigma_{12} \;\; \sigma_{13} \;\; \sigma_{23} \;\; \sigma_{33}]^{\mathrm{T}} \tag{2}$$

$$\boldsymbol{\varepsilon}_m = [\varepsilon_{11} \;\; \varepsilon_{22} \;\; \varepsilon_{12} \;\; \varepsilon_{13} \;\; \varepsilon_{23} \;\; \varepsilon_{33}]^{\mathrm{T}} \tag{3}$$

$$\boldsymbol{\sigma} = [\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{xy} \;\; \sigma_{xz} \;\; \sigma_{yz} \;\; \sigma_{zz}]^{\mathrm{T}} \tag{4}$$

$$\boldsymbol{\varepsilon} = [\varepsilon_{xx} \;\; \varepsilon_{yy} \;\; \varepsilon_{xy} \;\; \varepsilon_{xz} \;\; \varepsilon_{yz} \;\; \varepsilon_{zz}]^{\mathrm{T}} \tag{5}$$

The relations between the coefficient in the two reference system are:

$$\boldsymbol{\sigma} = \mathbf{T}\boldsymbol{\sigma}_m \tag{6}$$

$$\boldsymbol{\varepsilon}_m = \mathbf{T}^{\mathrm{T}}\boldsymbol{\varepsilon} \tag{7}$$

$$\boldsymbol{\sigma}_m = \mathbf{C}\boldsymbol{\varepsilon}_m \tag{8}$$

Upon substitution of Equation (7) into Equation (8) and by using Equation (6), one has

$$\boldsymbol{\sigma} = \mathbf{T}\mathbf{C}\mathbf{T}^{\mathrm{T}}\boldsymbol{\varepsilon} = \tilde{\mathbf{C}}\boldsymbol{\varepsilon} \tag{9}$$

3.2.2. Mixed form of Hooke's law. For our convenience, stresses and strains are grouped into two sets, in-plane and out-of-plane (transverse) components:

$$\boldsymbol{\sigma}_p = [\sigma_{xx} \;\; \sigma_{yy} \;\; \sigma_{xy}]^{\mathrm{T}}, \;\; \boldsymbol{\sigma}_n = [\sigma_{xz} \;\; \sigma_{yz} \;\; \sigma_{zz}]^{\mathrm{T}} \tag{10}$$

$$\boldsymbol{\varepsilon}_p = [\varepsilon_{xx} \;\; \varepsilon_{yy} \;\; \varepsilon_{xy}]^{\mathrm{T}}, \;\; \boldsymbol{\varepsilon}_n = [\varepsilon_{xz} \;\; \varepsilon_{yz} \;\; \varepsilon_{zz}]^{\mathrm{T}} \tag{11}$$

The same is done for the matrices:

$$\tilde{\mathbf{C}}_{pp} = \begin{bmatrix} \tilde{C}_{11} & \tilde{C}_{12} & \tilde{C}_{16} \\ \tilde{C}_{12} & \tilde{C}_{22} & \tilde{C}_{26} \\ \tilde{C}_{16} & \tilde{C}_{26} & \tilde{C}_{66} \end{bmatrix}, \quad \tilde{\mathbf{C}}_{pn} = \begin{bmatrix} 0 & 0 & \tilde{C}_{13} \\ 0 & 0 & \tilde{C}_{23} \\ 0 & 0 & \tilde{C}_{36} \end{bmatrix} \tag{12}$$

$$\tilde{\mathbf{C}}_{np} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \tilde{C}_{13} & \tilde{C}_{23} & \tilde{C}_{36} \end{bmatrix}, \quad \tilde{\mathbf{C}}_{nn} = \begin{bmatrix} \tilde{C}_{55} & \tilde{C}_{45} & 0 \\ \tilde{C}_{45} & \tilde{C}_{44} & 0 \\ 0 & 0 & \tilde{C}_{33} \end{bmatrix} \tag{13}$$

Hooke's law is therefore rewritten as

$$\begin{bmatrix} \boldsymbol{\sigma}_p \\ \boldsymbol{\sigma}_n \end{bmatrix} = \begin{bmatrix} \tilde{\mathbf{C}}_{pp} & \tilde{\mathbf{C}}_{pn} \\ \tilde{\mathbf{C}}_{np} & \tilde{\mathbf{C}}_{nn} \end{bmatrix} \begin{bmatrix} \boldsymbol{\varepsilon}_p \\ \boldsymbol{\varepsilon}_n \end{bmatrix} \tag{14}$$

That is,

$$\boldsymbol{\sigma}_p = \tilde{\mathbf{C}}_{pp}\boldsymbol{\varepsilon}_p + \tilde{\mathbf{C}}_{pn}\boldsymbol{\varepsilon}_n \tag{15}$$

$$\boldsymbol{\sigma}_n = \tilde{\mathbf{C}}_{np}\boldsymbol{\varepsilon}_p + \tilde{\mathbf{C}}_{nn}\boldsymbol{\varepsilon}_n \tag{16}$$

$$\boldsymbol{\sigma}_p = \mathbf{C}_{pp}\boldsymbol{\varepsilon}_p + \mathbf{C}_{pn}\boldsymbol{\sigma}_n \tag{17}$$

$$\boldsymbol{\varepsilon}_n = \mathbf{C}_{np}\boldsymbol{\varepsilon}_p + \mathbf{C}_{nn}\boldsymbol{\sigma}_n \tag{18}$$

Equations (17) and (18) represent the mixed form of Hooke's law. Such a form plays a fundamental role in the use of RMVT. The relations between the two forms of Hooke's law are

$$\begin{aligned} \mathbf{C}_{pp} &= \tilde{\mathbf{C}}_{pp} - \tilde{\mathbf{C}}_{pn}(\tilde{\mathbf{C}}_{nn})^{-1}\tilde{\mathbf{C}}_{np} \\ \mathbf{C}_{pn} &= \tilde{\mathbf{C}}_{pn}(\tilde{\mathbf{C}}_{nn})^{-1} \\ \mathbf{C}_{np} &= -(\tilde{\mathbf{C}}_{nn})^{-1}\tilde{\mathbf{C}}_{np} \\ \mathbf{C}_{nn} &= (\tilde{\mathbf{C}}_{nn})^{-1} \end{aligned} \tag{19}$$

3.3. Strain–displacement relations

As one remains within the small deformation field, the strain components $\boldsymbol{\varepsilon}_p, \boldsymbol{\varepsilon}_n$ are linearly related to the displacements $\mathbf{u}$ according to the differential, geometrical relations,

$$\boldsymbol{\varepsilon}_p = \mathbf{D}_p \mathbf{u} \tag{20}$$

$$\boldsymbol{\varepsilon}_n = \mathbf{D}_n \mathbf{u} = (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz})\mathbf{u} \tag{21}$$

where $\mathbf{u}$ denotes the array of the displacement components,

$$\mathbf{u} = [u_x \;\; u_y \;\; u_z]^{\mathrm{T}} \tag{22}$$

The differential matrices are

$$\mathbf{D}_p = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \end{bmatrix}, \quad \mathbf{D}_n = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \tag{23}$$

$$\mathbf{D}_{n\Omega} = \begin{bmatrix} 0 & 0 & \frac{\partial}{\partial x} \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & 0 & 0 \end{bmatrix}, \quad \mathbf{D}_{nz} = \begin{bmatrix} \frac{\partial}{\partial z} & 0 & 0 \\ 0 & \frac{\partial}{\partial z} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \end{bmatrix} \tag{24}$$

3.4. Finite element description and shape functions

Following standard FEM, the unknown variables in the element domain are expressed in terms of their values with correspondence to the element nodes. According to the isoparametric description, displacements or stresses are expressed as follows:

$$\mathbf{u}_\tau^k = N_i \mathbf{q}_{\tau i}^k \quad (i = 1, 2, \ldots, N_n) \tag{25}$$

where

$$\mathbf{q}_{\tau i}^k = [q_{u_x \tau i}^k \;\; q_{u_y \tau i}^k \;\; q_{u_z \tau i}^k]^{\mathrm{T}} \tag{26}$$

and

$$\boldsymbol{\sigma}_{n\tau}^k = N_i \mathbf{g}_{\tau i}^k \quad (i = 1, 2, \ldots, N_n) \tag{27}$$

where

$$\mathbf{g}_{\tau i}^k = [g_{xz\tau i}^k \;\; g_{yz\tau i}^k \;\; g_{zz\tau i}^k]^{\mathrm{T}} \tag{28}$$

N_n is the number of the node of the element and it is taken as free parameter of the model. N_i are the shape functions and $\mathbf{q}_{\tau i}^k$, $\mathbf{g}_{\tau i}^k$ are nodal variables. ξ, η are the natural co-ordinates. Explicit forms can be found in one of the many books on FEM; a few cases are detailed in Part 2 [103].

4. RMVT AND PVD AND THEIR USE TO DEVELOP FINITE ELEMENTS

4.1. PVD and RMVT

For a complete and rigorous understanding of the foundations of RMVT, reference can be made to the articles by Professor Reissner [104, 105]. Readers can refer to these works for a systematic comprehension of the mathematical/variational background of Reissner's theorem. Here the author's aim is to try to give a simple interpretation of RMVT, starting from the basic concept of continuum mechanics and the well-known statements of calculus of variations (see [106, 41, 53]).

In solid mechanics, it is well-known that the principle of virtual displacement (PVD) involves only a compatible displacement field as a variable, and has as its Euler–Lagrange the conditions of balance of momenta and traction boundary conditions. Likewise, the dual form of PVD, i.e. the principle of virtual forces (PVF), involves a stress field that is equilibrated and satisfies the traction boundary conditions, alone as a variable and has as its Euler–Lagrange equations the kinematic compatibility conditions and displacement boundary conditions. If in PVD kinematic compatibility and displacement boundary conditions are introduced as conditions of constraint through Lagrange multipliers, which turn out to be stresses and surface traction, respectively, one then obtains the so-called Hu–Washizu variational principle. Likewise, if the condition of equilibrium of stresses is introduced as a constraint condition through a Lagrange multipliers field (which turns out to be displacement) into PVF, one is led to the so-called Hellingher–Reissner principles. Thus, the Hu–Washizu and Hellingher–Reissner principles, which involve that one field in the continuum as variables (some of which play the role of Lagrange multipliers to enforce certain constraint conditions), are often referred to as mixed variational principles.

This is the scenario in which RMVT can be simply interpreted as a particular case of the previously mentioned mixed principles in which *only* compatibility of transverse strain $\boldsymbol{\varepsilon}_n = (\varepsilon_{13}, \varepsilon_{23}, \varepsilon_{33})$ is enforced by means of Lagrange multipliers which, in this case, turns out to be transverse stresses $\delta\boldsymbol{\sigma}_n = (\delta\sigma_{13}, \delta\sigma_{23}, \delta\sigma_{33})$ (δ is the variational symbol). The word *only* signifies Reissner's intuition: for multilayered structure analyses, it is sufficient to restrict the mixed assumptions to transverse stresses. It is for such stresses that an independent field is, in fact, required to *a priori* and *completely* fulfil the C_z^0-require ments.

PVD assumes a displacement field $\mathbf{u}$ and puts three-dimensional indefinite equilibrium (and related equilibrium conditions at the boundary surfaces which are, for the sake of brevity, not written here) into a variational form. In the static case, these equations are

$$\sigma_{ij_{,i}} = p_i, \quad i, j = 1, 2, 3 \tag{29}$$

$\mathbf{p} = (p_1, p_2, p_3)$ are volume loadings. The corresponding PVD integral, variational equation for a multilayered structure is written as

$$\int_V (\delta \boldsymbol{\varepsilon}_{pG}^{\mathrm{T}} \boldsymbol{\sigma}_{pH} + \delta \boldsymbol{\varepsilon}_{nG}^{\mathrm{T}} \boldsymbol{\sigma}_{nH}) \, \mathrm{d}V = \delta L_{\mathrm{e}} \tag{30}$$

V denotes the three-dimensional multilayered body volume while the subscript H underlines that stresses are computed via Hooke's law. The variation of the internal work has been split into in-plane and out-of-plane parts and involves stress from Hooke's law and strain from geometrical relations (subscript G). δL_{e} is the virtual variation of the work made by the external layer force $\mathbf{p}$.

RMVT can be simply constructed by adding the constraint equations for the transverse stresses to PVD. These equations can be built by evaluating transverse strains in two ways: by Hooke's law $\boldsymbol{\varepsilon}_{nH}$ and by geometrical relations $\boldsymbol{\varepsilon}_{nG}$. In formula

$$\boldsymbol{\varepsilon}_{nH} - \boldsymbol{\varepsilon}_{nG} = 0 \tag{31}$$

RMVT therefore states

$$\int_V (\delta \boldsymbol{\varepsilon}_{pG}^{\mathrm{T}} \boldsymbol{\sigma}_{pH} + \delta \boldsymbol{\varepsilon}_{nG}^{\mathrm{T}} \boldsymbol{\sigma}_{nM} + \delta \boldsymbol{\sigma}_{nM}^{\mathrm{T}} (\boldsymbol{\varepsilon}_{nG} - \boldsymbol{\varepsilon}_{nH})) \, \mathrm{d}V = \delta L_{\mathrm{e}} \tag{32}$$

The third 'mixed' term variationally enforces the compatibility of the transverse strain components. Subscript M underlines that transverse stresses are those of the assumed model.

4.2. Use of RMVT and PVD to develop finite elements

RMVT and PVD can be used to derive governing equations of plate problems in a strong form. Examples of the use of RMVT to derive governing differential equations are given in the already mentioned papers, see Murakami [56] and Carrera [101] as examples. In the present work, these two variational tools are used to establish the weak form of equilibrium and compatibility according to finite element approximations.

In the so-called axiomatic approach, a certain displacement and/or stress fields are postulated in the plate z-direction. An interesting discussion on the implications of the axiomatic character of a given theory has been provided by Antona [107]. According to PVD and RMVT variational statements, multilayered plate elements could be formulated according to the following five steps.

1. Displacement and/or stress distributions in the thickness z plate direction are *postulated* by referring to a certain set of base functions (Sections 5 and 6).
2. Material behaviour is assigned, i.e. Hooke's law is given (Section 3.2).
3. A geometrical relation is given, i.e. a strain–displacement relation is assumed (Section 3.3).
4. A finite element description and shape functions are introduced (Section 3.4).
5. Variational statements are then used to establish in weak sense finite element matrices (Sections 7 and 8).

These developments are presented in the most general cases of N-order for the expansion of the unknown variable in the z-thickness co-ordinate. The number of the nodes N_n is also taken as a free parameter of the present work.

