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# CLASSICAL AND QUANTUM GRAVITY EFFECTS FROM PLANCKIAN ENERGY SUPERSTRING COLLISIONS

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## A B S T R A C T

We argue that superstring collisions at Planckian energies and small deflection angles are calculable through an evaluation and resummation of string loops in flat space-time. This leads to a systematic expansion of the collision in classical general relativity and quantum string effects. Our results explicitly show how unitarity constraints are recovered through loop corrections and how flat metric calculations reveal the generation of non-trivial geometries by energetic particles. Finite-string-size effects can be evaluated in some regimes and are found to modify classical relativity expectations even at impact parameters much larger than the string scale. Conversely, we find that, through the resummation of large genus contributions, intermediate distance physics provides important classical relativity effects even in the hard, fixed angle regime.

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## 1. - INTRODUCTION AND OUTLINE

The subject of high energy scattering in string theory has been receiving increasing attention recently [1-5]. There are many motivations for such an interest.

1) High energy (i.e., energy much larger than $1/\sqrt{\alpha'}$) is a regime in which the string loop expansion parameter $\alpha_{SL}$ is enhanced by factors $O(\alpha' s) >> 1$. Thus loop corrections become non negligible even for $\alpha_{SL} << 1$ and are actually crucial in order to recover s-channel unitarity bounds which are violated at tree level [4,5].

2) High energies produce large gravitational fields (i.e., non-trivial space-time metrics). These should induce classical general relativity effects which are, however, absent in tree-level calculations around flat space-time. Do string loops account for these expected classical effects?

3) String theory is supposedly a scheme in which quantum gravity effects are calculable. Thus, we do not only expect to reproduce classical general relativity in some large-distance regime; we also expect to learn about quantum and string corrections to general relativity at not-so-large distances, in particular to learn if and how some of the ubiquitous singularities of classical general relativity are avoided in string theory.

4) Finally, if one could combine large-momentum transfers and high energies, one could perhaps understand the short-distance structure of string theory and of space-time itself.

In a recent letter [1] (hereafter referred to as I) we have discussed some results, especially on the issues 1) and 2) mentioned above. The purpose of this paper is twofold: (i) we wish to give a more complete account of the methods used and of the results given in I, and (ii) we shall discuss some new results especially about point 3). It will be evident that, at some point (i.e., at some scale) we shall reach the limit of our present understanding of what goes on in the scattering process. Beyond that limit could lie some very interesting new phenomena bearing on the fundamental issues alluded to above in point 4).

In order to be able to distinguish classical general relativity effects from those due to string and quantum corrections, it is useful to digress a moment on the various scales appearing in string theory in general and in our scattering problem in particular.

String theory possesses a fundamental classical constant, $\alpha' \equiv dJ/dM^2$, which in units $c = 1$ (but not $\hbar = 1$) has dimensions $\ell/E$. Its inverse is the string tension. A convenient system of units in string theory is one [6] in which $\alpha'$ is used to convert energies into lengths:

$$E \to 2\alpha' E \equiv \bar{E} \tag{1.1}$$

in analogy with (D=4) classical relativity ($E \to G_N E$).

In this system of units the usual quantum constant $\hbar$ is replaced by a fundamental quantum length $\lambda_s$ defined by:

$$2\alpha' \hbar \equiv \lambda_s^2 \tag{1.2}$$

and all observables become functions of $\lambda_s$, c and of dimensionless coupling constants.

In order to make contact with more familiar quantities, let us recall [7] that the double expansion of the string effective action in $\sigma$-model and string loops takes the form:

$$\frac{1}{\hbar} S_{eff} \equiv \frac{1}{\lambda_s^2} \bar{S}_{eff} = \frac{1}{\lambda_s^2} \left\{ \frac{1}{\alpha_D \lambda_s^{D-4}} \int d^D x \sqrt{-g} \left( a_1 R + a_2 \lambda_s^2 R^2 + \cdots \right) + \right.$$
$$\left. + \frac{1}{\lambda_s^{D-4}} \int d^D x \sqrt{-g} \left( b_1 R + b_2 \lambda_s^2 R^2 + \cdots \right) + \frac{\alpha_D}{\lambda_s^{D-4}} \int d^D x \sqrt{-g} \left( c_1 R + \cdots \right) + \cdots \right\} \tag{1.3}$$

where D is the number of effective (i.e., uncompactified) dimensions at the scale of the problem under study; $\lambda_s^2$ is the usual parameter of the $\sigma$-model loop-expansion (i.e., an expansion in derivatives of the background fields, see Ref. [7], Chap. 3.4); $\alpha_D$ is the effective string loop expansion parameter in D dimensions, taking into account the volume V of the compactified space through

$$\alpha_D = \alpha_{SL} \frac{\lambda_s^{10-D}}{V} \quad , \quad \alpha_{SL} \equiv \alpha_{10} \tag{1.4}$$

and $\alpha_{SL}$ is the already mentioned string loop, or topological expansion, parameter.

Both $\alpha_{SL}$ and $V\lambda_s^{D-10}$ are related to the expectation values of dimensionless scalar fields. Here we shall regard them as free parameters, to be adjusted in order to define convenient regimes, even if, ultimately, they might be dynamically determined [8]. We have also assumed that no cosmological constant appears in $S_{eff}$.

At sufficiently low energies and sufficiently small $\alpha_D$ the first term in (1.3), the usual Einstein term, dominates and the relevant classical dimensionful constant for gravitational interactions becomes:

$$g_D^2 \equiv \alpha_D \lambda_s^{D-4} = \frac{16\pi G_N}{\alpha'} \tag{1.5}$$

The relevant length associated with an energy E is the corresponding Schwarzschild (or gravitational) radius $R_S(E)$ given by:

$$R_s^{D-3}(E) \sim \frac{g_D^2 \bar{E}}{16\pi} = \frac{\alpha_D \lambda_s^{D-4} \bar{E}}{16\pi} = 2 G_N E \tag{1.6}$$

apart from numerical constants discussed further on.

The quantum gravity scale $\lambda_P$ is given by

$$\lambda_p^{D-2} = G_N \hbar = \frac{g_D^2}{32\pi} \lambda_s^2 = \frac{\alpha_D \lambda_s^{D-2}}{32\pi} \tag{1.7}$$

We are now ready to define the high energy regime that we shall analyze. We always assume that

$$\alpha_D \ll 1 \qquad i.e. \qquad \lambda_p \ll \lambda_s \tag{1.8a}$$

and we look at energies for which the tree amplitude is large, i.e., in the c.m. frame,

$$\alpha_D \frac{\bar{s}}{\lambda_s^2} \gtrsim 1 \qquad (\bar{s} = \bar{E}^2) \tag{1.8b}$$

or $\alpha_D(\alpha' s) \gtrsim 1$, in usual notation. We are therefore interested in the region

$$(\bar{E}/\lambda_s)^2 \gtrsim (\lambda_s/\lambda_p)^{D-2} \gg 1 \tag{1.9}$$

or

$$\alpha' s \gtrsim (M_P \sqrt{\alpha'})^{D-2} \gg 1$$

We also work (to start with) at fixed t, i.e.,

$$|t| \lesssim \frac{1}{\alpha'}, \frac{1}{R_c^2} \qquad \left(|\bar{t}| \lesssim \lambda_s^2, \frac{\lambda_s^4}{R_c^2}\right) \tag{1.10}$$

where the second restriction (in which $R_c \sim V^{1/10-D}$) is there to ensure that compactified momenta are not appreciably excited.

We shall see, however, that these conditions can be somewhat relaxed. The real limitations of our approach are better given, in impact parameter (b) space, by the conditions

$$b > \lambda_s \ , \qquad b > R_s(E) \tag{1.11}$$

which are weaker than (1.10) and cover also the region of small, but fixed, angles (Section 6).

The smallness of $\alpha_D$ allows us to define a hierarchy of contributions, in the high energy limit, corresponding to the various spin values (Regge trajectories) of the exchanged string states. Since the leading (graviton) trajectory is at $\alpha(t) = 2+(\alpha'/2)t$, we expect that graviton exchanges will dominate energetic, light-string scattering amplitudes for any number of loops, the gravitino, dilaton and "charged" state exchanges being subleading by at least one power of s.

The nature of this expansion is best clarified by going to impact parameter space and by observing that the leading loop corrections contain powers of the dimensionless quantity $\alpha_D\ (\bar{s}/\lambda_s^2) = \alpha_D(\alpha' s/\hbar)$. For a single string state, $\bar{s}$ is naturally quantized in units of $\lambda_s^2$ and the loop corrections are $O(\alpha_D)$. Instead, if we regard $\bar{s}$ as a classical energy, remaining $O(1)$ as $\lambda_s^2 \sim \hbar \to 0$, each loop contains an extra inverse power of $\lambda_s^2$ ($\hbar$), which is just the analogue of $e^2/4\pi\hbar c$ per QED loop. The sum over the leading corrections then exponentiates in a typical eikonal fashion, giving $\exp(i/\lambda_s^2\ \bar{S}) = \exp(i/\hbar\ S)$, with an eikonal "phase"*) that can be expanded as

$$\frac{i}{\lambda_s^2}\,\bar{S} = \frac{i}{\lambda_s^2}\,\bar{S}_0 \sum_{p,q,r\geq 0} a_{pqr}\left(\frac{\lambda_s^2}{b^2}\right)^{p}\left(\frac{R_s}{b}\right)^{q(D-3)}\alpha_D^{\,r} \tag{1.12}$$

Here $\bar{S}_0 \sim g_D^2\ \bar{s}\ b^{4-D} \sim \bar{E}b[R_s(\bar{E})/b]^{D-3}$ is the leading classical eikonal phase, while $p \neq 0$ and $q \neq 0$ terms correspond, respectively, to string-size and classical corrections. In the following, the $r \neq 0$ terms will be neglected throughout because of our assumption (1.8).

We shall be able to resum in closed operator form all terms with $p \neq 0$, $q = 0$, i.e., those corresponding to the string corrections to the leading classical term. For classical corrections (i.e., $q \neq 0$ terms) we shall only be able to discuss semiquantitatively the leading term, which turns out to have $q = 2$. This will enable us to identify the region in b in which our previous resummation can be trusted. Such region will depend on whether $R_s$ is smaller or larger than $\lambda_s$ (both

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*) We use inverted commas to underline the fact that this phase is occasionally complex.

cases being a priori possible in our regime) and it is obviously in the former case that our results will have a wider region of validity.

The outline of the paper is the following. In Section 2 we present a "standard" computation of the asymptotic behaviour of the one-string-loop diagram (the torus) in the region of interest, using and correcting a recent paper by Sundborg [9]. In Section 3 we show how the result of Section 2 can be exactly reproduced by using old-fashioned Regge-Gribov techniques. In Section 4 we generalize this method to the generic multiloop case and formally resum the loop series in closed operator form. In Section 5 we analyze the result - including string corrections - in impact parameter space, while in Section 6 we transform the result into momentum transfer (or scattering angle) space showing the emergence of classical relativity effects. This section describes the main features of the resummed amplitude in a self-contained form, for those who are not interested in the detailed calculations of Sections 2-5. In Section 7 we discuss subleading corrections, typically non-leading classical effects. Finally, in Section 8 we discuss our results - which include small fixed-angle scattering - recognizing the importance of string and classical effects at intermediate distances even in comparison with the short-distance contributions discussed by Gross and Mende [2].

## 2. - ASYMPTOTIC BEHAVIOUR OF THE ONE-LOOP AMPLITUDE

In this section we shall obtain the large s, fixed t asymptotic limit of the graviton-graviton scattering amplitude (Fig. 1) up to one loop in Type II superstring theory. Similar results will hold in any closed superstring theory which is one-loop finite.

If 10-D dimensions are compactified, this amplitude, in units $\alpha'_c = \frac{1}{2}\alpha' = \hbar = c = 1$, is given by [7]

$$A_4 = 2 g_0^2 K_{cl} \left\{ \frac{\Gamma(-\frac{s}{2})\,\Gamma(-\frac{t}{2})\,\Gamma(-\frac{u}{2})}{\Gamma(1+\frac{s}{2})\,\Gamma(1+\frac{t}{2})\,\Gamma(1+\frac{u}{2})} + \right.$$

$$\left. + \frac{g_{10}^2\, 2^{-3}}{(16\pi^3)^2}\, \frac{1}{2} \int_F \frac{d^2\tau}{(\mathrm{Im}\,\tau)^5}\, F_2(\tau, R_{c,i}) \int \prod_{r=a,b,c} d^2\nu_r \prod_{r<s} \chi_{rs}^{2 k_r \cdot k_s} \right\} \tag{2.1}$$

where the normalization of the loop term has been taken from Ref. [10].

In this equation, $K_{c\ell}$ is the standard superstring kinematical factor [7] whose asymptotic behaviour is

$$K_{cl} \sim \left(\frac{s}{2}\right)^4 \varepsilon_b \cdot \varepsilon_c \;\; \varepsilon_a \cdot \varepsilon_d \tag{2.2}$$

As already discussed, $g_D^2$ (denoted simply by $g^2$ in the following) and $g_{10}^2$ are the D-dimensional and ten-dimensional gauge couplings respectively, and are related by

$$g_D^2 = g_{10}^2 / V_{10-D} \tag{2.3}$$

where $V_{10-D}$ is the volume of the compact space.

In the integral representing the loop contribution, F is the fundamental region:

$$|\mathrm{Re}\,\tau| < \tfrac{1}{2} \quad , \quad |\mathrm{Re}\,\nu_r| < \tfrac{1}{2} \qquad (r=a,b,c)$$
$$0 \le \mathrm{Im}\,\nu_r \le \mathrm{Im}\,\tau \quad , \qquad 1 \le \mathrm{Im}\,\tau < \infty \tag{2.4}$$

while $F_2$ is the compactification factor for closed strings [7]. This depends on the details of the compact manifold and in particular, for toroidal compactification, on the radii $R_{c,i}$. However, in the limit we consider:

$$t \ll R_{c,i}^{-2} \tag{2.5}$$

the details of compactification do not enter. The relevant limit in parameter space is, as we shall see, Im $\tau \to \infty$ and there we find simply:

$$g_{10}^2 \; F_2(\tau, R_{c,i}) \xrightarrow[\mathrm{Im}\,\tau \to \infty]{} g_{10}^2 \left(2\pi\sqrt{2\,\mathrm{Im}\,\tau}\right)^{10-D} V_{10-D}^{-1} = g_D^2 \left(8\pi^2\,\mathrm{Im}\,\tau\right)^{5-D/2} \tag{2.6}$$

i.e., the dependence of $F_2$ on the radii is just the right one needed to convert $g_{10}^2$ into $g_D^2$ [or to make the leading loop effects we consider $O(g_D^2)$ as expected]. The residual D-dependence of (2.6) is just in the power of Im $\tau$ and will be given a momentum phase space interpretation below. Finally, $\chi_{rs} = \chi(e^{2\pi i(\nu_r - \nu_s)}, e^{2\pi i\tau})$ is given in terms of the Jacobi $\Theta$-function by

$$\chi\left(e^{2\pi i\nu}, e^{2\pi i\tau}\right) = 2\pi \left| \frac{\Theta_1(\nu|\tau)}{\Theta_1'(0|\tau)} \right| \exp\left(-\frac{\pi(\mathrm{Im}\,\nu)^2}{\mathrm{Im}\,\tau}\right) \xrightarrow[\substack{\tau \to i\infty \\ \frac{\nu}{\tau} > 0 \text{ fixed}}]{}$$
$$\to \mathrm{const.} \exp\left[-\pi\frac{(\mathrm{Im}\,\nu)^2}{\mathrm{Im}\,\tau} + \mathrm{Re}\left(i\pi\nu + e^{2\pi i\nu} + e^{2\pi i(\tau-\nu)}\right)\right] \tag{2.8}$$

where we have given the asymptotic behaviour of $\chi$ that will be relevant for our calculation.

Since the expression (2.1) for the amplitude on the mass shell (s+t+u=0) is properly convergent for imaginary s, we shall compute its $s \to i\infty$ asymptotic behaviour. The s and t dependence in the integrand of (2.1) appears in the factor

$$f = \left(\frac{\chi_{ab}\,\chi_{cd}}{\chi_{ac}\,\chi_{bd}}\right)^{-s}\left(\frac{\chi_{bc}\,\chi_{ad}}{\chi_{ac}\,\chi_{bd}}\right)^{-t} \qquad (\nu_d = \tau) \tag{2.9}$$

It is thus easy to realize that, for $s \to +i\infty$, the relevant integration region is the one around the points

$$\mathrm{Im}(\nu_b - \nu_c) = 0 \quad , \quad \mathrm{Im}(\tau - \nu_a) = 0 \tag{2.10}$$

and, by (2.8), of large Im $\tau$ = O(log s).

By using the asymptotic form of $\chi$ in (2.8), we find that the integral over $\nu_b - \nu_c$ and $\tau - \nu_a$ in the region (2.10) is governed by the factor*)

$$\int_F d\,\mathrm{Im}(\tau-\nu_a)\; d\,\mathrm{Im}(\nu_b-\nu_c)\; \exp\left[-\frac{2\pi s}{\mathrm{Im}\,\tau}\,\mathrm{Im}(\tau-\nu_a)\;\mathrm{Im}(\nu_b-\nu_c)\right] \xrightarrow[s\to i\infty]{}$$

$$\to \frac{\mathrm{Im}\,\tau}{i\pi s}\int_0^{\mathrm{const.}}\frac{dy}{y}\,\sin(isy\,\mathrm{Im}\,\tau) \;\to\; \frac{2i}{s}\,\mathrm{Im}\,\tau \tag{2.11}$$

Following Sundborg [9], we then introduce the variables a,b,c and x by

$$\mathrm{Re}\,\nu_a = \mathrm{Re}\,\tau - (b+c) \quad , \quad \mathrm{Re}\,\nu_b = \mathrm{Re}\,\tau - (a+c) \quad , \quad \mathrm{Re}\,\nu_c = \mathrm{Re}\,\tau - (a+b)$$

$$x = \frac{\mathrm{Im}(\nu_b+\nu_c)}{2\,\mathrm{Im}\,\tau} \tag{2.12}$$

so that, by setting $\mathrm{Im}(\nu_b - \nu_c) = \mathrm{Im}(\tau - \nu_a) = 0$, the factor (2.9) takes the form

$$f = \left[4\,\sigma_+\sigma_-\,\exp\left(2\pi\,x(1-x)\,\mathrm{Im}\,\tau\right)\right]^{-t}\exp\Big[s\Big(e^{-2\pi x\,\mathrm{Im}\,\tau}\cos 2\pi a \; +$$

$$+\; e^{-2\pi(1-x)\,\mathrm{Im}\,\tau}\cos 2\pi(\mathrm{Re}\,\tau - a)\Big)\,4\,\sigma_+\sigma_-\Big] \quad , \quad \sigma_\pm = |\sin \pi(b \pm c)| \tag{2.13}$$

Similarly, we perform the integrations over Re $\tau$ and a, which are simple and provide, each, a Bessel function.

