Commun. Math. Phys. 123, 177–254 (1989)

Communications in
Mathematical
Physics

Classical and Quantum Conformal Field Theory

Gregory Moore and Nathan Seiberg*

Institute for Advanced Study, Princeton, NJ 08540, USA

Abstract. We define chiral vertex operators and duality matrices and review the fundamental identities they satisfy. In order to understand the meaning of these equations, and therefore of conformal field theory, we define the classical limit of a conformal field theory as a limit in which the conformal weights of all primary fields vanish. The classical limit of the equations for the duality matrices in rational field theory together with some results of category theory, suggest that (quantum) conformal field theory should be regarded as a generalization of group theory.

1. Introduction and Conclusion

Although the classification of conformal field theory is an extremely interesting problem, of importance in mathematics, statistical mechanics, and string theory, it should not be mistaken for a fundamental problem in string theory. Conformal field theories are classical solutions of the string equations of motion. In string theory the basic physical laws which lead to the string equations are far more important than the classification of the solutions to those equations. However, the meaning of these equations is far from being understood. It seems that our lack of a full understanding of the meaning of conformal field theory prevents us from finding natural generalizations. One might hope that a proper formulation of conformal field theory will lead us to the principles underlying string theory and will allow us to generalize the "on-shell" results. The classification of conformal field theories should be viewed as a step in this direction – a concrete way of thinking about the more important problem of the meaning of conformal field theory.

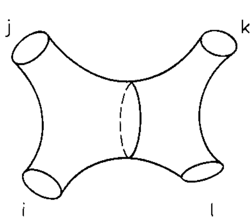
The classification of all conformal field theories is an enormous problem. To make it tractable, physicists have proceeded by solving the problem in stages. The

* On leave of absence from the Department of Physics, Weizmann Institute of Science, Rehovot 76100, Israel

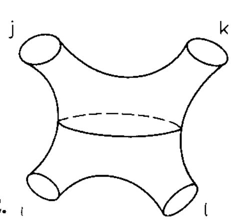
case of $c<1$ has been settled completely. It seems that imposing a finiteness condition, defining what are known as rational conformal field theories, offers the best prospects for further progress [1–19].

Consider a Riemann surface with punctures (more precisely, punctures with coordinates). Every puncture has a label corresponding to a representation space of some chiral algebra (e.g. Virasoro or Kac-Moody). To every such surface we assign a vector space. It is spanned by the different conformal blocks [1]. One definition of a *Rational Conformal Field Theory* (RCFT) is that this vector space has a finite dimension. (A different, but equivalent definition is given in Sect. 2 below.) In the language of FS [5], these blocks are interpreted as the holomorphic sections of a flat vector bundle over the moduli space of the surface. This point of view stresses the modular properties of the blocks. In [11] we stressed a more fundamental concept – that of duality. The Riemann surface can be formed by sewing a number of three holed spheres (a.k.a. trinions). Corresponding to this sewing, the conformal blocks are obtained by summing over the intermediate states passing through the sewn holes. The same Riemann surface can be obtained using different sewing procedures. For instance, the four point function on the sphere can be obtained by sewing as in Fig. 1 or as in Fig. 2. Similarly, the one point function on the torus can be obtained by sewing as in Figs. 3 or as in 4. Different ways of sewing the same surface lead to different blocks. The assumption of duality states that the vector space spanned by these blocks is independent of the way that the surface was sewn. Different sewing procedures lead to different bases of the same vector space. Hence, the conformal blocks obtained by one way of sewing the surface can be expressed as linear combinations of the blocks found by another way of sewing the surface. These linear transformations are the *duality matrices*. In a RCFT these matrices are finite dimensional.

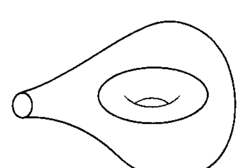
The duality matrices are not arbitrary matrices. They have to satisfy some consistency conditions. These are obtained as follows. We construct a simplicial complex whose sites correspond to the different ways of sewing a given surface. Clearly, this complex has an infinite number of sites. Some simple duality matrices are defined to be "simple moves." Other duality matrices are given by a product of



1.



2.


3.

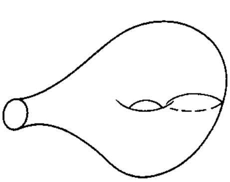

4.

Fig. 1. One sewing of the four-point function on the sphere

Fig. 2. Another sewing of the four-point function on the sphere

Fig. 3. One sewing of the one-point function on the torus

Fig. 4. Another sewing of the one-point function on the torus

the simple moves. We add edges to the complex connecting sites which are related by the simple duality matrices, i.e. by the simple moves. If the resulting complex is connected, every duality matrix can be represented as a product of the simple ones. However, often this can be done in more than one way. In order to have an unambiguous definition of all duality matrices, we have to make sure that all the different ways of defining a given duality matrix lead to the same matrix. This leads to consistency conditions on the matrices of the simple moves. Every closed loop of simple moves leads to such a consistency condition. It is important to find all the independent conditions. A convenient way of organizing the problem is the following. We impose some simple consistency conditions corresponding to some simple loops in the complex. Every such relation is the statement that a product of some duality matrices equals to one. We define these loops as the fundamental loops. Filling the faces of the fundamental loops we obtain a two complex. If the relations of these loops are complete, the resulting two complex is simply connected. In this case, every closed loop can be deformed to a product of fundamental loops. Then, every consistency condition on the duality matrices is satisfied by using the relations of the fundamental loops.

The set of transformations on the simplicial complex is a groupoid. We refer to it as the *duality groupoid*. The simple moves are the generators of the groupoid and the relations of the fundamental loops are its defining relations.

In [11] we argued that these equations for the duality matrices of rational conformal field theories are the defining equations for some kind of moduli space of rational conformal field theories. In this paper we attempt to understand the meaning of these equations better. It is exactly this kind of investigation which we hope will shed light on string theory. We will show that in a particular case, which we call "classical conformal field theory," the meaning of our equations and, therefore, of conformal field theory is well understood. Classical conformal field theory is defined as a conformal field theory where the conformal weights of all primary fields vanish. In this case, conformal field theory is nothing but group theory.

Some recent exciting work of Witten [20] has shown that $2+1$ dimensional generally covariant theories lead to some conformal field theories. The Hilbert space of the $2+1$ dimensional theory is the vector space of the conformal blocks. Some of our results have a simple and natural interpretation in that framework. It would be nice to make the correspondence between these two approaches more complete.

In Sects. 2–5 we review the formalism of references [11, 18]. In Sect. 6 we define classical conformal field theory and examine how it satisfies our equations. Section 7 shows how every compact group (either continuous or discrete) leads to a classical conformal field theory. The relation between the classical version of our formalism and ordinary group theory should be viewed, at the very least, as pedagogical. Also, it suggests that the more general case – that of quantum conformal field theory – corresponds to a generalization of group theory. In Sect. 8 we examine the possibility of defining a conformal field theory by our equations. We would like to know how to reconstruct the entire conformal field theory from a solution of our polynomial equations. This reconstruction problem is completely solved in the classical case. Using some results in category theory, the correspondence between classical conformal field theory and group theory is made more complete. Not only does every group lead to a classical conformal field theory but the converse of this statement is also true – every classical conformal field theory corresponds to a group. More mathematically, our framework and equations define a Tannakian category. Given some assumptions (mentioned in Sect. 8) which are satisfied in a classical conformal field theory, it was shown that such a category is the category of representation spaces of a group. Section 9 is more speculative. It suggests a possible connection between conformal field theory and quantum groups. Several appendices give some more technical details and examples of our formalism.

2. Basic Definitions

In this section we follow [11] and review the definitions of a rational conformal field theory and of chiral vertex operators. We begin with a review of the definition of a conformal field theory. Many definitions have been proposed, the most popular of these being the BPZ [1], FS [5], and Segal [21] definitions.

The BPZ definition of conformal field theory is that it is an inner product space $\mathscr{H}$ which can be decomposed into a direct sum

$$\mathscr{H} = \bigoplus_{h,\bar{h}} V(h,c) \otimes \bar{V}(\bar{h},\bar{c}) \tag{2.1}$$

of irreducible highest weight modules of $Vir_c \times \overline{Vir}_{\bar{c}}$ such that

1. There is a unique $SL_2(R) \times SL_2(R)$ invariant states $|0\rangle$ with $(h,\bar{h})=(0,0)$.
2. For each vector $\alpha \in \mathscr{H}$ there is an operator $\phi_\alpha(z)$ on $\mathscr{H}$, parametrized by $z \in C$. Also, for every operator ϕ_α there exists a conjugate operator $\phi_{\alpha^\vee}$ (partially) characterized by the requirement that the operator product expansion $\phi_\alpha \phi_{\alpha^\vee}$ contains a descendant of the unit operator.
3. For $\alpha = i$ a highest weight state we have $[L_n, \phi_i(z,\bar{z})] = \left(z^{n+1}\frac{d}{dz} + \Delta_i(n+1)z^n\right)\phi_i$.
4. The inner products $\langle 0|\phi_{i_1}(z_{i_1},\bar{z}_{i_1})\dots\phi_{i_n}(z_{i_n},\bar{z}_{i_n})|0\rangle$ exist for $|z_{i_1}| > \dots |z_{i_n}| > 0$ and admit an unambiguous real-analytic continuation, independent of ordering, to C^n minus diagonals. This is called the assumption of duality.
5. The one-loop partition function and correlation functions, computed as traces exist and are modular invariant.

In [5] Friedan and Shenker reformulated these axioms in terms of the geometry of vector bundles over moduli spaces of curves. We assume the reader has a nodding acquaintance with this formulation. An alternative definition has been proposed by Segal (many elements of which have been described by a number of physicists under the rubric of "the operator formulation" of conformal field theory [22–25]) in which a conformal field theory is a functor between categories satisfying certain "sewing axioms."[1] In this paper we use, strictly speaking, the

[1] In Segal's picture one defines a category whose objects are disjoint unions of (parametrized) circles and whose morphisms are Riemann surfaces interpolating between these. A conformal field theory is then a functor from this category to the category of Hilbert spaces whose morphisms are trace class operators. The functor is further required to satisfy certain axioms. The most important of these is the sewing axiom: By sewing together two holes one obtains another morphism in the category. Because we are working with Riemann surfaces the sewing is characterized by a modular parameter q. The conformal field theory functor should take this morphism to the q-trace of the previous morphism

axioms of BPZ, but our constructions are motivated by a mixture of the points of view of Friedan-Shenker and Segal. We discuss the geometry of compatible flat vector bundles over moduli space ("modular geometry") but in terms of duality and the consequence of the sewing axioms.

We now discuss the notion of chiral algebras, or vertex algebras [26]. The fields in a conformal field theory form a closed operator product expansion. An important subset of the fields are the holomorphic fields. Since the operator product expansion of two holomorphic fields is holomorphic, these form a closed subalgebra of the operator product algebra called the "chiral algebra," $\mathscr{A}$, of the theory.[2] Every conformal field theory has at least two holomorphic fields given by the unit operator and the stress tensor: 1, $T(z)$ and thus every chiral algebra contains the (enveloping algebra of the) Virasoro algebra. We can choose a basis $\{\mathcal{O}^i(z)\}$ for $\mathscr{A}$ such that each field has a well-defined dimension Δ_i. By the axiom of duality, fields in a conformal field theory have no relative monodromy, in particular, the weights Δ_i are integers. Defining modings $\mathcal{O}^i(z)=\sum_n \mathcal{O}^i_n z^{-n-\Delta_i}$ we can write the operator product algebra in two equivalent ways:

$$\begin{aligned} \mathcal{O}^i(z)\mathcal{O}^j(w) &= \sum_k \frac{c_{ijk}}{(z-w)^{\Delta_{ijk}}}\mathcal{O}^k(w), \\ [\mathcal{O}^i_n, \mathcal{O}^j_m] &= \sum_k c_{ijk}(n,m)\mathcal{O}^k_{n+m}, \end{aligned} \tag{2.2}$$

($\Delta_{ijk}\equiv\Delta_i+\Delta_j-\Delta_k$). Using the moding one can define Verma modules and irreducible quotients and, therefore, one can speak of the irreducible representations $\mathscr{H}_i$ of $\mathscr{A}$.

Since our discussion is general and somewhat formal, it is useful to keep in mind some simple examples. The simplest case is that where $\mathscr{A}$ is the enveloping algebra of Virasoro. A somewhat more involved example is given by Kac-Moody algebras. These two examples are particularly simple because $\mathscr{A}$ is the enveloping algebra of a Lie algebra. This is not always the case. The super Virasoro algebra has a (super) Lie algebra structure but its enveloping algebra is not a chiral algebra by the above definition. We should project out of it all the half integral weight operators. It is often the case that a complicated chiral algebra $\mathscr{A}$ can be expressed more simply by adding operators of fractional dimensions to it. These operators (together with some of the original operators in $\mathscr{A}$) form a Lie algebra like structure, $\tilde{\mathscr{A}}$ – they form a *finite* set of operators where the singular part of their expansion is closed. Then we form the enveloping algebra of these operators and $\mathscr{A}$ is found as a subalgebra. The super Virasoro case is a particular example of this procedure. Here $\tilde{\mathscr{A}}$ consists of the unit operator, the stress tensor and the supercharge. More complicated examples are given by coset constructions [27, 6] and by parafermion theories [4]. Orbifolds [28] provide us with another example of this phenomenon. In the simplest case of an orbifold of a torus, $\tilde{\mathscr{A}}$ is given by several factors of $U(1)$ Kac-Moody. Although typically none of these $U(1)$ currents

[2] It can happen that a further subset of the holomorphic fields forms a closed subalgebra (common examples are Virasoro or affine Kac-Moody subalgebras) and often people limit themselves to this restricted subset. In this paper the chiral algebra always means the maximally extended chiral algebra of the theory

are in $\mathscr{A}$, some bilinears of the currents are in $\mathscr{A}$. Sometimes several different $\tilde{\mathscr{A}}$'s lead to the same $\mathscr{A}$. Examples of this phenomenon are discussed in [29].

Perhaps the simplest example of a non-trivial algebra $\mathscr{A}$ is the "rational torus." This algebra is generated by ∂X and $e^{\pm i\sqrt{2N}X}$, where N is an integer. It can be understood by the procedure above as a sub-algebra of the enveloping algebra of various Kac-Moody algebras [e.g. of $SU(2N)$ level 1]. Its representations can easily be found. Since it includes $U(1)$ KM as a subalgebra, its representations are also representations of $U(1)$ KM. These are labeled by a real number k – the momentum. The presence of the operator $e^{\pm i\sqrt{2N}X}$ excludes most of the momenta. Only $k=\frac{m}{\sqrt{2N}}$ with m integer is local relative to $e^{\pm i\sqrt{2N}X}$. Furthermore, different values of m which correspond to different KM representations are combined into the same irreducible representations of this chiral algebra. There are only $2N$ irreducible representations: $m=-N+1,\ -N+2, \dots -1, 0, 1, \dots N$. The representation $m=0$ includes the identity operator and the whole chiral algebra. It is the "basic representation."

Rational conformal field theories are those for which the chiral algebra of the theory decomposes all the correlation functions into finite sums of holomorphic times antiholomorphic functions. In more detail, a rational CFT is a CFT such that the physical Hilbert space of the theory, $\mathscr{H}$, is given by a *finite* sum

$$\mathscr{H}=\bigoplus_{r,\bar{r}=0}^{N} h_{r,\bar{r}}\mathscr{H}_r\otimes\bar{\mathscr{H}}_{\bar{r}}, \tag{2.3}$$

where $\mathscr{H}_r$ is an irreducible representation of $\mathscr{A}$ and $h_{r,\bar{r}}$ is an integer counting the number of times $\mathscr{H}_{\bar{r}}\otimes\bar{\mathscr{H}}_{\bar{r}}$ occurs in $\mathscr{H}$. The representation $\mathscr{H}_0$ includes the identity operator and therefore all the operators in $\mathscr{A}$ (mod nulls!). Hence, $h_{r,0}=\delta_{r,0}$, $h_{0,\bar{r}}=\delta_{0,\bar{r}}$. Many definitions of RCFT have been proposed. Some of these are discussed in [7]. Another, based on finite factorization of the correlations and partition functions was proposed in the appendix of [6]. In [18] it was argued that the above definition is equivalent to that proposed in [6].

We now describe chiral vertex operators. In [11] we gave a constructive definition, which was, unfortunately tied to specific properties of known chiral algebras. Here we attempt a more intrinsic definition.

To motivate the definition, let us consider what properties we expect the "holomorphic part" $\Phi_\beta(z)$ of a vertex operator to have. First, its z-dependence should be governed by the Virasoro algebra, thus $\frac{d}{dz}\Phi_\beta(z)=\Phi_{L_{-1}\beta}(z)$ for all states β. Furthermore, contour integrals of operators in the chiral algebra cannot simply deform through Φ. Consider an insertion of Φ_β at a point z. Thus we think of the vacuum propagating out from the origin and hitting a disturbance at z. If we act on the resulting state with $\mathcal{O}_n$ from the chiral algebra we take a contour integral with $\mathcal{O}(z)$ which can be deformed in the standard way to a contour integral about z and about 0 (see Fig. 5). The contour integral about z can be expressed in terms of the modes $\mathcal{O}_n(z)$ of the local representation and Hilbert space at z. Finally, note that Φ_β is a linear operator, but it is also linear in β, and thus Φ should be thought of as a linear map of the tensor product of two representation spaces of the chiral algebra to a third. Therefore, we expect chiral vertex operators to take products of

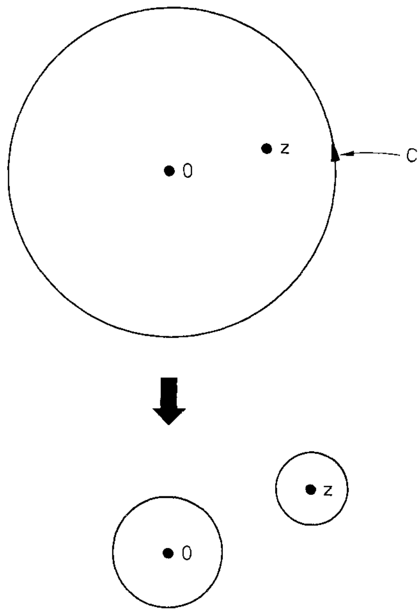


Fig. 5. Deformation of contours used to obtain a "tensor product" of representations

representations to a third representation. Hence, to begin with the general definition, one would like to make sense of $\mathscr{H}_j \otimes \mathscr{H}_k$ as a representation space of $\mathscr{A}$. One cannot take the standard tensor product representation because of the central terms in $\mathscr{A}$. Rather, one uses contour deformation to write a deformed tensor product representation parametrized by $z \neq 0$, ∞ in the complex plane $\Delta_{z,0}: \mathscr{A} \to \mathscr{A} \otimes \mathscr{A}$. To do this define:

$$\begin{aligned}\Delta_{z,0}(\mathcal{O}^i_n) &= \oint_z \zeta^{n+\Delta_i-1}\left(\sum_m (\zeta - z)^{-m-\Delta_i}\mathcal{O}^i_m\right) \otimes 1 + 1 \otimes \mathcal{O}^i_n \\ &= \sum_{k=0}^{\infty} \binom{n+\Delta_i-1}{k} z^{n+\Delta_i-1-k}\mathcal{O}_{1+k-\Delta_i} \otimes 1 + 1 \otimes \mathcal{O}^i_n . \end{aligned} \tag{2.4}$$

In particular, for the Virasoro algebra we have

$$\begin{aligned}\Delta_{z,0}(L_n) &= 1 \otimes L_n + (z^{n+1}L_{-1} + \ldots L_n) \otimes 1\,, \qquad n \geqq -1 \\ &= 1 \otimes L_n + (z^{n+1}L_{-1} + z^n(n+1)L_0 + \ldots) \otimes 1\,, \qquad n < -1\,. \end{aligned} \tag{2.5}$$

We could also define $\Delta_{0,z}$ from the above by permutation of the factors on the right-hand side. If ϱ_j, ϱ_k are two representations of $\mathscr{A}$, $\varrho_j: \mathscr{A} \to \mathrm{End}(\mathscr{H}_j)$ etc., then we can use $\Delta_{z,0}$ to define[3] a representation of $\mathscr{A}$ on $\mathscr{H}_j \underset{z,0}{\otimes} \mathscr{H}_k$. Specifically, $\mathcal{O}_n$ is represented by $\varrho_j \otimes \varrho_k(\Delta_{z,0}(\mathcal{O}_n))$.

We can now define *chiral vertex operators* [30, 31, 11] as intertwining operators (see Sect. 7) for the above representations, with an appropriate

[3] We thank E. Witten for pointing out subtleties in this definition which are related to the fact that representation of $\mathscr{A}$ obtained this way is not always a highest weight representation

dependence on z. That is, given three representations i, j, k a chiral vertex operator of type $\binom{i}{jk}$, can be thought of as a linear transformation

$$\binom{i}{jk}_z : (\mathscr{H}_i)^\vee \otimes \mathscr{H}_j \otimes \mathscr{H}_k \to C\,. \tag{2.6}$$

($\mathscr{H}_i^\vee$ is the dual of $\mathscr{H}_i$). Or, equivalently, as a linear transformation $\mathscr{H}_j \otimes \mathscr{H}_k \to (\mathscr{H}_i^\vee)^\vee$. Since $\mathscr{H}_k$ is infinite dimensional, the double-dual is in fact much larger than the representation, but we will ignore this subtlety in this paper. The chiral vertex operators are defined to be operators satisfying:

$$\begin{aligned} \varrho_i(\mathcal{O}_n)\binom{i}{jk}_z(\beta\otimes\gamma) &= \binom{i}{jk}_z(\varrho_j\otimes\varrho_k(\Delta_z(\mathcal{O}_n))(\beta\otimes\gamma))\,,\\ \frac{d}{dz}\binom{i}{jk}_z(\beta\otimes\gamma) &= \binom{i}{jk}_z(L_{-1}\beta\otimes\gamma)\,. \end{aligned} \tag{2.7}$$

The first of these conditions is the statement that the chiral vertex operator is an intertwining operator. This point will be discuss in detail in Sect. 7. Simply stated, the transformation laws of the state $\binom{i}{jk}_z(\beta\otimes\gamma)$ are determined by deforming the contour to the state γ at the origin and the operator β at z. The connection to the previous description of chiral vertex operators is given by

$$\Phi_{\beta,\binom{i}{jk}}(z)\,|\gamma\rangle = \binom{i}{jk}_z(\beta\otimes\gamma)\,. \tag{2.8}$$

We define V_{jk}^i to be the vector space of chiral vertex operators of type $\binom{i}{jk}$, and define the fusion rules to be $N_{jk}^i = \dim V_{jk}^i$. Picking a basis for V_{jk}^i, we denote the basis vectors by $\binom{i}{jk}_{z,a}$, where $a=1,\ldots,N_{jk}^i$. Notice that from (2.7) we see that for a Virasoro primary we have $[L_n, \Phi_\beta(z)] = \left(z^{n+1}\frac{d}{dz} + (n+1)z^n\Delta_\beta\right)\Phi_\beta(z)$. We assume that N_{jk}^i is always finite. This assumption is trivially satisfied if every $\mathscr{H}_i$ contains a finite number of Virasoro or KM representations.

In the operator formalism chiral vertex operators are the conformal blocks associated to the three-holed sphere. In particular, the first equation of (2.7), which we have presented as an intertwining property is known in the operator formalism as the Ward identity defining the state (or vector space of states) created by the three-holed sphere.

The chiral vertex operators can be given a more constructive definition as follows. Consider first the case where $\mathscr{A}$ is the Virasoro algebra. We can compute $\langle\alpha|\,\Phi_\beta(z)\,|\gamma\rangle$ for any states α, γ, and β primary by the following standard construction. The properties (2.7) allow us to commute creation operators $L_{-n}(n>0)$ to the left and annihilation operators $L_n(n>0)$ to the right. In this way the matrix element reduces to a differential operator acting on

$$\langle\alpha|\,\Phi_\beta(z)\,|\gamma\rangle \equiv \|\Phi_{\beta\gamma}^\alpha\|\, z^{-(\Delta_\beta+\Delta_\gamma-\Delta_\alpha)} \equiv \|\Phi_{\beta\gamma}^\alpha\|\, z^{-\Delta_t}\,, \tag{2.9}$$

where all the states $\alpha\beta\gamma$ are primary. If $\tilde{\beta}=L_{-n}\beta$, then

$$\langle\alpha|\,\Phi_{\tilde{\beta}}(z)\,|\gamma\rangle \equiv \oint_z (\zeta-z)^{-n+1}\langle\alpha|\,T(\zeta)\Phi_{\beta}(z)\,|\gamma\rangle\,, \tag{2.10}$$

and the contour integral can be deformed onto contours surrounding 0 and ∞. In this way the entire chiral vertex operator is determined from the single number $\|\Phi^{\alpha}_{\beta\gamma}\|$. Strictly speaking we have only defined the operators on Verma modules so far. Requiring that they pass to the irreducible quotient forces some $\|\Phi^{\alpha}_{\beta\gamma}\|$ to be zero, thus defining the fusion rules in the more usual way [1].

Suppose now the algebra $\mathscr{A}$ is an extension of the Virasoro algebra. If β is primary we must know how to compute the singular terms corresponding to the states $\mathcal{O}^i_{-\Delta_i+1}\beta, \ldots, \mathcal{O}^i_0\beta$ in the operator product expansion of $\mathcal{O}(\zeta)\in\mathscr{A}$ with $\Phi_\beta(z)$. These are partly determined by the chiral algebra, leaving behind a finite-dimensional space of chiral vertex operators. For example, in the case of Kac-Moody algebras we can reduce the computation of the matrix element to the case where β is a lowest L_0-weight state so the chiral vertex operators (before passing to the irreducible quotient) are determined by group invariant tensors $t:(\mathscr{H}^0_i)^{\vee}\otimes\mathscr{H}^0_j\otimes\mathscr{H}^0_k\to\mathscr{C}$.

Again, the operators Φ_β when β is not an L_0 lowest weight state are defined by contour integration. However, it is not in general true that the computation of a chiral vertex operator can always be reduced to its action on states with minimal L_0 [17]. Rather, it can happen that Φ vanishes on such states but is a nontrivial operator.

We give here three examples of chiral vertex operators

1. The rational torus. Above we discussed the representations of the chiral algebra of the rational torus. The algebra is labeled by the integer N and the representations are labeled by an integer m between $-N+1$ and N. The fusion rules are $N^{m_3}_{m_1m_2}=1$ when $m_1+m_2=m_3 \bmod 2N$ and zero otherwise. These fusion rules will be discussed in detail in Appendix E.

2. D_n modular invariants of $SU(2)$ current algebra of level $k=4n$. This chiral algebra has been discussed in [16–18]. The chiral algebra $\mathscr{A}$ is obtained by adding to the algebra of the $SU(2)$ currents the holomorphic $SU(2)$ primary fields of spin $\frac{k}{2}$. Labeling $SU(2)$ representations by half-integer spin $0\leqq j\leqq\frac{k}{2}$ the representations of $\mathscr{A}$ are given by $\mathscr{H}_j\oplus\mathscr{H}_{\frac{k}{2}-j}$, for $0\leqq j<n$ together with two different representations $\mathscr{H}_n$ and $\mathscr{H}_{n'}$. Thus, the chiral vertex operators of the new theory can be made out of chiral vertex operators of the $SU(2)$ theory provided the relative normalizations of the operators are correct. For example, consider a spin j with $2j\geqq n$ and a spin $l<n$. There will be 6 $SU(2)$ couplings of type:

$$\begin{pmatrix} & j & \\ j & & l\end{pmatrix},\quad \begin{pmatrix} & \frac{k}{2}-j & \\ j & & l\end{pmatrix},\quad \begin{pmatrix} & j & \\ j & & \frac{k}{2}-l\end{pmatrix},$$

$$\begin{pmatrix} & \frac{k}{2}-j & \\ \frac{k}{2}-j & & l\end{pmatrix},\quad \begin{pmatrix} & \frac{k}{2}-j & \\ j & & \frac{k}{2}-l\end{pmatrix},\quad \begin{pmatrix} & \frac{k}{2}-j & \\ \frac{k}{2}-j & & \frac{k}{2}-l\end{pmatrix}.$$

We should express these in terms of couplings of representations of the large algebra. Viewed this way, all these couplings are of type $\begin{pmatrix} j \\ jl \end{pmatrix}$. New Ward identities associated with the new chiral fields of spin $\frac{k}{2}$ and weight $\frac{k}{4}$ cut down the six-dimensional space spanned by the above vertex operators down to a two-dimensional space, determined by, say, the normalization of the first two vertex operators in the above list. Hence, $N^j_{jl}=2$. The first of these does not vanish on the states of lowest L_0 but the second does. Both lead to nonvanishing chiral vertex operators.

3. As a final example consider Zamolodchikov's W-algebra which is realized in the exceptional modular invariant of the $m=5$ ($c=4/5$) term in the discrete series. The algebra is obtained by adding to the stress tensor the field $\phi=\phi_{p,q}=\phi_{4,1}$ of weight $\Delta=3$. The Virasoro representations $[\phi_{(3,1)}]$ and $[\phi_{(3,5)}]$ are now linked by ϕ to form a single representation $\mathscr{H}_\psi$. The chiral vertex operators of type $\begin{pmatrix} \psi \\ \psi\ \psi \end{pmatrix}$ are formed from the Virasoro chiral vertex operators of type $\begin{pmatrix} (3,5) \\ (3,1)\ (3,1) \end{pmatrix}$ and of type $\begin{pmatrix} (3,1) \\ (3,1)\ (3,1) \end{pmatrix}$. In this case the Ward identity with ϕ shows that there is only a one-dimensional space of chiral vertex operators. Similarly, all spaces of chiral vertex operators for the W-algebra are finite dimensional.

In Examples 2 and 3 every representation of the chiral algebra contains a finite number of Virasoro or KM representations. Hence, it is obvious that the fusion rules are finite. In example 1 every representation of the chiral algebra contains an infinite number of $U(1)$ KM representations. Nevertheless, the fusion rules are finite.

As in the physical conformal field theory there is an isomorphism between states and chiral vertex operators obtained by $|\alpha\rangle=\Phi_{\alpha,k}(0)|0\rangle$, where k stands for the natural choice of coupling

$$\begin{pmatrix} k \\ k0 \end{pmatrix}:(\mathscr{H}_k)^\vee\otimes\mathscr{H}_k\to C \tag{2.11}$$

given by the evaluation map and 0 stands for the identity operator (we have ignored the last factor $\mathscr{H}_0$ in the chiral vertex operator, taking instead the canonical vector 1). Furthermore, by (BPZ's) axiom 2, for every conformal field there is a conjugate field so that the operator product with the conjugate field contains the unit operator. More mathematically, $(\mathscr{H}_i)^\vee\cong\mathscr{H}_j$ for an appropriate representation j which we denote by $j=i^\vee$. Selfdual representations have $i^\vee=i$. Since there is no natural choice of basis for the one-dimensional space of couplings

$$\begin{pmatrix} 0 \\ k^\vee k \end{pmatrix}:\mathscr{H}_{k^\vee}\otimes\mathscr{H}_k\to C\,,$$

we choose a metric $\mathscr{K}$. For L_0 eigenstates we have

$$\langle 0|\,\Phi_{\beta_k,k'}\,|\alpha_k\rangle=\mathscr{K}(\beta_{k^\vee}\otimes\alpha_k)z^{-(\Delta_{\alpha_k}+\Delta_{\beta_k})}$$

$\left[k' \text{ is the unique coupling of type } \begin{pmatrix} 0 \\ k^\vee k \end{pmatrix}\right]$ which enables us to relate "bras" to operators by choosing a basis α_k^I for each $\mathcal{H}_k$, defining $\mathcal{K}(\alpha_{k'}^I \otimes \alpha_k^J) \equiv \mathcal{K}_{IJ}$ and setting

$$\langle \alpha_k^I| = \lim_{z\to\infty} \sum_J z^{\Delta_{\alpha_k}+\Delta_{\alpha_k}} \mathcal{K}_{IJ}^{-1} \langle 0 \rangle \Phi_{\alpha_k, k'}(z). \tag{2.12}$$

This metric arises when the states have non-trivial quantum numbers as in the case of KM. More importantly, it also arises in the coupling of descendants. The proof that the metric involves only states with the same weight relies on the L_1 Ward identity. Hence, it applies only to primary fields of $SL(2,C)$. Descendants of $SL(2,C)$ obtained by the action of L_{-1} on the primary have complicated transformation laws under $z \to -\frac{1}{z}$ and have non-zero two-point functions with other states which are not degenerate with them. This can be seen explicitly by differentiating a two-point function of $SL(2,C)$ primaries. Clearly, the metric $\mathcal{K}$ is not always symmetric.

The metric we thus obtain[4] allows us to define several permutation operations on three point couplings. Define $\sigma_{23}: \mathcal{H}_a \otimes \mathcal{H}_b \otimes \mathcal{H}_c \to \mathcal{H}_a \otimes \mathcal{H}_c \otimes \mathcal{H}_b$ by $\sigma_{23}(a\otimes b\otimes c) = a\otimes c\otimes b$, $\sigma_{123}(a\otimes b\otimes c) = c\otimes a\otimes b$ so that $\sigma_{123} = \sigma_{13}\sigma_{12}$. For a chiral vertex operator t denote $\sigma(t) = t\circ\sigma^{-1}$. Thus we have

$$\sigma_{23}: V_{jk}^i \cong V_{kj}^i, \qquad \sigma_{13}: V_{jk}^i \cong V_{jk^\vee}^{i^\vee}. \tag{2.13}$$

Notice that $\sigma^2 = 1$, so when σ maps a space to itself the eigenvalues of σ are always plus or minus one. It is convenient to denote $t' = \sigma_{123}(t)$.