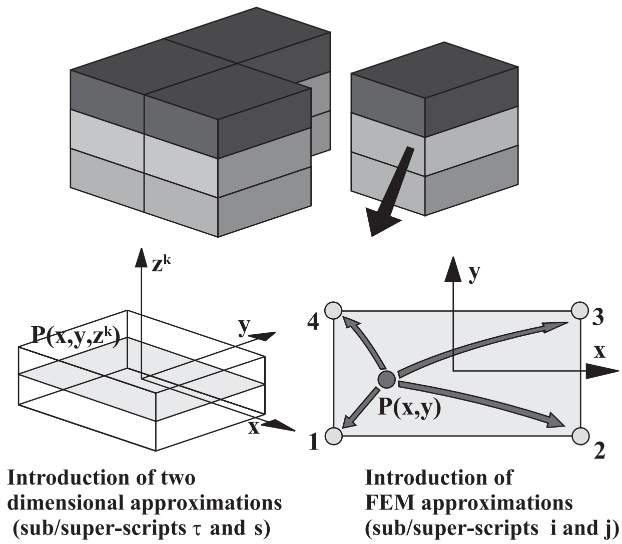

Figure 4. Summary of the introduced approximations and related indicial notations.

4.3. Summary of the introduced approximations, indicial notations and fundamental nuclei 3×3

In order to derive finite element matrices according to PVD or RMVT, the introduced approximations can be summarized in the following two points:

1. The three-dimensional problem is reduced to a two-dimensional one by postulating a certain behaviour in the plate thickness direction z. As a result, the unknown variables only depend on the x, y co-ordinates which are defined on Ω.
2. The unknowns on Ω are further expressed in terms of nodal variables via shape function assumptions.

Figure 4 shows a plate discretized in a certain number of finite elements. The dependence of the unknown variables on z is first eliminated via the introduction of two-dimensional approximations. The unknown variables are therefore only defined on a reference surface Ω. The dependence on x, y is further eliminated by introducing FE assumptions.

As far as indicial notations are concerned, one notices that the summands on the left-hand side of PVD and RMVT are products of stresses/strains times variations of stresses/strains and that each stress/strain array has three scalars. The two-dimensional and FE approximations (1) and (2) enumerated above are introduced in these stresses/strains as well as in their variations. In particular, in this paper the following sub/superscripts are used (Figure 4):

- τ, s sub–superscripts couple is used for the z-expansions for stresses/strains and their variation, respectively.
- i, j sub–superscripts couple is used for the number of nodes expansions for stresses/strains and their variations, respectively.

Hence PVD and RMVT statements lead to finite element matrices that could be written by means of simple arrays that have herein been called *fundamental nuclei*. These have 3×3 terms. By varying the introduced indexes, the generic term of the finite element matrices related to a given set of N and N_n values could be obtained. The used indicial notation has been designed for the computer implementations that are presented in the companion paper (Part 2) [103]. An example showing the way in which the indicial notation works is given in Appendix A.

5. DISPLACEMENT ASSUMPTIONS FOR PVD APPLICATIONS

In the framework of this paper, the behaviour of a displacement and/or transverse stress components f is postulated in the thickness plate z-directions according to a given expansion

$$f(x,y,z) = F_i(z)\ f_i(x,y), \quad i = 1, N^\star \tag{33}$$

It is intended that repeated indexes are summed over their ranges. The polynomials $F_i(z)$ constitute a set of independent functions. Such a base can be arbitrarily chosen: power of z, and combinations of Legendre polynomials will be considered in this paper. $N^\star$ denotes the number of the introduced terms.

In the case of classical models, formulated on the basis of PVD, the assumptions of Equation (33) are restricted to the displacement variables. Traditionally z power expansion is employed,

$$\mathbf{u} = \mathbf{u}_0 + z^r \mathbf{u}_r, \quad r = 1, 2, \ldots, N \tag{34}$$

The subscript 0 denotes displacement values with correspondence to the plate/shell reference surface Ω, not necessarily corresponding to the middle layer or multilayered surface. Linear and higher-order distributions in the z-direction are introduced by the r-polynomials. *N remains a free parameter of the model*. Different N values could be used for different variables.

In order to write the whole modellings in a unified notation, the above expansion is rewritten as

$$\mathbf{u} = F_t \mathbf{u}_t + F_b \mathbf{u}_b + F_r \mathbf{u}_r = F_\tau \mathbf{u}_\tau, \quad \tau = t, b, r, \quad r = 2, \ldots, N-1 \tag{35}$$

By comparing Equation (35) to Equation (34), one finds that subscript b denotes values corresponding to Ω ($\mathbf{u}_b = \mathbf{u}_0$) while subscript t refers to the highest-order term ($\mathbf{u}_t = \mathbf{u}_N$). The F_τ polynomials assume the following explicit form:

$$F_b = 1, \quad F_t = z^N, \quad F_r = z^r, \quad r = 2, \ldots, N-1 \tag{36}$$

where b and t subscripts will also signify, see below, values of the displacement and/or stress variables with correspondence to layer bottom and top surfaces, respectively.

The assumptions written at previous expansions can be made at layer or multilayered level. Layer-wise LW and equivalent single-layer (ESLM) descriptions correspond to the first and second cases, respectively. These are discussed separately in the following two subsections.

5.1. Equivalent single-layer models (ESLM) with zig-zag function

The displacement variables are the same in each of the N_l-layers. Resulting theories are well-known classical plate models.

It is possible to introduce zig-zag effects in the previous expansion and in the PVD framework by referring to Murakami's idea which was originally introduced in the framework of RMVT. Murakami [56] proposed to add a zig-zag function to Equation (33),

$$\mathbf{u} = \mathbf{u}_0 + (-1)^k \zeta_k \mathbf{u}_Z + z^{r-1}\mathbf{u}_r, \quad r = 2, \ldots, N \tag{37}$$

Subscript Z refers to the introduced zig-zag term. Note that the unknown variables $\mathbf{u}_0, \mathbf{u}_Z, \mathbf{u}_r$ are k-independent. The geometrical meaning of the zig-zag function is explained in Figure 3 of Part 2 of this paper. $\zeta_k = 2z_k/h_k$ is a non-dimensional layer co-ordinate (z_k is the physical co-ordinate of the k-layer whose thickness is h_k). The exponent k changes the sign of the zig-zag term in each layer. Such a trick permits one to reproduce the discontinuity of the first derivative of the displacement variables in the z-directions which physically comes from the intrinsic transverse anisotropy (TA) of multilayered structures (as depicted in Figure 2). By employing a unified notation, Equation (37) becomes

$$\mathbf{u} = F_t\mathbf{u}_t + F_b\mathbf{u}_b + F_r\mathbf{u}_r = F_\tau\mathbf{u}_\tau, \quad \tau = t, b, r, \quad r = 2, \ldots, N \tag{38}$$

Subscript t has been chosen to denote the zig-zag term ($\mathbf{u}_t = \mathbf{u}_Z$, $F_t = (-1)^k\zeta_k$).

5.2. Layer-wise models (LWM)

By assuming the expansion in Equation (34) in each layer, layer-wise description is obtained. Nevertheless, Taylor-type expansion of Equation (34) is not convenient for a layer-wise description. In fact, the fulfilment of continuity requirements for the displacement at interfaces, i.e. the C_z^0-requirements, could be easily introduced by using the interface variables as unknowns. A convenient combination of Legendre polynomials [56–58, 18] could be used as base functions:

$$\mathbf{u}^k = F_t\mathbf{u}_t^k + F_b\mathbf{u}_b^k + F_r\mathbf{u}_r^k = F_\tau\mathbf{u}_\tau^k \quad \tau = t, b, r, \quad r = 2, 3, \ldots, N, \quad k = 1, 2, \ldots, N_l \tag{39}$$

It is now intended that the subscripts t and b denote values related to the layer top and bottom surfaces, respectively. These two terms consist of the linear part of the expansion. The thickness functions $F_\tau(\zeta_k)$ have now been defined at the k-layer level,

$$F_t = \frac{P_0 + P_1}{2}, \quad F_b = \frac{P_0 - P_1}{2}, \quad F_r = P_r - P_{r-2}, \quad r = 2, 3, \ldots, N \tag{40}$$

in which $P_j = P_j(\zeta_k)$ is the Legendre polynomial of j-order defined in the ζ_k-domain: $-1 \leqslant \zeta_k \leqslant 1$. The chosen functions have the following properties:

$$\zeta_k = \begin{cases} 1, & F_t = 1, \ F_b = 0, \ F_r = 0 \\ -1, & F_t = 0, \ F_b = 1, \ F_r = 0 \end{cases} \tag{41}$$

The continuity of the displacement at each interface is easily linked,

$$\mathbf{u}_t^k = \mathbf{u}_b^{(k+1)}, \quad k = 1, N_l - 1 \tag{42}$$

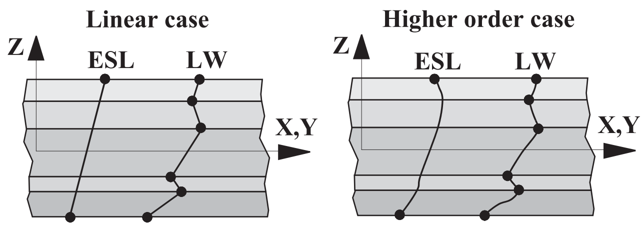

Figure 5. Examples of linear and higher-order field for both ESLM and LW variable description.

Examples of linear and higher-order fields in the multilayer for ESLM and LW description are shown in Figure 5.

6. DISPLACEMENT AND TRANSVERSE STRESS ASSUMPTIONS FOR RMVT APPLICATIONS

6.1. ESLM case

RMVT consists of a variational tool designed for multilayered structures. Appropriate applications of RMVT demand displacement fields which describe a zig-zag effect and transverse stresses which are continuous at the interfaces. The zig-zag effect can be included by referring to displacement fields quoted in Equations (37) and (39) for ESL and LW description, respectively. Equation (37) is not appropriate for ESL description of transverse stresses. Its extension to transverse shear and normal stress would violate Reissner's aims. In fact, the resulting stresses model does not fulfil homogeneous and non-homogeneous conditions at the plate top/bottom surface. The use of RMVT therefore demands layer-wise description of transverse stresses even though ESLM expansions are used for displacements. It is intended that in the presented derivations ESLM description is only related to displacement fields in RMVT applications.

Transverse stresses are assumed independent in each layer. The layer-wise description already used for displacements is extended to transverse stresses,

$$\boldsymbol{\sigma}_{n\mathrm{M}}^k = F_t \boldsymbol{\sigma}_{nt}^k + F_b \boldsymbol{\sigma}_{nb}^k + F_r \boldsymbol{\sigma}_{nr}^k = F_\tau \boldsymbol{\sigma}_{n\tau}^k, \quad \tau = t, b, r, \quad r = 2, 3, \ldots, N; \quad k = 1, 2, \ldots, N_l \tag{43}$$

The interlaminar transverse shear and normal stress continuity IC can therefore be linked by simply writing

$$\boldsymbol{\sigma}_{nt}^k = \boldsymbol{\sigma}_{nb}^{(k+1)}, \quad k = 1, N_l - 1 \tag{44}$$

In those cases in which the top/bottom plate/shell stress values are prescribed (zero or imposed values), the following additional equilibrium conditions must be accounted for:

$$\boldsymbol{\sigma}_{nb}^1 = \bar{\boldsymbol{\sigma}}_{nb}, \quad \boldsymbol{\sigma}_{nt}^{N_l} = \bar{\boldsymbol{\sigma}}_{nt} \tag{45}$$

where the over-bar denotes the imposed values in correspondence to the plate boundary surfaces.

6.2. *Layer-wise models* (*LWM*)

Full layer-wise description can be introduced by simply extending the stress assumptions of the previous paragraph to displacement variables,

$$\begin{aligned}\mathbf{u}^k &= F_t\mathbf{u}_t^k + F_b\mathbf{u}_b^k + F_r\mathbf{u}_r^k = F_\tau\mathbf{u}_\tau^k, \qquad \tau = t,b,r, \quad r=2,3,\dots,N \\ \boldsymbol{\sigma}_{n\mathrm{M}}^k &= F_t\boldsymbol{\sigma}_{nt}^k + F_b\boldsymbol{\sigma}_{nb}^k + F_r\boldsymbol{\sigma}_{nr}^k = F_\tau\boldsymbol{\sigma}_{n\tau}^k, \quad k=1,2,\dots,N_l\end{aligned} \tag{46}$$

In addition to Equation (44) the compatibility of the displacement reads

$$\mathbf{u}_t^k = \mathbf{u}_b^{(k+1)}, \quad k=1,N_l-1 \tag{47}$$

Note that LW description does not require any zig-zag function for the simulation of zig-zag effects. C_z^0-requirements are completely and *a priori* fulfilled by Equations (44)–(47).

7. ESL AND LW FINITE ELEMENTS DEVELOPED ON THE BASIS OF PVD

7.1. *FEM matrices for the k-layer*

The assumed displacement field is first introduced in the expression for the strains, leading to

$$\boldsymbol{\varepsilon}_p^k = \mathbf{D}_p\mathbf{u}^k = \mathbf{D}_p(F_\tau\mathbf{u}_\tau^k) \tag{48}$$

$$\boldsymbol{\varepsilon}_n^k = \mathbf{D}_n\mathbf{u}^k = (\mathbf{D}_{n\Omega} + \mathbf{D}_{nz})(F_\tau\mathbf{u}_\tau^k) = \mathbf{D}_{n\Omega}(F_\tau\mathbf{u}_\tau^k) + F_{\tau,z}\mathbf{u}_\tau^k \tag{49}$$

in which the notation

$$F_{\tau,z} = \frac{\partial F_\tau}{\partial z} \tag{50}$$

has been introduced.