Finally, by inserting the various factors (2.2), (2.6), (2.11) and (2.13) into the loop representation (2.1), we obtain

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*) The spurious log s factor present in the computation of Ref. [9] was due to an improper subdivision of the integration region.

$$A^{(h=1)} \underset{s\to i\infty}{\longrightarrow} (g^2)^2\, \varepsilon_a\cdot\varepsilon_d\;\; \varepsilon_b\cdot\varepsilon_c\;\; i\left(\frac{s}{2}\right)^3 I\,,$$

$$I = \int_0^1 dx \int_0^\infty d\,\mathrm{Im}\,\tau\;\; (\mathrm{Im}\,\tau)^{\frac{4-D}{2}} \int_0^1 \frac{d\sigma_+ d\sigma_-\,(4\sigma_+\sigma_-)^{-t}}{[(1-\sigma_+^2)(1-\sigma_-^2)]^{1/2}}\;\cdot$$

$$\cdot\frac{2^{2-D/2}}{\pi^2(2\pi)^{D-4}}\; e^{2\pi t x(1-x)\,\mathrm{Im}\,\tau}\; J_0\!\left(4is\sigma_+\sigma_-\, e^{-2\pi x\,\mathrm{Im}\,\tau}\right)\; J_0\!\left(4is\sigma_+\sigma_-\, e^{-2\pi(1-x)\,\mathrm{Im}\,\tau}\right) \tag{2.14}$$

Here the integrations over $\mathrm{Im}\,\tau$ and $x$ will be dominated, for $s \to i\infty$, by $\mathrm{Im}\,\tau = O(\log s) \to \infty$ and by a saddle point at $x = \frac{1}{2}$. We prefer, however, to recast Eq. (2.14) into a form directly involving powers of $s$, that will then be interpreted as a Regge cut. Indeed, the $\sigma_+, \sigma_-$ and $\mathrm{Im}\,\tau$ integrations can be explicitly performed by a series expansion of the Bessel functions, yielding the expression

$$I = \frac{1}{2}\int \frac{d^{D-2}\underset{\sim}{k}}{(2\pi)^{D-2}} \sum_{n,m} \frac{(-1)^{n+m}\left(\frac{is}{2}\right)^{2(n+m)}}{n!^2\, m!^2}\; \frac{4^{2n+2m-t}\, B^2\!\left(1-\frac{t}{2}+m+n,\, \frac{1}{2}\right)}{\pi^2\left[2n+\left(\frac{1}{2}\underset{\sim}{q}+\underset{\sim}{k}\right)^2\right]\left[2m+\left(\frac{1}{2}\underset{\sim}{q}-\underset{\sim}{k}\right)^2\right]} \tag{2.15}$$

where $\underset{\sim}{q}^2 = |t|$, $B$ is the usual beta-function, and the $(D-2)$-dimensional transverse integration $d\underset{\sim}{k}$ replaces the $x$-integration by the identity

$$\int_0^1 \frac{dx\; \pi^{\frac{D-2}{2}}\, \Gamma\!\left(\frac{6-D}{2}\right)}{[q^2 x(1-x) + 2nx + 2m(1-x)]^{\frac{6-D}{2}}} = \int \frac{d^{D-2}\underset{\sim}{k}}{\left[2n+\left(\frac{1}{2}\underset{\sim}{q}+\underset{\sim}{k}\right)^2\right]\left[2m+\left(\frac{1}{2}\underset{\sim}{q}-\underset{\sim}{k}\right)^2\right]} \tag{2.16}$$

Finally, we can sum the series in (2.15) by a double Sommerfeld-Watson transform [11] in the n,m complex planes [i.e., picking up the poles at $2n = -(\frac{1}{2}\underset{\sim}{q}+\underset{\sim}{k})^2 = +t_1$, $2m = -(\frac{1}{2}\underset{\sim}{q}-\underset{\sim}{k})^2 = +t_2$] and we obtain

$$A^{(h=1)} \xrightarrow[s\to\infty]{} \varepsilon_a\cdot\varepsilon_d\ \varepsilon_b\cdot\varepsilon_c\ i\left(\frac{s}{2}\right)^3 (g^2)^2\ \frac{1}{2}\int \frac{d^{D-2}\underset{\sim}{k}}{(2\pi)^{D-2}} \left(\frac{s\,e^{-i\pi/2}}{2}\right)^{t_1+t_2} \cdot$$

$$\cdot\ \frac{\Gamma(-\frac{t_1}{2})}{\Gamma(1+\frac{t_1}{2})}\ \frac{\Gamma(-\frac{t_2}{2})}{\Gamma(1+\frac{t_2}{2})}\ \frac{\Gamma^2(1+t_1+t_2-t)}{\Gamma^4(1+\frac{t_1+t_2-t}{2})} \tag{2.17}$$

where we have continued the phase in the integrand to the physical s-axis.

Equations (2.16) and (2.17) show that the factor $(\sqrt{\mathrm{Im}\tau})^{10-D}$ arising in (2.6) from the compactified dimensions corresponds to reducing the transverse phase space integration to (D-2) physical dimensions.

The final expression (2.17) contains a superposition of powers of s, typical of a Regge cut, with a maximum power (given by $t_1=t_2=t/4$) at $J = 3+t/2$. Its leading asymptotic behaviour is

$$A^{(h=1)} \xrightarrow[\substack{s\to\infty\\ t<0}]{} \varepsilon_a\cdot\varepsilon_d\ \varepsilon_b\cdot\varepsilon_c\ \frac{g^4}{2}\ \frac{\left(e^{-\frac{i\pi}{2}}\frac{s}{2}\right)^{3+\frac{t}{2}}}{(8\pi\log s)^{\frac{D-2}{2}}} \left(\frac{\Gamma(-\frac{t}{8})}{\Gamma(1+\frac{t}{8})}\right)^2 \frac{\Gamma^2(1-\frac{t}{2})}{\Gamma^4(1-\frac{t}{4})} \tag{2.18}$$

This behaviour is valid for $|t|\log s \gg 1$ and comes from the region $x \simeq \frac{1}{2}$ in (2.16), or $t_1 = t_2 = t/4$ in (2.17). In the opposite situation, i.e., $t \to 0$, the region of small $t_1,t_2$ (or $x\to 0,1$) dominates and we obtain

$$A^{(h=1)} \xrightarrow[t\to 0]{} 2i\left(\frac{s}{2}\right)^3 g^4 \int \frac{d^{D-2}\underset{\sim}{k}}{(2\pi)^{D-2}\, t_1 t_2} \sim i\,\frac{\left(\frac{g^2 s}{2}\right)^3}{|t|^{\frac{6-D}{2}}}\ \frac{4^{4-D}\,\Gamma(D-4)\,\Gamma(\frac{6-D}{2})}{\Gamma(\frac{D-3}{2})\,\pi^{\frac{D-3}{2}}}$$

$$(D<6) \tag{2.19}$$

This shows the existence of a singular $t \to 0$ behaviour for $D < 6$, and of an infra-red singularity at $D = 4$. Both are related to the (massless) graviton exchanges in the loop diagram.

## 3. - THE REGGE-GRIBOV METHOD AT ONE-LOOP LEVEL

Having obtained the large s behaviour of the one-loop (closed) superstring amplitude, we want to propose here a direct construction of its asymptotic form through a simpler method which can be extended to multiloops.

Let us first see how the Regge-Gribov method [12] - that was originally proposed in strong interaction physics - can be applied to strings, which are known to be Regge-behaved at tree level. We know that the one-loop amplitude can be obtained by a sewing procedure [13] (with a complete set of string intermediate states) in which each element is a tree (Fig. 1). The same element gives rise to, say, a six-point graviton tree amplitude (Fig. 2a).

When $s \to \infty$ and t are kept fixed, this sewing procedure simplifies, because only the (non-degenerate) leading Regge trajectory contributes*) (Figs. 2b and 3).

The Regge-Gribov method for strings consists therefore of extracting, by factorization, this leading behaviour from the explicit tree expansion (Fig. 2b) in order to construct the loop behaviour (Fig. 3).

The fact that this method correctly gives the asymptotic behaviour of loop amplitudes rests on the very structure of string theory. We will be nevertheless conforted by reobtaining in this way the one-loop result of Section 2.

It is known that the tree amplitude is Regge-behaved. For graviton-graviton scattering we have, from Eq. (2.1),

$$A_{tree}(ab \to cd) = \varepsilon_a \cdot \varepsilon_d \; \varepsilon_b \cdot \varepsilon_c \; a_{tree}(s,t)$$

$$a_{tree}(s,t) \xrightarrow[s \to \infty]{} \beta(t)\, s^{\alpha(t)} = 2g^2 \frac{\Gamma(-t/2)}{\Gamma(1+\frac{t}{2})} \left(\frac{s}{2}\right)^{2+t} e^{-\frac{i\pi}{2}t} \tag{3.1}$$

where $\varepsilon_i$ are the graviton polarization tensors.

The corresponding Regge amplitude is a simple pole in the t-channel angular momentum at $J = \alpha(t) = 2+t$, i.e.,

$$a_J(t) \sim \frac{g^2}{J - \alpha(t)} \tag{3.2}$$

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*) Note that only physical states contribute to the leading Regge trajectory.

corresponding to the graviton trajectory exchange. It is a simple matter to show [5] that the impact parameter transform of (3.1) [given below, Eq. (5.2)] violates partial wave unitarity at high energy, thus indicating that loop corrections are important.

The asymptotic form of the loop amplitude is given by the double graviton Regge exchange of Fig. 3, which reads, in the J plane,

$$A_J^{(h=1)}(t) = \int ds\, s^{-J-1}\, \mathrm{Im}\, A^{(h=1)}(s,t)$$

$$\mathrm{Im}\, A^{(h=1)}(s,t) = 2\int \frac{dq_1\, dq_2}{(2\pi)^{D-2}}\, \delta(q_1+q_2-q)\; s^{\alpha(t_1)+\alpha(t_2)}\; Re(\beta(t_1)\beta(t_2))\, .$$

$$\cdot \int \frac{dM_u^2\, dM_d^2}{s}\; \theta(s-M_u^2 M_d^2)\; \mathrm{Im}\, A^{\alpha_1\alpha_2}(M_u^2, t)\, .\, \mathrm{Im}\, A^{\alpha_1\alpha_2}(M_d^2, t) \qquad (3.3)$$

where we have related the s-discontinuity of $A^{(h=1)}$ to the ones of the gravi-Reggeon (GR) amplitudes $A^{\alpha_1\alpha_2}$. The latter are defined by the asymptotic behaviour of the six-point function (Fig. 2b), as

$$A_6 \xrightarrow[s_1, s_2 \to \infty]{} \varepsilon_1.\varepsilon_2\; \varepsilon_3.\varepsilon_4\; \beta(t_1)\, s_1^{\alpha(t_1)}\, \beta(t_2)\, s_2^{\alpha(t_2)}\; A^{\alpha_1\alpha_2}(M^2, t)$$

$$t_1 = -q_1^2 = -(k_1+k_2)^2 \quad , \quad t_2 = -q_2^2 = -(k_3+k_4)^2 \qquad (3.4)$$

By inserting the definition (3.3a) into (3.3b) and taking into account the phase space boundary, we obtain the J-plane amplitude

$$A_J^{(h=1)}(t) = \int dq_1\; Re(\beta(t_1)\beta(t_2))\left[A^{\alpha_1\alpha_2}_{J-\alpha_1-\alpha_2}(t)\right]^2 \qquad (3.5)$$

where $dq_1$ denotes the transverse momentum integration in D-2 dimensions, and $q_2 = q - q_1$. Note the shift $J \to J-\alpha_1-\alpha_2$ in the angular momentum of the GR amplitudes.

Our connection with string theory is provided by the expression of the six-point function from which, by (3.5), we shall calculate $A^{\alpha_1\alpha_2}$. As is well known, $A_6$ is given in terms of the matrix element (Fig. 4)

$$\langle \varepsilon_a, k_a |\, W_{\varepsilon_1}(k_1, 1)\; W_{\varepsilon_2}(k_2, z_1)\; W_{\varepsilon_3}(k_3, \xi z_1)\; W_{\varepsilon_4}(k_4, \xi z_1 z_2)\, | \varepsilon_d, k_d \rangle \qquad (3.6)$$

where $W_\varepsilon$ is the string vertex for graviton emission [7]

$$W_\varepsilon(k,z) = \varepsilon_{ij} \, B^i \, \tilde{B}^j \, e^{ik.\left(X(z) + \tilde{X}(\bar{z})\right)} \tag{3.7}$$

and, in light-cone notation, the expressions

$$\begin{aligned} B^i &= P^i - R^{ij} k_j = z \frac{\partial}{\partial z} X^i(z) - \frac{1}{4}\left(\gamma^{ij}\right)_{ab} S^a(z) \, S^b(z) \\ X^i(z) &= \frac{1}{2} x^i - i p^i \log z + i \sum_{n \neq 0} \frac{\alpha_n^i}{n} z^{-n} \\ S^a(z) &= \sum_n S_n^a z^{-n} \end{aligned} \tag{3.8}$$

with

$$\begin{aligned} \left[\alpha_n^i, \alpha_m^j\right] &= n \, \delta_{n+m,0} \, \delta^{ij} \\ \left\{S_n^a , S_m^b\right\} &= \delta_{n+m,0} \, \delta^{ab} \end{aligned} \tag{3.9}$$

represent the usual left-moving operators, and similar expressions hold for the right-moving ones in terms of $\tilde{\alpha}^i_n$ and $\tilde{S}^a_n$.

The $s_1$ and $s_2$ asymptotic behaviour of $A_6$ in Fig. 4 is determined, as usual, by the $z_1 \to 1$ and $z_2 \to 1$ behaviour of the matrix element in Eq. (3.6). This is given in turn by the leading singularity in the operator product expansion (OPE) of pairs of vertices $W_\varepsilon$, that we shall now determine.

From their expression in (3.8) we find the OPE

$$\begin{aligned} P^i(z) \, P^j(w) &\underset{z \to w}{\sim} \delta_{ij} \frac{zw}{(z-w)^2} \\ P^i(z) \, e^{ih.X(w)} &\underset{z \to w}{\sim} k_i \frac{z}{z-w} \\ R^{ij}(z) \, R^{kl}(w) &\underset{z \to w}{\sim} \left(-\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right) \frac{zw}{(z-w)^2} \end{aligned} \tag{3.10}$$

Then, from their definition in Eq. (3.7), we obtain

$$W_{\varepsilon_1}(k_1,z)\, W_{\varepsilon_2}(k_2,w) \sim \varepsilon_1\cdot\varepsilon_2\,(1-k_1\cdot k_2)^2 \frac{|z-w|^2}{|z-w|^{4-(k_1+k_2)^2}}\, W_0(k_1+k_2,z)$$

$$W_0(q,z) = :e^{iq\cdot[X(z)+\tilde{X}(\bar{z})]}: \tag{3.11}$$

Let us briefly discuss the physical content of Eq. (3.11). The operator $W_0$ is an off-shell scalar vertex: it occurs here because the helicity of the external gravitons (1,2) is conserved, so that the exchanged graviton (q) is emitted in a well-defined helicity state [++, say, if $k_1$ is along the (D-1) axis]. The $(1-k_1k_2)^2$ factor cancels the would-be tachyon pole of the scalar vertex and is due to a compensation*) of the bosonic and fermionic ($\dot{X}$ and R) contributions to $W_\varepsilon$ in Eq. (3.7).

We finally note that

$$|z-w|^{-4+(k_1+k_2)^2} = |z-w|^{-2-\alpha((k_1+k_2)^2)} \tag{3.12}$$

so that the behaviour (3.11), inserted in (3.6), correctly gives the asymptotic behaviour (3.4). This provides the expression for $A^{\alpha_1\alpha_2}$:

$$A^{\alpha_1\alpha_2} = \varepsilon_a\cdot\varepsilon_d\; a^{\alpha_1\alpha_2}(M^2;q_1,q_2) = \int\frac{d^2\zeta}{\pi|\zeta|^2}\langle\varepsilon_a,k_a|\,W_0(q_1,1)\,|\zeta|^{q_2^2}\,W_0(q_2,\zeta)\,|\varepsilon_d,k_d\rangle =$$

$$= \varepsilon_a\cdot\varepsilon_d\int\frac{d^2\zeta}{\pi}|\zeta|^{-\alpha(M^2)}|1-\zeta|^{2q_1\cdot q_2} \tag{3.13}$$

which is simply tree-like, with poles in $M^2$ starting at $M^2 = 0$.

In order to go to the t-channel angular momentum, used in Eq. (3.5), it is convenient to define, for dual amplitudes like (3.13), the beta-transform [14]

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*) A similar compensation holds for the heterotic string, with a single power of $(1-k_1k_2)$.

$$\tilde{a}_J(t) = \int_{-i\infty}^{i\infty} \frac{ds}{2\pi i}\, a(s,t)\, B(s+1, t+1) \qquad (a)$$

$$a(s,t) = \int_{-i\infty}^{i\infty} \frac{dJ}{2\pi i}\, \tilde{a}_J(t) \left[ B\left(-J, -\frac{s}{2}\right) + B\left(-J, -\frac{u}{2}\right)\right] \qquad (b) \tag{3.14}$$

where B(x,y) is the usual beta function. This transform, for the leading singularities, is simply related to the Mellin transform (3.3a) by

$$\hat{a}_J = a_J\, \Gamma(J+1) \tag{3.15}$$

The beta transform of (3.13) can then be given in closed form

$$\tilde{a}_J^{\alpha_1\alpha_2}(q_1,q_2) = \frac{\Gamma\left(J+\alpha(t_1)+\alpha(t_2)-\alpha(t)\right)\ \Gamma(J+1)}{\Gamma^2\left(J+1+\frac{\alpha(t_1)+\alpha(t_2)-\alpha(t)}{2}\right)} \tag{3.16}$$

showing a Regge pole at

$$J = \alpha(t)-\alpha(t_1)-\alpha(t_2) = -2+t-t_1-t_2 = -2+2q_1\cdot q_2 \tag{3.17}$$

and a fixed pole at J = -1.