Physical vertex operators are obtained by combining left and right chiral vertex operators for left and right chiral algebras $\mathcal{A}$, $\bar{\mathcal{A}}$ and are obtained by introducing "operator product coefficients." These will be discussed in Sect. 5.

3. Duality and Conformal Blocks: Basic Data

In this section we introduce the basic data defining the certain equivalence classes of conformal field theories, defined by equivalent Friedan-Shenker vector bundles. These data are certain duality matrices for conformal blocks [11].

One of the main reasons for introducing chiral vertex operators is that their correlation functions are conformal blocks for the physical correlation functions. Chiral vertex operators $\Phi(z)$ and $\Phi(w)$ may be multiplied to form $\Phi(z)\Phi(w)$ which makes sense for $|z| > |w|$, and the composition can be analytically continued outside that region once a choice of cuts has been made. We pick the cut in the w plane to start at z and to run parallel to the positive real axis. We must also choose a cut in the z, w planes from the origin since matrix elements of $\Phi(z)$ involve fractional powers of z. We choose this cut to run along the negative real axis. Thus, if we denote by R_- the nonpositive real numbers, then $\Phi(z_1)\Phi(z_2)$ is defined for z_1, z_2, $z_{12} \notin R_-$. Physical correlation functions decompose into sums over analytic times anti-analytic functions – known as conformal blocks – which may be computed as

$$\mathcal{F}(z) = \langle i, \alpha| \begin{pmatrix} i \\ j_1 p_1 \end{pmatrix}_{z_1, a_1} (\beta_1 \otimes \cdot) \begin{pmatrix} p_1 \\ j_2 p_2 \end{pmatrix}_{z_2, a_2} (\beta_2 \otimes \cdot) \dots \begin{pmatrix} p_n \\ j_n k \end{pmatrix}_{z_n, a_n} (\beta_n \otimes \cdot) |k, \gamma\rangle. \tag{3.1}$$

[4] As we mentioned above we ignore the fact that $(\mathcal{H}_k)^\vee$ is bigger than $\mathcal{H}_k$

We define the axiom of duality, provisionally,[5] to be the property of Axiom 4 in the BPZ definition of conformal field theory. Geometrically, the composition of chiral vertex operators corresponds to the sewing of three-holed spheres. We must, therefore, expect that we could also form conformal blocks composing chiral vertex operators of type:

$$\begin{pmatrix} i_1 \\ p'k \end{pmatrix}_{z_2,c} \begin{pmatrix} p' \\ j_1 j_2 \end{pmatrix}_{z_{12},d}.$$

This is indeed the case, and will play a role in the statement of the full axiom of duality.

To derive some consequences of duality we remind the reader of an elementary mathematical fact.[6] If $\sum_{i=1}^{N} f_i \bar{g}_i = \sum_{i=1}^{M} h_i \bar{k}_i$ and each set of the analytic functions $\{f\}$, $\{g\}$, $\{h\}$, and $\{k\}$ is separately linearly independent, then $N=M$, $f_i = \sum_j A_{ij} h_j$ and $g_i = \sum_j A_{ij}^{-1*} k_j$. The statement is easily proved by completing each of the collections f, g, h, k to a full basis of analytic functions $\{f_i, F_I\}$ etc. We may certainly express

$$f_i = A_{ij} h_j + B_{iJ} H_J, \qquad \bar{g}_i = C_{iJ} \bar{K}_J + D_{ij} \bar{k}_j \tag{3.2}$$

from which we learn that $A^{tr} D = 1_{M \times M}$ while $A^{tr} C = B^{tr} C = B^{tr} D = 0$. From the first equation we learn that the rank of A, D is equal to M, and hence $N \geqq M$. We could have expanded the other way, so $N = M$. Thus, A, D are invertible, from which it follows that $B = C = 0$.

We may apply the axiom of duality (that the correlation function is independent of the order of ϕ) together with the above remark to obtain our first piece of data: the braiding matrix. Either way, we see that conformal blocks computed with

$$\begin{pmatrix} i \\ j_1 p \end{pmatrix}_{z_1,a} (\beta_1 \otimes \cdot) \begin{pmatrix} p \\ j_2 k \end{pmatrix}_{z_2,b} (\beta_2 \otimes \cdot)$$

must be linear combinations of blocks computed with

$$\begin{pmatrix} i \\ j_2 p' \end{pmatrix}_{z_2,c} (\beta_2 \otimes \cdot) \begin{pmatrix} p' \\ j_1 k \end{pmatrix}_{z_1,d} (\beta_1 \otimes \cdot).$$

By using contour integration and the intertwining property we can transform a highest weight state into any descendant state, hence the linear transformation is independent of β_1, β_2. Choosing a basis for the space of chiral vertex operators which we label by $a, b, \ldots$, we therefore have:

$$\begin{pmatrix} i \\ j_1 p \end{pmatrix}_{z_1,a} \begin{pmatrix} p \\ j_2 k \end{pmatrix}_{z_2,b} = \sum_{p';c,d} B_{pp'} \begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix}_{ab}^{cd} (\varepsilon) \begin{pmatrix} i_1 \\ j_2 p' \end{pmatrix}_{z_2,c} \begin{pmatrix} p' \\ j_1 k \end{pmatrix}_{z_1,d}. \tag{3.3}$$

---

[5] At the end of this section we give the full axiom of duality

[6] This fact, which is well-known, was worked out in this context with C. Vafa and T. Banks

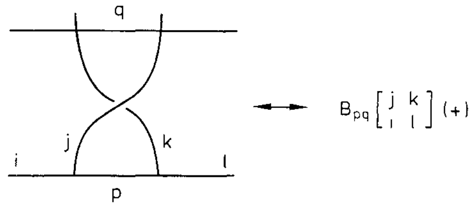


Fig. 6. A geometric picture of braiding

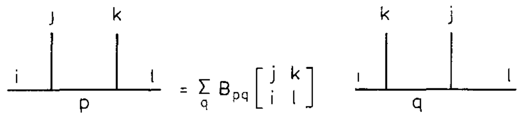


Fig. 7. Braiding matrix between blocks

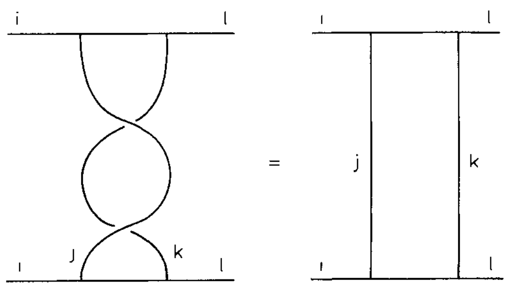


Fig. 8. Pictorial proof of a simple identity

The above equation holds for $z_1-z_2\equiv z_{12}\in H_\varepsilon$, where $H_{\varepsilon=\pm}$ is the upper/lower half plane. As we said above, we have chosen cuts so that the composition of chiral vertex operators on the left-hand side of the equation makes sense as long as $z_{12}\notin R_-$, while for the right-hand side to be well defined we must have $z_{21}\notin R_-$, in other words, $z_{12}\notin R_+$. In each connected region of the common domain of definition the transformation B is independent of z. Since there are two regions there are two transformations $B(\pm)$. Each is an invertible linear transformation

$$B\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} : \bigoplus_p V^i_{j_1 p}\otimes V^p_{j_2 k} \to \bigoplus_q V^i_{j_2 q}\otimes V^q_{j_1 k}\,. \tag{3.4}$$

It is very useful to work with the pictorial notation for the B-matrices shown in Figs. 6, 7. From Fig. 8 it is, e.g. obvious that $B(\varepsilon)B(-\varepsilon)=1$.

It is crucial for our approach that the transformations B depend only on the couplings which are involved (and therefore on the representations which are coupled) and not on the particular states in the representation. Let us examine a simple consequence of this fact. Consider the two point function

$$\left\langle \begin{pmatrix} 0 \\ i' i \end{pmatrix}_z (\beta^I_{i'}\otimes\cdot) \begin{pmatrix} i \\ i0 \end{pmatrix}_w (\alpha^J_i\otimes\cdot) \right\rangle = \langle \Phi^I_{\beta_{i'},i'}(z)\Phi^J_{\alpha_i,i}(w)\rangle = \frac{\mathscr{K}_{IJ}}{(z-w)^{\Delta_{\alpha^J_i}+\Delta_{\beta^I_{i'}}}}\,. \tag{3.5}$$

Since the couplings with the identity operators are unique, there is no need to perform the sum in Eq. (3.3) and we find

$$\langle\Phi^{I}_{\beta_{i^\vee},i'}(z)\Phi^{J}_{\alpha_i,i}(w)\rangle=\frac{\mathcal{K}_{IJ}}{(z-w)^{\Delta_{\alpha_i^J}+\Delta_{\beta_{i^\vee}^I}}}=B_{ii'}\begin{bmatrix} i^\vee & i\\ 0 & 0\end{bmatrix}\langle\Phi_{\alpha_i^J,i'}(w)\Phi_{\beta_{i^\vee}^I,i}(z)\rangle$$
$$=B_{ii'}\begin{bmatrix} i^\vee & i\\ 0 & 0\end{bmatrix}\frac{\mathcal{K}_{JI}}{(w-z)^{\Delta_{\alpha_i^J}+\Delta_{\beta_{i^\vee}^I}}} \tag{3.6}$$

[where we use $B(+)$ or $B(-)$ depending on the sign of $\mathrm{Im}(z-w)$]. Therefore

$$B_{ii'}\begin{bmatrix} i^\vee & i\\ 0 & 0\end{bmatrix}(\pm)=\frac{\mathcal{K}_{IJ}}{\mathcal{K}_{JI}}e^{\mp i\pi(\Delta_{\alpha_i^J}+\Delta_{\beta_{i^\vee}^I})}. \tag{3.7}$$

The left-hand side of this equation is independent of I and J. As I and J are varied, the exponential factor on the right-hand side of this equation can change at most by a sign. Therefore, the sign of $\frac{\mathcal{K}_{IJ}}{\mathcal{K}_{JI}}$ changes accordingly (notice that $\mathcal{K}_{IJ}$ can be antisymmetric in I and J). This result can be checked explicitly. As we said above, we should only examine the situation when I (and J) change within a representation of $SL(2,C)$. In this case, the change in I is easily computed by differentiating the two-point function. This trivially leads to the necessary change in sign of $\frac{\mathcal{K}_{IJ}}{\mathcal{K}_{JI}}$.

From the three-point function we deduce, for t of type $\binom{i}{jk}$:

$$B(\varepsilon)(t\otimes k)=(\Omega(-\varepsilon)t)\otimes j,$$
$$B(\varepsilon)(i'\otimes t)=(j^\vee)'\otimes(\Theta(-\varepsilon)\sigma_{123}(t)). \tag{3.8}$$

[Remember that k is the unique coupling of type $\binom{k}{k0}$ and k' is the unique coupling of type $\binom{0}{kk^\vee}$.] Where we introduce $\Omega(\pm)=\Omega^i_{jk}(\pm): V^i_{jk}\cong V^i_{kj}$ and $\Theta(\pm): V^i_{jk}\cong V^{k^\vee}_{ji^\vee}$ defined by $\Omega(\pm)(t)=e^{\pm i\pi\Delta_t}\sigma_{23}(t)$ and $\Theta(\pm)(t)=\sigma_{13}(e^{\pm i\pi\Delta_t}t)$. Note that $\Omega(+)\Omega(-)=1$, but the analogous statement is not true for Θ. We see that Ω and Θ are special cases of B – they are the B's of the simple couplings associated with the unit operator.

Our second piece of data is the fusing matrix, which can also be deduced from the assumption of duality, from sewing, or from the operator product expansion. In terms of sewing of three-holed spheres we are comparing the two sewing procedures of Figs. 1, 2. We give the derivation of the fusing matrix using the operator product expansion. Consider the operator product expansion:

$$\Phi_{\alpha,a}(z_1)\Phi_{\beta,b}(z_2)=\sum_k\sum_{c\in V^l_{kr};\,d\in V^k_{ij}}F_{pk}\begin{bmatrix} i & j\\ l & r\end{bmatrix}^{cd}_{ab}$$
$$\times\sum_{K\in\mathcal{H}_k}\xi^{\alpha,\beta}_{p,K,d}\begin{bmatrix} i & j\\ l & r\end{bmatrix}(z_1,z_2,z_3)\Phi_{K,c}(z_3) \tag{3.9}$$

($a \in V^l_{ip}$, $b \in V^p_{jr}$, $c \in V^l_{kr}$, $d \in V^k_{ij}$ label the couplings). We define F and hence normalize ξ by requiring that the leading nonvanishing ξ be one. This expansion is an asymptotic expansion which is believed to have a finite radius of convergence. It is valid for $z_1 \sim z_2 \sim z_3$. By translation and scaling invariance we can rewrite the right-hand side as

$$\sum_{kcd} F_{pk}\begin{bmatrix} i & j \\ l & r \end{bmatrix}^{cd}_{ab} \sum_{K \in \mathscr{H}_k} \frac{\xi^{\alpha,\beta}_{p,K,d}\begin{bmatrix} i & j \\ l & r \end{bmatrix}(z_{13}/z_{23}, 1)}{z_{23}^{\Delta_\alpha + \Delta_\beta - \Delta_K}} \Phi_{K,c}(z_3). \tag{3.10}$$

We can evaluate the coefficients ξ as follows. Consider first the case $r=0$, $p=j$, $l=k$. Taking the inner product between $\langle K|$ and $|0\rangle$, then taking $z_3 \to 0$ and using the isomorphism of states and vertex operators we get

$$\xi^{\alpha,\beta}_{j,K,d}\begin{bmatrix} i & j \\ k & 0 \end{bmatrix}(z, 1) = \langle K|\, \Phi_{\alpha,d}(z)\Phi_{\beta,j}(1)\, |0\rangle . \tag{3.11}$$

To obtain the general expression we consider the product

$$\Phi_{\alpha,a}(z_1)\Phi_{\beta,b}(z_2)\Phi_{\gamma,r}(z_4) . \tag{3.12}$$

We can compare the definition (3.9) with the result of braiding first β, γ then α, γ, then applying (3.11) and then braiding once more to obtain the general expression

$$\xi^{\alpha,\beta}_{p,K,d}\begin{bmatrix} i & j \\ l & r \end{bmatrix}(z_1, z_2, z_3) = \langle K|\, \Phi_{\alpha,d}(z_{13})\Phi_{\beta,j}(z_{23})\, |0\rangle . \tag{3.13}$$

The expansion (3.9) holds for all α, β so we may rewrite it as a statement about intertwiners. Taking $z_3 \to z_2$ we find the relation:

$$\begin{pmatrix} i_1 \\ j_1 p \end{pmatrix}_{z_1,a} \begin{pmatrix} p \\ j_2 k \end{pmatrix}_{z_2,b} = \sum_{p';c,d} F_{pp'}\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix}^{cd}_{ab} \begin{pmatrix} i_1 \\ p' k \end{pmatrix}_{z_2,c} \begin{pmatrix} p' \\ j_1 j_2 \end{pmatrix}_{z_{12},d}, \tag{3.14}$$

which is the statement of st-duality. As for the B's it is important that F depends only on representations, and is independent of z_1 and z_2. The left-hand side of (3.14) is defined for $z_1, z_2, z_{12} \notin R_-$. On the right-hand side we have $z_{12}, z_2 \notin R_-$, and since $z_1 \sim z_2$ we see that the domain of definition of both sides is the same. Thus there is only one fusing matrix.

Similar to the case with B, we can regard F as a linear transformation

$$F\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} : \bigoplus_r V^i_{j_1 r} \otimes V^r_{j_2 k} \to \bigoplus_s V^i_{sk} \otimes V^s_{j_1 j_2} . \tag{3.15}$$

If we specify the initial and final term in the direct sums in (3.4) and (3.15), then we denote by

$$\begin{aligned} B_{pq}\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} &: V^i_{j_1 p} \otimes V^p_{j_2 k} \to V^i_{j_2 q} \otimes V^q_{j_1 k} , \\ F_{rs}\begin{bmatrix} j_1 & j_2 \\ i & k \end{bmatrix} &: V^i_{j_1 r} \otimes V^r_{j_2 k} \to V^i_{sk} \otimes V^s_{j_1 j_2} , \end{aligned} \tag{3.16}$$

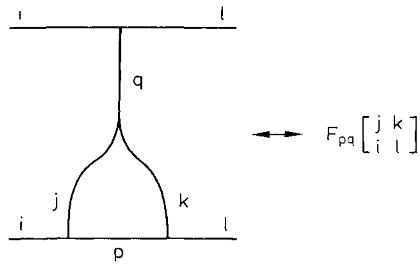


Fig. 9. A geometric picture of fusing

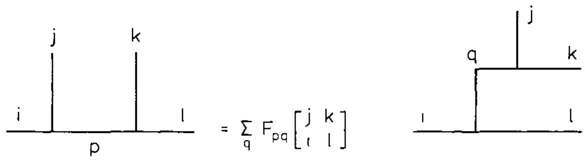


Fig. 10. Fusing matrix between blocks

the corresponding linear transformations. Pictorially, fusion F can be represented as in Fig. 9 or as in Fig. 10. Using this representation, it is easy to keep track of the many indices which are involved.

The third piece of data is obtained from the conformal blocks for the one-point functions on the torus. These blocks may be expressed in terms of the modular parameter τ together with the chiral vertex operators of type

$$\begin{pmatrix} i \\ ji \end{pmatrix}_z : (\mathscr{H}_i)^\vee \otimes \mathscr{H}_j \otimes \mathscr{H}_i \to C \tag{3.17}$$

by taking a trace on the first and third spaces:

$$\begin{aligned} \chi_i^j(q,z) &\equiv \mathrm{Tr}_i \left[q^{L_0 - c/24} \begin{pmatrix} i \\ ji \end{pmatrix}_z \right] (dz)^{\Delta_j} \\ &= \mathrm{Tr}_i \left[q^{L_0 - c/24} \begin{pmatrix} i \\ ji \end{pmatrix}_1 \right] \left(\frac{dz}{z} \right)^{\Delta_j} : \mathscr{H}_j \to C \,. \end{aligned} \tag{3.18}$$

These are the conformal blocks for one-point functions on a torus obtained by dividing the complex plane by $z \sim qz$. If we put $j=0$ and evaluate the characters on the unit operator, we obtain the familiar vacuum characters $\chi_i(q)$.

Two remarks are in order regarding the definition of χ:

1. If $\beta_1, \beta_2 \in \mathscr{H}_j$ are both primaries for the Virasoro algebra, then they will be related by contour integrals with Virasoro-primary fields in the chiral algebra:

$$\begin{pmatrix} i \\ ji \end{pmatrix}_z (\beta_2 \otimes \cdot) = \sum_a \prod \oint_z (\zeta - z)^{m_a} \mathcal{O}^a(\zeta) \begin{pmatrix} i \\ ji \end{pmatrix}_z (\beta_1 \otimes \cdot) \,.$$

These contour integrals may be deformed off of z, thus expressing $\chi_i^j(q,z)(\beta_1)$ in terms of $\chi_i^j(q,z)(\beta_2)$ times a (q,z)-independent coefficient (which might vanish or be infinite). A related problem is that the character $\chi_i^j(\beta)$ might vanish for primary fields β. For example, in $SU(2)$ current algebra ($k>1$) the one-point function of the primary field in the spin one representation must vanish, even when the fusion rule

is not zero. However, if the *operator* $\chi_i^j(q,z):\mathscr{H}_j\to C$ is not zero, then there must be some field β for which $\chi_i^j(q,z)(\beta)$ is not zero. Furthermore, since matrix elements of Virasoro descendants are related to Virasoro primaries by differential operators, we can find such a β which is a Virasoro primary. This fact will be used below. In $SU(2)$ example the character does not vanish for the state $|\beta\rangle=\sum_a j_{-1}^a|a\rangle$, where $|a\rangle$ are the three states at the lowest L_0 grading of the spin one representation.
2. It sometimeshappens that different couplings or representations lead to the same vacuum character. For example in $SU(3)$ current algebra the vacuum characters for the 3 and 3* are the same. More generally, a representation r and its conjugate $r^\vee$ will have the same vacuum character. This follows from CPT. However, we will assume that for two different couplings $r_1 \neq r_2$ the operator $\chi_{r_1}^j(q,z):\mathscr{H}_j\to C$ is different from the operator $\chi_{r_2}^j(q,z):\mathscr{H}_j\to C$, although they might have the same value on, say, a primary state in $\mathscr{H}_j$. Returning to the $SU(3)$ example, although the one point function of the identity operator in 3 and 3* are the same, there are $SU(3)$ descendants of the identity which are Virasoro primaries $\left(\text{e.g. } \sum_{abc} d_{abc}:j^aj^bj^c:\right)$ for which the one point functions are different.

The net result of the above remarks is that the vector space of conformal blocks for the one-point function on the torus of representation j may be identified with $\bigoplus_i V_{ji}^i$. In general, if we evaluate χ for a conformal field β then we may apply the axiom of duality or, equivalently, the sewing axiom for the two sewing procedures in Figs. 3, 4 to obtain a linear transformation

$$S(j):\bigoplus_i V_{ji}^i\to\bigoplus_i V_{ji}^i. \tag{3.19}$$

Choosing a basis $\binom{i}{ji}_{z,a}$ we thus have, for every Virasoro primary in $\mathscr{H}_j$, for $\tau'=-1/\tau$, $\log z'=\log z/\tau$,

$$\chi_{i,a}^j(q';z')=\sum_r\sum_{b\in V_{jr}^r} S(j)_{ab}\chi_{r,b}^j(q,z). \tag{3.20}$$

The assumption that for two different couplings $r_1\neq r_2$ the operators $\chi_{r_1}^j(q,z)$ and $\chi_{r_2}^j(q,z)$ are different guarantees that $S(j)$ is defined unambiguously.

As we explained above, when the fusion rule is not zero there exists a Virasoro primary $\beta\in\mathscr{H}_j$ (not necessarily in the lowest L_0 grading) for which χ is non-zero. It is important that β be a Virasoro primary since under the modular transformation $\log z'=\log z/\tau$, an external state β which is a Virasoro descendant mixes with other descendants in $\mathscr{H}_j$. However, Virasoro primaries transform like tensors and do not mix with other fields. The differential form in (3.18) is necessary for χ to transform linearly as in (3.20). Since $\left(\frac{dz'}{z'}\right)^\Delta=\frac{1}{\tau^\Delta}\left(\frac{dz}{z}\right)^\Delta$, without this differential form χ transforms like a modular form.

The final piece of data is given by T, which we will consider as a scalar transformation $V_{ji}^i\to V_{ji}^i$ acting as multiplication by $e^{2\pi i(\Delta_i-c/24)}$.

The above data is a complete set of data for describing the moduli space of Friedan-Shenker vector bundles. The reason is that we may form duality transformations for all other conformal blocks in the theory out of the above

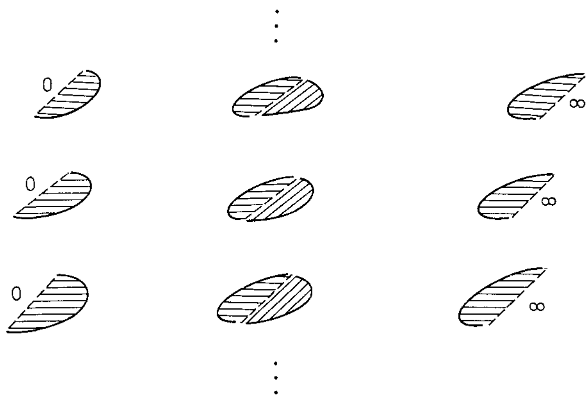


Fig. 11. Disjoint regions of Teichmuller space defined by the four-point function with fixed operators at zero, one, and infinity

transformations. This follows from the construction of conformal blocks by sewing, or, operatorially, by composing chiral vertex operators. (The traces used to form loops can be considered as composing a chiral vertex operator with itself.) Different bases of blocks can always be related by the transformations F, Ω. The FS bundles are described by representations of the modular group (if the surface has punctures we must use the modular group of the surface with holes [32]) and the representation of any Dehn twist can be computed by considering the appropriate asymptotic region of moduli space and using the appropriate basis of conformal blocks. More precisely, consider a ϕ^3 diagram $\mathscr{D}$. We can thicken $\mathscr{D}$ to obtain a partition of the n-holed surface into pants together with a Fenchel-Nielsen coordinate system (l_i, θ_i), where θ is normalized to lie between 0 and 2π. For small enough l_i, and θ_i restricted to intervals of length π, the different diagrams define disjoint regions in moduli space which may be lifted to Teichmuller space T, thus providing disjoint regions in T which we will denote by v. An example of such regions is shown in Fig. 11. To each region v associate a basis of sections $\mathscr{F}_v(\tau) \equiv \mathscr{F}_v(z_1, \dots, z_n)$ by associating an operator Φ to each trinion (a three holed sphere), composing operators according to the partition of the surface. The sections $\mathscr{F}_v$ may be analytically continued from the region v to all of T.

We are now ready to state the full axiom of duality: *The physical correlation functions obtained by combining the left movers and the right movers are independent of the sewing procedure.* Hence, for any pair v, v' there is a duality matrix $A(v, v')$ with $\mathscr{F}_v = A(v, v')\mathscr{F}_{v'}$ throughout T [in this expression the matrix $A(v, v')$ multiplies the vector $\mathscr{F}_{v'}$ and there is no summation over v']. The duality matrices we have discussed are particular examples, and, by suitable decomposition of the Riemann surface one sees that all other duality matrices can be expressed in terms of these. Further consequences of duality are described in Sect. 5.

4. Duality Identities

We would like to find the minimal number of equations on our basic data which guarantee that the axiom of duality is satisfied. We do this as follows. We consider a complex $\mathscr{C}$ defined by taking a vertex for each of the regions $v \in T$ described

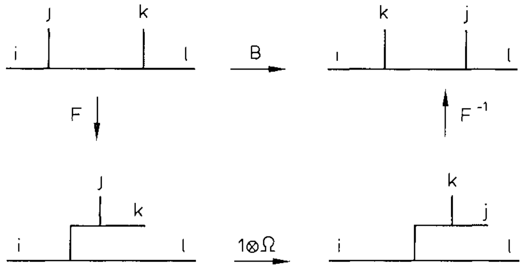


Fig. 12. A simple loop of transformations for the four-point function

above, that is, vertices are pairs $(\mathscr{D},\gamma)$ with $\mathscr{D}$ an ordered ϕ^3-diagram (see Appendix B) and $\gamma\in\Gamma$, the modular group. We define the edges of the complex by joining verticles related by "simple moves" associated to F, Ω, S, T (for more details see Appendix B). The edge-path groupoid of $\mathscr{C}$ – the set of transformations generated by these simple moves – defines the *duality groupoid D*. Notice that Γ acts on $\mathscr{C}$ so Γ is a subgroup of D. In this section we will describe the defining relations of D.

In Appendix A we describe in more detail one way of sewing trinions to obtain high genus characters in the "multiperipheral basis."

We begin by writing explicitly the conformal blocks for the three-point function and imposing duality. In this way we may evaluate B, F explicitly and obtain:

$$\begin{aligned} F(t\otimes k)&=i\otimes t\,,\\ F(i'\otimes t)&=k'\otimes\sigma_{123}t\,,\\ B(\varepsilon)(t\otimes k)&=(\Omega(-\varepsilon)t)\otimes j\,,\\ B(\varepsilon)(i'\otimes t)&=(j^{\vee})'\otimes(\Theta(-\varepsilon)\sigma_{123}(t))\,. \end{aligned}\tag{4.1}$$

Next, we consider the four and five point functions and derive some consistency conditions on the above data by performing "closed loops" of transformations on the complex $\mathscr{C}$. These may be regarded as consistency relations for the operator algebra of the Φ's, as consistency conditions for sewing of amplitudes, or as relations in the duality groupoid. Beginning with the four-point function we see that we can accomplish the braiding of two chiral vertex operators in two different ways illustrated in Fig. 12. These must give the same answer, so we have:[7]

$$B(\varepsilon)=F^{-1}(1\otimes\Omega(-\varepsilon))F\,. \tag{4.2}$$

[7] Note that this allows us to deduce the eigenvalues of B as half-monodromies. For fields which have only two fields on the right-hand side of the fusion rules, like the N representation of $SU(N)$ level k current algebra, B is a 2×2 matrix. In this example its eigenvalues are

$$\lambda_1=e^{-i\pi(2\Delta_N-\Delta_{N(N+1)/2})}=q^{-\frac{N-1}{N}}$$

and

$$\lambda_2=-e^{-i\pi(2\Delta_N-\Delta_{N(N-1)/2})}=-q^{\frac{N+1}{N}}\quad\text{for}\quad q=e^{\frac{i\pi}{N+k}}.$$

The minus sign in λ_2 arises from the anti-symmetry of the coupling. This is an example where the eigenvalue of the permutation σ discussed after Eq. (2.13) is -1. It follows that the B matrices obey a Hecke-algebra: $(B-\lambda_1)(B-\lambda_2)=0$

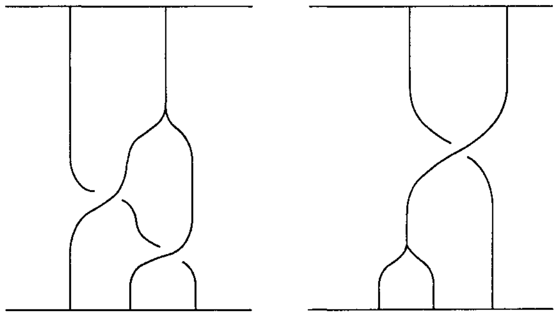


Fig. 13. A geometric version of the braiding/fusing identity

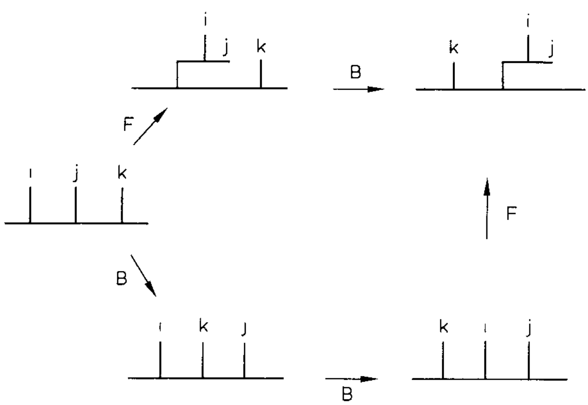


Fig. 14. Braiding/fusing in terms of moves on the complex

We see that B can be expressed in terms of F and Ω and therefore it should not be viewed as independent data. Similarly, Θ is a particular case of B.

Now consider the five point function. We begin with the fundamental observation, discussed above, that the braiding matrices for descendant fields are the same as for the primaries. Therefore, if in the product of three fields we first take an operator product expansion and then braid the resultant field, we should obtain the same answer by first braiding two fields and then taking the operator product expansion. Combining Fig. 6 and Fig. 9 this remark is illustrated in Fig. 13. Alternatively, we can use Fig. 7 and Fig. 10 to obtain Fig. 14. In equations we have the braiding/fusing identity:

$$P_{23}B_{13}(\varepsilon)F_{12}=F_{23}B_{12}(\varepsilon)B_{23}(\varepsilon). \tag{4.3}$$

When there are three or more vector spaces we denote, $B(u\otimes v\otimes w)=B(u\otimes v)\otimes w$, etc. by $B_{12}: V^{\otimes 3}\rightarrow V^{\otimes 3}$ etc. $P_{\mu\nu}$ is the permutation operator on vector spaces μ, ν. Evaluating (4.3) on the coupling $t_1\otimes t_2\otimes k$ (Fig. 15) we find that B and F are related by diagonal matrices of phases:

$$B(\varepsilon)=(\Omega(-\varepsilon)\otimes 1)F(1\otimes\Omega(\varepsilon)). \tag{4.4}$$

This equation expresses the fact that there is essentially only one duality matrix. The "su-duality" matrix B differs from the "st-duality" matrix F only by a phase.