Secondly, finite element approximations are used to express the displacement in terms of their nodal values, via shape functions,

$$\mathbf{u}_\tau^k = N_i\mathbf{q}_{\tau i}^k \quad (i=1,2,\dots,N_n) \tag{51}$$

where N_n denotes the numbers of the nodes in the element while

$$\mathbf{q}_{\tau i}^k = [q_{u_x\tau i}^k \;\; q_{u_y\tau i}^k \;\; q_{u_z\tau i}^k]^{\mathrm{T}} \tag{52}$$

The base functions F_τ being independent of x and y, the strains can be written as

$$\boldsymbol{\varepsilon}_p^k = F_\tau\mathbf{D}_p(N_i\mathbf{I})\mathbf{q}_{\tau i}^k \tag{53}$$

$$\boldsymbol{\varepsilon}_n^k = F_\tau\mathbf{D}_{n\Omega}(N_i\mathbf{I})\mathbf{q}_{\tau i}^k + F_{\tau,z}N_i\mathbf{q}_{\tau i}^k \tag{54}$$

in which $\mathbf{I}$ is the identity matrix. By introducing the written strain–displacement relations (Equations (53) and (54)) along with Hooke's law (Equations (15) and (16)), the RHS of the PVD statement is

$$
\begin{aligned}
\delta L^k_{\text{int}} = & \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{D}^{\text{T}}_p (N_i \mathbf{I}) \tilde{\mathbf{C}}^k_{pp} \left[\int_{A_k} (F_\tau F_s)\, \mathrm{d}z \right] \mathbf{D}_p (N_j \mathbf{I}) \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{D}^{\text{T}}_p (N_i \mathbf{I}) \tilde{\mathbf{C}}^k_{pn} \left[\int_{A_k} (F_\tau F_s)\, \mathrm{d}z \right] \mathbf{D}_{n\Omega} (N_j \mathbf{I}) \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{D}^{\text{T}}_p (N_i \mathbf{I}) \tilde{\mathbf{C}}^k_{pn} \left[\int_{A_k} (F_\tau F_{s,z})\, \mathrm{d}z \right] N_j \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{D}^{\text{T}}_{n\Omega} (N_i \mathbf{I}) \tilde{\mathbf{C}}^k_{np} \left[\int_{A_k} (F_\tau F_s)\, \mathrm{d}z \right] \mathbf{D}_p (N_j \mathbf{I}) \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{D}^{\text{T}}_{n\Omega} (N_i \mathbf{I}) \tilde{\mathbf{C}}^k_{nn} \left[\int_{A_k} (F_\tau F_s)\, \mathrm{d}z \right] \mathbf{D}_{n\Omega} (N_j \mathbf{I}) \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{D}^{\text{T}}_{n\Omega} (N_i \mathbf{I}) \tilde{\mathbf{C}}^k_{nn} \left[\int_{A_k} (F_\tau F_{s,z})\, \mathrm{d}z \right] N_j \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} N_i \tilde{\mathbf{C}}^k_{np} \left[\int_{A_k} (F_{\tau,z} F_s)\, \mathrm{d}z \right] \mathbf{D}_p (N_j \mathbf{I}) \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} N_i \tilde{\mathbf{C}}^k_{nn} \left[\int_{A_k} (F_{\tau,z} F_s)\, \mathrm{d}z \right] \mathbf{D}_{n\Omega} (N_j \mathbf{I}) \mathbf{q}^k_{sj}\, \mathrm{d}\Omega \\
& + \int_\Omega \delta \mathbf{q}^{k\text{T}}_{\tau i} N_i \tilde{\mathbf{C}}^k_{nn} \left[\int_{A_k} (F_{\tau,z} F_{s,z})\, \mathrm{d}z \right] N_j \mathbf{q}^k_{sj}\, \mathrm{d}\Omega
\end{aligned} \tag{55}
$$

Note again that subscripts τ and i have been used for the finite values of unknown variables while subscripts s and j have been introduced for their variations.

As usual in two-dimensional modellings the integration in the thickness direction can be made *a priori* by introducing the following layer integrals:

$$(\tilde{\mathbf{Z}}^{k\tau s}_{pp}, \tilde{\mathbf{Z}}^{k\tau s}_{pn}, \tilde{\mathbf{Z}}^{k\tau s}_{np}, \tilde{\mathbf{Z}}^{k\tau s}_{nn}) = (\tilde{\mathbf{C}}^k_{pp}, \tilde{\mathbf{C}}^k_{pn}, \tilde{\mathbf{C}}^k_{np}, \tilde{\mathbf{C}}^k_{nn}) E_{\tau s}$$

$$(\tilde{\mathbf{Z}}^{k\tau s_{,z}}_{pn}, \tilde{\mathbf{Z}}^{k\tau s_{,z}}_{nn}, \tilde{\mathbf{Z}}^{k\tau_{,z} s}_{np}, \tilde{\mathbf{Z}}^{k\tau_{,z} s}_{nn}, \tilde{\mathbf{Z}}^{k\tau_{,z} s_{,z}}_{nn}) = (\tilde{\mathbf{C}}^k_{pn} E_{\tau s_{,z}}, \tilde{\mathbf{C}}^k_{nn} E_{\tau s_{,z}}, \tilde{\mathbf{C}}^k_{np} E_{\tau_{,z} s}, \tilde{\mathbf{C}}^k_{nn} E^k_{\tau_{,z} s}, \tilde{\mathbf{C}}^k_{nn} E_{\tau_{,z} s_{,z}})$$

$$(E_{\tau s}, E_{\tau s_{,z}}, E_{\tau_{,z} s}, E_{\tau_{,z} s_{,z}}) = \int_{A_k} (F_\tau F_s, F_\tau F_{s,z}, F_{\tau,z} F_s, F_{\tau,z} F_{s,z})\, \mathrm{d}z$$

Equation (55) is therefore written in the following form:

$$\delta L^k_{\text{int}} = \delta \mathbf{q}^{k\text{T}}_{\tau i} \mathbf{K}^{k\tau s i j} \mathbf{q}^k_{sj} \tag{56}$$

where the following finite element matrix has been introduced:

$$\begin{aligned}\mathbf{K}^{k\tau sij} = \triangleleft \mathbf{D}_p^{\mathrm{T}}(N_i\mathbf{I})[\tilde{\mathbf{Z}}_{pp}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{pn}^{k\tau s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{pn}^{k\tau s_{,z}}N_j] \\ + \mathbf{D}_{n\Omega}^{\mathrm{T}}(N_i\mathbf{I})[\tilde{\mathbf{Z}}_{np}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau s_{,z}}N_j] \\ + N_i[\tilde{\mathbf{Z}}_{np}^{k\tau_{,z}s}\mathbf{D}_p(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau_{,z}s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}) + \tilde{\mathbf{Z}}_{nn}^{k\tau_{,z}s_{,z}}N_j]\triangleright_\Omega\end{aligned} \tag{57}$$

The symbol $\triangleleft \ldots \triangleright_\Omega$ has been introduced to denote integrals on Ω.

Note that the matrix $\mathbf{K}^{k\tau sij}$ is made by triplicate products of 3×3 arrays, so that $\mathbf{K}^{k\tau sij}$ is itself a 3×3 array. Such an array consists of the fundamental nucleus of finite element matrices related to PVD applications. The nine terms $\mathbf{K}^{k\tau sij}$ are:

$$\begin{aligned}
K_{xx}^{k\tau sij} &= \tilde{Z}_{pp11}^{k\tau s} \triangleleft N_{i,x}N_{j,x}\triangleright_\Omega + \tilde{Z}_{pp16}^{k\tau s} \triangleleft N_{i,y}N_{j,x}\triangleright_\Omega + \tilde{Z}_{pp16}^{k\tau s} \triangleleft N_{i,x}N_{j,y}\triangleright_\Omega \\
&\quad + \tilde{Z}_{pp66}^{k\tau s} \triangleleft N_{i,y}N_{j,y}\triangleright_\Omega + \tilde{Z}_{nn55}^{k\tau_{,z}s_{,z}} \triangleleft N_iN_j\triangleright_\Omega \\
K_{xy}^{k\tau sij} &= \tilde{Z}_{pp12}^{k\tau s} \triangleleft N_{i,x}N_{j,y}\triangleright_\Omega + \tilde{Z}_{pp26}^{k\tau s} \triangleleft N_{i,y}N_{j,y}\triangleright_\Omega + \tilde{Z}_{pp16}^{k\tau s} \triangleleft N_{i,x}N_{j,x}\triangleright_\Omega \\
&\quad + \tilde{Z}_{pp66}^{k\tau s} \triangleleft N_{i,y}N_{j,x}\triangleright_\Omega + \tilde{Z}_{nn45}^{k\tau_{,z}s_{,z}} \triangleleft N_iN_j\triangleright_\Omega \\
K_{xz}^{k\tau sij} &= \tilde{Z}_{pn13}^{k\tau s_{,z}} \triangleleft N_{i,x}N_j\triangleright_\Omega + \tilde{Z}_{pn36}^{k\tau s_{,z}} \triangleleft N_{i,y}N_j\triangleright_\Omega + \tilde{Z}_{nn55}^{k\tau_{,z}s} \triangleleft N_iN_{j,x}\triangleright_\Omega \\
&\quad + \tilde{Z}_{nn45}^{k\tau_{,z}s} \triangleleft N_iN_{j,y}\triangleright_\Omega \\
K_{yx}^{k\tau sij} &= \tilde{Z}_{pp12}^{k\tau s} \triangleleft N_{i,y}N_{j,x}\triangleright_\Omega + \tilde{Z}_{pp16}^{k\tau s} \triangleleft N_{i,x}N_{j,x}\triangleright_\Omega + \tilde{Z}_{pp26}^{k\tau s} \triangleleft N_{i,y}N_{j,y}\triangleright_\Omega \\
&\quad + \tilde{Z}_{pp66}^{k\tau s} \triangleleft N_{i,x}N_{j,y}\triangleright_\Omega + \tilde{Z}_{nn45}^{k\tau_{,z}s_{,z}} \triangleleft N_iN_j\triangleright_\Omega \\
K_{yy}^{k\tau sij} &= \tilde{Z}_{pp22}^{k\tau s} \triangleleft N_{i,y}N_{j,y}\triangleright_\Omega + \tilde{Z}_{pp26}^{k\tau s} \triangleleft N_{i,x}N_{j,y}\triangleright_\Omega + \tilde{Z}_{pp26}^{k\tau s} \triangleleft N_{i,y}N_{j,x}\triangleright_\Omega \\
&\quad + \tilde{Z}_{pp66}^{k\tau s} \triangleleft N_{i,x}N_{j,x}\triangleright_\Omega + \tilde{Z}_{nn44}^{k\tau_{,z}s_{,z}} \triangleleft N_iN_j\triangleright_\Omega \\
K_{yz}^{k\tau sij} &= \tilde{Z}_{pn23}^{k\tau s_{,z}} \triangleleft N_{i,y}N_j\triangleright_\Omega + \tilde{Z}_{pn36}^{k\tau s_{,z}} \triangleleft N_{i,x}N_j\triangleright_\Omega + \tilde{Z}_{nn45}^{k\tau_{,z}s} \triangleleft N_iN_{j,x}\triangleright_\Omega \\
&\quad + \tilde{Z}_{nn44}^{k\tau_{,z}s} \triangleleft N_iN_{j,y}\triangleright_\Omega \\
K_{zx}^{k\tau sij} &= \tilde{Z}_{nn55}^{k\tau s_{,z}k} \triangleleft N_{i,x}N_j\triangleright_\Omega + \tilde{Z}_{nn45}^{k\tau s_{,z}k} \triangleleft N_{i,y}N_j\triangleright_\Omega + \tilde{Z}_{np13}^{k\tau_{,z}s} \triangleleft N_iN_{j,x}\triangleright_\Omega \\
&\quad + \tilde{Z}_{np36}^{k\tau_{,z}s} \triangleleft N_iN_{j,y}\triangleright_\Omega \\
K_{zy}^{k\tau sij} &= \tilde{Z}_{nn45}^{k\tau s_{,z}k} \triangleleft N_{i,x}N_j\triangleright_\Omega + \tilde{Z}_{nn44}^{k\tau s_{,z}k} \triangleleft N_{i,y}N_j\triangleright_\Omega + \tilde{Z}_{np23}^{k\tau_{,z}s} \triangleleft N_iN_{j,y}\triangleright_\Omega \\
&\quad + \tilde{Z}_{np36}^{k\tau_{,z}s} \triangleleft N_iN_{j,x}\triangleright_\Omega \\
K_{zz}^{k\tau sij} &= \tilde{Z}_{nn55}^{k\tau sk} \triangleleft N_{i,x}N_{j,x}\triangleright_\Omega + \tilde{Z}_{nn45}^{k\tau sk} \triangleleft N_{i,y}N_{j,x}\triangleright_\Omega + \tilde{Z}_{nn45}^{k\tau sk} \triangleleft N_{i,x}N_{j,y}\triangleright_\Omega \\
&\quad + \tilde{Z}_{nn44}^{k\tau sk} \triangleleft N_{i,y}N_{j,y}\triangleright_\Omega + \tilde{Z}_{nn33}^{k\tau_{,z}s_{,z}} \triangleleft N_iN_j\triangleright_\Omega
\end{aligned} \tag{58}$$

By varying N and N_n, the finite element matrices of the k-layer corresponding to the implemented two-dimensional theories and number of nodes are obtained.

By introducing the external work of applied loadings, one has (see Appendix B for an example)

$$\delta \mathbf{q}_{\tau i}^{k\mathrm{T}} \mathbf{K}^{k\tau sij} \mathbf{q}_{sj}^{k} = \delta \mathbf{q}_{\tau i}^{k\mathrm{T}} \mathbf{P}_{\tau i}^{k}$$

By imposing the definition of virtual variations, PVD leads for each k-layer to the following equilibrium conditions:

$$\delta \mathbf{q}_{\tau i}^{k\mathrm{T}}: \quad \mathbf{K}^{k\tau sij} \mathbf{q}_{sj}^{k} = \mathbf{P}_{\tau i}^{k} \tag{59}$$

7.2. Assembly from layer to multilayer

In order to write the finite element matrix for the multilayered plate, for a given set of parameters N, N_n and N_l, the following steps must be implemented (global/local approaches mentioned in Section 2.2.1 should be taken into account at this stage):

1. The 3×3 fundamental nucleus of the matrix $\mathbf{K}^{k\tau sij}$ should be expanded according to the indexes τ, s and i, j. The expansion according to τ, s indexes is shown in Figure 6 (a four-noded element has been considered in this figure in conjunction to $N = 2$ expansions in z).
2. The obtained matrix must be written for each of the N_l-layers.
3. Resulting matrices are assembled from layer to multilayer level depending on the used variables descriptions.
   (a) In the case of ESLM, the variables and their variations being the same for each layer, these matrices are simply summed. That is, layer stiffness is accumulated layer by layer. Assemblage related to a three-layered plate is depicted in Figure 7.
   (b) Displacement variables are independent in each layer in the LW cases which require only continuity of displacement variables at the interface. This is formally shown in Figure 8.

8. ESL AND LW FINITE ELEMENTS DEVELOPED ON THE BASIS OF RMVT

The same steps made in the PVD case could be extended to RMVT formulated finite elements. Transverse normal stress variables along with displacement ones will now lead to four 3×3 fundamental nuclei. Three of them are related to equilibrium conditions; the other establishes compatibility conditions.

