Classical and quantum results on logarithmic terms in the soft theorem in four dimensions

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ABSTRACT: We explore the logarithmic terms in the soft theorem in four dimensions by analyzing classical scattering with generic incoming and outgoing states and one loop quantum scattering amplitudes. The classical and quantum results are consistent with each other. Although most of our analysis in quantum theory is carried out for one loop amplitudes in a theory of (charged) scalars interacting via gravitational and electromagnetic interactions, we expect the results to be valid more generally.

KEYWORDS: Gauge Symmetry, Scattering Amplitudes

Contents

1 Introduction

Soft theorems [1–12] have been analyzed recently from different perspectives, both using asymptotic symmetries [13–36] and also by direct analysis of amplitudes in field theory and string theory [37–87]. There are general arguments establishing their validity in any space-time dimensions in any theory as long as one maintains the relevant gauge symmetries — general coordinate invariance for soft graviton theorem and U(1) gauge invariance for soft photon theorem [81, 83, 84]. However these arguments break down in four space-time dimensions due to infrared divergences [41] where more care may be needed [27, 44]. Indeed, since the S-matrix itself is infrared divergent, it is not *a priori* clear how to interpret a relation whose both sides are divergent.

Although soft theorem is a relation between quantum scattering amplitudes — amplitudes with soft photon or graviton to amplitudes without soft photon or graviton — one can also relate soft theorem to classical scattering amplitudes. In four space-time dimensions this can be done via asymptotic symmetries [16, 31, 34]. Ref. [88] produced a more direct relation between soft theorem and classical scattering in generic space-time dimensions by directly taking the classical limit of a quantum scattering amplitude. This relates various

terms in soft theorem to appropriate terms in the radiative part of the electromagnetic and gravitational fields in classical scattering in generic space-time dimensions. Reversing the logic, one can use the classical scattering data to give an alternative definition of the soft factors.

Since classical scattering is well defined even in four space-time dimensions, one can hope to use the classical definition of soft factors to understand soft theorem in four dimensions. Since in higher dimensions the soft theorem expresses the low frequency radiative part of the electromagnetic and gravitational fields in terms of momenta and angular momenta of incoming and outgoing finite energy particles, the naive guess will be that the same formula will continue to hold in four dimensions. However in carrying out this procedure we encounter an obstacle [89, 90]. The subleading terms in the soft theorem contain a factor of angular momentum $j^{\mu\nu}$ of the individual particles involved in the scattering, with the orbital contribution to the angular momentum given by $x^\mu p^\nu - x^\nu p^\mu$, where $x^\mu(\tau)$ and $p^\mu(\tau)$ label the asymptotic coordinates and momenta of the particle as a function of the proper time. In dimensions larger than four, p^μ approaches a constant and x^μ approaches the form $c^\mu + \alpha\, p^\mu \tau$ with constant c^μ and α as $\tau \to \infty$. Therefore $j^{\mu\nu}$ is independent of τ as $\tau \to \infty$ and we can use the asymptotic value of $j^{\mu\nu}$ computed this way to evaluate the soft factor. However in four space-time dimensions the long range gravitational and/or electromagnetic forces acting on the particles produce an additional term of the form $b^\mu \ln \tau$ in the expression for x^μ. This gives a logarithmically divergent term of the form $(b^\mu p^\nu - b^\nu p^\mu) \ln \tau$ in the expression for $j^{\mu\nu}$, making the subleading soft factor divergent.

Since we do not expect the radiative component of the metric or gauge fields to diverge in classical scattering in four space-time dimensions, this suggests that the divergence in the subleading soft factor is a breakdown of the power series expansion in the energy ω of the soft particle. Therefore the soft factor must contain non-analytic terms in ω. The natural guess is that the soft factor at the subleading order is given by replacing the factors of $\ln \tau$ in the naive expression by $\ln \omega^{-1}$. This has been tested in [89] by considering several examples of classical scattering in four space-time dimensions.[1]

The purpose of this paper is two fold. In all the examples considered in [89] the scattering process considered had one heavy center producing the long range Coulomb or gravitational field, and other particles carrying smaller masses were taken to be moving under the influence of the long range fields produced by the heavy center. In this paper we relax this assumption and consider a general scattering process where all particles involved in the scattering have masses of the same order, and then determine the logarithmic terms in the classical soft factor using the $\ln \tau \to \ln \omega^{-1}$ replacement rule. We also analyze directly the quantum subleading soft factor by considering one loop scattering of charged scalar fields in the presence of gravitational and electromagnetic interaction. The difference with the previous analysis, e.g. in [41], is that we do not insist on a power series expansion in ω and calculating the coefficients of the power series expansion. Instead we allow for possible non-analytic terms of order $\ln \omega^{-1}$ in the soft expansion. This analysis yields results consistent

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[1]The existence of various logarithmic terms in classical scattering has been known earlier [91–94]. Soft theorem provides a systematic procedure for computing the coefficient of the logarithmic term in the subleading soft factor without detailed knowledge of the forces responsible for the scattering.

with the classical results, although the quantum results contain additional real part which we interpret as the result of back reaction of the radiation on the motion of the particles.

Since [81, 83, 84] gave a general derivation of soft theorem including loop corrections as long as 1PI vertices do not generate soft factor in the denominator, one could ask to what extent we could derive the results of the current paper using the result of [81, 83, 84]. To this end we note that there are two distinct sources of logarithmic terms in the soft theorem. The first is the region of integration in which the loop momentum is large compared to the energy of the external soft particle. In this region we expect the arguments of [81, 83, 84] to be valid, and we find that the contribution from this region can indeed be obtained by applying the usual soft operator on the amplitude without the soft graviton. The other source is the region of integration in which the loop momentum is small compared to the external soft momentum. The contribution from this region cannot be derived using the usual soft theorem, and need to be computed explicitly.

The rest of the paper is organized as follows. Section 2 contains a summary and a discussion of our results where we also discuss various special cases of our classical result. Section 3 describes the analysis of the logarithmic terms in the soft expansion for general classical scattering. Section 4 describes some general strategy for dealing with the infrared divergent part of the S-matrix and extracting the quantum soft factor by making use of momentum conservation. Section 5 describes one loop quantum computation of the logarithmic terms in the soft photon theorem in scalar quantum electrodynamics (scaler QED). Section 6 describes a similar computation in the soft graviton theorem in a theory of charge neutral scalars interacting with the gravitational field. In section 7 we consider charged scalars interacting via both gravitational and electromagnetic interaction, and determine the one loop contribution to the quantum soft graviton factor due to electromagnetic interaction and one loop contribution to the quantum soft photon factor due to gravitational interaction.

Classical gravitational radiation during a high energy scattering process has been analyzed in [95, 96]. We have been informed by G. Veneziano that for massless particle scattering, results related to the ones described here were found in [97], and also that the logarithmic terms in the classical scattering have been derived in [98] by taking the soft limit of the results of [95, 96].

Note added. The papers quoted above have now appeared in the arXiv [97, 98]. In particular the results of [98] can be shown to be in perfect agreement with our results for scattering of massless particles.

2 Summary and analysis of the results

In this section we shall first summarize our results and then discuss various aspects of the results. Finally we shall consider some special limits and compare with known results. We shall use $\hbar = c = 8\pi G = 1$ units.

2.1 Summary of the results

In order to give a uniform treatment of the classical soft photon and soft graviton theorem, we shall denote by $\phi(\vec{x}, t)$ the radiative part of the metric or electromagnetic field at a point $\vec{x}$ at time t for a scattering event around the origin. For electromagnetic field, ϕ can be directly identified with the gauge field. For the gravitational field we define

$$h_{\mu\nu} = (g_{\mu\nu} - \eta_{\mu\nu})/2, \qquad e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h_\rho{}^\rho, \qquad (2.1)$$

and take ϕ to be $e_{\mu\nu}$. For both electromagnetism and gravity we define classical soft factor $S(\varepsilon, k)$ in D space-time dimensions via the relation:

$$\int dt\, e^{i\omega t}\, \varepsilon.\phi(\vec{x}, t) = e^{i\omega R} \left(\frac{\omega}{2\pi i R}\right)^{(D-2)/2} \frac{1}{2\omega} S(\varepsilon, k)$$

$$= -\frac{i}{4\pi R} e^{i\omega R} S(\varepsilon, k) \quad \text{for} \quad D = 4, \qquad (2.2)$$

where ε is the polarization tensor of the soft particle so that $\varepsilon.\phi = \varepsilon^\mu A_\mu$ for gauge fields and $\varepsilon^{\mu\nu} e_{\mu\nu}$ for gravity, and

$$k = -\omega(1, \hat{n}), \quad \hat{n} \equiv \vec{x}/|\vec{x}|, \quad R = |\vec{x}|. \qquad (2.3)$$

On the other hand the quantum soft factor $S(\varepsilon, k)$ is the ratio of an amplitude with an outgoing soft photon or graviton with momentum k and polarization ε and an amplitude without such a soft particle. It was shown in [88] that in the classical limit the quantum soft factor reduces to the classical soft factor for $D > 4$. Our interest will be in analyzing the situation in $D = 4$.

We consider the scattering of n particles carrying electric charges $\{q_a\}$ and momenta $\{p_a\}$ for $a = 1, \cdots n$. In our convention the momenta/charges carry extra minus sign if they are outgoing. The particles are taken to interact via electromagnetic and gravitational interactions besides other short range interactions whose nature we need not know. The symbol η_a takes value $+1$ (-1) if the a-th particle is ingoing (outgoing). Then the classical result for the soft photon factor $S_{\rm em}(\varepsilon, k)$, containing terms of order ω^{-1} and $\ln \omega^{-1}$, is[2]

$$S_{\rm em} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a - i \ln\omega^{-1} \sum_a \frac{q_a \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2 \{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}$$

$$+ \frac{i}{4\pi} (\ln\omega^{-1} + \ln R^{-1}) \sum_{\substack{b \\ \eta_b = -1}} k.p_b \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a$$

$$+ \frac{i}{8\pi} \ln\omega^{-1} \sum_a \frac{q_a \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho) \{2(p_b.p_a)^2 - 3m_a^2 m_b^2\}.$$

$$(2.4)$$

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[2]In this and subsequent expressions R arises as an infrared cut-off. For the classical result the $\ln R$ terms arise due to long range gravitational force on the soft photon or graviton during its journey from the scattering center to the detector over a distance R. For the quantum part, the natural infrared cut-off is provided by the resolution of the detector. For a detector placed at a distance R from the scattering center, the best energy resolution possible is of order $1/R$. Therefore it is again natural to take R as the infrared regulator.

Since for real polarization the subleading contribution is purely imaginary, it does not affect the flux to this order. However the flux for circular polarization and/or the wave-form of the electromagnetic field do receive subleading contribution. An identical situation prevails for gravity.

The quantum result for $S_{\rm em}$ has additional terms:[3]

$$\Delta S_{\rm em} = \frac{1}{16\pi^2} \, \ln \omega^{-1} \, \sum_a q_a \, \frac{\varepsilon_\mu k_\nu}{p_a.k} \left\{ p_a^\mu \frac{\partial}{\partial p_{a\nu}} - p_a^\nu \frac{\partial}{\partial p_{a\mu}} \right\}$$

$$\sum_{b \neq a} \left[\frac{\left\{ 2 \, q_a q_b p_a.p_b + 2 \, (p_a.p_b)^2 - p_a^2 p_b^2 \right\}}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \ln \left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \right) \right]$$

$$- \frac{1}{8\pi^2} \left(\ln \omega^{-1} + \ln R^{-1} \right) \sum_a \frac{q_a \varepsilon_\mu p_a^\mu}{p_a.k} \sum_b (p_b.k) \, \ln \left(\frac{m_b^2}{(p_b.\hat{k})^2} \right). \tag{2.5}$$

The classical results are universal, independent of the theory and the nature of external particles. We expect that the quantum results are also universal, but we have derived them by working with one loop amplitudes in scalar QED coupled to gravity. It is easy to check that (2.4), (2.5) are invariant under gauge transformation $\varepsilon_\mu \to \varepsilon_\mu + \xi \, k_\mu$ for any constant ξ.

As will be discussed in section 2.2, the quantum correction (2.5) should not be directly added to (2.4) and substituted into (2.2) to compute the radiative component of the classical electromagnetic field. Rather, when the contribution (2.5) is small compared to (2.4), we can substitute (2.4) into (2.2) to compute the classical electromagnetic field produced by a scattering event.

As discussed in section 4, the quantum results are ambiguous and are defined up to addition of a term to $S_{\rm em}$ of the form $\ln R^{-1} \, k.U \, S_{\rm em}^{(0)}$ where $S_{\rm em}^{(0)}$ is the leading soft factor given by the first term on the right hand side of (2.4) and U is a vector constructed out of the p_a's. By choosing $U = (8\pi^2)^{-1} \sum_b p_b \ln(m_b^2/\mu^2)$, we can replace the $\ln m_b^2$ term in the coefficient of $\ln R^{-1}$ in the last line of (2.5) by $\ln \mu^2$ for any mass parameter μ. This makes manifest the fact that the coefficient is not divergent in the $m_b \to 0$ limit. The coefficient of $\ln \omega^{-1}$ cannot be changed this way, but in this case the finiteness of $m_b \to 0$ limit follows as a consequence of cancellation between the second and third line of (2.5) and momentum conservation.

If we want to consider the situation where we ignore the effect of gravity, then we need to set the terms proportional to $\ln \omega^{-1}$ that are linear in q_c's to zero. On the other hand if we want to consider the situation where we ignore the effect of electromagnetic interaction between the particles during scattering (but still use electromagnetic interaction to compute soft photon emission process), we have to set the terms proportional to $\ln \omega^{-1}$ that are cubic in the q_c's to zero.

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[3]Note however that when we express the results in terms of the frequency/wavelength of the soft photon/graviton and momenta of the finite energy particles, neither the classical nor the quantum result has any power of $\hbar$. We shall discuss later the conditions under which we expect the quantum results to be small compared to the classical results.

The classical result for soft graviton factor takes the form

$$S_{\rm gr} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} - i \ln \omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2 \{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}$$

$$+ \frac{i}{4\pi} (\ln \omega^{-1} + \ln R^{-1}) \sum_{\substack{b \\ \eta_b = -1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k}$$

$$+ \frac{i}{8\pi} \ln \omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\} \,. \tag{2.6}$$

The quantum result has additional terms

$$\Delta S_{\rm gr} = \frac{1}{16\pi^2} \ln \omega^{-1} \sum_a \frac{\varepsilon_{\mu\rho} p_a^\rho k_\nu}{p_a.k} \left\{ p_a^\mu \frac{\partial}{\partial p_{a\nu}} - p_a^\nu \frac{\partial}{\partial p_{a\mu}} \right\}$$

$$\sum_{b \neq a} \left[\frac{\{2 q_a q_b p_a.p_b + 2 (p_a.p_b)^2 - p_a^2 p_b^2\}}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \ln \left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \right) \right]$$

$$- \frac{1}{8\pi^2} (\ln \omega^{-1} + \ln R^{-1}) \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \sum_b p_b.k \, \ln \frac{m_b^2}{(p_b.\hat{k})^2} \,, \tag{2.7}$$

where $\hat{k} = -k/\omega = (1, \hat{n})$. Again the classical results are valid universally. The quantum results are obtained from one loop calculation in scalar QED coupled to gravity, but we expect them to be universal. As in the case of (2.5), the $\ln m_b^2$ term in the coefficient of $\ln R^{-1}$ in the last line of (2.7) can be replaced by $\ln \mu^2$ by exploiting the ambiguity in the definition of the soft factor discussed in section 4. One can check that (2.6), (2.7) are invariant under gauge transformations $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu$ for any constant vector ξ_μ.

If we want to consider the situation where we ignore the effect of electromagnetic interactions, then we need to set the terms proportional to $\ln \omega^{-1}$ that are quadratic in q_c's to zero. On the other hand if we want to consider the situation where we ignore the effect of gravitational interaction between the particles during scattering (but still use gravitational interaction to compute soft graviton emission process), we have to set the q_c independent terms in the coefficient of $\ln \omega^{-1}$ to zero.

2.2 Discussion of results

First we shall briefly outline how these results are derived; more details can be found in later sections. The classical results (2.4) and (2.6) are the result of direct application of classical soft theorem to subleading order. As described in [89], the soft factor involves orbital angular momenta of initial and final particles and these diverge logarithmically in the elapsed time τ in four dimensions due to the long range gravitational/electromagnetic force on the incoming and outgoing particles that generates a term proportional to $\ln |\tau|$ in the trajectory. We follow the prescription of [89] of replacing $\ln |\tau|$ by $\ln \omega^{-1}$ to arrive at the first and third lines of the classical results (2.4), (2.6). The second lines of (2.4)

and (2.6) arise from additional phases that are not directly determined by soft theorem. They represent the effect of long range gravitational force on the outgoing soft photon or graviton which causes the soft particle to slow down and also backscatter.

Quantum results are the result of direct one loop computation in a field theory of multiple charged scalars, coupled to electromagnetic and gravitational fields. We simply evaluate the order ω^{-1} and $\ln \omega^{-1}$ terms in the scattering amplitude of multiple finite energy scalars and an outgoing soft photon or graviton of energy ω, and express this as the product of the amplitude without the soft photon or graviton and a multiplicative factor that we call the soft factor. The latter is given by the sum of (2.4) and (2.5) for soft photon and the sum of (2.6) and (2.7) for the soft graviton. Even though the S-matrix elements with and without the soft particle are infrared divergent, much of this cancels when we take the ratio of the two. The remaining infrared divergent part is regulated by the infra-red length cut-off R and is responsible for the terms proportional to $\ln R$ in these expressions. This is related to the quantity σ'_n introduced in [41].

The different terms proportional to $\ln \omega^{-1}$ in (2.4), (2.5) and in (2.6), (2.7) have different origin. We shall explain them in the context of the soft graviton factor, but the case of soft photon factor is very similar.

1. We begin with the classical result (2.6). The term proportional to $q_a q_b$ in the first line represents the effect of late time gravitational radiation due to the late time acceleration of the particles via long range electromagnetic interaction. The term in the last line of (2.6) represents the effect of late time gravitational radiation due to the late time acceleration of the particles via long range gravitational interaction. We expect the scale of these logarithms to be set by the largest length scale involved in the classical scattering process, e.g. the typical distance of closest approach between the particles involved in the scattering. This is taken to be larger than or of the order of the Schwarzschild radii of the particles and much larger than the Compton wave-lengths of the particles involved in the scattering. In the quantum one loop computation both these terms arise from the region of loop momentum integration where the loop momentum is large compared to ω but small compared to the energies of the other particles. In this case the scale of these logarithms is again set by the largest length scale involved in the quantum scattering which is the inverse of the typical energy carried by the finite energy external states. For one loop result to be reliable, this needs to be taken to be large compared to the Schwarzschild radii of these particles.

2. The term in the second line of (2.6) proportional to $(\ln \omega^{-1} + \ln R^{-1})$ represents the effect of gravitational drag on the soft graviton due to the other finite energy particles in the final state. This has the effect of causing a time delay, represented by the $\ln R^{-1}$ term, for the soft graviton to travel to a distance R. This also has the effect of inducing backscattering of the soft graviton, represented by the $\ln \omega^{-1}$ term. In the quantum computation these terms arise from region of loop momentum integration where the loop momentum is smaller than ω and larger than the infrared cut-off R^{-1}. This term has appeared e.g. in [91, 93, 94]. As mentioned in footnote 2,

the scale of these logarithms is set by the effective infrared cut-off, e.g. the distance R to the detector for the classical scattering and the resolution of the detector for the quantum scattering. The latter in turn has a lower limit set by R^{-1} since we cannot measure the energy of the outgoing particle with an accuracy better than R^{-1} if the detector is placed at a distance R from the scattering center.