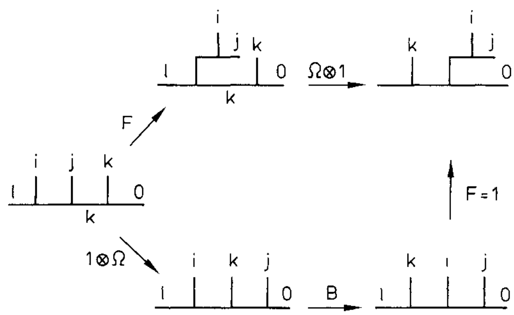


Fig. 15. A special case of the braiding/fusing identity

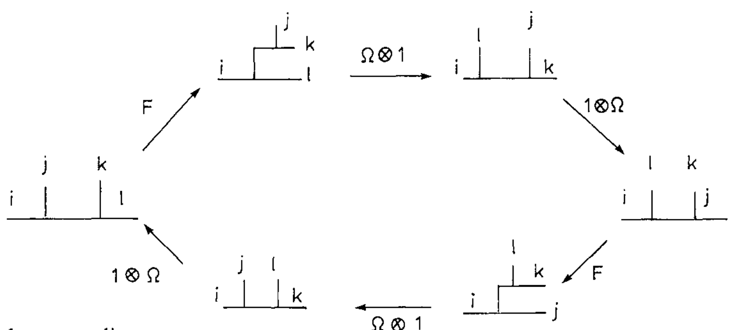


Fig. 16. A hexagon diagram

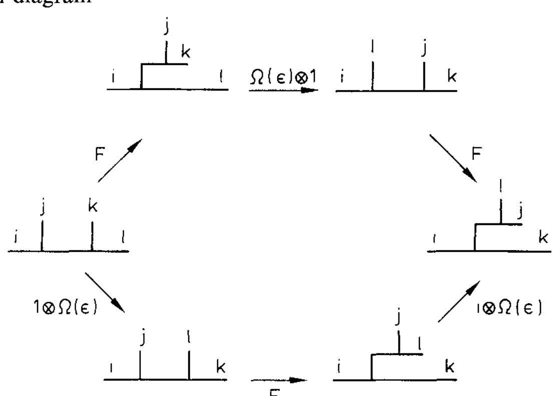


Fig. 17. Two other hexagon diagrams

From this relation we obtain the three fundamental $g=0$ relations on Ω and F. First, from the evident relation $B(\varepsilon)B(-\varepsilon)=1$ we obtain[8]

$$F(\Omega(\varepsilon)\otimes\Omega(\varepsilon))F(\Omega(-\varepsilon)\otimes\Omega(-\varepsilon))=F(\sigma_{23}\otimes\sigma_{23})F(\sigma_{23}\otimes\sigma_{23})=1\,, \qquad (4.5)$$

which is illustrated in Fig. 16. Substituting (4.4) back into (4.2) we obtain

$$F(\Omega(\varepsilon)\otimes 1)F=(1\otimes\Omega(\varepsilon))F(1\otimes\Omega(\varepsilon))\,, \qquad (4.6)$$

illustrated in Fig. 17. The physical interpretation of this equation is similar to the interpretation of the braiding/fusing identity (4.3): Fused intermediate lines are transformed by the same transformation Ω as the external legs. The external states

[8] Equivalently, we can use (4.4) to find $B(\varepsilon)(\Omega(\varepsilon)^2\otimes 1)B(\varepsilon)(1\otimes\Omega(-\varepsilon)^2)=1$. This relation was noticed in [30] and was used in [30] and [10]

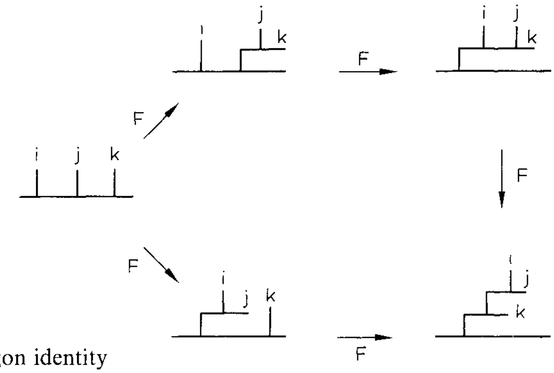


Fig. 18. The pentagon identity

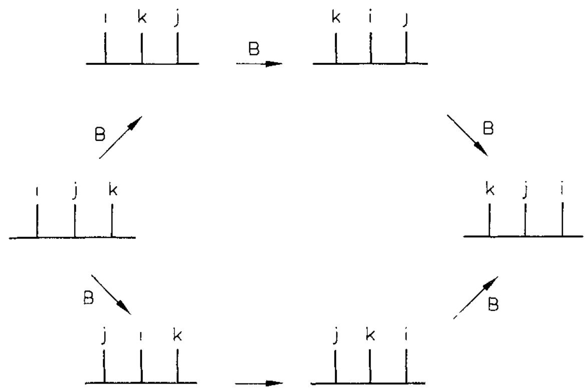


Fig.19. The loop giving the Yang-Baxter identity

and the intermediate states in all possible channels are of the same nature. In fact, (4.5) and (4.6) are not independent equations. Considering (4.6) as two equations for $\varepsilon = \pm$ we have three hexagon configurations. Any one of these can be "tiled" by the other two. We will take (4.6) as the fundamental equations. Finally, using (4.4) in (4.3) we find the pentagon identity:

$$P_{23}F_{13}F_{12} = F_{23}F_{12}F_{23}, \tag{4.7}$$

which can be deduced diagrammatically as in Fig. 18.

Many authors have noticed that braid groups and the Yang-Baxter equation play a role in conformal field theory. Certainly, from Fig. 19 or, alternatively, from considering the product of three chiral vertex operators, one can deduce that the braiding matrices satisfy the Yang-Baxter equation:

$$B_{12}(\varepsilon)B_{23}(\varepsilon)B_{12}(\varepsilon) = B_{23}(\varepsilon)B_{12}(\varepsilon)B_{23}(\varepsilon). \tag{4.8}$$

As with (4.5) this is in fact not a new equation, but already follows from Eqs. (4.6) and (4.7). To see this bring all the factors of B to one side, rewrite in terms of F, Ω and use the above equations to reduce the number of factors of F. Note that evaluating (4.8) on $t_1 \otimes t_2 \otimes k \left[k \text{ is the unique coupling of type } \begin{pmatrix} k \\ k0 \end{pmatrix} \right]$ yields (4.6).

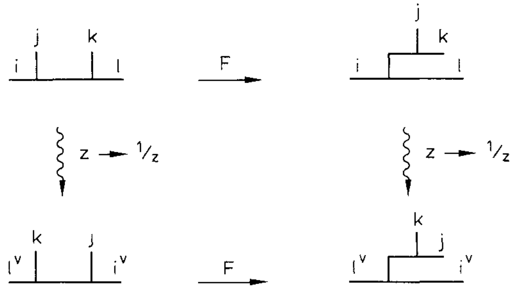


Fig. 20. A symmetry of F related to the Mobius transformation $z \to 1/z$

As a consequence of the above equations the transformations F satisfy several useful symmetry properties:

$$\begin{aligned} F_{\mathrm{pr}}\begin{bmatrix} j & k \\ i & l \end{bmatrix} &= \sigma_{13} \otimes \sigma_{23} F_{\mathrm{p'r}}\begin{bmatrix} k & j \\ l^\vee & i^\vee \end{bmatrix} \sigma_{13} \otimes \sigma_{13} P, \\ F_{\mathrm{pr}}\begin{bmatrix} j & k \\ i & l \end{bmatrix} &= \sigma_{12} \otimes \sigma_{12} P F_{\mathrm{pr'}}\begin{bmatrix} i^\vee & l \\ j^\vee & k \end{bmatrix} \sigma_{12} \otimes \sigma_{23}, \\ F_{\mathrm{pr}}\begin{bmatrix} j & k \\ i & l \end{bmatrix} &= \sigma_{123} \otimes \sigma_{132} P F_{\mathrm{p'r'}}\begin{bmatrix} l & i^\vee \\ k^\vee & j \end{bmatrix} P \sigma_{123} \otimes \sigma_{132}. \end{aligned} \tag{4.9}$$

These can easily be understood in pictures. For example, see Fig. 20. These identities reflect the Mobius invariance on the plane. In the four point function this invariance is fixed by putting three of the states at 0, 1 and ∞ and the fourth at z. The freedom to put different states at these points is equivalent to (4.9).

In Appendix B we show that there are no further independent identities on F and Ω arising from considerations at genus zero.

There are three further equations involving the data S and T. The first two may be deduced from the requirement that $\chi_i^j(q, z)$ represent the modular group of the one-holed torus when evaluated on conformal states β. If we iterate the transformation S we obtain the transformation $\tau \to \tau$, $\log z \to -\log z$, which is a $180°$ rotation of the local parameter $\log z$ at the insertion of the state β, or, equivalently $z \to 1/z$ on the complex plane. We may compute the representation of this transformation on the characters as follows. By the assumption of duality it suffices to calculate the behavior of a particular matrix element. Choose a basis $\{\gamma_a\}$ for the space of Virasoro-primary fields of lowest weight which contribute to the trace. From conformal invariance we have, for $\beta \in \mathscr{H}_j$ a Virasoro-primary and $w = 1/z$:

$$\left\langle \gamma_a \middle| \begin{pmatrix} i \\ ji \end{pmatrix}_w (\beta \otimes \gamma_a) \right\rangle (dw)^{\Delta_\beta} = \left\langle \gamma_a^\vee \middle| \Theta_{ji}^i(-) \begin{pmatrix} i \\ ji \end{pmatrix}_z (\beta \otimes \gamma_a^\vee) \right\rangle (dw)^{\Delta_\beta}. \tag{4.10}$$

Thus we have

$$(S(j))^2 = \bigoplus_i \Theta_{ji}^i(-). \tag{4.11}$$

For $j = 0$ this reduces to $S^2 = C$, where C is the charge conjugation matrix. The transformation $T: \tau \to \tau + 1$, $\log z \to \log z$ is a diagonal transformation on the characters $T: \otimes V_{ji}^i \to \otimes V_{ji}^i$ with value $e^{2\pi i(\Delta_i - c/24)}$ on V_{ji}^i. [The only new information

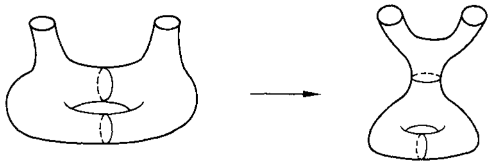


Fig. 21. Two sewing procedures giving two-point functions on the torus

in T is the central extension c, since $\Omega^{-1}(\Theta(-))^2\Omega: V^i_{ji}\to V^i_{ji}$ is multiplication by $e^{-2\pi i\Delta_i}$.] The other relation of the modular group then forces $(ST)^3=S^2$, that is:

$$S(j)TS(j)=T^{-1}S(j)T^{-1}. \tag{4.12}$$

There are no other relations from the modular group or the duality groupoid for the one-point function. See Appendix B.

The final equation is obtained from the two-point function on the torus. The conformal blocks for the two-point function of $\beta_1\in\mathscr{H}_{j_1}$, $\beta_2\in\mathscr{H}_{j_2}$ are given by

$$\mathrm{Tr}_i\left[q^{L_0-c/24}\begin{pmatrix} i \\ j_1 p\end{pmatrix}_{z_1}(\beta_1\otimes\cdot)\begin{pmatrix} p \\ j_2 i\end{pmatrix}_{z_2}(\beta_2\otimes\cdot)\right](dz_1)^{\Delta_{\beta_1}}(dz_2)^{\Delta_{\beta_2}}. \tag{4.13}$$

As before we can regard the two-point function as an operator $\mathscr{H}_{j_1}\otimes\mathscr{H}_{j_2}\to C$ and as before, the space of conformal blocks can, therefore, be thought of as $\bigoplus_{i,p} V^i_{j_1p}\otimes V^p_{j_2i}$. The transformation of the blocks under $\tau\to-1/\tau$ $\log z_i\to\log z_i/\tau$ may be obtained in terms of (3.19) by first fusing the two vertex operators to obtain a one-point function. (In terms of sewing we have Fig. 21.) Thus we may rewrite (4.13) for basis couplings a, b as

$$\sum_{p',c,d} F_{pp'}\begin{bmatrix} j_1 & j_2 \\ i & i\end{bmatrix}^{cd}_{ab}\chi^{p'}_{i,c}(q,z_2)\begin{pmatrix} p' \\ j_1 j_2\end{pmatrix}_{z_{12},d}(\beta_1\otimes\beta_2). \tag{4.14}$$

In the above equation we compose the operators $\chi^{p'}_i$ and $\begin{pmatrix} p' \\ j_1 j_2\end{pmatrix}$ by summing over the intermediate descendants. The composition makes sense for z_{12} small. Under a modular transformation $z_{12}\to z_{12}+\mathcal{O}(z_{12}^2/\tau)$, so z_{12} remains small if it began small, so, again by the assumption of duality we see that S is represented by

$$\begin{array}{ccc} \bigoplus_{i,p} V^i_{j_1p}\otimes V^p_{j_2i} & \xrightarrow{S(j_1,j_2)} & \bigoplus_{i,p} V^i_{j_1p}\otimes V^p_{j_2i} \\ \downarrow F & & \uparrow F^{-1} \\ \bigoplus_{i,p} V^i_{pi}\otimes V^p_{j_1j_2} & \xrightarrow{\bigoplus_p S(p)\otimes 1} & \bigoplus_{i,p} V^i_{pi}\otimes V^p_{j_1j_2} \end{array} \tag{4.15}$$

denoting $\bigoplus_p S(p)\otimes 1$ by $S\otimes 1$ we may write simply, $S(j_1,j_2)=F^{-1}(S\otimes 1)F$. In addition to T there are two other generators of the modular group namely $a: z_2\to e^{-2\pi i}z_2$, which on the block of Fig. 22 is simply the diagonal matrix with

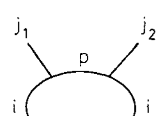


Fig. 22. A basis of characters for the two-point function on the torus

$e^{2\pi i(\Delta_i - \Delta_p)}$, and in terms of our basic transformations can be written as

$$1\otimes\Theta(-)\Theta(+)=\Theta(+)\Theta(-)\otimes 1:\bigoplus_{i,p} V^i_{j_1,p}\otimes V^p_{j_2,i}.$$

Finally b, corresponding to $z\to q^{-1}z$ is simply $PB(+)$. The transformations a and b and S satisfy the relation in the modular group $SaS^{-1}=b$. Expressing S of the two point function in terms of S of the one point function and $B(+)$ in terms of F and Ω, this equation becomes

$$(S\otimes 1)F(1\otimes\Theta(-)\Theta(+))F^{-1}(S^{-1}\otimes 1)=FPF^{-1}(1\otimes\Omega(-)). \tag{4.16}$$

The set of equations we have discussed so far is complete. Obviously, one can find infinitely many equations which are satisfied by the duality matrices. One of the main results of [11] is that all these equations can be derived from the equations above. The proof of this statement, which we call the completeness theorem is given in Appendix B. The above equations are, therefore, the defining relations of the duality groupoid. As we have seen, the modular group is a subgroup of the duality groupoid. Therefore, the generators of the modular group can be expressed in terms of the generators of the groupoid. The generators of the modular group are subject to an infinite number of defining relations (new relations arise whenever the genus of the surface or the number of punctures or holes is increased). Since the modular group is embedded in the duality groupoid, all the defining relations of the modular group are automatically satisfied. Therefore, by the completeness theorem the moduli space of Friedan-Shenker vector bundles for RCFT's can be characterized by the following data and conditions:

Data:

1. A finite index set I and a one to one map of I to itself written $i\mapsto i^\vee$.
2. Vector spaces: $V^i_{jk} i, j, k\in I$, with $\dim V^i_{jk}=N^i_{jk}<\infty$.
3. Isomorphisms:

$$\Theta^i_{jk}(\pm): V^i_{jk}\cong V^{k^\vee}_{ji^\vee},$$

$$\Omega^i_{jk}(\pm): V^i_{jk}\cong V^i_{kj},$$

$$F\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix}: \bigoplus_r V^{i_1}_{j_1 r}\otimes V^r_{j_2 k_2}\cong \bigoplus_s V^{i_1}_{s k_2}\otimes V^s_{j_1 j_2}, \tag{4.17}$$

$$S(j): \bigoplus_i V^i_{ji}\cong \bigoplus_i V^i_{ji},$$

$$T: \bigoplus_i V^i_{ji}\cong \bigoplus_i V^i_{ji}.$$

Conditions:

1. $(i^\vee)^\vee=i$.
2. $V^i_{0j}\cong\delta_{ij}C$, $V^0_{ij}\cong\delta_{ij^\vee}C$, $V^i_{jk}\cong V^{k^\vee}_{ji^\vee}$, $(V^i_{jk})^\vee\cong V^{i^\vee}_{j^\vee k^\vee}$.
3. $\Omega^2(+)=\Omega^i_{jk}(+)\Omega^i_{kj}(+)$ is multiplication by a phase. Similarly, the action of T on V^i_{ji} is a diagonal matrix of phases independent of the external index j.
4. The identities:

$$F(\Omega(\varepsilon)\otimes 1)F=(1\otimes\Omega(\varepsilon))F(1\otimes\Omega(\varepsilon)), \tag{4.18a}$$

$$F_{23}F_{12}F_{23}=P_{23}F_{13}F_{12}, \tag{4.18b}$$

$$S^2(j)=\bigoplus_i \Theta^i_{ji}(-), \tag{4.18c}$$

$$S(j)TS(j)=T^{-1}S(j)T^{-1}, \tag{4.18d}$$

$$(S\otimes 1)(F(1\otimes\Theta(-)\Theta(+))F^{-1})(S^{-1}\otimes 1)=FPF^{-1}(1\otimes\Omega(-)). \tag{4.18e}$$

These equations are equivalent to the equations in [11]. A few remarks are in order. From Ω one can recover the data $e^{2\pi i\Delta_j}$, and, as a consequence of (4.18a) one finds that the eigenvalues of Ω are square roots of mutual locality factors. [Remember that $\Omega(\pm)(t)=e^{\pm i\pi\Delta_t}\sigma_{23}(t)$ and $\Theta(\pm)(t)=\sigma_{13}(e^{\pm i\pi\Delta_t}t)$.] One can then obtain Θ from F, by (4.1). Finally, one defines T in terms of $e^{2\pi i\Delta_i}$ and the new data $e^{-2\pi ic/24}$ as multiplication by $e^{2\pi i(\Delta_i-c/24)}$. Furthermore, notice that the isomorphisms Ω and Θ imply that $N_{jk}^{i}=N_{jki^\vee}$ have to be totally symmetric in j, k, and $i^\vee$. The isomorphism F implies that the fusion rules N_{jk}^{i} form a commutative and associative algebra [9]. In practice, when one solves these equations, it is easier to replace one of the equations in (4.18a) which is cubic in F by Eq. (4.5) which is quadratic in F. In (4.18e) recall that $S\otimes 1$ means $\bigoplus_p S(p)\otimes 1$. As explained in [11, 18], (4.18e) and (4.18b) together imply that the modular transformation law of vacuum characters diagonalizes the fusion rules, as was conjectured by Verlinde [9]. One may use this observation to express S in terms of F, Ω [11, 18].

5. Physical Considerations

In the previous sections we studied the geometrical constraints on the conformal blocks. We made sure that they transform properly under the duality groupoid and therefore also under the modular group. In this section we will discuss some physical constraints on the blocks and we will use them to construct a consistent conformal field theory.

We can view our formalism from two different points of view. Each corresponds to different ways of finding the conformal blocks. We can start with some chiral algebra and its chiral vertex operators and use them to compute the conformal blocks. Alternatively, we can start with some fusion rules (without specifying the chiral algebra), set up Eqs. (4.18) and solve for the duality matrices. Then we can look for sections of the FS vector bundle transforming under the duality groupoid as the representation we found. It is not obvious that every solution leads to a conformal field theory. That is, the above equations characterize the bundles occurring in conformal field theory, and therefore, the sections should satisfy further physical requirements. Such sections might or might not exist. We must demand, for example that the asymptotic behavior of the sections near the boundaries of moduli space has to be consistent with factorization – the order of the poles in different channels should be consistent. This is determined by the *integral* parts of the weights Δ_i and by the integer part of $c/8$. Another condition (which we will discuss later) is that there exists a Hilbert space interpretation. In particular, it is not clear why the equations guarantee that the coefficients of the q expansion of the characters on the torus are integral.

Another physical requirement is CPT invariance in unitary theories. In Lagrangian conformal field theories, it follows from the hermiticity of the Lagrangian in Minkowski space (reflection positivity). As explained in [18], this leads to several consequences. In particular, to some reality constraints on the conformal blocks. Labeling the conformal block by the external legs $i_1, \ldots, i_n$ and the kind of couplings $s_1, \ldots, s_m$ (on the plane $n=m$ and s_1 and s_n are simple because they involve the identity operator) we find

$$\mathscr{F}_{s_1,\ldots,s_n}(\tau)=\mathscr{F}_{s_1^*,\ldots,s_n^*}(\tau^*)^*, \tag{5.1}$$

where we have used the following notation. If s is a coupling of type $\binom{i}{jk}$, s^* is a coupling of type $\binom{i^\vee}{j^\vee k^\vee}$ obtained by an antilinear isomorphism between V_{jk}^i and $V_{j^\vee k^\vee}^{i^\vee}$. (To define it we must make a choice of basis in $\mathscr{H}_i$ and then define the complex conjugate intertwiner having complex conjugated matrix elements.) In (5.1) τ stands for all the moduli including the locations of the punctures. Because of the two complex conjugations in (5.1), we effectively treat τ as a real parameter – CPT does not change τ to τ^*. Considering (5.1) for the four point function and using the fusion transformation we find

$$F_{pq}\begin{bmatrix} i & j \\ k & l \end{bmatrix}_{ab}^{cd} = \left(F_{p^\vee q^\vee}\begin{bmatrix} i^\vee & j^\vee \\ k^\vee & l^\vee \end{bmatrix}_{a^*b^*}^{c^*d^*} \right)^*. \tag{5.2}$$

Similarly, by considering the implication of CPT for the one point function on the torus, we learn that

$$(S(j)_a^b)^* = (S(j^\vee)^{-1})_{a^*}^{b^*}, \tag{5.3}$$

where a and b label a basis of $\oplus_i V_{ji}^i$ and are related by conjugation to a^* and b^* which label a basis of $\oplus_{i^\vee} V_{j^\vee i^\vee}^{i^\vee}$.

Another consequence of CPT [which follows from (5.1)] is that $\mathscr{A}$ is a real algebra. That is, there exists a basis for its operators $\mathscr{O}^i$ in which they are self conjugate and all the operator product coefficients c_{ijk} in (2.2) are real. Therefore, for every representation r of $\mathscr{A}$, there exists a conjugate representation $r^\vee$. Using this observation, it is easy to derive Eqs. (5.2) and (5.3) directly.

The considerations above ensure that the conformal blocks satisfy all the necessary geometrical and some of the physical constraints to be consistent sections of the FS vector bundle. However, a conformal field theory is not just a set of conformal blocks. The important objects in a conformal field theory are the *physical correlation functions.* These are the correlation functions of the *physical vertex operators* which should be distinguished from the chiral vertex operators. The conformal blocks contain all the information about the symmetries of the theory. Therefore, using the Wigner-Eckart theorem every physical correlation function f which is a function of the moduli τ and τ^* can be written as $f(\tau,\tau^*) = \sum_m d^m \mathscr{F}_m(\tau)\bar{\mathscr{F}}_m(\tau^*)$, where $\mathscr{F}_m(\tau)$ are the conformal blocks. They depend only on the quantum numbers of the external states. The d^m's are "reduced matrix elements." Notice that since the symmetries of the theory control the dependence on the moduli, the d^m's are constants independent of the moduli.

In typical quantum mechanics problems, the reduced matrix elements cannot be determined by symmetry considerations. They contain the non-trivial information about the dynamics of the system. This is not the case in a conformal field theory. First, the requirement of physical factorization of the correlation functions determines all d^m in terms of the d^m's of all three-point functions. Furthermore, as we said above, Eqs. (4.18) are the defining relations of the duality groupoid. They guarantee that the conformal blocks $\mathscr{F}_m(\tau)$ transform covariantly under this groupoid. We should also impose the condition that the physical correlation functions f are *invariant* under the duality groupoid – i.e. they are local (modular invariant) and dual. Clearly, it is enough to check invariance under the generators of the groupoid. This leads to some conditions on the "operator product coefficients" which were discussed in detail in [18]. The transformations of the generators of the groupoid can be realized at the four-point function on the sphere and the one-point function on the torus. Hence, it is enough to check the consistency of these correlation functions. This leads to the consistency of all other correlation functions, e.g. higher n-point functions on the plane and on the torus or correlation functions at high genus [11, 18]. A similar result (from a somewhat different point of view) has been established independently in [33].

In a RCFT the constraints of invariance under the generators of the groupoid are powerful enough to determine all the d^m's. Since we have discussed these extensively elsewhere [18], here we will simply summarize the main results of that paper for the sake of completeness.

In [18] (see also [17, 34]) it was shown that although $\mathscr{A}$ and $\bar{\mathscr{A}}$ need not be the same, the requirements of duality force the fusion rule algebras of $\mathscr{A}$ and $\bar{\mathscr{A}}$ to be the same.

Let us recall that the physical Hilbert space $\mathscr{H}_{\text{phys}}$ is given as a direct sum over representations of $\mathscr{A}\otimes\bar{\mathscr{A}}$. Hence, every state in $\mathscr{H}_{\text{phys}}$ transforms as $(r,\bar{r})$ for some representations of the two chiral algebras. We take $\mathscr{A}\otimes\bar{\mathscr{A}}$ to be the maximal chiral algebra in the spectrum – it includes *all* the holomorphic and the antiholomorphic fields in $\mathscr{H}_{\text{phys}}$. It was shown in [18] that in a rational conformal field theory every representation r and every representation $\bar{r}$ occurs exactly once. Therefore, the Hilbert space decomposes as

$$\mathscr{H}_{\text{phys}} = \bigoplus_r \mathscr{H}_r \otimes \bar{\mathscr{H}}_{\bar{r}=\sigma(r)} . \tag{5.4}$$

Moreover, the pairing of representations in (5.4) $\bar{r}=\sigma(r)$ must define an automorphism $r\to\bar{r}$ of the fusion rule algebra [17, 18]. Denoting by $V=\oplus V^i_{jk}$ the vector space of chiral vertex operators for $\mathscr{A}$ and by $\bar{V}=\oplus \bar{V}^i_{jk}$ the space of chiral vertex operators for $\bar{\mathscr{A}}$, we define, for each triplet i, j, k of representations of $\mathscr{A}$ an "operator product coefficient" to be a bilinear form

$$d^i_{jk}: V^\vee \times \bar{V}^\vee \to C . \tag{5.5}$$

d^i_{jk} can only be non-zero on $V^i_{jk}\times \bar{V}^{\sigma(i)}_{\sigma(j)\sigma(k)}$. Denoting by $\{t_a\}$ a basis for V, the *physical vertex operator* for the state $|n\rangle\otimes|\bar{n}\rangle\in\mathscr{H}_j\otimes\bar{\mathscr{H}}_{\sigma(j)}$ is[9]

$$\phi_{j,n,\bar{n}} = \sum_{i,k}\sum_{t_a,\bar{t}_a} d^i_{jk}(t^\vee_a,\bar{t}^\vee_a)\begin{pmatrix} i \\ jk\end{pmatrix}_{z,a}(n\otimes\cdot)\overline{\begin{pmatrix} i \\ jk\end{pmatrix}_{\bar{z},\bar{a}}(\bar{n}\otimes\cdot)}, \tag{5.6}$$

[9] Without the results that every r occurs exactly once and that the pairing of the left moving and the right moving representations is so simple, a more complicated notation is necessary [18]. However, after this result has been established, this notation suffices

where $\overline{\begin{pmatrix} i \\ jk \end{pmatrix}_{\bar{z},\bar{a}}(\bar{n}\otimes\cdot)}$ is an intertwiner for the right movers and where the sum on t_a, $\bar{t}_a$ runs over a basis for $V^i_{jk}\otimes \bar{V}^{\sigma(i)}_{\sigma(j)\sigma(k)}$ and $\{t^\vee_a\}$ is a basis dual to $\{t_a\}$. It can be shown [18] from the requirements of duality that $d^i_{jk}=0$ if and only if the fusion rules $N^i_{jk}=0$. We refer to this result as the naturality theorem.

Physical correlation functions can be calculated using Eq. (5.6) for the physical vetex operators. The equations for invariance under the duality groupoid

$$
\begin{aligned}
e^{2\pi i \Delta_i} &= e^{2\pi i \Delta_{\sigma(i)}},\\
d^i_{jk} &= d^i_{kj}(\Omega(+)\times\bar{\Omega}(-))^\vee,\\
d^i_{jk} &= d^{i^\vee}_{j^\vee k}(\Theta(+)\times\bar{\Theta}(-))^\vee,\\
\bigoplus_p d^p_{jl}\otimes d^i_{pk} &= \bigoplus_p d^i_{jp}\otimes d^p_{lk}(F\times\bar{F})^\vee,\\
\bigoplus_i d^i_{ji} &= \bigoplus_i d^i_{ji}(S(j)\times\bar{S}^{-1}(j))^\vee,
\end{aligned}
\tag{5.7}
$$

(where $\bar{\Omega}$, $\bar{\Theta}$, $\bar{F}$, and $\bar{S}$ act on the $\bar{V}$'s) then guarantee that the physical correlation functions are dual and modular invariant on any genus.

Even if the duality matrices satisfy all our equations and the sections have all the desired physical properties, it is still not obvious that the overdetermined equations (5.7) have a consistent solution. If there is no solution for the d's, a consistent conformal field theory cannot be constructed. However, in the simple case where $N^i_{jk}<2$ and all the fields are self-conjugate it can be shown [18] that the diagonal solution ($\sigma(r)=r$) satisfies the equations on the plane and the $S(0)$ equation on the torus.

It is often the case that the diagonal theory ($\sigma(r)=r^\vee$) exists and the operator product coefficients satisfy $d(t^\vee_a, \bar{t}^\vee_a)=\lambda_a\delta_{a,a^*}$ and all λ_a are real and positive. In this case, it is possible to pick the gauge (pick bases for the V's) $\lambda_a=1$. In this gauge all the duality matrices are unitary.

6. The Classical Limit

We would like to have a better understanding and a more elegant characterization of the mathematical object described by the axioms (4.17), (4.18). A good answer to this question should deepen our understanding of conformal field theory, and, we hope, string theory.

A natural way to approach this problem is to study the equations in a simplifying limiting case. The Yang-Baxter is well known to have an interesting limit in which $B=P(1+\varepsilon r)$ for small ε where r satisfies what is known as the "classical Yang-Baxter equation." Unfortunately, the pentagon identity has no such limit: Although $F=1$ and $F=P$ solve the pentagon, substitution of $F=1+r$, or $F=P(1+r)$ shows that $r=0$. Thus we must expand around other nontrivial solutions which might be far away.

A more interesting limit is the following. We define a *classical conformal field theory* as a conformal field theory in which the weights of all primary fields Δ_i vanish. As we will show below, in this limit some of our equations are simplified and their interpretation is understood.

We are also interested in theories which are "approximately classical." For that we define the classical limit of a sequence of conformal field theories. Suppose we have a sequence of conformal field theories CFT_k with the property that the set of primary fields P_k of CFT_k forms a sequence of nested sets $\ldots \subset P_k \subset P_{k+1} \subset P_{k+2} \subset \ldots$ such that, for any field i, $\lim_{k\to\infty} \Delta_i(k) \to 0$ and for k large enough the fusion rules of any $i \times j$ stabilizes. In such a limit some of the duality matrices are well-defined, for example, $\Omega^i_{jk} \to \sigma_{23}$ and (4.18a, b) stabilize for $k \to \infty$. We may then refer to Ω, F as duality matrices in the classical limit. An example of such a limit is provided by the WZW models of current algebra at level k where

$$\Delta_j = \frac{C_j}{k+h^\vee} \to 0$$

(where C_j is the Casimir in the j representation and $h^\vee$ is the dual Coxeter number) since for fixed j, C_j is finite. It is well-known that this is the classical limit of the conformal field theory since in the WZW model the coupling constant is $g^2 = 8\pi/k$ [35]. Another well known example is the Gaussian model – a boson on a circle of radius R. This theory is "quasirational" [18] and most of our formalism can still be used (quasirational theories will be defined and discussed below). Its classical limit is obtained when the radius of the boson is taken to infinity. A related example is that of the rational torus. Its algebra, representations and fusion rules were discussed above. It is labeled by an integer N. Taking N to infinity (in the diagonal modular invariant) we find a classical conformal field theory. Some rational conformal field theories do not have a classical limit. For instance, the $c \to 1$ limit of the $c < 1$ discrete series has an infinite number of zero weight states, but there are also primary fields with non-zero weight which cannot be excluded, by the fusion rules. Therefore, the $c \to 1$ limit of the discrete series is not a classical conformal field theory (according to our definition).

Let us now study the classical limit of (4.18). The first thing to notice is that the isomorphism $\Omega \to \sigma_{23} = \pm 1$ so that $\Omega(+) = \Omega(-)$ and $\Omega^2 = 1$. Thus the two equations in (4.18a) are no longer independent and we are left with two basic equations at $g = 0$.

Since the conformal weights vanish, there is a null vector in any representation generated by the action of L_{-1} on the primary field. These null vectors guarantee that the correlation functions are independent of z. Therefore, it is not surprising that the conformal blocks do not have monodromies ($\Omega^2 = 1$).

The situation with the torus equations is more delicate. It is clear that they cannot all be satisfied as finite matrix equations, for if $T \to e^{-2\pi i c/24} 1$, then the first two equations force $S(0) = \pm 1$, $c = 0 \bmod 4$ and this cannot satisfy the third equation. The bad behavior of the torus equations in the classical limit should come as no surprise, since this is exactly the limit in which stringy effects should go away. (Nevertheless, we will show below that some of the torus relations remain.) This argument assumes that the classical theory has a finite number of representations. This assumption is not satisfied in the classical limit of the WZW model. There, the classical theory is consistent on the torus because all the equations are satisfied for every finite k, and therefore, it is also consistent for infinite k. However, the $k \to \infty$ limit on the torus is not smooth and the argument leading to an inconsistency which was given above is not valid.

Quasirational conformal field theories [18] are defined to be those conformal field theories where the number of representations might be infinite, but the right-hand side of every fusion rule is finite, i.e. $\sum_i N^i_{jk}$ is finite for every j and k. For such theories, the formalism on the plane [and in particular Eqs. (4.18a) and (4.18b)] is applicable. Since most of what we say about the classical theory is independent of the equations on the torus (which have a subtle classical limit even in the rational case), it applies also to all quasirational classical conformal field theories.