Note that, because of the shift $\Delta J = -(\alpha_1+\alpha_2)$ of Eq. (3.17) (due to the peculiar helicities of the exchanged gravitons in Fig. 1b), the fixed pole is leading over the moving one at small $t_i$. Its residue, that we shall call the two-gravi-Reggeon vertex $V_2(q_1,q_2)$, is given by

$$V_2 = \frac{\Gamma(1+t_1+t_2-t)}{\Gamma^2\left(1+\frac{t_1+t_2-t}{2}\right)} = \sum_n \left(\frac{1}{n!}\,\frac{\Gamma(-q_1\cdot q_2+n)}{\Gamma(-q_1\cdot q_2)}\right)^2 = \int_{-i\infty}^{i\infty} \frac{dM^2}{2\pi i}\, a^{\alpha_1\alpha_2}(M^2;q_1,q_2) \tag{3.18}$$

Here the sum over string states reproduces the usual sum rule of the last equality which is just $a^{\alpha_1\alpha_2}_{J=-1}$. The latter shows that $V_2$ is obtained from the integral expression (3.13) by setting $|\zeta| = 1$ and has therefore the equivalent form

$$V_2 = \int_0^{2\pi} \frac{d\sigma}{2\pi} \langle 0 | : e^{i q_1 \cdot \hat{X}(1)} :: e^{i q_2 \cdot \hat{X}(\sigma)} : | 0 \rangle = \int_0^{2\pi} \frac{d\sigma}{2\pi} \left| 1 - e^{i\sigma} \right|^{2 q_1 \cdot q_2} \tag{3.19}$$

where $x(\sigma)$ is the non-zero mode contribution to the usual closed string position operator*) at $\tau = 0$, i.e.,

$$\hat{X}(\sigma) = i \sum_{n \neq 0} \frac{1}{n} \left( \alpha_n^i e^{i\sigma n} + \tilde{\alpha}_n^i e^{-i\sigma n} \right) \tag{3.20}$$

We then see from (3.19) that $V_2$ is operatorially factorized, with a spectrum of massive excitations given in (3.18). For $q_1, q_2 \to 0$, $V_2 = 1 + O((q_1 q_2)^2)$, and thus the inelastic channels decouple in the soft graviton limit.

We are now ready to discuss the loop asymptotic behaviour. By inserting (3.15) and (3.16) into (3.5) and by using the explicit form of $\beta(t)$ in (3.1), we find

$$A_J^{(h=1)}(t) = \frac{g^4}{(2\pi)^{D-2}} \varepsilon_a \cdot \varepsilon_d \, \varepsilon_b \cdot \varepsilon_c \int \frac{d^{D-2} q_1}{J - 3 - t_1 - t_2} \frac{\Gamma(-t_1/2)}{\Gamma(1 + t_1/2)} \frac{\Gamma(-t_2/2)}{\Gamma(1 + t_2/2)} \cdot$$

$$\cdot \, 2^{-\alpha(t_1) - \alpha(t_2)} \cos\left(\pi \frac{t_1 + t_2}{2}\right) \Gamma^2(J - 2 - t) \, \Gamma^{-4}\left(J - 2 - \frac{t_1 + t_2 + t}{2}\right) \tag{3.21}$$

We recognize in Eq. (3.21) the two-graviton cut at $J = \alpha_c = \alpha(t_1) + \alpha(t_2) - 1$, with tip at $J = 3 + t/2$. It is easy to check that its J-plane discontinuity is given by the square of the two-gravi-Reggeon vertex $V_2(q_1, q_2)$ just discussed. We find also simple and double poles at $J = \alpha(t) = 2 + t$, which are thus subleading. We notice that the pole residues vanish at $2 q_1 q_2 = t_1 + t_2 - t = 0$ due to the $\Gamma^{-4}$ factor in (3.21), consistently with the non-renormalization of the graviton mass**).

Note, however, that cut and pole contributions will become comparable for $q_1 q_2 = O(1)$, i.e., away from the cut tip. Indeed, for $2 q_1 q_2 = t_1 + t_2 - t = -1$, i.e., $\alpha_c = \alpha$, the pole contributions happen to cancel the spurious singularity, appearing

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*) The zero mode at $\tau = 0$, i.e. $|\zeta| = 1$, provides the momentum and helicity conserving factor $\varepsilon_a \cdot \varepsilon_d \; \delta(k_a + k_d + \Sigma q_i)$.

**) The exact vanishing at $t = 0$ of the pole residues according to superstring non-renormalization theorems needs a compensation with subleading terms, at $J_i = \alpha(t_i) - n_i$, that we have not considered here.

in the cut discontinuity according to Eq. (3.18), and absent in (3.21). A similar mechanism is responsible for the cancellation of the spurious poles at positive t, appearing in the asymptotic expression (2.18).

The cut generates, for $t < O(1)$, the leading asymptotic behaviour of the loop which [inverting the Mellin transform (3.3a)] reads

$$A^{(h=1)}(s,t) \to \varepsilon_a\cdot\varepsilon_d\;\varepsilon_b\cdot\varepsilon_c\;\frac{i}{2s}\int\frac{d^{D-2}q_1\,d^{D-2}q_2}{2(2\pi)^{D-2}}\,\delta^{D-2}(q-q_1-q_2)\;.$$
$$.\;a_{tree}(s,q_1)\;a_{tree}(s,q_2)\left[V_2(q_1,q_2)\right]^2 \tag{3.22}$$

By recalling the explicit expressions for $a_{tree}$ [Eq. (3.1)] and $V_2$ [Eq. (3.18)] we see that this result is identical to the one obtained in Section 2 from the integral representation over the torus.

We have obtained here the additional information that the cut discontinuity has an operatorially factorized form. In fact, by interpreting the $V_2{}^2$ in (3.22) as a $V_2$ factor for each of the two strings exchanging graviReggeons in Fig. 3, we can write from Eq. (3.19)

$$V_2^2(q_1,q_2) = \langle 0|\int\frac{d\sigma_u\,d\sigma_d}{(2\pi)^2}:e^{iq_1(\hat{X}_u(0)-\hat{X}_d(0))}::e^{iq_2(\hat{X}_u(\sigma_u)-\hat{X}_d(\sigma_d))}:|0\rangle \tag{3.23}$$

where u(d) refers to the upper (lower) string in Fig. 3.

Therefore, the torus asymptotic behaviouir (3.22) reduces to a convolution in q-space of tree amplitudes times a string vertex $\exp[iq\cdot(X_u(\sigma_u)-X_d(\sigma_d))]$, averaged over string angles $\sigma$'s. The fact that this vertex is evaluated at $\tau_u = \tau_d = 0$ stems from the $|\zeta| = 1$ constraint on each of the tree amplitudes described by Eq. (3.13). This is equivalent to the restrictions $\mathrm{Im}(\nu_b-\nu_c) = \mathrm{Im}(\tau-\nu_a) = 0$ found on the torus [Eq. (2.10)].

## 4. - MULTILOOPS, RESUMMATION AND UNITARITY

### 4.1 Extension to multiloops

The Regge-Gribov method of Section 3 can be simply extended to h loops (Fig. 5) . The sewing procedure involves, in this case, the analysis of the 2N+2-point tree amplitude, where N = h+1 gravi-Reggeon trajectories are exchanged

(Fig. 6). Analogously to Eq. (3.4), this Regge limit defines the N-GR amplitude $A^{\alpha_1\ldots\alpha_N}$ by

$$A_{2N+2} \xrightarrow[s_i\to\infty]{} \left(\prod_{i=1}^{N} \beta(t_i)\, s_i^{\alpha(t_i)}\right) A^{\alpha_1\ldots\alpha_N}(M_1^2,\ldots M_{N-1}^2; q_1,\ldots q_N) \qquad (4.1)$$

Such amplitudes have then to be inserted in the h-loop integral over the momenta $q_i$ of Fig. 5. In the c.m. frame, for large s and fixed t, we have

$$p_a^+,\ p_d^+,\ p_b^-,\ p_c^- = O(\sqrt{s}) \qquad (4.2)$$

where $p^\pm = p_o \pm p_{D-1}$ denote the longitudinal momentum components, and the intermediate masses are

$$\left(M_i^u\right)^2 = \sqrt{s}\sum_{j=1}^{i} q_j^+ + O(q_+q_-, q\cdot q)$$

$$\left(M_i^d\right)^2 = \sqrt{s}\sum_{j=1}^{i} q_j^- + O(q_+q_-, q\cdot q) \qquad (4.3)$$

Since the loop integral in Fig. 5b will turn out to be dominated by values of $M_{i+1}^2 - M_i^2 \sim O(1)$, this implies that $q_i^+,\ q_i^- = O(1/\sqrt{s})$, and therefore that the momentum transfers are essentially transverse. In the loop integrations we can then use the replacement

$$\prod_{i=1}^{h} dq_i^+ dq_i^- \to \frac{1}{s^h}\prod_i d(M_i^u)^2\, d(M_i^d)^2 \qquad (4.4)$$

to obtain

$$A^h(s,t) \xrightarrow[s\to\infty]{} \frac{1}{(h+1)!}\int \prod_{i=1}^{h} \frac{d(M_i^u)^2 d(M_i^d)^2}{i\, s(2\pi)^2} \cdot \prod_{i=1}^{h+1} \frac{dq_i}{(2\pi)^{D-2}}\, \delta(\Sigma q_i - q)\,.$$

$$\cdot\, s^{\sum_{i=1}^{h+1}\alpha(t_i)}\, \beta(q_1^2)\ldots\beta(q_{h+1}^2)\, A^{\alpha_1\ldots\alpha_{h+1}}(M_1^{u2}\ldots M_h^{u2}; q_i)\; A^{\alpha_1\ldots\alpha_{h+1}}(M_1^{d2}\ldots M_h^{d2}; q_i)$$

(4.5)

where the 1/(h+1)! in front of the integral has the meaning of the Bose counting factor of (h+1) identical Reggeons.

The $M_i^2$ integrals in (4.5) run over the physical regions of the right-hand and left-hand cuts of $A^{\alpha_1\ldots\alpha_N}$, and can therefore be distorted, for $s\to\infty$, around the right-hand cut, to yield

$$\int \prod_{j=1}^{N-1} \left( \frac{d M_j^2}{2\pi i} \right) . A^{\alpha_1 \cdots \alpha_N} = \int_0^\infty \prod_j \frac{d M_j^2}{2\pi} \, \mathrm{Disc}_{M_j^2} A^{\alpha_1 \cdots \alpha_N} = V_N(q_1 \cdots q_N) \tag{4.6}$$

This $M^2$ integral defines the general N-GR vertex. As in the one-loop case it is related to the residue of the t-channel partial wave amplitudes $A_{J_1 \ldots J_{N-1}}^{\alpha_1 \ldots \alpha_N}$ at the multiple fixed pole $J_1 = J_2 = \ldots J_{N-1} = -1$, $J_i$ being conjugate to $M_i^2$.

The explicit calculation of $A^{\alpha_1 \ldots \alpha_N}$ and $V_N$ completely parallels the one of Section 3. We can write $A_{2N+2}$ in terms of 2N graviton emission vertex operators as in Eq. (3.6). The $s_i \to \infty$ behaviour of Eq. (4.1) is again given by a chain of N off-shell scalar vertices $W_0(\zeta_i)$, as in Eq. (3.13) (Fig. 6).

Finally, the $M^2$ integrations defining $V_N$ in Eq. (4.6) fix $|\zeta_i| = 1$ in the integral representation of $A^{\alpha_1 \ldots \alpha_N}$, as was the case in Eq. (3.19) for $V_2$. We thus obtain

$$V_N(q_1, \cdots q_N) = \langle 0 | \int \prod_{i=1}^N \frac{d\sigma_i}{2\pi} : e^{i q_i \cdot \hat{X}(\sigma_i)} : | 0 \rangle =$$

$$= \int \prod_{i=1}^N \frac{d\sigma_i}{2\pi} \, . \prod_{1 \le j \le \ell \le N} \left| e^{i\sigma_j} - e^{i\sigma_\ell} \right|^{2 q_\ell \cdot q_j} \tag{4.7}$$

By then replacing the expressions (4.5) and (4.7) of $V_N$ into the loop integral (4.4) we obtain the h-loop asymptotic behaviour

$$A^{(h)}(s,t) \underset{s \to \infty}{\longrightarrow} \varepsilon_a \cdot \varepsilon_d \; \varepsilon_b \cdot \varepsilon_c \left( \frac{i}{2s} \right)^h \int \frac{d^{D-2} q_1 \ldots d^{D-2} q_{h+1}}{(h+1)! \, (2\pi)^{(D-2)h}} \, \delta^{(D-2)}(q - \varepsilon q_i) \, .$$

$$. \prod_{j=1}^{h+1} a_{tree}(s, t_j) \langle 0 | \int \prod_j \frac{d\sigma_j^u d\sigma_j^d}{(2\pi)^2} : e^{i q_j \cdot \left( \hat{X}^u(\sigma_u) - \hat{X}^d(\sigma_d) \right)} : | 0 \rangle \tag{4.8}$$

Equations (4.7) and (4.8) for the N-GR vertex and the h-loop amplitude, being written in operatorial form, show very interesting properties.

To start with, the N-GR vertex $V_N$ is operatorially factorized as in the one-loop case, and is symmetrical in the N Reggeon momenta. The latter property is a simple consequence of the fact that the equal time operators $X(\sigma)$ commute, and is anyway expected in a theory of closed strings. Furthermore, $V_N$ shows the infra-red (IR) decoupling, in the sense that

$$V_{N+1}(q_1 \cdots q_{N+1}) \xrightarrow[q_{N+1}\to 0]{} V_N(q_1 \cdots q_N) \quad , \quad V_1 = 1$$

$$V_N \sim 1 + O\Big(\sum_{j<\ell} (q_j \cdot q_\ell)^2\Big) \tag{4.9}$$

4.2 Resummation in impact parameter space

The operator factorization implies that $A^{(h)}$ in Eq. (4.8) is a multiple convolution in q-space, thus diagonalized by an impact parameter transform. By defining in general

$$\frac{1}{s} A(s,t) = \varepsilon_a \cdot \varepsilon_d \; \varepsilon_b \cdot \varepsilon_c \; 4 \int d^{D-2} \underset{\sim}{b} \; e^{i q \cdot b} \, a(s,b) \tag{4.10}$$

we obtain

$$a^h(s,b) = \frac{(2i)^h}{(h+1)!} \langle 0 | \left[ \hat{\delta}(s, b, \hat{X}^u, \hat{X}^d) \right]^{h+1} | 0 \rangle \tag{4.11}$$

with

$$\hat{\delta} = \int \frac{d q}{(2\pi)^{D-2}} \frac{a_{tree}(s,t)}{s} \int \frac{d\sigma_u d\sigma_d}{(2\pi)^2} : e^{i q \cdot (\underset{\sim}{b} + \hat{\underset{\sim}{X}}^u(\sigma_u) - \hat{\underset{\sim}{X}}^d(\sigma_d))} : =$$

$$= \int \frac{d\sigma_u d\sigma_d}{(2\pi)^2} : a_{tree}(s, \underset{\sim}{b} + \hat{\underset{\sim}{X}}^u(\sigma_u) - \hat{\underset{\sim}{X}}^d(\sigma_d)) : \tag{4.12}$$

being an operator functional of $\hat{X}(\sigma)$.

Summing Eq., (4.11) over loops yields the final result for the amplitude

$$a(s,b) = \sum_{h=0}^{\infty} a^{(h)}(s,b) = \langle 0 | \frac{1}{2i} \left( \hat{S}(s,b,\hat{X}^u,\hat{X}^d) - 1 \right) | 0 \rangle$$

$$\hat{S} = \exp 2i\hat{\delta} \tag{4.13}$$

which has an operator eikonal form. This fact, together with the simple dependence of the "phase" $\hat{\delta}$ on $\hat{X}$, shows that the collision process can be interpreted as a rescattering series at displaced impact parameter $(\underset{\sim}{b}+\underset{\sim}{X}^u-\underset{\sim}{X}^d)$, as depicted in Fig. 7.

## 4.3 Unitarity

We have discussed so far graviton-graviton scattering (or, similarly, gravitino and dilaton scattering). It is, however, clear that an analogous treatment could be performed for two-body scattering of string excitations of masses $M_a$ and $M_b$, as long as $M_a, M_b \ll \sqrt{s}$. The resummed amplitude will be a matrix element like the one in Eq. (4.13), but between the corresponding excited states.

It is clear, on the other hand, that the Regge-Gribov construction does not apply to high energy processes in which the produced states have sum of masses of order $\sqrt{s}$, and therefore small velocities. Both kind of states appear of course in string amplitudes and will be correlated by unitarity relations. It is therefore legitimate to ask ourselves how (s-channel) unitarity is satisfied by our result (4.13) that concerns only a limited class of amplitudes.

This problem is solved by the AGK [12] discontinuity rules which connect s-channel cuts corresponding to sum of masses of $O(\sqrt{s})$ ("inelastic" cuts) to two-body ("diffractive") cuts, i.e., with small masses, which are under control, i.e., are directly treated by Eq. (4.13).

In our treatment, such inelastic states involved by unitarity give rise to an imaginary part of the $\hat{\delta}$ operator (4.12) ($\hat{\delta}-\hat{\delta}^+ \neq 0$). If for some value of b such states do not contribute, $\hat{\delta}$ is hermitian, and the $\hat{S}$ operator is explicitly unitary. As we shall see, this is the case for $b > O(\log s)$, where unitarity is satisfied by diffractive intermediate states only.

For b values for which $\hat{\delta}$ is not hermitian, its discontinuity is given in terms of a cut Reggeon, corresponding to inelastic string states [with sum of masses $O(\sqrt{s})$]. For h loops, the number $N_c$ of cut Reggeons ranges from zero up to h+1, and the AGK rules (generalized to our operator case) give the amplitude discontinuity as a sum over $N_c$, as follows (Fig. 8):

$$\frac{2}{i}\left(a_N - a_N^+\right) = {\sum}' S_{N_-}^+ \frac{\left[\frac{2}{i}\left(\delta - \delta^+\right)\right]^{N_c}}{N_c!} S_{N_+} \delta_{N,\, N_+ + N_- + N_c} \tag{4.14}$$

Here, $S_{N_+}$ ($S^+_{N_-}$) is the rescattering operator with $N_+$($N_-$) exchanged Reggeons to the right (left) of the discontinuity cut, and $\Sigma'$ runs over the non-negative values of $N_c$, $N_+$, $N_-$, satisfying $N_c+N_++N_- = N$ except for the values $N_c = N_+ = 0$ ($N_-=N$) and the symmetric one, which corresponds to "disconnected" terms.