3. We emphasize that the classical results are obtained by replacing in the classical soft theorem the logarithmically divergent terms by $\ln \omega^{-1}$ and not by direct calculation of electromagnetic and gravitational radiation during classical scattering. In special cases the equivalence of these two procedures was tested in [89] by direct classical computation. In principle similar tests can be done for the general formulae (2.4) and (2.6), but we have not done this.

4. We now turn to the additional terms (2.7) that arise in the quantum computation. First note that both these terms are real for real polarizations unlike the classical result where the coefficients of $\ln \omega^{-1}$ terms are imaginary for real polarizations. The terms in the first two lines come from regions of loop momentum integration where the loop momentum is large compared to ω but small compared to the energies of the other particles, while the term in the third line arise from region of loop momentum integration where the loop momentum is small compared to ω and large compared to the infrared cut-off R^{-1}.

5. In the quantum computation the terms that arise from loop momenta large compared to ω, namely the terms in the first and third line of (2.6) and the first two lines of (2.7), can be generated using a simple algorithm. As discussed earlier, the amplitude without the soft graviton has an infrared divergent factor multiplying it. Let us call this the IR factor. If in the integration over loop momenta of this IR factor we restrict the loop momentum integration to be large compared to ω and apply the usual subleading soft differential operator that arises in higher dimensions to this IR factor, we recover precisely the results given in the first and third line of (2.6) and the first two lines of (2.7). The rest of the contribution that arises from integration region where the loop momentum is small compared to ω cannot be recovered this way. This indicates that the general argument of [81, 83], based on general coordinate invariance of 1PI effective action and power counting assuming that loops do not generate inverse power of soft momentum, remain valid in four dimensions as well as long as the loop momentum is large compared to the external soft momentum.

Since the real infrared divergent part of the amplitude reflects the effect of real graviton emission, our interpretation of the extra contributions (2.7) in the quantum theory is that they reflect the effect of backreaction of soft radiation on the classical trajectories. To this end note that the validity of the classical limit described in [88] requires that the total energy carried by soft radiation should remain small compared to the energies of the finite energy objects taking part in the scattering. Here 'soft radiation' represents those particles which are not included in the sum over a in (2.6). Therefore we should expect that the extra terms arising in the quantum theory should be small in the limit when the total energy carried by the soft radiation is small.

In order to test this hypothesis we need to consider a scattering where the energy carried away by soft radiation remains small compared to the energies of finite energy objects. One way to achieve this is to consider scattering at large impact parameter so that each incoming particle gets deflected by a small amount and the energy radiated during this process remains small. In this case the momenta $\{p_a\}$ come in approximately equal and opposite pairs — the incoming and the corresponding outgoing particle. Now in eq. (2.7) the last term changes sign under $p_b \to -p_b$ and also under $p_a \to -p_a$. This shows that it is small for small deflection scattering. The first term on the right hand side of (2.7) changes sign under $(p_b, q_b) \to -(p_b, q_b)$ and also under $(p_a, q_a) \to -(p_a, q_a)$, due to the argument of the log getting inverted under each of these operations. This shows that the terms approximately cancel making the result small. There is one exception to this that arises when $q_b = -q_a$, $p_b \simeq -p_a$, i.e. the pairs (a, b) represent the incoming and the corresponding outgoing particle. In this case there is no other term that cancels this since the sum does not include the $b = a$ term, and we need to explicitly evaluate this and show that it vanishes. This can be checked explicitly by first evaluating the derivatives in the second line of (2.7), then setting $p_b = -p_a + \epsilon$ and then carefully evaluating the result in the $\epsilon \to 0$ limit. Even though individual terms diverge in the $\epsilon \to 0$ limit, a careful analysis shows that the result vanishes. This confirms that quantum corrections are small in this limit.

Another situation discussed in [88], where the radiated energy remains small compared to the energies of the hard particles, is the probe limit in which one of the particles has a large mass M and the other particles are lighter carrying energy small compared to M. We shall now verify that in this case too the quantum corrections (2.7) are small compared to the classical result (2.6). For this we shall work in a frame in which the heavy particle is initially at rest, and using gauge invariance choose the polarization tensor ε to have only spatial components. After the scattering the heavy particle acquires a momentum but it is small compared to M. In this case the dominant contribution to (2.6), of order M, comes from choosing a to be one of the light particles and b to be the heavy particle in the second and third line of (2.6). However in the quantum correction (2.7) similar contribution cancels between the choice of b as the initial state heavy particle and the final state heavy particle, and we do not get any contribution proportional to M. This again shows that quantum corrections are small compared to the classical result in this limit.

We must emphasize however that the quantum analysis is carried out for single soft graviton emission. If we want to relate the quantum result to the radiative component of the classical gravitational field as in [88], then we need to first consider multiple soft graviton emission and then take the classical limit. The analysis of [88] relied on the fact that the soft factors associated with different bins in the phase space are independent of each other, i.e. the probability of emitting certain number of soft particles in one bin does not depend on how many soft particles are emitted in the other bin. This independence breaks down when the total energy carried by the soft particles becomes comparable to the energies of the hard particles — precisely when the quantum correction (2.7) becomes comparable to the classical result (2.6). Therefore we should not use (2.7) to modify the classical result (2.6). Instead we should use the smallness of (2.7) as a test of when the classical result (2.6) is valid. An identical discussion holds for electromagnetism.

2.3 Special cases

As a special case we can consider the situation described in [90] where a neutral massive object of mass M at rest decays into a heavy object of mass $M_0 \simeq M$ and a set of neutral light objects carrying mass $m_a \ll M$ and momentum $p_a = -e_a(1, \vec{\beta}_a)$ with $e_a \ll M$ for $a = 1, \cdots N$. Our goal will be to write down the classical soft graviton factor for this case. We shall take the polarization tensor of the soft graviton to have components only along the spatial direction, since the result for the other components may be found by using invariance under the gauge transformation $\varepsilon_{\mu\nu} \to \varepsilon_{\mu\nu} + \xi_\mu k_\nu + \xi_\nu k_\mu$ for any vector ξ. If we denote the momentum carried by final state heavy object of mass M_0 by p_{N+1}, then we have $p_{N+1}^0 \simeq -M_0$ and $|p_{N+1}^i| \ll M_0$. Examining (2.6) with $q_a = q_b = 0$ we see that dominant term proportional to $\ln \omega^{-1}$ comes from the terms where we choose $b = N + 1$ and a labels any of the N finite energy states. Using the relation $e_a^2 = m_a^2/(1 - \vec{\beta}_a^2)$, the net contribution takes the form:

$$
\begin{aligned}
&\frac{i}{4\pi} \ln \omega^{-1} M_0 \sum_{a=1}^N e_a \frac{\varepsilon^{ij} \beta_{ai} \beta_{aj}}{1 - \hat{n}.\vec{\beta}_a} + \frac{i}{8\pi} \ln \omega^{-1} M_0 \sum_{a=1}^N e_a \frac{\varepsilon^{ij} \beta_{ai} \beta_{aj}}{1 - \hat{n}.\vec{\beta}_a} \frac{(-e_a)(2e_a^2 - 3m_a^2)}{(e_a^2 - m_a^2)^{3/2}} \\
&= \frac{i}{8\pi} \ln \omega^{-1} M_0 \sum_{a=1}^N e_a \frac{\varepsilon^{ij} \beta_{ai} \beta_{aj}}{1 - \hat{n}.\vec{\beta}_a} \frac{2\vec{\beta}_a^3 + 1 - 3\vec{\beta}_a^2}{|\vec{\beta}_a|^3} + \cdots,
\end{aligned}
\tag{2.8}
$$

where $\cdots$ contain terms without a factor of M_0 and are therefore smaller in the limit of large M_0. This agrees with the results of [90]. As discussed in [90], this produces a late time tail in the gravitational wave-form that falls off as inverse power of time.

Note that when all the final state light particles are massless, so that $|\vec{\beta}_a| = 1$ for $1 \leq a \leq N$, the expression (2.8) vanishes. This would be the situation during binary black hole merger when the final state particles are only gravitons. However since in such processes the radiation carries away an appreciable fraction of the mass of the parent system, the $\cdots$ terms in (2.8) could be significant even though their contribution will be suppressed by the ratio of the total energy carried away by radiation to the mass of the parent system. We shall now evaluate the result without making any approximation. In this case in the sum over a and b in (2.6), either a or b (or both) represents a massless particle. Recalling that when p_a and p_b are both outgoing then $p_a.p_b$ is negative, we can express the terms in (2.6) proportional to $\ln \omega^{-1}$ as

$$
\frac{i}{4\pi} \ln \omega^{-1} \sum_{a=1}^{N+1} \varepsilon^{ij} p_{ai} p_{aj} + \frac{i}{4\pi} \ln \omega^{-1} \sum_{a=1}^{N+1} \varepsilon^{ij} p_{ai} \sum_{\substack{b=1 \\ b \neq a}}^{N+1} p_{bj} = 0,
\tag{2.9}
$$

where in the last step we have used conservation of spatial momentum $\sum_{b=1}^{N+1} p_{bj} = 0$. Therefore we see that even without making any approximation, the coefficient of the $\ln \omega^{-1}$ term in the classical soft graviton factor continues to vanish.

Another special case we can consider is when a charge neutral object of mass M at rest breaks apart into two charge neutral objects of masses m_1 and m_2, spatial momenta $\vec{p}$ and $-\vec{p}$ and energies $e_1 = \sqrt{m_1^2 + \vec{p}^2}$ and $e_2 = \sqrt{\vec{p}^2 + m_2^2}$. In this case if we take the polarization tensor of the soft graviton to have components only along the spatial direction,

then the contribution from the initial state to (2.6) vanishes and we need to only compute the contribution from a pair of final states. This can be easily evaluated and the terms proportional to $\ln \omega^{-1}$ take the form

$$\frac{i}{8\pi} \ln \omega^{-1} \varepsilon_{ij} p^i p^j (e_1 + e_2) \left\{ \frac{1}{e_1 - \hat{n}.\vec{p}} + \frac{1}{e_2 + \hat{n}.\vec{p}} \right\}$$
$$\times \left[\frac{e_1 e_2 + \vec{p}^2}{\{(e_1 e_2 + \vec{p}^2)^2 - m_1^2 m_2^2\}^{3/2}} \left\{ 2(e_1 e_2 + \vec{p}^2)^2 - 3m_1^2 m_2^2 \right\} - 2 \right]. \qquad (2.10)$$

Next special case we shall analyze is that of scattering of massless particles, again focussing on the classical result (2.6). Defining

$$P \equiv \sum_{\eta_a = 1} p_a = - \sum_{\eta_a = -1} p_a, \qquad (2.11)$$

and the fact that $p_a.p_b$ is negative for $\eta_a \eta_b = 1$, we can express the term proportional to $\ln \omega^{-1}$ in (2.6) for massless particles as

$$-\frac{i}{2\pi} \ln \omega^{-1} k.P \sum_{\substack{a,b \\ b \neq a, \eta_a = \eta_b = 1}} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{i}{2\pi} \ln \omega^{-1} \varepsilon_{\mu\nu} P^\mu P^\nu. \qquad (2.12)$$

Note that this involves only the momenta of the initial state particles and is insensitive to the momenta of the final state particles. This asymmetry is related to the fact that in our analysis we are considering soft particle only in the final state and not in the initial state.

More generally one can show that for a general scattering process involving both massive and massless particles, the terms proportional to $\ln \omega^{-1}$ in the classical formula (2.6) is not sensitive to the details of the final state massless particles except through overall momentum conservation. To see this let us first consider terms that could involve a final state massless particle momenta and the initial state momenta. These come from choosing a to be an initial state and b to be a final state massless state in the term in the second line of (2.6). The net contribution from such terms is given by

$$\frac{i}{4\pi} \ln \omega^{-1} \sum_{\substack{b \, \text{massless} \\ \eta_b = -1}} k.p_b \sum_{\substack{a \\ \eta_a = 1}} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{k.p_a} = -\frac{i}{4\pi} \ln \omega^{-1} k.(P - P_{\text{massive}}) \sum_{\substack{a \\ \eta_a = 1}} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{k.p_a}, \qquad (2.13)$$

where $-P$ denotes total outgoing momentum as defined in (2.11) and $-P_{\text{massive}}$ denotes the total outgoing momentum carried by the massive particles. Therefore this does not depend explicitly on the momenta of the outgoing massless states except through momentum conservation.

Next we consider terms that involve a pair of final state momenta at least one of which is massless. This term receives contribution from all three lines on the right hand side of (2.6) with the restriction $\eta_a = 1$, $\eta_b = 1$, and either m_a or m_b or both zero. Therefore the term proportional to $q_a q_b$ vanishes. Also the coefficient of $\ln \omega^{-1}$ in the summand in the last two lines simplifies to

$$\frac{i}{4\pi} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} p_b.k - \frac{i}{4\pi} \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} p_b.k + \frac{i}{4\pi} \varepsilon_{\mu\nu} p_a^\mu p_b^\nu. \qquad (2.14)$$

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In the first term the sum over a and b includes the term where $b = a$, but in the second and the third term the sum excludes the $b = a$ term. Therefore the first two terms almost cancel, leaving behind a contribution where we set $b = a$. This left over contribution $\frac{i}{4\pi} \varepsilon_{\mu\nu} p_a^\mu p_a^\nu$ can now be added to the last term to include in the sum over a or b also the contribution where $b = a$. The net contribution from the terms where either a or b or both represent massless state is then

$$\frac{i}{4\pi} \ln \omega^{-1} \sum_{\substack{a,b;\eta_a=\eta_b=-1 \\ \text{either } a \text{ or } b \text{ massless}}} \varepsilon_{\mu\nu}\, p_a^\mu p_b^\nu \,. \tag{2.15}$$

This can be rewritten as

$$\frac{i}{4\pi} \ln \omega^{-1} \varepsilon_{\mu\nu} \left(\sum_{a,b;\eta_a=\eta_b=-1} p_a^\mu p_b^\nu - \sum_{\substack{a,b;\eta_a=\eta_b=-1 \\ a \text{ and } b \text{ massive}}} p_a^\mu p_b^\nu \right) = \frac{i}{4\pi} \ln \omega^{-1} \varepsilon_{\mu\nu} \left(P^\mu P^\nu - P_{\text{massive}}^\mu P_{\text{massive}}^\nu \right). \tag{2.16}$$

This also does not depend on the details of the momenta of massless final state particles except for the total momentum carried by these particles.

3 Classical analysis

The goal of this section will be to calculate the logarithmic terms in the soft factors in four space-time dimensions by examining them in the classical limit.

In dimensions larger than 4, the soft factors for photons and gravitons are given respectively by

$$S_{\text{em}} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\, q_a + i \sum_a q_a\, \frac{\varepsilon_\mu k_\rho \mathbf{J}_a^{\rho\mu}}{p_a.k} \,, \tag{3.1}$$

and

$$S_{\text{gr}} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + i \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho \mathbf{J}_a^{\rho\mu}}{p_a.k} \,. \tag{3.2}$$

Here the sum over a runs over all the incoming and outgoing particles, and q_a, p_a and $\mathbf{J}_a$ denote the charge, momentum and angular momentum of the a-th particle, counted with positive sign for an ingoing particle and negative sign for an outgoing particle. S_{em} may also contain a non-universal term at the subleading order. For S-matrix elements in quantum theory, $\mathbf{J}_a$ is a differential operator involving derivatives with respect to the external momenta. However in the classical limit in which the external finite energy states are macroscopic, $\mathbf{J}_a$ represents the classical angular momenta carried by the external particles. In this limit the soft factors describe the radiative part of the low frequency electromagnetic and gravitational fields during a classical scattering [88] as described in (2.2).

In applying (3.1), (3.2) to four dimensional theories, the complication arises from the contribution to $\mathbf{J}_a^{\mu\nu}$ from the orbital angular momentum. They are computed from the form of the asymptotic trajectories:

$$r_a^\mu(\sigma) = \eta_a\, \frac{1}{m_a} p_a^\mu \sigma + c_a^\mu \ln |\sigma| + \cdots \,, \tag{3.3}$$

where η_a is positive for incoming particles and negative for outgoing particles, m_a is the mass of the a-th particle and the proper time σ is large and negative for incoming particles and large and positive for outgoing particles. The term proportional to $\ln |\sigma|$ represents the effect of long range electromagnetic and/or gravitational interaction between the particles. This gives, for large $|\sigma|$,

$$\mathbf{J}_a^{\mu\nu} \simeq r_a^\mu(\sigma)p_a^\nu - r_a^\nu(\sigma)p_a^\mu + \text{spin} = (c_a^\mu p_a^\nu - c_a^\nu p_a^\mu) \ln |\sigma| + \cdots . \tag{3.4}$$

Here and in the following we shall use the convention that when a variable is followed by an argument (σ) it denotes the value of the variable at proper time σ, but when a variable is written without an argument, we take it to be its σ independent asymptotic value. Therefore in (3.3), (3.4) the p_a^μ's denote the asymptotic values of p_a^μ, reflecting the fact that the difference between $p_a^\mu(\sigma) = m_a\eta_a dr_a^\mu/d\sigma$ and p_a^μ approaches zero asymptotically.

Analysis of [89] indicates that if we substitute (3.4) into (3.1) and (3.2) and replace $\ln |\sigma|$ by $\ln \omega^{-1}$ — where $\omega = k_0$ is the frequency of the outgoing soft radiation — we can recover the logarithmic terms in the soft factors up to overall phases. This gives, up to overall phases:

$$S_{\text{em}} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a + i \ln \omega^{-1} \sum_a q_a \frac{\varepsilon_\mu k_\rho(c_a^\rho p_a^\mu - c_a^\mu p_a^\rho)}{p_a.k} , \tag{3.5}$$

and

$$S_{\text{gr}} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + i \ln \omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho(c_a^\rho p_a^\mu - c_a^\mu p_a^\rho)}{p_a.k} . \tag{3.6}$$

Note that although S_{em} may contain a non-universal term at the subleading order, the term proportional to $\ln \omega^{-1}$ comes from orbital angular momentum and is universal.

Irrespective of what forces are operative during the scattering, the coefficient c_a^μ are determined only by the long range forces acting on the incoming and the outgoing particles. These will be taken to be electromagnetic and/or gravitational interaction. We shall now compute c_a^μ due to electromagnetic and gravitational interactions. We know from explicit comparison with known results that in the case of scattering via electromagnetic interactions there are no additional phases in the soft factor, but in the case of gravitational long range interaction there is an additional phase reflecting the effect of backscattering of the soft photon or soft graviton in the background gravitational field [91, 93, 94]. This phase will also be determined below.