8.1. FEM matrices for the k-layer

The mixed form of Hooke's law for the k-layer is here rewritten as

$$\boldsymbol{\sigma}_{p\mathrm{H}}^{k} = \mathbf{C}_{pp}^{k} \boldsymbol{\varepsilon}_{p\mathrm{G}}^{k} + \mathbf{C}_{pn}^{k} \boldsymbol{\sigma}_{n\mathrm{M}}^{k} \tag{60}$$

$$\boldsymbol{\varepsilon}_{n\mathrm{H}}^{k} = \mathbf{C}_{np}^{k} \boldsymbol{\varepsilon}_{p\mathrm{G}}^{k} + \mathbf{C}_{nn}^{k} \boldsymbol{\sigma}_{n\mathrm{M}}^{k} \tag{61}$$

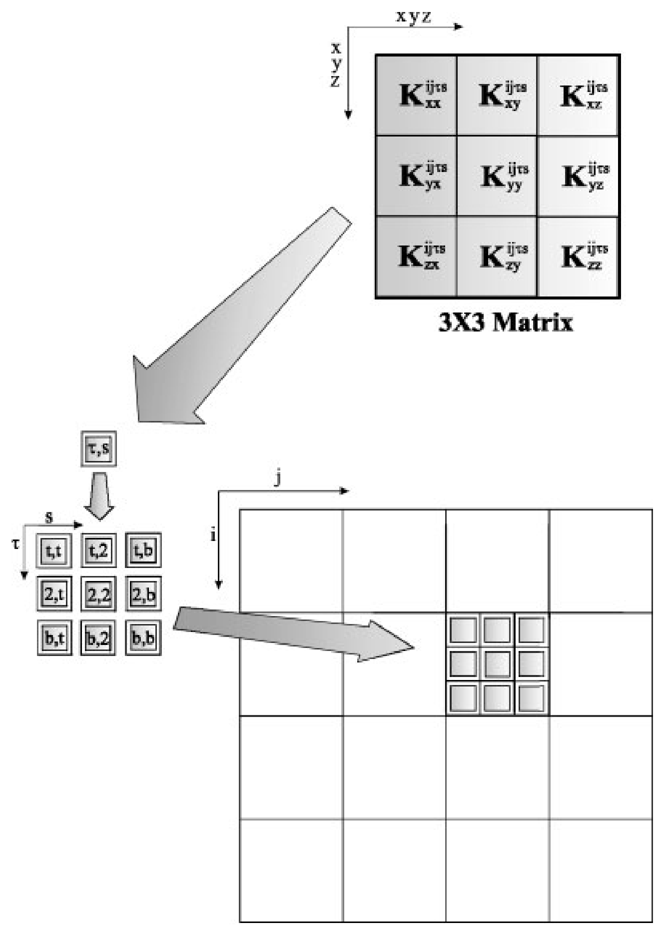

Figure 6. Expansion of the layer matrix from the correspondent 3×3 fundamental nuclei via τ and s indexes.

Transverse stress variables are expressed in terms of shape functions as done for the displacement ones,

$$\boldsymbol{\sigma}_{n\tau}^{k} = N_i \mathbf{g}_{\tau i}^{k} \quad (i = 1, 2, \ldots, N_n) \tag{62}$$

where

$$\mathbf{g}_{\tau i}^{k} = [g_{xz\tau i}^{k} \;\; g_{yz\tau i}^{k} \;\; g_{zz\tau i}^{k}]^{\mathrm{T}} \tag{63}$$

By introducing

$$\boldsymbol{\sigma}_{n\mathrm{M}}^{k} = F_\tau N_i \mathbf{g}_{\tau i}^{k} \tag{64}$$

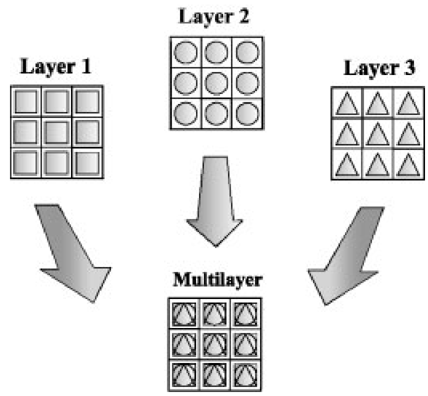

Figure 7. Assemblage from layer to multilayered level in ESLM description for a three-layered plate.

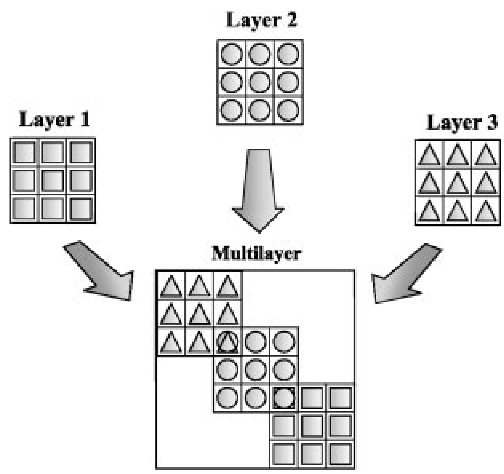

Figure 8. Assemblage from layer to multilayered level in LW description for a three-layered plate.

the left-hand side of RMVT becomes

$$\delta L_{\text{int}} = \int_v [\delta \boldsymbol{\varepsilon}_{p\text{G}}^{k\text{T}} \mathbf{C}_{pp}^k \boldsymbol{\varepsilon}_{p\text{G}}^{k\text{T}} + \delta \boldsymbol{\varepsilon}_{p\text{G}}^{k\text{T}} \mathbf{C}_{pn}^k \boldsymbol{\sigma}_{n\text{M}}^k + \delta \boldsymbol{\varepsilon}_{n\text{G}}^{k\text{T}} \boldsymbol{\sigma}_{n\text{M}}^k + \delta \boldsymbol{\sigma}_{n\text{M}}^{k\text{T}} \boldsymbol{\varepsilon}_{n\text{G}}^k - \delta \boldsymbol{\sigma}_{n\text{M}}^{k\text{T}} \mathbf{C}_{np}^k \boldsymbol{\varepsilon}_{p\text{G}}^{k\text{T}} - \delta \boldsymbol{\sigma}_{n\text{M}}^k \mathbf{C}_{nn}^k \boldsymbol{\sigma}_{n\text{M}}^{k\text{T}}]\, \mathrm{d}v \tag{65}$$

Upon substitution of (64), (53) and (54) one has

$$\begin{aligned}
\delta L_{\text{int}}^k = {} & \triangleleft \{\delta \mathbf{q}_{\tau i}^{k\text{T}} [\mathbf{D}_p^{\text{T}} (N_i \mathbf{I}) \mathbf{Z}_{pp}^{k\tau s} \mathbf{D}_p (N_j \mathbf{I})] \mathbf{q}_{sj}^k\} \triangleright_\Omega + \triangleleft \{\delta \mathbf{q}_{\tau i}^{k\text{T}} [\mathbf{D}_p^{\text{T}} (N_i \mathbf{I}) \mathbf{Z}_{pn}^{k\tau s} N_j] \mathbf{g}_{sj}^k\} \triangleright_\Omega \\
& + \triangleleft \{\delta \mathbf{q}_{\tau i}^{k\text{T}} [\mathbf{D}_{n\Omega}^{\text{T}} (N_i \mathbf{I}) E_{\tau s} N_j + E_{\tau_{,z} s} N_i N_j \mathbf{I}] g_{sj}^k\} \triangleright_\Omega \\
& + \triangleleft \{\delta \mathbf{g}_{\tau i}^{k\text{T}} [N_i E_{\tau s} \mathbf{D}_{n\Omega} (N_j \mathbf{I}) + E_{\tau s_{,z}} N_i N_j \mathbf{I}] q_{sj}^k\} \triangleright_\Omega \\
& - \triangleleft \{\delta \mathbf{g}_{\tau i}^{k\text{T}} [N_i \mathbf{Z}_{np}^{k\tau s} \mathbf{D}_p (N_j \mathbf{I})] \mathbf{q}_{sj}^k\} \triangleright_\Omega - \triangleleft \{\delta \mathbf{g}_{\tau i}^{k\text{T}} [N_i \mathbf{Z}_{nn}^{k\tau s} N_j] \mathbf{g}_{sj}^k\} \triangleright_\Omega
\end{aligned}$$

where the following layer stiffness and compliance have been introduced:

$$(\mathbf{Z}_{pp}^{k\tau s}, \mathbf{Z}_{pn}^{k\tau s}, \mathbf{Z}_{np}^{k\tau s}, \mathbf{Z}_{nn}^{k\tau s}) = (\mathbf{C}_{pp}^{k}, \mathbf{C}_{pn}^{k}, \mathbf{C}_{np}^{k}, \mathbf{C}_{nn}^{k}) E_{\tau s}$$

so that

$$\delta L_{\text{int}}^{Rk} = \delta \mathbf{q}_{\tau i}^{k\text{T}} [\mathbf{K}_{uu}^{k\tau sij} \mathbf{q}_{sj}^{k} + \mathbf{K}_{u\sigma}^{k\tau sij} \mathbf{g}_{sj}^{k}] + \delta \mathbf{g}_{\tau i}^{k\text{T}} [\mathbf{K}_{\sigma u}^{k\tau sij} \mathbf{q}_{sj}^{k} + \mathbf{K}_{\sigma\sigma}^{k\tau sij} \mathbf{g}_{sj}^{k}] \tag{66}$$

where

$$\begin{aligned}
\mathbf{K}_{uu}^{k\tau sij} &= \triangleleft [\mathbf{D}_p^{\text{T}}(N_i \mathbf{I}) \mathbf{Z}_{pp}^{k\tau s} \mathbf{D}_p(N_j \mathbf{I})] \triangleright_{\Omega} \\
\mathbf{K}_{u\sigma}^{k\tau sij} &= \triangleleft [\mathbf{D}_p^{\text{T}}(N_i \mathbf{I}) \mathbf{Z}_{pn}^{k\tau s} N_j + \mathbf{D}_{n\Omega}^{\text{T}}(N_i \mathbf{I}) E_{\tau s} N_j + E_{\tau,_z s} N_i N_j \mathbf{I}] \triangleright_{\Omega} \\
\mathbf{K}_{\sigma u}^{k\tau sij} &= \triangleleft [N_i E_{\tau s} \mathbf{D}_{n\Omega}(N_j \mathbf{I}) + E_{\tau s,_z} N_i N_j \mathbf{I} - N_i \mathbf{Z}_{np}^{k\tau s} \mathbf{D}_p(N_j \mathbf{I})] \triangleright_{\Omega} \\
\mathbf{K}_{\sigma\sigma}^{k\tau sij} &= \triangleleft [-N_i \mathbf{Z}_{nn}^{k\tau s} N_j] \triangleright_{\Omega}
\end{aligned} \tag{67}$$

By imposing the definition of virtual variations, the RMVT leads to the following equilibrium and compatibility equations:

$$\begin{aligned}
\delta \mathbf{q}_{\tau i}^{k\text{T}}: &\quad \mathbf{K}_{uu}^{k\tau sij} \mathbf{q}_{sj}^{k} + \mathbf{K}_{u\sigma}^{k\tau sij} \mathbf{g}_{sj}^{k} = \mathbf{P}_{\tau i}^{k} \\
\delta \mathbf{g}_{\tau i}^{k\text{T}}: &\quad \mathbf{K}_{\sigma u}^{k\tau sij} \mathbf{q}_{sj}^{k} + \mathbf{K}_{\sigma\sigma}^{k\tau sij} \mathbf{g}_{sj}^{k} = \mathbf{0}
\end{aligned} \tag{68}$$

As anticipated, four 3×3 fundamental nuclei have been obtained. In explicit form these hold:

$$\begin{aligned}
K_{uuxx}^{k\tau sij} &= Z_{pp11}^{k\tau s} \triangleleft N_{i,x} N_{j,x} \triangleright_{\Omega} + Z_{pp16}^{k\tau s} \triangleleft N_{i,y} N_{j,x} \triangleright_{\Omega} + Z_{pp16}^{k\tau s} \triangleleft N_{i,x} N_{j,y} \triangleright_{\Omega} + Z_{pp66}^{k\tau s} \triangleleft N_{i,y} N_{j,y} \triangleright_{\Omega} \\
K_{uuxy}^{k\tau sij} &= Z_{pp12}^{k\tau s} \triangleleft N_{i,x} N_{j,y} \triangleright_{\Omega} + Z_{pp26}^{k\tau s} \triangleleft N_{i,y} N_{j,y} \triangleright_{\Omega} + Z_{pp16}^{k\tau s} \triangleleft N_{i,x} N_{j,x} \triangleright_{\Omega} + Z_{pp66}^{k\tau s} \triangleleft N_{i,y} N_{j,x} \triangleright_{\Omega} \\
K_{uuxz}^{k\tau sij} &= 0 \\
K_{uuyx}^{k\tau sij} &= Z_{pp12}^{k\tau s} \triangleleft N_{i,y} N_{j,x} \triangleright_{\Omega} + Z_{pp16}^{k\tau s} \triangleleft N_{i,x} N_{j,x} \triangleright_{\Omega} + Z_{pp26}^{k\tau s} \triangleleft N_{i,y} N_{j,y} \triangleright_{\Omega} + Z_{pp66}^{k\tau s} \triangleleft N_{i,x} N_{j,y} \triangleright_{\Omega} \\
K_{uuyy}^{k\tau sij} &= Z_{pp22}^{k\tau s} \triangleleft N_{i,y} N_{j,y} \triangleright_{\Omega} + Z_{pp26}^{k\tau s} \triangleleft N_{i,x} N_{j,y} \triangleright_{\Omega} + Z_{pp26}^{k\tau s} \triangleleft N_{i,y} N_{j,x} \triangleright_{\Omega} + Z_{pp66}^{k\tau s} \triangleleft N_{i,x} N_{j,x} \triangleright_{\Omega} \\
K_{uuyz}^{k\tau sij} &= 0 \\
K_{uuzx}^{k\tau sij} &= 0 \\
K_{uuzy}^{k\tau sij} &= 0 \\
K_{uuzz}^{k\tau sij} &= 0
\end{aligned} \tag{69}$$

$$
\begin{aligned}
K^{k\tau sij}_{u\sigma xx} &= E_{\tau_{,z}s} \triangleleft N_i N_j \triangleright_\Omega \\
K^{k\tau sij}_{u\sigma xy} &= 0 \\
K^{k\tau sij}_{u\sigma xz} &= Z^{k\tau s}_{pn13} \triangleleft N_{i,x} N_j \triangleright_\Omega + Z^{k\tau s}_{pn36} \triangleleft N_{i,y} N_j \triangleright_\Omega \\
K^{k\tau sij}_{u\sigma yx} &= 0 \\
K^{k\tau sij}_{u\sigma yy} &= E_{\tau_{,z}s} \triangleleft N_i N_j \triangleright_\Omega \\
K^{k\tau sij}_{u\sigma yz} &= Z^{k\tau s}_{pn23} \triangleleft N_{i,y} N_j \triangleright_\Omega + Z^{k\tau s}_{pn36} \triangleleft N_{i,x} N_j \triangleright_\Omega \\
K^{k\tau sij}_{u\sigma zx} &= E_{\tau s} \triangleleft N_{i,x} N_j \triangleright_\Omega \\
K^{k\tau sij}_{u\sigma zy} &= E_{\tau s} \triangleleft N_{i,y} N_j \triangleright_\Omega \\
K^{k\tau sij}_{u\sigma zz} &= E_{\tau_{,z}s} \triangleleft N_i N_j \triangleright_\Omega
\end{aligned} \tag{70}
$$

$$
\begin{aligned}
K^{k\tau sij}_{\sigma uxx} &= E_{\tau s_{,z}} \triangleleft N_i N_j \triangleright_\Omega \\
K^{k\tau sij}_{\sigma uxy} &= 0 \\
K^{k\tau sij}_{\sigma uxz} &= E_{\tau s} \triangleleft N_i N_{j,x} \triangleright_\Omega \\
K^{k\tau sij}_{\sigma uyx} &= 0 \\
K^{k\tau sij}_{\sigma uyy} &= E_{\tau s_{,z}} \triangleleft N_i N_j \triangleright_\Omega \\
K^{k\tau sij}_{\sigma uyz} &= E_{\tau s} \triangleleft N_i N_{j,y} \triangleright_\Omega \\
K^{k\tau sij}_{\sigma uzx} &= -Z^{k\tau s}_{np13} \triangleleft N_i N_{j,x} \triangleright_\Omega - Z^{k\tau s}_{np36} \triangleleft N_i N_{j,y} \triangleright_\Omega \\
K^{k\tau sij}_{\sigma uzy} &= -Z^{k\tau s}_{np23} \triangleleft N_i N_{j,y} \triangleright_\Omega - Z^{k\tau s}_{np36} \triangleleft N_i N_{j,x} \triangleright_\Omega \\
K^{k\tau sij}_{\sigma uzz} &= E_{\tau s_{,z}} \triangleleft N_i N_j \triangleright_\Omega
\end{aligned} \tag{71}
$$

$$
\mathbf{K}^{k\tau sij}_{\sigma\sigma} = \begin{bmatrix} -Z^{k\tau s}_{nn55} \triangleleft N_i N_j \triangleright_\Omega & -Z^{k\tau s}_{nn45} \triangleleft N_i N_j \triangleright_\Omega & 0 \\ -Z^{k\tau s}_{nn45} \triangleleft N_i N_j \triangleright_\Omega & -Z^{k\tau s}_{nn44} \triangleleft N_i N_j \triangleright_\Omega & 0 \\ 0 & 0 & -Z^{k\tau s}_{nn33} \triangleleft N_i N_j \triangleright_\Omega \end{bmatrix} \tag{72}
$$

As done for the PVD case, by expanding the (τ, s) as well as (i, j) couples of indices, the finite element matrix for the given k-layer is obtained.