In our case, we can set operatorially, as in Eq. (4.11)

$$2\, i\, a_N = S_N = \frac{\left(2\, i \hat{\delta}\right)^N}{N!} \tag{4.15}$$

and it is a simple matter to check that Eq. (4.14) is identically satisfied. A simplifying point is the fact that $\delta$ and $\delta^+$, being functionals of the $\tau = 0$ field $X(\sigma)$, commute, and therefore (4.14), by Eq. (4.15), reduces to the identity:

$$0 = \left[2\, i\, \delta - 2\, i\, \delta^+ + \frac{2}{i}\left(\delta - \delta^+\right)\right]^N = \sum_{N_c + N_+ + N_- = N} \frac{\left(-2\, i\, \delta^+\right)^{N_-}}{N_-!} \frac{\left(2\, i\, \delta\right)^{N_+}}{N_+!} \frac{\left[\frac{2}{i}\left(\delta - \delta^+\right)\right]^{N_c}}{N_c!}$$

In particular, in the one-loop case, Eq. (4.14) follows from the q-space identity

$$\cos \frac{\pi}{2}(\alpha_1 + \alpha_2) = \cos \frac{\pi}{2}(\alpha_1 - \alpha_2) + \left(-4 \sin \frac{\pi}{2}\alpha_1 \sin \frac{\pi}{2}\alpha_2\right) + 2 \sin \frac{\pi}{2}\alpha_1 \sin \frac{\pi}{2}\alpha_2 \tag{4.16}$$

in which the three terms in the right-hand side correspond to $N_c = 0,1,2$, respectively. We recognize here the relation between the phase of the two-Reggeon cut [left-hand side of (4.16)] and the sum over different s-channel intermediate states (diffractive for $N_c = 0$, inelastic otherwise).

From the example just given it is clear that the various terms in the right-hand side of the AGK rules (4.14) do not have a definite sign. This feature is due to the fact that they refer to a fixed loop number $h = N-1$, and disappears by summing over N. If we do that, by isolating the $N_c = 0$ term and using Eq. (4.15), we obtain

$$1-\hat{S}^{+}\hat{S} = \sum_{N_c=1}^{\infty} \hat{S}^{+} \frac{\left[\frac{2}{i}(\hat{\delta}-\hat{\delta}^{+})\right]^{N_c}}{N_c!} \hat{S} \tag{4.17}$$

which gives indeed the unitary defect (in the space of diffractive states) as a sum of non-negative inelastic terms, provided Im$\delta$ is semi-positive definite.

In our case, by (4.12), the functional form of $\hat{\delta}$ depends on the Bessel transform of the tree amplitude $a_{tr}(s,b)$ which has, as it should, a positive imaginary part [Eq. (5.2)]. It can be seen that, from this, Im$\hat{\delta} \geq 0$ follows.

We therefore conclude that our $\hat{\delta}$ operator (4.13) satisfies inelastic s channel unitarity in the sense of Eqs. (4.14) and (4.17). In particular, it is explicitly unitary for those b-values for which $\hat{\delta}$ is hermitian, i.e., when diffractive intermediate states are the only important ones.

## 5. - IMPACT PARAMETER ANALYSIS

In this section we shall discuss the physical content of the basic expressions (4.13) and (4.12) for the resummed S-matrix in impact parameter space. We shall leave to Section 6 the task of converting the results into momentum transfer or scattering angle variables.

### 5.1 Semi-classical field theory limit

From the point of view of our general considerations of Section 1, this limit, corresponding to $b \gg \lambda_s$, is obtained by keeping only the p = r = 0 terms of Eq. (1.12), i.e., neglecting string-size corrections. Furthermore, our leading s resummation is only able to take into account terms with q = 0. The higher order terms in the parameter $R_s(\sqrt{s})/b$ - that may be interpreted as classical corrections - will be discussed in Section 7, where we show that they arise from a particular class of subleading corrections to a(s,b). The p = q = r = 0 terms of (1.12) corresponding to $b \gg R_s$ and $b \gg \lambda_s$, correspond to neglecting $\hat{X}_u-\hat{X}_d$ with respect to b in the eikonal operator*) $\delta(s,b+\hat{X}_u-\hat{X}_d)$.

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*) The eikonal representation in this pure field theory framework has been independently discussed in Ref. [3].

Let us start by noticing that indeed the eikonal representation (4.13) takes a semi-classical form

$$S = e^{2i\delta} = e^{\frac{i}{\hbar} S_{c\ell}} = e^{\frac{i}{\lambda_s^2} \bar{S}_{c\ell}} \tag{5.1}$$

where $S_{c\ell} \sim G_N s/b^{D-4} \gg \hbar$ plays the rôle of a classical gravitational action.

In fact, by using the results of Section 4, we find

$$\delta(s,b) = \hat{\delta}\Big|_{\hat{X}=0} = a_{tree}(s,b) = \frac{g^2 s}{16\pi^{D/2-1}}\, b^{4-D} \int_0^{\frac{1}{4} b^2/(Y - i\pi/2)} dt\; e^{-t}\, t^{D/2-3}$$

$$\xrightarrow[b \gg \sqrt{Y}]{} \left(\frac{b_c}{b}\right)^{D-4} + \frac{i\pi g^2 s}{8(4\pi Y)^{D/2-1}}\, e^{-b^2/4Y} \quad , \quad Y \equiv \log s \tag{5.2}$$

where

$$b_c^{D-4} = \frac{g^2 s}{8\pi\,\Omega_{D-4}} \tag{5.3}$$

and $\Omega_d$ is the solid angle in d dimensions

$$\Omega_d = 2\pi^{d/2}/\Gamma(d/2) \tag{5.4}$$

Note that $\mathrm{Re}\delta$ is power-behaved for large b (due to the long-range graviton exchange), but is also finite for $b \to 0$, being cut off by the string soft behaviour for large q, so that $\mathrm{Re}\delta \underset{b\to 0}{\to} g^2 s(\alpha' Y)^{-(D-4)/2}$. Therefore, string effects play a crucial rôle in cutting-off short distances, so that the classical limit can be recovered for $\bar{S}_{c\ell} \gg \hbar$. The magnitude of $\mathrm{Re}\delta$ at large b's is governed by $b_c$, which, according to (5.3), has a pole at D = 4. This pole is connected with the well-known [15] infinite Coulomb phase and will give rise to a $\mathrm{Re}\delta \sim -2\, g^2 s \log b$ behaviour after subtraction (see Section 6).

On the other hand, $\mathrm{Im}\delta$ has an exponential form, dying out for $b > b_I = 2 \log s$. Since clearly $b_c > b_I$ one can define three regions in impact parameter space (Fig. 9):

(i) $b > b_c$ (D>4). Here $\delta$ is small and perturbation theory is valid. It corresponds to very small t physics as discussed in Section 6.

(ii) $b_I < b < b_c$. In this intermediate region $\mathrm{Re}\delta$ is large while $\mathrm{Im}\delta$ is small. A large phase signals, as usual, a semi-classical limit and, indeed, in Section 6, we shall see how to recover in this region some known (or expected) classical relativity results.

(iii) $b < b_I$. In this region, $\mathrm{Im}\delta$ is large and inelastic absorption dominates ($S \sim e^{2i\delta} \to 0$, $a \to i/2$) leading to an absorbing, logarithmically expanding black disc, with an inelastic cross-section given by:

$$\sigma_{in}(s) = 4\int d^{D-2}\underset{\sim}{b}\left(1-|S(s,b)|^2\right) \simeq \frac{2}{D-2}\,\Omega_{D-2}\left(2\ln s\right)^{D-2} \tag{5.5}$$

which increases therefore with s.

It is clear, however, that this is a region where string corrections are bound to be substantial. We shall have control over this region for $R_s \ll \lambda_s$.

## 5.2 String corrections

Let us now take into account the string corrections to the eikonal operator (4.12), whose expansion in powers of X corresponds, as we shall see, to the terms with $p \neq 0$ in Eq. (1.12). It is intuitive that, for $b \gg 2Y \equiv b_I$, where $\hat{\delta}$ has a power dependence on $(b+\hat{X}_u-\hat{X}_d)$, a Taylor series in $(\hat{X}_u-\hat{X}_d)$ can be meaningful, being controlled by the small parameter $\hat{X}/b \sim \lambda_s/b$.

Although this seems to be indeed the case (see Section 5.5 below) we hasten to point out that effects $O(\hat{X}^2/b^2)$ can be very important, even at $b \gg \lambda_s$, if they effectively induce imaginary parts in $\delta$, which are absent in the leading c-number term. For Im$\delta$ the competition will rather exist between string-size effects and the classical relativity corrections to be discussed in Section 7.

With this in mind, we now consider the first non-trivial term of the expansion:

$$\hat{\delta}\left(s,\underset{\sim}{b}+\hat{\underset{\sim}{X}}^u-\hat{\underset{\sim}{X}}^d\right) = \delta(s,b) + \frac{1}{2}\frac{\partial^2\delta}{\partial b_i\partial b_j}\,\overline{\left(\hat{X}_i^u\hat{X}_j^u+\hat{X}_i^d\hat{X}_j^d\right)}+\cdots \tag{5.6}$$

where, in general, we define:

$$\overline{O_{u,d}} \equiv \frac{1}{2\pi}\int_0^{2\pi} d\sigma : O_{u,d} : \tag{5.7}$$

and we here used the fact that $\overline{X^{u,d}} = \overline{X_u X_d} = 0$. It is easy to see that (for $b \gg b_I$):

$$\frac{\partial^2\delta}{\partial b_i\partial b_j} = \Delta_\perp\left(\delta_{ij}-\frac{b_i b_j}{\underset{\sim}{b}^2}\right) + \Delta_{//}\,\frac{b_i b_j}{\underset{\sim}{b}^2} \equiv \Delta_\perp^{ij}+\Delta_{//}^{ij} \tag{5.8}$$

$$\Delta_\perp = 2\frac{\partial\delta}{\partial \underset{\sim}{b}^2} \to -\frac{D-4}{b^2}\left(\frac{b_c}{b}\right)^{D-4} \equiv -\Delta \quad ; \quad \Delta_{//} = (D-3)\Delta \tag{5.9}$$

By inserting (5.8) into (5.6), the $O(\hat{X}^2)$ terms take the form

$$\frac{1}{2}\left(\Delta_\perp \hat{X}^u_\perp \cdot \hat{X}^u_\perp + \Delta_\parallel \hat{X}^u_\parallel \cdot \hat{X}^u_\parallel + (u \leftrightarrow d)\right) \equiv \frac{1}{2}\sum_j \Delta_j \left(\hat{X}^u_j \hat{X}^u_j + \hat{X}^d_j \hat{X}^d_j\right) \tag{5.10}$$

where the index j runs over the (1+D-3) X-components parallel and transverse to b. We then obtain

$$\exp 2i\hat{\delta}(s, b+\hat{X}^u-\hat{X}^d) \simeq e^{2i\delta(s,b)}\,\hat{S}^{(2)} = e^{2i\delta(s,b)} \cdot$$

$$\cdot \prod_{j=1}^{D-2}\prod_{n=1}^{\infty}\prod_{a=u,d} \exp\left\{-i\frac{\Delta_j}{n^2}\left(\alpha^{a,j}_{-n}\alpha^{a,j}_{n} + \tilde{\alpha}^{a,j}_{-n}\tilde{\alpha}^{a,j}_{n} - \tilde{\alpha}^{a,j}_{-n}\alpha^{a,j}_{-n} - \tilde{\alpha}^{a,j}_{n}\alpha^{a,j}_{n}\right)\right\} \tag{5.11}$$

In order to evaluate matrix elements of S we have to normal order the operators in (5.11). This we do by noticing that, for every j and n, the operators

$$I_{\mp} = \frac{1}{n}\alpha^i_{\pm n}\tilde{\alpha}^i_{\pm n} \quad , \quad 2I_0 = 1 + \frac{1}{n}\left(\alpha^i_{-n}\alpha^i_n + \tilde{\alpha}^i_{-n}\tilde{\alpha}^i_n\right) \tag{5.12}$$

satisfy the SU(1,1) algebra:

$$[I_0, I_\pm] = \pm I_\pm \quad , \quad [I_+, I_-] = -2I_0 \tag{5.13}$$

Using the known identity:

$$\exp iw(2I_0 - I_+ - I_-) = e^{-\frac{iw}{1-iw}I_+}(1-iw)^{-2I_0}\,e^{-\frac{iw}{1-iw}I_-} \tag{5.14}$$

we finally get

$$\hat{S}^{(2)} = \prod_{n,j,a}\exp\left(-i\frac{\Delta_j/n}{n-i\Delta_j}\alpha^j_{-n}\tilde{\alpha}^j_{-n}\right)\cdot\left(1-i\frac{\Delta_j}{n}\right)^{-\frac{\alpha^j_{-n}\alpha^j_n + \tilde{\alpha}^j_{-n}\tilde{\alpha}^j_n}{n}} \cdot$$

$$\cdot \exp\left(-i\frac{\Delta_j/n}{n-i\Delta_j}\alpha^j_n\tilde{\alpha}^j_n\right)\cdot\frac{e^{-i\frac{\Delta_j}{n}}}{1-i\frac{\Delta_j}{n}} \tag{5.15}$$

where the label a = u,d has been suppressed for notational simplicity. We also find the action of S on the ground (i.e., two-graviton) state to be:

$$\hat{S}\,|0\rangle = e^{2i\delta(s,b)}\prod_{n,j}\frac{e^{-i\Delta_j/n}}{1-i\frac{\Delta_j}{n}}\,e^{-i\frac{\Delta_j}{n-i\Delta_j}\frac{\tilde{\alpha}^{u,j}_{-n}\alpha^{u,j}_{-n} + \tilde{\alpha}^{d,j}_{-n}\alpha^{d,j}_{-n}}{n}}|0\rangle \tag{5.16}$$

Note that only states with equal left and right occupation numbers, i.e., with $\nu_n = \tilde{\nu}_n$ are produced by $\hat{S}^{(2)}$. This restriction is a feature of the second-order expansion which leads to bilinear expression in creation operators.

5.3 Elastic channel

Two-graviton scattering is governed by the ground state expectation value of (5.16). By performing the product over parallel and transverse modes we find

$$\langle 0|S|0\rangle \equiv e^{2i\delta_{el}} = e^{2i\delta(b,s)} \prod_{n=1}^{\infty} \prod_{j=1}^{D-2} \left( \frac{e^{-i\Delta_j/n}}{1 - \frac{i\Delta_j}{n}} \right)^2 =$$

$$= e^{2i\delta} \, \Gamma^{2(D-3)}_{(1-i\Delta_\perp)} \, \Gamma^{2}_{(1-i\Delta_{\parallel})} \tag{5.17}$$

and

$$|\langle 0|S|0\rangle| = e^{-2\,\mathrm{Im}\,\delta_{el}} = e^{-2\,\mathrm{Im}\,\delta} \left( \frac{\pi\Delta_\perp}{\mathrm{sh}\,\pi\Delta_\perp} \right)^{D-3} \frac{\pi\Delta_{\parallel}}{\mathrm{sh}\,\pi\Delta_{\parallel}} \tag{5.18}$$

$\mathrm{Im}\delta$ being given in Eq. (5.2). Recall that for $b \gg b_I$, $\Delta_{\parallel} = (D-3)|\Delta_\perp| = (D-3)\Delta(b,s)$ and that $\Delta = \delta'(s,b)/b$.

We can see from (5.18) that, even in the region where $\mathrm{Im}\delta$ has died out (i.e., $b > b_I = 2\log s$), still $\mathrm{Im}\delta_{e\ell} > 0$ because $\Delta(b,s)$ only falls off like a power. Quantitatively, by replacing in Eq. (5.18) the explicit value of $\Delta$ in Eq. (5.9), we obtain

$$e^{-2\,\mathrm{Im}\,\delta_{el}} \underset{b_I < b < b_D}{\simeq} (D-3)(\pi\Delta)^{D-2} e^{-2\pi(D-3)\Delta}$$

$$\Delta(b,s) = \frac{g_s^2}{4 b^{D-2} \Omega_{D-2}} \xrightarrow[D\to 4]{} \frac{G_N s}{8\pi b^2} \tag{5.19}$$

This absorption of the elastic channel is due to diffractive excitation of massive string states induced by the X-dependent part in the S operator (5.15) which, in this region, is explicitly unitary, due to the hermiticity of $\delta$. By (5.19b), it extends, in impact parameter, up to a radius $b_D$ such that

$$\Delta(b_D, s) \sim 1 \Rightarrow \Omega_{D-2}\, b_D^{D-2} \sim G_N s \tag{5.20}$$

The corresponding diffractive excitation cross-section is given by the surface of radius $b_D$ *), i.e.,

$$\sigma_D(s) = 4\int d^{D-2}\underset{\sim}{b}\left(1 - e^{-4\,\mathrm{Im}\,\delta_{el}}\right) \sim G_N s \tag{5.21}$$

and has therefore the same energy dependence as the tree amplitude.

The t-dependence corresponding to the amplitude (5.17) will be discussed in the next section. We note now, however, that the absorption radius $b_D(s)$ becomes large for $G_N s \gg 1$. For instance, in four dimensions, $b_D = \sqrt{G_N s} = \alpha_4^{-\frac{1}{2}} R_s(E)$, which is larger than the Schwarzschild radius $R_s(E) = G_N E$ corresponding to the c.m. energy $E = \sqrt{s}$. Therefore, at very high energies, string effects may be felt at large distances. This is because in this regime, graviton exchange provides a long-range (and strong-coupling) excitation vertex.

A final point to notice is that the large absorption provided by (5.19) is due to the cumulative effect of a large number $\langle h \rangle$ of loops, i.e., of rescatterings contributing to the absolute value of the amplitude (5.18). Since the loop series is exponential, we have, by (5.2) and (5.9)

$$\langle h \rangle_I \simeq 2\,\mathrm{Im}\,\delta \simeq G_N s \quad , \quad b < b_I$$

$$\langle h \rangle_D \simeq \Delta(b,s) \simeq \frac{G_N s}{\Omega_{D-2}\, b^{D-2}} \quad , \quad b > b_I \tag{5.22}$$

for inelastic and diffractive absorption respectively. Thus $\langle h \rangle_I$ can be interpreted as the average number of cut gravi-Reggeons, while $\langle h \rangle_D$ represents the average number of loops with excited diffractive states. The (large) values (5.22) help clarifying the features of the corresponding spectra.