3.1 Effect of electromagnetic interactions

We shall first study the effect of logarithmic correction to the trajectory due to long range electromagnetic interaction. For this we need to compute the gauge potential $A_\mu^{(b)}(x)$ at space-time point x due to particle b. We have

$$A_\mu^{(b)}(x) = \frac{1}{2\pi} \int d\sigma\, \eta_b\, q_b\, V_{b\mu}(\sigma)\, \delta_+(-(x - r_b(\sigma))^2), \quad V_b^\mu(\sigma) \equiv \frac{dr_{b\mu}(\sigma)}{d\sigma} \simeq \eta_b \frac{p_b^\mu}{m_b} , \tag{3.7}$$

where δ_+ denotes the usual Dirac delta function with the understanding that we have to choose the zero of the argument for which $x^0 > r_b^0(\sigma)$. V_b denotes the asymptotic four

velocity of the b-th particle. In evaluating (3.7) we shall ignore the logarithmic corrections to the trajectory and take $r_b(\sigma) \simeq V_b \sigma$. This gives, using $V_b^2 = -1$,

$$\delta_+(-(x-r_b(\sigma))^2) = \delta_+(-x^2+2\,V_b.x\,\sigma+\sigma^2+\cdots) \simeq \frac{1}{2|V_b.x+\sigma|}\,\delta(\sigma+V_b.x+\sqrt{(V_b.x)^2+x^2})\,, \tag{3.8}$$

where the sign in front of the square root has been chosen to ensure that $x^0 > x_b^0(\sigma)$ at the solution. Substituting this into (3.7) we get

$$A_\mu^{(b)}(x) \simeq \frac{1}{4\pi}\,\frac{\eta_b\,q_b V_{b\mu}}{\sqrt{(V_b.x)^2+x^2}}\,. \tag{3.9}$$

From this we calculate

$$F_{\mu\nu}^{(b)}(x) = \partial_\mu A_\nu^{(b)}(x) - \partial_\nu A_\mu^{(b)}(x) \simeq -\frac{\eta_b\,q_b}{4\pi}\,\frac{x_\mu V_{b\nu} - x_\nu V_{b\mu}}{\{(V_b.x)^2+x^2\}^{3/2}}\,. \tag{3.10}$$

At the location $r_a = V_a\sigma = -V_a|\sigma|\eta_a$ of the a-th particle we get, using $V_a^2 = -1$

$$F_{\mu\nu}^{(b)}(r_a(\sigma)) \simeq \eta_a\,\eta_b\,\frac{q_b}{4\pi\,\sigma^2}\,\frac{V_{a\mu}V_{b\nu} - V_{a\nu}V_{b\mu}}{\{(V_b.V_a)^2-1\}^{3/2}}\,. \tag{3.11}$$

Now the a-th particle will feel the field produced by the b-th particle if either both a-th and the b-th particle are outgoing or if both particles are ingoing. Therefore the equation of motion for the a-th particle takes the form

$$\frac{dp_{a\mu}(\sigma)}{d\sigma} = q_a \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} F_{\mu\nu}^{(b)}(r_a(\sigma))\,V_a^\nu(\sigma) \simeq \frac{1}{\sigma^2} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \eta_a\,\eta_b\,\frac{q_a q_b}{4\pi}\,\frac{V_a.V_b V_{a\mu} + V_{b\mu}}{\{(V_b.V_a)^2-1\}^{3/2}}\,. \tag{3.12}$$

On the other hand we have

$$\frac{dp_{a\mu}(\sigma)}{d\sigma} = \frac{m_a}{\eta_a}\frac{d^2 r_{a\mu}}{d\sigma^2} = -\frac{m_a}{\eta_a}\frac{c_{a\mu}}{\sigma^2}\,, \tag{3.13}$$

where in the last step we used (3.3). Comparing (3.12), (3.13) we get

$$c_a^\mu = -\frac{1}{m_a} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \eta_b\,\frac{q_a q_b}{4\pi}\,\frac{V_a.V_b V_a^\mu + V_b^\mu}{\{(V_b.V_a)^2-1\}^{3/2}} = -\sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\,\frac{m_b^2\,p_a.p_b\,p_a^\mu + m_a^2 m_b^2\,p_b^\mu}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,, \tag{3.14}$$

and

$$c_a^\mu p_a^\nu - c_a^\nu p_a^\mu = -\sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\,\frac{m_a^2 m_b^2\,\{p_b^\mu p_a^\nu - p_b^\nu p_a^\mu\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,. \tag{3.15}$$

Eqs. (3.5) and (3.6) now give[4]

$$S_{\text{em}} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k}\,q_a - i\,\ln\omega^{-1} \sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\,\frac{m_a^2 m_b^2\,\{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}\,, \tag{3.16}$$

---

[4]Note that even if we assume that the logarithmic corrections to the trajectories are generated predominantly by electromagnetic interaction, the resulting acceleration can generate logarithmic corrections to the gravitational radiation during the scattering.

and

$$S_{\text{gr}} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} - i \ln \omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi} \frac{m_a^2 m_b^2 \{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}. \tag{3.17}$$

3.2 Effect of gravitational interactions

Let us now suppose that the logarithmic correction to the trajectories arise due to gravitational interaction. We introduce the graviton field $h_{\mu\nu}$ and its trace reversed version $e_{\mu\nu}$ via the equations

$$h_{\mu\nu} \equiv (g_{\mu\nu} - \eta_{\mu\nu})/2, \qquad e_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}\eta_{\mu\nu} h_\rho{}^\rho. \tag{3.18}$$

Then the analog of (3.7) for the gravitational field produced at x due to the b-th particle is

$$e_{\mu\nu}^{(b)}(x) = \frac{1}{2\pi} \int d\sigma \, m_b \, V_{b\mu}(\sigma) \, V_{b\nu}(\sigma) \, \delta_+(-(x - r_b(\sigma))^2). \tag{3.19}$$

Using $r_b(\sigma) = V_b \, \sigma + \cdots$ we get the analog of (3.9)

$$e_{\mu\nu}^{(b)}(x) \simeq \frac{1}{4\pi} \frac{m_b \, V_{b\mu} \, V_{b\nu}}{\sqrt{(V_b.x)^2 + x^2}}. \tag{3.20}$$

The associated Christoffel symbol is given by, in the weak field approximation,

$$\Gamma_{\rho\tau}^{(b)\alpha}(x) = -\frac{m_b}{4\pi} \frac{1}{\{(V_b.x)^2 + x^2\}^{3/2}} \eta^{\alpha\mu} \left[\left\{ V_{b\mu} V_{b\tau} + \frac{1}{2}\eta_{\mu\tau} \right\} \{x_\rho + V_b.x \, V_{b\rho}\} \right.$$
$$\left. + \left\{ V_{b\mu} V_{b\rho} + \frac{1}{2}\eta_{\mu\rho} \right\} \{x_\tau + V_b.x \, V_{b\tau}\} - \left\{ V_{b\rho} V_{b\tau} + \frac{1}{2}\eta_{\rho\tau} \right\} \{x_\mu + V_b.x \, V_{b\mu}\} \right]. \tag{3.21}$$

From this we can write down the equation of motion of the a-th particle

$$\frac{d^2 r_a^\alpha(\sigma)}{d\sigma^2} = - \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \Gamma_{\rho\tau}^{(b)\alpha}(r_a(\sigma)) \, V_a^\rho(\sigma) \, V_a^\tau(\sigma) \tag{3.22}$$

$$\simeq -\eta_a \frac{1}{4\pi\sigma^2} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} m_b \frac{1}{\{(V_b.V_a)^2 - 1\}^{3/2}} \left[-\frac{1}{2}V_a^\alpha + \frac{1}{2}V_b^\alpha \left\{ 2(V_b.V_a)^3 - 3V_b.V_a \right\} \right].$$

On the other hand using (3.3) the left hand side is given by $-c_a^\alpha/\sigma^2$. This gives

$$c_a^\alpha = \eta_a \frac{1}{4\pi} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} m_b \frac{1}{\{(V_b.V_a)^2 - 1\}^{3/2}} \left\{ -\frac{1}{2}V_a^\alpha + \frac{1}{2}V_b^\alpha \left(2(V_b.V_a)^3 - 3V_b.V_a \right) \right\}, \tag{3.23}$$

and

$$c_a^\rho p_a^\mu - c_a^\mu p_a^\rho = \frac{1}{8\pi\sigma^2} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} m_a m_b \frac{1}{\{(V_b.V_a)^2 - 1\}^{3/2}} (V_b^\rho V_a^\mu - V_b^\mu V_a^\rho) \left\{ 2(V_b.V_a)^3 - 3V_b.V_a \right\}$$

$$= \frac{1}{8\pi} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho) \left\{ 2(p_b.p_a)^2 - 3m_a^2 m_b^2 \right\}. \tag{3.24}$$

Substituting this into (3.5) and (3.6) we get,[5] up to overall phases:

$$S_{\rm em} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} \, q_a + \frac{i}{8\pi} \, \ln \omega^{-1} \sum_a \frac{q_a \, \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \, (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)$$

$$\times \left\{ 2(p_b.p_a)^2 - 3m_a^2 m_b^2 \right\} , \tag{3.25}$$

and

$$S_{\rm gr} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{i}{8\pi} \, \ln \omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a \eta_b = 1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \, (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)$$

$$\times \left\{ 2(p_b.p_a)^2 - 3m_a^2 m_b^2 \right\} . \tag{3.26}$$

In this case we expect the wave-form of the gauge field/metric to also have an additional phase factor reflecting the effect of the gravitational drag on the soft particle due to the other particles. For this let us characterize the asymptotic trajectory of the soft particle as

$$x^\mu(\tau) = n^\mu \tau + m^\mu \ln|\tau| , \tag{3.27}$$

where τ is the affine parameter associated with the trajectory, $n = (1, \hat{n})$ is a null vector along the asymptotic direction of motion of the soft particle and m^μ is a four vector to be determined. Now substituting (3.27) into the equation of motion

$$\frac{d^2 x^\mu}{d\tau^2} = -\Gamma^\mu_{\nu\rho} \frac{dx^\nu}{d\tau} \frac{dx^\rho}{d\tau} , \tag{3.28}$$

and using the form (3.21) of $\Gamma^\mu_{\nu\rho}$, we get the following expression for m^μ by comparing the $1/\tau^2$ terms on the two sides of the equations of motion:

$$m^\alpha = -\frac{1}{4\pi} \sum_{\substack{b \\ \eta_b = -1}} \frac{m_b}{|n.V_b|^3} V_b^\alpha \, (V_b.n)^3 = \frac{1}{4\pi} \sum_{\substack{b \\ \eta_b = -1}} m_b V_b^\alpha = -\frac{1}{4\pi} \sum_{\substack{b \\ \eta_b = -1}} p_b^\alpha . \tag{3.29}$$

Now eliminating τ in terms of $t \equiv x^0$ using (3.27), we can express (3.27) as

$$x^i = n^i t + (m^i - n^i m^0) \ln|t| + \text{finite} . \tag{3.30}$$

Therefore if we denote by $k = (k^0, k) = -\omega(1, \hat{n})$ the four momentum of the soft particle, the overall $-$ sign reflecting the fact that it is an outgoing particle, the wave-function of the particle will be proportional to

$$\exp\left[-i\vec{k}. \left\{ \vec{x} - \hat{n} t - (\vec{m} - \hat{n} \, m^0) \ln|t| \right\} \right] = \exp[-i\omega t + i\omega \hat{n}.\vec{x}] \, \exp[i(\vec{k}.\vec{m} + \omega \, m^0) \ln|t|] . \tag{3.31}$$

_____

[5]Even if the logarithmic correction to the trajectory is generated by gravitational interaction, the particles can emit electromagnetic waves. This happens for example if we have a scattering of a charged particle and a neutral particle.

The second factor can be regarded as an additional infrared divergent contribution to the soft factor. Using $|t| \sim R$ where R is the distance of the soft particle from the scattering center, and eq. (3.29), we can express the second factor in (3.31) as

$$\exp[ik.m \ln R] = \exp\left[-\frac{i}{4\pi} \ln R \sum_{\substack{b \\ \eta_b=-1}} k.p_b\right].$$ (3.32)

Since this is a pure phase it does not affect the flux. However it does produce observable effect on the electromagnetic/gravitational wave-form [90].

It follows from the analysis of [91, 93, 94] that the effect of gravitational backscattering of the soft photon/graviton actually converts $\ln R$ in (3.32) to $\ln(R\omega)$. This has been reviewed in [89]. It is natural to absorb this multiplicative factor in the wave-form into the definition of the soft factors. Expanding the exponential in a power series, picking up the term of order $\omega \ln(\omega R)$ in the expansion, and multiplying this by the leading soft factor, we get additional contributions to the soft photon and soft graviton factor at the subleading order

$$\frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) S_{\text{em}}^{(0)} \sum_{\substack{b \\ \eta_b=-1}} k.p_b, \qquad \text{and} \qquad \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) S_{\text{gr}}^{(0)} \sum_{\substack{b \\ \eta_b=-1}} k.p_b.$$ (3.33)

Adding these to (3.25) and (3.26) we get the net soft factors to be

$$S_{\text{em}} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a + \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) \sum_{\substack{b \\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a$$
$$+ \frac{i}{8\pi} \ln\omega^{-1} \sum_a \frac{q_a \varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\},$$ (3.34)

and

$$S_{\text{gr}} = \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} + \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) \sum_{\substack{b \\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k}$$
$$+ \frac{i}{8\pi} \ln\omega^{-1} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\nu k_\rho}{p_a.k} \sum_{\substack{b \neq a \\ \eta_a\eta_b=1}} \frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} (p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\}.$$ (3.35)

3.3 Effect of electromagnetic and gravitational interactions

We now combine the results of last two subsections to write down the general expression for the soft factor when both gravitational interaction and electromagnetic interactions are responsible for the logarithmic corrections to the trajectory. The logarithmic terms get

added up, yielding the results:

$$
\begin{aligned}
S_{\mathrm{em}} = \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a &+ \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right) \sum_{\substack{b \\ \eta_b = -1}} k.p_b \sum_a \frac{\varepsilon_\mu p_a^\mu}{p_a.k} q_a \\
&- i\ln\omega^{-1} \sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k} \sum_{\substack{b\neq a \\ \eta_a\eta_b=1}} \frac{q_a q_b}{4\pi}\frac{m_a^2 m_b^2\{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \\
&+ \frac{i}{8\pi}\ln\omega^{-1}\sum_a \frac{q_a\,\varepsilon_\mu k_\rho}{p_a.k}\sum_{\substack{b\neq a \\ \eta_a\eta_b=1}}\frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\},
\end{aligned}
$$

(3.36)

and

$$
\begin{aligned}
S_{\mathrm{gr}} = \sum_a \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k} &+ \frac{i}{4\pi}\left(\ln\omega^{-1} + \ln R^{-1}\right)\sum_{\substack{b \\ \eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k} \\
&- i\ln\omega^{-1}\sum_a \frac{\varepsilon_{\mu\nu}p_a^\nu k_\rho}{p_a.k}\sum_{\substack{b\neq a \\ \eta_a\eta_b=1}}\frac{q_a q_b}{4\pi}\frac{m_a^2 m_b^2\{p_b^\rho p_a^\mu - p_b^\mu p_a^\rho\}}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}} \\
&+ \frac{i}{8\pi}\ln\omega^{-1}\sum_a \frac{\varepsilon_{\mu\nu}p_a^\nu k_\rho}{p_a.k}\sum_{\substack{b\neq a \\ \eta_a\eta_b=1}}\frac{p_b.p_a}{\{(p_b.p_a)^2 - m_a^2 m_b^2\}^{3/2}}(p_b^\rho p_a^\mu - p_b^\mu p_a^\rho)\{2(p_b.p_a)^2 - 3m_a^2 m_b^2\}.
\end{aligned}
$$

(3.37)

These reproduce (2.4) and (2.6) respectively.

Note that the soft factors given in (3.36) and (3.37) depend only on the charges and momenta carried by the external states. Therefore these can be reinterpreted as multiplicative soft factors in the full quantum theory — since there is no angular momentum there is no derivative with respect to the external momenta. In the next few sections we shall carry out some explicit quantum computations to examine to what extent this holds.

4 How to treat momentum conservation and infrared divergences

In quantum theory, single soft theorem is expected to relate an amplitude $\Gamma^{(n,1)}$ with n finite energy external states carrying momenta $p_1, \cdots p_n$ and one soft particle of momentum k to an amplitude $\Gamma^{(n)}$ with just n finite energy external states carrying momenta $p_1, \cdots p_n$. This relation takes the form

$$
\Gamma^{(n,1)}(p_1, \cdots p_n, k) \simeq S(\varepsilon, k; \{p_a\})\,\Gamma^{(n)}(p_1, \cdots p_n),
$$

(4.1)

where $S(\varepsilon, k; \{p_a\})$ is the soft factor S_{em} or S_{gr}. There is however a potential problem. While the amplitude $\Gamma^{(n,1)}$ has momentum conservation $\sum_a p_a + k = 0$, the amplitude $\Gamma^{(n)}$ has momentum conservation $\sum_a p_a = 0$. Therefore we cannot keep the p_a's and k as independent variables in (4.1). Usually this problem is overcome by including the momentum conserving delta-functions in the definition of the amplitudes $\Gamma^{(n,1)}$ and $\Gamma^{(n)}$ and

treating (4.1) as a relation between distributions. The soft factor $S(\varepsilon, k; \{p_a\})$ appearing in (4.1) is treated as a differential operator that also acts on the delta function and generates the Taylor series expansion of $\delta\left(\sum_a p_a + k\right)$ in power series of the momentum k of the soft particle. The subleading term in this expansion, given by $k^\mu \{\partial/\partial p_b^\mu\} \delta\left(\sum_a p_a\right)$ for any b, is included in the full subleading soft theorem in dimensions $D > 4$. However since in $D = 4$ we only analyze subleading terms containing $\ln \omega^{-1}$ factors, the term proportional to derivative of the delta function will not appear in our analysis.

In four space-time dimensions there are additional issues due to infrared divergence. Both the amplitudes $\Gamma^{(n,1)}$ and $\Gamma^{(n)}$ have infrared divergences which can be represented as overall multiplicative factors multiplying infrared finite amplitudes. For electromagnetic interactions these factors are common and can be factored out of the amplitudes but for gravity there is a residual infrared divergent factor in $\Gamma^{(n,1)}$ besides the ones that appear in $\Gamma^{(n)}$. In any case we shall denote by $\exp[K]$ the infrared divergent factor of $\Gamma^{(n)}$ and define regulated amplitudes via the relation:

$$\Gamma^{(n)} = \exp[K] \, \Gamma^{(n)}_{\mathrm{reg}}, \quad \Gamma^{(n,1)} = \exp[K] \, \Gamma^{(n,1)}_{\mathrm{reg}} \,. \tag{4.2}$$

K is in general a function of the momenta p_a of the finite energy particles. This makes $\Gamma^{(n)}_{\mathrm{reg}}$ free from infrared divergences, but $\Gamma^{(n,1)}_{\mathrm{reg}}$ still contains some residual infrared divergences for gravitational interaction. Eq. (4.1) is now replaced by[6]

$$\Gamma^{(n,1)}_{\mathrm{reg}}(p_1, \cdots p_n, k) \simeq S(\varepsilon, k; \{p_a\}) \, \Gamma^{(n)}_{\mathrm{reg}}(p_1, \cdots p_n) \,. \tag{4.3}$$

The residual infrared divergences in $\Gamma^{(n,1)}_{\mathrm{reg}}$ will be reflected in the infrared divergent contributions to $S(\varepsilon, k; \{p_a\})$.