8.2. Assembly of matrices from layer to multilayer level

In order to obtain multilayer matrices, the procedure already described for the PVD case must be applied. The LW case perfectly follows what is written for the PVD case.

Some difference arises because the ESL–RMVT formulation demands LW description for the stresses. In these cases, $\mathbf{K}^{k\tau sij}_{\sigma\sigma}$ follows the LW PVD case while $\mathbf{K}^{k\tau sij}_{uu}$ follows the ESL PVD case. A mixed LW and ESL assembly procedure has to be implemented for the other two matrices $\mathbf{K}^{k\tau sij}_{u\sigma}$ and $\mathbf{K}^{k\tau sij}_{\sigma u}$ This is described in Figure 9 for a three-layer case and the $\mathbf{K}^{k\tau sij}_{\sigma u}$ case (N and N_l are fixed to the values 2 and 3, respectively).

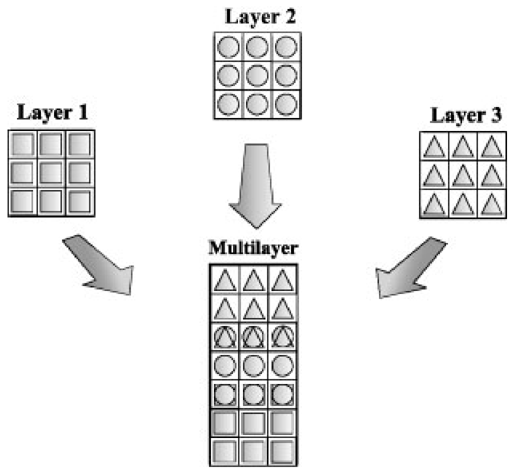

Figure 9. Assemblage from layer to multilayered level related to $K_{\sigma u}$-type matrix in the mixed case and ESLM description, for a three-layered plate.

9. TREATMENT OF STRESS VARIABLES

Mixed formulation offers several possibilities as far as the treatment of stress variables is concerned. Stress variables can be expressed in terms of the displacement ones. This can be done at layer level, multilayer level or structure level. As an alternative, stress variables can be retained and full mixed implementation is then obtained.

These methods are discussed in the following sections. For the sake of simplicity, attention has been restricted to the particular case of homogeneous boundary conditions, that is no transverse stresses are applied at any interface.

9.1. Elimination of stress variables at layer level

Let us consider a plate loaded by concentrated loadings. After expansion of (τ, s) and (i, j) indexes, the four mixed matrices

$$\mathbf{K}_{uu}^{k\tau sij}, \mathbf{K}_{u\sigma}^{k\tau sij}, \mathbf{K}_{\sigma u}^{k\tau sij}, \mathbf{K}_{\sigma\sigma}^{k\tau sij}$$

lead to corresponding layer matrices that are denoted by

$$\mathbf{K}_{uu}^{k}, \mathbf{K}_{u\sigma}^{k}, \mathbf{K}_{\sigma u}^{k}, \mathbf{K}_{\sigma\sigma}^{k}$$

RMVT can therefore be written as

$$\delta\mathbf{q}^{k\mathrm{T}}[\mathbf{K}_{uu}^{k}\mathbf{q}^{k} + \mathbf{K}_{u\sigma}^{k}\mathbf{g}^{k}] + \delta\mathbf{g}^{k\mathrm{T}}[\mathbf{K}_{\sigma u}^{k}\mathbf{q}^{k} + \mathbf{K}_{\sigma\sigma}^{k}\mathbf{g}^{k}] = \delta L_{\mathrm{est}}^{k} \tag{73}$$

This leads to, for each layer, the following set of governing equations:

$$\begin{aligned} \mathbf{K}_{uu}^k \mathbf{q}^k + \mathbf{K}_{u\sigma}^k \mathbf{g}^k &= \mathbf{P}^k \\ \mathbf{K}_{\sigma u}^k \mathbf{q}^k + \mathbf{K}_{\sigma\sigma}^k \mathbf{g}^k &= \mathbf{0} \end{aligned} \tag{74}$$

The second equation is then solved in terms of displacements by means of the so-called static-condensation technique. The first equation becomes

$$[\mathbf{K}_{uu}^k - \mathbf{K}_{u\sigma}^k (\mathbf{K}_{\sigma\sigma}^k)^{-1} \mathbf{K}_{\sigma u}^k] \mathbf{q}^k = \mathbf{P}^k \tag{75}$$

By introducing

$$\mathbf{K}_{\text{mixed}}^k = [\mathbf{K}_{uu}^k - \mathbf{K}_{u\sigma}^k (\mathbf{K}_{\sigma\sigma}^k)^{-1} \mathbf{K}_{\sigma u}^k] \tag{76}$$

one has

$$\mathbf{K}_{\text{mixed}}^k \mathbf{q}^k = \mathbf{P}^k \tag{77}$$

Such a matrix assumes the role that stiffness matrix $\mathbf{K}$ plays in PVD applications. It is to be pointed out that $\mathbf{K}^k$ and $\mathbf{K}_{\text{mixed}}^k$ consist of two completely different matrices; the differential operator in them as well as stiffness/compliances are completely different. Nevertheless, as will be demonstrated in Part 2 of this work, PVD and RMVT will lead to the same results if applied to one-layered structures (see Table IV of Part 2 [103]).

The matrix $\mathbf{K}_{\text{mixed}}^k$ must be assembled in a similar manner as those used for the PVD case. Loadings require to be assembled too. At the very end, for the whole multilayer one has

$$\mathbf{K}_{\text{mixed}} \mathbf{q} = \mathbf{P} \tag{78}$$

Assembly from element to structure level is made as usual in the finite element technique. At the structure level, the governing finite element system is

$$\mathbf{K}_{\text{mixed}}^{\text{S}} \mathbf{q}^{\text{S}} = \mathbf{P}^{\text{S}} \tag{79}$$

where superscript S denotes that arrays are those at the structure level.

9.2. Elimination of stress variables at element level

Let us expand the layer matrices

$$\mathbf{K}_{uu}^k, \mathbf{K}_{u\sigma}^k, \mathbf{K}_{\sigma u}^k, \mathbf{K}_{\sigma\sigma}^k$$

to multilayered element level, according to the procedure described in Section 8.2,

$$\mathbf{K}_{uu}, \mathbf{K}_{u\sigma}, \mathbf{K}_{\sigma u}, \mathbf{K}_{\sigma\sigma}$$

RMVT is then written as

$$\delta \mathbf{q}^{\text{T}} [\mathbf{K}_{uu} \mathbf{q} + \mathbf{K}_{u\sigma} \mathbf{g}] + \delta \mathbf{g}^{\text{T}} [\mathbf{K}_{\sigma u} \mathbf{q} + \mathbf{K}_{\sigma\sigma} \mathbf{g}] = \delta L_{\text{est}} \tag{80}$$

This leads to the following governing equations at element level:

$$\begin{aligned} \mathbf{K}_{uu} \mathbf{q} + \mathbf{K}_{u\sigma} \mathbf{g} &= \mathbf{P} \\ \mathbf{K}_{\sigma u} \mathbf{q} + \mathbf{K}_{\sigma\sigma} \mathbf{g} &= \mathbf{0} \end{aligned} \tag{81}$$

Static condensation is then applied at this stage,

$$[\mathbf{K}_{uu} - \mathbf{K}_{u\sigma}(\mathbf{K}_{\sigma\sigma})^{-1}\mathbf{K}_{\sigma u}]\mathbf{q} = \mathbf{P} \tag{82}$$

By introducing

$$\mathbf{K}^{\star}_{\text{mixed}} = [\mathbf{K}_{uu} - \mathbf{K}_{u\sigma}(\mathbf{K}_{\sigma\sigma})^{-1}\mathbf{K}_{\sigma u}] \tag{83}$$

one has the governing equations written in terms of only displacement variables,

$$\mathbf{K}^{\star}_{\text{mixed}}\mathbf{q} = \mathbf{P} \tag{84}$$

At structure level, one has

$$\mathbf{K}^{\star\,\mathrm{S}}_{\text{mixed}}\mathbf{q}^{\mathrm{S}} = \mathbf{P}^{\mathrm{S}} \tag{85}$$

Transverse stresses are then calculated *a posteriori*.

9.3. Elimination of stress variables at structure level

Following similar steps that have been discussed above, the layer matrix is assembled at multilayered level and then at structure level,

$$\mathbf{K}^{\mathrm{S}}_{uu}, \mathbf{K}^{\mathrm{S}}_{u\sigma}, \mathbf{K}^{\mathrm{S}}_{\sigma u}, \mathbf{K}^{\mathrm{S}}_{\sigma\sigma}$$

RMVT is then written as

$$\delta\mathbf{q}^{\mathrm{ST}}[\mathbf{K}^{\mathrm{S}}_{uu}\mathbf{q}^{\mathrm{S}} + \mathbf{K}^{\mathrm{S}}_{u\sigma}\mathbf{g}^{\mathrm{S}}] + \delta\mathbf{g}^{\mathrm{ST}}[\mathbf{K}^{\mathrm{S}}_{\sigma u}\mathbf{q}^{\mathrm{S}} + \mathbf{K}^{\mathrm{S}}_{\sigma\sigma}\mathbf{g}^{\mathrm{S}}] = \delta L^{\mathrm{S}}_{\text{est}} \tag{86}$$

which leads to the following governing equations:

$$\begin{aligned} \mathbf{K}^{\mathrm{S}}_{uu}\mathbf{q}^{\mathrm{S}} + \mathbf{K}^{\mathrm{S}}_{u\sigma}\mathbf{g}^{\mathrm{S}} &= \mathbf{P}^{\mathrm{S}} \\ \mathbf{K}^{\mathrm{S}}_{\sigma u}\mathbf{q}^{\mathrm{S}} + \mathbf{K}^{\mathrm{S}}_{\sigma\sigma}\mathbf{g}^{\mathrm{S}} &= \mathbf{0} \end{aligned} \tag{87}$$

Static condensation can be applied at this stage,

$$[\mathbf{K}^{\mathrm{S}}_{uu} - \mathbf{K}^{\mathrm{S}}_{u\sigma}(\mathbf{K}^{\mathrm{S}}_{\sigma\sigma})^{-1}\mathbf{K}^{\mathrm{S}}_{\sigma u}]\mathbf{q}^{\mathrm{S}} = \mathbf{P}^{\mathrm{S}} \tag{88}$$

By introducing

$$\mathbf{K}^{\star\star\,\mathrm{S}}_{\text{mixed}} = [\mathbf{K}^{\mathrm{S}}_{uu} - \mathbf{K}^{\mathrm{S}}_{u\sigma}(\mathbf{K}^{\mathrm{S}}_{\sigma\sigma})^{-1}\mathbf{K}^{\mathrm{S}}_{\sigma u}] \tag{89}$$

One has

$$\mathbf{K}^{\star\star\,\mathrm{S}}_{\text{mixed}}\mathbf{q}^{\mathrm{S}} = \mathbf{P}^{\mathrm{S}} \tag{90}$$

which has only displacement variables as nodal unknowns.

9.4. Full mixed case

For the full mixed case, the governing equations with stress variables are obtained. This leads to the following system of governing equations:

$$\begin{aligned} \mathbf{K}_{uu}^{S}\mathbf{q}^{S} + \mathbf{K}_{u\sigma}^{S}\mathbf{g}^{S} = \mathbf{P}^{S} \\ \mathbf{K}_{\sigma u}^{S}\mathbf{q}^{S} + \mathbf{K}_{\sigma\sigma}^{S}\mathbf{g}^{S} = \mathbf{0} \end{aligned} \tag{91}$$

For convenience, the following arrays are introduced:

$$\mathbf{h} = \begin{bmatrix} \mathbf{q} \\ \mathbf{g} \end{bmatrix} \tag{92}$$

$$\mathbf{P}^{f} = \begin{bmatrix} \mathbf{P} \\ \mathbf{0} \end{bmatrix} \tag{93}$$

$$\mathbf{K}_{f} = \begin{bmatrix} \mathbf{K}_{uu} & \mathbf{K}_{u\sigma} \\ \mathbf{K}_{\sigma u} & \mathbf{K}_{\sigma\sigma} \end{bmatrix} \tag{94}$$

The resulting global system of linear algebraic equations is

$$\mathbf{K}_{f}\mathbf{h} = \mathbf{P}^{f} \tag{95}$$

in which both displacement and stress variables appear as problem unknowns.

Different stress treatments would lead to different governing FE matrices and to different results. For the sake of completeness the so-called weak form of Hooke's law, established by Carrera [18], should be mentioned as a further possibility for the treatment of stress variables.

10. SUMMARY OF THE PRESENTED FINITE ELEMENTS AND CONCLUDING REMARKS

This paper has formulated multilayer finite plate elements according to the following statements:

- Two-dimensional theories that consider each layer as an independent plate (layerwise) as well as plate theories that consider all the layers as a single plate (equivalent single layer) have been considered.
- Linear and higher-order fields are considered for the two-dimensional expansion in the thickness direction. The order N of such an expansion has been taken as a free parameter of the derived formulations.
- The number of element nodes N_n has also been taken as a free parameter of the considered plate finite elements.
- Classical formulations with only displacement variables have been addressed in the framework of the principle of virtual displacements.