We could have used a weaker definition for a classical conformal field theory. Rather than saying that *all* primary fields are of zero weight, it is enough to require this for a subset of them. Such a subset is trivially closed under the operator product expansion, i.e. for every i and j in this set and t not in this set, $N^t_{ij}=0$. This follows from the null vector created by acting with L_{-1} on the zero weight field. Often we are also interested in the vicinity of the classical conformal field theory – the semi-classical theory – in which the weights are very small (but not exactly zero). Then, the state generated by L_{-1} on an almost zero weight field is not null and the fusion rule N^t_{ij} is typically non-zero. When $N^t_{ij}\neq 0$ the decoupling of the null vector at the classical limit is not obtained by a vanishing fusion rule. Instead, as we approach the classical limit the relevant operator product coefficient approaches zero. An example of this phenomenon occurs in the $c<1$ discrete series. There, the operator product coefficients of two states which approach zero weight and a state whose weight does not approach zero asymptotes to zero as $c\to 1$. Another example is provided by the conformal field theory of a sigma model on a Calabi-Yau manifold. As the radius of the manifold becomes large, the theory becomes more and more "classical" in the standard sense. Although the infinite radius theory is classical (according to our definition), the large (but not infinite) radius theory is not "approximately classical."

Since in the classical theory all fields are of zero weight, our definition of the chiral algebra has to be slightly more precise. We cannot define the chiral algebra as the set of holomorphic (or anti-holomorphic) fields in the spectrum because there are such fields in every representation. In this case we define the chiral algebra to be the set of all fields which couple like the identity in the right (left) movers. This definition is equivalent to the standard one when the identity is the only $\Delta=0$ representation.

7. Group Theory as Classical Conformal Field Theory

Consider a collection $\mathbf{C}$ of representations of a group G which satisfies the property that for all $X, Y\in\mathbf{C}$, $X\otimes Y$ is isomorphic to a finite sum of representations in $\mathbf{C}$. For example we may consider the finite dimensional representations of any group. For an appropriate class of groups we can always build such a collection starting with the collection of irreducible representations $\{R_i\}$. These satisfy

$$R_i\otimes R_j\cong \bigoplus_k V^k_{ij}\otimes R_k\,, \tag{7.1}$$

where the vector spaces V^i_{jk} will be n-dimensional, if a representation appears n times. These vector spaces can be identified with a space of intertwining operators.

Recall that if $\varrho_1 : G \to \mathrm{End}(W_1)$, $\varrho_2 : G \to \mathrm{End}(W_2)$ are two representations of a group, an intertwining operator $T : W_1 \to W_2$ is a vector space homomorphism such that

$$\begin{array}{ccc} W_1 & \xrightarrow{T} & W_2 \\ {\scriptstyle \varrho_1(g)}\downarrow & & \downarrow{\scriptstyle \varrho_2(g)} \\ W_1 & \xrightarrow{T} & W_2 \end{array} \tag{7.2}$$

commutes for all $g \in G$. The vector spaces V_{jk}^i above can be identified[10] with the space of intertwining operators $\begin{pmatrix} i \\ jk \end{pmatrix} : R_j \otimes R_k \to R_i$. Moreover the natural isomorphisms between representations

$$\begin{aligned} R_i \otimes R_j &\cong R_j \otimes R_i , \qquad v \otimes w \mapsto w \otimes v , \\ (R_i \otimes R_j) \otimes R_k &\cong R_i \otimes (R_j \otimes R_k) , \qquad (u \otimes v) \otimes w \mapsto u \otimes (v \otimes w) , \end{aligned} \tag{7.3}$$

imply the existence of isomorphism

$$\begin{aligned} &\Omega : V_{jk}^i \cong V_{kj}^i , \\ &F : \bigotimes_r V_{j_1 r}^{i_1} \otimes V_{j_2 k_2}^r \cong \bigoplus_s V_{s k_2}^{i_1} \otimes V_{j_1 j_2}^s . \end{aligned} \tag{7.4}$$

Note that $\Omega^2 = 1$, the sign of Ω measuring the symmetry or antisymmetry of the coupling. Furthermore, considering the pentagon commutative diagram:

$$\begin{array}{ccccc} R_1 \otimes (R_2 \otimes (R_3 \otimes R_4)) & \xrightarrow{F} & (R_1 \otimes R_2) \otimes (R_3 \otimes R_4) & \xrightarrow{F} & ((R_1 \otimes R_2) \otimes R_3) \otimes R_4 \\ \downarrow{\scriptstyle 1 \otimes F} & & & & \downarrow{\scriptstyle F \otimes 1} \\ R_1 \otimes ((R_2 \otimes R_3) \otimes R_4) & & \xrightarrow{\qquad F \qquad} & & (R_1 \otimes (R_2 \otimes R_3)) \otimes R_4 \end{array} \tag{7.5}$$

for representations $R_1, \ldots, R_4$ yields the pentagon relation on F while the hexagon diagram:

$$\begin{array}{ccccc} R_1 \otimes (R_2 \otimes R_3) & \xrightarrow{F} & (R_1 \otimes R_2) \otimes R_3 & \xrightarrow{\Omega} & R_3 \otimes (R_1 \otimes R_2) \\ \downarrow{\scriptstyle 1 \otimes \Omega} & & & & \downarrow{\scriptstyle F} \\ R_1 \otimes (R_3 \otimes R_2) & \xrightarrow{F} & (R_1 \otimes R_3) \otimes R_2 & \xrightarrow{\Omega \otimes 1} & (R_3 \otimes R_1) \otimes R_2 \end{array} \tag{7.6}$$

yields a relation similar to (4.6).

Thus we learn that any group defines a solution to the classical limit of our equations. In the case of $SU(2)$ the mapping F is in fact well-known in physics and the matrix elements are nothing other than Racah coefficients (or $6j$ symbols). For example, in the case of $SU(2)$, the pentagon identity is known as the "Biedenharn sum rule" [36]. In more detail, one usually chooses a basis for the representation space by diagonalizing J_3 in the Cartan subalgebra: $\{|j, m\rangle\}$, $m = -j, \ldots, j$ and defines the Clebsch-Gordan coefficients $\langle J, M | j_1 m_1 j_2 m_2 \rangle$ which are related to intertwiners via

$$\begin{pmatrix} J \\ j_1 j_2 \end{pmatrix} = \sum_{M, m_1, m_2} |J, M\rangle \langle J, M | j_1 m_1 j_2 m_2 \rangle \langle j_1 m_1 j_2 m_2 | . \tag{7.7}$$

[10] More precisely, we are discussing the dual to the space of intertwiners

(Note that in this case the space of intertwiners is always zero or one-dimensional.) Racah coefficients are more traditionally defined by relating products of Clebsch-Gordan coefficients, or equivalently, by writing:

$$\begin{pmatrix} J_0 \\ j_3 J \end{pmatrix}\begin{pmatrix} J \\ j_1 j_2 \end{pmatrix} = \sum_{J'} F_{JJ'}\begin{bmatrix} j_3 & j_1 \\ J_0 & j_2 \end{bmatrix}\begin{pmatrix} J_0 \\ J' j_2 \end{pmatrix}\begin{pmatrix} J' \\ j_3 j_1 \end{pmatrix}. \tag{7.8}$$

Clearly, a similar equation exists for any group.

Although, as we have seen, the torus equations are not well-behaved in the classical limit, many aspects of group-theoretic characters parallel the quantum situation.[11] We may define "one-point functions" by

$$\chi_i^j(g)(\beta) = \mathrm{tr}_i\left[g\begin{pmatrix} i \\ ji \end{pmatrix}(\beta\otimes\cdot)\right]. \tag{7.9}$$

Similarly the "two-point function"

$$\mathrm{tr}_i\left[g\begin{pmatrix} i \\ j_1 p \end{pmatrix}(\beta_1\otimes\cdot)\begin{pmatrix} p \\ j_2 i \end{pmatrix}(\beta_2\otimes\cdot)\right] \tag{7.10}$$

(where g is any group element) has a "b"-monodromy relating characters at (β_1, β_2) to those at $(\beta_1, g\beta_2)$. Considering $\chi_j^i(g)$ as a linear operator $\mathscr{H}_j \to C$ we can transform to the basis of characters $\chi_i^p(g)\begin{pmatrix} p \\ j_1 j_2 \end{pmatrix}(\beta_1\otimes\beta_2)$. As in the quantum case, the pentagon shows that the b-monodromy for the subspace of characters with $p=0$ (and hence $j_1=k$, $j_2=k^\vee$) is proportional to the fusion rules:

$$\chi_i^0(g)\begin{pmatrix} 0 \\ kk^\vee \end{pmatrix}(\beta_1\otimes g\beta_2) = \sum_j F_k N_{ik}^j \chi_j^0(g)\begin{pmatrix} 0 \\ kk^\vee \end{pmatrix}(\beta_1\otimes\beta_2), \tag{7.11}$$

where we defined

$$F_k \equiv (\Omega_{0k}^k \otimes \Omega_{kk^\vee}^0) F_{00}\begin{bmatrix} k & k^\vee \\ k & k \end{bmatrix}. \tag{7.12}$$

Taking $g=1$ we get:

$$\frac{1}{F_k}\dim R_i = \sum_p \dim R_p N_{ik}^p, \tag{7.13}$$

so we must identity

$$\dim R_k = \frac{1}{F_k}, \tag{7.14}$$

a formula which will prove interesting later. Thus, one may take as a classical version of S the unitary matrix which diagonalizes the fusion rule algebra (which, classically, is just the representation ring). For finite groups one may enumerate representations by conjugacy classes $\{C_J\}$ of order $|C_J|$, in which case we have[12]

$$S_{jJ} = \sqrt{\frac{|C_J|}{|G|}}\,\chi_j(C_J). \tag{7.15}$$

---

[11] The reader should compare the following discussion with that of Verlinde [9]

[12] This expression for S was pointed out to us by I. Frenkel and P. Ginsparg

($|G|$ is the order of the group.) It is clear that this matrix is a unitary matrix and it diagonalizes the fusion rules. This equation, appropriately interpreted, can be used also for continuous groups. For instance, for $SU(2)$ we can label the conjugacy classes by the continuous parameter θ and the representations by the discrete parameter j. The volume of the conjugacy class is $\frac{|C_\theta|}{|G|}=\frac{1}{2\pi}(1-\cos\theta)=\frac{1}{\pi}\sin^2(\theta/2)$. Using $\chi_j(\theta)=\frac{\sin((2j+1)\theta/2)}{\sin(\theta/2)}$, we find

$$S_{j\theta}=\frac{1}{\sqrt{\pi}}\sin((2j+1)\theta/2), \tag{7.16}$$

which is unitary in the following sense: $\int_0^{2\pi} d\theta S_{j\theta}S^*_{j'\theta}=\delta_{jj'}$ and $\sum_j S_{j\theta}S^*_{j\theta'}=\delta(\theta-\theta')$. Also, it is easy to check that it diagonalizes the fusion rules. If we discretize θ in this expression as $\theta_i=\frac{(2i+1)}{k+2}2\pi$ (and normalize S appropriately) we recover the standard $SU(2)$ level k modular transformation law, now with a slightly novel interpretation.

The different eigenvalues of the fusion rule matrices form the different one dimensional representations of the fusion rule algebra. These are given at the classical level by

$$\lambda_j^{(J)}=\chi_j(C_J). \tag{7.17}$$

These can be thought of as related to the classical version of the a-monodromy which is given by $\lambda_j^{(J)}/F_j$. Notice that since at the classical level the string becomes effectively a point, the b-monodromy is well defined and makes sense (see above) but the a-monodromy (as well as S) which depend on the "stringy" nature of the quantum theory are not straightforward.

It is interesting that the classical S depends on two different kinds of indices. J labels conjugacy classes and j labels the representations of the group. Although the number of conjugacy classes is the same as the number of representations (S is a square matrix), there is no natural map between them. Therefore, formulae like $S^\dagger S=1$, $SS^\dagger=1$ and $(S^\dagger N^k S)_{IJ}=\lambda_k^{(J)}\delta_{IJ}$ which do not depend on the contractions of the two different kinds of indices make sense. On the other hand, the relations of the modular group involve terms like S^2 which need a correspondence between the two different kinds of indices, and therefore do not in general make sense. There are some cases where such a natural correspondence (related to "self-dual groups") does exist and these should be investigated more thoroughly.

Finally, let us consider the relation of the algebra of intertwiners to the algebra of functions Fun(G) on the group manifold G, at least for the case of compact groups. From the Peter-Weyl theorem we know that $L^2(G)\cong\bigoplus_R D^R\otimes D^R$, where the sum runs over the irreducible representations of the group. This means that the matrix elements $D^R_{\mu,\nu}$ form a basis for the algebra of functions on G. This algebra can be written explicitly as:

$$D^{R_1}_{\mu_1\nu_1}D^{R_2}_{\mu_2\nu_2}=\sum_{R,\gamma_1,\gamma_2}\sum_{a\in V^R_{R_1R_2}} D^R_{\gamma_1\gamma_2} \times\left\langle R,\gamma_1\left|\begin{pmatrix} R \\ R_1R_2\end{pmatrix}_a(\mu_1\otimes\mu_2)\right.\right\rangle\left\langle R,\gamma_2\left|\begin{pmatrix} R^\vee \\ R_1^\vee R_2^\vee\end{pmatrix}_{a^\cdot}(\nu_1\otimes\nu_2)\right.\right\rangle,$$

where we sum over a basis of intertwiners and $a^{\vee}$ is a basis dual to a. Given the Peter-Weyl theorem we may easily prove this statement as follows. Consider the sum

$$\sum_{\mu_1,\mu_2,\nu_1,\nu_2} D^{R_1}_{\mu_1\nu_1}(g) D^{R_2}_{\mu_2\nu_2}(g) \langle \mu_1\mu_2 | R\gamma_1\rangle_a \langle \nu_1\nu_2 | R\gamma_2\rangle_b .$$

This must be given by $\langle T|a\rangle \otimes |b\rangle D^{R}_{\gamma_1\gamma_2}(g)$, where T is a tensor on $V \times V^{\vee}$. Recall that the space of intertwiners is itself a representation space. By the transformation properties of the above quantities we find that T is an invariant tensor, from which we deduce the above. Each function $f \in \mathrm{Fun}(G)$ defines a linear operator $U(f)$ on the vector space $L^2(G)$ corresponding to the operation of multiplying by that function and we have

$$U(D^R_{\mu\nu}) = \sum_{R_1,R_2} \sum_{a \in V^{R_1}_{RR_2}} \begin{pmatrix} R_1 \\ RR_2 \end{pmatrix} (\mu \otimes \cdot)_a \begin{pmatrix} R_1^{\vee} \\ R^{\vee}R_2^{\vee} \end{pmatrix} (\nu \otimes \cdot)_{a^{\vee}} . \tag{7.18}$$

Comparing this with (5.6) we see that $U(D^R)$ is similar to the conformal field in the $(R, R^{\vee})$ representation of $\mathscr{A} \times \mathscr{A}$. Hence, in this case we naturally find the "diagonal theory." From this expression it is clear that in the gauge $\begin{pmatrix} R_1^{\vee} \\ R^{\vee}R_2^{\vee} \end{pmatrix} = \begin{pmatrix} R_1 \\ RR_2 \end{pmatrix}^*$ we have $d^{R_1}_{RR_2}(t^{\vee}_a, \bar{t}^{\vee}_a) = \delta_{t_a,\bar{t}_a}$. As remarked at the end of Sect. 5, in this case the duality matrices are all unitary.

This fact can be easily understood as follows. In the classical theory the tensor product $R_1 \otimes R_2$ space is isomorphic to the sum of representations $\bigoplus_j V^j_{12} \otimes R_j$. Therefore, some normalizations of the Clebsch-Gordan coefficients are more natural than others. In particular, it is standard to impose

$$\sum_{JM} \langle j_1 m_1' j_2 m_2' | JM\rangle \langle JM | j_1 m_1 j_2 m_2\rangle = \delta_{m_1 m_1'} \delta_{m_2 m_2'}$$

for $SU(2)$ and a similar expression for other groups. With such a normalization (gauge choice) the Clebsch-Gordan coefficients are unitary. They represent unitary transformations between different bases of $R_1 \otimes R_2$. In this gauge the Racah coefficients are also unitary transformations between different bases of the space of tensors coupling four representations. Since the F's are unitary transformations the solution $d^{R_1}_{RR_2}(t^{\vee}_a, \bar{t}^{\vee}_a) = \delta_{t_a, \bar{t}^*_a}$ solves Eq. (5.7).

The correspondence $f \leftrightarrow U(f)$ is the classical version of the isomorphism of states and vertex operators. In the limik $k \to \infty$ of current algebra the correspondence can be made much more explicit [35]. Wavefunctions are functions on the loop group [37, 38] LG and the wavefunction corresponding to an operator ϕ is computed by the path integral on the disk:

$$\Psi[g(\sigma)] = \int_{g|_{\partial D} = g(\sigma)} [dg]\, \phi(0)\, e^{-kS_{\mathrm{WZW}}} \tag{7.19}$$

as $k \to \infty$ the path integral is concentrated on constant loops, so the wavefunctions become functions on the group manifold.

This result is more general and applies also to discrete groups. At the classical level the conformal fields ϕ_i are independent of z. We can think of them as forming a basis for the functions on the group with the correspondence, $\phi_i \leftrightarrow f_i(g)$. The operator product expansion is simply the product of two functions on the group

$$f_i(g) f_j(g) = \sum_k c^k_{ij} f_k(g) . \tag{7.20}$$

The calculation of correlation functions is reduced to a sum over the group

$$\langle \phi_i \phi_j \dots \phi_k \rangle = \sum_g f_i(g) f_j(g) \dots f_k(g), \tag{7.21}$$

where for continuous groups the sum is replaced by an integral. This result is simple to understand in the WZW model. There it arises from performing the functional integral at the classical limit.

It is not obvious that compact Lie groups give the only examples of classical theories which can be extended to full quantum conformal field theories. Therefore, it might be of use to learn how to do semi-classical perturbation theory. As an example of how this works, consider $SU(2)$. Classically we have

$$F\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{pmatrix} -\frac{1}{2} & \left(\frac{\sqrt{3}}{2}\right)\lambda \\ -\frac{\sqrt{3}}{(2\lambda)} & -\frac{1}{2} \end{pmatrix},$$

where the parameter λ depends on the gauge. Expanding the equations around this solution one finds that

$$\frac{2\Delta_{1/2}}{\Delta_1 - 2\Delta_{1/2}} = 3,$$

which can be checked in the exact theory.

To summarize, we have shown that every compact group (discrete or continuous) leads to a classical conformal field theory on the plane. The correspondence between familiar concepts in group theory and conformal field theory is the following:

Group	Chiral algebra
Representations	Representations
Clebsch-Gordan coefficients/intertwiners	Chiral vertex operators
Invariant tensors	Conformal blocks
Symmetry of couplings	Ω
Racah coefficients ($6j$ symbols)	Fusion matrix
Functions on the group	Physical fields
Product of functions on the group	Operator product expansion
Average over the group of a product of functions	Physical correlation function

8. Classical and Quantum Reconstruction

We have seen that a classical conformal field theory is a realization in terms of collections of vector spaces R_i satisfying (7.1) and the Axioms 1, 2, 3 (with $\Omega^2 = 1$ and $|I|$ possibly infinite) and (4.18a, b) of Sect. 4. In the previous section we showed that the collection of representations of a group always provide such a realization, and hence define a classical conformal field theory. It is natural to ask if the converse is true: is every classical conformal field theory defined by the representations of a group? This question should be viewed as the classical version of the reconstruction problem, which asks for a minimal set of data and conditions on that data from which one can reconstruct a conformal field theory (or an equivalence class of conformal field theories). The classical reconstruction problem has been solved in the context of the Tannaka-Krein approach to group theory [39–42].

The basic philosophy of the Tannaka-Krein approach is that the knowledge of a group is equivalent to the knowledge of its representations. Roughly speaking, given vector spaces R_i, V^i_{jk} and isomorphisms F, Ω subject to (7.1), (4.18a, b) etc., the theorems of Tannaka and Krein show how to reconstruct a group, whose finite dimensional representations are constructed from the R_i in terms of direct sums, tensor products, duals, and quotients.

The Tannaka-Krein viewpoint can be considerably deepened using some concepts from category theory, namely those of tensor and Tannakian categories[13]. Roughly speaking, the axioms of a tensor category can be interpreted as the axioms we have stated for classical conformal field theory, namely, the axioms of Sect. 4 for V, F, Ω with $\Omega^2=1$. The axioms of a Tannakian category include those of a tensor category but include an additional axiom which amounts to the assumption of the existence of finite-dimensional vector spaces R_i obeying (7.1). Hence the theorems of Tannaka-Krein essentially state that a Tannakian category is the category of representations of a group, and the classical reconstruction problem is the problem of passing from a tensor category to a Tannakian category. Since there are examples of tensor categories which are not Tannakian [41], it is clear that the reconstruction problem is nontrivial.

A recent result of Deligne [42] provides a criterion for deciding when a tensor category is Tannakian. In Appendix C we describe Deligne's condition and show that in terms of the data V, F, Ω, it is simply the condition that F_i^{-1} is a nonnegative integer. [Recall that when we are given a group we may deduce (7.14).] It follows that if we supplement the axioms of Sect. 4 by the axiom $F_i^{-1}\in Z_+$, then we can use Deligne's result to conclude that every classical conformal field theory is associated with a group. That is, we can identify classical conformal field theory as group representation theory, and the classification of classical conformal field theory is, therefore, the classification of groups with finite dimensional representations. This is a paradigm which should be emulated in the quantum case.

Unfortunately very little is known about quantum reconstruction, so we can only make a few remarks and speculations. As we have seen, if we start with a chiral algebra $\mathscr{A}$ and its representations $\mathscr{H}_k$ we con construct a conformal field theory using the chiral vertex operators. Then, the duality matrices can be computed and must satisfy the identities of Sect. 4. Alternatively, one can consider the reverse process. Namely, one begins with a fusion rule algebra N^i_{jk}, sets up the equations for the duality matrices and looks for solutions. In Appendices D, E we illustrate how one can do this for some simple fusion rule algebras. As we discussed above, it is not obvious that every solution to our equations leads to a conformal field theory. One should make sure that conformal blocks with the right asymptotic

[13] This section is only meant to be a very rough sketch of the relevant category-theoretic results. A more precise account can be found in Appendix C, and the real thing appears in [41–43]

behavior exist. Furthermore there must be a sensible Hilbert space interpretation of the results implying, for example, that the one-loop vacuum characters have integral coefficients in the q-expansion. One might want to impose further physical requirements of CPT, etc. Nevertheless, given the success of the Tannaka-Krein approach to group theory we may speculate that the axioms of Sect. 4, or some extension of these might define rational conformal field theories.

Inspired by the group-theory example we can offer the following speculation on what the appropriate extension of the axioms of Sect. 4 will look like. Deligne's extra condition is simply an integrality condition on $1/F_i$. In the quantum case the same quantity $1/F_i$ appears to be of fundamental importance. An interpretation of $1/F_k$ as a dimension in the quantum case appears in [17], where it was noticed that for unitary theories

$$\frac{1}{F_k}=\frac{S_{k0}}{S_{00}}=\lim_{q\to 1}\frac{\chi_k}{\chi_0}=\frac{\text{“dim”}\mathscr{H}_k}{\text{“dim”}\mathscr{H}_0}.$$

Now $\mathscr{H}_0$ is both an operator algebra and a Hilbert space, and it acts on another Hilbert space $\mathscr{H}_i$. Instead of decomposing into an integral number of copies of $\mathscr{H}_0$, we instead find some real number. This situation is very reminiscent of that of subfactors of finite index [44] and F_i^{-1} seems to play the role of an index. Jones found that only for special values of the index is it possible to construct appropriate subfactors. It might be that here, too, only for special values of F_i^{-1} can the $\mathscr{H}_i$ exist (guaranteeing integer coefficients in the q-expansion of characters). Some further evidence for this speculation is provided by noting that for $SU(2)$ level k we find

$$\frac{1}{F_j}=\frac{\sin\left(\dfrac{(2j+1)\pi}{k+2}\right)}{\sin\left(\dfrac{\pi}{k+2}\right)}=\frac{q^{2j+1}-q^{-(2j+1)}}{q-q^{-1}}=[2j+1]_q,$$

which coincides with the definition of [45, 46] of the "q dimension" of quantum $SU(2)$ for $q=e^{\frac{\pi i}{k+2}}$.

This example suggests that the quantum analog of the integrality condition is that F_i^{-1} is a q-integer, where q is some root of unity, perhaps related to the value of c.

From the $\mathscr{H}_i$ we might hope to reconstruct the chiral algebra and hence the entire conformal field theory as in the classical case. Nevertheless, since very different chiral algebras can have the same fusion rules, and since there are holomorphic conformal field theories with no monodromy, the quantum version of reconstruction is likely to be subtle.

One could have defined a classical conformal field theory as a theory where all the Δ_i are integers. In such a theory the equations simplify precisely as for $\Delta_i=0$. However, such a theory is not a classical field theory. Although the sections do not have monodromies, they have poles reflecting the quantum nature of the problem. Also, even if there is a finite number of blocks in any process, an infinite number of states propagate at intermediate channels. Therefore, there is no description of the theory in terms of *finite* vector spaces. Hence, Deligne's theorem does not apply.

9. Quantum Groups and the Meaning of z

This section is a very speculative attempt to understand chiral algebras as generalizations of quantum groups. The chief obstacle to progress is a good understanding of the abstract meaning of the parameter z on the complex plane.

Let us begin by recalling some well-known facts about quantum groups [47]. Quantum groups are defined by their algebra of functions, which is a Hopf algebra [47]. One of the distinguishing characteristics of a Hopf algebra is the existence of a comultiplication $\Delta:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$ satisfying:

$$\begin{array}{ccc} & \mathscr{A}\otimes\mathscr{A} & \\ \nearrow & & \searrow \\ \mathscr{A} & & \mathscr{A}\otimes\mathscr{A}\otimes\mathscr{A}\,, \\ \searrow & & \nearrow \\ & \mathscr{A}\otimes\mathscr{A} & \end{array} \tag{9.1}$$

where the top route is $(1\otimes\Delta)\Delta$ and the bottom is $(\Delta\otimes 1)\Delta$. The existence of the comultiplication allows us to take tensor products of representations, for if $\varrho_{1,2}:\mathscr{A}\to\mathrm{End}(V_{1,2})$, are representations then $\varrho_1\otimes\varrho_2\circ\Delta:\mathscr{A}\to\mathrm{End}(V_1\otimes V_2)$ is a tensor product representation. In general $V_1\otimes V_2$ and $V_2\otimes V_1$ are not isomorphic representations, but if there exists an invertible $R\in\mathscr{A}\otimes\mathscr{A}$ such that

$$\sigma\circ\Delta(a)=R\Delta(a)R^{-1} \tag{9.2}$$

[where $\sigma(a\otimes b)=b\otimes a$] then we may define the isomorphism $\Omega: V_1\otimes V_2\to V_2\otimes V_1$ by $x\otimes y\mapsto P\varrho_1\otimes\varrho_2(R)(x\otimes y)$, where P is the permutation operator. The associativity isomorphism is still the usual one $x\otimes(y\otimes z)\mapsto(x\otimes y)\otimes z$. From the hexagon diagram one then deduces $\Delta\otimes\mathrm{id}(R)=R_{13}R_{23}$, from which one may deduce the Yang-Baxter equation. The existence of R implies that there is an isomorphism between spaces of intertwiners: $\Omega: V_{ij}^k\to V_{ji}^k$ defined by

$$\begin{pmatrix} k \\ ij \end{pmatrix}\mapsto\begin{pmatrix} k \\ ij \end{pmatrix}P\varrho_j\otimes\varrho_i(R)\,. \tag{9.3}$$

Let us now consider the case of conformal field theory. One of the distinguishing features of conformal field theory is contour deformation which allows us to define a map $\Delta_{0,z}:\mathscr{A}\to\mathscr{A}\otimes\mathscr{A}$, which satisfies

$$\begin{array}{ccc} & \mathscr{A}\otimes\mathscr{A} & \\ \nearrow & & \searrow \\ \mathscr{A} & & \mathscr{A}\otimes\mathscr{A}\otimes\mathscr{A}\,, \\ \searrow & & \nearrow \\ & \mathscr{A}\otimes\mathscr{A} & \end{array} \tag{9.4}$$

where the top route is $(1\otimes\Delta_{0,z_2-z_1})\Delta_{0,z_1}$ and the bottom is $(\Delta_{0,z_1}\otimes 1)\Delta_{0,z_2}$. The existence of Δ allows us to take tensor products of representations, as explained in Sect. 2. Once again there is an isomorphism Ω between spaces of intertwiners $V_{ij}^k\to V_{ji}^k$. If we wish to emphasize the z-dependence of the intertwiners we may denote $V_{ij}^k(z)$. Then we can define $\Omega: V_{ij}^k(z) \to V_{ji}^k(-z)$ by

$$\begin{pmatrix} k \\ ij \end{pmatrix}_z \mapsto \begin{pmatrix} k \\ ij \end{pmatrix}_z \circ P\varrho_j \otimes \varrho_i(R) \in V_{ji}^k(-z), \tag{9.5}$$

where $R = e^{-zL_{-1}} \otimes e^{-zL_{-1}}$.

This formula must be treated with care. In verifying the properties of the chiral vertex operators one easily checks that the derivative with respect to z is correct. However in checking the intertwining condition [Eq. (2.7)] one finds the composition of the chiral vertex operator with $R\Delta_{0,-z}(\cdot)R^{-1} = \Delta_{z,0}(\cdot)$ which is in close analogy with (9.2). For the case of conformal field theory the formula can be demonstrated formally because L_{-1} is the translation operator, but in fact, without the composition with $\begin{pmatrix} k \\ ji \end{pmatrix}$ the equation is only true for half the modes of the vertex operators. Related to this is the consideration that R matrices of the form $A \otimes A$ are usually considered to be trivial solutions of the Yang-Baxter equation (corresponding to noninteracting particles in factorizable S-matrix theory). Nevertheless, (9.5) is one way of stating the existence of Ω and has nontrivial consequences.

We should also note that many people have noticed that in particular conformal field theories the braiding matrix is closely related to the R matrix of certain quantum groups. Most notably, in $SU(N)$ level k current algebra the braiding matrix

$$B_{pq}\begin{bmatrix} j_1 & j_2 \\ i & r \end{bmatrix}$$

seems to be the same as $\varrho_{j_1} \otimes \varrho_{j_2}(R)_{pi|qr}$, where R is the R-matrix for $sl(N)_q$ with $q = e^{\frac{i\pi}{N+k}}$, and we have used the q-deformed intertwiners to pass from the "vertex-representation" to the "IRF" representation in which R depends only on representations [45, 46]. Note also that the analogs of our $g=0$ equations for quantum groups have been discussed in [48]. It would be very interesting if one could connect these observations with the above proposals.

Appendix A. Characters at High Genus

In this appendix we describe an *ansatz* for high genus characters which is motivated by sewing constructions in string field theory. It is very important to note that the validity of this *ansatz* does not affect the correctness of the representation of the high-genus modular group given in Appendix B. The latter can be deduced from the constancy of the monodromy matrices and the factorization properties of the characters.

We first describe the basic strategy for deriving the *ansatz*. We coordinatize moduli space using the Schottky parametrization. In this parametrization the Riemann surface is represented as the quotient of C by a Schottky group $\langle \gamma_1, \ldots, \gamma_n \rangle \subset PSL(2, C)$. We may write the generators in the form $U_{a_i, b_i}(\gamma(z)) = q_i^{-1} U_{a_i, b_i}(z)$, where $U_{a,b}(z) = (z-a)/(z-b)$. The Riemann surface is obtained by identifying the g pairs of isometric circles associated with each of the generators. [Isometric circles associated to γ are the circles C_γ, $\tilde{C}_\gamma$ on which $|\gamma(z)'| = 1$ and

$|\gamma^{-1}(z)'|=1$ respectively.] Next, following the methods of [50] we define a $2g$-string vertex $\langle V|$ by identifying each of the $2g$-isometric icrcles $C_{\gamma_i}, \tilde{C}_{\gamma_i}$ with the unit circle via mappings $h_{\gamma_i}, \tilde{h}_{\gamma_i}$. That is, defining $h[\Phi] \equiv U(h)\Phi(0)U(h)^{-1}$, where $U(h)$ is the operator representative of the Mobius transformation h, we have

$$\langle V|I_1\rangle|\tilde{I}_1\rangle \ldots |I_g\rangle|\tilde{I}_g\rangle \equiv \langle h_{\gamma_1}R[\Phi^{I_1}]\tilde{h}_{\gamma_1}[\Phi^{\tilde{I}_1}] \ldots h_{\gamma_g}R[\Phi^{I_g}]\tilde{h}_{\gamma_g}[\Phi^{\tilde{I}_g}]\rangle, \quad \text{(A.1)}$$

where R is the transformation $R(z)=-1/z$. We may then sew the circles together to get the blocks by contracting with the identity operator, that is, the blocks are given by $\mathscr{F}=\langle V|1_{12}\rangle \ldots |1_{2g-1,2g}\rangle$ where $|1\rangle=\sum|I\rangle|I\rangle$, the sum running over an orthonormal basis of nonnull descendants. The main problem is to find an appropriate set of mappings h_γ.