## 5.4 Diffractive channels and excitation spectrum

In the region $b_I < b < b_D$, the unitary S operator (5.15) completely describes the physics of diffractive states, i.e., those with masses $M_u, M_d \ll \sqrt{s}$. For an incoming ground state (of two gravitons) we can compute from Eq. (5.16) the excitation probability of a configuration of upper (lower) string states with occupation numbers $\{\nu^{j_u}_{n_u}\}$ $(\{\nu^{j_d}_{n_d}\})$ which are the same for left and right movers,

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*) The integral in Eq. (5.21) is convergent because of the behaviour $\mathrm{Im}\,\delta_{e\ell} \sim \Delta^2(b,s)$ given by (5.18) in the large b region $b \gg b_D$.

i.e.,

$$P(\{\nu_n^j\}_u, \{\nu_n^j\}_d) = |\langle \{\nu_n^j\}_u, \{\nu_n^j\}_d | S | 0 \rangle|^2 = P(\{\nu_n^j\}_u)\, P(\{\nu_n^j\}_d)$$

$$P(\{\nu_n^j\}) = |\langle 0 | S | 0 \rangle| \prod_{n,j} \left( \frac{\Delta_j^2}{n^2 + \Delta_j^2} \right)^{\nu_n^j} \tag{5.23}$$

From this, one obtains the probability distribution of masses $M_u$ ($M_d$) in the form

$$P(M_u, M_d) = P(M_u)\, P(M_d)$$

$$P(M) = \sum_{\{\nu_n^j\}} P(\{\nu_n^j\})\, \delta\Big(M^2 - \sum_{n,j} n\, \nu_n^j\Big) =$$

$$= \int \frac{d\lambda}{2\pi i}\, e^{i\lambda M^2} \prod_n \left[1 + \frac{\Delta^2 (D-3)^2}{n^2} (1 - e^{-i\lambda n})\right]^{-1} \left[1 + \frac{\Delta^2}{n^2}(1 - e^{-i\lambda n})\right]^{-(D+3)} \tag{5.24}$$

where we have used the expression of $|\langle 0|S|0\rangle|$ in (5.17) and introduced the $\delta$-function constraint by a Fourier transform.

The behaviour of (5.24) can be evaluated by saddle point methods in the two limiting cases $M \ll \Delta$, $M \gg \Delta$, to yield

$$P(M) \xrightarrow[M \ll \Delta]{} \left(2\pi \sqrt{\frac{6}{D-2}}\, M^3\right)^{-1/2} \cdot e^{2\pi\sqrt{\frac{D-2}{6}} M} \cdot \Delta \cdot e^{-2\pi\Delta(D-3)}$$

$$\xrightarrow[M \gg \Delta]{} \frac{\sqrt{2}}{\pi}\, \Delta^{-3/2} \left[(D-2)(D-3)\right]^{-1/2} \exp \frac{-M^2/\Delta^2}{2(D-2)(D-3)} \tag{5.25}$$

We thus see that, up to $M = O(\Delta)$, the diffractive spectrum increases exponentially, i.e., like the number of closed string states with identical right and left occupation numbers, but is sharply cut off at larger M values. Therefore, the average excitation mass is $\langle M \rangle \simeq \Delta \simeq G_N s/(\Omega_{D-2} b^{D-2})$, which increases quite rapidly with energy.

One can understand this increase of $\langle M^2 \rangle = \langle \nu_n \rangle \langle n \rangle$ for $\Delta > 1$, as due to both $\langle n \rangle$ and $\langle \nu_n \rangle$ being $O(\Delta)$, the first one because of the excitation probability

$\sim\Delta^2/(\Delta^2+n^2)$, and the second one because of the number of rescatterings which, according to (5.22) is again $\langle h\rangle_D \simeq \Delta$. In fact, the $\delta$ operator in (5.11), which governs the elementary scattering, can only provide transitions with $|\Delta\nu_n| = 1$, corresponding to $|\Delta M| = 1$: hence, $\langle \nu_n\rangle \simeq \Delta$ because of the number of transitions, which is $\sim\Delta$ too.

When $\Delta$ becomes large, we have to require, for internal consistency, that $\langle M\rangle \simeq \Delta(b,s) \ll \sqrt{s}$, and therefore that

$$\Omega_{D-2}\, b^{D-2} \gg G_N\sqrt{s} \sim R_S(E)^{D-3} \tag{5.26}$$

This condition is automatically satisfied if $b > 1$, $b > R_s(E)$, i.e., in the region (1.11) of validity of our basic expansion (1.12). We shall also see in the next section that Eq. (5.26) is indeed satisfied by the b values that dominate the scattering amplitude, provided the associated scattering angle is not too large.

One may wonder at this point whether, for mass values which are close to $\Delta$, the individual scatterings which build up the resummed amplitude (5.17) may still occur at small t-values, as assumed before resumming. One can see that this is indeed the case, due to the previous remark that $|\Delta\nu_n| \simeq |\Delta M| \simeq 1$, so that the phase space boundary is $|t_{min}| \simeq (M\Delta M)^2/s \simeq \Delta^2/s$, i.e., small if (5.26) is satisfied.

## 5.5 Higher string corrections

In order to estimate the magnitude of the neglected terms in the expansion (5.7) of the string corrections, let us first recall that the second-order $\hat{S}^{(2)}$ operator (5.11) commutes with $(\alpha_{-n}\alpha_n - \tilde{\alpha}_{-n}\tilde{\alpha}_n)$, i.e., conserves the difference of the numbers of right- and left-moving modes for each n and each string (upper or lower). This means that matrix elements involving, besides $\hat{S}^{(2)}$, an odd number of $\hat{X}_u$'s or $\hat{X}_d$'s must vanish, because R and L modes are to be excited in pairs. Therefore, the first neglected terms in (5.7) are of the type $\overline{X_u^4}$, $\overline{X_d^4}$ or $\overline{X_u^2 X_d^2}$. By expanding the exponential, we find fourth-order matrix elements of type

$$M^{(4)} = \Delta^{(4)} \langle 0|\, \hat{S}^{(2)} \sum_{m,n} \frac{\alpha^i_{-n}\tilde{\alpha}^j_{-n}}{n}\, \frac{\alpha^k_{-m}\tilde{\alpha}^\ell_{-m}}{m}\, |0\rangle \tag{5.27}$$

where $\Delta^{(4)} \sim g^2 s/b^D$ gives the magnitude of the fourth-order derivatives of $\delta$, and the superscripts $i,j,k,\ell$ refer to a choice of parallel and transverse modes to be identified in pairs.

From the normal-ordered form (5.15) it is easy to prove the commutation relation

$$\hat{S}^{(2)} \frac{\alpha_{-n}^{j} \tilde{\alpha}_{-n}^{j}}{n} |0\rangle = \frac{i\Delta_j}{n - i\Delta_j} \left( 1 + \frac{i\Delta_j}{n - i\Delta_j} \frac{\alpha_{-n}^{j} \tilde{\alpha}_{-n}^{j}}{n} \right) \hat{S}^{(2)} |0\rangle \tag{5.28}$$

which implies

$$M^{(4)} = \langle 0 | \hat{S}^{(2)} | 0 \rangle \, \Delta^{(4)} \left( \sum_n \frac{i\Delta_j}{n - i\Delta_j} \frac{1}{n} \right)^2 \tag{5.29}$$

We thus realize that $M^{(4)}$ is $O(\Delta^{(4)}\log\Delta)$, and is therefore negligible, with respect to the corrections exponentiated in $S^{(2)}$, if $\Delta^{(4)}/\Delta \simeq 1/b^2 \ll 1$, as naively expected.

We thus see that, while the quadratic string terms played a crucial rôle by providing the first absorptive contributions for $b > b_I$, higher orders just give rise to small corrections.

## 6. - MOMENTUM TRANSFER ANALYSIS AND SMALL ANGLE SCATTERING

Here we want to discuss the physical picture of light string scattering as function of the momentum transfer t, by keeping in mind that the rough relation $\sqrt{|t|} = q \simeq 1/b$ is not always valid. Thus we shall identify, for various t-values, the regions in b-space that give the most important contribution. To this purpose, we start recalling the main features of the amplitude in b-space, found in Section 5.

### 6.1 Summary of impact parameter properties (Section 5)

The resummed amplitude in b-space takes an operator eikonal form [Eq. (5.11)]

$$\hat{S} = e^{2i\hat{\delta}(b,s,\hat{X}^u,\hat{X}^d)} \simeq e^{2i\delta(s,b)} \, \hat{S}^{(2)} \tag{6.1}$$

where, from Eq. (5.2), δ is the Bessel transform of the tree amplitude given by (Fig. 9):

$$\delta(s,b) = \frac{g^2 s}{16\pi^{\frac{D}{2}-1}} b^{4-D} \int_0^{\frac{b^2}{4(Y-\frac{i\pi}{2})}} dt\, t^{\frac{D}{2}-3} e^{-t} \underset{b>2\sqrt{Y}}{\simeq} \left(\frac{b_c}{b}\right)^{D-4} +$$

$$+ \frac{i}{8} \frac{\pi g^2 s}{(\pi 4Y)^{\frac{D}{2}-1}} \exp -\frac{b^2}{4Y}$$

$$b_c^{D-4} = \frac{g^2 s}{8\pi \Omega_{D-4}} \quad , \quad Y \equiv \log s \tag{6.2}$$

and the operator $\hat{S}^{(2)}$, given in Eq. (5.11), represents the string corrections in $\hat{\delta}$, to second-order in the string position operator $\hat{X}(\sigma)$. This is a valid approximation for $b \gg \lambda_s = 1$, i.e., outside a radius of the order of the string size.

In the elastic channel, string corrections are given by the expectation values [Eqs. (5.17), (5.18)],

$$S_{el} = e^{2i\delta} \langle 0 | \hat{S}^{(2)} | 0 \rangle = e^{2i\delta} \left(\Gamma(1-i\Delta_\perp)\right)^{2(D-3)} \left(\Gamma(1-i\Delta_\parallel)\right)^2$$

$$e^{-2\,\mathrm{Im}\,\delta_{el}} = |\langle 0 | \hat{S}^{(2)} | 0 \rangle| = \begin{cases} (\pi\Delta)^{D-2} (D-3)\, e^{-2\pi\Delta(D-3)} & , \quad \Delta \gg 1 \\ 1 - (D-2)(D-3)\frac{\Delta^2}{6} & , \quad \Delta \ll 1 \end{cases} \tag{6.3}$$

and are thus governed by the quantity

$$\Delta(b,s) \equiv -2 \frac{\partial \delta(bs)}{\partial b^2} \underset{b>b_I}{\simeq} \frac{g^2 s\, \alpha_c'}{2\, \Omega_{D-2}\, b^{D-2}} \tag{6.4}$$

From (6.1)-(6.3), the following features were derived:

(a) The elastic channel is absorbed (i.e., $\mathrm{Im}\delta_{e\ell} \gg 1$) (Fig. 9), because of production of (i) mostly "inelastic" (i.e., many-body or very massive) string states for $b < b_I$ and (ii) mostly "diffractive" (i.e., energetic two-body) states for $b_I < b < b_D$ where $b_I \simeq 2\log s$ and $b_D$ is the diffractive excitation radius given by $\Delta(b,s) = 1$, i.e.,

$$b_D^{D-2} = \frac{g^2 s}{\Omega_{D-2}} \tag{6.5}$$

(b) The corresponding production cross-sections are

$$\sigma_I \simeq \frac{\Omega_{D-2}}{D-2}\left(2 \ell_g s\right)^{D-2} \quad , \quad \sigma_D \simeq g^2 s \tag{6.6}$$

The large value of $\sigma_D$, compared to $\sigma_I$, is due to the fact that diffractive excitation goes through graviton exchange and dies out as a power in b [Eq. (6.4)], while Im$\delta$ due to inelastic production dies out exponentially [Eq. (6.2)].

(c) For $b_c > b > b_D$, elastic scattering dominates ($|S_{e\ell}| \approx 1$) and the amplitude has a large real phase $\delta_{e\ell} \simeq \delta(b,s)$, decreasing as a power of b, for $D > 4$. The phase becomes small, and therefore perturbation theory is valid, only for $b > b_c \sim (g^2 s)^{1/(D-4)}$. For $D = 4$, the phase shows an IR singularity, thus requiring the well-known subtraction of its infinite part [1] and therefore becomes logarithmic ($\sim \log b$) and is never small for $g^2 s > 1$.

(d) The region $b_I < b < b_c$, including the diffractive excitation region, is characterized by a phase whose real part is large and dominating ($\mathrm{Im}\delta_{e\ell}/\mathrm{Re}\delta_{e\ell} \sim \lambda_s^2/b^2$) and will be called the eikonal region in the following.

The scales $b_c$, $b_D$ and $b_I$ (the latter two involving powers of the string size $\lambda_s = \sqrt{\alpha'\hbar}$), are introduced naturally by our leading s resummation results. Let us point out, however, that the analysis of the subleading terms in Section 7 introduces a further scale, the classical Schwarzschild radius $R_s$, corresponding to the c.m. energy $\sqrt{s}$, i.e.,

$$R_s^{D-3} = \frac{16\pi G_N \sqrt{s}}{(D-2)\,\Omega_{D-1}}$$

$$\text{i.e.} \qquad R_s = 2 G_N \sqrt{s} \qquad \text{for} \quad D=4 \tag{6.7}$$

Since this scale parametrizes the most important subleading terms - which are of relative order $(R_s/b)^{2(D-3)}$ - we have to require that $b > R_s$ for our results to be fully reliable [cf. Eq. (1.11)].

## 6.2 Small t region

Let us evaluate the elastic scattering amplitude in q-space, given by the Bessel transform (4.10). It is convenient to perform an integration by parts, to get:

$$\frac{1}{s} a(s,q) = 4\int d^{D-2} \underset{\sim}{b}\; e^{i \underset{\sim}{q}\cdot \underset{\sim}{b}}\, \frac{S_{el}(s,b)-1}{2i} = \qquad a)$$

$$= 8\pi\left(\frac{2\pi}{q}\right)^{\frac{D}{2}-2}\int_0^\infty db\, b^{\frac{D}{2}-1} J_{\frac{D}{2}-2}(qb)\, \frac{e^{2i\delta_{el}}-1}{2i} =$$

$$= 4\left(\frac{2\pi}{q}\right)^{\frac{D}{2}-1}\int_0^\infty db\, b^{\frac{D}{2}-1} J_{\frac{D}{2}-1}(qb)\, e^{2i\delta_{el}(s,b)}\left(-\frac{\partial \delta_{el}}{\partial b}\right) \qquad b) \tag{6.8}$$

For $D > 4$ and $q < b_c^{-1} \sim (Gs)^{-1/(D-4)}$, i.e., in a region of very small $|t| = q^2$, shrinking with s as a power, Eq. (6.8) is dominated by large impact parameters $b > b_c$, where $\delta_{e\ell} \simeq \delta(s,b)$ is small. By isolating the leading terms for $t \to 0$, we obtain, by (6.1),

$$\frac{1}{s} a(s,q) \underset{t\to 0}{\simeq} \frac{g^2 s}{|t|}\left[1 + 2^{\frac{D}{2}-2}\,\Gamma\!\left(\frac{D}{2}-1\right)\int_0^\infty dx\, x^{-\frac{D}{2}+2}\,\cdot\right.$$

$$\left.\cdot\, J_{\frac{D}{2}-1}(x)\left(e^{2i\delta_{el}\left(\frac{x}{q},s\right)}-1\right) + O(\alpha' t)\right] \tag{6.9}$$

where we have also indicated the order of magnitude of the small string contributions obtained from Eq. (6.3).

We can see from (6.9) that the graviton pole is still there, with unrenormalized residue, thus showing that unitarity is consistent with gauge invariance, which guarantees the 1/t pole for spin-2 exchange (masslessness of the graviton). As we have seen, the effect of unitarity is to shrink with energy the region of t in which this pole dominates $[t \ll (g^2 s)^{-2/(D-4)}]$.

The loop expansion (in powers of $\delta_{e\ell}$) of the subleading terms of (6.9) is complicated by the occurrence of IR singularities in t, starting at $D = 4+2/h$ at h-loop level. In particular, it is never valid for $D = 4$, a case that requires a specific treatment (Section 6.3).

For $4 < D < 6$, the one-loop term [$\sim\delta^2_{e\ell}$ in (6.8)] gives the leading contribution to Ima, which reads

$$\frac{1}{s}\,\mathrm{Im}\,a(s,q) \underset{t\to 0}{\simeq} \frac{\left(g^2\frac{s}{2}\right)^2}{2\,|t|^{3-\frac{D}{2}}}\;\frac{2^{-2(D-4)}\,\Gamma\left(\frac{D-4}{2}\right)\Gamma\left(\frac{6-D}{2}\right)}{\pi^{\frac{D-3}{2}}\,\Gamma\left(\frac{D-3}{2}\right)} \tag{6.10}$$

and is therefore singular at $t = 0$. This is not a problem, however, because $\sigma_{e\ell}$ is also infinite due to the $1/t$ pole in Rea. The result (6.10) checks of course the one obtained from the explicit one-loop calculation [Eq. (2.19)].

For $D > 6$, the $q = 0$ limit in the integration of Eq. (6.8) can be directly done, to yield

$$\frac{1}{s}\,\mathrm{Im}\,a(s,0) = \sigma_{tot}(s) = 2\int d^{D-2}\underset{\sim}{b}\left(1 - e^{-2\,\mathrm{Im}\,\delta_{e\ell}}\cdot\cos(2\,\mathrm{Re}\,\delta_{e\ell})\right) \tag{6.11}$$

Let us remark that the loop expansion of (6.11) is not dominated by large b only. In fact, by using the behaviour of Re$\delta$ in (5.2), we obtain from (6.11), for odd h.

$$\frac{1}{s}\,\mathrm{Im}\,a^{(h)}(s,0) \simeq -\int_{\sqrt{b_I}}^{\infty} d^{D-2}\underset{\sim}{b}\,\frac{(-1)^{\frac{h+1}{2}}}{(1+h)!}\left(2\,\mathrm{Re}\,\delta(s,b)\right)^{h+1} =$$

$$= -\left(2g^2 s\right)^{h+1}\frac{(-1)^{\frac{h+1}{2}}}{(1+h)!}\left(2\alpha'\log s\right)^{\frac{2-(D-4)(h)}{2}} \qquad D>6 \tag{6.11'}$$

This shows the expected power behaviour for $N = h+1$ gravi-Reggeon exchange. Notice, however, the inverse power of $\alpha'$, which exhibits the important rôle of the soft small distance behaviour of the string in cutting off the field theory behaviour at $b = \sqrt{\alpha'}$, instead of $b = 1/\sqrt{s}$.