- Advanced formulations have been developed in the framework of Reissner's mixed variational theorem which consists of a variational tool designed for multilayered structure applications. Displacements and interlaminar continuous transverse stresses (shear and normal components) are assumed in the RMVT case.

Depending on the used variational statement (PVD or RMVT), variable descriptions (LWM or ESLM), order of the used expansion N, number of nodes N_n, a large number of multilayered plate elements have been derived. In order to lower the number of the formulas related to the different finite element matrices as much as possible, extensive use of indicial notations has been proposed in this work. Such an indicial notation, which was mainly invented for computer implementation, has permitted the authors to write all the considered FE matrices in terms of only five arrays (one for PVD and four for RMVT applications). These arrays have been called fundamental nuclei and were derived at a layer level. Each of them is of dimension 3×3. Multilayer arrays are constructed by imposing the continuity requirements for stresses and/or displacements, according to the variational statements that have been used.

Implementation of some of the derived finite elements is given in Part 2 of this work [103], where it is mainly concluded that RMVT can be considered a natural tool to analyse multilayered structures. RMVT, in fact, leads to a quasi-three-dimensional description of the stress fields of layered plates.

APPENDIX A: AN EXAMPLE SHOWING HOW THE INDICIAL NOTATION WORKS

This appendix shows how indicial notation works for a simple case. A particular plate element related to PVD applications has been chosen. These elements will be denoted by the acronym ED1 in Part 2 of this work: E for equivalent single layer, D for displacement approaches based upon PVD and 1 to signify that it consists of first-order linear expansion in z. ED1 finite element consists of one of the most popular plate elements. It is in fact the closest to the Reissner–Mindlin plate theories (ED1 cases include transverse normal strain/stress effects).

The functions used in the z-expansions are Equation (43),

$$F_b = 1, \quad F_t = z$$

The corresponding layer integrals are in this case

$$\begin{aligned}
E_{bb} &= \int_{A_k} F_t F_t \, \mathrm{d}z = \int_{A_k} \mathrm{d}z = z_{k-1} - z_k = h_k \\
E_{bt} &= \int_{A_k} F_t F_b \, \mathrm{d}z = \int_{A_k} z \, \mathrm{d}z = \frac{1}{2}(z_{k-1}^2 - z_k^2) \\
E_{tb} &= \int_{A_k} F_b F_t \, \mathrm{d}z = \int_{A_k} z \, \mathrm{d}z = \frac{1}{2}(z_{k-1}^2 - z_k^2) \\
E_{tt} &= \int_{A_k} F_b F_b \, \mathrm{d}z = \int_{A_k} z^2 \, \mathrm{d}z = \frac{1}{3}(z_{k-1}^3 - z_k^3)
\end{aligned} \tag{A1}$$

ED1 corresponds to the ESLM case of Figure 5 (left), that is stiffness terms in the correspondent fundamental nuclei are summed over k-range. In other words, layer stiffness is accumulated from layer to multilayered level. At this stage, we can denote the accumulated multilayer stiffness by referring to notations usually used for laminate analysis [27, 31]:

$$\begin{aligned}
A_{IJ} &= \sum_{k=1}^{N_l} C_{ij}^k \int_{A_k} F_b F_b \,\mathrm{d}z &&= \sum_{k=1}^{N_l} C_{ij}^k h_k \\
B_{IJ} &= \sum_{k=1}^{N_l} C_{ij}^k \int_{A_k} F_b F_t \,\mathrm{d}z &&= \sum_{k=1}^{N_l} C_{ij}^k \tfrac{1}{2}(z_{k-1}^2 - z_k^2), \quad I,J = 1,6 \\
D_{IJ} &= \sum_{k=1}^{N_l} C_{ij}^k \int_{A_k} F_t F_t \,\mathrm{d}z &&= \sum_{k=1}^{N_l} C_{ij}^k \tfrac{1}{3}(z_{k-1}^3 - z_k^3)
\end{aligned} \tag{A2}$$

which correspond to well-known in-plane, coupling and bending plate stiffness.

By using these stiffnesses, the fundamental nuclei related to ED1 finite element can be expanded as far as τ and s superscripts are concerned. As a result, the finite element stiffness matrix related to i,j node is obtained. It takes the form of $(2 \times 3) \times (3 \times 2)$ arrays in which the terms

$$Z_{pp}^{k\tau s}, Z_{pn}^{k\tau s}, Z_{np}^{k\tau s}, Z_{nn}^{k\tau s}$$

are opportunely replaced by the plate stiffnesses

$$A_{IJ}, B_{IJ}, D_{IJ}$$

according to the following substitution:

$$\left.\begin{aligned}
Z_{pp}^{kbb}, Z_{pn}^{kbb}, Z_{np}^{kbb}, Z_{nn}^{kbb} &\rightsquigarrow A_{IJ} \\
Z_{pp}^{kbt}, Z_{pn}^{kbt}, Z_{np}^{kbt}, Z_{nn}^{kbt} &\rightsquigarrow B_{IJ} \\
Z_{pp}^{ktb}, Z_{pn}^{ktb}, Z_{np}^{ktb}, Z_{nn}^{ktb} &\rightsquigarrow B_{IJ}, \\
Z_{pp}^{ktt}, Z_{pn}^{ktt}, Z_{np}^{ktt}, Z_{nn}^{ktt} &\rightsquigarrow D_{IJ}
\end{aligned}\right. \quad I,J = 1,6 \tag{A3}$$

The resulting 36 terms of this matrix are:

$$\begin{aligned}
K_{xx}^{ttij} &= \int_\Omega (D_{11} N_{i,x} N_{j,x} + D_{16} N_{i,y} N_{j,x} + D_{16} N_{i,x} N_{j,y} +; + D_{66} N_{i,y} N_{j,y} + A_{55} N_i N_j) \,\mathrm{d}\Omega \\
K_{xx}^{tbij} &= \int_\Omega (B_{11} N_{i,x} N_{j,x} + B_{16} N_{i,y} N_{j,x} + B_{16} N_{i,x} N_{j,y} +; + B_{66} N_{i,y} N_{j,y}) \,\mathrm{d}\Omega \\
K_{xx}^{btij} &= \int_\Omega (B_{11} N_{i,x} N_{j,x} + B_{16} N_{i,y} N_{j,x} + B_{16} N_{i,x} N_{j,y} +; + B_{66} N_{i,y} N_{j,y}) \,\mathrm{d}\Omega \\
K_{xx}^{bbij} &= \int_\Omega (A_{11} N_{i,x} N_{j,x} + A_{16} N_{i,y} N_{j,x} + A_{16} N_{i,x} N_{j,y} +; + A_{66} N_{i,y} N_{j,y}) \,\mathrm{d}\Omega
\end{aligned}$$

$$
\begin{aligned}
K_{xy}^{ttij} &= \int_{\Omega} (D_{12}N_{i,x}N_{j,y} + D_{26}N_{i,y}N_{j,y} + D_{16}N_{i,x}N_{j,x} + D_{66}N_{i,y}N_{j,x} + A_{45}N_iN_j)\,\mathrm{d}\Omega \\
K_{xy}^{tbij} &= \int_{\Omega} (B_{12}N_{i,x}N_{j,y} + B_{26}N_{i,y}N_{j,y} + B_{16}N_{i,x}N_{j,x} + B_{66}N_{i,y}N_{j,x})\,\mathrm{d}\Omega \\
K_{xy}^{btij} &= \int_{\Omega} (B_{12}N_{i,x}N_{j,y} + B_{26}N_{i,y}N_{j,y} + B_{16}N_{i,x}N_{j,x} + B_{66}N_{i,y}N_{j,x})\,\mathrm{d}\Omega \\
K_{xy}^{bbij} &= \int_{\Omega} (A_{12}N_{i,x}N_{j,y} + A_{26}N_{i,y}N_{j,y} + A_{16}N_{i,x}N_{j,x} + A_{66}N_{i,y}N_{j,x})\,\mathrm{d}\Omega \\
K_{xz}^{ttij} &= \int_{\Omega} (B_{13}N_{i,x}N_j + B_{36}N_{i,y}N_j + B_{55}N_iN_{j,x} + B_{45}N_iN_{j,y})\,\mathrm{d}\Omega \\
K_{xz}^{tbij} &= \int_{\Omega} (A_{55}N_iN_{j,x} + A_{45}N_iN_{j,y})\,\mathrm{d}\Omega \\
K_{xz}^{btij} &= \int_{\Omega} (B_{13}N_{i,x}N_j + B_{36}N_{i,y}N_j)\,\mathrm{d}\Omega \\
K_{xz}^{bbij} &= 0
\end{aligned}
\qquad (A4)
$$

$$
\begin{aligned}
K_{yx}^{ttij} &= \int_{\Omega} (D_{12}N_{i,y}N_{j,x} + D_{16}N_{i,x}N_{j,x} + D_{26}N_{i,y}N_{j,y} + D_{66}N_{i,x}N_{j,y} + A_{45}N_iN_j)\,\mathrm{d}\Omega \\
K_{yx}^{tbij} &= \int_{\Omega} (B_{12}N_{i,y}N_{j,x} + B_{16}N_{i,x}N_{j,x} + B_{26}N_{i,y}N_{j,y} + B_{66}N_{i,x}N_{j,y})\,\mathrm{d}\Omega \\
K_{yx}^{btij} &= \int_{\Omega} (B_{12}N_{i,y}N_{j,x} + B_{16}N_{i,x}N_{j,x} + B_{26}N_{i,y}N_{j,y} + B_{66}N_{i,x}N_{j,y})\,\mathrm{d}\Omega \\
K_{yx}^{bbij} &= \int_{\Omega} (A_{12}N_{i,y}N_{j,x} + A_{16}N_{i,x}N_{j,x} + A_{26}N_{i,y}N_{j,y} + A_{66}N_{i,x}N_{j,y})\,\mathrm{d}\Omega \\
K_{yy}^{ttij} &= \int_{\Omega} (D_{22}N_{i,y}N_{j,y} + D_{26}N_{i,x}N_{j,y}D_{26}N_{i,y}N_{j,x} + D_{66}N_{i,x}N_{j,x} + A_{44}N_iN_j)\,\mathrm{d}\Omega \\
K_{yy}^{ttij} &= \int_{\Omega} (B_{22}N_{i,y}N_{j,y} + B_{26}N_{i,x}N_{j,y}B_{26}N_{i,y}N_{j,x} + B_{66}N_{i,x}N_{j,x})\,\mathrm{d}\Omega \\
K_{yy}^{ttij} &= \int_{\Omega} (B_{22}N_{i,y}N_{j,y} + B_{26}N_{i,x}N_{j,y}B_{26}N_{i,y}N_{j,x} + B_{66}N_{i,x}N_{j,x})\,\mathrm{d}\Omega \\
K_{yy}^{ttij} &= \int_{\Omega} (A_{22}N_{i,y}N_{j,y} + A_{26}N_{i,x}N_{j,y}A_{26}N_{i,y}N_{j,x} + A_{66}N_{i,x}N_{j,x})\,\mathrm{d}\Omega \\
K_{yz}^{ttij} &= \int_{\Omega} (B_{23}N_{i,y}N_j + B_{36}N_{i,x}N_j + B_{45}N_iN_{j,x} + B_{44}N_iN_{j,y})\,\mathrm{d}\Omega \\
K_{yz}^{tbij} &= \int_{\Omega} (B_{23}N_{i,y}N_j + B_{36}N_{i,x}N_j)\,\mathrm{d}\Omega \\
K_{yz}^{btij} &= \int_{\Omega} (B_{45}N_iN_{j,x} + B_{44}N_iN_{j,y})\,\mathrm{d}\Omega \\
K_{yz}^{bbij} &= 0
\end{aligned}
\qquad (A5)
$$

$$
\begin{aligned}
K_{zx}^{ttij} &= \int_{\Omega} (B_{55} N_{i,x} N_j + B_{45} N_{i,y} N_j + B_{13} N_i N_{j,x} + B_{36} N_i N_{j,y})\, \mathrm{d}\Omega \\
K_{zx}^{ttij} &= \int_{\Omega} (B_{55} N_{i,x} N_j + B_{45} N_{i,y} N_j)\, \mathrm{d}\Omega \\
K_{zx}^{ttij} &= \int_{\Omega} (B_{13} N_i N_{j,x} + B_{36} N_i N_{j,y})\, \mathrm{d}\Omega \\
K_{zx}^{ttij} &= 0 \\
K_{zy}^{ttij} &= \int_{\Omega} (B_{45} N_{i,x} N_j + B_{44} N_{i,y} N_j + B_{23} N_i N_{j,y} + Z_{36} N_i N_{j,x})\, \mathrm{d}\Omega \\
K_{zy}^{ttij} &= \int_{\Omega} (B_{45} N_{i,x} N_j + B_{44} N_{i,y} N_j)\, \mathrm{d}\Omega \\
K_{zy}^{ttij} &= \int_{\Omega} (+B_{23} N_i N_{j,y} + Z_{36} N_i N_{j,x})\, \mathrm{d}\Omega \\
K_{zy}^{ttij} &= 0 \\
K_{zz}^{\tau sij} &= \int_{\Omega} (D_{55} N_{i,x} N_{j,x} + D_{45} N_{i,y} N_{j,x} + D_{45} N_{i,x} N_{j,y} + D_{44} N_{i,y} N_{j,y} + A_{33} N_i N_j)\, \mathrm{d}\Omega \\
K_{zz}^{\tau sij} &= \int_{\Omega} (B_{55} N_{i,x} N_{j,x} + B_{45} N_{i,y} N_{j,x} + B_{45} N_{i,x} N_{j,y} + B_{44} N_{i,y} N_{j,y})\, \mathrm{d}\Omega \\
K_{zz}^{\tau sij} &= \int_{\Omega} (B_{55} N_{i,x} N_{j,x} + B_{45} N_{i,y} N_{j,x} + B_{45} N_{i,x} N_{j,y} + B_{44} N_{i,y} N_{j,y})\, \mathrm{d}\Omega \\
K_{zz}^{\tau sij} &= \int_{\Omega} (A_{55} N_{i,x} N_{j,x} + A_{45} N_{i,y} N_{j,x} A_{45} N_{i,x} N_{j,y} + A_{44} N_{i,y} N_{j,y})\, \mathrm{d}\Omega
\end{aligned}
\tag{A6}
$$

The integral on Ω has been explicitly written. Note that it is a multilayered level matrix. By varying the superscripts i, j over the element node N_n the full $6N_n \times 6N_n$ matrix is obtained.

The written explicit expression of stiffness matrix will never be used in computer implementations. In fact, these implementations will build stiffness matrices by making appropriate loops around the five derived fundamental nuclei (see Part 2).