We will demand that the characters satisfy the following set of reasonableness criteria:

1. The characters must only depend on $3g-3$ moduli. In particular, an overall conjugation $\gamma_i \to N\gamma_i N^{-1}$ changes $(a_i, b_i, q_i) \to (\tilde{a}_i, \tilde{b}_i, q_i)$ where

$$N(z)=\frac{Az+B}{Cz+D}, \qquad \tilde{a}_i=\frac{B-a_iD}{A-a_iC}, \qquad \tilde{b}_i=\frac{B-b_iD}{A-b_iC},$$

so the characters must be invariant under this transformation.

2. The characters χ must be holomorphic in the moduli. (They will typically be power series which only converge in some region of moduli space and hence will have monodromy and not be globally well-defined.)

3. The characters must reduce to known expressions in obvious limits. Especially, if we ignore all pairs of circles but one, then $\langle V|1_{12}\rangle$ must correspond to gluing a pair of concentric circles related by $z \to qz$, hence computing a torus amplitude.

4. Finally, the h_γ must satisfy the (admittedly nebulous) criterion of being naturally constructed out of the transformation γ.

In deciding how to identify two isometric circles with a standard copy of S^1 we may use the following

Lemma. *Given two nonintersecting circles C_1, C_2 there is a Mobius transformation h and a unique radius $R<1$ such that*

$$h: R\cdot S^1 \to C_1, \qquad S^1 \to C_2.$$

Moreover, h is unique up to right multiplication by a rotation. Thus if we specify basepoints (C_i, p_i) and specify $h(1)=p_2$ there is a unique complex number q, $|q|<1$ such that $h(q)=p_1$.

The proof is straightforward and will be omitted. Using the lemma, let h_γ be the map determined by choosing C_1, C_2 to be the two isometric circles of γ. This map may be written as $h_\gamma=U_{a,b}^{-1}h_q$, where h_q depends only on q and maps

$$\begin{aligned} h_q:(qS^1,q) &\to \left(q^{1/2}\left[\frac{1-q^{-1/2}e^{i\theta}}{q^{-1/2}-e^{i\theta}}\right], q^{1/2}\right),\\ (S^1,1) &\to \left(q^{-1/2}\left[\frac{q^{-1/2}-e^{i\theta}}{1-q^{-1/2}e^{i\theta}}\right], q^{-1/2}\right). \end{aligned} \quad \text{(A.2)}$$

Using the maps h_q, U and the dilation by q, $D_q: z \to qz$, one can identify a pair of unit circles with C_γ, $\tilde{C}_\gamma$ in many ways. However, since h_q is *not* holomorphic in q the criteria (1), (2) above put strong constraints on the possible choices. We have only found the choice

$$S^1 \xrightarrow{h_q} U(\tilde{C}) \xrightarrow{D_q} U(C) \xrightarrow{U^{-1}} C, \quad S^1 \xrightarrow{h_\gamma} \tilde{C}, \tag{A.3}$$

to give a good answer, $\langle V_{\gamma_1, \ldots, \gamma_g}|$. Since h_q is not holomorphic the characters are not manifestly holomorphic, but after some algebra we find that they can be expressed as

$$\mathscr{F}_v = \sum \prod_{j=1}^{g} q_j^{|I_j| - c/24} \left\langle \prod_{j=1}^{g} U_{a_j, b_j}^{-1} R[\Phi_{t_j}^{I_j}] U_{a_j, b_j}^{-1} [\Phi_{s_j}^{I_j}] \right\rangle. \tag{A.4}$$

The factors of $q_i^{-c/24}$ were put in by hand to assure a correct limit at the boundary of moduli space. Furthermore, the *same* basis of characters can be written in the form of a sewn chain of traces for two-point functions on the torus as follows. Let $|I|$ be the weight of the descendant I, and define $\mathscr{K}$ as a metric on descendants by $\langle \Phi_{i'}^I(z) \Phi_i^{\tilde{I}}(w) \rangle = \mathscr{K}_{I, \tilde{I}}^{-1} (z-w)^{-(|I| + |\tilde{I}|)}$. [Remember, $\mathscr{K}$ can be antisymmetric. It is necessary for the coupling of $SL(2, C)$ descendants.] We may formally prove that the projection operator on the representation $\mathscr{H}_i$ is given by

$$\Pi_i = \sum_{I, \tilde{I}} \Phi_{i0}^I(0) |0\rangle \mathscr{K}_{I\tilde{I}} \langle 0| \Phi_{0i}^{\tilde{I}}(1), \tag{A.5}$$

where the sum is only over descendants in representation i. From this we find that (A.4) can also be written as

$$\mathscr{F} = \sum \prod \mathscr{K}_{P_{k-1} \tilde{P}_{k-1}} \mathrm{Tr}(q_k^{L_0 - c/24} U_{a_k, b_k} [\Phi_{t_k}^{\tilde{P}_{k-1}}(1)] U_{a_k, b_k} [\Phi_{s_k}^{P_k}]). \tag{A.6}$$

The first and the last operators in this expression should be set to the identity operators – the first and last handles have only one external leg. The representation of the generators of the modular group provided by these characters is that given in Appendix B, although this representation can be deduced by more general reasoning.

It is an easy matter to generalize the above discussion to the case of n-point function characters at genus g. We also remark that, as a consequence of Verlinde's conjecture one can count the number of independent characters [9]. The $n=0$ formula of [9] is easily generalized to[14]

$$\dim V(g, i_1, \ldots, i_n) = \sum_p \frac{S_{i_1 p} \ldots S_{i_n p}}{S_{0p} \ldots S_{0p}} \left(\frac{1}{S_{0p}} \right)^{2g-2}, \tag{A.7}$$

where $i_1, \ldots, i_n$ are the representations on the external lines and S is the transformation law on vacuum characters. Note that the implications of sewing for the number of characters follow from this equation because $S^2 = C$.

Appendix B. The Completeness Theorem

In this appendix we give the proof that the set of equations given in Sect. 4 give a complete set of relations for the duality groupoid. We divide the problem into three steps considering genus zero, one and larger than one in turn.

---

[14] This formula was worked out together with T. Banks. The $g=0$, $n=3$ formula is equivalent to Verlinde's conjecture, and was also noted in [17]

I. Genus Zero: The Quantum Coherence Theorem

This is a generalization of the MacLane [51] coherence theorem in category theory. MacLane's result applies to the equations of a classical conformal field theory. Hence, we will refer to our result as "the quantum coherence theorem." (The completeness of these equations for the $6j$ symbols of $SU(2)$ was first proven by Racah [36].) We will first show that the only identity involving F only, and not Ω, is the pentagon (4.7). This part of our proof is similar (but not identical) to the corresponding part in MacLane's proof. Then we will show that the only other identities at genus zero are the two hexagons (4.6).

We start with the situation at genus zero. We construct a simplical complex $\mathscr{C}(i_1, \ldots, i_n)$ whose vertices are the different tree-level ϕ^3 diagrams where the external legs are ordered as $i_1, \ldots, i_n$. It is convenient (but not essential) to add two more auxiliary external legs i_0 and i_{n+1} of the identity operators. This does not change the relevant conformal blocks. For example, $\mathscr{C}(i_1, i_2, i_3, i_4)$ consists of the two vertices corresponding to the diagrams of Fig. 23 but the diagram of Fig. 24 is not in $\mathscr{C}(i_1, i_2, i_3, i_4)$ because of the different order of the external legs. Thus we work not with ϕ^3 diagrams but with diagrams together with an ordering of the external

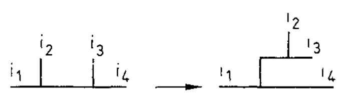


Fig. 23. A simple complex for the four-point function

Fig. 24. A diagram with a different ordering

lines. We connect two vertices by an edge if the corresponding diagrams can be connected by a fusing "simple move" F. Since at this point we limited ourselves to ordered tree-level ϕ^3 diagrams of a given order, the other transformations of the duality groupoid (like Ω and S) do not act on $\mathscr{C}(i_1, \ldots, i_n)$. It is clear that with these edges $\mathscr{C}(i_1, \ldots, i_n)$ is connected. However, it is not simply connected. Every closed loop on the resulting one-complex corresponds to a consistency condition on F. For example, Fig. 18 is a closed loop in the complex of the five point function and it leads to the pentagon equation (4.7),

$$P_{23}F_{13}F_{12}=F_{23}F_{12}F_{23}. \tag{B.1}$$

We will now show that filling every face corresponding to the pentagon makes the two-complex simply connected. Consequently there are no new independent equations involving only F.

We prove this statement by induction on n. Two particularly important configurations are the "multi-pheripheral" diagram (Fig. 25) and the "staircase"

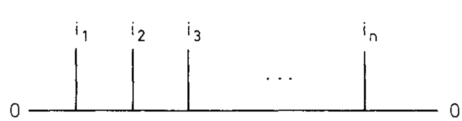


Fig. 25. Multiperipheral basis for genus zero characters

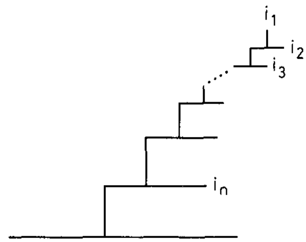


Fig. 26. The staircase configuration

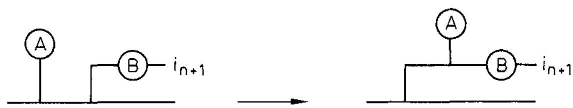


Fig. 27. The last operation in which i_{n+1} participates must be of this form

diagram (Fig. 26). We have to show that every closed loop is homotopically trivial. To begin, we borrow a trick from MacLane which allows us to consider only paths which go through the stair-case configuration and in which every move involves F and not F^{-1}. MacLane's trick is the following: consider an arbitrary closed path of diagrams $\mathscr{D}_1, \dots, \mathscr{D}_p, \dots, \mathscr{D}_1$. Let us denote the application of F by an arrow (an application of F^{-1} is denoted by the reverse arrow). From any diagram it is always possible to find a path to the stair-case using F only, and not F^{-1}. Denoting by $\mathscr{D}_s$ the staircase diagram we can therefore decompose any loop in $\mathscr{C}$ as follows:

$$\begin{array}{ccccccccc} \mathscr{D}_1 & \longrightarrow & \mathscr{D}_2 & \longrightarrow & \dots\ \mathscr{D}_k & \leftarrow & \mathscr{D}_{k+1} & \dots & \mathscr{D}_1 \\ \downarrow & & \downarrow & & \dots\ \downarrow & & \downarrow & \dots & \downarrow \\ \mathscr{D}_s & = & \mathscr{D}_s & = & \dots\ \mathscr{D}_s & = & \mathscr{D}_s & \dots & \mathscr{D}_s \end{array} \quad . \tag{B.2}$$

If any two paths with the same endpoints involving only F and not F^{-1} can be deformed into one another then each of the subloops in (B.2) is trivial, hence the entire loop is trivial. Therefore, assume (B.1) is enough for diagrams with up to n lines and consider a diagram $\mathscr{D}_0$ with $n+1$ external lines, and fix some path γ_0 from $\mathscr{D}_0$ to $\mathscr{D}_s$. We would like to show that any other such path γ can be deformed to γ_0. Define the component of an external line i_k to be the collection of lines connected to i_k with only a single line joining to the base of the diagram. We say that an external line i_k "participates" in a move if that move changes the collection of external lines in the component of i_k. The last operation in γ_0, γ in which i_{n+1} participates must be of the form of Fig. 27 (every blob stands for an arbitrary configuration) and without loss of generality we can take the blob B for the path γ_0 to have zero external lines. Let k be the number of external lines in the blob B for γ. If $k=0$ (the blob B is trivial), our proof ends by the induction hypothesis. The reason is that two paths with $k=0$ must look as in Fig. 28. Consider the top loop in Fig. 28. Since the paths do not involve F^{-1} the loop cannot have i_{n+1} participating, and is therefore null homotopic, by the induction hypothesis. Furthermore, after the last participation of i_{n+1} any two paths from B to the stair-case must be homotopic, again by the induction hypothesis. The middle loop is null homotopic because the moves act on separate lines and do not "interfere." (In category theory the

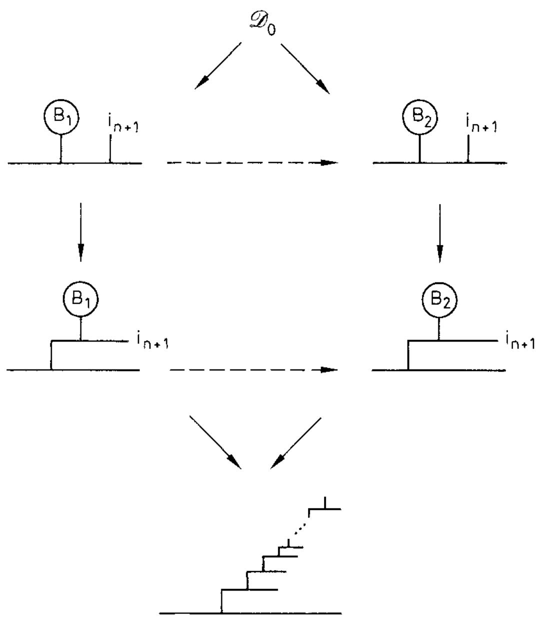


Fig. 28. A deformation of paths in a special case

statement that this loop is trivial is part of the statement of functoriality.) Therefore, it is enough to show that our curve is homotopic to another curve for which the number of external legs in B is smaller than k. To see this, we should look at the last two operations which change the number of lines in the first step in the staircase. The end of the path must be of the form of Fig. 29 (a prime on a letter means that some transformations were performed inside the blob). In asserting this we have used once more the property that the path involves only F and not F^{-1}. We can now deform the curve as in Fig. 29. The closed loop (1) is simply connected by the induction hypothesis. Loop (2) is simply connected since we have performed two operations on two disjoint sets of lines. Finally loop (3) is simply connected by the pentagon identity. Therefore, we succeeded to deform the curve to a curve with a smaller k (since D is non-trivial). Finally, we must consider special cases that arise because the initial point of γ_0, γ is so close to the staircase that i_{n+1} participates fewer than two times. In this case the paths must look like those in Fig. 30, and by adding the dashed line in that figure we see that the loop is homotopically trivial by the induction hypothesis. This completes the proof that there are no new identities involving only F.

Now that we have established that there are no new equations involving only F, we should find all the equations satisfied by F and Ω. These can be found by enlarging the previous complex. Let $\Gamma_{0,n}$ be the modular group of the n-holed sphere. It is sometimes convenient to regard it as the modular group for the space of n punctures on the sphere with a choice of coordinate at the puncture [32]. We consider an infinite complex $\mathscr{C}_n$ and organize it by finite "layers." Every layer is isomorphic to the previously discussed finite complex $\mathscr{C}(i_1, \ldots, i_n)$ for some permutation of $i_1, \ldots, i_n$. The different layers differ by the action of $\Gamma_{0,n}$. Notice that

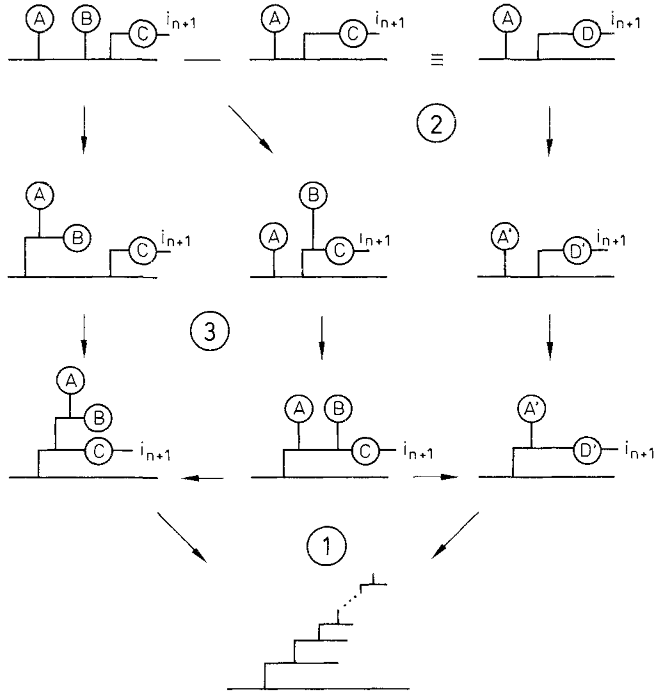


Fig. 29. The last two operations in which i_{n+1} participates must be of the form given in the top right part of this diagram

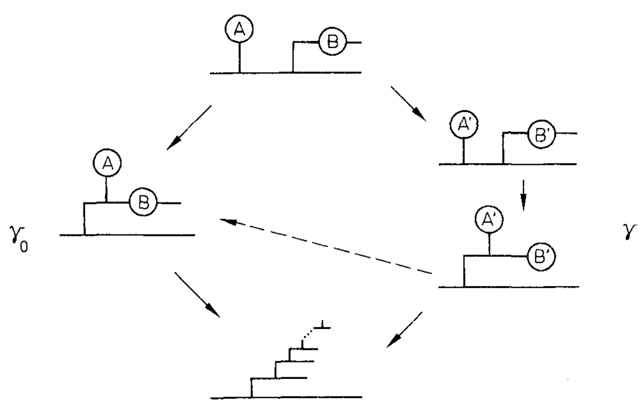


Fig. 30. An exceptional case

we do not move the two auxiliary external legs i_0 and i_{n+1}. Every layer has exactly one multi-peripheral diagram and one staircase diagram. Since $\Gamma_{0,n}$ is an infinite group, there is an infinite number of layers. However, since the symmetric group of n objects is finite, there are only $n!$ different orderings, and hence ϕ^3 diagrams in different layers might look the same. They differ by 2π rotations of one vertex around another. The transformations in $\Gamma_{0,n}$ which permute the holes change the form of the ϕ^3 diagram. The edges of $\mathscr{C}_n$ correspond to the simple moves F and Ω.

We use $\Omega(\pm)$ according to whether the move corresponds to $\omega(\pm)$ [$\omega(\pm)$ is defined below] in the modular group. Clearly, this complex is connected. We would like to show that if we fill the faces corresponding to the pentagon (B.1) and the two hexagons (4.6)

$$F(\Omega(\varepsilon)\otimes 1)F=(1\otimes\Omega(\varepsilon))F(1\otimes\Omega(\varepsilon)) \tag{B.3}$$

with $\varepsilon=\pm 1$, the resulting two complex is simply connected. Using the previous result, every layer of the complex is simply connected. We should only examine the closed loops which go between the layers. It is convenient to define the two braiding moves of Eq. (4.2),

$$B(\varepsilon)=F^{-1}(1\otimes\Omega(-\varepsilon))F \tag{B.4}$$

and to view them as basic moves,[15] i.e. to add the corresponding edges to $\mathscr{C}_n$. The braiding/fusing identity (4.3) [which follows from (B.1) and (B.3)]

$$P_{23}B_{13}(\varepsilon)F_{12}=F_{23}B_{12}(\varepsilon)B_{23}(\varepsilon) \tag{B.5}$$

expresses the fact that braiding of fused lines is performed by the same B as for external lines. Therefore, by using this equation, every closed loop of F's and Ω's can be deformed to a closed loop of the following form. Every link between two different layers is a B move and it starts and ends at a multi-peripheral diagram (even though the braiding of the first two external lines i_1 and i_2 is represented by Ω we can equivalently represent it by B). Using the fact that every layer is simply connected, we can use the pentagon to shrink that part of the loop which starts and ends at the multi-peripheral diagram and stays within the layer. Thus, we have deformed every closed loop in $\mathscr{C}_n$ to a closed loop involving only B's between multiperipheral diagrams. Every such loop corresponds to a relation in the modular group of the n-holed sphere, $\Gamma_{0,n}$. Therefore, it is enough to check that our equations guarantee that this group is properly represented. Thus, the next task is to describe the generators and relations of $\Gamma_{0,n}$.

To begin, order the points so that $|z_1|>|z_2|>\ldots>|z_n|$. We may then take generators to be Dehn twists R_i around each of the holes, or, rotations of the local parameters z_i by 2π, together with interchanges $\omega_i(\varepsilon)$ of holes i, $i+1$ defined as follows. Cut out a disk containing z_i, z_{i+1} and no other points and choose a diffeomorphism which is one outside this disk and rotates the points by π. Finally, undo any rotation of the local parameter that may have occurred in the process (this is well-defined for small disks). The sign $\varepsilon=\pm 1$ determines the orientation of the interchange, clearly, $\omega_i\equiv\omega_i(+)=\omega_i(-)^{-1}$. The relations satisfied by these generators are[16]

(A) $\omega_i\omega_j=\omega_j\omega_i$, $\quad |i-j|>1$.
$\omega_i R_j=R_j\omega_i$,
(B) $\omega_i\omega_{i+1}\omega_i=\omega_{i+1}\omega_i\omega_{i+1}$.
(C) $\omega_1\ldots\omega_{n-1}^2\ldots\omega_1=R_1^2$.
(D) $(\omega_1\ldots\omega_{n-1})^n=\prod_i R_i$.

---

[15] We could have alternatively defined the braiding moves by Eq. (4.4): $B(\varepsilon)=(\Omega(-\varepsilon)\otimes 1)F(1\otimes\Omega(\varepsilon))$. Because of the two hexagons (B.3), these two definitions are equivalent

[16] Relation (C) was incorrectly stated in [11]

We may derive these relations as follows. Relations (A) are obvious. To make further progress we relate $\Gamma_{0,n}$ to the modular group Γ_0^n of n punctures on the sphere by the exact sequence:[52]:

$$1 \to \mathbf{Z}^n \to \Gamma_{0,n} \to \Gamma_0^n \to 1\,,$$

where the right arrow corresponds to shrinking the holes to points. We then lift the relations of Γ_0^n to $\Gamma_{0,n}$. The relations of Γ_0^n may be found in [53] and are closely related to the braid group of the sphere, defined as π_1 of the space $(S^2 \times \ldots \times S^2 - \text{diag})/\Sigma_n$, where Σ_n is the permutation group. The correspondence is given by associating to ω_i the rotation of points i, $i+1$ induced by a homotopy of the diffeomorphism to one through diffeomorphisms which do not preserve the set of points $z_1, \ldots, z_n$. The relations (B) are the famous braid relations. Relation (C) corresponds to the braiding shown in Fig. 31. On the sphere the element of π_1 is given by a circuit of z_1 around all the other points and is therefore homotopically trivial since it can be deformed off the back of the sphere. Thus, in Γ_0^n the right-hand side of (C) is 1. When lifting this relation to $\Gamma_{0,n}$ we must be careful since the punctures now have tangent vectors attached and the tangent bundle on the sphere is nontrivial. Changing $z \to w = 1/z$ with $w=0$ at infinity shows that for a loop around ∞ the tangent vector $\partial/\partial z = -w^2 \partial/\partial w$ rotates by 4π proving relation (C). Similarly, relation (D), which corresponds to the "barber-pole" braidings in Fig. 32 has 1 on the right-hand side as a relation in Γ_0^n and so must be given by factors of R in $\Gamma_{0,n}$. Following the behaviour of local parameters we arrive at (D).

Before proceeding we remark that if we discuss the modular group of labeled holes we obtain the analog of the pure braid group – defined as the group of braidings that does not permute the strings. This modular group is the subgroup $\tilde{\Gamma}_{0,n}$ of $\Gamma_{0,n}$ such that $\Sigma_n = \Gamma_{0,n}/\tilde{\Gamma}_{0,n}$ is the symmetric group. In a classical conformal

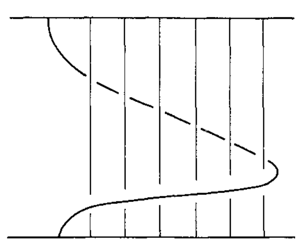


Fig. 31. A pictorial version of relation C on the sphere

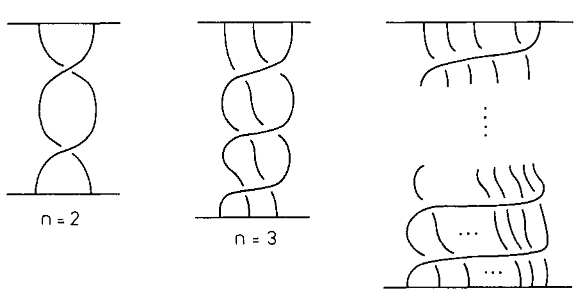


Fig. 32. A pictorial version of relation D on the sphere

field theory $\tilde{\Gamma}_{0,n}$ is realized trivially. Hence, Σ_n is the classical version of $\Gamma_{0,n}$. For the completeness of the classical equations, it is enough to check that the closed loops realize correctly the defining relations of the symmetric group Σ_n. This was done by MacLane [51].

We now return to our proof. The advantages of deforming every closed loop to the multi-peripheral diagrams are that there every B transformation is in $\Gamma_{0,n}$ and every generator of $\Gamma_{0,n}$ has a simple realization. In the multiperipheral basis we identify the space of characters with

$$\oplus V^0_{i_1 i_1} \otimes V^{i_1}_{i_2 p_2} \otimes \ldots \otimes V^{i_n}_{i_n 0}\,. \tag{B.6}$$

The representation of the generators is then given by:

$$\begin{aligned} \varrho(\omega_l(\varepsilon)) &= B_{p_l}\begin{bmatrix} i_l & i_{l+1} \\ p_{l-1} & p_{l+1} \end{bmatrix}(\varepsilon)\,, \\ \varrho(R_k) &= 1\otimes\ldots\otimes e^{-2\pi i \Delta_{i_k}}\otimes\ldots\otimes 1\,, \end{aligned} \tag{B.7}$$

where the indices indicate which subspace of (B.6) the transformation acts on. Relation (A) is trivially satisfied. Relation (B) follows from the Yang-Baxter equation (4.8) applied to three successive lines,

$$B_{12}(\varepsilon)B_{23}(\varepsilon)B_{12}(\varepsilon) = B_{23}(\varepsilon)B_{12}(\varepsilon)B_{23}(\varepsilon)\,. \tag{B.8}$$

Recall that (B.8) follows from (B.1) and (B.3). Relation (C) may be easily proved by iterating the identity

$$B(\varepsilon)\,(\Omega^2(\varepsilon)\otimes 1)B(\varepsilon)\,(1\otimes\Omega^2(-\varepsilon)) = 1\,. \tag{B.9}$$

A better proof uses (B.5) repeatedly to express the sequence of braidings

$$B\begin{bmatrix} i_1 & i_2 \\ & \end{bmatrix}(\varepsilon)B\begin{bmatrix} i_1 & i_3 \\ & \end{bmatrix}(\varepsilon)\ldots B\begin{bmatrix} i_1 & i_k \\ & \end{bmatrix}(\varepsilon) \tag{B.10}$$

in terms of a single braiding, that is, as

$$\underbrace{F\begin{bmatrix} i_2 & i_3 \\ & \end{bmatrix}\ldots F\begin{bmatrix} * & i_k \\ & \end{bmatrix}}_{k} B\begin{bmatrix} i_i & q \\ & \end{bmatrix} \underbrace{F\begin{bmatrix} * & i_k \\ & \end{bmatrix}^{-1}\ldots F\begin{bmatrix} i_2 & i_3 \\ & \end{bmatrix}^{-1}}_{k} \tag{B.11}$$

as in Fig. 33.

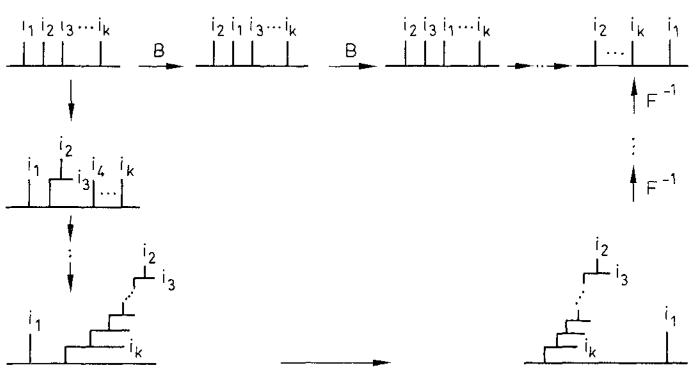


Fig. 33. Expressing a product of braidings in terms of a single braiding

Thus we have

$$\begin{aligned}\varrho(\omega_1)\ldots\varrho(\omega_{n-1}^2)\ldots\varrho(\omega_1) &= B\begin{bmatrix} i_1 & i_2 \\ & \end{bmatrix}\ldots B\begin{bmatrix} i_1 & i_n \\ & \end{bmatrix} B\begin{bmatrix} i_n & i_1 \\ & \end{bmatrix}\ldots B\begin{bmatrix} i_2 & i_1 \\ & \end{bmatrix}\\ &= (\textstyle\prod F) B\begin{bmatrix} i_1 & i_1 \\ 0 & 0 \end{bmatrix}(+)^2(\textstyle\prod F^{-1})\\ &= (\textstyle\prod F)\, e^{-4\pi i \Delta_{i_1}}(\textstyle\prod F^{-1})\\ &= e^{-4\pi i \Delta_{i_1}} = \varrho(R_1^2)\,. \end{aligned} \tag{B.12}$$

Thus proving relation (C). Intuitively we may represent the proof as in Fig. 34.

We now prove (D) by induction. For $n=2$ it is the same as (C). For $n=3$ we may also easily compute that on $V_{ii'}^0 \otimes V_{jk}^{i'} \otimes V_{k0}^k$ the representation of $(\omega_1\omega_2)^3$ is (define $\Delta_{ijk} \equiv \Delta_i + \Delta_j - \Delta_k$)

$$e^{-i\pi\Delta_{ijk}} e^{-i\pi\Delta_{ikj}} e^{-i\pi\Delta_{jki}} e^{-i\pi\Delta_{jik}} e^{-i\pi\Delta_{kij}} e^{-i\pi\Delta_{kji}} = e^{-2\pi i(\Delta_i+\Delta_j+\Delta_k)}\,. \tag{B.13}$$

For simplicity, we consider in this appendix the case where the eigenvalues of the permutations σ are all $\xi = +1$, and hence $\Omega_{jk}^i = e^{i\pi\Delta_{jki}}$. The generalization to arbitrary $\xi = \pm 1$ is straightforward. To proceed we may use the same trick as for (C) by fusing the last two strings and using (B.5) repeatedly to obtain Fig. 35. From

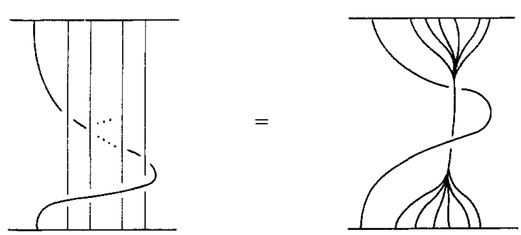


Fig. 34. A proof of relation C on the sphere

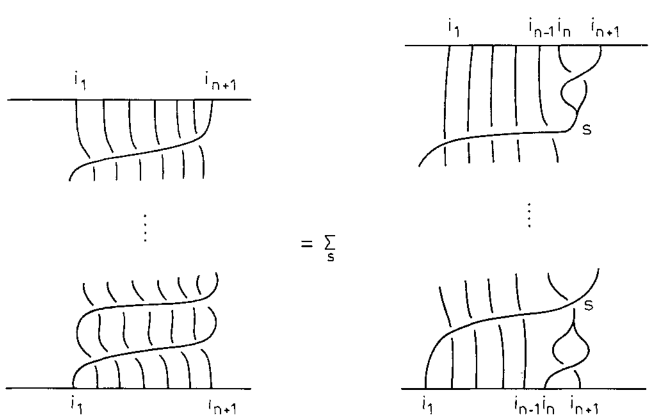


Fig. 35. An inductive proof of relation D on the sphere

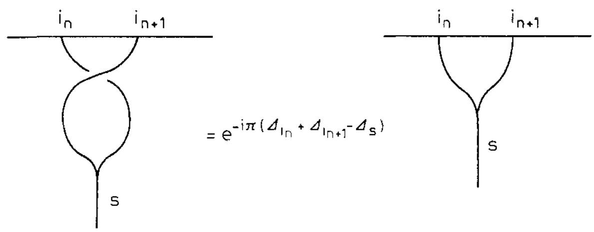


Fig. 36. A relation used in the proof of D on the sphere

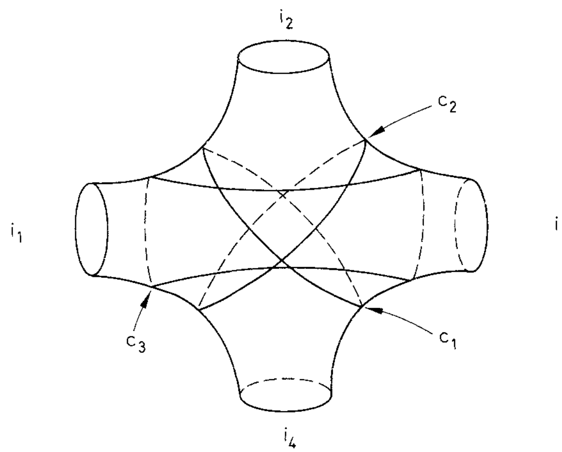


Fig. 37. A four holed sphere for the lantern identity

the relation of B to F and (4.6) we obtain Fig. 36. Therefore, the induction hypothesis and Fig. 35 gives:

$$e^{-2\pi i\varepsilon(\Delta_{i_{n+1}}+\Delta_{i_n}-\Delta_s)}e^{-2\pi i\varepsilon(\Delta_{i_1}+\ldots+\Delta_{i_{n-1}}+\Delta_s)}$$
$$=e^{-2\pi i\varepsilon(\Delta_{i_1}+\ldots+\Delta_{i_n}+\Delta_{i_{n+1}})}. \tag{B.14}$$

This completes the proof of the modular relations at $g=0$.