There is however a potential ambiguity in the definition of $\Gamma^{(n,1)}_{\mathrm{reg}}$ and hence of $S(\varepsilon, k; \{p_a\})$. This is due to the fact that in the definition of K we can add a term of the form $Q.\sum_a p_a$ for any vector Q (which could be a function of the p_a's) since by the momentum conserving delta function in $\Gamma^{(n)}$, $\sum_a p_a$ vanishes. However addition of such a term changes the definition of $\Gamma^{(n,1)}_{\mathrm{reg}}$ in (4.2) by a multiplicative factor of $\exp[k.Q]$ since the momentum conserving delta function in $\Gamma^{(n,1)}$ gives $k + \sum_a p_a = 0$. This has the effect of multiplying $S(\varepsilon, k; \{p_a\})$ by $\exp[k.Q]$. Expanding $\exp(k.Q)$ as $(1 + k.Q)$ we see that the additional contribution appears at the subleading order, and has the form of $k.Q$ multiplying the leading soft factor. It does not affect the $\ln \omega^{-1}$ terms that we are after since the leading soft factor has no $\ln \omega^{-1}$ term and Q is ω independent. However this can affect the genuine infrared divergent terms proportional to $\ln R$ in the expression for $\Gamma^{(n,1)}_{\mathrm{reg}}$, since in the definition of Q we can include terms proportional to $\ln R$. Choosing $Q = -U \ln R$ for some vector U constructed from the p_a's amounts to having an additive contribution to $S^{(1)}$ of the form

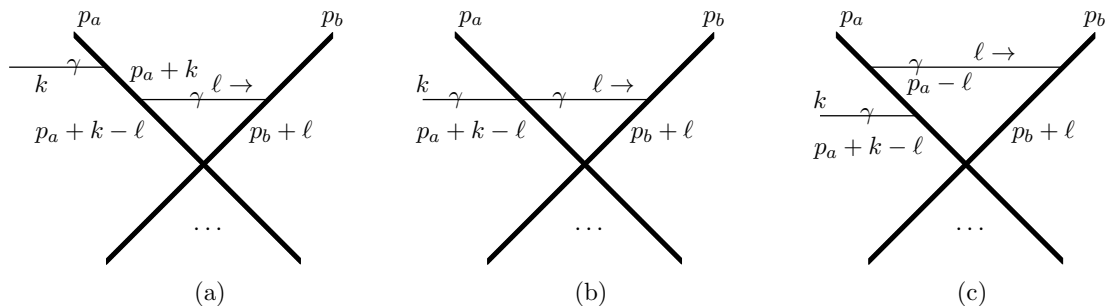$$- \ln R \, k.U \, S^{(0)}(\varepsilon, k; \{p_a\}) \,. \tag{4.4}$$

Figure 1. One loop contribution to $\Gamma^{(n,1)}$ involving internal photon line connecting two different legs. The thick lines represent scalar particles and the thin lines carrying the symbol γ represent photons. There are other diagrams related to this by permutations of the external scalar particles.

5 Soft photon theorem in scalar QED

Consider a theory containing a U(1) gauge field A_μ and n scalars $\phi_1, \cdots \phi_n$ of masses $m_1, \cdots m_n$ and carrying U(1) charges $q_1, \cdots q_n$, satisfying $\sum_{a=1}^{n} q_a = 0$. We further assume that there is a non-derivative contact interaction between the n-scalars. Then the relevant part of the action takes the form

$$\int d^4x \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \sum_{a=1}^{n} \left\{ (\partial_\mu \phi_a^* + iq_a A_\mu \phi_a^*)(\partial^\mu \phi_a - iq_a A^\mu \phi_a) + m_a^2 \phi_a^* \phi_a \right\} \right.$$
$$\left. + \lambda\, \phi_1 \cdots \phi_n + \lambda\, \phi_1^* \cdots \phi_n^* \right]. \tag{5.1}$$

We consider in this theory an amplitude with one external outgoing photon of momentum k and n external states corresponding to the fields $\phi_1, \cdots \phi_n$, carrying momenta $p_1, \cdots p_n$. All momenta are counted as positive if ingoing so that if the a-th particle is outgoing it will have negative p_a^0. Our goal will be to analyze this amplitude at one loop order, involving an internal photon connecting two matter lines. The relevant diagrams have been shown in figures 1 and 2. We denote by $\Gamma^{(n,1)}$ the sum over tree and one loop contribution to this amplitude. $\Gamma^{(n)}$ will denote the amplitude without the external soft photon to one loop order. One loop contribution to $\Gamma^{(n)}$ has been shown in figure 3.

In our analysis we shall ignore graphs with self energy insertions on external legs and assume that we follow on-shell renormalization so that the mass parameters appearing in the tree level propagators are the physical masses. The wave-function renormalization of the external scalars cancel between $\Gamma^{(n)}$ and $\Gamma^{(n,1)}$.

We shall use Feynman gauge and decompose the photon propagator of momentum ℓ, connecting the leg a to the leg b for $b \neq a$, with ℓ flowing from the a-th leg to the b-th leg, as [99]

$$-i \frac{\eta^{\mu\nu}}{\ell^2 - i\epsilon} = -\frac{i}{\ell^2 - i\epsilon} \left\{ K_{(ab)}^{\mu\nu} + G_{(ab)}^{\mu\nu} \right\} \tag{5.2}$$

where,

$$K_{(ab)}^{\mu\nu} = \ell^\mu \ell^\nu \frac{(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)(2p_b.\ell + \ell^2 - i\epsilon)}, \quad G_{(ab)}^{\mu\nu} = \eta^{\mu\nu} - K_{(ab)}^{\mu\nu}. \tag{5.3}$$
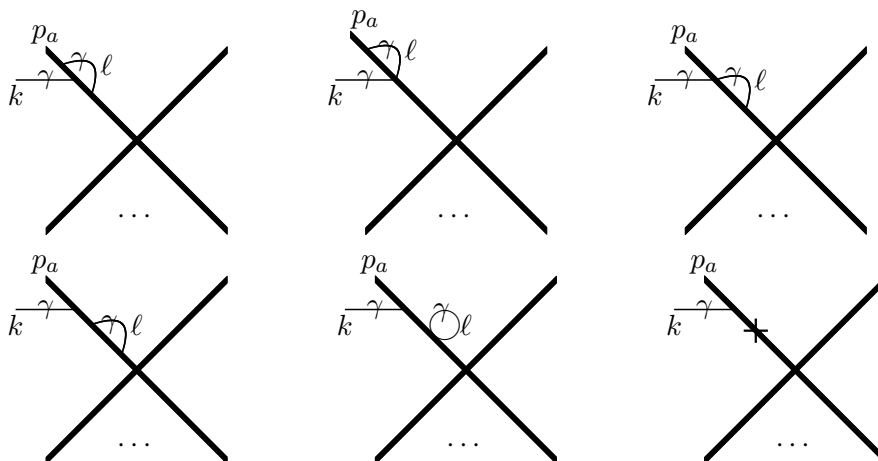
Figure 2. One loop contribution to $\Gamma^{(n,1)}$ involving internal photon line connecting two different points on the same leg. There are other diagrams related to this by permutations of the external scalar particles. In the last term the $+$ on the scalar line represents a counterterm associated with mass renormalization that has to be adjusted to cancel the net contribution proportional to $1/(p_a.k)^2$.
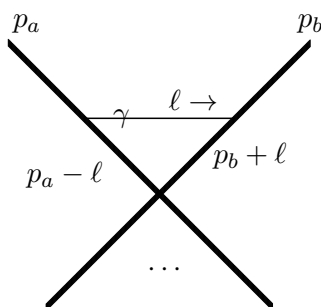

Figure 3. One loop contribution to $\Gamma^{(n)}$. There are other diagrams related to this by permutations of the external scalar particles.

Note that p_a and p_b refer to the external momenta flowing into the legs a and b, and not necessarily the momenta of the lines to which the photon propagator attaches (which may have additional contribution from external soft momentum, e.g. in figures 1(a)). ℓ denotes the momentum flowing from leg a to leg b. For $a = b$ we do not carry out any decomposition.

Since the K-photon polarization is proportional to $\ell^\mu \ell^\nu$, it is pure gauge. This allows us to sum over K-photon insertions using Ward identities

$$\frac{-i}{p_c^2 + m_c^2}\, \ell^\mu\, i\, q_c\, (2p_{c\mu} + \ell_\mu)\, \frac{-i}{(p_c + \ell)^2 + m_c^2} = -q_c \left[\frac{-i}{(p_c + \ell)^2 + m_c^2} - \frac{-i}{p_c^2 + m_c^2} \right], \quad (5.4)$$

and

$$q_c \left[i\, q_c\, \varepsilon.(2p_c + 2\ell + k) - i\, q_c\, \varepsilon.(2p_c + k) \right] - 2\, i\, q_c^2\, \varepsilon.\ell = 0, \quad (5.5)$$

whose diagrammatic representations have been shown in figure 4. Sum over all insertions
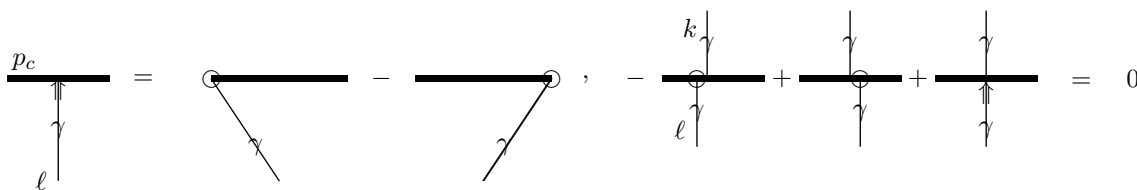
Figure 4. Diagrammatic representations of (5.4) and (5.5). The arrow on the photon line represents that the polarization of the photon is taken to be equal to the momentum entering the vertex. The circle denotes a simple vertex $-q_c$ with the polarization of the incoming photon stripped off.

of the K-photons to either $\Gamma^{(n)}$ or $\Gamma^{(n,1)}$ produces an exponential factor [99]

$$\exp\left[i\sum_{a<b}q_a\,q_b\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2-i\epsilon}\frac{(2p_a-\ell).(2p_b+\ell)}{(2p_a.\ell-\ell^2+i\epsilon)(2p_b.\ell+\ell^2-i\epsilon)}\right].\qquad(5.6)$$

Therefore we may write

$$\Gamma^{(n)}=\exp\left[K_{\text{em}}\right]\left\{\Gamma^{(n)}_{\text{tree}}+\Gamma^{(n)}_{\text{G}}\right\},\qquad\Gamma^{(n,1)}=\exp\left[K_{\text{em}}\right]\left\{\Gamma^{(n,1)}_{\text{tree}}+\Gamma^{(n,1)}_{\text{G}}+\Gamma^{(n,1)}_{\text{self}}\right\},$$

$$K_{\text{em}}\equiv\frac{i}{2}\sum_{\substack{a,b\\b\neq a}}q_a\,q_b\int\frac{d^4\ell}{(2\pi)^4}\,\frac{1}{\ell^2-i\epsilon}\frac{(2p_a-\ell).(2p_b+\ell)}{(2p_a.\ell-\ell^2+i\epsilon)(2p_b.\ell+\ell^2-i\epsilon)},\qquad(5.7)$$

where $\Gamma^{(n)}_{\text{G}}$ and $\Gamma^{(n,1)}_{\text{G}}$ are computed by replacing the internal photons by the G-photons in figures 3 and 1 respectively and $\Gamma^{(n,1)}_{\text{self}}$ denotes the sum of diagrams in figure 2 for which we use the full photon propagator. Therefore a relation of the form $\Gamma^{(n,1)}=S_{\text{em}}\Gamma^{(n)}$ takes the form

$$\Gamma^{(n,1)}_{\text{tree}}+\Gamma^{(n,1)}_{\text{G}}+\Gamma^{(n,1)}_{\text{self}}=S_{\text{em}}\left\{\Gamma^{(n)}_{\text{tree}}+\Gamma^{(n)}_{\text{G}}\right\}.\qquad(5.8)$$

Now it is easy to see that figure 3 vanishes when we replace the internal photon by G-photon. Therefore $\Gamma^{(n)}_{\text{G}}=0$, and we have:[7]

$$\Gamma^{(n)}_{\text{tree}}+\Gamma^{(n)}_{\text{G}}=\Gamma^{(n)}_{\text{tree}}=i\,\lambda.\qquad(5.9)$$

If we write $S_{\text{em}}=S^{(0)}_{\text{em}}+S^{(1)}_{\text{em}}$ where $S^{(0)}_{\text{em}}$ is the leading soft factor $\sum_{a=1}^{n}q_a\,\varepsilon.p_a/k.p_a$ and $S^{(1)}_{\text{em}}$ is the subleading multiplicative factor containing logarithmic terms, then eq. (5.8) can be written as

$$\Gamma^{(n,1)}_{\text{tree}}+\Gamma^{(n,1)}_{\text{G}}+\Gamma^{(n,1)}_{\text{self}}=i\lambda\sum_{a=1}^{n}q_a\frac{\varepsilon.p_a}{k.p_a}+i\lambda\,S^{(1)}_{\text{em}},\qquad(5.10)$$

to one loop order. Now $\Gamma^{(n,1)}_{\text{tree}}$ is equal to the first term on the right hand side up to terms involving Taylor series expansion of the momentum conserving delta function in powers of

---

[7]Note that we are not explicitly writing the momentum conserving delta function, but are implicitly assuming that both sides of (5.8) are multiplied by the appropriate delta functions. We also implicitly assume that the delta function $\delta(\sum_a p_a+k)$ on the left hand side has been expanded in a power series in k.

k, but the latter are subleading contributions without any logarithmic terms and can be ignored in our analysis. Therefore (5.10) can be rewritten as:

$$\Gamma_{\text{self}}^{(n,1)} + \Gamma_{\text{G}}^{(n,1)} = i\lambda\, S_{\text{em}}^{(1)}\,. \tag{5.11}$$

This is a simple algorithm for determination of $S_{\text{em}}^{(1)}$.

Therefore we need to focus on the evaluation of the one loop contribution to $\Gamma_{\text{G}}^{(n,1)}$ and $\Gamma_{\text{self}}^{(n,1)}$ by summing the diagrams in figures 1 and 2, with the internal photon replaced by G-photon in figure 1. We first consider the diagrams in figure 1. It is easy to see that the G-photon contribution to figure 1(c) vanishes. Therefore we need to focus on figures 1(a) and (b). The contribution from figure 1(a) is given by

$$
\begin{aligned}
\mathcal{I}_1 = \lambda\, q_a^2\, q_b\, \frac{\epsilon.p_a}{k.p_a}\, \int \frac{d^4\ell}{(2\pi)^4} &\left[2k.(2p_b + \ell) - \frac{2k.\ell\,(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)} \right] \\
&\times \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{2p_a.(k-\ell) + (k-\ell)^2 - i\epsilon}\, \frac{1}{2p_b.\ell + \ell^2 - i\epsilon}\,,
\end{aligned} \tag{5.12}
$$

and the contribution from figure 1(b) is given by

$$
\begin{aligned}
\mathcal{I}_2 = -\lambda\, q_a^2\, q_b\, \int \frac{d^4\ell}{(2\pi)^4} &\left[2\epsilon.(2p_b + \ell) - \frac{2\epsilon.\ell\,(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)} \right] \\
&\times \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{2p_a.(k-\ell) + (k-\ell)^2 - i\epsilon}\, \frac{1}{2p_b.\ell + \ell^2 - i\epsilon}\,.
\end{aligned} \tag{5.13}
$$

Both $\mathcal{I}_1$ and $\mathcal{I}_2$ are infrared finite since for small ℓ the integrands diverge as $1/\ell^3$. The terms involving logarithm of k come from the region of ℓ integration where the components $|\ell^\mu|$ are large compared to $\omega \equiv k_0$ but small compared to the p_a's. In this range we can approximate $\mathcal{I}_1$ and $\mathcal{I}_2$ as

$$
\begin{aligned}
\mathcal{I}_1 &\simeq -\lambda\, q_a^2\, q_b\, \frac{\epsilon.p_a}{k.p_a}\, \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \left[k.p_b - \frac{k.\ell\, p_a.p_b}{p_a.\ell + i\epsilon} \right] \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon} \\
&= -\lambda\, q_a^2 q_b\, \frac{\epsilon.p_a}{k.p_a}\, \left[k.p_b + p_a.p_b\, k^\mu \frac{\partial}{\partial p_a^\mu} \right] \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}\,,
\end{aligned} \tag{5.14}
$$

and

$$
\begin{aligned}
\mathcal{I}_2 &\simeq \lambda\, q_a\, q_a q_b\, \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \left[\epsilon.p_b - \frac{\epsilon.\ell\, p_a.p_b}{p_a.\ell + i\epsilon} \right] \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon} \\
&= \lambda\, q_a^2 q_b\, \left[\epsilon.p_b\, + p_a.p_b\, \epsilon^\mu \frac{\partial}{\partial p_a^\mu} \right] \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4}\, \frac{1}{\ell^2 - i\epsilon}\, \frac{1}{p_a.\ell + i\epsilon}\, \frac{1}{p_b.\ell - i\epsilon}\,,
\end{aligned} \tag{5.15}
$$

where the subscript reg indicates that the integration needs to be carried out over the region where $|\ell^\mu|$ is large compared to ω but small compared to the energies of the finite energy particles. Adding $\mathcal{I}_1$ and $\mathcal{I}_2$ and summing over a, b we get the total contribution to
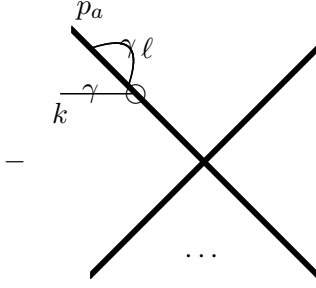
Figure 5. Sum of the first four diagrams in figure 2 with ε replaced by k.

$\Gamma_{\rm G}^{(n,1)}$ to one loop order:

$$
\Gamma_G^{(n,1)} = -\lambda \sum_{\substack{a,b \\ b \neq a}} (q_a)^2 q_b \left[\frac{\epsilon.p_a}{k.p_a} \ k.p_b + \frac{\epsilon.p_a}{k.p_a} \ p_a.p_b \ k^\mu \frac{\partial}{\partial p_a^\mu} - \epsilon.p_b \ - \ p_a.p_b \ \epsilon^\mu \frac{\partial}{\partial p_a^\mu} \right]
$$

$$
\int_{\rm reg} \frac{d^4\ell}{(2\pi)^4} \ \frac{1}{\ell^2 - i\epsilon} \ \frac{1}{p_a.\ell + i\epsilon} \ \frac{1}{p_b.\ell - i\epsilon}
$$

$$
= -\lambda \sum_{\substack{a,b \\ b \neq a}} (q_a)^2 \ q_b \frac{\varepsilon_\mu k_\nu}{p_a.k} \left\{ p_a^\mu \frac{\partial}{\partial p_{a\nu}} - p_a^\nu \frac{\partial}{\partial p_{a\mu}} \right\} \int_{\rm reg} \frac{d^4\ell}{(2\pi)^4} \ \frac{1}{\ell^2 - i\epsilon} \frac{p_a.p_b}{(p_a.\ell + i\epsilon)\,(p_b.\ell - i\epsilon)} \,.
$$

$$(5.16)$$

The contribution to $\Gamma_{\rm self}^{(n,1)}$ from figure 2 can be analyzed using the following indirect method. First of all we note that the net dependence on ε and k from the first four diagrams must be of the form $\varepsilon.p_a \, f(p_a.k)$ for some function f. To determine f, we can set $\varepsilon = k$ and sum over all insertions of the external photon using the Ward identities shown in figure 4. The final result, given in figure 5, has the form:

$$
\frac{C_1}{p_a.k} \,, \tag{5.17}
$$

for some constant C_1. Therefore we get

$$
p_a.k \, f(p_a.k) = \frac{C_1}{p_a.k} \quad \Rightarrow \quad f(p_a.k) = \frac{C_1}{(p_a.k)^2} \,. \tag{5.18}
$$

The fifth and sixth diagrams also have the form

$$
\frac{C_2}{(p_a.k)^2} \quad \text{and} \quad \frac{C_3}{(p_a.k)^2} \,, \tag{5.19}
$$

for appropriate constants C_2 and C_3. Now since we are using on-shell renormalization the counterterm proportional to C_3 is to be adjusted precisely so that the net contribution proportional to $1/(p_a.k)^2$ vanishes. Therefore we must choose $C_3 = -C_1 - C_2$, and the total contribution to $\Gamma_{\rm self}^{(n,1)}$ from all the diagrams in figure 2 vanishes. We have verified this by explicitly computing the Feynman diagrams in figure 2.