On the other hand, the eikonal resummation turns out to suppress short distances automatically. In fact, the large s behaviour of the resummed amplitude can be obtained directly from (6.11), by rescaling $b = b_c x$. By noticing that $\mathrm{Im}\delta_{e\ell}(b_c x,s) \sim b_c^{-2} \to 0$, we obtain the finite expression

$$\sigma_{Tot}(s) \sim 2\left(\frac{G_N s}{\Omega_{D-4}}\right)^{\frac{D-2}{D-4}}\int d^{D-2}x\,\left(1-\cos x^{4-D}\right) \tag{6.12}$$

which is dominated by $x = O(1)$, i.e., $b = O(b_c) \gg 1$.

This high energy behaviour violates the Froissart bound [11], or, more precisely, higher-dimensional bounds based on polynomial boundedness and on the existence of a mass gap [16]. Again, this is not surprising here in view of the exchange of the graviton, which remains massless.

### 6.3 Intermediate t region. Eikonal scattering

For larger values of t, i.e., $b_c^{-1} \ll q \lesssim 1$, the diffractive phase of the Bessel function becomes large enough to yield a saddle point of the integral in (6.8b) at

$$q = -2 \left. \frac{\partial \delta_{e\ell}(b,s)}{\partial b} \right|_{b_s} \simeq -2 \left. \frac{\partial \delta(b,s)}{\partial b} \right|_{b_s} \tag{6.13}$$

where we have neglected the string corrections to $(d/db)\delta_{e\ell}$ by assuming that, at the saddle point, $b_s \gg 1$.

Equation (6.13) defines, in general, a classical relation between impact parameters and momentum transfers, if $2\delta$ is identified as the classical Hamilton-Jacobi function. In our case, due to (6.1), we obtain

$$q = \frac{g^2 s}{2\Omega_{D-2} b_s^{D-3}} = \frac{8\pi G_N \, k_a \cdot k_b}{b_s^{D-3} \Omega_{D-2}} \tag{6.14}$$

In a general collinear frame for the incoming gravitons, with

$$k_a = (E_a, \underset{\sim}{0}, E_a) \quad , \quad k_b = (E_b, \underset{\sim}{0}, -E_b) \tag{6.15}$$

we have $\frac{1}{2}s = k_a k_b = 2E_a E_b$. Therefore, Eq. (6.14) corresponds to the deflection angles

$$\theta_a = \frac{16\pi G_N E_b}{\Omega_{D-2} b_s^{D-3}} \quad , \quad \theta_b = \frac{16\pi G_N E_a}{\Omega_{D-2} b_s^{D-3}} \tag{6.16}$$

for the two gravitons a, b respectively. This result for $\theta_a(\theta_b)$ happens to coincide, for every D, with the classical deflection suffered by particle a(b) in a Schwarzschild metric generated by a particle at rest of mass $2E_b(2E_a)$, to <u>first</u> non-trivial order in the corresponding radii, defined in Eq. (6.7). Indeed, the well-known result for D = 4 [17]

$$\theta = \frac{2R_s}{b} \tag{6.17}$$

can be easily generalized to arbitrary D, to yield Eq. (6.16).

The correct classical interpretation of Eq. (6.16) is presumably different, however: in first approximation, each particle should experience the metric of a massless particle with given energy, the Aichelburg-Sexl (AS) metric [18] rather than the static one of Schwarzschild.

The problem of classical gravitational scattering of two relativistic particles has actually been considered very recently by 't Hooft [19], who, reducing the problem to the one in an external AS metric, implicitly obtains our Eq. (6.17). A confirmation that our particles do not feel a Schwarzschild metric comes from the absence of $O(\theta^2)$ corrections to Eqs. (6.16) and (6.17), as discussed in Section 7. This absence of $O(\theta^2)$ corrections is very natural in the AS metric, and, furthermore, can be used to find an amusing link [20] between the mutual focusing of two graviton beams and the singularities found in the classical collision of two infinite gravitational waves [21,22].

It is not clear, however, that even the AS metric is fully adequate for describing the collisions. The results of Section 7 on the $O(\theta^3)$ corrections to (6.16) indicate that other non-linear phenomena, which cannot be described by scattering in an external field, set in beyond the leading approximation.

Let us now discuss the expression of the scattering amplitude at the saddle point (6.13). A standard analysis shows that

$$\frac{1}{s}\, a(s,q)\Big|_{eik.} = \frac{2\, e^{i\phi_{eik.}}}{\sqrt{D-3}} \left( \frac{2\pi\, b_s(q,s)\, e^{-i\pi/2}}{q} \right)^{D/2-1} =$$

$$= \frac{g^2 s}{|t|}\, e^{i\phi_{eik.}} \left( \frac{g^2 s\, |t|^{\frac{D}{2}-2}}{2\,\Omega_{D-2}} \right)^{-\frac{(D-4)}{2(D-3)}} (-i) \left(2\, e^{-i\pi/2}\right)^{\frac{D}{2}-2} \frac{\Gamma(D/2-1)}{\sqrt{D-3}} \tag{6.18}$$

where the phase

$$\phi_{eik.} = \left( \frac{g^2 s\, |t|^{\frac{D}{2}-2}}{2\,\Omega_{D-2}} \right)^{\frac{1}{D-3}} \frac{D-3}{D-4} \tag{6.19}$$

is IR divergent at D = 4.

Note that this amplitude, apart for the phase (6.19), has a fixed power behaviour in the quantity $g^2 s|t|^{(D/2)-2}$ which is the effective loop expansion parameter. Furthermore, in the overlap region with small t's, i.e., $t = O[(g^2 s)^{-2/(D-4)}]$, the eikonal amplitudes and the small t expressions (6.10) and (6.12) join smoothly, being all of the same order $\sim (g^2 s)^{(D-2)/(D-4)}$.

The D = 4 limit of (6.18) is obtained by subtracting the infinite part of $\phi_{eik}$, to get

$$\delta(s,b) = G_N s \log c/b \qquad , \ D=4$$

$$s^{-1} a(s,q)\Big|_{eik.} = \frac{8\pi G_N s}{|t|} \exp\left(i\left(G_N s \log|t| + \text{phase}\right)\right) \tag{6.20}$$

Therefore, for D = 4, higher order effects just build up an s- and t-dependent phase, as in the well-known Coulomb scattering case [23].

The cross-section corresponding to (6.18) and (6.20) turns out to be of Rutherford type for any D, since one can check that

$$d\sigma_{el} = \frac{1}{4}\left|\frac{1}{s} a(s,q)\right|^2_{eik} \frac{d^{D-2}\underset{\sim}{q}}{(2\pi)^{D-2}} =$$

$$= \frac{g^2 s}{2q} \frac{\partial b_s}{\partial q} dq \cdot \frac{d\Omega_{D-2}}{\Omega_{D-2}} = d^{D-2} b_s(q) \tag{6.21}$$

where the last expression indicates the surface element in (D-2) dimensions implied by the relation $b_s(q)$ in (6.14).

We have neglected in the preceding discussion of fixed-t eikonal scattering the string contributions to both $\mathrm{Re}\,\delta_{e\ell}$ and $\mathrm{Im}\,\delta_{e\ell}$. We have already noticed that the first ones are always negligible, but the second ones are to be looked at, because they give rise to $\mathrm{Im}\,\delta_{e\ell} \neq 0$ for $b_I < b < b_D$. It turns out that for fixed t, they are also small, because the b integral is dominated by the saddle point $b_s(q,s) > b_D(s)$. In fact one has

$$b_s^{D-3} = \frac{g^2 s}{2q\,\Omega_{D-2}} > b_D^{D-3} = \frac{1}{b_D} \frac{g^2 s}{\Omega_{D-2}}$$

provided

$$q < \frac{b_D(s)}{2} = \frac{1}{2}\left[\frac{g^2 s}{\Omega_{D-2}}\right]^{\frac{1}{D-2}} \tag{6.22}$$

which includes the fixed-t region ($q \lesssim 1$).

## 6.4 Extension to small fixed angles. Diffractive string absorption

The fixed-t condition that we have imposed on each loop in order to guarantee the leading graviton behaviour can be somewhat relaxed for the resummed amplitude. We have indeed realized in Section 5 that our eikonal scattering is due to a large number of rescatterings. For the elastic scattering amplitude, which is dominated by the saddle point (6.13), such a number is

$$\langle h \rangle_{eik} \simeq 2\,\delta(s, b_s) \sim b_s(q,s)\, q \tag{6.23}$$

Therefore, the average momentum transfer $\langle q_i \rangle$ of each individual scattering will be

$$\langle q_i \rangle = \frac{q}{\langle h \rangle_{eik}} = b_s^{-1}(q,s) \tag{6.24}$$

which is small, provided $b_s(q,s) > 1$, i.e., larger than the string size.

On the other hand, let us recall that the condition implied by the absence of (non-leading) classical corrections was $b_s > R_s(\sqrt{s})$, with $R_s$ given by Eq. (6.7). Therefore, our procedure is justified if

$$\begin{aligned} &b_s(q,s) > 1 \qquad ; \quad R_s < 1 \\ &b_s(q,s) > R_s \qquad ; \quad R_s > 1 \end{aligned} \tag{6.25}$$

By inserting in (6.25) the value of $b_s$ in Eq. (6.16), we find the conditions for the c.m. scattering angle*):

$$\theta < R_s^{D-3}(\sqrt{s}) \qquad , \quad R_s < 1 \qquad \text{(a)}$$

$$\theta < o(1) \qquad , \quad R_s > 1 \qquad \text{(b)} \qquad (6.26)$$

---

*) The condition (6.26a) will be better discussed in Section 6.5 where we take into account a modification of the saddle point for $R_s < 1$.

Thus, we see that our analysis applies to small angle, but large t, scattering. We must, however, supplement the results of Section 6.3 by considering also string effects (diffractive excitations) that will occur if $b_I < b_s < b_D$, i.e., inverting the inequality (6.22), if

$$\frac{g^2 s}{(2Y)^{D-3}} > q > \frac{1}{2} b_D(s) \qquad \text{or} \qquad \frac{b_D(s)}{\sqrt{s}} < \theta < \mathrm{Min}\left[1, \left(\frac{R_s}{2Y}\right)^{D-3}\right] \tag{6.27}$$

The saddle point procedure applies to this case too. The string contribution to $\mathrm{Re}\delta_{e\ell}$ and to its variation is, for $b > b_I$ still negligible, so that the saddle point position $b_s$ is not appreciably modified. Therefore, the amplitude is simply multiplied by the string correction factor (6.3), evaluated at $b = b_s$:

$$a(s,q) = a(s,q)\Big|_{eik} \langle 0 | \hat{S}^{(2)}_{(b_s(q,s))} | 0 \rangle ,$$

$$\left| a(s,q) \right|_{1 < b_s < b_D} \simeq \left| a(s,q) \right|_{eik} \exp - \frac{2\pi q (D-3)}{b_s(q,s)} =$$

$$= \left| a(s,q) \right|_{eik} \exp\left[ - \mathrm{const} \left( \frac{\theta^{D-2} s^{\frac{D-4}{2}}}{G_N} \right)^{\frac{1}{D-3}} \right] \tag{6.28}$$

where we have used the fact that, at the saddle point,

$$\Delta(s, b_s) = \frac{g^2 s}{2 b_s^{D-2} \Omega_{D-2}} = \frac{q}{b_s(q,s)} \tag{6.29}$$

The exponential factor $\exp(-cq/b_s)$ occurring in (6.28) represents the absorption due to production of diffractive states. Note that, in the region (6.25), it is always larger than $\exp(-cq)$ and therefore it is larger than the short-distance contribution [2], which, loop by loop is $\sim\exp(-q^2/N)$. This point will be discussed further in Section 8, i.e., after analysis of the classical corrections.

### 6.5 The $R_s(\sqrt{s}) < \lambda_s$ case. Inelastic string absorption

We have considered so far string absorptive effects due to diffractive excitation, which occurs at intermediate distances, i.e., $b_I < b_s < b_D$, corresponding to the angular region (6.27). If $R_s(\sqrt{s}) > \lambda_s$ (i.e., at extremely high energies) the relevant impact parameters are around $b_s \sim R_s \theta^{-1/D-3}$, and therefore are more sensitive to classical corrections (Section 7) which occur around $b = R_s$, rather than to string inelastic effects, which occur for $b < b_I = 2\lambda_s \log s$.

On the other hand, one can also consider energies which are high, but not extreme, such that

$$R_s(\sqrt{s}) < \lambda_s \quad , \quad \text{or} \quad \frac{1}{g} < \sqrt{s} < \frac{1}{g^2} \quad , \quad (\lambda_s = 1) \tag{6.30}$$

In this regime, our approach allows us to investigate the angular region $1 > \theta > (R_s/2\log s)^{D-3}$, for which the saddle point value enters the inelastic region $b < b_I$. Here the phase $\delta(b,s)$ in (6.2) starts being sensitive to the soft behaviour $\sim\exp(-Y\underline{q}^2)$ of the string tree amplitude which turns the power behaviour of $\mathrm{Re}\,\delta$ for $b \gtrsim b_I$ into the smooth small b behaviours

$$\mathrm{Re}\,\delta(b,s) \underset{b<\sqrt{Y}}{\simeq} \frac{g^2 s}{8\pi} \frac{1}{(4\pi Y)^{\frac{D}{2}-2}} \left( \frac{1}{D-4} - \frac{b^2}{(D-2)4Y} \right) \qquad (Y = \log s)$$

$$\mathrm{Im}\,\delta(b,s) \simeq \frac{\pi g^2 s (\pi)^{1-D/2}}{8\,(4Y)^{D/2-1}} e^{-b^2/4Y} \tag{6.31}$$

Correspondingly, the saddle point relation (6.13) changes from the classical formula (6.14) to the relation

$$\theta \underset{b<\sqrt{Y}}{=} \frac{g^2\sqrt{s}\,\pi^{1-D/2}}{(D-2)(2\sqrt{Y})^{D-2}} \cdot b \quad \sim \left( \frac{R_s}{2\lambda_s\sqrt{Y}} \right)^{D-3} \cdot \frac{b}{2\lambda_s\sqrt{Y}} \tag{6.32}$$

which is typical of scattering by an extended object of size $\lambda_s\sqrt{Y}$ [20].

Therefore, the deflection $\theta(b,s)$ (Fig. 10a), interpolating between (6.14) and (6.32), has around $b \sim 2\sqrt{Y}$ a maximum value

$$\theta_M \simeq \left( \frac{R_s}{2\lambda_s\sqrt{Y}} \right)^{D-3} \tag{6.33}$$

This means that, for $\theta < \theta_M$, there are two saddle points in (6.8), the one at larger b dominating because the absorption is of diffractive type; while for $\theta > \theta_M$ there is none, and the integral in (6.8) is controlled by the endpoint $b = 0$, i.e., by inelastic absorption. By using Eq. (6.28) for $\theta < \theta_M$ and Eq. (6.31) for $\theta > \theta_M$, we then obtain

$$|a(s,q)| \sim \begin{cases} e^{-\text{const.}\frac{\sqrt{s}}{R_s}\theta^{\frac{D-2}{D-3}}} & , \left(\frac{R_s}{2Y}\right)^{D-3} < \theta < \left(\frac{R_s}{2\sqrt{Y}}\right)^{D-3} \\ e^{-g^2 s / Y^{\frac{D}{2}-1}} & , \theta \geq \theta_M = \left(\frac{R_s}{2\sqrt{Y}}\right)^{D-3} \end{cases} \tag{6.34}$$

The last expression in (6.34) gives a form of limiting absorption, reached already at $\theta = \theta_M$, because $\mathrm{Im}\delta$ in (6.31) is roughly b-independent for $b < 2\sqrt{Y}$. The distinction between inelastic and diffractive absorption becomes meaningless at $\theta = \theta_M$, because at $b \simeq 2\sqrt{Y}$, $\Delta(b,s)$ and $\mathrm{Im}\delta$ are of the same order $\sim g^2 s(Y)^{-\frac{1}{2}(D-2)}$.

We thus see how the finite string size effects modify classical gravity for $R_s < \lambda_s$. Diffractive excitation gives absorption along the classical trajectory at rather large distances $2\sqrt{Y} < b < b_D(s)$. However, at shorter distances $R_s < b < 2\sqrt{Y}$ inelastic absorption takes over and the deflection angle deviates from the classical one, reaching its maximal value precisely at $b = 2\sqrt{Y}$. The maximal (inelastic) absorption, occurring for $b \leqslant 2\sqrt{\log s}$, is given by (6.34b) and gives an amplitude which, for $R_s < \lambda_s$, i.e., $g^2\sqrt{s} < 1$, is still larger than $\sim\exp(-cq)$, i.e., of the Cerulus-Martin bound [11].

This picture is confirmed by the analysis of inelastic unitary in Eq. (4.17). For $b \lesssim 2\sqrt{\log s}$ the latter is saturated by inelastic cuts, with $\langle n_c\rangle \simeq 4\mathrm{Im}\delta \simeq g^2 s/(2\sqrt{Y})^{D-2}$, corresponding to an average energy per cut Reggeon

$$\langle E_c \rangle = \frac{\sqrt{s}}{\langle n_c \rangle} \simeq \frac{(2\sqrt{Y})^{D-2}}{g^2\sqrt{s}} \underset{R_s < \lambda_s}{\gg} \frac{1}{\lambda_s}$$

which is still consistent with Regge behaviour.

The previous arguments show that, although our method is adequate only for $b \lesssim \sqrt{Y}$ neither string short-distance effects [2] nor classical corrections (Section 7) are likely to modify, for $R_s < b < \lambda_s$, the basic soft behaviour of the string in (6.34). For $\theta > \theta_M$, this is due to inelastic absorption which, by shrinkage effects, is already maximal at $b \simeq 2\lambda_s\sqrt{\log s}$.