One of the relations of $\Gamma_{0,4}$, known as the Chinese lantern identity, will be useful in verifying one of the modular relations at high genus, so we describe this one in detail. The relation states that for the Dehn twists C_1, C_2, C_3 in Fig. 37 we have

$$C_3C_2C_1=R_1^{-1}R_2^{-1}R_3^{-1}R_4^{-1}. \tag{B.15}$$

This can be derived from (A–D) for $n=4$ since we have

$$R_1R_2R_3R_4=(\omega_1\omega_2\omega_3)^4=\omega_2^2\omega_1^2\omega_1^{-1}\omega_2^2\omega_1\omega_3\omega_2\omega_1^2\omega_2\omega_3,$$

hence

$$(\omega_2^2)(\omega_1^2)(\omega_1^{-1}\omega_2^2\omega_1)=R_1R_2R_3R_4^{-1}, \tag{B.16}$$

but we have $C_1 = R_2^{-1}R_3^{-1}\omega_2^2$, $C_2 = R_1^{-1}R_2^{-1}\omega_1^2$ and $C_3 = R_1^{-1}R_3^{-1}\omega_1^{-1}\omega_2^2\omega_1$. In fact, (B.16) is easily checked directly since on $\oplus V_{i_2 p}^{i_1} \otimes V_{i_3 i_4}^{p}$, we have

$$
\begin{aligned}
&\varrho((\omega_2)^2(\omega_1^2)(\omega_1^{-1}\omega_2^2\omega_1))\\
&= B\begin{bmatrix} i_2 & i_3 \\ i_1 & i_4 \end{bmatrix}(+)B\begin{bmatrix} i_3 & i_2 \\ i_1 & i_4 \end{bmatrix}(+)(\Omega^2(+)\otimes 1)B\begin{bmatrix} i_2 & i_3 \\ i_1 & i_4 \end{bmatrix}(+)\\
&\quad \times(\Omega^2(+)\otimes 1)B\begin{bmatrix} i_3 & i_2 \\ i_1 & i_4 \end{bmatrix}(+)e^{-4\pi i(\Delta_{i_2}+\Delta_{i_3})}\\
&= B\begin{bmatrix} i_2 & i_3 \\ i_1 & i_4 \end{bmatrix}(+)(\Omega^2(+)\otimes\Omega^2(+))B\begin{bmatrix} i_3 & i_2 \\ i_1 & i_4 \end{bmatrix}(+)e^{-4\pi i(\Delta_{i_2}+\Delta_{i_3})}.
\end{aligned}
\tag{B.17}
$$

But we may replace $(\Omega^2(+)\otimes\Omega^2(+))$ by $e^{2\pi i(\Delta_{i_3}+\Delta_{i_2}+\Delta_{i_1}-\Delta_{i_4})}$ in (B.17), giving (B.16).

II. Genus One

The strategy here will be similar to that at genus zero. We will define a simplicial complex and then show that we can deform any loop to a loop in the multiperipheral basis. Loops in the multiperipheral basis correspond to relations in the modular group $\Gamma_{1,n}$. Therefore, we begin by checking the relations of $\Gamma_{1,n}$ in the multiperipheral basis.

If (τ, z) are coordinates on the Teichmuller space of the one-holed torus (where z lies in the complex plane identified by $z \sim qz$) then $\Gamma_{1,1}$ is generated by

$$
\begin{aligned}
S&:\tau\to -1/\tau\,, \qquad \log z\to \log z/\tau\,,\\
T&:\tau\to\tau+1\,, \qquad R: z\to e^{2\pi i}z\,.
\end{aligned}
\tag{B.18}
$$

The relations are $S^4 = R^{-1}$ and $(ST)^3 = S^2$. We have described the representation of these generators in Sect. 4, and the relations are guaranteed by (4.18a, b).

For $n \geqq 2$ we choose and ordering of points so that $1 > |z_1| > |z_2| > \ldots > |z_n| > |q|$. Then $\Gamma_{1,n}$ is generated by:

$$
\begin{aligned}
&S:\tau\to -1/\tau\,, \qquad \log z_i\to \log z_i/\tau\,,\\
&T:\tau\to\tau+1\,, \qquad a_i: z_j\to z_j\,, \qquad 1\leqq j\leqq (i-1)\,,\\
&z_j\to e^{-2\pi i}z_j\,, \qquad i\leqq j\leqq n\,, \qquad b_i: z_j\to z_j\,, \qquad 1\leqq j\leqq (i-1)\,,\\
&z_j\to q^{-1}z_j\,, \qquad i\leqq j\leqq n\,, \qquad R_i: dz_i\to e^{2\pi i}dz_i\,.
\end{aligned}
\tag{B.19}
$$

As on the plane, the relations of $\Gamma_{1,n}$ are obtained from the case with punctures Γ_1^n [54] by lifting the relations using the exact sequence $1\to \mathbf{Z}^n \to \Gamma_{1,n}\to\Gamma_1^n\to 1$.

For $n=2$ the relations are (set $a_2 = a$ and $b_2 = b$, since $a_1 = b_1 = 1$):

$$
\begin{aligned}
(ST)^3 &= S^2\,, \qquad S^4 = b^{-1}aba^{-1}R_1^{-1}R_2^{-1}\,,\\
SaS^{-1} &= b\,, \qquad SbS^{-1} = b^{-1}a^{-1}b\,,\\
TaT^{-1} &= a\,, \qquad TbT^{-1} = ab\,.
\end{aligned}
\tag{B.20}
$$

As explained in Sect. 4 we may choose to represent these generators in the multiperipheral basis:

$$
\operatorname{Tr} q^{L_0 - c/24}[\Phi_{ip}^{j_1}(z_1)\Phi_{pi}^{j_2}(z_2)]\,(dz_1)^{\Delta_{j_1}}(dz_2)^{\Delta_{j_2}}\,,
\tag{B.21}
$$

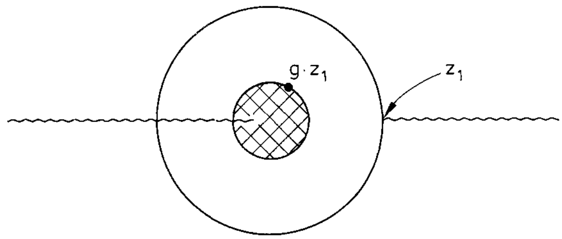


Fig. 38. Cuts used for the torus conformal blocks. The cuts inside the hatched region are complicated but irrelevant

where we choose cuts as in Fig. 38. Thus we think of the space of characters as $\oplus V^i_{j_1 p} \otimes V^p_{j_2 i}$. In the notation used for the calculations of this appendix we have the representation

$$\begin{aligned} &\varrho(a) = D_i^2 D_p^{-2}, \quad \varrho(b^{-1}) = B_p\begin{bmatrix} j_1 & j_2 \\ i & i \end{bmatrix}(-)P_{ip}, \\ &\varrho(T) = D_i^2 e^{-2\pi i c/24}, \quad \varrho(S) = F_p\begin{bmatrix} j_1 & j_2 \\ i & i \end{bmatrix} S_i(p) F_p\begin{bmatrix} j_1 & j_2 \\ i & i \end{bmatrix}^{-1}, \end{aligned} \tag{B.22}$$

where $D_i = e^{i\pi\Delta_i}$ and P_{ip} is a permutation operator on the indicated indices. The relations are easily checked from (4.18a, b) once one notices that for the two-point function we have the simple expression:

$$\begin{aligned} \varrho(S)^2 &= P_{ip} e^{-i\pi(\Delta_{j_1}+\Delta_{j_2})} \sigma_{ik} \otimes \sigma_{ik} B_p\begin{bmatrix} j_2 & j_1 \\ i & i \end{bmatrix}(-) \\ &= B_p\begin{bmatrix} j_1 & j_2 \\ i & i \end{bmatrix}(-)P_{ip} e^{-i\pi(\Delta_{j_1}+\Delta_{j_2})} \sigma_{ik} \otimes \sigma_{ik} . \end{aligned} \tag{B.23}$$

For example one can compute S^4 and compare this with $b^{-1}aba^{-1}$ which gives a monodromy of z_1 around z_2, just as on the plane. Similarly, the other relations are straightforwardly checked.

For the case $n \geqq 3$ we have the relations $Sa_iS^{-1} = b_i$, $Sb_iS^{-1} = b_i^{-1}a_i^{-1}b_i$, etc. In addition we have the relations

$$\begin{aligned} &[b_k^{-1}, a_k a_j^{-1}] = [(b_k^{-1}b_j), a_k] \equiv N_{jk}, \quad 1 \leqq j < k \leqq n, \\ &[a_i, N_{jk}] = [b_i, N_{jk}] = 1, \quad 1 \leqq i \leqq j < k \leqq n, \\ &S^4 = N_{12}N_{23}N_{34}\ldots N_{n-1,n}R_1^{-1}\ldots R_n^{-1}, \end{aligned} \tag{B.24}$$

where $[\,,]$ denotes the commutator: $[g_1, g_2] = g_1 g_2 g_1^{-1} g_2^{-1}$. We check the above relations in the multiperipheral basis, identifying the space of characters with

$$\oplus V^i_{j_1 p_1} \otimes V^{p_1}_{j_2 p_2} \otimes \ldots \otimes V^{p_{n-2}}_{j_{n-1} p_{n-1}} \otimes V^{p_{n-1}}_{j_n i}$$

(see Fig. 39). In this basis we represent $\varrho(a_l) = D_i^2 D_{p_{l-1}}^{-2}$, we represent S by fusing all strings, using the S of the one-point function and then defusing. For b_l we fuse lines 1 to $l-1$ and l to n separately and then use the representation of b from the two-point function. With this representation we can check the first two relations in (B.24) by fusing lines to obtain three-point functions on the torus, where they may

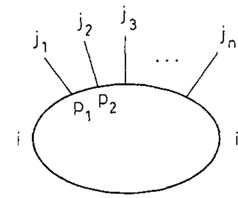


Fig. 39. The multiperipheral basis for the torus

be explicitly checked. Each of the N_{jk} has a representation purely in terms of operations on the plane-namely, of braiding the group of lines j to $k-1$ around k to n. From this we may conclude that the right-hand side of the last equation in (B.24) is the planar representation of a 2π Dehn twist at "infinity" as is consistent with the value of S^4 from the one-point function and the definition of S for the n-point function.

We now proceed to the full duality complex $\mathscr{C}_{1,n}$. The vertices of $\mathscr{C}_{1,n}$ are pairs $(\mathscr{D},\gamma)$, where $\mathscr{D}$ is an ordered one-loop ϕ^3 diagram and $\gamma\in\Gamma_{1,n}$. $\mathscr{C}_{1,n}$ is connected if we add to F, $\Omega(\varepsilon)$ four new simple moves (three suffice for connectivity). We say a diagram $\mathscr{D}$ is of tyke k if k lines join directly onto the loop. The first two moves are defined for diagrams of type 1 and connect $(\mathscr{D},\gamma)$ to $(\mathscr{D},\gamma\cdot S)$ or $(\mathscr{D},\gamma\cdot T)$, where S, T are the standard generators of $\Gamma_{1,0}$. Furthermore, choosing a marking for the torus as above we can define a and b simple moves for diagrams of type 2 which correspond to the modular transformations defined above. Next we connect $(\mathscr{D},\gamma)$ to $(\mathscr{D},\gamma\cdot a)$ and $(\mathscr{D},\gamma)$ to $(\mathscr{D}',\gamma\cdot b)$, where $\mathscr{D}'$ is obtained by cycling an external line around the loop. By fusing or braiding we can always deform a loop in $\mathscr{C}_{1,n}$ to a loop in the multiperipheral basis *without using the cyclicity move.* There is no ambiguity in this deformation since – by avoiding the cyclicity move all such manipulations can be performed on the plane, where we have demonstrated the lack of ambiguity. Thus all loops can be deformed to loops entirely within the multiperipheral basis, which, as we have said correspond to relations in the modular group. Since all the relations of $\Gamma_{1,n}$ are satisfied, we have shown that $\pi_1(\mathscr{C}_{1,n})=1$, so all one-loop constraints are summarized by (4.18).

III. High Genus

Again the strategy will be the same. We first check the relations of the modular group in the "multiperipheral" basis for the high genus characters. After that we define a complex and argue that we can always deform a loop in that complex to a loop in the multiperipheral basis. For simplicity we will consider the modular group for genus g with no punctures. Remarks on the case of punctures are at the end of this appendix.

One set of generators of $\Gamma_{g,0}$ is given by Dehn twists about the curves $\langle\alpha_i,\beta_i,\delta_2\rangle$ shown in Fig. 40. It is convenient to define auxiliary curves ε_i, δ_i, see Fig. 41. Identifying the span of $\mathscr{F}$ with

$$\oplus V^{0}_{i_1 i_1}\otimes V^{i_1}_{i_1 j_1}\otimes V^{j_1}_{i_2 k_2}\otimes V^{k_2}_{i_2 j_2}\otimes\ldots\otimes V^{j_{n-1}}_{i_n i_n}\otimes V^{i_n}_{i_n 0} \qquad \text{(B.25)}$$

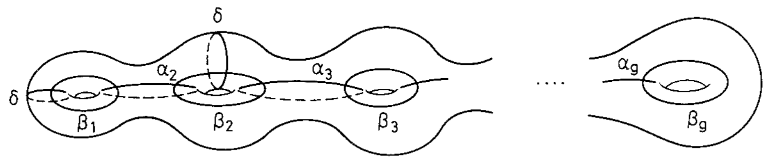


Fig. 40. A choice of Dehn twists generating the modular group

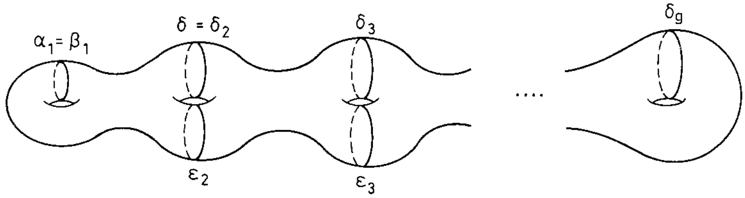


Fig. 41. Other Dehn twists used in the proof

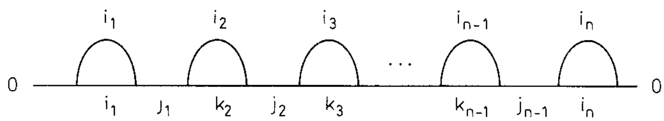


Fig. 42. Characters in the multiperipheral basis at high genus

for the basis of characters in Fig. 42, we have the following representation of the generators:

$$\varrho(\alpha_1) = e^{-2\pi i(\Delta_{i_1} - c/24)},$$

$$\varrho(\alpha_2) = D_{i_1}^{-2}\left(B\begin{bmatrix} i_1 & i_2 \\ i_1 & k_2 \end{bmatrix}(-)B\begin{bmatrix} i_2 & i_1 \\ i_1 & k_2 \end{bmatrix}(-)\right)D_{i_2}^{-2},$$

$$\begin{aligned}\varrho(\alpha_l) &= \mathrm{D}_{i_{l-1}}^{-2}\left(B_{j_{l-1}}\begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix}(-)B_{j_{l-1}}\begin{bmatrix} i_l & i_{l-1} \\ k_{l-1} & k_l \end{bmatrix}(-)\right)D_{i_l}^{-2} \\ &= F_{j_{l-1}}\begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix} e^{-2\pi i \Delta_{j_{l-1}}} F_{j_{l-1}}\begin{bmatrix} i_{l-1} & i_l \\ k_{l-1} & k_l \end{bmatrix}^{-1},\end{aligned} \tag{B.26}$$

$$\varrho(\beta_l) = T_{k_l} S_{k_l i_l}(j_{l-1}, j_l) T_{k_l} \equiv T_{k_l} F_{i_l}\begin{bmatrix} j_{l-1} & j_l \\ k_l & k_l \end{bmatrix} S_{k_l}(i_l) F_{i_l}\begin{bmatrix} j_{l-1} & j_l \\ k_l & k_l \end{bmatrix}^{-1} T_{k_l},$$

$$\varrho(\delta_2) = e^{-2\pi i(\Delta_{i_2} - c/24)},$$

where, as usual, the indices on the linear transformations B, F, S, T tell which of the subspaces of (B.25) it acts on.

The relations of $\Gamma_{g,0}$ are [55]:
(A) $\alpha\beta\alpha = \beta\alpha\beta$ if α, β intersect, otherwise they commute.
(B) $(\delta_2\beta_2\alpha_2\alpha_1^2\beta_1\alpha_2\delta_2)^2 = 1$ at $g=2$.
(C) $(\alpha_1\beta_1\alpha_2)^4 = \varepsilon_2\delta_2$.
(D) $b_2\delta_2 b_1 = \alpha_1\alpha_2\alpha_3\delta_3$, where

$$b_1 = (\beta_2\alpha_2\alpha_3\beta_2)^{-1}\delta_2(\beta_2\alpha_2\alpha_3\beta_2),$$
$$b_2 = (\beta_1\alpha_1\alpha_2\beta_1)^{-1} b_1(\beta_1\alpha_1\alpha_2\beta_1).$$

(E) $I\delta_g I^{-1} = \varepsilon_g$, where $I = \beta_g\alpha_g \ldots \beta_1\alpha_1^2\beta_1 \ldots \alpha_g\beta_g$.

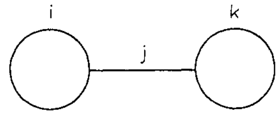


Fig. 43. A basis of two-loop characters

The relations (A) follow by definition or from corresponding relations for the two- and three-point functions on the torus.

We now check (B) as follows. We use the basis of characters of Fig. 43, identifying their span with $V^{2\text{-loop}} = \bigoplus_{i,j,k} V^i_{ji} \otimes V^k_{jk}$. Denoting by $S_i(j)$ the transformation $\bigoplus_{j,k} S(j) \otimes 1$ etc. The above representation may be written:

$$
\begin{aligned}
&\varrho(\delta) = T_k^{-1}, \qquad \varrho(\beta_2) = T_k S_k(j) T_k, \\
&\varrho(\beta_1) = T_i S_i(j) T_i, \qquad \varrho(\alpha_1) = T_i^{-1}, \\
&\qquad \varrho(\alpha_2) = D_i^{-2} B^2(-) D_k^{-2}.
\end{aligned}
\tag{B.27}
$$

Writing out the representation of the generators we therefore find

$$
\begin{aligned}
&(e^{-2\pi i c/24})^{/8} S_k \left(B\begin{bmatrix} i & k \\ i & k \end{bmatrix}(-) B\begin{bmatrix} k & i \\ i & k \end{bmatrix}(-) \right) \\
&\quad \times (S_i)^2 \left(B\begin{bmatrix} i & k \\ i & k \end{bmatrix}(-) B\begin{bmatrix} k & i \\ i & k \end{bmatrix}(-) \right) (S_k)^2 , \\
&\quad \times \left(B\begin{bmatrix} i & k \\ i & k \end{bmatrix}(-) B\begin{bmatrix} k & i \\ i & k \end{bmatrix}(-) \right) (S_i)^2 \left(B\begin{bmatrix} i & k \\ i & k \end{bmatrix}(-) B\begin{bmatrix} k & i \\ i & k \end{bmatrix}(-) \right) S_k .
\end{aligned}
\tag{B.28}
$$

Using $S_i^2 = S_k^2 = \sigma_{13} D_j^{-1}$ and (4.6) (more conveniently expressed in terms of B) we get:

$$
\begin{aligned}
&\pm (e^{-2\pi i c/24})^8 S_k(j) D_j^- \left(B_j \begin{bmatrix} i & k \\ i & k \end{bmatrix}(-) D_j^{-2} B_j \begin{bmatrix} k & i \\ i & k \end{bmatrix}(-) \right) \\
&\quad \times D_j^{-2} \left(B_j \begin{bmatrix} i & k \\ i & k \end{bmatrix}(-) D_j^{-2} B_j \begin{bmatrix} k & i \\ i & k \end{bmatrix}(-) \right) e^{4\pi i(\Delta_i + \Delta_k)} S_k(j) .
\end{aligned}
\tag{B.29}
$$

Repeatedly using (B.9) in the form

$$
D_j^{-2} B_j \begin{bmatrix} j & l \\ i & k \end{bmatrix} D_j^{-2} B_j \begin{bmatrix} l & j \\ i & k \end{bmatrix} = e^{-2\pi i(\Delta_i + \Delta_k)} ,
$$

we may reduce this to

$$
\begin{aligned}
&\pm (e^{-2\pi i c/24})^8 S_k(j) D^+ e^{-2\pi i(\Delta_i + \Delta_k)} e^{-2\pi i(\Delta_i + \Delta_k)} e^{4\pi i(\Delta_i + \Delta_k)} S_k(j) \\
&\quad = (e^{-2\pi i c/24})^8 .
\end{aligned}
\tag{B.30}
$$

The phase factor from the central extension is to be expected since the presence of a central term indicates we only have a projective representation, as expected from general considerations [5]. This verifies condition (B).

Before proceeding, note that from (A.4) (or, more generally, from factorization) it follows that $\varrho(\delta_l)=e^{-2\pi i \Delta_{i_l}}$ and $\varrho(\varepsilon_l)=e^{-2\pi i \Delta_{k_l}}$. Since δ_l, ε_l may be expressed in terms of the generators α_i, β_i, δ_2 we obtain nontrivial relations on F, S. Our first task is to show that these relations are implied by those of Sect. 4. Such relations all follow from the three-loop identity $\delta_{l+1}=J\delta_{l-1}J^{-1}$, where

$$J=\beta_{l+1}\alpha_{l+1}(\beta_l\delta_l\alpha_l\beta_l)\alpha_{l+1}\beta_{l+1}\cdot\beta_{l-1}\alpha_l(\beta_l\delta_l\alpha_{l+1}\beta_l)\alpha_l\beta_{l-1}\,.$$

(The reader may easily verify this pictorially.) If we verify this for an arbitrary two-point character at three loops, then, since α_1, δ are represented as in (B.26) we can use this identity to prove iteratively the relations implied by the above equations for $\varrho(\delta_l)$, $\varrho(\varepsilon_l)$. To verify the three-loop identity we use $1\otimes F\otimes F\otimes 1$ to pass from the multiperipheral basis

$$V^{3\text{-loop}}=\bigoplus V^{j_0}_{i_1k_1}\otimes V^{k_1}_{i_1j_1}\otimes V^{j_1}_{i_2k_2}\otimes V^{k_2}_{i_2j_2}\otimes V^{j_2}_{i_3k_3}\otimes V^{k_3}_{i_3j_3} \tag{B.31}$$

to the basis

$$V^{3\text{-loop}}=\bigoplus V^{j_0}_{i_1k_1}\otimes V^{k_1}_{p_1k_2}\otimes V^{p_1}_{i_1i_2}\otimes V^{k_2}_{p_2k_3}\otimes V^{p_2}_{i_2i_3}\otimes V^{k_3}_{i_3j_3}\,. \tag{B.32}$$

This basis of characters is illustrated in Fig. 44. We denote, e.g. $S_{k_1i_1p_1}$ to be the action of S on the embedded one-loop three-point function in Fig. 44 with those indices. Mathematically, we pass from $\bigoplus V^{j_0}_{i_1k_1}\otimes V^{k_1}_{p_1k_2}\otimes V^{p_1}_{i_1i_2}$ to the isomorphic space $\bigoplus V^{k_1}_{j_0i_1}\otimes V^{i_1}_{i_2p_1}\otimes V^{p_1}_{k_2k_1}$ by rewriting characters expressed as sewn planar amplitudes as characters expressed as a trace of three chiral vertex operators. We then use fusing to define S to be

$$S_{k_1i_1p_1}=F_{p_1}\begin{bmatrix} i_2 & k_2 \\ i_1 & k_1 \end{bmatrix}F_{i_1}\begin{bmatrix} j_0 & p_1 \\ k_1 & k_1 \end{bmatrix}S_{k_1}(i_1)F_{i_1}\begin{bmatrix} j_0 & p_1 \\ k_1 & k_1 \end{bmatrix}^{-1}F_{p_1}\begin{bmatrix} i_2 & k_2 \\ i_1 & k_1 \end{bmatrix}^{-1}, \tag{B.33}$$

using the path illustrated in Fig. 45.

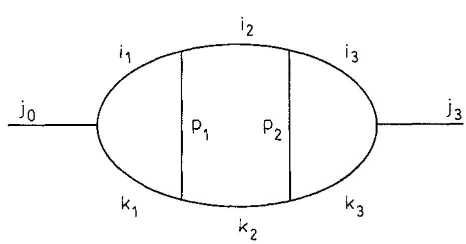


Fig. 44. Alternate basis of characters for a two-point function at three loops

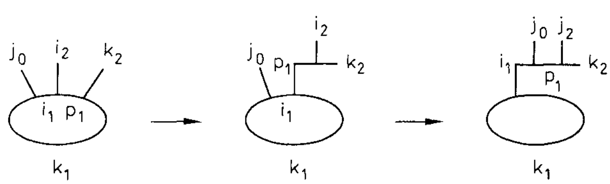


Fig. 45. Defining S for a subloop

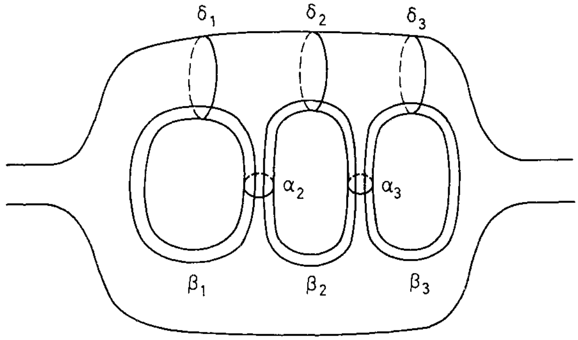


Fig. 46. Dehn twists in a two holed three loop surface

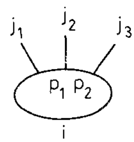


Fig. 47. A three point function at one loop

With this understood we represent the Dehn twists in Fig. 46 by

$$\begin{gathered}\beta_3 = T_{k_3} S_{k_3 p_2 i_3} T_{k_3}, \qquad \beta_2 = T_{k_2} S_{k_2 p_1 i_2 p_2} T_{k_2}, \\ \beta_1 = T_{k_1} S_{k_1 i_1 p_1} T_{k_1}, \qquad \delta_2 = \mathrm{T}_{i_2}^{-1}, \\ \alpha_2 = T_{p_1}^{-1}, \qquad \alpha_3 = \mathrm{T}_{p_2}^{-1}.\end{gathered} \tag{B.34}$$

We now evaluate $J = \beta_3\alpha_3(\beta_2\delta_2\alpha_2\beta_2)\alpha_3\beta_3\beta_1\alpha_2(\beta_2\alpha_3\delta_2\beta_2)\alpha_2\beta_1$ by first writing out $\beta_1\alpha_2(\beta_2\alpha_3\delta_2\beta_2)\alpha_2\beta_1$. To begin we need an identity from the torus. Beginning with characters $\oplus V^i_{j_1 p} \otimes V^p_{j_2 i}$ for the two-point function we have

$$S T_i T_p^{-1} S = P_{ip} e^{-i\pi(\Delta_{j_1} + \Delta_{j_2})}, \tag{B.35}$$

where P_{ip} permutes the indicated indices, and

$$S^2 = B_p \begin{bmatrix} j_1 & j_2 \\ i & i \end{bmatrix} (-) P_{ip} \sigma_{13} \otimes \sigma_{13} e^{-i\pi(\Delta_{j_1} + \Delta_{j_2})}. \tag{B.36}$$

Using these one-loop equations and fusing we obtain an equation for S for the three-point function of Fig. 47,

$$S T_i T_{p_1}^{-1} T_i T_{p_2}^{-1} S = e^{-i\pi(\Delta_{j_1} + \Delta_{j_2} + \Delta_{j_3})} e^{-i\pi\Delta_i} B_i \begin{bmatrix} j_3 & j_1 \\ p_2 & p_1 \end{bmatrix} (+) e^{i\pi\Delta_i} P_{p_1 p_2}. \tag{B.37}$$

Using the relation, and similar torus relations we may reduce the transformation $\beta_1\alpha_2(\beta_2\alpha_3\delta_2\beta_2)\alpha_2\beta_1$ to

$$\begin{aligned} &T_{k_1} T_{k_2} D_{i_1} D^-_{k_1} D^-_{p_1} D^-_{j_0} D^-_{i_2} D^-_{k_2} P_{k_1 p_1} P_{i_2 p_2} F_{i_1} \begin{bmatrix} k_1 & p_2 \\ j_0 & p_1 \end{bmatrix} F_{p_1} \begin{bmatrix} k_1 & k_2 \\ i_1 & p_2 \end{bmatrix} F_{k_2} \begin{bmatrix} p_2 & p_1 \\ k_3 & i_2 \end{bmatrix} \\ &\times F_{i_1} \begin{bmatrix} p_1 & k_1 \\ k_1 & j_0 \end{bmatrix} F_{p_1} \begin{bmatrix} i_2 & k_2 \\ i_1 & k_1 \end{bmatrix} F_{i_1} \begin{bmatrix} p_1 & i_2 \\ j_0 & k_1 \end{bmatrix} P_{k_1 p_1} D_{i_1} D_{p_2} D_{p_1} D_{k_1} D^-_{k_2} D^-_{k_3} D^-_{i_3}. \end{aligned} \tag{B.38}$$

We may simplify this greatly by using the pentagon repeatedly. The net result is that $J^{-1} = WP_{i_1k_2}P_{i_2p_2}P_{i_3k_2}P_{i_2p_1}$, where W is a complicated transformation, but one which does not affect the representation i_1. Because of the permutation operators, it follows that J conjugates the representation of a Dehn twist on the line with i_3 to that on the line with i_1, completing the proof of the three-loop identity. Thus, the Dehn twists around δ_l are indeed just diagonal matrices of phases in our representation, and in using this fact we do not need any new identities beyond (4.18).

For the next step we show that

$$\beta_l\alpha_l\ldots\beta_1\alpha_1^2\beta_1\ldots\alpha_l\beta_l$$

is represented on the characters of Fig. 42 by

$$e^{-i\pi\Delta_{jl}+2\pi i(\Delta_{il}+\Delta_{kl})}P_{i_2k_2}P_{i_3k_3}\ldots P_{i_lk_l}.$$

(We ignore factors of $e^{-2\pi ic/24}$ since they drop out of the conjugation.) To prove this start with $\beta_1\alpha_1^2\beta_1 = D_{j_1}^-$, so from (B.35) we get

$$\beta_2\alpha_2\beta_1\alpha_1^2\beta_1\alpha_2\beta_2 = T_{k_2}S_{k_2i_2}D_{j_1}T_{k_2}T_{i_2}^{-1}S_{k_2i_2}T_{k_2} = D_{j_2}^-D_{i_2}^2D_{k_2}^2P_{i_2k_2}.$$

From here one proceeds by induction, using (4.6) and (B.35). Because of the permutation operators we can use this identity to establish $I\delta_lI^{-1} = \varepsilon_l$, so the curves ε_l are also represented by the appropriate phases, and then relation (C) is satisfied because it is a relation for the two-point function on the torus. Clearly, relation (E) is also satisfied.

Finally, relation (D) is simply the Chinese lantern identity (B.15) for the embedded four-holed sphere shown in Fig. 48. Since we have already checked the relation for the four-holed sphere, and since δ_2, δ_3, α_1, α_2, α_3 are represented as at

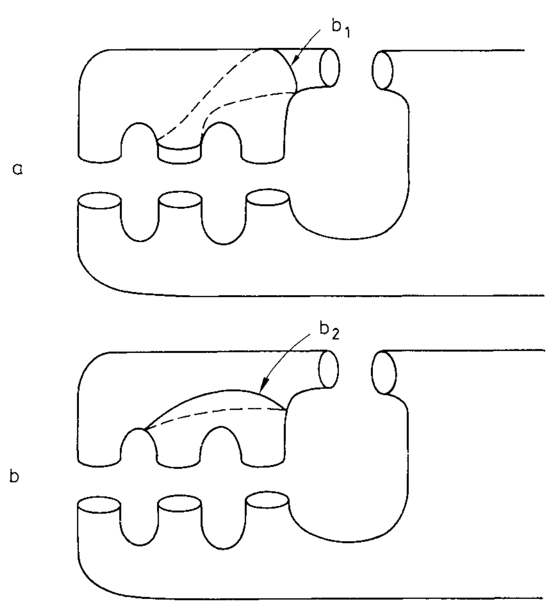


Fig. 48. Embedding of the four holed sphere in a high genus surface

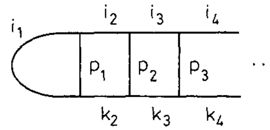


Fig. 49. Characters used in the verification of the Chinese lantern identity

$g=0$ we need only check that the twists b_1, b_2 defined in (D) and illustrated in Fig. 48 are also correctly represented. To do this, we use fusing to pass to the basis of characters in Fig. 49 such that the fourpoint function characters are given by $\oplus V^{i_3}_{p_2 i_2} \otimes V^{i_2}_{i_1 p_1}$. Thus we must check that in this basis we have

$$\varrho(b_1)=F_{i_2}\begin{bmatrix} p_1 & i_3 \\ i_1 & p_2 \end{bmatrix} e^{-2\pi i \Delta_{i_2}} F_{i_2}\begin{bmatrix} p_1 & i_3 \\ i_1 & p_2 \end{bmatrix}^{-1} \tag{B.39}$$

and

$$\varrho(b_2)=e^{-i\pi\Delta_{i_2}} F_{i_2}\begin{bmatrix} p_2 & p_1 \\ i_3 & i_1 \end{bmatrix} e^{-2\pi i \Delta_{i_2}} F_{i_2}\begin{bmatrix} p_2 & p_1 \\ i_3 & i_1 \end{bmatrix}^{-1} e^{i\pi\Delta_{i_2}}. \tag{B.40}$$

To do this we first evaluate $\beta_2\alpha_2\alpha_3\beta_2$ to get

$$D_{p_2} D_{p_1} D^-_{i_1} D^-_{i_3} D^-_{k_2} F_{i_2}\begin{bmatrix} p_2 & i_1 \\ i_3 & p_1 \end{bmatrix} P_{p_1 p_2} T_{i_2}$$

from this one easily recovers (B.39). Now, using a similar representation of $\beta_1\alpha_1\alpha_2\beta_1$ and the above result for $\varrho(b_1)$ we recover (B.40). This completes the proof of the high genus relations.