From (5.11) we now see that the net contribution to the logarithmic terms in $S_{\text{em}}^{(1)}$ is obtained by dividing $\Gamma_{\text{G}}^{(n,1)}$ given in (5.16) by $i\,\lambda$. This can be written as

$$S_{\text{em}}^{(1)} = \sum_c q_c \frac{\varepsilon_\mu k_\nu}{p_c.k} \left\{ p_c^\mu \frac{\partial}{\partial p_{c\nu}} - p_c^\nu \frac{\partial}{\partial p_{c\mu}} \right\} K_{\text{em}}^{\text{reg}}, \tag{5.20}$$

where $K_{\text{em}}^{\text{reg}}$ is the factor K_{em} defined in (5.7) with the understanding that integration over the loop momentum ℓ will run over the range where $|\ell^\mu|$ is larger than ω but small compared to the momenta of the finite energy external states:

$$K_{\text{em}}^{\text{reg}} \equiv \frac{i}{2} \sum_{\substack{a,b \\ b \neq a}} q_a \, q_b \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{(2p_a - \ell).(2p_b + \ell)}{(2p_a.\ell - \ell^2 + i\epsilon)(2p_b.\ell + \ell^2 - i\epsilon)}. \tag{5.21}$$

So essentially we need to evaluate $K_{\text{em}}^{\text{reg}}$. For this we need to evaluate the integral:[8]

$$\begin{aligned}
\mathcal{I}_{ab} &\equiv \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{\ell^2 - i\epsilon} \frac{1}{p_a.\ell + i\epsilon} \frac{1}{p_b.\ell - i\epsilon} \tag{5.22}\\
&= -\frac{1}{E_a E_b} \int_{\text{reg}} \frac{d^3\ell}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\ell^0}{2\pi} \frac{1}{(\ell^0 - |\vec{\ell}| + i\epsilon)(\ell^0 + |\vec{\ell}| - i\epsilon)} \frac{1}{\ell^0 - \vec{v}_a.\vec{\ell} - i\epsilon} \frac{1}{\ell^0 - \vec{v}_b.\vec{\ell} + i\epsilon},
\end{aligned}$$

where $E_a = p_a^0$, $E_b = p_b^0$, $\vec{v}_a = \vec{p}_a/E_a$ and $\vec{v}_b = \vec{p}_b/E_b$. In writing down the above equation we have assumed that E_a and E_b are positive, i.e. both lines represent incoming states. The integrand has simple poles at,

$$\ell^0 = (|\vec{\ell}| - i\epsilon), \ -(|\vec{\ell}| - i\epsilon), \ (\vec{v}_a.\vec{\ell} + i\epsilon), \ (\vec{v}_b.\vec{\ell} - i\epsilon). \tag{5.23}$$

So now if we close the contour in the lower half plane we have to take the pole contributions from $\ell^0 = (|\vec{\ell}| - i\epsilon)$ and $\ell^0 = (\vec{v}_b.\vec{\ell} - i\epsilon)$. This gives

$$\begin{aligned}
\mathcal{I}_{ab} &= \frac{i}{E_a E_b} \int_{\text{reg}} \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{1}{2|\vec{\ell}|} \frac{1}{|\vec{\ell}| - \vec{v}_a.\vec{\ell}} \frac{1}{|\vec{\ell}| - \vec{v}_b.\vec{\ell}} \\
&\quad + \frac{i}{E_a E_b} \int_{\text{reg}} \frac{d^3\vec{\ell}}{(2\pi)^3} \frac{1}{(\vec{v}_b.\vec{\ell})^2 - |\vec{\ell}|^2} \frac{1}{(\vec{v}_b - \vec{v}_a).\vec{\ell} - i\epsilon}. \tag{5.24}
\end{aligned}$$

Note that we have removed the $i\epsilon$'s from the denominators that never vanish.

Let us first analyze the second term. Since the result should be Lorentz invariant, it should not depend on the chosen frame. For simplicity choose a frame in which $\vec{v}_b$ and $\vec{v}_a$ are along the positive z-axis with $|\vec{v}_b| > |\vec{v}_a|$. Denoting by θ the angle between $\vec{\ell}$ and the z-axis, we can express the second term in (5.24) as

$$\mathcal{I}_{ab}' = \frac{i}{E_a E_b (2\pi)^2} \frac{1}{|\vec{v}_a - \vec{v}_b|} \int_{\text{reg}} \frac{d|\vec{\ell}|}{|\vec{\ell}|} \int_{-1}^{1} d(cos\theta) \frac{1}{|\vec{v}_b|^2 cos^2\theta - 1} \frac{1}{cos\theta - i\epsilon}. \tag{5.25}$$

---

[8]Since the ℓ^μ integration runs over a limited range, one might wonder why we are choosing the ℓ^0 integration range from $-\infty$ to ∞. To this end, note that once we have imposed the range restriction on $|\vec{\ell}|$, we can let the ℓ^0 integral in (5.22) run over the entire real axis since the regions outside the allowed range do not generate any logarithmic contribution.

Without the $i\epsilon$ piece of the last term the integral vanishes since the integrand is an odd function of $\cos\theta$. However the imaginary part of the last term makes the integral non-vanishing. Using $1/(x - i\epsilon) = i\pi\delta(x) + P(1/x)$ in the integral, and using the fact that the value of $|\vec{\ell}|$ for which our approximation of the integrand is valid ranges from ω to some finite energy, we get,

$$\mathcal{I}'_{ab} \simeq \frac{1}{4\pi \, E_a E_b} \, \ln\omega^{-1} \, \frac{1}{|\vec{v}_a - \vec{v}_b|} = \frac{1}{4\pi} \, \ln\omega^{-1} \, \frac{1}{\sqrt{(p_a.p_b)^2 - m_a^2 m_b^2}} \,, \qquad (5.26)$$

where in the intermediate stage we used $|\vec{p}_a||\vec{p}_b| = |\vec{p}_a.\vec{p}_b|$, since $\vec{p}_a$ and $\vec{p}_b$ are parallel.

If both the legs a and b are outgoing instead of ingoing, then E_a and E_b are negative and the signs of the $i\epsilon$ in the last two terms in (5.22) are reversed. But this can be brought back to the form given in (5.22) by making a change of variables $\ell^\mu \to -\ell^\mu$. Therefore the net result for the residue at $\ell^0 = \vec{v}_b.\vec{\ell} - i\epsilon$ will continue to be described by (5.26). Finally if one of the momenta is outgoing and the other is ingoing, then both the $i\epsilon$'s in the last two terms of (5.22) come with the same sign. By changing variables from ℓ^μ to $-\ell^\mu$ if necessary, we can ensure that both the poles are in the upper half plane and close the contour to the lower half plane. In this case there will be no analog of the contribution given in (5.26).

We now turn to the contribution from the first term on the right hand side of (5.24), which we will call $\mathcal{I}''_{ab}$. We will again evaluate this integral in the frame in which $\vec{v}_a$ and $\vec{v}_b$ are parallel to the z-axis with $|\vec{v}_b| > |\vec{v}_a|$. We get

$$
\begin{aligned}
\mathcal{I}''_{ab} &= \frac{i}{E_a E_b} \, \int_{\text{reg}} \frac{d^3\vec{l}}{(2\pi)^3} \, \frac{1}{2|\vec{l}|} \, \frac{1}{|\vec{l}| - \vec{v}_a.\vec{l}} \, \frac{1}{|\vec{l}| - \vec{v}_b.\vec{l}} \\
&= \frac{i}{8\pi^2 E_a E_b} \, \ln\omega^{-1} \int_{-1}^{1} d(\cos\theta) \, \frac{1}{v_b - v_a} \left[\frac{v_b}{1 - v_b \cos\theta} - \frac{v_a}{1 - v_a \cos\theta} \right] \\
&= \frac{i}{8\pi^2} \, \ln\omega^{-1} \, \frac{1}{|\vec{p}_b|E_a - |\vec{p}_a|E_b} \ln\left[\frac{(E_a - |\vec{p}_a|)(E_b + |\vec{p}_b|)}{(E_a + |\vec{p}_a|)(E_b - |\vec{p}_b|)} \right] \\
&= -\frac{i}{8\pi^2} \, \ln\omega^{-1} \, \frac{1}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \ln\left[\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \right] \,.
\end{aligned}
\qquad (5.27)
$$

It is easy to check that the form of the contribution remains unchanged even when both legs are outgoing or one leg is incoming and the other leg is outgoing.

Combining these results we get

$$K_{\text{em}}^{\text{reg}} = \frac{i}{2} \sum_{\substack{a,b \\ b \neq a}} q_a \, q_b \, \frac{1}{4\pi} \, \ln\omega^{-1} \, \frac{p_a.p_b}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \left\{ \delta_{\eta_a \eta_b, 1} - \frac{i}{2\pi} \ln\left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}} \right) \right\}.$$

$$(5.28)$$

Using (5.20) we can now write down the expression for the logarithmic term in the sub-

leading soft factor $S_{\rm em}^{(1)}$

$$
-\frac{i}{4\pi}\,\ln\omega^{-1}\sum_{a=1}^{n}\sum_{\substack{b\neq a\\ \eta_a\eta_b=1}}q_a^2\,q_b\,\frac{\epsilon_\mu k_\rho}{p_a.k}\,\frac{m_a^2 m_b^2\big[p_a^\mu p_b^\rho-p_b^\mu p_a^\rho\big]}{\big[(p_a.p_b)^2-m_a^2 m_b^2\big]^{3/2}}
$$

$$
-\frac{1}{8\pi^2}\,\ln\omega^{-1}\sum_{\substack{a,b\\ b\neq a}}q_a^2\,q_b\,\ln\left[\frac{p_a.p_b+\sqrt{(p_a.p_b)^2-p_a^2 p_b^2}}{p_a.p_b-\sqrt{(p_a.p_b)^2-p_a^2 p_b^2}}\right]\frac{p_a^2 p_b^2}{\{(p_a.p_b)^2-p_a^2 p_b^2\}^{3/2}}\left\{-\epsilon.p_b+\frac{\epsilon.p_a}{k.p_a}k.p_b\right\}
$$

$$
+\frac{1}{4\pi^2}\,\ln\omega^{-1}\sum_{\substack{a,b\\ b\neq a}}q_a^2 q_b\,\frac{p_a.p_b}{(p_a.p_b)^2-p_a^2 p_b^2}\left\{-\epsilon.p_b+\frac{\epsilon.p_a}{k.p_a}k.p_b\right\}. \tag{5.29}
$$

The term in the first line agrees with the classical expression for $S_{\rm em}^{(1)}$ given by the second term of (3.16). The rest of the contribution is extra.

We have also checked that (5.29) holds if instead of scalars we have interacting fermions. This confirms that the logarithmic correction to the soft factor is independent of the spin of the particle.

We end this section by making some observation on the results derived above:

1. Suppose we assume the validity of the naive version of the subleading soft photon theorem:[9]

$$
\Gamma^{(n,1)}=\big\{S_{\rm em}^{(0)}+\widehat{S}_{\rm em}^{(1)}\big\}\,\Gamma^{(n)}\,, \tag{5.30}
$$

where the 'hat' on $S^{(1)}$ denotes that we are using the differential operator form that arises in the quantum theory:

$$
S_{\rm em}^{(0)}=\sum_a q_a\,\frac{\varepsilon.p_a}{p_a.k},\qquad \widehat{S}_{\rm em}^{(1)}=\sum_a q_a\,\frac{\varepsilon_\mu k_\nu}{p_a.k}\left\{p_a^\mu\frac{\partial}{\partial p_{a\nu}}-p_a^\nu\frac{\partial}{\partial p_{a\mu}}\right\}. \tag{5.31}
$$

Then using (5.7) and the fact that $\Gamma_{\rm G}^{(n)}$ vanishes at one loop order, we get

$$
\Gamma_{\rm tree}^{(n,1)}+\Gamma_{\rm self}^{(n,1)}+\Gamma_{\rm G}^{(n,1)}=S_{\rm em}^{(0)}\Gamma_{\rm tree}^{(n)}+\{\widehat{S}_{\rm em}^{(1)}K_{\rm em}\}\,\Gamma_{\rm tree}^{(n)}+\widehat{S}_{\rm em}^{(1)}\Gamma_{\rm tree}^{(n)}\,. \tag{5.32}
$$

Using $\Gamma_{\rm tree}^{(n)}=i\,\lambda$, using (5.10) to replace the left hand side, and throwing away terms like $\widehat{S}_{\rm em}^{(1)}\Gamma_{\rm tree}^{(n)}$ which vanishes, we get

$$
S_{\rm em}^{(1)}=\widehat{S}_{\rm em}^{(1)}\,K_{\rm em}\,. \tag{5.33}
$$

In the definition of $K_{\rm em}$ the integration over loop momentum runs over all range and we have an infrared divergence from the region of small ℓ. However if we make an ad hoc restriction that the loop momentum integral will run in the range much larger than the energy ω of the external soft photon, then $K_{\rm em}$ reduces to $K_{\rm em}^{\rm reg}$ defined in (5.21) and we recover the correct logarithmic terms in $S_{\rm em}^{(1)}$ as given in (5.20). This suggests an ad hoc rule for computing the logarithmic terms in the soft expansion

---

[9]Since the presence of the logarithmic term makes the finite part ambiguous, we consider only the logarithmic terms in the subleading factor.

in quantum theory — begin with the usual soft expansion and explicitly evaluate the action of the differential operator on the amplitude, restricting the region of loop momentum integration to lie in a range larger than the soft momenta but smaller than the momenta of the finite energy particles. With hindsight, this prescription can be justified by noting that the general arguments of [81, 83], that assumes existence of 1PI effective action with no powers of soft momenta coming from the vertices, breaks down for the contribution where the loop momentum is smaller than the external soft momenta. On the other hand we do not expect any large contribution from the region of integration where the loop momentum is of the order of the external momenta or larger.

This argument also suggests that although we have carried out the explicit calculation only at one loop order, the result may be valid to all orders in perturbation theory, since K_{em} is known to be valid to all orders in perturbation theory [99].

2. The second observation concerns the relation between the classical and the quantum results. As already noted, compared to the classical result that agrees with the first line of (5.29), the quantum result found here has an extra term given in the second and third line of (5.29). If however we replace in (5.22) the Feynman propagator for the photon by the retarded propagator, we get only the contribution from the first line of (5.29), since the contribution from the pole at $\ell^2 = 0$ can then be avoided by appropriate choice of contour. Therefore at least for the soft photon theorem in quantum electrodynamics, the rule for relating the quantum and the classical result seems to be to replace the Feynman propagator of the photon in the loop in the quantum result by retarded propagator.

We shall now write down the results for the other cases and test if the generalization of observation 1 works. We shall also explore if the results satisfy the generalization of observation 2.

6 Soft graviton theorem in gravitational scattering

We now turn to the analysis of the soft graviton theorem in the scattering of scalar particles, interacting via gravity, to one loop order. The action is taken to be

$$\int d^4x \sqrt{-\det g} \left[\frac{1}{16\pi G} R - \sum_{a=1}^{n} \left\{ g^{\mu\nu} \partial_\mu \phi_a^* \partial_\nu \phi_a + m_a^2 \phi_a^* \phi_a \right\} + \lambda \phi_1 \cdots \phi_n + \lambda \phi_1^* \cdots \phi_n^* \right]. \tag{6.1}$$

Even though in this case we could take the scalar fields to be real, we have kept them complex in order to extend the analysis to the case where the scalars have both electromagnetic and gravitational interaction. As in section 5, we shall postulate a relation of the form

$$\Gamma^{(n,1)} = \left\{ S_{gr}^{(0)} + S_{gr}^{(1)} \right\} \Gamma^{(n)}, \tag{6.2}$$

and try to determine the logarithmic terms in $S_{gr}^{(1)}$ by comparing the two sides up to one loop order.