Another, possibly related, consequence of the smoother $b \to 0$ limit of $\delta(s,b)$ in string theory is that the curious poles found [19] at $G_N s = -iN$ ($N=1,2,\ldots$; $D=4$) can be shown to disappear as long as $\lambda_s \neq 0$.

## 7. - CLASSICAL CORRECTIONS: A SEMIQUANTITATIVE ANALYSIS

In this section we turn to the description of classical, general relativistic corrections to the expressions used in the preceding sections.

Our analysis is, for the moment, rather limited in scope, aiming essentially at establishing the regions where these corrections can be neglected and at finding the leading qualitative effects when they cannot.

### 7.1 Regge-Gribov diagrams for classical corrections

Because of eikonalization (i.e., of exponentiation) it is not immediately obvious how to classify a given non-leading contribution to a(s,b). In a sense, one has to define suitable 2PI (two-particle irreducible) diagrams, i.e., diagrams, or better contributions, that do not correspond to two-body-diffractive intermediate states (in the s-u channels). Let us elucidate this point at the one-loop level where one or two Reggeons can be exchanged in the t-channel.

The (obviously 2PI) single GR exchanges (Fig. 11a) are absent because of the non-renormalization of the Newton constant and of the graviton mass in superstring theory (see discussion in Section 3). Non-leading Regge poles are negligible in our high-energy limit. Turning to two-Reggeon exchanges (Fig. 11b), one finds, besides the already considered 2GR cut, contributions from non-leading Regge cuts such as the one due to two gravitinos. This exchange behaves as

$$a(s,t)_{2\,\text{gravitinos}} \sim g^4 s^{2\left(\frac{3}{2}\right)-1} = \alpha_D^2 s^2 \tag{7.1}$$

Being 2PI, it has to be compared to the GR Born term (3.1) and is non-leading by a power of $\alpha_D$. In order to identify a classical correction at order $\alpha_D^2$, we would need terms $O(\alpha_D s^{\frac{1}{2}})$ relative to Born, but such terms appear to be excluded by relativistic invariance.

This fact allows us to conclude that there are no corrections $O((R_s/b)^{D-3})$ to the leading deflection formula (6.16) confirming the expectation that the effective metric experienced by each graviton is <u>not</u> of a Schwarzschild type. It is instead consistent, to this order, with scattering in an AS metric [18].

At two-loop level (Fig. 12), we find several possibilities corresponding to one (Fig. 12a), two (Figs. 12b,d) or three (Fig. 12c) Reggeons being exchanged in the t-channel. Diagrams of type (a) can be discarded by the same non-renormalization arguments invoked at the one-loop level. As for the three Reggeon exchanges of Fig. 12c, these are either reducible (e.g., the 3GR exchange) or non-leading.

We are thus left with two GR cuts at the two-loop level (Figs. 12b,d). These give contributions to a(s,t) which are $O(G_N^3 s^3) = O(G_N s^2 (G_N s)^2)$, i.e., $O(R_s^{2(D-3)})$ relative to the Born term. The corresponding corrections to the eikonal phase can be $O(R_s/b)^{2(D-3)})$, i.e., typical of a classical correction [see Eq. (1.12)].

Extracting the classical correction, however, turns out to be far from trivial. The point is that long-distance, classical, effects get intertwined with short-distance quantum effects on which we only have, at present, limited control.

In the rest of this section, we shall examine, as far as we shall be able to go, the leading corrections due to Figs. 12b,d.

## 7.2 The H diagram

The two diagrams of Figs. 12b,d are there to represent the fact that the one-loop correction to $V_2$ (denoted by $V_2^{(1)}$) can itself contain a 2GR intermediate state in the t channel, giving rise to the diagram 12d. All other contributions are lumped into Fig. 12b, which contains, for instance, contributions with the same structure as in Fig. 12d, but where the two upper GRs are replaced by non-leading Regge poles. It is obvious that the only genuine, long-distance, classical contribution must come from the diagram containing four graviton poles (Fig. 12d). Considering diagram 12d in the field theory limit gives the H diagram of Fig. 13, where the horizontal lines stand for massless particles only. On general (Regge-Gribov) factorization grounds, we expect such a diagram to have an imaginary part given asymptotically by:

$$\mathrm{Im}\, a^H(s,t) \sim G_N^3 s^3 \int_0^Y dy \int \frac{dq_1\, dq_2\, P_4}{q_1^2 q_2^2 (q-q_1)^2 (q-q_2)^2} \exp\left[-(q_1^2+(q-q_1)^2)y - (q_2^2+(q-q_2)^2)(Y-y)\right] \tag{7.2}$$

where y is the rapidity of the particle produced in the middle and $P_4$ is a fourth order polynomial in the $q_i$'s arising from the derivative self-interaction of the gravitons. The detailed structure of $P_4$ is not very crucial. We have arguments suggesting that $P_4 \sim q_1(q-q_1)q_2\cdot(q-q_2)$ in which case Eq. (7.2), transformed into b-space, gives:

$$\mathrm{Im}\, a^{H}(s,b) \sim G_N^3 s^2 \int_0^Y dY db_1\; \partial_i \alpha(y,\underset{\sim}{b}_1)\; \partial_i \alpha(y,\underset{\sim}{b}_1)\; \partial_j \alpha(Y-y,\underset{\sim}{b}-\underset{\sim}{b}_1)\cdot$$
$$\cdot\, \partial_j \alpha(Y-y,\underset{\sim}{b}-\underset{\sim}{b}_1) \qquad (7.3)$$

where $\alpha(y,b)$ is $a_{tree}(s,b)$ of Eq. (5.2) with the factor $g^2s$ removed and $\partial_i \equiv \partial/\partial b_i$.

However, even the expression (7.3) contains important short-distance contributions to be subtracted out. In fact, if we replace in (7.3) the long-distance part of $\alpha \sim b^{4-D}$, we get the formal result:

$$\mathrm{Im}\, a^{H} \;\propto\; \frac{\Gamma\left(\frac{D}{2}-2\right)}{\Gamma\left(\frac{3-D}{2}\right)}\left(b^2\right)^{5-\frac{3}{2}D} \qquad (7.4)$$

which shows poles for even $D \geqslant 4$ and zeros for odd $D \geqslant 5$. When performing the subtraction procedure (which can be explicitly done in terms of known integrals) the poles will turn into $\sim \log b$ behaviour, while the zeros signal that no long-distance part is present for odd D*). For even $D \geqslant 4$, the finite, long-distance contribution of (7.3) turns out to be of the form:

$$\mathrm{Im}\, a^{H}(s,b) \underset{\text{large } b}{\sim} \log s\; g^6 s^2 b^{10-3D} \log\frac{b^2}{Y} = Y\left(\log\frac{b^2}{Y}\right) b.\,E\left(\frac{R_s}{b}\right)^{3(D-3)} \qquad (7.5)$$

On the other hand, the subtraction terms [which come from $\underset{\sim}{b}_1^{\,2}$ or $(\underset{\sim}{b}-\underset{\sim}{b}_1)^2 \sim O(\alpha' Y)$ are of the form

$$(g^3 s)^2\left[(b^2)^{3-D}\int_0^Y (\alpha' y)^{2-\frac{D}{2}} dy + (b^2)^{2-D}\int (\alpha' y)^{3-\frac{D}{2}} dy + \cdots\right] \qquad (7.6)$$

and start appearing for $D \geqslant 4$, $D \geqslant 6$, and so on.

They correspond, therefore, to divergent**) terms in the limit $\alpha' \to 0$, which will be interpreted as renormalizations of the two-particle 2GR vertices at one-loop level.

---

*) This is because (7.3) admits an asymptotic series in inverse powers of $b^2$, while (7.4) would contain a fractional exponent for odd D.

**) This divergence is not actually present in Ima, because of the phase space cut-off $b \gtrsim 1/\sqrt{s}$. However, the energy dependence would be substantially different for $\alpha' \to 0$ and true divergences would occur in Rea.

If we compare the contributions (7.6) to the one-loop 2GR cut $\sim g^4 s^2 (b^2)^{4-D}$, we find that they give corrections of relative order $g^2 \sim \alpha_D$, possibly times a power of $(\alpha'/b^2)$. They correspond, therefore, to a redefinition of the diffractive states, to take into account two phenomena: (i) instability of string states at $O(\alpha_D)$, and (ii) graviton bremsstrahlung, which is particularly important for $D = 4$.

We see therefore that, due to the string cut-off, the short-distance terms (7.6), which for large b are larger than (7.5), can be interpreted as reducible contributions, and are therefore under control.

A similar interpretation would be much more difficult in a pure field theoretical framework, due to the fact that the phase space cut-off $b > 1/\sqrt{s}$ gives rise for $\alpha' \to 0$ to drastically different energy dependences in Eq. (7.6), of type $\sim s^{D/2-2}$. In other words, due to the unquenched large momenta circulating in virtual loops, large corrections may occur for sufficiently large D, altering perhaps even the large-distance classical results.

On the contrary, taking into account the string cut-off, the only irreducible, large-distance effect left over is the contribution (7.5) which, for $b > R_s$, gives rise to small, but non-trivial classical corrections. String theory may well be the only way to rescue classical relativity as a limit of the full quantum theory!

## 7.3 Physical consequences of classical corrections

In order to discuss the effects implied by a term of the type (7.5) on our scattering process it is essential to determine the phase of such a contribution. This can be done by using dispersion relations and crossing. More simply we can notice that the only way to have (7.5) out of a real analytic, $s \leftrightarrow u$ symmetric amplitude is to have:

$$a^{H}(s,t) \sim G_N^3 s^3 \log^2 s + (s \leftrightarrow u) \simeq \pi G_N^3 s^3 \left( i \log s + \frac{\pi}{2} \right) + O(s^2) \tag{7.7}$$

i.e., a real part which is down by a single power of logs.

Exponentiating this ansatz gives the modified, field-theory-limit eikonal formula

$$a(s,q) = \int d\underline{b}\, \exp\left\{ \frac{i}{\lambda_s^2} \left[ \bar{q}\cdot\underline{b} + \bar{E}\, b \left(\frac{R_s}{b}\right)^{D-3} \left[ 1 + O\left(\left(\frac{R_s}{b}\right)^{2(D-3)}\right) \right] + i \bar{E} b \left(\frac{R_s}{b}\right)^{3(D-3)} \ln s \, \ln b^2 \right] \right\} \tag{7.8}$$

Equation (7.8) shows that, whereas the deflection angle is practically unchanged at $b \gg R_s > \lambda_s$, inserting the saddle point value $b_s(q)$ in the imaginary part of the eikonal phases induces an exponential damping factor given by:

$$\exp\left(-\frac{\bar{E}\, b_s}{\lambda_s^2}\,\theta^3\right) \sim \exp\left(-\frac{\bar{E} R_s}{\lambda_s^2}\,\theta^{\frac{3D-10}{D-3}}\right) \tag{7.9}$$

This classical absorption becomes strong at large enough scattering angles, i.e., for:

$$\theta > \theta_{c.a.} = \left(\frac{\lambda_s^{D-2}}{\alpha_D\, \bar{E}^{D-2}}\right)^{\frac{1}{3D-10}} \tag{7.10}$$

The origin of this absorption is similar to the one due to diffractive string excitations. They both come from the appearance of a large imaginary part to be added to a leading, but purely real, eikonal phase.

## 7.4 Comparison with string diffractive absorption

It is interesting to compare the result (7.9) with the one obtained in Sections 5 and 6 on string-induced absorption:

$$\exp\left[-\frac{\bar{E}}{b}\left(\frac{R_s}{b}\right)^{D-3}\right] \sim \exp\left(-\frac{\bar{E}}{R_s}\,\theta^{\frac{D-2}{D-3}}\right) \tag{7.11}$$

giving strong damping as soon as:

$$\theta > \theta_{s.a.} = \alpha_D^{\frac{1}{D-2}}\left(\frac{\lambda_s}{\bar{E}}\right)^{\frac{D-4}{D-2}} \tag{7.12}$$

One thus finds:

$$\theta_{s.a.}/\theta_{c.a.} = \left[\alpha_D^2\left(\lambda_s/\bar{E}\right)^{D-6}\right]^{\frac{2(D-3)}{(D-2)(3D-10)}} \tag{7.13}$$

As a consequence, for $D \geq 6$ and even, string absorption becomes important before classical absorption (recall that $\alpha_D \ll 1$, $\bar{E} \gg \lambda_s$) while, in $D = 4$, this only happens for $R_s < \lambda_s$. Thus in $D = 4$ classical absorption dominates at fixed $\alpha_D$ and sufficiently large energies. For odd $D$, string effects are always dominant with respect to the leading classical correction since the latter vanishes. We can also ask which of the two effects dominates at a given scattering angle. One finds that, at $D = 4$ (and at any $\theta$), string (classical) absorption dominates for $R_s < \lambda_s$ ($R_s > \lambda_s$), while, for $D \geq 6$, the condition for string dominance becomes

$$\theta < \left(\lambda_s / R_s\right)^{\frac{D-3}{D-4}} \sim \left(\frac{\sqrt{s}}{\lambda_s}\right)^{-\frac{1}{D-4}} \tag{7.14}$$

which includes the fixed-t region. At fixed small angle, classical effects dominate for $s \to \infty$.

## 8. - DISCUSSION

Let us now try to summarize what we have learnt by listing our main results.

1) We have found no apparent difficulty in recovering unitarity at high energy. As expected, very high genus surfaces play a crucial rôle in that recovery.

2) After loop corrections, the graviton pole still dominates at sufficiently small t, being distorted by a typical Coulomb phase at D = 4.

3) Elastic scattering is the dominant process at fixed t and also at sufficiently small angles. In this region, one obtains relations between deflection angles and impact parameter which are typical of particle propagation in an external metric. Again, loops are essential for producing the effect.

4) Absorption - via opening of inelastic channels - starts already at a small, critical angle or, equivalently, at a large, critical impact parameter. There is both classical absorption and string absorption and the relative importance of the two crucially depends on whether the classical scale $R_s$ is larger or smaller than the quantum, string scale $\lambda_s$ (see Section 1). Let us thus distinguish two cases.
a) $R_s > \lambda_s$ - In this case we are limited by our ignorance to $b > R_s$. As $b \to R_s^+$, classical corrections (not quantitatively studied as yet) become important and we expect the amplitude to be roughly absorbed as $\exp(-R_s\sqrt{s})$. At smaller angles, diffractive string absorption can be evaluated and, for $D > 4$, is important even at angles where classical corrections are negligible.
b) $R_s < \lambda_s$ - In this case (which excludes infinite energy at fixed coupling), string effects - that we are able to estimate - dominate over classical corrections and actually modify classical general relativity expectations. In particular, the diffractive excitation region $\lambda_s\sqrt{Y} < b < b_D$ is followed, at smaller b, by an inelastic absorption region in which string effects "soften" classical gravity, preventing deflection angles larger than a critical value.

Let us finally discuss the possible implications of our results on the work by Gross and Mende [2] on high energy fixed-angle scattering. These authors discuss the asymptotic limit of the four-point function at fixed genus obtaining a simple and suggestive result:

$$A = \sum_{h=0}^{\infty} A^{(h)} \quad , \quad A^{(h)} \sim c_h G_N^{h+1} s^{P_h} \exp - \frac{s f(\theta)}{h+1} \tag{8.1}$$

where f(θ) is a simple function of θ extracted by the tree-level calculation and $C_h$, $P_h$ are numbers whose evaluation is not straightforward.

As pointed out in Ref. [2], this result implies that, at fixed angle, large genus (h) contributions become very important. This is analogous to our conclusions in the small-angle regime ($\langle h\rangle \sim b_s q$). Indeed, a simple saddle point approximation to (8.1) would suggest that

$$\langle h \rangle \sim \sqrt{s} \quad , \quad A \sim e^{-\sqrt{s}} \tag{8.2}$$

Unfortunately, if ⟨h⟩ grows like $\sqrt{s}$ the average momentum transfer per collision is of order $\sqrt{s}/\langle h\rangle \simeq O(1)$ and it is therefore a fixed t, and not a fixed angle, dynamics that dominates. Such a conclusion is confirmed by our results: precisely because of the extreme softness of string theory, the most efficient way to reach a finite scattering angle is through an enormous number of soft processes. As we have seen, these can be estimated and lead to a fixed (small) angle damping of the type

$$\exp - \mathrm{Max}\left( \alpha_D^{\frac{1}{D-3}} E^{\frac{D-2}{D-3}} f_1(\theta) \, , \, \alpha_D^{-\frac{1}{D-3}} E^{\frac{D-4}{D-3}} f_2(\theta) \right) \tag{8.3}$$

Typically, at fixed θ and $\alpha_D(\sim G_N)$ and at sufficiently high energy, the first term (classical absorption, Section 7) dominates and the damping goes like $\exp(-E^{(D-2/D-3)} f_1(\theta))$ giving an elastic amplitude larger than Born but still smaller than the Cerulus-Martin bound $\exp(-\sqrt{s})$.

In our view, the behaviour that we have found reflects some very physical, classical gravitational phenomena occurring already at large distances ($R_s \gg \lambda_s$) because of the high energies involved. It is not obvious to us that an approach based on order by order fixed angle calculations can effectively take these phenomena into account.

REFERENCES

[1] D. Amati, M. Ciafaloni and G. Veneziano - Phys.Lett. 197B (1987) 81.

[2] D.J. Gross and P.F. Mende - Phys.Lett. 197B (1987) 129.

[3] I. Muzinich and M. Soldate - Santa Barbara Preprint NSF-ITP-87-93 (1987).

[4] M.B. Green - Nobel Lectures (1986).

[5] M. Soldate - Phys.Lett. 186B (1987) 321.

[6] G. Veneziano - Europhys.Lett. 2 (1986) 199.

[7] See, e.g.: M.B. Green, J.H. Schwarz and E. Witten - Superstring Theory, Cambridge University Press (1987).

[8] E. Witten - Phys.Lett. 149B (1984) 1351.

[9] B. Sundborg - Göteborg Preprint 86-45 (1986) and Erratum (1987).

[10] N. Sakai and Y. Tanii - Tokyo Preprint TIT/HEP-103 (1986).

[11] See, e.g.: P.D.B. Collins and E.J. Squires - Regge Poles in Particle Physics, Berlin, Springer (1968).

[12] See, e.g.: V.N. Gribov, Sov.Phys.JETP 26 (1968) 414;
V.A. Abramovskii, V.N. Gribov and O.V. Kancheli - Sov.J.Nucl.Phys. 18 (1974) 308.