Next we turn to the definition of the duality complex. It is intuitively clear that there are no further *duality* relations so we will content ourselves with the following heuristic argument. The vertices of the complex $\mathscr{C}_{g,n}$ are again pairs $(\mathscr{D}, \gamma)$, where $\mathscr{D}$ is a g-loop ϕ^3-diagram. The simple moves S, T can be defined for loops connected by a single line to the remainder of $\mathscr{D}$. For $\mathscr{C}_{g,0}$ to be connected one must add moves for Dehn twists around the propagators. There should be no new relations because any loop can once again be deformed to a loop in the multiperipheral basis. If, in a ϕ^3 diagram, we cut g lines so that the diagram has no loops (i.e. we consider a sewn $2g$-holed sphere amplitude) then using braidings and fusings we can bring any loop to a loop in the multiperipheral basis, where loops correspond to relations in the modular group. Since we are discussing braidings and fusings for a sphere amplitude there is no ambiguity in deforming the loop. This completes the proof for the duality complex with no external lines.

For the case of (g, n) with $n>0$ the above argument can be generalized. One needs to introduce generators analogous to the a_i, b_i needed in the $g=1$ case and lift relations in the n-string braid group at genus g and in $\Gamma_{g,0}$ to relations in $\Gamma_{g,n}$. Representations of the new generators are easily obtained as in the genus one case. We have not checked all the relations in this case, but it is not really necessary since one can obtain the transformation laws of blocks for surfaces with punctures from those of surfaces with no punctures.

This finally ends the proof of the completeness theorem.

Appendix C. Tannakian Categories for Pedestrians

In this appendix we describe some of the concepts mentioned in Sect. 8 with some more precision. We will need to use some very simple notions of category theory, an esoteric subject noted for its difficulty and irrelevance. We have attempted to make the presentation readable, sometimes at the cost of taking short cuts. The real thing can be found in [43]. Tannakian categories are described in [40–42]. The only part of this section which might have any remote claims to novelty is the expression of Deligne's normalization condition in terms of the classical fusing matrix in Eq. (C.15) below. We are grateful to P. Deligne and D. Kazhdan for explaining some of the category-theoretic constructions.

We begin by recalling (for the convenience of the reader) some of the elementary definitions from [43]. Category theory is an attempt to make precise generalizations about mathematical concepts and constructions, and is impossible to understand without examples. The reader is urged to consult [43] for lists of examples. A *category* **C** consists of two sets **Obj** and **Arr** of objects and arrows (also called "morphisms"), with two functions dom and cod from arrows to objects. An arrow f with $\mathrm{dom}(f)=x$ and $\mathrm{cod}(f)=y$ is said to be an arrow from x to y and written $f:x\to y$. There is also a function called composition $\mathbf{Arr}\times\mathbf{Arr}\to\mathbf{Arr}$ defined for (f,g) when $\mathrm{dom}(g)=\mathrm{cod}(f)$ such that

1. $\mathrm{dom}(g\circ f)=\mathrm{dom}(f)$ and $\mathrm{cod}(g\circ f)=\mathrm{cod}(g)$.
2. For three morphisms $(f\circ g)\circ h=f\circ(g\circ h)$.
3. For every object x there is an arrow $1_x:x\to x$ and $f\circ 1_x=f$ and $1_y\circ f=f$.

An important (albeit trivial) concept in category theory is that of hom-sets which is simply the set of arrows between two objects: $\mathrm{hom}(x,y)=\{f\in\mathbf{Arr}\,|\,\mathrm{dom}(f)=x, \mathrm{cod}(f)=y\}$. Examples include the category of groups whose arrows are group homomorphisms, the category of topological spaces whose arrows are continuous maps, the category of Hilbert spaces whose arrows are bounded operators, etc.

There is an alternative (but more complicated) definition of categories in terms of hom-sets, which we will use below when we construct a category "from nothing." Namely, we will begin with a set of objects $a, b, \ldots$ and for every pair of objects we will define a set, which with hindsight we may denote $\mathrm{hom}(a,b)$. Next we define a function $\mathrm{hom}(b,c)\times\mathrm{hom}(a,b)\to\mathrm{hom}(a,c)$ with the usual property of the composition law. In addition we must have, for each b an element $1_b\in\mathrm{hom}(b,b)$ with the properties of the identity morphism, and finally we must insist that if $a\neq a'$ or $b\neq b'$ then $\mathrm{hom}(a,b)\cap\mathrm{hom}(a',b')=\emptyset$. The reader may check that these axioms define a category. An important class of categories are those for which the hom-sets are in fact abelian groups such that composition is bilinear. These categories are called preadditive categories.

Often one wishes to speak of relations between categories, and to this end we define a *functor* between categories $\mathbf{C}\to\mathbf{B}$ to be a pair of functions, $\mathbf{Obj(C)}\to\mathbf{Obj(B)}$ and $\mathbf{Arr(C)}\to\mathbf{Arr(B)}$ (somewhat sloppily, we denote both by T) such that if $f:c\to c'$ is an arrow, then the arrow $T(f)$ is an arrow $T(f):T(c)\to T(c')$ and we have $T(1_c)=1_{T(c)}$, and $T(g\circ f)=T(g)\circ T(f)$. In the category of categories a functor is a morphism between categories. Examples of functors include homology and homotopy groups, GL_n (from rings to groups) and conformal field theory (according to Segal). Below we will use a functor called the forgetful functor.

Finally, we may wish to compare functors, and to this end we define a *natural transformation* between two functors $S, T: \mathbf{C} \to \mathbf{B}$. The basic idea is that each of S, T maps commutative diagrams in $\mathbf{C}$ to commutative diagrams in $\mathbf{B}$, and we would like to have a transformation of the images of these diagrams. Precisely, a natural transformation $\tau: S \to T$ is defined to be a collection of morphisms in $\mathbf{B}$: $\tau_c: S(c) \to T(c)$ such that for any arrow $f: c \to c'$ in $\mathbf{C}$ the corresponding arrows $S(f): S(c) \to S(c')$ and $T(f): T(c) \to T(c')$ are related by

$$\begin{array}{ccc} S(c) & \xrightarrow{\tau_c} & T(c) \\ {\scriptstyle S(f)} \downarrow & & \downarrow {\scriptstyle T(f)} \\ S(c') & \xrightarrow{\tau_{c'}} & T(c') . \end{array} \tag{C.1}$$

If each arrow τ_c has an inverse then the natural transformation is called a *natural isomorphism* or, sometimes, a *functorial isomorphism*. Below we will define, F, Ω in terms of certain functorial isomorphisms. This completes our rehash of the elementary definitions of category theory.

The basic premise of Tannakian category theory is that a knowledge of the representations of a group is equivalent to a knowledge of the group. For example, we will indicate how, given the category of finite dimensional representations of a group we can recover the group itself by a purely category-theoretic definition. Recall that the category of finite dimensional representations of a group G, $\mathbf{Rep}(G)$ has as objects those vector spaces X over a field k which are representation spaces of G. That is, there is a group homomorphism $\varrho_X: G \to \mathrm{Aut}(X)$. The morphisms of $\mathbf{Rep}$ are vector space homomorphisms $T_{YX}: X \to Y$ which are also intertwining operators, i.e. satisfy (7.2). Notice that $\mathbf{Rep}$ satisfies the following (rather evident) properties (which will soon be generalized):

A1) The zero vector space is an object, and has a unique intertwiner with any other representation.

A2) The direct sum of two representations is a representation.

A3) The kernel and cokernel (= quotient of range by the image) of any intertwiner is a representation.

Thus $\mathbf{Rep}(G)$ is an abelian category. Moreover:

T1) The tensor product of two objects is an object and there are intertwining operators defining isomorphisms $\Omega: X \otimes Y \cong Y \otimes X$ and $F: (X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z)$.

T2) The isomorphisms F, Ω satisfy the pentagon and hexagon axioms.

T3) There is an identity object, namely k itself with the trivial representation, which satisfies $X \otimes k \cong X$ for all X.

Thus $\mathbf{Rep}(G)$ is a tensor category.

RT1) The set of intertwining operators $\mathrm{Hom}(X, Y)$ from X to Y is itself a representation,

RT2) In particular the dual space $X^\vee$ is a representation and $(X^\vee)^\vee \cong X$, that is, X is "reflexive."

RT3) Moreover we have:

$$\mathrm{Hom}(X_1, Y_1) \otimes \mathrm{Hom}(X_2, Y_2) \cong \mathrm{Hom}(X_1 \otimes X_2, Y_1 \otimes Y_2) .$$

Thus $\mathbf{Rep}(G)$ is a rigid tensor category. (Sometimes the name "tensor category" is used for "rigid abelian tensor category" as in [42]. We adopted that terminology in Sect. 8, for brevity.) If we replace "intertwiner" by "linear transformation" and "representation" by "vector space" then the above axioms are still true. Thus, the category **Vec**, whose objects are vector spaces and whose morphisms are linear transformations is a rigid abelian tensor category.

Consider now the forgetful functor ω: **Rep**$\to$**Vec** which assigns to X the vector space $\omega(X)=X$, but considered only as a vector space and which assigns to morphisms T_{YX} the vector space homomorphisms $\omega(T_{YX})=T_{YX}$, but considered only as a linear transformation of vector spaces. This functor again satisfies some evident properties, namely,

FF) If two intertwiners are equal as linear transformations they are equal as intertwiners.

EF) ω takes exact sequences to exact sequences.

LF) ω takes direct sums and tensor products to direct sums and tensor products.

Thus, ω is faithful, exact, and k-linear, Moreover

TF1) ω takes the isomorphism F in **Rep** to the isomorphism F in **Vec**.

TF2) ω takes the isomorphism Ω in **Rep** to the isomorphism Ω in **Vec**.

TF3) ω takes the identity object k in **Rep** to the identity object k in **Vec**.

Thus, ω is a tensor functor.

The group G is recovered in category theory by considering "automorphisms of the tensor functor ω" (see below). Concretely, such automorphisms are families (λ_X) of invertible linear transformations $\lambda_X: X\to X$ for $X\in \mathrm{Obj}(\mathbf{Rep})$ such that $\lambda_k=1$, $\lambda_{X\otimes Y}=\lambda_X\otimes\lambda_Y$, and λ commutes with intertwiners $T_{YX}\lambda_X=\lambda_Y T_{YX}$. First note that the set of families of such transformations forms a group: The identity element is the family of identity transformations and multiplication is defined by $(\lambda_X)\cdot(\mu_X)\equiv(\lambda_X\circ\mu_X)$ and every family has an inverse family for this multiplication. Second, note that any group element g defines such a family by $\lambda_X=\varrho_X(g)$. In fact, it is not hard to show that the converse holds, and every such family is defined by a group element. Let us check this, at least in a special case. Denote the group generated by the families $\{\lambda\}$ by $\mathrm{Aut}^{\otimes}$. We have that G is a subgroup. Moreover, any representation of G extends to a representation of $\mathrm{Aut}^{\otimes}$. Next, note that any vector $\vec{v}$ in some representation X which is fixed by G is also fixed by $\mathrm{Aut}^{\otimes}$. For, suppose a vector $\vec{v}\in X$ is fixed by $\varrho_X(G)$. Then define an intertwiner $T: k\to X$ between the trivial representation k and X by $T(z)=z\cdot\vec{v}$ for $z\in k$. We then have $\lambda(\vec{v})=\lambda(T(1))=T\lambda_1(1)=T(1)=\vec{v}$. So $\vec{v}$ is fixed by $\mathrm{Aut}^{\otimes}$. In the case of continuous groups this clearly means that $\mathrm{Aut}^{\otimes}$ cannot have any "broken generators," so the Lie algebras must be the same, and since G is a subgroup we must have $G\cong\mathrm{Aut}^{\otimes}$. This argument can be generalized [41] to conclude more generally that $G\cong\mathrm{Aut}^{\otimes}$.

The above rather trivial observations have a nontrivial generalization [41, 42]: in some cases one encounters categories which are defined with no reference to a group. If these categories satisfy certain axioms, one can conclude that in fact the category is equivalent to the category of representations of a group. To state the theorem one must generalize each of the above notions.

Briefly, an abelian tensor category is an abelian category [43] with a bilinear functor $\otimes: \mathbf{C}\times\mathbf{C}\to\mathbf{C}$. Using $\otimes$ we can in fact construct two functors $\mathbf{C}\times\mathbf{C}\to\mathbf{C}$ and several $\mathbf{C}\times\mathbf{C}\times\mathbf{C}\to\mathbf{C}$. We would like these to be "the same," so we require that there be two functorial isomorphisms $F_{X,Y,Z}: X\otimes(Y\otimes Z)\cong(X\otimes Y)\otimes Z$ and

$\Omega_{X,Y}: X \otimes Y \cong Y \otimes X$ called associativity and commutativity constraints. These are required to satisfy the hexagon and pentagon axioms as in Sect. 7, where we replace representations R_i by arbitrary objects. Moreover, we require that there be a unit object $\underline{1}$ which has the property that $\underline{1} \otimes X \cong X$ for all objects X. We impose further axioms which make the tensor category "rigid." Rigidity means that an internal hom exists and satisfies certain axioms. Since the internal hom plays an important role in the calculation of Eq. (C.15) below, we spell out the meaning of this axiom. The existence of internal hom means that for every pair of objects X, Y there is an object $\mathrm{Hom}(X, Y)$ and a morphism $ev_{X,Y}: \mathrm{Hom}(X, Y) \otimes X \to Y$, which satisfy the following property: For every object T and every morphism $g: T \otimes X \to Y$ there is a *unique* morphism $: T \to \mathrm{Hom}(X, Y)$ such that the following diagram commutes:

$$\begin{array}{ccc} T\otimes X & & \\ {\scriptstyle f\otimes \mathrm{id}_X}\downarrow & \searrow^{g} & \\ \mathrm{Hom}(X,Y)\otimes X & \xrightarrow{\mathrm{ev}_{X,Y}} & Y. \end{array} \tag{C.2}$$

The existence of internal hom canonically leads to certain morphisms [the most important being f in (C.3) below], and the rigidity axiom further states that these are isomorphisms. For example, one defines $X^\vee \cong \mathrm{Hom}(X, \underline{1})$ and requires $(X^\vee)^\vee \cong X$. As we have seen **Rep**(G) and **Vec** are rigid tensor categories.

The reconstruction of the group in the general case proceeds by studying certain functors. A tensor functor between tensor categories is a functor which transforms the tensor product from one category to the other, so e.g. for every pair of objects X, Y there is an isomorphism $c_{X,Y}: \omega(X) \otimes \omega(Y) \to \omega(X \otimes Y)$ (which is functorial). Moreover, the functor takes the commutativity and associativity constraints of one category to the other, and also transforms a unit object of one category to the other. The key construction is of the automorphisms of a tensor functor of rigid tensor categories. Suppose $\omega: \mathbf{C}_1 \to \mathbf{C}_2$ is a tensor functor between two rigid tensor categories. An automorphism of ω is a family $\{\lambda_X\}$ of morphisms $\lambda_X: \omega(X) \to \omega(X)$ with the property that $\lambda_1 = \mathrm{id}_1$, $\lambda_{X \otimes Y} c'_{X,Y} = c_{X,Y} \lambda_X \otimes \lambda_Y$, and if $\alpha_{YX}: X \to Y$ is a morphism in $\mathbf{C}_1$ then $\lambda_Y \omega(\alpha_{XY}) = \omega(\alpha_{XY}) \lambda_X$. One can show that the rigidity axiom forces the λ's to be invertible [41]. Thus, as in the example before, the collection $\mathrm{Aut}^{\otimes}$ of such families (λ_X) is a group.

We are now in a position to paraphrase:

Theorem 1 (Deligne-Milne [41]). *If* $\mathbf{C}$ *is a rigid abelian tensor category, with* $k = \mathrm{Hom}(\underline{1}, \underline{1})$, *and* $\omega: \mathbf{C} \to \mathbf{Vec}$ *is an exact faithful tensor functor, then* $\mathbf{C}$ *is equivalent to the category of representations of the group* $\mathrm{Aut}^{\otimes}$.

For the proof see [41]. A functor of the type described in the theorem is called a fiber functor. Recently, Deligne has shown that one can dispense with the assumption of the existence of a fiber functor, and replace it with an integrality condition on the rank $\mathrm{rk}(X)$ of objects of $\mathbf{C}$. The rank is defined as follows. From the universal property of Hom(,) we obtain for each g a unique isomorphism f satisfying

$$\begin{array}{ccc} (\mathrm{Hom}(X_1, Y_1)\otimes \mathrm{Hom}(X_2, Y_2))\otimes(X_1\otimes X_2) & & \\ {\scriptstyle f\otimes \mathrm{id}_{X_1\otimes X_2}}\downarrow & \searrow^{g} & \\ \mathrm{Hom}(X_1\otimes X_2, Y_1\otimes Y_2)\otimes(X_1\otimes X_2) & \xrightarrow{\mathrm{ev}} & (Y_1\otimes Y_2). \end{array} \tag{C.3}$$

Moreover, using the associativity and commutativity constraints and the evaluation maps ev_{X_1, Y_1} there is a canonical choice of g which therefore defines a canonical choice of f. In particular, taking $X_2 = \underline{1}$ and $Y_1 = \underline{1}$ we obtain a canonical isomorphism $\mathrm{Hom}(X, Y) \cong X^{\vee} \otimes Y$. Similarly, the universal property of Hom shows that we may uniquely associate a morphism $f: \underline{1} \to \mathrm{Hom}(X, X)$ given a morphism $g: X \to X$ since with g we can form the diagram

$$\begin{array}{ccc} \underline{1} \otimes X & \longrightarrow & X \\ {\scriptstyle f \otimes \mathrm{id}_X} \downarrow & & \downarrow {\scriptstyle g} \\ \mathrm{Hom}(X, X) \otimes X & \xrightarrow{\mathrm{ev}} & X. \end{array} \tag{C.4}$$

Again, there is a canonical choice for g, namely, $g = \mathrm{id}_X$, with this choice we obtain a uniquely determined f_0 and can therefore form the composition

$$\underline{1} \xrightarrow{f_0} \mathrm{Hom}(X, X) \cong X^{\vee} \otimes X \xrightarrow{\mathrm{ev}_{X, 1}} \underline{1}\,. \tag{C.5}$$

Since $\mathrm{Hom}(\underline{1}, \underline{1}) = k$ we can take $\mathrm{Hom}(\underline{1}, \cdot)$ of the above sequence to obtain a map of a one dimensional vector space to itself which can therefore be identified with an element of k, called the rank of X and denoted $\mathrm{rk}(X)$. Notice that for **Rep**(G), following through this construction shows that $\mathrm{rk}(X)$ is just the dimension of the vector space X. We may now quote

Theorem 2 (Deligne [42]). *If* **C** *is a rigid abelian tensor category such that* $k = \mathrm{Hom}(\underline{1}, \underline{1})$ *and* $\mathrm{rk}(X)$ *is a nonnegative integer for every* $X \in \mathrm{Obj}(\mathbf{C})$, *then there exists a fiber functor* ω.

We are finally in a position to describe the application of these ideas to conformal field theory. Recall that the axioms of classical rational conformal field theory are the following:

Data:

1. An index set I and a bijection of I to itself written $i \mapsto i^{\vee}$.
2. Vector spaces: V^i_{jk}, $i, j, k \in I$, with $\dim V^i_{jk} = N^i_{jk} < \infty$.
3. Isomorphisms:

$$\begin{gathered} \Omega^i_{jk} : V^i_{jk} \cong V^i_{kj}, \\ F\begin{bmatrix} j_1 & j_2 \\ i_1 & k_2 \end{bmatrix} : \bigoplus_r V^{i_1}_{j_1 r} \otimes V^r_{j_2 k_2} \cong \bigoplus_s V^{i_1}_{s k_2} \otimes V^s_{j_1 j_2}\,. \end{gathered} \tag{C.6}$$

Conditions:

1. $(i^{\vee})^{\vee} = i$.
2. $V^i_{0j} \cong \delta_{ij} C$, $V^0_{ij} \cong \delta_{ij^{\vee}} C$, $V^i_{jk} \cong V^{k^{\vee}}_{ji^{\vee}}$, $(V^i_{jk})^{\vee} \cong V^{i^{\vee}}_{j^{\vee}k^{\vee}}$.
3. $\Omega^i_{jk} \Omega^i_{kj} = 1$.
4. The identities:

$$F(\Omega \otimes 1)F = (1 \otimes \Omega)F(1 \otimes \Omega)\,, \tag{4.18a}$$

$$F_{23}F_{12}F_{23} = P_{23}F_{13}F_{12}\,. \tag{4.18b}$$

With the above data we can construct a rigid abelian tensor category as we now show. We begin by defining a preadditive category $\mathbf{C}_0$. Its objects are simple objects S_i for each $i\in I$. The S_i are not defined in terms of any more elementary concepts. We can define a preadditive category by defining a collection of hom-sets with abelian group structure and compositions which are bilinear. The hom-sets are defined by

$$\begin{aligned}\mathrm{Hom}(S_i,S_j)&=\{O_{ji}\}\quad\text{if}\quad i\neq j\,,\\ \mathrm{Hom}(S_i,S_i)&=\{k\cdot\mathrm{id}_{S_i}\}\,.\end{aligned}\tag{C.8}$$

Here O_{ji} is the unique morphism $S_i\to S_j$ for $i\neq j$ and the abelian group structure is defined by $O_{ji}+O_{ji}=O_{ji}$. We then define composition of arrows so that it is bilinear.

This preadditive category can be turned into an additive category [43] $\mathbf{C}_1$ whose objects are n-tuples of objects of $\mathbf{C}_0$ (the null object is the 0-tuple). Morphisms in $\mathbf{C}_1$ are matrices of morphisms in $\mathbf{C}_0$, and composition is given by matrix multiplication together with composition and addition of the elements of (C.8). It can be shown that $\mathbf{C}_1$ is an additive and in fact an abelian category. So far we have not used any of the data other than the index set I. The data 2, 3 are used to turn $\mathbf{C}_1$ into a rigid tensor category.

To begin, in any abelian category with vector space hom-sets we can to an object X and a vector space $V\in\mathrm{Obj}(\mathbf{Vec})$ associate an object (more precisely, an isomorphism class of objects) $V\otimes X$ [41]. One considers the transitive system $((X^n)_\alpha,\phi_{\beta\alpha})$, where α runs over vector space isomorphisms $\alpha:k^n\to V$, $X^n=X\oplus X\oplus\ldots\oplus X$ with n factors (this is only defined up to isomorphism), and $\phi_{\beta\alpha}$: $(X^n)_\alpha\to(X^n)_\beta$ is defined by $\beta^{-1}\alpha\in GL(n,C)$. An object $V\otimes X$ is any direct or inverse limit (the two coinciding in this case) of the system, and can be taken to be any of the X^n. The definition of $V\otimes X$ is thus analogous to the definition of tensors adopted in most books on general relativity. One defines tensors to be multi-indexed objects and simply specifies their transformation laws. Morphisms $Y\to V\otimes X$ are families of morphisms $\zeta_\alpha:Y\to(X^n)_\alpha$ which are compatible with the system, i.e., $\phi_{\beta\alpha}\zeta_\alpha=\zeta_\beta$, and similarly for morphisms $V\otimes X\to Y$. In particular, note that a morphism $V\otimes X\to W\otimes X$ canonically defines a vector space morphism $V\to W$. In terms of this construction and the simple objects we can define the tensor functor by taking

$$S_j\otimes S_k\cong\bigoplus_i V^i_{jk}\otimes S_i\,,\tag{C.9}$$

and then extending $\otimes$ to other objects by linearity. The data (C.6) now provide us with associativity constraints, defining, for simple objects, isomorphisms $\Omega:S_j\otimes S_k\cong S_k\otimes S_j$ and $F:S_i\otimes(S_j\otimes S_k)\cong(S_i\otimes S_j)\otimes S_k$. For example, Ω is defined by the matrix of morphisms

$$\begin{pmatrix}\ddots & & \\ & \Omega^l_{jk}\otimes\mathrm{id}_{S_l} & \\ & & \ddots\end{pmatrix}.\tag{C.10}$$

We can take $V^0_{00}=C$ and then S_0 with an obvious isomorphism is an identity object. To show rigidity we begin by showing that if we take $\mathrm{Hom}(S_i,S_0)=(S_i)^\vee\cong S_{i^\vee}$, then $S_{i^\vee}$ satisfies the universal property, the evaluation map being given by

$$S_{i^\vee}\otimes S_i\cong\bigoplus_l V^l_{i^\vee i}\otimes S_l\to V^0_{i^\vee i}\otimes S_0\cong S_0\,.\tag{C.11}$$

(We must choose a basis of $1^0_{i^\vee i} \in V^0_{i^\vee i}$ to define the last isomorphism.) We may then extend the dual by linearity to obtain $X^\vee$ for all objects X. Clearly, every object is reflexive. Finally, we take $\mathrm{Hom}(S_i, S_j) \cong S_{i^\vee} \otimes S_j$, defining the evaluation map similarly to (C.11), that is, by the following composition of morphisms:

$$\begin{aligned}(S_{i^\vee} \otimes S_j) \otimes S_i &\xrightarrow{\Omega \otimes 1} (S_j \otimes S_{i^\vee}) \otimes S_i \xrightarrow{F^{-1}} S_j \otimes (S_{i^\vee} \otimes S_i) \\ &\xrightarrow{\Omega} (S_{i^\vee} \otimes S_i) \otimes S_j \xrightarrow{\mathrm{ev}_{i,0} \otimes \mathrm{id}_{S_j}} S_0 \otimes S_j \cong S_j ,\end{aligned} \tag{C.12}$$

we can check the universality property and extend by linearity to other objects.

It remains to express Deligne's integrality condition in terms of a condition on the above data. Following through the above definitions we see that the computation of the morphism f_0 in (C.5) (with $X = S_i$) follows from the diagram:

$$\begin{array}{ccc} S_0 \otimes S_i \cong V^i_{0i} \otimes S_i & \longrightarrow & k \otimes S_i \cong S_i \\ & & \uparrow \\ \Big\downarrow {\scriptstyle f_0 \otimes \mathrm{id}_{S_i}} & & V^i_{0i} \otimes S_i \\ & & \uparrow {\scriptstyle \mathrm{ev}_{i,0} \otimes \mathrm{id}_{S_i}} \\ (S_{i^\vee} \otimes S_i) \otimes S_i \cong \bigoplus\limits_r \Big(\bigoplus\limits_l V^r_{li} \otimes V^l_{i^\vee i} \Big) \otimes S_r & \xrightarrow{\Omega F^{-1} (\Omega \otimes 1)} & \bigoplus\limits_r \Big(\bigoplus\limits_l V^r_{li} \otimes V^l_{i^\vee i} \Big) \otimes S_r \end{array} \tag{C.13}$$

Recall that morphisms are given by matrices of more elementary morphisms. Considering $f_0 : V^i_{0i} \to \bigoplus\limits_l V^i_{li} \otimes V^l_{i^\vee i}$ the only nonvanishing component is in the space with $l = 0$. This is determined by a single nonvanishing element of k, call it z, which is in fact $\mathrm{rk}(S_i)$. It remains to evaluate the matrix along the bottom row of (C.13) the id_{S_i} component of which is

$$\bigoplus_{r,l} (\Omega^i_{ir} \otimes 1) F_{rl} \begin{bmatrix} i & i \\ i & i^\vee \end{bmatrix}^{-1} (1 \otimes \Omega^l_{i^\vee i}) .$$

For the computation of z we need only look at the term with $l = r = 0$. The commutativity of the diagram forces

$$z (\Omega^i_{i0} \otimes 1) F_{00} \begin{bmatrix} i & i \\ i & i^\vee \end{bmatrix}^{-1} (1 \otimes \Omega^0_{i^\vee i}) = 1 . \tag{C.14}$$

Using the identity $F^{-1} = \Omega \otimes \Omega F \Omega \otimes \Omega$ finally gives

$$\mathrm{rk}(S_i) = \frac{1}{(\Omega^i_{0i} \otimes \Omega^0_{ii^\vee}) F_{00} \begin{bmatrix} i & i^\vee \\ i & i \end{bmatrix}} \equiv \frac{1}{F_i} . \tag{C.15}$$

Note that the quantity in the denominator is a linear map of a one-dimensional vector space to itself and is therefore a canonically defined complex number. Equation (C.15) is in agreement with Eq. (7.14). Thus, by Theorems 1 and 2, when (C.15) is a nonnegative integer for all i we can identify our category with the category of representations of a group, and the isomorphisms F are the analogs of the $6j$ symbols as in Sect. 7.

Appendix D. Examples of Solutions

For simple fusion rule algebras one can solve the equations of Sect. 4 by hand. We give here a few examples of such solutions in order to illustrate some points about the integral parts of Δ, $c/8$ and about the algebraic nature of operator product coefficients.

1. *Ising Model.* In this case we have three representations 1, ψ, σ with the famous fusion rule algebra:

$$\psi\times\psi=1\,,\qquad \psi\times\sigma=\sigma\,,\qquad \sigma\times\sigma=1+\psi\,. \tag{D.1}$$

To solve for the linear transformations we must choose bases for the various vector spaces (i.e. we must "choose a gauge" in the terminology of [11]) and we do this by demanding that $\|\Phi^i_{jk}\|$ [recall the notation of (2.9)] be totally symmetric in i, j, k, and that

$$F\begin{bmatrix}\sigma & \psi\\ \sigma & \psi\end{bmatrix}=F\begin{bmatrix}\psi & \sigma\\ \psi & \sigma\end{bmatrix}=F\begin{bmatrix}\psi & \psi\\ \sigma & \sigma\end{bmatrix}=F\begin{bmatrix}\sigma & \sigma\\ \psi & \psi\end{bmatrix}=1\,. \tag{D.2}$$

As discussed above, when two identical representations occur in V^i_{jk}, some of the permutations σ could have eigenvalues $\xi=-1$. To simplify the analysis in this example, we will assume the eigenvalues are all $+1$. This is clearly the case in the Ising model. Define $x=e^{i\pi\Delta_\psi}$, $y=e^{i\pi\Delta_\sigma}$ and

$$\alpha=F\begin{bmatrix}\psi & \psi\\ \psi & \psi\end{bmatrix},\qquad \beta=F\begin{bmatrix}\sigma & \psi\\ \psi & \sigma\end{bmatrix}. \tag{D.3}$$

From (4.5) we obtain $\alpha^2=\beta^2=1$ and

$$F\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}^2=1\,. \tag{D.4}$$

From the pentagon we have

$$\begin{aligned}F_{00}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}&=F_{0\psi}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}F_{\sigma 0}\begin{bmatrix}\psi & \psi\\ \sigma & \sigma\end{bmatrix},\\ F_{00}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}&=F_{0\sigma}\begin{bmatrix}\sigma & \psi\\ \sigma & \psi\end{bmatrix}F_{\psi 0}\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}.\end{aligned} \tag{D.5}$$

From (D.4) and (D.5) we deduce that

$$F\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}=\frac{\gamma}{\sqrt{2}}\begin{pmatrix}1 & 1\\ 1 & -1\end{pmatrix}$$

with $\gamma^2=1$. Defining

$$D=\begin{pmatrix}1 & 0\\ 0 & x\end{pmatrix},$$

we have from (4.6)

$$DF\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}DF\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}D=F\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}y^4\,, \tag{D.6}$$

which implies that $x^2=-1$ and $y^4=\gamma\frac{1+x}{\sqrt{2}}$ from which we find $y^8=x$ and $y^{16}=-1$, and therefore $\gamma=(y^4+y^{-4})/\sqrt{2}$. Applying (4.6) once more shows that $\alpha=1$, $\beta=-1$. The equations on the torus lead to

$$S(0)=\frac{1}{2}\begin{pmatrix}1 & \sqrt{2} & 1\\ \sqrt{2} & 0 & -\sqrt{2}\\ 1 & -\sqrt{2} & 1\end{pmatrix}; \qquad S(\psi)=y^{-4} \tag{D.7}$$

and

$$e^{\frac{i\pi c}{4}}=y^2; \qquad \gamma=1\,. \tag{D.8}$$

Thus, all the matrices are determined from a single choice of y such that $y^4=e^{\pm\frac{i\pi}{4}}$. For example, one may then easily compute

$$B\begin{bmatrix}\sigma & \sigma\\ \sigma & \sigma\end{bmatrix}(+)=\frac{y^2}{\sqrt{2}}\begin{pmatrix}1 & x^{-1}\\ x^{-1} & 1\end{pmatrix}. \tag{D.9}$$

The operator product coefficients are seen to be $(d_{\psi\sigma\sigma})^2=(d_{0\sigma\sigma})^2=1$. Giving a complete solution on the plane. Naively, it seems that each of these eight solutions has a physical realization in terms of $O(2m+1)$ level 1 current algebra in which there are three primary fields 1, ψ, σ with weights 0, 1/2, $\frac{1}{16}(2m+1)$ respectively. However, the assumption that all the σ's are realized as $\xi=+1$ is not satisfied for all m but only for $m=0 \bmod 4$.