We shall carry out our computation in the de Donder gauge in which the propagator of a graviton of momentum ℓ is given by:

$$-\frac{i}{\ell^2 - i\epsilon}\frac{1}{2}\left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right). \tag{6.3}$$

For our analysis we also need the vertices involving the graviton. The scalar-scalar-graviton vertex, with the scalars carrying ingoing momenta p_1, p_2 and the graviton carrying ingoing momentum $-p_1 - p_2$ and Lorentz index $(\mu\nu)$, is given by

$$-i\,\kappa\left[p_{1\mu}p_{2\nu} + p_{1\nu}p_{2\mu} - \eta_{\mu\nu}(p_1.p_2 - m^2)\right], \tag{6.4}$$

where $\kappa = \sqrt{8\pi G} = 1$ in our convention. The vertex involving two scalars carrying ingoing momenta p_1, p_2, and two gravitons carrying ingoing momenta k_1, k_2 and Lorentz indices $(\alpha\beta)$ and $(\mu\nu)$ is given by[10]

$$2\,i\,\kappa^2\left[-\eta_{\alpha\mu}\,\eta_{\beta\nu}\,p_1.p_2 + \frac{1}{2}\eta_{\alpha\beta}\,\eta_{\mu\nu}\,p_1.p_2 - \eta_{\alpha\beta}\,p_{1\mu}p_{2\nu} - \eta_{\mu\nu}\,p_{1\alpha}p_{2\beta}\right.$$
$$\left. + 2\,\eta_{\alpha\mu}\{p_{1\beta}p_{2\nu} + p_{2\beta}p_{1\nu}\} + m^2\left(\eta_{\mu\alpha}\eta_{\nu\beta} - \frac{1}{2}\eta_{\mu\nu}\eta_{\alpha\beta}\right)\right]. \tag{6.5}$$

If we label the ingoing graviton momenta by k_1, k_2 and $k_3 = -k_1 - k_2$ and the Lorentz indices carried by them by $(\mu\alpha)$, $(\nu\beta)$ and $(\sigma\gamma)$ respectively, then the 3-graviton vertex takes the form:

$$i\,\kappa\,\left[\left(k_1.k_2\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma} + k_2.k_1\eta_{\nu\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma} + k_1.k_3\eta_{\mu\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma}\right.\right.$$
$$\left. + k_3.k_1\eta_{\sigma\gamma}\eta_{\mu\nu}\eta_{\alpha\beta} + k_2.k_3\eta_{\nu\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma} + k_3.k_2\eta_{\sigma\gamma}\eta_{\mu\nu}\eta_{\alpha\beta}\right)$$
$$- 2\left(k_{1\sigma}k_{2\gamma}\eta_{\mu\nu}\eta_{\alpha\beta} + k_{2\mu}k_{3\alpha}\eta_{\nu\sigma}\eta_{\beta\gamma} + k_{3\nu}k_{1\beta}\eta_{\mu\sigma}\eta_{\alpha\gamma}\right)$$
$$- 4\left(k_1.k_2 + k_2.k_3 + k_3.k_1\right)\eta_{\alpha\nu}\eta_{\beta\sigma}\eta_{\gamma\mu}$$
$$+ \left(k_1.k_2\eta_{\mu\nu}\eta_{\alpha\beta}\eta_{\sigma\gamma} + k_2.k_3\eta_{\nu\sigma}\eta_{\beta\gamma}\eta_{\mu\alpha} + k_3.k_1\eta_{\mu\sigma}\eta_{\alpha\gamma}\eta_{\nu\beta}\right)$$
$$+ 2\left(k_{1\sigma}k_{2\mu}\eta_{\alpha\nu}\eta_{\beta\gamma} + k_{2\mu}k_{3\nu}\eta_{\sigma\alpha}\eta_{\gamma\beta} + k_{3\nu}k_{1\sigma}\eta_{\mu\beta}\eta_{\alpha\gamma}\right.$$
$$\left. + k_{2\sigma}k_{1\nu}\eta_{\mu\beta}\eta_{\alpha\gamma} + k_{3\mu}k_{2\sigma}\eta_{\nu\gamma}\eta_{\beta\alpha} + k_{1\nu}k_{3\mu}\eta_{\sigma\beta}\eta_{\gamma\alpha}\right)$$
$$\left. - \frac{1}{2}\left(k_1.k_2 + k_2.k_3 + k_3.k_1\right)\eta_{\mu\alpha}\eta_{\nu\beta}\eta_{\sigma\gamma}\right]. \tag{6.6}$$

In (6.5) and (6.6) it is understood that the vertices need to be symmetrized under the exchange of the pair of Lorentz indices carried by each external graviton, e.g. $\mu \leftrightarrow \nu$ and $\alpha \leftrightarrow \beta$ in (6.5) and $\mu \leftrightarrow \alpha$, $\nu \leftrightarrow \beta$ and $\sigma \leftrightarrow \gamma$ in (6.6). Even though (6.6) has a complicated form, we shall need the form of the vertex when one of the external momenta (say k_3) is small compared to the others. In this limit it simplifies.

The vertex where a graviton carrying Lorentz index $(\mu\nu)$ attaches to n scalar fields is given by:

$$i\kappa\lambda\,\eta_{\mu\nu}. \tag{6.7}$$

---

[10]In writing this and other vertices we already include the symmetry factor related to exchange of identical particles. Therefore if we were to use this vertex to compute tree level two graviton, two scalar amplitude, no further symmetry factor is necessary.
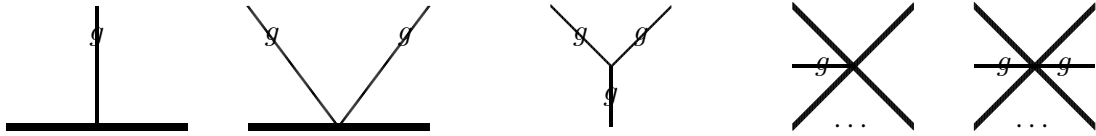
Figure 6. This diagram shows various vertices induced from the action (6.1) that are needed for our computation. Here the thinner lines carrying the symbol g denote gravitons and the thicker lines denote scalars.
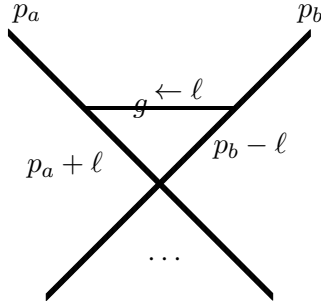

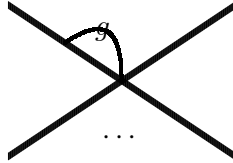**Figure 7**. Diagram contributing to $\Gamma^{(n)}$.

Figure 8. Another diagram contributing to $\Gamma^{(n)}$. We can also have a diagram where both ends of the internal graviton are attached to the n-scalar vertex, but this vanishes in dimensional regularization and so we have not displayed them.

The vertex where two gravitons carrying Lorentz index $(\mu\nu)$ and $(\rho\sigma)$ attach to n scalar fields is given by:

$$- i\,\kappa^2\lambda \left(\eta_{\mu\nu}\eta_{\rho\sigma} - \eta_{\mu\rho}\eta_{\nu\sigma} - \eta_{\mu\sigma}\eta_{\nu\rho}\right) . \tag{6.8}$$

We also need the vertex containing two scalars and three gravitons for evaluating the fifth diagram of figure 10. However even without knowing the form of this vertex one can see that this diagram does not generate contributions proportional to $\ln\omega^{-1}$. Therefore we have not written down the expression for this vertex.

We can use these vertices to compute one loop contribution to the n scalar amplitude $\Gamma^{(n)}$ and n-scalar and one soft graviton amplitude $\Gamma^{(n,1)}$. At one loop order $\Gamma^{(n)}$ receives contribution from diagrams shown in figure 7 that are analogous to figure 3 with the internal photon replaced by a graviton. There are also some additional diagrams shown in figure 8.

The relevant diagrams for $\Gamma^{(n,1)}$ include the analogs figures 1 and 2 with all photons replaced by gravitons. This have been shown in figures 9 and 10. However there are also some extra diagrams that we shall list below:
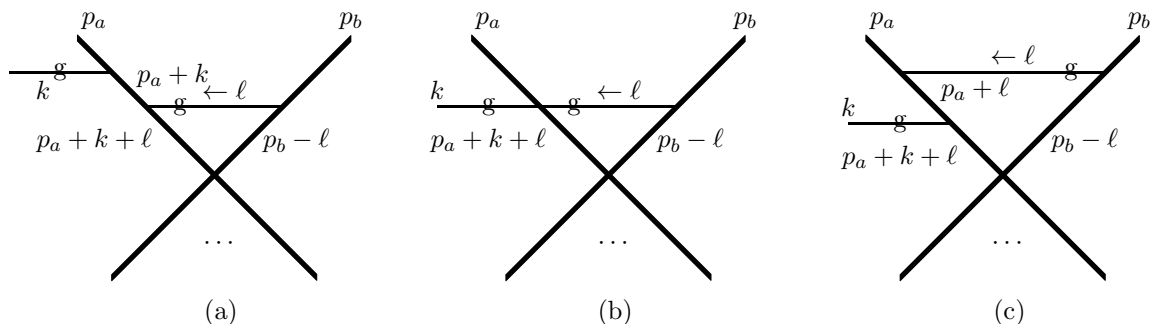
Figure 9. One loop contribution to $\Gamma^{(n,1)}$ involving internal graviton line connecting two different legs. The thicker lines represent scalar particles and the thinner lines represent gravitons.
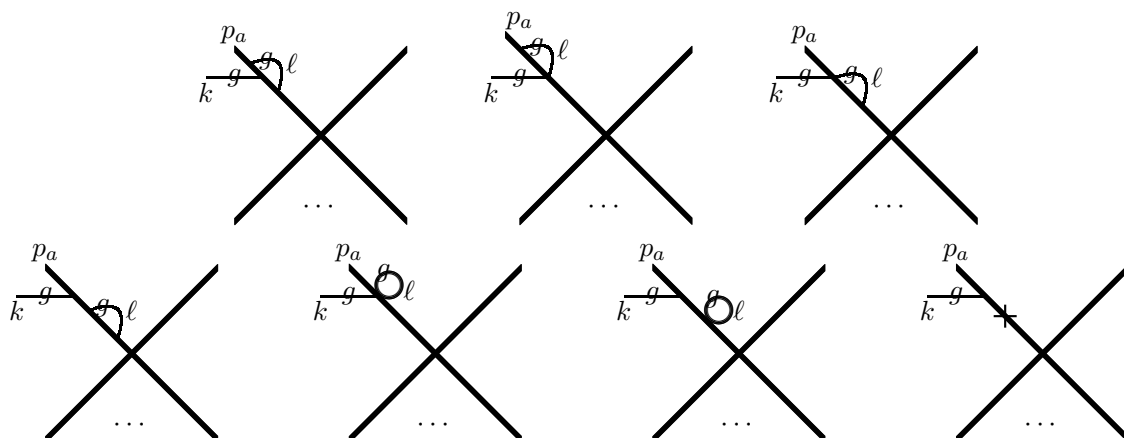

Figure 10. One loop contribution to $\Gamma^{(n,1)}$ involving internal graviton line connecting two different points on the same leg.

1. There are diagrams where the external graviton couples to the internal graviton via the cubic coupling (6.6). Examples of these are shown in figure 11.

2. There are diagrams where one end of the internal graviton attaches to the n-scalar vertex via the coupling (6.7). These have been shown in figures 12.

3. There are diagrams where the external graviton attaches to the scalar n-point vertex via the coupling (6.7) or (6.8). These have been shown in figure 13. The first diagram can be made to vanish by taking the external graviton polarization to be traceless: $\varepsilon_\rho{}^\rho = 0$. The second diagram has no logarithmic terms. Therefore we shall ignore these diagrams in subsequent discussions.

4. There are diagrams of the type shown in figure 14 where two ends of the internal graviton attach to the n-scalar vertex. In dimensional regularization these diagrams vanish. Therefore we shall ignore these diagrams in our analysis.

Our analysis of these diagrams will proceed as in section 5, but there will be some important differences that we shall point out below. For an internal graviton of momentum
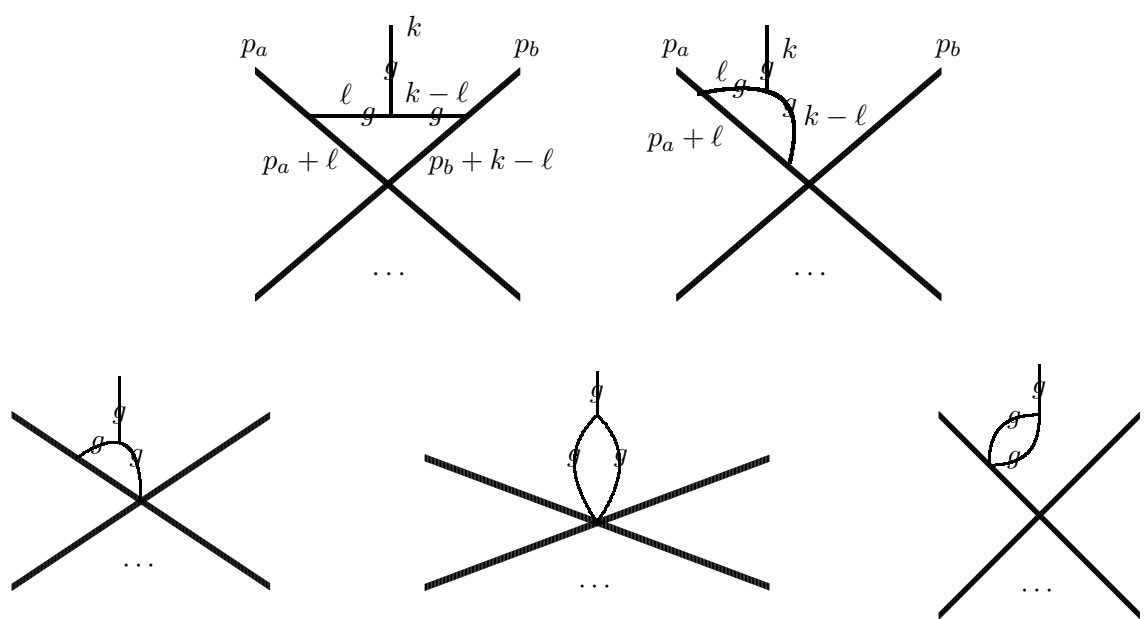
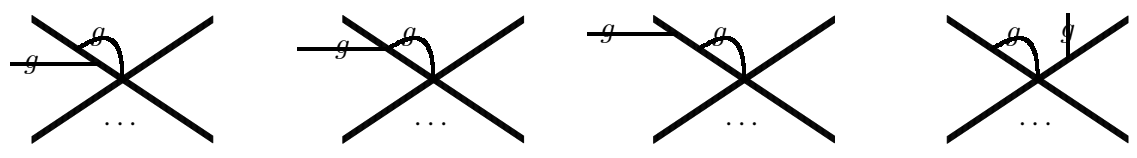**Figure 11**. Diagrams involving 3-graviton vertex.

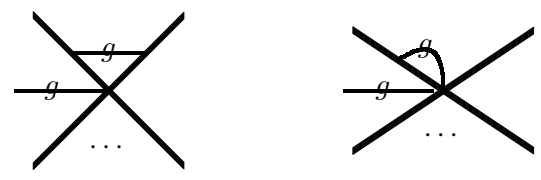**Figure 12**. Diagrams where the internal graviton attaches to the n-point vertex.

Figure 13. Diagrams where the external graviton attaches to the n-point vertex. The first diagram vanishes if we take the external graviton polarization to be traceless. The second diagram has no logarithmic terms.

Figure 14. Diagrams where both ends of the internal graviton attach to the n-point vertex. In dimensional regularization these diagrams vanish. Even if we use momentum cut-off, these diagrams cannot have any contribution proportional to $\ln \omega^{-1}$ since the soft momentum k does not flow through any loop.

ℓ, whose two ends are attached to two scalar lines a and b with ℓ flowing from the leg b towards the leg a, as in figures 7, 9, the analog of Grammer-Yennie decomposition of the graviton propagator will be taken to be

$$G_{(ab)}^{\mu\nu,\rho\sigma}(\ell,p_a,p_b) = \left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) - K_{(ab)}^{\mu\nu,\rho\sigma}(\ell,p_a,p_b)\,, \tag{6.9}$$

$$K_{(ab)}^{\mu\nu,\rho\sigma}(\ell,p_a,p_b) = \mathcal{C}(\ell,p_a,p_b) \left[(p_a+\ell)^\mu\ell^\nu + (p_a+\ell)^\nu\ell^\mu\right] \left[(p_b-\ell)^\rho\ell^\sigma + (p_b-\ell)^\sigma\ell^\rho\right]\,,$$

where

$$\mathcal{C}(\ell,p_a,p_b) = \frac{(-1)}{\{p_a.(p_a+\ell)-i\epsilon\}\ \{p_b.(p_b-\ell)-i\epsilon\}\{\ \ell.(\ell+2p_a)-i\epsilon\}\ \{\ell.(\ell-2p_b)-i\epsilon\}}$$
$$\left[2(p_a.p_b)^2 - p_a^2 p_b^2 - \ell^2(p_a.p_b) - 2(p_a.p_b)(p_a.\ell) + 2(p_a.p_b)(p_b.\ell)\right]\,. \tag{6.10}$$

If one end of an internal graviton is attached to the n-scalar vertex and the other end is attached to the a'th scalar leg as in figures 8, 12, with ℓ flowing from the vertex towards the a'th leg, we express the propagator as:

$$-\ \frac{i}{\ell^2-i\epsilon}\ \frac{1}{2}\ \left\{G_{(a)}^{\mu\nu,\rho\sigma}(\ell,p_a) + K_{(a)}^{\mu\nu,\rho\sigma}(\ell,p_a)\right\}\,, \tag{6.11}$$

where

$$G_{(a)}^{\mu\nu,\rho\sigma}(\ell,p_a) = \left(\eta^{\mu\rho}\eta^{\nu\sigma} + \eta^{\mu\sigma}\eta^{\nu\rho} - \eta^{\mu\nu}\eta^{\rho\sigma}\right) - K_{(a)}^{\mu\nu,\rho\sigma}(\ell,p_a)\,, \tag{6.12}$$

$$K_{(a)}^{\mu\nu,\rho\sigma}(\ell,p_a) = \widetilde{\mathcal{C}}(\ell,p_a) \left[(p_a+\ell)^\mu\ell^\nu + (p_a+\ell)^\nu\ell^\mu\right] \eta^{\rho\sigma}\,, \tag{6.13}$$

and

$$\widetilde{\mathcal{C}}(\ell,p_a) \ = \ -\frac{2\,p_a.(p_a-\ell)}{\{p_a.(p_a+\ell)-i\epsilon\}\ \{\ \ell.(\ell+2p_a)-i\epsilon\}}\,. \tag{6.14}$$

For internal gravitons whose one end is attached to a 3-graviton vertex instead of a scalar, as in figure 11, we do not carry out any Grammer-Yennie decomposition.

The decomposition into G and K-gravitons is not arbitrary but has been chosen to ensure two properties:

1. The K-graviton polarization, being proportional to ℓ, is pure gauge and allows us to sum over K-graviton insertions using Ward identities. The relevant Ward identities have been shown in figure 15, with the quantity $A(p,k,\ell,\xi,\zeta)$ is given by

$$A(p,k,\ell,\xi,\zeta) = 2\,i\,\xi.p\,\zeta^{\mu\nu}\left[2\,(2p_\mu+\ell_\mu)\,k_\nu + 2\,k_\mu k_\nu - \eta_{\mu\nu}\left\{k.(2p+\ell)+k^2\right\}\right]$$
$$+ 2\,i\,\xi.(k+\ell)\,\zeta^{\mu\nu}\left[-2\,p_\mu\,(p+\ell)_\nu + \eta_{\mu\nu}\left\{p.(p+\ell)+m^2\right\}\right]$$
$$+ 2\,i\,(\xi^\alpha k^\beta + \xi^\beta k^\alpha)\,\zeta^{\mu\nu}\left[\eta_{\alpha\mu}\,\eta_{\beta\nu}\,p.(p+k+\ell) - \frac{1}{2}\,\eta_{\alpha\beta}\,\eta_{\mu\nu}\,p.(p+k+\ell)\right.$$
$$+ \eta_{\alpha\beta}\,p_\mu\,(p+k+\ell)_\nu + \eta_{\mu\nu}\,p_\alpha\,(p+k+\ell)_\beta - 2\,\eta_{\alpha\mu}\,p_\beta(p+k+\ell)_\nu$$
$$\left. - 2\,\eta_{\alpha\mu}\,p_\nu(p+k+\ell)_\beta + m^2\left(\eta_{\mu\alpha}\,\eta_{\nu\beta} - \frac{1}{2}\,\eta_{\mu\nu}\,\eta_{\alpha\beta}\right)\right]\,. \tag{6.15}$$

Due to this additional term, the sum over K-gravitons will leave behind some residual terms that will be discussed below.
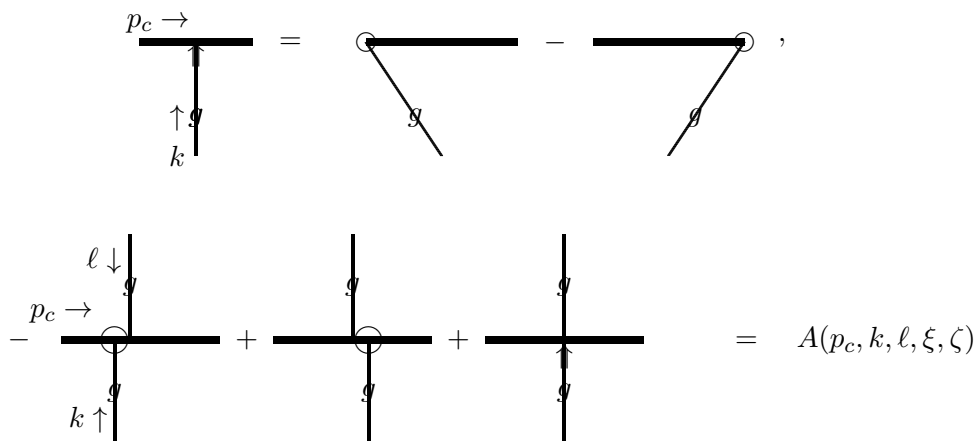
Figure 15. Analog of figure 4 for gravity. The arrow on the graviton line represents that the polarization of the graviton carrying momentum k is taken to be equal to $\xi_\mu k_\nu + \xi_\nu k_\mu$. The polarization of the graviton carrying momentum k is taken to be $\zeta_{\rho\sigma}$. In the first diagram the circle on the left denotes a vertex $-2\xi.(p_c + k)$ while the circle on the right denotes a vertex $-2\xi.p_c$. $A(p_c, k, \ell, \xi, \zeta)$ appearing on the right hand side of the second diagram is given in eq. (6.15).