APPENDIX B: APPLIED LOADING VECTORS

The technique employed to derive finite element stiffness/compliance matrices can be applied to derive consistent loading arrays. An example is given in this appendix, which deals with a distribution of pressure acting on the k-layer and applied on a plane parallel to the reference surface Ω which is distant $\zeta^k = \zeta_1^k$. The external work made by these pressure distributions is

$$\delta L_P^k = \int_{\Omega_1} \delta \mathbf{u}^{k\mathrm{T}}(x, y, \zeta_1^k) \mathbf{P}^k(x, y, \zeta_1^k)\, \mathrm{d}\Omega_1 \tag{B1}$$

where Ω_1 is the domain on which the pressure acts and $\mathbf{P}^k(x, y, \zeta_1^k)$ is the array which denotes the pressure.

Since $\Omega_1 = \Omega$ for plates, Equation (B1) becomes

$$\delta L_P^k = \delta \mathbf{q}_{\tau i}^{k\mathrm{T}} \; F_\tau^1 \; F_s^1 \; {}_i N_j \mathbf{p}_s^k \triangleright_\Omega \mathbf{a}_{sj}^k \tag{B2}$$

In case more pressure loadings are applied corresponding to more than one plane, the related terms must be summed. In formula

$$\delta L_P^k = \delta \mathbf{q}_{\tau i}^{k\mathrm{T}} \; F_\tau^m \; F_s^m \triangleleft N_i N_j \mathbf{p}_s^k \triangleright_\Omega \mathbf{a}_{sj}^k, \quad m = t, r, b, \quad r = 2, 3, \ldots, N \tag{B3}$$

At least top and bottom layer surface pressure are included in the previous equation. By introducing

$$\mathbf{D}^{k\tau sij} = F_\tau^m \; F_s^m \triangleleft N_i N_j \mathbf{p}_s^k \triangleright_\Omega \tag{B4}$$

one has

$$\delta L_P^k = \delta \mathbf{q}_{\tau i}^{k\mathrm{T}} \; \mathbf{D}^{k\tau sij} \mathbf{a}_{sj}^k \tag{B5}$$

$\mathbf{D}^{k\tau sij}$ plays the role of fundamental nucleus. In explicit form, it holds that

$$\mathbf{D}^{k\tau sij} = F_\tau^m \; F_s^m \begin{bmatrix} \triangleleft N_i N_j \, p_{xs}^k \triangleright_\Omega & 0 & 0 \\ 0 & \triangleleft N_i N_j \, p_{ys}^k \triangleright_\Omega & 0 \\ 0 & 0 & \triangleleft N_i N_j \, p_{zs}^k \triangleright_\Omega \end{bmatrix} \tag{B6}$$

At the very end, one notices that by introducing

$$\mathbf{P}_{\tau i}^{k\mathrm{eq}} = \mathbf{D}^{k\tau sij} \mathbf{a}_{sj}^k \tag{B7}$$

Equation (B5) becomes

$$\delta L_P^k = \delta \mathbf{q}_{\tau i}^{k\mathrm{T}} \mathbf{P}_{\tau i}^{k\mathrm{eq}} \tag{B8}$$

The array $\mathbf{P}_{\tau i}^{k\mathrm{eq}}$ therefore assumes the meaning of the loading array variationally equivalent to the applied pressure.

APPENDIX C: IDENTIFICATION OF TERMS RELATED TO TRANSVERSE STRESSES AND STRAINS

Owing to numerical reasons, such as shear locking mechanisms [108], it is essential to distinguish stiffness/compliance terms related to different transverse stress components. These terms are, in fact, treated with different numerical integration schemes in the companion paper (Part 2), where numerical evaluations are given.

C.1. PVD cases

The Hooke's law matrix can be conveniently arranged in the following form:

$$\boldsymbol{\sigma}^k = (\tilde{\mathbf{C}}^{kp} + \tilde{\mathbf{C}}^{k\ddagger} + \tilde{\mathbf{C}}^{k\dagger}) \boldsymbol{\varepsilon}^k \tag{C1}$$

By splitting in-plane and out-of-plane contribution, one has

$$\boldsymbol{\sigma}_p^k = \tilde{\mathbf{C}}_{pp}^k \boldsymbol{\varepsilon}_p^k + \tilde{\mathbf{C}}_{pn}^k \boldsymbol{\varepsilon}_n^k \tag{C2}$$

$$\boldsymbol{\sigma}_n^k = \delta_\dagger \tilde{\mathbf{C}}_{np}^k \boldsymbol{\varepsilon}_p^k + (\delta_\ddagger \tilde{\mathbf{C}}_{nn}^{k\ddagger} + \delta_\dagger \tilde{\mathbf{C}}_{nn}^{k\dagger}) \boldsymbol{\varepsilon}_n^k \tag{C3}$$

where

$$\tilde{\mathbf{C}}_{nn}^{k\ddagger} = \begin{bmatrix} \tilde{C}_{55}^k & \tilde{C}_{45}^k & 0 \\ \tilde{C}_{45}^k & \tilde{C}_{44}^k & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \tilde{\mathbf{C}}_{nn}^{k\dagger} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \tilde{C}_{33}^k \end{bmatrix}$$

For our convenience, the symbols $\delta_\dagger$ $\delta_\ddagger$ have been introduced. Such symbols permit one to evaluate in a different manner shear and normal components.

In order to outline transverse strain contribution ε_{zz}^k, geometrical relations are then written in the following form:

$$\boldsymbol{\varepsilon}_p^k = F_\tau \mathbf{D}_p (N_i \mathbf{I}) \mathbf{q}_{\tau i}^k \tag{C4}$$

$$\boldsymbol{\varepsilon}_n^k = F_\tau \mathbf{D}_{n\Omega} (N_i \mathbf{I}) \mathbf{q}_{\tau i}^k + F_{\tau,z} (N_i \mathbf{I}_\delta) \mathbf{q}_{\tau i}^k \tag{C5}$$

where

$$\mathbf{I}_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta \end{bmatrix} \tag{C6}$$

ε_{zz}^k is therefore written as

$$\varepsilon_{zz}^k = F_{\tau,z} \delta (N_i q_{u_z \tau i}^k) \tag{C7}$$

$\varepsilon_{zz}^k = 0$ is simply obtained by forcing $\delta = 0$.

The stiffness matrix can be written as

$$\begin{aligned} \mathbf{K}^{k\tau sij} = \triangleleft \mathbf{D}_p^{\mathrm{T}}(N_i \mathbf{I}) [\tilde{\mathbf{Z}}_{pp}^{k\tau s} \mathbf{D}_p (N_j \mathbf{I}) + \tilde{\mathbf{Z}}_{pn}^{k\tau s} \mathbf{D}_{n\Omega} (N_j \mathbf{I}) + \tilde{\mathbf{Z}}_{pn}^{k\tau s_{,z}} (N_j \mathbf{I}_\delta)] \\ + \mathbf{D}_{n\Omega}^{\mathrm{T}} (N_i \mathbf{I}) [\delta_\dagger \tilde{\mathbf{Z}}_{np}^{k\tau s} \mathbf{D}_p (N_j \mathbf{I}) + (\delta_\ddagger \tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau s} + \delta_\dagger \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau s}) \mathbf{D}_{n\Omega} (N_j \mathbf{I}) \\ + (\delta_\ddagger \tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau s_{,z}} + \delta_\dagger \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau s_{,z}}) (N_j \mathbf{I}_\delta)] + (N_i \mathbf{I}_\delta) [\delta_\dagger \tilde{\mathbf{Z}}_{np}^{k\tau_{,z} s} \mathbf{D}_p (N_j \mathbf{I}) \\ + (\delta_\ddagger \tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau_{,z} s} + \delta_\dagger \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau_{,z} s}) \mathbf{D}_{n\Omega} (N_j \mathbf{I}) + (\delta_\ddagger \tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau_{,z} s_{,z}} + \delta_\dagger \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau_{,z} s_{,z}}) (N_j \mathbf{I}_\delta)] \triangleright_\Omega \end{aligned} \tag{C8}$$

where

$$(\tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau s}, \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau s}) = (\tilde{\mathbf{C}}_{nn}^{k\ddagger} E_{\tau s}, \tilde{\mathbf{C}}_{nn}^{k\dagger} E_{\tau s})$$

$$(\tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau s_{,z}}, \tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau_{,z}s}, \tilde{\mathbf{Z}}_{nn}^{k\ddagger\tau_{,z}s_{,z}}) = (\tilde{\mathbf{C}}_{nn}^{k\ddagger}E_{\tau s_{,z}}, \tilde{\mathbf{C}}_{nn}^{k\ddagger}E_{\tau_{,z}s}^{k}, \tilde{\mathbf{C}}_{nn}^{k\ddagger}E_{\tau_{,z}s_{,z}})$$

$$(\tilde{\mathbf{Z}}_{nn}^{k\dagger\tau s_{,z}}, \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau_{,z}s}, \tilde{\mathbf{Z}}_{nn}^{k\dagger\tau_{,z}s_{,z}}) = (\tilde{\mathbf{C}}_{nn}^{k\dagger}E_{\tau s_{,z}}, \tilde{\mathbf{C}}_{nn}^{k\dagger}E_{\tau_{,z}s}^{k}, \tilde{\mathbf{C}}_{nn}^{k\dagger}E_{\tau_{,z}s_{,z}})$$

As far as the reduced/selective integration technique is concerned, it is intended that:

- Normal integration denoted by the IN scheme (see Part 2) signifies that all the terms are fully integrative (full integration by using 3×3 and 2×2 Gaussian points for eight- or nine- and four-noded plates, respectively).
- Selective integration denoted by the IS scheme signifies that terms that have been put in a single rectangle must be calculated by reduced integration (it is intended that the reduced integration scheme is obtained by the full one by reducing the grid of one unity).
- Selective integration denoted by the IS2 scheme signifies that both terms that have been put in single and double rectangles must be calculated according to reduced integration.

C.2. RMVT cases

Following what was done above, $\sigma_{xz}^k, \sigma_{yz}^k; \sigma_{zz}^k$ (the fundamental array related to RMVT applications) are obtained in the following forms:

$$\begin{aligned}
\mathbf{K}_{uu}^{k\tau sij} &= \triangleleft [\mathbf{D}_p^{\mathrm{T}}(N_i\mathbf{I})\mathbf{Z}_{pp}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I})]\triangleright_\Omega \\
\mathbf{K}_{u\sigma}^{k\tau sij} &= \triangleleft [\mathbf{D}_p^{\mathrm{T}}(N_i\mathbf{I}_z)\mathbf{Z}_{pn}^{k\tau s}N_j + \mathbf{D}_{n\Omega}^{\mathrm{T}}(N_i\mathbf{I}_z)E_{\tau s}N_j + E_{\tau_{,z}s}N_iN_j\mathbf{I}_z\mathbf{I}_\delta]\triangleright_\Omega \\
\mathbf{K}_{\sigma u}^{k\tau sij} &= \triangleleft [N_iE_{\tau s}\mathbf{D}_{n\Omega}(N_j\mathbf{I}_z) + E_{\tau s_{,z}}N_iN_j\mathbf{I}_z\mathbf{I}_\delta - N_i\mathbf{Z}_{np}^{k\tau s}\mathbf{D}_p(N_j\mathbf{I}_z)]\triangleright_\Omega \\
\mathbf{K}_{\sigma\sigma}^{k\tau sij} &= \triangleleft [-(N_i\mathbf{I}_z)\mathbf{Z}_{nn}^{k\tau s}(N_j\mathbf{I}_z)]\triangleright_\Omega
\end{aligned} \tag{C9}$$

where

$$\mathbf{I}_z = \begin{bmatrix} \delta_T & 0 & 0 \\ 0 & \delta_T & 0 \\ 0 & 0 & \delta_z \end{bmatrix}, \quad \mathbf{I}_\delta = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & \delta \end{bmatrix} \tag{C10}$$

Note that $\mathbf{K}_{uu}^{k\tau sij}$ is not influenced by $\sigma_{xz}^k, \sigma_{yz}^k; \sigma_{zz}^k$.

REFERENCES

1. Pagano NJ. Stress fields in composite laminates. *International Journal of Solids and Structures* 1978; **14**: 385–400.
2. Pagano NJ. Exact solutions for composite laminates in cylindrical bending. *Journal of Composite Materials* 1969; **3**:398–411.
3. Pagano NJ. Exact solutions for rectangular bi-direction composites and sandwich plates. *Journal of Composite Materials* 1970; **4**:20–34.
4. Pagano NJ, Hatfield SJ. Elastic behavior of multilayered bidirectional composites. *American Institute of Aeronautics and Astronautics Journal* 1972; **10**:931–933.
5. Vel SS, Batra RC. Analytical solution for rectangular thick plates subjected to arbitrary boundary conditions. *American Institute of Aeronautics and Astronautics Journal* 1999; **37**:1464–1473.
6. Vel SS, Batra RC. A generalized plane strain deformation of thick anisotropic composite laminates plates. *International Journal of Solids and Structures* 2000; **37**:715–733.