Let us compare the above matrices with the explicit blocks known for the Ising model. From the 4-spin correlation function [1] we see that with the normalization to obtain (D.9) we have

$$\begin{aligned}\langle\sigma|\begin{pmatrix}\sigma\\ \sigma 0\end{pmatrix}(1)\begin{pmatrix}0\\ \sigma\sigma\end{pmatrix}(x)|\sigma\rangle&=(x(1-x))^{-1/8}\sqrt{\frac{1+\sqrt{1-x}}{2}}\,,\\ \langle\sigma|\begin{pmatrix}\sigma\\ \sigma\psi\end{pmatrix}(1)\begin{pmatrix}\psi\\ \sigma\sigma\end{pmatrix}(x)|\sigma\rangle&=(x(1-x))^{-1/8}\sqrt{\frac{1-\sqrt{1-x}}{2}}\,.\end{aligned} \tag{D.10}$$

Hence in this basis the chiral vertex operators have a nontrivial normalization $\|\Phi^{\psi}_{\sigma\sigma}\|^2=1/2$. We may use this to derive the physical operator product expansion coefficient $c_{\sigma\sigma\varepsilon}$ (ε is the energy operator) defined by

$$\sigma(z_1,\bar{z}_1)\sigma(z_2,\bar{z}_2)=|z_{12}|^{-1/4}+c_{\sigma\sigma\varepsilon}|z_{12}|^{3/4}\varepsilon(z_2,\bar{z}_2)+\dots.$$

The physical vertex operators are

$$\begin{aligned}\sigma(z,\bar{z})&=\sum_{i,k}d_{ik\sigma}\begin{pmatrix}i\\ \sigma k\end{pmatrix}(z)\overline{\begin{pmatrix}i\\ \sigma k\end{pmatrix}(z)}\,,\\ \varepsilon(z,\bar{z})&=\sum_{i,k}d_{ik\psi}\begin{pmatrix}i\\ \psi k\end{pmatrix}(z)\overline{\begin{pmatrix}i\\ \psi k\end{pmatrix}(z)}\,.\end{aligned}$$

From these equations one may deduce $c^2_{\sigma\sigma\varepsilon}=1/4$ in agreement with [56].

In this case it is very easy to show how analytic constraints on the sections restrict the integer parts of Δ_σ, Δ_ψ, $c/8$. Choosing the root $y=e^{i\pi/16}$ and defining $\Delta_\sigma=1/16+2n_\sigma$, $\Delta_\psi=1/2+2n_\psi$, $c=1/2+8n_c$ (with $n_\sigma, n_\psi, n_c\in Z$) we see that $(\chi^\psi_\sigma(\tau))^{24}$ is a modular form of weight $12+48n_\psi$ whose q-expansion begins with $q^{1+48n_\sigma-8n_c}[1+\ldots]$. If we make the (physically reasonable) assumption that the partition function only has a zero at $\tau\to i\infty$, then it must be a power of η^{24} and we obtain $6n_\sigma-n_c=6n_\psi\geqq 0$. Similar considerations might fix the integers n_σ, n_c up to tensor products with $c=24$ theories. (Note that in the Ising case all these integers are zero, from which one immediately obtains the one-point block $\eta(\tau)(dz)^{1/2}$. This block can also be derived by factorizing the ψ-channel two-point block for $\langle\sigma\sigma\rangle$ on the torus:

$$\frac{1}{\eta^{1/8}}\frac{\theta_1\left(\frac{z_1-z_2}{2}\right)^{1/2}}{\theta_1(z_1-z_2)^{1/8}},$$

where θ are Jacobi theta functions, which may in turn be extracted from [56].)

2. Another simple but instructive example is provided by the fusion rules for two fields 1, ϕ with $\phi\times\phi=1+\phi$. Defining $x=e^{i\pi\Delta_\phi}$ one easily finds:

$$F\begin{bmatrix}\phi & \phi\\ \phi & \phi\end{bmatrix}=\begin{pmatrix}a_\pm & 1\\ a_\pm & -a_\pm\end{pmatrix},$$
$$a_\pm=x^4(1+x^2)=\frac{-1\pm\sqrt{5}}{2},\qquad x^5=-\Omega^0_{\phi\phi}=\pm 1\,. \tag{D.11}$$

The operator product coefficients may be found, e.g.

$$(d^\phi_{\phi\phi})^2=\frac{1}{a_\pm}d^0_{\phi\phi}\,. \tag{D.12}$$

As discussed in [18] the choice $a_+>0$ should correspond to a unitary theory and $a_-<0$ to a nonunitary theory, in accord with the observations in [9]. This example is also useful for studying the reconstruction problem. Namely, it is easily shown that the conformal blocks $\mathscr{F}_1$, $\mathscr{F}_\phi$ for the four-point function of ϕ are *uniquely* determined from the above monodromies together with (1) a choise of the integral part of Δ_ϕ and (2) the assumption that the conformal blocks have no common zeroes. It would be very interesting to see whether the generalization of the latter assumption leads to any general uniqueness theorems.

3. $SU(2)$, $k=3$. This example clarifies some of the issues associated with the transcendental nature of operator product coefficients.[17] The primary fields in this theory have spins 0, $\frac{1}{2}$, 1, $\frac{3}{2}$ with the usual $SU(2)$ current algebra fusion rules, e.g. $\frac{1}{2}\times\frac{1}{2}=0+1$, one can solve for the duality matrices up to a few discrete choices of

[17] To the best of our knowledge this issue was first addressed in unpublished work of D. Friedan and S. Shenker. Based on the example of the $c=7/10$ model they suggested, in various seminars, that all transcendental numbers could be absorbed into the normalization of the conformal blocks. Our results give a first step towards proving this assertion

signs $\left(\text{a solution for } B\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ * & * \end{bmatrix} \text{ for any level appears in [30]} \right)$. In this example some of the eigenvalues of the permutations σ are -1. Choosing the signs appropriate to this example we find:

$$F\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} = \begin{pmatrix} \frac{1-\sqrt{5}}{2} & 1 \\ \frac{5\sqrt{5}-11}{2\sqrt{5}-7} & \frac{1-\sqrt{5}}{2} \end{pmatrix}, \qquad d^2_{\frac{1}{2},\frac{1}{2},1} = \frac{1+\sqrt{5}}{2}. \tag{D.13}$$

These are algebraic numbers and should be compared to the transcendental numbers found by Knizhnik and Zamalodchikov [2]. By comparing two singularities of the four-point function of the spin $\frac{1}{2}$ field we obtain

$$\begin{gathered} d^2_{\frac{1}{2},\frac{1}{2},1} \|\Phi^{\frac{1}{2}}_{1,\frac{1}{2}}\|^2 = \frac{\Gamma(\frac{1}{5})(\Gamma(\frac{3}{5}))^3}{\Gamma(\frac{4}{5})(\Gamma(\frac{2}{5}))^3}, \\ \|\Phi^{\frac{1}{2}}_{\frac{1}{2}1}\|^2 \left[1 + d^2_{\frac{1}{2}\frac{1}{2}1} \left(F_{11}\begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix} \right)^2 \right] = \frac{\Gamma(\frac{3}{5})^4}{\Gamma(\frac{2}{5})^2 \Gamma(\frac{4}{5})^2} + \frac{1}{4} \frac{\Gamma(\frac{1}{5})\Gamma(\frac{3}{5})^5 \Gamma(\frac{7}{5})^2}{\Gamma(\frac{2}{5})^3 \Gamma(\frac{4}{5})^3 \Gamma(\frac{6}{5})^2}, \end{gathered} \tag{D.14}$$

from which one may recover $d^2_{\frac{1}{2}\frac{1}{2}1}$ in (D.13). The difference in operator product expansion coefficients is, therefore, a difference in normalization of the chiral vertex operators. The above examples (and others) suggest that a given fusion rule algebra always has a discrete set of solutions for F, d. In this case the numbers F, d must be algebraic, and hence any "gauge-invariant" combination of operator product coefficients will be algebraic. It would be of interest to see if this has any implications for the algebraic nature of Γ-functions at rational arguments.

Appendix E. The Pentagon as a 3-Cocycle Condition

There is one class of fusion rules for which the polynomial equations admit a very general solution, namely, when there is only one representation on the right-hand side of every fusion rule. A little thought reveals that such fusion rules are in one-one correspondence with abelian groups. In this appendix we show that for such fusion rules the polynomial equations on the plane admit an interpretation in terms of group cohomology. This might be a hint at another interpretation of the equations, and might prove a useful starting point for finding a general solution to the equations.

If A is the abelian group defining the fusion rules, then all spaces of intertwiners are zero or one-dimensional, and the one dimensional spaces are always of the form $V^{g_1 g_2}_{g_1, g_2}$. Therefore, choosing a basis $1_{g_1,g_2}$ for these spaces we interpret our basic data in terms of C^*-valued functions on A, $A \times A$, $A \times A \times A$:

$$\begin{gathered} w(g) = e^{i\pi\Delta_g}, \qquad \sigma(g_1, g_2) = \pm 1, \\ \Omega(g_1, g_2) = \sigma(g_1, g_2) \frac{w(g_1) w(g_2)}{w(g_1 g_2)}, \\ F(g_1, g_2, g_3), \qquad S(g_1, g_2) = S(0)_{g_1, g_2}. \end{gathered} \tag{E.1}$$

Under a change of basis $1_{g_1,g_2} \to \lambda(g_1,g_2) 1_{g_1,g_2}$, we have to change

$$\begin{aligned} \Omega \to \tilde{\Omega}(g_1,g_2) &= \frac{\lambda(g_1,g_2)}{\lambda(g_2,g_1)} \Omega(g_1,g_2), \\ F \to \tilde{F}(g_1,g_2,g_3) &= \frac{\lambda(g_1 g_2, g_3)\lambda(g_1,g_2)}{\lambda(g_1, g_2 g_3)\lambda(g_2,g_3)} F(g_1,g_2,g_3). \end{aligned} \tag{E.2}$$

We can interpret some of our equations in terms of the cohomology groups $H^k(A, C^*)$. Recall that these are defined as follows [57, 58]. A k-cochain is a function

$$c: \underbrace{A \times \ldots A}_{k\text{-times}} \to C^* .$$

The coboundary operation on a k-chain is defined by

$$\delta c(g_1, \ldots, g_{k+1}) = \frac{c(g_2, \ldots, g_{k+1})}{c(g_1 \cdot g_2, g_3, \ldots, g_{k+1})} \frac{c(g_1, g_2 \cdot g_3, \ldots, g_{k+1})}{c(g_1, g_2, g_3 \cdot g_4, \ldots, g_{k+1})} \ldots c(g_1, \ldots, g_k)^{(-1)^{k+1}} .$$

For example, for zero and one-cochains we have

$$\begin{aligned} c(g), \qquad \delta c(g_1,g_2) &= \frac{c(g_1)c(g_2)}{c(g_1 g_2)}, \\ c(g_1,g_2), \qquad \delta c(g_1,g_2,g_3) &= \frac{c(g_2,g_3)c(g_1,g_2\cdot g_3)}{c(g_1\cdot g_2,g_3)c(g_1,g_2)}. \end{aligned} \tag{E.3}$$

As usual the cohomology groups are the groups of cocycles modulo coboundaries.

We may interpret F and Ω as 2 and 3 cochains respectively. Note that under a change of basis (="gauge transformation") the fusing matrix F changes by a coboundary. Moreover, it is easy to see that the pentagon condition is the statement that F is a three-cocycle:[18]

$$\frac{F(g_2,g_3,g_4)F(g_1,g_2g_3,g_4)F(g_1,g_2,g_3)}{F(g_1g_2,g_3,g_4)F(g_1,g_2,g_3g_4)} = 1 . \tag{E.4}$$

So the gauge-invariant information in F is a class in $H^3(A, C^*)$.[19]

The two hexagon equations give

$$\left(\frac{w(g_1)w(g_2)w(g_3)w(g_1g_2g_3)}{w(g_1g_2)w(g_1g_3)w(g_2g_3)}\right)^2 = 1 \tag{E.5}$$

and

$$\frac{F(g_3,g_1,g_2)F(g_1,g_2,g_3)}{F(g_1,g_3,g_2)} = \frac{\sigma(g_2,g_3)\sigma(g_1,g_3)}{\sigma(g_1g_2,g_3)} \left(\frac{w(g_1)w(g_2)w(g_3)w(g_1g_2g_3)}{w(g_1g_2)w(g_1g_3)w(g_2g_3)}\right). \tag{E.6}$$

---

[18] A similar remark was made (independently!) in the context of category theory by N. Saavedra, *voir* [40]

[19] In [59] three-cocycles where used to discuss the *failure* of associativity of operators. Here the opposite is true, since the pentagon is derived from the associativity of the operator product expansion

All the above considerations apply to an arbitrary abelian group (defining a quasirational conformal field theory). If A is a finite abelian group we can write $A=Z/m_1Z\times\ldots\times Z/m_lZ$ with $m_i|m_{i+1}$. Writing a group element $\vec{r}=(r_1,\ldots,r_l)$ (with an additive group law) the most general solution of (E.5) is $\Delta_r=\vec{r}\cdot a\cdot\vec{r}+b\cdot r+n(r)$ where $n(r)$ is an integer with $n(0)=0$, $n(r)=n(-r)$ and a is a quadratic form with

$$a_{ij}=\frac{l_{ij}}{\min(m_i,m_j)},\quad l_{ij}=l_{ji}\in Z,\quad \left(\frac{l_{ii}}{2}+b_i\right)m_i\in Z. \tag{E.7}$$

Further restrictions on the weights can be deduced from the torus equations. From the expression for S in terms of duality matrices we have

$$S_{g_1,g_2}=S_{00}\frac{e^{-2\pi i[\Delta_{g_1}+\Delta_{g_2}-\Delta_{g_1g_2}]}}{F_{g_1}F_{g_2}}. \tag{E.8}$$

Putting this back into the equation $SaS^{-1}=b$, we find that the quadratic form a must be diagonal with $a=\operatorname{diag}\left\{\frac{k_i}{2m_i}\right\}$ and $(k_i,m_i)=1$. Moreover, F_q must be a one-cocycle. No further information is obtained from $S^2=C$. The remaining equation constrains c. Defining

$$\sum_{r=0}^{m-1}e^{i\pi\frac{k}{m}r^2}\equiv m^{1/2}\,e^{i\phi(k,m)}, \tag{E.9}$$

the phase can be evaluated in terms of Gauss sums [60]. The condition on c is

$$e^{i\sum_j\phi(k_j,m_j)-2\pi ic/8}=1. \tag{E.10}$$

Since the phase of a Gauss sum is always an eight root of unity, we see that c must be an integer.

Although one could use the above remarks to give a complete solution for arbitrary finite abelian groups, we will henceforth restrict ourselves to the case of $A=Z/mZ$ and give a complete solution. Consider first the restrictions on c. These are expressed in terms of the Jacobi symbol $\left(\frac{n}{m}\right)$ defined as follows. For p prime $\left(\frac{n}{p}\right)=+1$ if there is an r with $n=r^2\bmod p$. If no such r exists then $\left(\frac{n}{p}\right)=-1$. If $m=p_1\ldots p_t$ is the prime factorization of m, then for $(n,m)=1$ we define

$$\left(\frac{n}{m}\right)=\prod_{r=1}^{t}\left(\frac{n}{p_r}\right).$$

For m odd and k even we may distinguish four cases:

$$\begin{aligned}
m&=1(4), & \left(\frac{k/2}{m}\right)&=1, & c&=0(8),\\
m&=1(4), & \left(\frac{k/2}{m}\right)&=-1, & c&=4(8),\\
m&=3(4), & \left(\frac{k/2}{m}\right)&=1, & c&=2(8),\\
m&=3(4), & \left(\frac{k/2}{m}\right)&=-1, & c&=6(8).
\end{aligned} \tag{E.11}$$

When m is even we write $m=2^l\tilde{m}$ with $\tilde{m}$ odd and distinguish twelve cases:

$$
\begin{aligned}
&\tilde{m}=1(4), \quad \left(\frac{k2^{l+1}}{m}\right)=1, \quad (l+1)=1(2), \quad c=k\tilde{m}(8),\\
&\tilde{m}=1(4), \quad \left(\frac{k2^{l+1}}{m}\right)=1, \quad (l+1)=0(2), \quad \tilde{m}k=1(4), \quad c=1(8),\\
&\tilde{m}=1(4), \quad \left(\frac{k2^{l+1}}{m}\right)=1, \quad (l+1)=0(2), \quad \tilde{m}k=3(4), \quad c=7(8),\\
&\tilde{m}=1(4), \quad \left(\frac{k2^{l+1}}{m}\right)=-1, \quad (l+1)=1(2), \quad c=k\tilde{m}+4(8),\\
&\tilde{m}=1(4), \quad \left(\frac{k2^{l+1}}{m}\right)=-1, \quad (l+1)=0(2), \quad \tilde{m}k=1(4), \quad c=5(8),\\
&\tilde{m}=1(4), \quad \left(\frac{k2^{l+1}}{m}\right)=-1, \quad (l+1)=0(2), \quad \tilde{m}k=3(4), \quad c=3(8),\\
&\tilde{m}=3(4), \quad \left(\frac{k2^{l+1}}{m}\right)=1, \quad (l+1)=1(2), \quad c=k\tilde{m}+2(8),\\
&\tilde{m}=3(4), \quad \left(\frac{k2^{l+1}}{m}\right)=1, \quad (l+1)=0(2), \quad \tilde{m}k=1(4), \quad c=2(8),\\
&\tilde{m}=3(4), \quad \left(\frac{k2^{l+1}}{m}\right)=1, \quad (l+1)=0(2), \quad \tilde{m}k=3(4), \quad c=0(8),\\
&\tilde{m}=3(4), \quad \left(\frac{k2^{l+1}}{m}\right)=-1, \quad (l+1)=1(2), \quad c=k\tilde{m}+5(8),\\
&\tilde{m}=3(4), \quad \left(\frac{k2^{l+1}}{m}\right)=-1, \quad (l+1)=0(2), \quad \tilde{m}k=1(4), \quad c=6(8),\\
&\tilde{m}=3(4), \quad \left(\frac{k2^{l+1}}{m}\right)=-1, \quad (l+1)=0(2), \quad \tilde{m}k=3(4), \quad c=4(8).
\end{aligned}
\tag{E.12}
$$

We can use group cohomology to solve for F.[20] The cohomology groups of Z/mZ are

$$
\begin{aligned}
H^{2k+1}(A, C^*)&=Z/mZ, \quad H^{2k}(A, C^*)=0,\\
H^{2k+1}(A, Z)&=0, \quad H^{2k}(A, Z)=Z/mZ,
\end{aligned}
\tag{E.13}
$$

the two coefficient systems being related by the exponential sequence. From the representative of the generator $f(r)=e^{2\pi ir/m}$ of $H^1(A, C^*)$ we may use the long exact sequence in cohomology to obtain a generator

$$f(r_1, r_2)=\frac{1}{m}(\bar{r}_1+\bar{r}_2-\overline{r_1+r_2})$$

[20] We would like to thank D. Kazhdan for some help with group cohomology

for $H^2(A,Z)$, and therefore, we obtain

$$f(r_1,r_2,r_3,r_4)=\frac{1}{m^2}(\bar{r}_1+\bar{r}_2-\overline{r_1+r_2})(\bar{r}_3+\bar{r}_4-\overline{r_3+r_4})$$

for $H^4(A,Z)$, where $\bar{r}$ is the residue of r modulo m. From this, going backwards in the long exact sequence we obtain a representative of the generator for $H^3(A,C^*)$:

$$c(r_1,r_2,r_3)=\exp\left(\frac{2\pi i}{m^2}\bar{r}_1(\bar{r}_2+\bar{r}_3-\overline{r_2+r_3})\right). \quad \text{(E.14)}$$

From the hexagon equation we find $F(g_1,g_2,g_3)F(g_3,g_2,g_1)=\delta\sigma(g_1,g_2,g_3)$, and since $F=\gamma^s$ up to a coboundary, and $\gamma(r_3,r_2,r_1)=\gamma(r_1,r_2,r_3)$ up to a coboundary we find that $2s=0(m)$, and hence F is pure gauge when m is odd and is pure gauge or cohomologous to $\gamma^{m/2}$ when m is even. In gauging F to one must be careful about σ, since the change of gauge might spoil the physical requirement that σ be ± 1 with $+1$ for distinct group elements. Demanding that there exist a gauge satisfying this physical requirement, and using some further group cohomology one can show that F can always be gauged to one. Moreover, the complete solution in this case is given by any choice of $n(r)$ with onè of

$$\begin{aligned} F=1\,, \quad & \sigma(r,r)=e^{i\pi n(2r)} \quad m \text{ even or odd}\,, \\ F=1\,, \quad & \sigma(r,r)=e^{i\pi n(2r)-i\pi r^2} \quad m \text{ even}\,. \end{aligned} \quad \text{(E.15)}$$

Based on this example we might speculate that one can define some kind of nonabelian cohomology theory for general fusion rule algebras in which the pentagon condition is a 3-cocycle condition. Such an interpretation might be very helpful in fidinging the general solution to the equations.

Acknowledgements. After we wrote this paper we learned of a construction of deformations of Virasoro (and chiral) algebras closely related to (2.5) which appears in [49]. The close relation of our work to category theory was brought to our attention by P. Deligne, D. Kazhdan, and E. Witten, and we thank them for useful discussions. It is also a pleasure to thank T. Banks, D. Bernard, V. Bazhanov, J. Birman, I. Frenkel, D. Friedan, V. F. R. Jones, J. Lepowsky, E. Martinec, V. Pasqiuer, M. Peskin, J. Polchinski, S. Shenker, Z. Qui, E. Verlinde, H. Verlinde, E. Witten, and S. Wolpert for many interesting remarks. G. M. would like to thank the Aspen Center of Physics and N.S. would like to thank C.C.N.Y. for hospitality when part of this work was done. This work was supported in part by NSF grant PHY-86-20266 and DOE contract DE-AC02-76ER02220.

References

1. Belavin, A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two dimensional quantum field theory. Nucl. Phys. B **241**, 33 (1984)
2. Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess-Zumino model in two dimensions. Nucl. Phys. B **247**, 83 (1984)
3. Zamolodchikov, A.B.: Infinite additional symmetries in two-dimensional conformal quantum field theory. Theor. Math. Phys. **65**, 1205 (1986)
4. Zamolodchikov, A.B., Fateev, V.A.: Parafermionic currents in the two-dimensional conformal quantum field theory and self dual critical points in $Z(n)$ invariant statistical systems. Sov. Phys. JETP **62**, 215 (1985)

5. Friedan, D., Shenker, S.: The analytic geometry of two-dimensional conformal field theory. Nucl. Phys. B **281**, 509 (1987)
Friedan, D.: A new formulation of string theory. Physica Scripta T **15**, 72 (1987)
Friedan, D, Shenker, S.: Talks at Cargese and I.A.S. (unpublished) (1987)
6. Kastor, D., Martinec, E., Qiu, Z.: Current algebra and conformal discrete series. Phys. Lett. **200** B, 434 (1988)
Bagger, J., Nemeschansky, D., Yankielowicz, S.: Virasoro algebras with central charge $c > 1$. Phys. Rev. Lett. **60**, 389 (1988)
Douglas, M.R.: G/H conformal field theory. CALT-68-1453
Ravanini, F.: An infinite class of new conformal field theories with extended algebras. Nordita-87/56-P
7. Harvey, J.A., Moore, G., Vafa, C.: Quasicrystalline compactification. Nucl. Phys. B **304**, 269 (1988)
8. Anderson, G., Moore, G.: Rationality in conformal field theory. Commun. Math. Phys. **117**, 441 (1988)
9. Verlinde, E.: Fusion rules and modular transformations in 2-D conformal field theory. Nucl. Phys. B **300**, 360 (1988)
10. Vafa, C.: Toward classification of conformal theories. Phys. Lett. **206** B, 421 (1988)
11. Moore, G., Seiberg, N.: Polynomial equations for rational conformal field theories. Phys. Lett. (in press)
12. Bais, F.A., Bouwknegt, P., Surridge, M., Schoutens, K.: Extensions of the Virasoro algebra constructed from Kac-Moody algebra using higher order casimir invariants. Nucl. Phys. B **304**, 348 (1988); Coset constructions for extended Virasoro algebras. Nucl. Phys. B **304**, 371 (1988)
13. Mathur, S.D., Mukhi, S., Sen, A.: Differential equations for correlators in arbitrary rational conformal field theories. TIFR/TH/88-32; On the classification of rational conformal field theories. TIFR/TH/88-39
14. Bakas, I.: Higher spin fields and the Gelfand-Dicke Algebra, the Hamiltonian structure of the spin 4 operator algebra. University of Texas at Austin preprints
15. Lukyanov, S.L., Fateev, V.A.: Additional symmetries in two dimensional conformal field theory and exactly solvable models, Parts I, II, III. (In Russian) Institute for Theoretical Physics (Kiev) preprint ITP-88-74P
16. Brustein, R., Yankielowicz, S., Zuber, J.-B.: Factorization and selection rules of operator product algebras in conformal field theory. TAUP-1647-88
17. Dijkgraaf, R., Verlinde, E.: Modular invariance and the fusion algebra. Presented at Annecy Conf. on conformal field theory
18. Moore, G., Seiberg, N.: Naturality in conformal field theory. Nucl. Phys. (in press)
19. Blok, B., Yankielowicz, S.: Extended algebras and the coset construction of conformal field theories. TAU-1661-88
20. Witten, E.: Quantum field theory and the Jones polynomials. To appear in the proc. of the IAMP Congress, Swansea, July, 1988
21. Segal, G.: Talks at IAS 1987
22. Ishibashi, N., Matsuo, Y., Ooguri, H.: Soliton equations and free fermions on Riemann surfaces. UT-499-Tokyo
23. Alvarez-Gaumé, L., Gomez, C., Reina, C.: Loop groups, Grassmannians and string theory. Phys. Lett. **190** B, 55 (1987)
24. Vafa, C.: Operator formulation on Riemann surfaces. Phys. Lett. **190** B, 47 (1987)
25. Also see, Alvarez-Gaumé, L., Gomez, C., Moore, G., Vafa, C.: Strings in the operator formalism. Nucl. Phys. B **303**, 455 (1988)
Alvarez-Gaumé, L., Gomez, C., Nelson, P., Sierra, G., Vafa, C.: Fermionic strings in the operator formalism. BUHEP-88-11, and refs. therein
26. In the mathematical literature these have been discussed In: Borcherds, R.: Proc. Nat. Acad. Sci. USA, **83**, 3068 (1986)
Lepowsky, J.: Perspectives on the Monster. Lepowsky, J., Frenkel, I., Meurman, A.: Vertex operators and the Monster. Academic Press, New York to appear; Borcherds, R.: Berkeley preprints

27. Goddard, P., Kent, A., Olive, D.: Unitary representations of the Virasoro and super Virasoro algebras. Commun. Math. Phys. **103**, 105 (1986)
28. Dixon, L., Harvey, J.A., Vafa, C., Witten, E.: Strings on orbifolds. Nulc. Phys. B **261**, 678 (1985); Strings on orbifolds. II. Nucl. Phys. B **274**, 285 (1986)
29. Goddard, P., Schwimmer, A.: Unitary constructions of extended conformal algebras. Phys. Lett. **206** B, 62 (1988)
30. Schroer, B.: Quasiprimary fields: An approach to positivity of 2-D conformal quantum field theory. Nucl. Phys. B **295**, 4 (1988); Algebraic aspects of non-perturbative quantum field theories. Como lectures; Rehren, K.-H.: Locality of conformal fields in two-dimensions: Exchange algebras on the light cone. Commun. Math. Phys. **116**, 675 (1988); Fröhlich, J.: Statistics of fields, the Yang-Baxter equation, and the theory of knots and links. Lectures at Cargese 1987, to appear In: Nonperturbative quantum field theory, Plenum Press: NewYork. Felder, G., Fröhlich, J.: Unpublished lectures notes
Rehren, K.-H., Schroer, B.: Einstein causality and artin braids. FU preprint 88-0439
31. Tsuchiya, A., Kanie, Y.: Vertex operators in the conformal field theory on *P*1 and monodromy representations of the braid group. In: Conformal field theory and solvable lattice models. Adv. Stud. Pure Math. **16**, 297 (1988); Vertex operators in the conformal field theory on *P*1 and monodromy representations of the braid group. Lett. Math. Phys. **13**, 303 (1987)
32. Vafa, C.: Conformal theories and punctured surfaces. Phys. Lett. **199** B, 195 (1987)
33. Frenkel, I.: Talk at Canadian Society of Math. Vancouver, Nov. 1987 H. Sonoda. Nucl. Phys. B **311**, 417 (1988)
34. DiFrancesco, P.: Structure constants for rational conformal field theories. Saclay preprint, PhT-88/139
35. Witten, E.: Non-Abelian bosonization. Commun. Math. Phys. **92**, 455 (1984)
36. Rose, M.E.: Elementary theory of angular momentum. New York: Wiley 1957
37. Pressley, N., Segal, G.: Loop groups. Oxford: Oxford Univ. Press 1986
38. Felder, G., Gawedzki, K., Kupiainen, A.: The spectrum of Wess-Zumino-Witten models. IHES/p/87/35
39. Kirillov, A.A.: Elements of the theory of representations. Berlin, Heidelberg, New York: Springer 1976
40. Saavedra, N.: Catégories tannakiennes. Lecture Notes in Mathematics, Vol. 265. Berlin, Heidelberg, New York: Springer 1972
41. Deligne, P., Milne, J.S.: Tannakian categories. In: Hodge cycles, motives and Shimura varieties. Lecture Notes in Mathematics, Vol. 900. Berlin, Heidelberg, New York: Springer 1982
42. Deligne, P.: Catégories tannakiennes,. IAS preprint
43. MacLane, S.: Categories for the Working Mathematician. GTM 5
44. Jones, V.F.R.: Invent. Math. **72**, 1 (1983)
45. Pasquier, V.: Operator content of the ADE lattice models. J. Phys. A **20**, 5707 (1987); Continuum limit of lattice models built on quantum groups. Nucl. Phys. B **295** 491 (1988); Etiology of IRF models. Saclay-SPhT/88/20
46. Kirillov, A.N., Reshetikhin, N.Y.: Representations of the algebra $U_q(sl(2))$. q-orthogonal polynomials and invariants of links. LOMI preprint E-9-88
47. See, for example, Drinfeld, V.: Quantum Groups. In: Proc. at the Intl. Cong. of Math. 1986, p. 798, and references therein
48. Reshetikhin, N.Y.: Quantized Universal Enveloping Algebras. The Yang-Baxter Equation and Invariants of Links. LOMI preprint E-4-87, E-17-87
49. Beilinson, A.A., Schechtman, V.V.: Determinant bundles and Virasoro algebras. Commun. Math. Phys. **118**, 651–701 (1988)
50. LeClair, A., Peskin, M., Preitschopf, C.: String field theory on the conformal plane. SLAC-PUB-4306; SLAC-PUB-4307
51. MacLane, S.: Natural associativity and commutativity. Rice University Studies, Vol. 49, 4, 28 (1963)
52. Harer, J.: The second homology group of the mapping class group of an orientable surface. Invent. Math. **72**, 221 (1983)

53. Birman, J.: Braids, links, and mapping class groups. Ann. Math. Studies, Vol. 82. Princeton, NJ: Princeton University Press 1974
54. Birman, J.: On braid groups. Commun. Pure App. Math. **22**, 41 (1969); Mapping class groups and their relationship to braid groups. Commun. Pure App. Math. **22**, 213 (1969)
55. Wajnryb, B.: A simple presentation for the mapping class group of an orientable surface. Israel J. Math. **45**, 157 (1983); See also the review: Birman, J.: Mapping class group of surfaces. In: Proc. of "Braids" conference, Contemporary Math. (to appear)
56. DiFrancesco, P., Saleur, H., Zuber, J.-B.: Critical Ising correlations in the plane and on the torus. Nucl. Phys. B **290**, 527 (1987)
57. Eilenberg, S., MacLane, S.: Cohomology theory in abstract groups. I. Ann. Math. **48**, 51 (1947)
58. Brown, K.: Cohomology of groups. Berlin, Heidelberg, New York: Springer 1982
59. Jackiw, R.: Three cocycles in mathematics and physics. Phys. Rev. Lett. **54**, 159 (1985) Grossman, B.: A three cocycle in quantum mechanics. Phys. Lett. **152** B, 93 (1985)
60. See, e.g., Loo-Keng, Hua: Introduction to Number Theory, p. 162. Berlin, Heidelberg, New York: Springer 1982

Communicated by L. Alvarez-Gaumé

Received September 28, 1988

Note added in proof. After completing this work, we noticed that the solution for S in terms of F and Ω [11, 18] satisfies Eq. (4.18e). Hence, this equation can be replaced by a definition of S in terms of F and Ω.