2. In any one loop diagram contributing to the amplitude $\Gamma^{(n)}$ without external soft graviton, the result vanishes if we replace the internal graviton by G-graviton.

With this convention the K-graviton contribution to figure 7 for gravity can be computed as in section 5, leading to a contribution of the form $i\lambda K_{\mathrm{gr}}$ to $\Gamma^{(n)}$, where K_{gr} is the gravitational counterpart of K_{em}. The relevant part of the expression for K_{gr} will be described later. The K-graviton contribution to figure 8 can be carried out similarly, leading to an expression of the form $i\lambda \widetilde{K}_{\mathrm{gr}}$. $\widetilde{K}_{\mathrm{gr}}$ has no infrared divergence and we shall not write down its expression explicitly although it is straightforward to do so. The G-graviton contributions to figures 7 and 8 vanish by construction. Therefore the net contribution to $\Gamma^{(n)}$ to one loop order may be written as $i\lambda \exp[K_{\mathrm{gr}} + \widetilde{K}_{\mathrm{gr}}]$.

The K-graviton contributions to figures 9 and 12 may be evaluated similarly, with the factorized term giving $i\lambda S_{\mathrm{gr}}^{(0)} \exp[K_{\mathrm{gr}} + \widetilde{K}_{\mathrm{gr}}]$. There are however some left-over terms arising as follows:

1. As shown in figure 15, in the sum over K-graviton insertions in $\Gamma^{(n,1)}$ there is a residual contribution A that comes from lack of complete cancellation among terms where a K-graviton is inserted to the two sides of a scalar-scalar-graviton vertex and into the scalar-scalar-graviton vertex.

2. As explained in the caption of figure 15, the circled vertices are momentum dependent. Therefore the two circled vertices shown in figure 16 are not the same, one carries a factor of $\xi.p_a$ while the other carries a factor of $\xi.(p_a + k)$. The left hand figure is relevant for $\Gamma^{(n)}$ while the right-hand figure is relevant for $\Gamma^{(n,1)}$. Therefore, even after factoring out $\exp[K_{\mathrm{gr}} + \widetilde{K}_{\mathrm{em}}]$ factor multiplying $\Gamma^{(n)}$, we are left with an additional contribution to $\Gamma^{(n,1)}$ from sum over K-gravitons that must be accounted for.
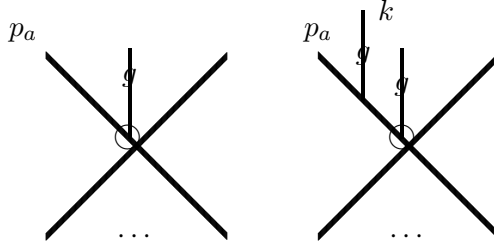
Figure 16. Figure illustrating the difference in the factorized K-graviton contribution to $\Gamma^{(n)}$ and $\Gamma^{(n+1)}$.

We shall denote the sum of these two types of residual contributions as $\Gamma^{(n,1)}_{\text{residual}}$. The G-graviton contributions to figures 9 and 12 will be denoted by $\Gamma^{(n,1)}_G$ and the net contribution from figure 10 will be called $\Gamma^{(n,1)}_{\text{self}}$. Finally the contribution to the diagrams in figure 11 involving 3-graviton coupling will be denoted by $\Gamma^{(n,1)}_{3-\text{graviton}}$. In principle we should also include the contributions from figure 13 and figure 14, but we ignore them since they do not generate logarithmic terms. In this case the analog of (5.11) takes the form:

$$\Gamma^{(n,1)}_{\text{self}} + \Gamma^{(n,1)}_{\text{G}} + \Gamma^{(n,1)}_{3-\text{graviton}} + \Gamma^{(n,1)}_{\text{residual}} = i\lambda\, S^{(1)}_{\text{gr}}\,. \tag{6.16}$$

We shall now briefly describe how we evaluate these contributions and then give the final result. First let us consider $\Gamma^{(n,1)}_{\text{residual}}$. This receives contribution from figure 9 and figure 12. As explained above, there are two kinds of terms: one due to the right hand side of the second figure of figure 15, and the other due to the momentum dependence of the circled vertices in figure 15. It turns out that the residual part of the K-graviton contribution from figure 12 does not have any logarithmic term. On the other hand the residual part of the K-graviton contribution from figure 9 receives logarithmic contribution only from the region where the loop momentum is large compared to ω. The result takes the form:

$$\Gamma^{(n,1)}_{\text{residual}} = -(i\lambda)\,\frac{i}{2}\,\sum_{a=1}^{n}\sum_{\substack{b=1 \\ b\neq a}}^{n}\big[2(p_a.p_b)^2 - p_a^2 p_b^2\big]\,\frac{p_a.\varepsilon.p_a}{p_a^2}\int_{\text{reg}}\frac{d^4l}{(2\pi)^4}\,\frac{1}{[p_a.l-i\epsilon]\,[p_b.l+i\epsilon]\,[l^2-i\epsilon]}\,. \tag{6.17}$$

This contribution may be evaluated following a procedure similar to the one used in section 5.

Contribution to $\Gamma^{(n,1)}_{3-\text{graviton}}$ arises from the five diagrams in figure 11, but only the first two give terms proportional to $\ln\omega^{-1}$. Individually these diagrams suffer from collinear divergence from region of integration where the momenta of the internal gravitons become parallel to that of the external graviton, but these divergences cancel in the sum over such graphs after using momentum conservation. Therefore we always work with sum of these diagrams. The net contribution from these diagrams receive logarithmic contribution from two regions — one where the loop momentum is large compared to ω and the other where the loop momentum is small compared to ω. We shall analyze the contribution from the region of small loop momentum later. Contribution from the region where the loop

momentum is large compared to ω may be approximated as

$$
- (i\lambda) \frac{i}{4} \sum_{a=1}^{n} \sum_{\substack{b=1 \\ b \neq a}}^{n} \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[p_a.\ell - i\epsilon] \ [p_b.\ell + i\epsilon] \ [\ell^2 - i\epsilon]}
$$

$$
\left[-8 \, (p_a.\varepsilon.p_b) \, (p_a.p_b) + 2 \, (p_a.\varepsilon.p_a) \, p_b^2 + 2 \, (p_b.\varepsilon.p_b) \, p_a^2 - 2 \left\{ 2(p_a.p_b)^2 - p_a^2 p_b^2 \right\} \frac{\ell.\varepsilon.\ell}{\ell^2 - i\epsilon} \right]
$$

$$
- (i\lambda) \frac{i}{2} \sum_{a=1}^{n} \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[p_a.\ell - i\epsilon]^2} \frac{1}{[\ell^2 - i\epsilon]} \left[-2 \, p_a^2 \, (p_a.\varepsilon.p_a) + \frac{(p_a^2)^2}{p_a.\ell - i\epsilon} \, (p_a.\varepsilon.\ell) \right].
$$
(6.18)

In arriving at this result we have used integration by parts and also conservation of total momentum $\sum_{a=1}^{n} p_a = 0$. We have also used the fact that in the expression for the graviton propagator carrying momentum $(k - \ell)$ in the second diagram of figure 11, we can use the identity

$$
\frac{1}{(k-\ell)^2 - i\epsilon} = \frac{2\ell.k}{\{(k-\ell)^2 - i\epsilon\}\{\ell^2 - i\epsilon\}} + \frac{1}{\ell^2 - i\epsilon} \,,
$$
(6.19)

and ignore the contribution from the $(\ell^2 - i\epsilon)^{-1}$ term, since the expression for the amplitude involving this term has no k-dependent denominator and therefore cannot have a $\ln \omega^{-1}$ term.[11] Similar manipulations will be used in other terms as well.

Contribution to $\Gamma_{\text{self}}^{(n,1)}$ given in figure 10 may be analyzed following the argument given below (5.16). We assume a general form $\varepsilon_{\mu\nu} p_a^\mu p_a^\nu f(p_a.k)$ for this amplitude based on Lorentz invariance and replace $\varepsilon_{\mu\nu}$ by $\xi_\mu k_\nu + \xi_\nu k_\mu$ for an arbitrary vector ξ satisfying $k.\xi = 0$. Then the amplitude reduces to $2 \, p_a.\xi \, p_a.k \, f(p_a.k)$. On the other hand the diagrams in figure 10 for this choice of polarization may be evaluated using the Ward identity given in figure 15. Due to the presence of the non-vanishing right-hand side in figure 15, the result does not vanish. Comparing this with the expected result $2 \, p_a.\xi \, p_a.k \, f(p_a.k)$, we can compute $f(p_a.k)$ and hence $\Gamma_{\text{self}}^{(n,1)}$. It turns out that it receives logarithmic contribution from region of integration where the loop momentum is large compared to ω. The result is:

$$
\Gamma_{\text{self}}^{(n,1)} = -(i\lambda) \frac{i}{2} \sum_{a=1}^{n} p_a^2 \, p_a.\varepsilon.p_a \int_{\text{reg}} \frac{d^4\ell}{(2\pi)^4} \frac{1}{[p_a.\ell - i\epsilon]^2 \, [\ell^2 - i\epsilon]} \,.
$$
(6.20)

This cancels the term in the last line of (6.18).

One loop contribution from the diagrams involving G-gravitons in figures 9 and 12 may be evaluated following the procedure described in section 5. We find that the G-graviton contribution to figure 12 has no logarithmic contribution. Therefore we are left with the G-graviton contributions to figure 9. These diagrams have the same structure as in scalar QED and can be evaluated similarly. As in the case of scalar QED, these diagrams receive significant contribution only from the region where the loop momentum is large compared

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[11] This manipulation can be carried out only for terms containing at least two powers of ℓ in the numerator so that each of the terms in (6.19) generates infrared finite integral.

to ω and small compared to the momenta of finite energy particles. The net logarithmic contributions from these diagrams is given by

$$\Gamma_G^{(n,1)} = -(i\lambda)\frac{i}{2}\sum_{a=1}^{n}\sum_{\substack{b=1\\b\neq a}}^{n}\int\frac{d^4l}{(2\pi)^4}\frac{1}{[p_a.l-i\epsilon]\;[p_b.l+i\epsilon]\;[l^2-i\epsilon]}$$

$$\left[8(p_a.p_b)\;(p_a.\varepsilon.p_b) - 2p_b^2\;(p_a.\varepsilon.p_a) - \left[2(p_a.p_b)^2 - p_a^2 p_b^2\right]\left(\frac{p_a.\varepsilon.p_a}{p_a^2} + 2\;\frac{p_a.\varepsilon.l}{p_a.l}\right)\right]$$

$$+ (i\lambda)\frac{i}{2}\left(\frac{p_a.\varepsilon.p_a}{p_a.k}\right)\int\frac{d^4l}{(2\pi)^4}\frac{1}{[p_a.l-i\epsilon]\;[p_b.l+i\epsilon]\;[l^2-i\epsilon]}$$

$$\left[4(p_a.p_b)\;(p_b.k) - \left[2(p_a.p_b)^2 - p_a^2 p_b^2\right]\frac{k.l}{p_a.l}\right]. \tag{6.21}$$

The total logarithmic terms in $\Gamma_{\mathrm{G}}^{(n,1)}$, $\Gamma_{\mathrm{self}}^{(n,1)}$, $\Gamma_{\mathrm{residual}}^{(n,1)}$ and $\Gamma_{3-\mathrm{graviton}}^{(n,1)}$ from the region of integration where the loop momentum is large compared to ω, can be expressed as[12]

$$(i\lambda)\;\widehat{S}_{\mathrm{gr}}^{(1)}\;K_{\mathrm{gr}}^{\mathrm{reg}}, \tag{6.22}$$

where $\widehat{S}_{\mathrm{gr}}^{(1)}$ is the quantum subleading soft graviton operator

$$\widehat{S}_{\mathrm{gr}}^{(1)} = \sum_a \frac{\varepsilon_{\mu\rho}p_a^\rho k_\nu}{p_a.k}\left\{p_a^\mu\frac{\partial}{\partial p_{a\nu}} - p_a^\nu\frac{\partial}{\partial p_{a\mu}}\right\}, \tag{6.23}$$

and

$$K_{\mathrm{gr}}^{\mathrm{reg}} \equiv \frac{i}{2}\sum_{\substack{a,b\\b\neq a}}\left\{(p_a.p_b)^2 - \frac{1}{2}p_a^2 p_b^2\right\}\int_{\mathrm{reg}}\frac{d^4\ell}{(2\pi)^4}\frac{1}{\ell^2-i\epsilon}\frac{1}{(p_a.\ell-i\epsilon)(p_b.\ell+i\epsilon)}. \tag{6.24}$$

$K_{\mathrm{gr}}^{\mathrm{reg}}$ is the analog of $K_{\mathrm{em}}^{\mathrm{reg}}$ for gravitational scattering, namely it is the factor that appears in the exponent of the soft factor in the scattering of n scalars, with the understanding that the integration over loop momentum is restricted to the region larger than ω. We note however that the full expression for K_{gr} has more terms — (6.24) already involves an approximation that the loop momentum is small compared to the energies of external lines since this is the region that generates $\ln\omega^{-1}$ terms. Explicit evaluation gives the following expression for the terms involving $\ln\omega^{-1}$:

$$K_{\mathrm{gr}}^{\mathrm{reg}} = \frac{i}{2}\sum_{\substack{a,b\\b\neq a}}\frac{1}{4\pi}\;\ln\omega^{-1}\;\frac{\left\{(p_a.p_b)^2 - \frac{1}{2}p_a^2 p_b^2\right\}}{\sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}\left\{\delta_{\eta_a\eta_b,1} - \frac{i}{2\pi}\ln\left(\frac{p_a.p_b + \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}{p_a.p_b - \sqrt{(p_a.p_b)^2 - p_a^2 p_b^2}}\right)\right\}. \tag{6.25}$$

At this stage the only remaining terms are the contributions to $\Gamma_{3-\mathrm{graviton}}^{(n,1)}$ from regions of loop momentum integration where the loop momentum is small compared to ω. These

---

[12]It is natural to conjecture that this pattern continues to hold also for subsubleading soft graviton theorem, i.e. the universal part of the subsubleading contribution is given by the action of the subsubleading soft graviton operator $\widehat{S}_{\mathrm{gr}}^{(2)}$ acting on $\exp[K_{\mathrm{gr}}^{\mathrm{reg}}]$. But we have not verified this by explicit computation.

come from the first two diagrams in figure 11. In the first diagram there are two relevant regions: when ℓ is small and when $k - \ell$ is small, but they are related to each other by $\ell \to k - \ell$ and $a \leftrightarrow b$ symmetry. In the second diagram the relevant region is when ℓ is small. The net contribution from these regions may be approximated by

$$\lambda \sum_{a=1}^{n} \sum_{b=1}^{n} \int \frac{d^4\ell}{(2\pi)^4} \frac{1}{[2k.\ell + \ell^2 + i\epsilon][p_a.\ell - i\epsilon][\ell^2 + i\epsilon]} \left[2(p_a.\varepsilon.p_b)\,(p_a.k) - 2(p_b.\varepsilon.p_b)\frac{(p_a.k)^2}{p_b.k} \right],$$
(6.26)

with the understanding that the integration over ℓ runs in the region where the components of ℓ are small compared to ω. The result may be expressed as

$$i\lambda\,(\ln\omega^{-1} + \ln R^{-1}) \left[\frac{i}{4\pi} \sum_{\substack{b \\ \eta_b = -1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} - \frac{1}{8\pi^2} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \sum_b p_b.k \ln \frac{m_b^2}{(p_b.\hat{k})^2} \right],$$
(6.27)

where $1/R$ is an infrared lower cut-off on momentum integration and $\hat{k} = -k/\omega = (1, \hat{n})$.

Adding (6.22) to (6.27) and dividing by $i\lambda$ we get the terms involving $\ln\omega^{-1}$ and $\ln R$ in $S_{\mathrm{gr}}^{(1)}$:

$$S_{\mathrm{gr}}^{(1)} = \widehat{S}_{gr}^{(1)}\, K_{\mathrm{gr}}^{\mathrm{reg}}$$

$$+ \frac{1}{4\pi}(\ln\omega^{-1} + \ln R^{-1}) \left[i \sum_{\substack{b \\ \eta_b = -1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} - \frac{1}{2\pi} \sum_a \frac{\varepsilon_{\mu\nu} p_a^\mu p_a^\nu}{p_a.k} \sum_b p_b.k \ln \frac{m_b^2}{(p_b.\hat{k})^2} \right].$$
(6.28)

7 Generalizations

In this section we shall consider the case where the scalars interact via both electromagnetic and gravitational interaction via the action:

$$\int d^4x \sqrt{-\det g} \left[-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{16\pi G} R - \sum_{a=1}^{n} \left\{ g^{\mu\nu}(\partial_\mu \phi_a^* + iq_a A_\mu \phi_a^*)(\partial_\nu \phi_a - iq_a A_\nu \phi_a) \right. \right.$$

$$\left. \left. + m_a^2 \phi_a^* \phi_a \right\} + \lambda\,\phi_1 \cdots \phi_n + \lambda\,\phi_1^* \cdots \phi_n^* \right].$$
(7.1)

For this analysis we need two new vertices, the graviton-photon-photon vertex and the graviton-photon-scalar-scalar vertex. If the graviton carries an ingoing momentum q and Lorentz index $(\rho\sigma)$, and the two photons carry ingoing momenta k_1 and k_2 and Lorentz indices μ and ν respectively, then the graviton-photon-photon vertex is given by:

$$- i\,\kappa \left[\eta_{\rho\sigma} \left(-k_1.k_2\,\eta_{\mu\nu} + k_{1\nu}k_{2\mu} \right) + \eta_{\mu\nu} \left(k_{1\rho}k_{2\sigma} + k_{2\rho}k_{1\sigma} \right) + k_1.k_2 \left(\eta_{\mu\rho}\eta_{\nu\sigma} + \eta_{\mu\sigma}\eta_{\nu\rho} \right) \right.$$

$$\left. - \left(k_{1\sigma}k_{2\mu}\eta_{\rho\nu} + k_{2\sigma}k_{1\nu}\eta_{\rho\mu} + k_{1\rho}k_{2\mu}\eta_{\sigma\nu} + k_{2\rho}k_{1\nu}\eta_{\sigma\mu} \right) \right].$$
(7.2)

On the other hand the vertex with a pair of scalars carrying charges q, $-q$ and momenta p_1 and p_2, a graviton carrying Lorentz indices $(\mu\nu)$ and momentum k_1 and a photon carrying Lorentz index ρ and momentum k_2, all counted ingoing, is given by

$$- i\,\kappa\,q \left[\eta_{\mu\rho}(p_1 - p_2)_\nu + \eta_{\nu\rho}(p_1 - p_2)_\mu - \eta_{\mu\nu}(p_1 - p_2)_\rho \right]. \tag{7.3}$$

In this theory we shall analyze the extra terms in both the soft graviton theorem and the soft photon theorem.