7. Whitney JM. The effects of transverse shear deformation on the bending of laminated plates. *Journal of Composite Materials* 1969; **3**:534–547.
8. Noor AK, Burton WS. Stress and free vibration analyses of multilayered composite plates. *Composite Structures* 1989; **11**:183–204.
9. Noor AK, Burton WS. Assessment of shear deformation theories for multilayered composite plates. *Applied Mechanics Review* 1989; **41**:1–18.
10. Noor AK, Burton WS, Peters JM. Predictor corrector procedures for stress and free vibration analysis of multilayered composite plates and shells. *Computer Methods in Applied Mechanics and Engineering* 1990; **82**:341–363.
11. Malik M. Differential quadrature method in computational mechanics: new development and applications. *Ph.D. Dissertation*, University of Oklahoma, Oklahoma, 1994.
12. Malik M, Bert CW. Differential quadrature analysis of free vibration of symmetric cross-ply laminates with shear deformation and rotatory inertia. *Shock and Vibration* 1995; **2**:321–338.
13. Teo TM, Liew KM. Three-dimensional elasticity solutions to some orthotropic plate problems. *International Journal of Solids and Structures* 1999; **36**:5301–5326.
14. Davì G. A general boundary integral formulation for numerical solutions of bended multilayer sandwich plates. *Proceedings of the 11th International Conference on Boundary Element Method* 1989; **1**:23–35.
15. Davì G. Stress field in general composite laminates. *American Institute of Aeronautics and Astronautics Journal* 1996; **34**:2604–2608.
16. Davì G, Milazzo A. Bending stress fields in composite laminate beams by a boundary integral formulation. *Composite Structures* 1999; **71**:267–276.
17. Milazzo A. Interlaminar stress in laminated composite beam-type structures under shear/bending. *American Institute of Aeronautics and Astronautics Journal* 2000; **38**:687–694.
18. Carrera E. A class of two-dimensional theories for anisotropic multilayered plates analysis. *Accademia delle Scienze di Torino*, Memorie Scienze Fisiche, 19–20 (1995–1996); 1–39.
19. Carrera E. Mixed layer-wise models for multilayered plates analysis. *Composite Structures* 1998; **43**:57–70.
20. Carrera E. Evaluation of layer-wise mixed theories for laminated plates analysis. *American Institute of Aeronautics and Astronautics Journal* 1998; **26**:830–839.
21. Carrera E. Layer-wise mixed models for accurate vibration analysis of multilayered plates. *Journal of Applied Mechanics* 1998; **65**:820–828.
22. Librescu L, Reddy JN. A critical review and generalization of transverse shear deformable anisotropic plates. *Euromech Colloquium 219*, Kassel, September 1986; *Refined Dynamical Theories of Beams*, *Plates and Shells and their Applications*. Elishakoff, Irretier (eds). Springer: Berlin, 1987; 32–43.
23. Kapania RK, Raciti S. Recent advances in analysis of laminated beams and plates. *American Institute of Aeronautics and Astronautics Journal* 1989; **27**:923–946.
24. Reddy JN, Robbins DH. Theories and computational models for composite laminates. *Applied Mechanics Review* 1994; **47**:147–165.
25. Noor AK, Burton S, Bert CW. Computational model for sandwich panels and shells. *Applied Mechanics Review* 1996; **49**:155–199.
26. Librescu L. *Elasto-statics and Kinetics of Anisotropic and Heterogeneous Shell-Type Structures*. Noordhoff Int.: Leyden, Netherlands, 1975.
27. Reddy JN. *Mechanics of Laminated Composite Plates. Theory and Analysis.* CRC Press: Boca Raton FL, 1997.
28. Cauchy AL. Sur l'equilibre et le mouvement d'une plaque solide. *Exercises de Matematique* 1828; **3**:328–355.
29. Poisson SD. Memoire sur l'equilibre et le mouvement des corps elastique. *Memoires de l'Academie des Sciences* 1829; **8**:357.
30. Kirchhoff G. Über das Gleichgewicht und die Bewegung einer elastishen Scheibe. *Zeitscheift fuer Angewandte Mathematik* 1850; **40**:51–88.
31. Jones RM. *Mechanics of Composite Materials*. McGraw-Hill: New York, 1975.
32. Koiter WT. A consistent first approximations in the general theory of thin elastic shells, *Proceedings of Symposium on the Theory of Thin Elastic Shells*, August 1959, North-Holland: Amsterdam, 1959; 12–23.
33. Reissner E. The effect of transverse shear deformation on the bending of elastic plates. *Journal of Applied Mechanics* 1945; **12**:69–76.
34. Mindlin. Influence of rotatory inertia and shear in flexural motions of isotropic elastic plates. *Journal of Applied Mechanics* 1951; **18**:1031–1036.
35. Yang PC, Norris CH, Stavsky Y. Elastic Wave propagation in heterogenous plates. *International Journal of Solids and Structures* 1966; **2**:665–684.
36. Hildebrand FB, Reissner E, Thomas GB. Notes on the foundations of the theory of small displacements of orthotropic shells. *NACA TN*-1833, Washington, DC, 1938.
37. Sun CT, Whitney JM. On the theories for the dynamic response of laminated plates. *American Institute of Aeronautics and Astronautics Journal* 1973; **11**:372–398.

38. Lo KH, Christensen RM, Wu EM. A higher-order theory of plate deformation, Part 2: laminated plates. *Journal of Applied Mechanics* 1977; **44**:669–676.
39. Carrera E. C_z^0 requirements—models for the two-dimensional analysis of multilayered structures. *Composite Structures* 1997; **37**:373–384.
40. Vlasov BF. On the equations of bending of plates. *Doklady Akademii Nauk Azerbeidzhanskoi SSR* 1957; **3**:955–979.
41. Reddy JN. *Energy and Variational Methods in Applied Mechanics.* Wiley: NY, 1984.
42. Reddy JN. A simple higher order theories for laminated composites plates. *Journal of Applied Mechanics* 1984; **52**:745–742.
43. Reddy JN, Phan ND. Stability and vibration of isotropic, orthotropic, and laminated plates according to a higher order shear deformation theory. *Journal of Sound and Vibration* 1985; **98**:157–170.
44. Srinivas S. A refined analysis of composite laminates. *Journal of Sound and Vibration* 1973; **30**:495–507.
45. Cho KN, Bert CW, Striz AG. Free vibrations of laminated rectangular plates analyzed by higher order individual-layer theory. *Journal of Sound and Vibration* 1991; **145**:429–442.
46. Nosier A, Kapania RK, Reddy JN. Free vibration analysis of laminated plates using a layer-wise theory. *American Institute of Aeronautics and Astronautics Journal* 1993; **31**:2335–2346.
47. Lekhnitskii SG. Strength calculation of composite beams. *Vestnik inzhen. i tekhnikov* 1935; (9).
48. Lekhnitskii SG. *Anisotropic Plates*. 2nd edn. Translated from the 2nd Russian edn, Tsai SW, Cheron (eds). Gordon and Breach: New York, 1968.
49. Ambartsumian SA. *Theory of Anisotropic Plates.* Translated from Russian by T. Cheron and Edited by J.E. Ashton Tech. Pub. Co., 1969.
50. Ren JG. A new theory for laminated plates. *Composite Science and Technology* 1986; **26**:225–239.
51. Ren JG, Owen DRJ. Vibration and buckling of laminated plates. *International Journal of Solids and Structures* 1989; **25**:95–106.
52. Rath BK, Das YC. Vibration of layered shells. *Journal of Sound and Vibration* 1973; **28**:737–757.
53. Atluri SN, Tong P, Murakawa H. Recent studies in hybrid and mixed finite element methods in mechanics. In *Hybrid and Mixed Finite Element Methods*. Atluri SN, Callagher RH, Zienkiewicz O (eds). Wiley: New York, 1983; 51–71.
54. Murakami H. A laminated beam theory with interlayer slip. *Journal of Applied Mechanics* 1984; **51**:551–559.
55. Murakami H. Laminated composite plate theory with improved in-plane responses. *ASME Proceedings of PVP Conference*, New Orleans, June 24–26, PVP-Vol. 98-2, 1985; 257–263.
56. Murakami H. Laminated composite plate theory with improved in-plane responses. *Journal of Applied Mechanics* 1986; **53**:661–666.
57. Toledano A, Murakami H. A high-order laminated plate theory with improved in-plane responses. *International Journal of Solids and Structures* 1987; **23**:111–131.
58. Toledano A, Murakami H. A composite plate theory for arbitrary laminate configurations. *Journal of Applied Mechanics* 1987; **54**:181–189.
59. Carrera E. A refined multilayered finite element model applied to linear and nonlinear analysis of sandwich structures. *Composite Science and Technology* 1998; **58**:1553–1569.
60. Carrera E. Transverse normal stress effects in multilayered plates. *Journal of Applied Mechanics* 1999; **66**: 1004–1012.
61. Carrera E. A study of transverse normal stress effects on vibration of multilayered plates and shells. *Journal of Sound and Vibration* 1999; **225**:803–829.
62. Carrera E. Single-layer vs. multi-layers plate modelings on the basis of Reissner's mixed theorem. *American Institute of Aeronautics and Astronautics Journal* 2000; **38**:342–343.
63. Carrera E. *A priori* vs *a posteriori* evaluation of transverse stresses in multilayered orthotropic plates. *Composite Structures* 2000; **48**:245–260.
64. Carrera E. An assessment of mixed and classical theories for thermal stress analysis of orthotropic plates. *Journal of Thermal Stress* 2000; **23**:797–831.
65. Carrera E. Developments, ideas and evaluations based upon the Reissner's mixed theorem in the modeling of multilayered plates and shells. *Applied Mechanics Review* 2001; **54**:301–329.
66. Pryor CW, Barker RM. A finite element analysis including transverse shear effect for applications to laminated plates. *American Institute of Aeronautics and Astronautics Journal* 1971; **9**:912–917.
67. Noor AK. Finite element analysis of anisotropic plates. *American Institute of Aeronautics and Astronautics Journal* 1972; **11**:289–307.
68. Noor AK, Mathers MD. Finite element analysis of anisotropic plates. *International Journal for Numerical Methods in Engineering* 1977; **11**:289–370.
69. Panda SC, Natarayan R. Finite element analysis of laminated composite plates. *International Journal for Numerical Methods in Engineering* 1979; **14**:69–79.
70. Reddy JN. A penalty plate-bending element for the analysis of laminated anisotropic composites plates. *International Journal for Numerical Methods in Engineering* 1980; **12**:1187–1206.

71. Kant T, Kommineni JR. Large amplitude free vibration analysis of cross-ply composite and sandwich laminates with a refined theory and C^0 finite elements. *Computers and Structures* 1989; **50**:123–134.
72. Kant T, Owen DRJ, Zienkiewicz OC. Refined higher order C^0 plate bending element. *Computers and Structures* 1982; **15**:177–183.
73. Bathe KJ, Dvorkin EN. A four node plate bending element based on Mindlin/Reissner plate theory and mixed interpolation. *International Journal for Numerical Methods in Engineering* 1985; **21**:367–383.
74. Brank B, Carrera E. Multilayered shell finite element with interlaminar continuous shear stresses: a refinement of the Reissner–Mindlin formulation. *International Journal for Numerical Methods in Engineering* 2000; **48**:843–874
75. Pandya BN, Kant T. Higher-order shear deformable for flexural of sandwich plates. Finite element evaluations. *International Journal of Solids and Structures* 1988; **24**:1267–1286.
76. Reddy JN. On computational models for composite laminate. *International Journal for Numerical Methods in Engineering* 1989; **27**:361–382.
77. Barboni R, Gaudenzi P. A class of C^0 finite elements for the static and dynamic analysis of laminated plates. *Computers and Structures* 1992; **44**:1169–1178.
78. Cho M, Parmerter RR. Efficient higher order composite plate theory for general lamination configurations. *American Institute of Aeronautics and Astronautics Journal* 1993; **31**:1299–1305.
79. Aitharaju VR, Averill RC. C^0 zig-zag kinematic displacement models for the analysis of laminated composites. *Mechanics of Composite Materials and Structures* 1996; **6**:31–56.
80. Idlbi A, Karama M, Touratier M. Comparison of various laminated plate theories. *Composite Structures* 1997; **37**:173–184.
81. Cho YB, Averill RC. First order zig-zag sublaminate plate theory and finite element model for laminated composite and sandwich panels. *Computers and Structures* 2000; **50**:1–15.
82. Polit O, Touratier M. Higher order triangular sandwich plate finite elements for linear and nonlinear analyses. *Computer Methods in Applied Mechanics and Engineering* 2000; **185**:305–324.
83. Pinsky PM, Kim KK. A multi-director formulation for elastic-viscoelastic layered shells. *International Journal for Numerical Methods in Engineering* 1986; **23**:2213–2224.
84. Reddy JN. An evaluation of equivalent single layer and layer-wise theories of composite laminates. *Composite Structures* 1993; **25**:21–35.
85. Robbins DH Jr, Reddy JN. Modeling of thick composites using a layer-wise theory. *International Journal for Numerical Methods in Engineering* 1993; **36**:655–677.
86. Gaudenzi P, Barboni R, Mannini A. A finite element evaluation of single-layer and multi-layer theories for the analysis of laminated plates. *Computers and Structures* 1995; **30**:427–440.
87. Botello S, Onate E, Canet JM. A layer-wise triangle for analysis of laminated composite plates and shells. *Computers and Structures* 1999; **70**:635–646.
88. Babuska I, Szabo, Actis. Hierarchy models for laminated composites. *International Journal for Numerical Methods in Engineering* 1992; **33**:503–535.
89. Fish J, Markolefas S. The *s*-version of the finite element method for multilayer laminates. *International Journal for Numerical Methods in Engineering* 1992; **33**:1081–1105.
90. Turn J-Q, Wang Y-B, Wang Y-M. Three-dimensional asymptotic finite element method for anisotropic inhomogeneous and laminated plates. *International Journal of Solids and Structures* 1996; **33**:1939–1960.
91. Pian THH, Mau ST. Some recent studies in assumed-stress hybrid models. In *Advances in Computational Methods in Structural Mechanics and Design*, Oden, Clought, Yamamoto (eds), 1972.
92. Spilker RL, Orringer O, Witmer O. Use of hybrid/stress finite element model for the static and dynamic analysis of multilayer composite plates and shells. *MIT ASRL TR* 181-2, 1976.
93. Spilker RL, Chou SC, Orringer O. Alternate hybrid-stress elements for analysis of multilayer composite plates. *Journal of Composite Materials* 1977; **11**:51–70.
94. Moriya K. Laminated plate and shell elements for finite element analysis of advanced fiber reinforced composite structure, laminated composite plates. *Transactions of the Society of Mechanical Engineers* 1986; **52**: 1600–1607 (in Japanese).
95. Liou WJ, Sun CT. A three-dimensional hybrid stress isoparametric element for the analysis of laminated composite plates. *Computers and Structures* 1987; **25**:241–249.
96. Jing H, Liao ML. Partial hybrid stress element for the analysis of thick laminate composite plates. *International Journal for Numerical Methods in Engineering* 1989; **28**:2813–2827.
97. Auricchio F, Sacco E. Partial mixed formulation and refined models for the analysis of composite laminates within and FSDT. *Composite Structures* 1999; **46**:103–113.
98. Rao KM, Meyer-Piening HR. Analysis of thick laminated anisotropic composites plates by the finite element method. *Composite Structures* 1990; **15**:185–213.
99. Carrera E. C^o Reissner–Mindlin multilayered plate elements including zig-zag and interlaminar stresses continuity. *International Journal for Numerical Methods in Engineering* 1996; **39**:1797–1820.
100. Carrera E, Kröplin B. Zig-zag and interlaminar equilibria effects in large deflection and postbuckling analysis of multilayered plates. *Mechanics of Composite Materials and Structures* 1997; **4**:69–94.

101. Carrera E, Krause H. An investigation on nonlinear dynamics of multilayered plates accounting for C_z^0 requirements. *Computers and Structures* 1998; **69**:463–486.
102. Carrera E. An improved Reissner–Mindlin-type model for the electromechanical analysis of multilayered plates including piezo-layers. *Journal of Intelligent Materials System and Structures* 1997; **8**:232–248.
103. Carrera E, Demasi L. Classical and advanced multilayered plate elements based upon PVD and RMVT. Part 2. Numerical implementations. 2000; **55**:(in press).
104. Reissner E. On a certain mixed variational theory and a proposed applications. *International Journal for Numerical Methods in Engineering* 1984; **20**:1366–1368.
105. Reissner E. On a mixed variational theorem and on a shear deformable plate theory. *International Journal for Numerical Methods in Engineering* 1986; **23**:193–198.
106. Washizu K. *Variational Method in Elasticity and Plasticity*. Pergamon Press: Oxford, 1968.
107. Antona E. *Mathematical Model and their Use in Engineering*. Miele A, Salvetti A (eds). Applied Mathematics in the Aerospace Science/Engineering, vol. 44, 1991; 395–433.
108. Zienkiewicz OC. *The Finite Element Method.* McGraw-Hill: London, 1986.