[13] See, e.g.: V. Alessandrini, D. Amati, M. Le Bellac and D. Olive - Physics Reports 1C (1971) 269 and Ref. [7] for superstrings.

[14] See: G. Veneziano - Physics Reports 9C (1974) 199.

[15] Cf.: S. Weinberg - Phys.Rev. 140B (1965) 516.

[16] M. Chaichian and J. Fischer - Helsinki Preprint 87/29 (1987).

[17] See, e.g.: L.D. Landau and E.M. Lifshitz - The Classical Theory of Fields, Pergamon Press (1971).

[18] P.C. Aichelburg and R.U. Sexl - Gen.Rel. and Grav. 2 (1971) 303.

[19] G. 't Hooft - Phys.Lett. 198B (1987) 61.

[20] G. Veneziano - Mod.Phys.Lett. 2A (1987) 899.

[21] P. Szekeres - Nature (London) 228 (1970) 1183;
K. Khan and R. Penrose - Nature (London) 229 (1971) 185.

[22] T. Dray and G. 't Hooft - Class.Quant.Grav. 3 (1986) 825.

[23] See, e.g.: L.D. Landau and E.M. Lifshitz - Quantum Mechanics, Pergamon Press (1971).

FIGURE CAPTIONS

Fig. 1 Representation of the torus as a sum over t-channel intermediate states.

Fig. 2 The six-point tree amplitude as given by:
(a) sum over t-channel states, and
(b) 2GR amplitude $A^{\alpha_1\alpha_2}$ in its asymptotic form.

Fig. 3 The torus s-channel discontinuity, as given asymptotically by 2-gravi-Reggeon exchange. $M_u$ ($M_d$) denote the masses of the upper (lower) string intermediate states.

Fig. 4 Factorization pattern and corresponding Koba-Nielsen variables for the integrand (3.6) of the six-point tree amplitude.

Fig. 5 Asymptotic form of the genus h contribution in terms of $N = h+1$ gravi-Reggeon exchanges, with transverse momentum transfers $\underset{\sim}{q}_1 \ldots \underset{\sim}{q}_N$.

Fig. 6 Asymptotic form for the 2N+2-point tree amplitude for $s_i \to \infty$, and corresponding Koba-Nielsen variables $\zeta_i$ for the gravi-Reggeon amplitude $A^{\alpha_1 \ldots \alpha_N}$.

Fig. 7 Impact parameter picture of the genus $h = N-1$ contribution to asymptotic string scattering, as given by N-GR vertices $V_N^u$, $V_N^d$ in terms of the b-displacement $X_u(\sigma_u)-X_d(\sigma_d)$.

Fig. 8 Notation for the AGK cutting rules of Eq. (4.14). $N_c$ is the number of cut Reggeons and $N_+(N_-)$ is the one of those to the right (left) of the cutting plane.

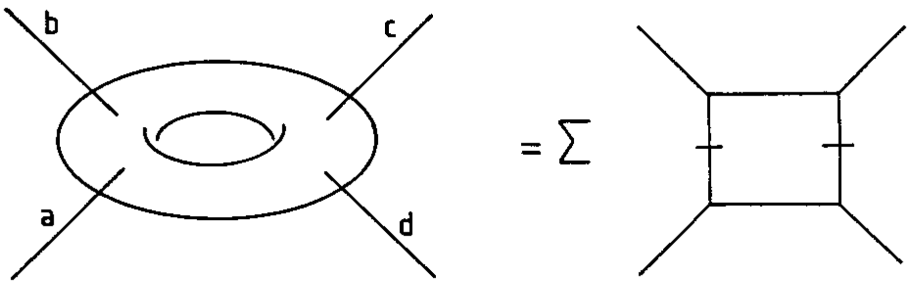
Fig. 9 Qualitative impact parameter behaviour for the Bessel transform of the tree amplitude $\delta(s,b)$ ($\mathrm{Re}\delta, \mathrm{Im}\delta$) and for $\mathrm{Im}\delta_{e\ell}$, which includes the diffractive string corrections. At the critical values $b_c$, $b_I$, $b_D$ such amplitudes exceed the unitarity bound.

Fig. 10 Qualitative modification of the saddle point deflection angle for $b < \mathrm{Max}(b_I, R_s)$ for the cases: (a) $R_s < \lambda_s$ and (b) $R_s > \lambda_s$. The full (dotted) line includes (excludes) the string or classical corrections for the cases (a) or (b) respectively.

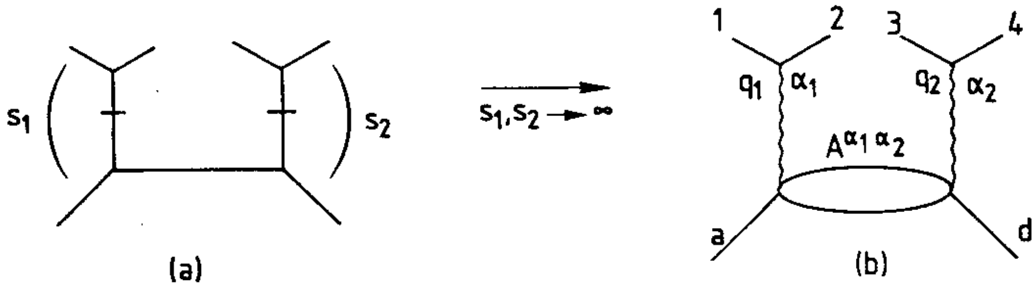
Fig. 11 Classification of one-loop diagrams in terms of: (a) vertex corrections to single-GR exchange and (b) 2GR exchanges.

Fig. 12 Classification of two-loop diagrams in terms of: (a-b) subleading vertex corrections to 1GR and 2GR exchanges, (c) leading 3GR exchange, and (d) new, "irreducible" 2GR contribution providing the classical corrections.

Fig. 13 Diagram 12d in the field theory limit. Solid lines represent the massless sector of superstring theory and the dashes on them correspond to taking the imaginary part of the amplitude.




- Fig. 1 -

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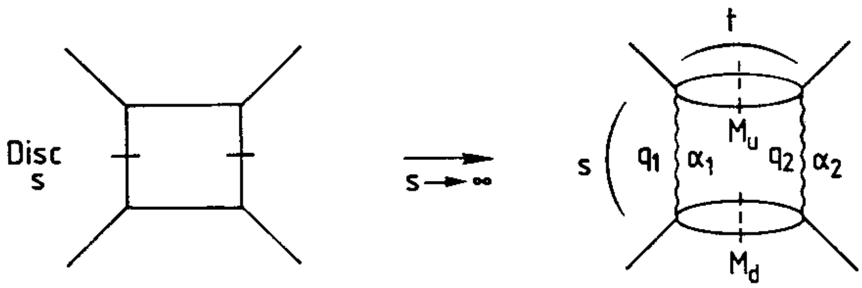


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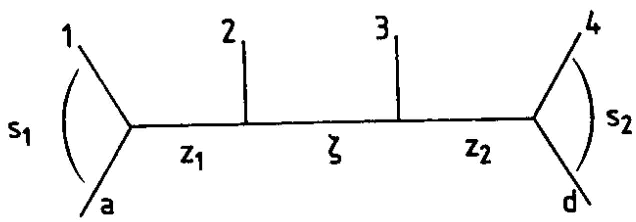


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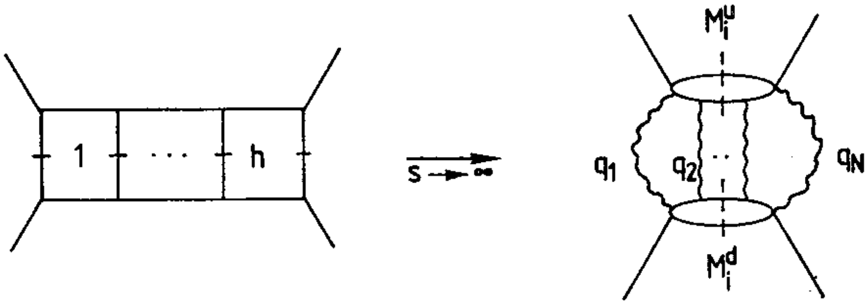


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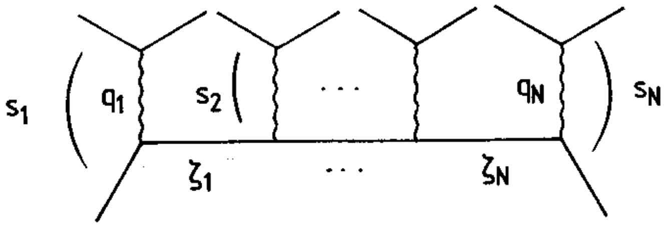


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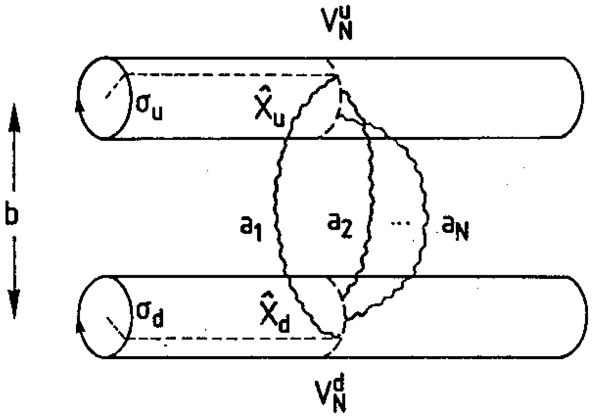


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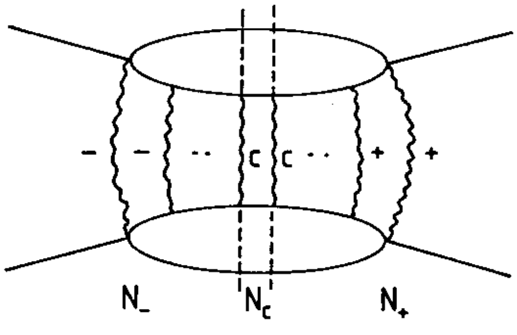


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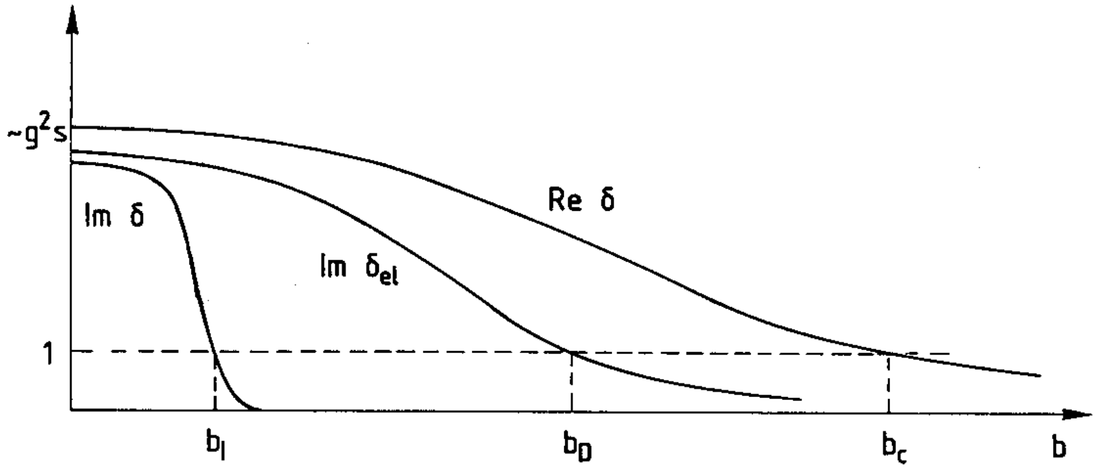


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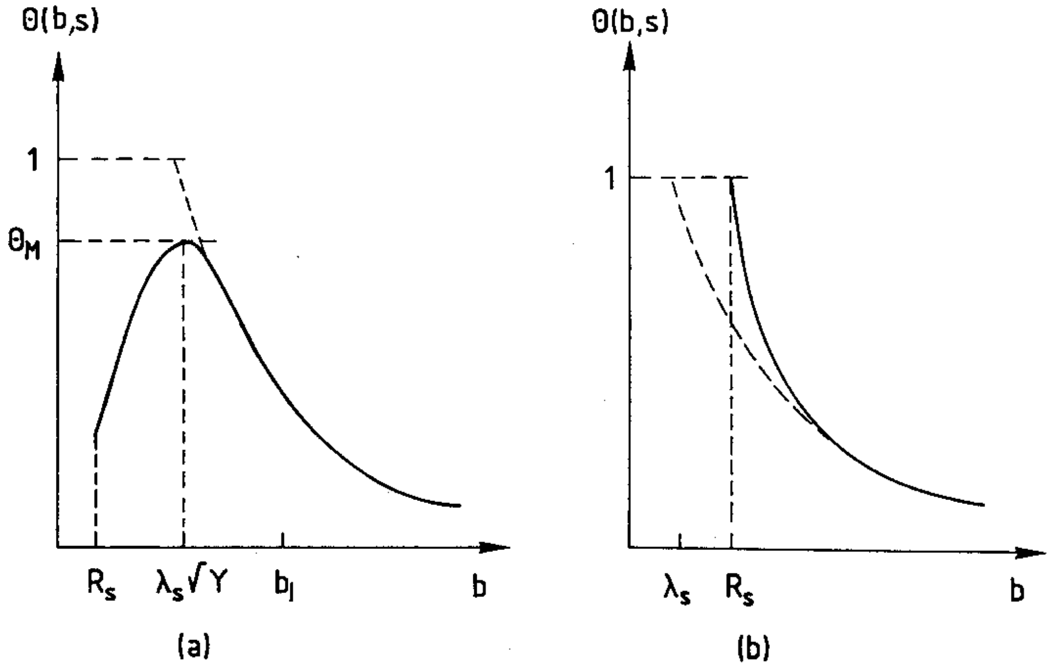


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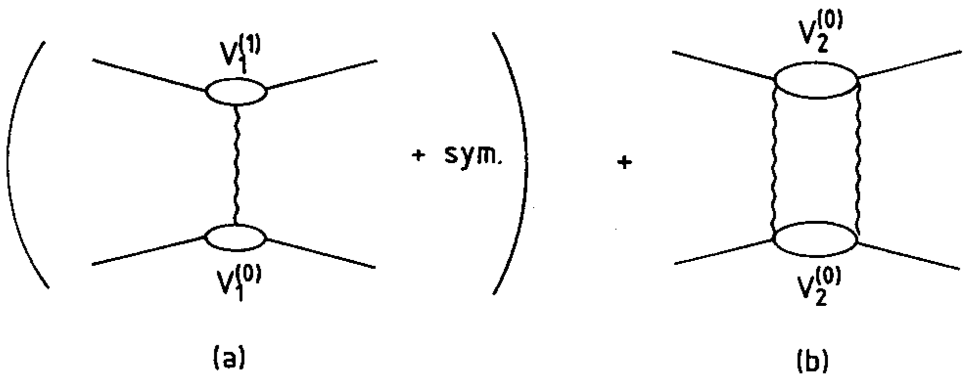


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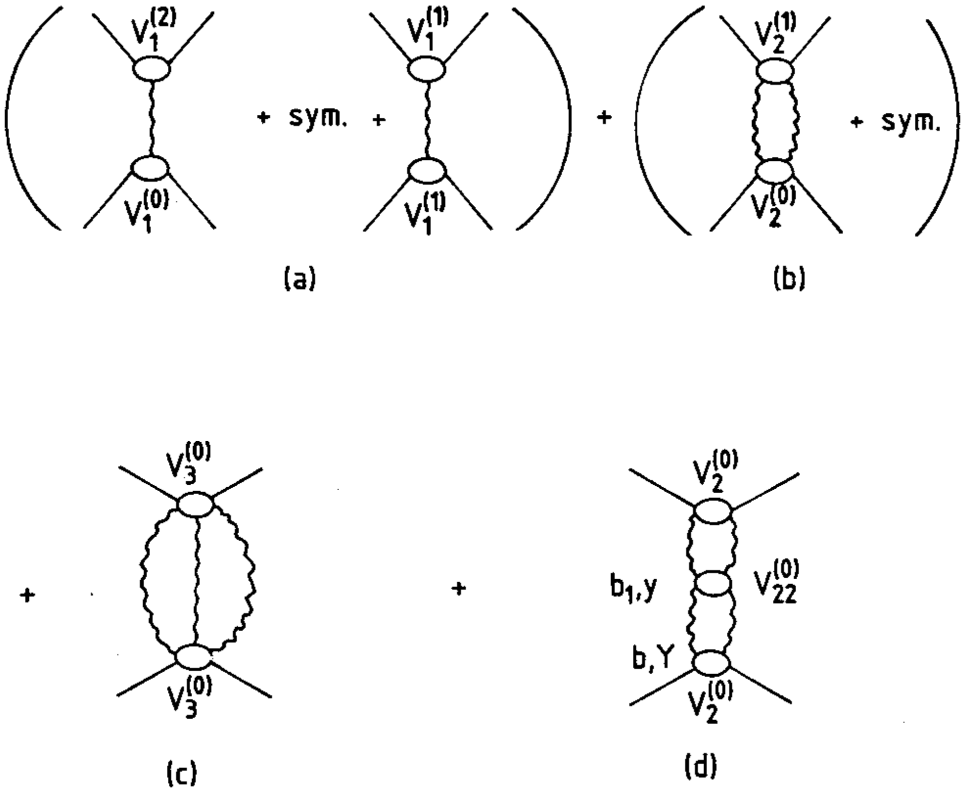
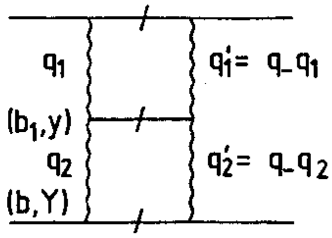


$V_1^{(2)}$
$V_1^{(0)}$
+ sym.
+
$V_1^{(1)}$
$V_1^{(1)}$
+
$V_2^{(1)}$
$V_2^{(0)}$
+ sym.
(a)
(b)
+
$V_3^{(0)}$
$V_3^{(0)}$
+
$V_2^{(0)}$
$b_1, y$
$V_{22}^{(0)}$
$b, Y$
$V_2^{(0)}$
(c)
(d)


- Fig. 12 -



$q_1$
$(b_1,y)$
$q_2$
$(b,Y)$
$q_1' = q - q_1$
$q_2' = q - q_2$


- Fig. 13 -