There are two other vertices that are needed for our analysis. For example the sixth diagram of figure 18 needs the vertex containing two scalars, two photons and one graviton, whereas the sixth diagram of figure 23 requires the two scalar, two graviton and one photon vertex. However even without knowing the form of these vertices one can see that these diagrams do not generate contributions proportional to $\ln\omega^{-1}$. Therefore we have not written down the expressions for these vertices.

7.1 Soft graviton theorem

We first consider the soft graviton theorem. In this case besides the contributions analyzed in section 6, we also have the diagrams of figure 17 and figure 18, obtained by replacing, in the diagrams in section 5, the external photon by a graviton but keeping the internal line as a photon. We also have an additional set of diagrams shown in figure 19 where the external graviton connects to the internal photon. Diagrams in which the external graviton attaches to the n-scalar vertex vanish for $\varepsilon_\rho{}^\rho = 0$ and have not been displayed. We carry out Grammer-Yennie decomposition for the internal photons in figure 17 following (5.3), but not for diagrams of the form shown in figure 18 and figure 19. The sum over K-photons factorize as in section 5 and gives the factor of $\exp[K_{\mathrm{em}}]$ that cancels between $\Gamma^{(n)}$ and $\Gamma^{(n,1)}$. In this case there is no residual contribution since the analog of figure 4 holds with the upper photon in the second identity replaced by a graviton (see figure 20). This leads to the analog of (5.11) with an additional contribution to the left hand side given by diagrams of the form shown in figure 19. Denoting this contribution by $\Gamma^{(n,1)}_{\gamma\gamma g}$ we arrive at the relation

$$\Gamma^{(n,1)}_{\mathrm{self}} + \Gamma^{(n,1)}_{\mathrm{G}} + \Gamma^{(n,1)}_{\gamma\gamma g} = i\,\lambda\,S^{(1)}_{\mathrm{gr}}, \tag{7.4}$$

with the understanding that both sides represent contributions in addition to what already appear in (6.16). None of the terms have any infrared divergence, and therefore there are no logarithmic terms from the region of integration in which the loop momentum is small compared to ω. We shall describe below the organization of the various terms and then state the final result:

1. One can analyze $\Gamma^{(n,1)}_{\mathrm{self}}$ represented by the graphs in figure 18 by following the procedure described below (5.16). We replace the external graviton polarization by a pure gauge form $(\xi_\mu k_\nu + \xi_\nu k_\mu)$ and apply Ward identity to evaluate the sum over the graphs in figure 18. In this case the Ward identity has an additional contribution as shown in the right hand side of the second diagram in figure 21. It turns out however that its contribution to the amplitude does not have any logarithmic term. Therefore $\Gamma^{(n,1)}_{\mathrm{self}}$ does not generate any logarithmic contribution.
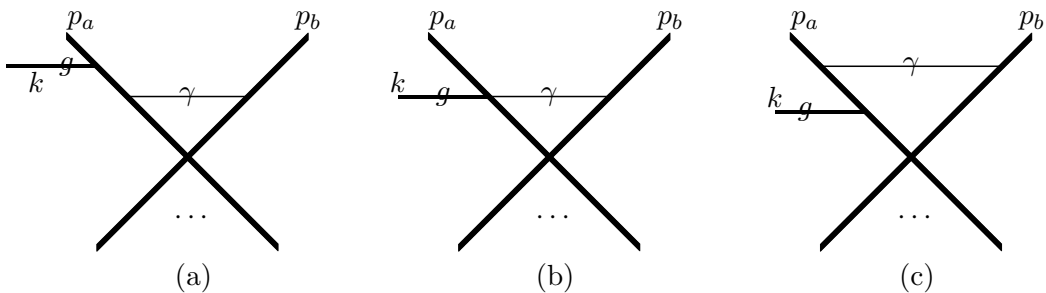
Figure 17. Contribution to soft graviton amplitude due to internal photon whose two ends are connected to two different scalar lines. Here the thickest lines denote scalars, lines of medium thickness carrying the symbol g denote gravitons and the thin lines carrying the symbol γ denote photons.
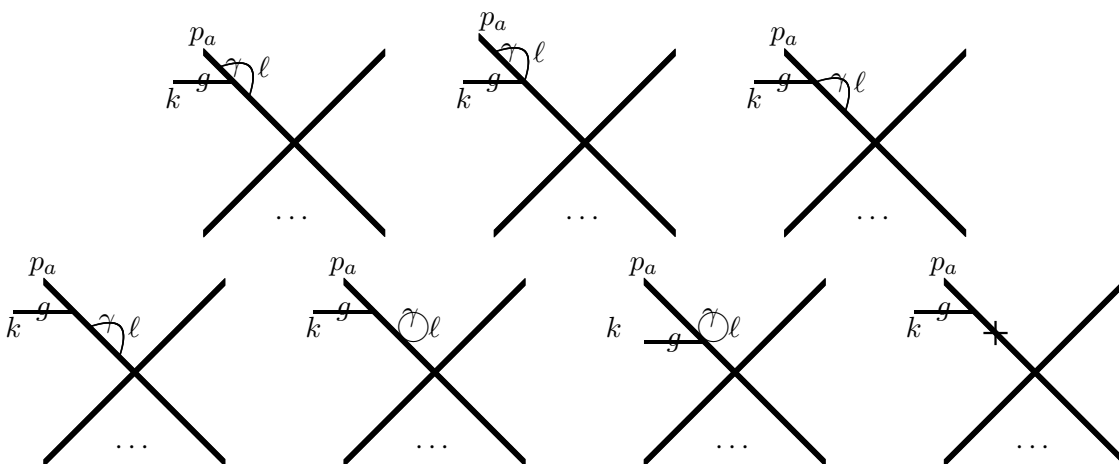

Figure 18. One loop contribution to soft graviton amplitude involving internal photon line connecting two points on the same leg.

2. $\Gamma_{\gamma\gamma g}^{(n,1)}$ receives contribution proportional to $\ln \omega^{-1}$ from the first two diagrams of figure 19, from the region where the loop momentum is large compared to ω.

3. Finally, the G-photon contribution $\Gamma_{\mathrm{G}}^{(n,1)}$ from the first two diagrams in figure 17 also has terms proportional to $\ln \omega^{-1}$ from the region where the loop momentum is large compared to ω.

The net logarithmic contribution from $\Gamma_{\gamma\gamma g}^{(n,1)}$ and $\Gamma_{\mathrm{G}}^{(n,1)}$ is given by:

$$(i\lambda)\,\widehat{S}_{\mathrm{gr}}^{(1)}\;K_{\mathrm{em}}^{\mathrm{reg}}. \tag{7.5}$$

After removing the $i\,\lambda$ factor, we have to add this to (6.28) to get the total logarithmic contribution to $S_{\mathrm{gr}}^{(1)}$:

$$
\begin{aligned}
S_{\mathrm{gr}}^{(1)} = {}& \widehat{S}_{gr}^{(1)}\;\left(K_{\mathrm{em}}^{\mathrm{reg}} + K_{\mathrm{gr}}^{\mathrm{reg}}\right) \\
& + \frac{1}{4\pi}(\ln\omega^{-1} + \ln R^{-1})\left[i\sum_{\substack{b\\\eta_b=-1}} k.p_b \sum_a \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k} - \frac{1}{2\pi}\sum_a \frac{\varepsilon_{\mu\nu}p_a^\mu p_a^\nu}{p_a.k}\sum_b p_b.k\,\ln\frac{m_b^2}{(p_b.\hat{k})^2}\right].
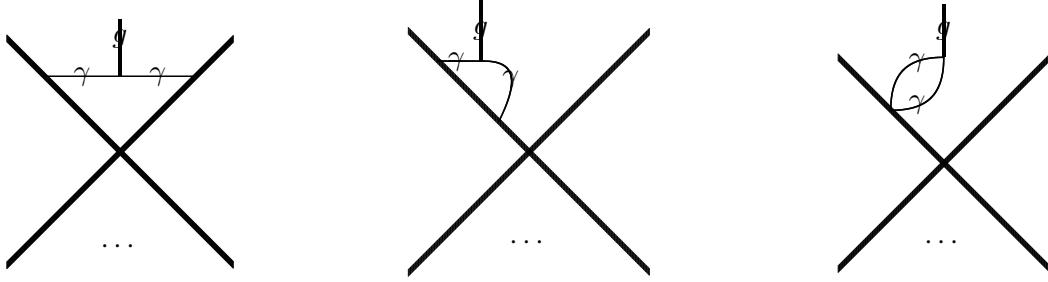\end{aligned}
\tag{7.6}
$$

Figure 19. Diagrams containing graviton-photon-photon vertex that contribute to the soft photon contribution to the soft graviton theorem.

Figure 20. The Ward identity for the photon in the presence of a graviton-scalar-scalar vertex.

Figure 21. Analog of figure 4 for graviton in the presence of a photon. The graviton carries a polarization $(\xi_\mu \ell_\nu + \xi_\nu \ell_\mu)$ and the photon carries a polarization ϵ. The circled vertex has been explained in the caption of figure 15.

This reproduces terms proportional to $\ln \omega^{-1}$ in the sum of (2.6) and (2.7) after using (5.28) and (6.25).

7.2 Soft photon theorem

Next we shall consider the soft photon theorem. In this case we have all the diagrams considered in section 5, but also extra diagrams where the internal photon of figures 1 and 2 is replaced by an internal graviton, as shown in figures 22 and 23, and two additional sets of diagrams: one where one end of the internal graviton connects to the external photon as in figure 24 and the other where one end of the internal graviton is attached to the *n*-scalar vertex as in figure 25. There is also an additional diagram obtained by
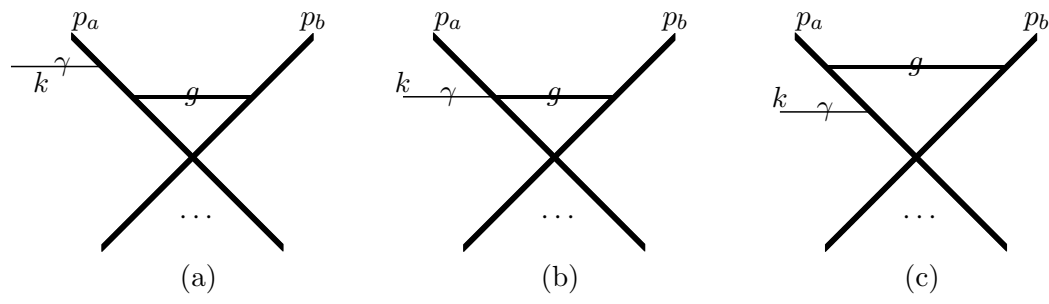
Figure 22. One loop contribution to soft photon amplitude involving internal graviton line connecting two different legs.
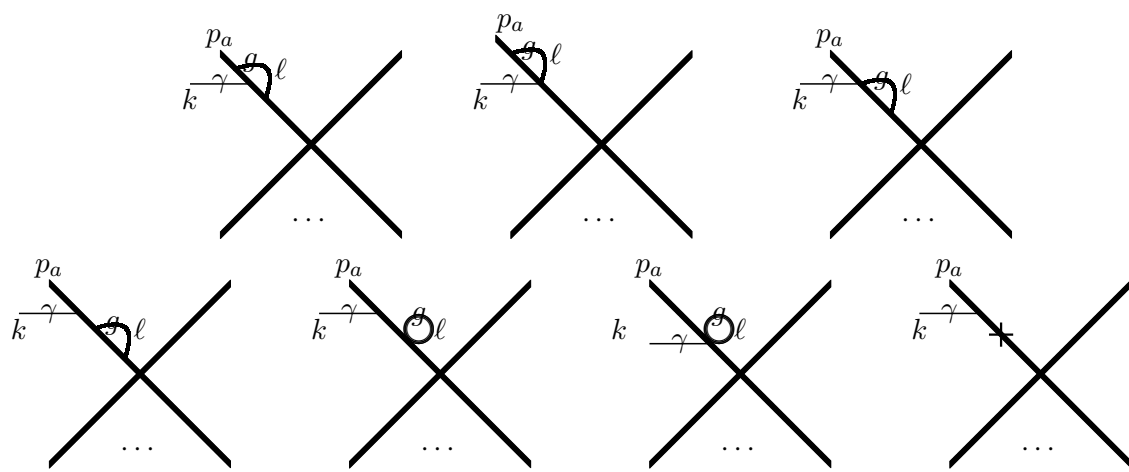

Figure 23. Diagrams in which the external photon and both ends of the internal graviton attach to the same scalar leg.
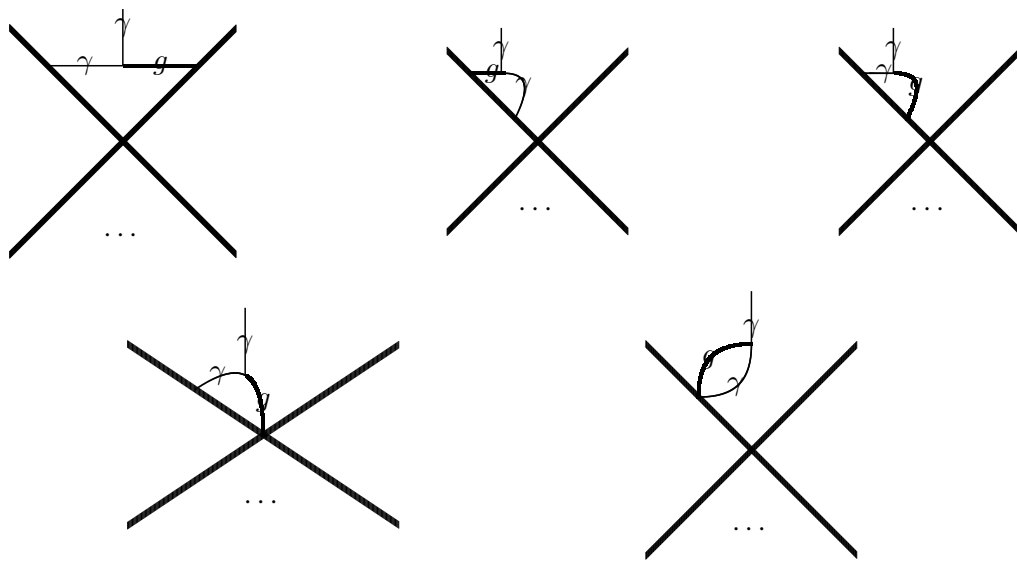

Figure 24. Diagrams involving graviton-photon-photon vertex that need to be included in computing the soft graviton contribution to the soft photon theorem.
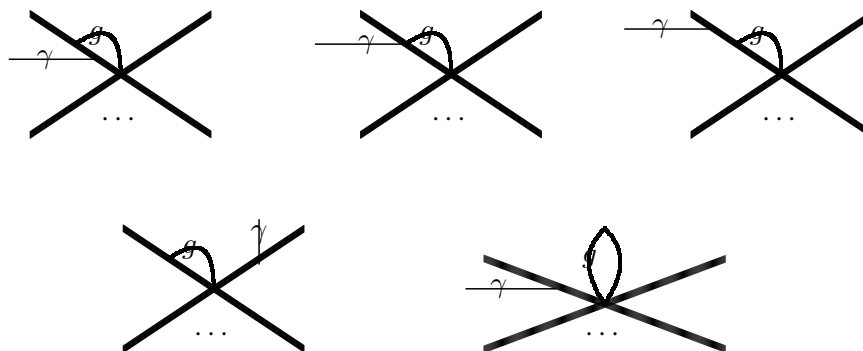
Figure 25. Diagrams with external soft photon and an internal graviton where the internal graviton attaches to the n-point vertex.

replacing in the first diagram of figure 14 the external graviton by the external photon, but this vanishes in dimensional regularization.

We shall analyze the diagrams in figures 22 and 25 using Grammer-Yennie decomposition for the internal graviton following the rules described in (6.9)–(6.14). The result of summing over K-gravitons in $\Gamma^{(n,1)}$ will generate the factor of $\exp[K_{\mathrm{gr}} + \widetilde{K}_{\mathrm{gr}}]$ which cancels a similar factor in the expression of $\Gamma^{(n)}$. However there will be residual part that will be left over due to non-cancellation of the sum over K-graviton insertions reflected in the right-hand side of figure 21. Another residual contribution arises due to the momentum dependence of the circled vertices; as illustrated in figure 16, the factorized contribution of K-gravitons for $\Gamma^{(n)}$ and $\Gamma^{(n,1)}$ differ. The only difference in the present case is that the external graviton carrying momentum k in figure 16 is replaced by an external photon. As in section 6, we shall denote these residual contributions in the sum over K-gravitons by $\Gamma^{(n,1)}_{\mathrm{residual}}$. The G-graviton contribution to figures 22 and 25 will be denoted by $\Gamma^{(n,1)}_{\mathrm{G}}$. The contribution from diagrams involving the coupling of graviton to photon, as shown in figure 24, will be denoted by $\Gamma^{(n,1)}_{\gamma\gamma g}$, and the contributions from figure 23 will be denoted by $\Gamma^{(n,1)}_{\mathrm{self}}$. Then the generalization of (5.11) takes the form:

$$\Gamma^{(n,1)}_{\mathrm{self}} + \Gamma^{(n,1)}_{\mathrm{G}} + \Gamma^{(n,1)}_{\gamma\gamma g} + \Gamma^{(n,1)}_{\mathrm{residual}} = i\,\lambda\,S^{(1)}_{\mathrm{gr}}\,, \tag{7.7}$$

again with the understanding that both sides represent additional contribution besides those described in section 5.

Analysis of various terms on the left hand side of (7.7) goes as follows:

1. $\Gamma^{(n,1)}_{\mathrm{self}}$ can be shown to vanish using the same argument given below (5.16). In this case the relevant Ward identities given in figures 4 and 20 do not have any left-over extra contributions.

2. It turns out that $\Gamma^{(n,1)}_{\mathrm{residual}}$, given by the left-over contribution after summing over K-graviton insertions in figures 22 and 25, does not receive any logarithmic terms either from the region of loop momentum integration small compared to ω or from regions of loop momentum integration large compared to ω.

3. $\Gamma_{\text{G}}^{(n,1)}$ receives contributions proportional to $\ln \omega^{-1}$ only from the G-graviton contribution to figure 22, from region of integration where the loop momentum is larger than ω.

4. The individual diagrams contributing to $\Gamma_{\gamma\gamma g}^{(n,1)}$ have collinear divergence from the region where the momenta of the internal graviton and photon are parallel to the momentum of the external photon. This cancels in the sum over all diagrams in figure 24. The second and third diagrams of figure 24 each has contribution proportional to $\ln \omega^{-1}$ from the region of integration where the loop momentum is large compared to ω, but the sum of these contributions vanishes. Finally, $\Gamma_{\gamma\gamma g}^{(n,1)}$ receives contributions proportional to $\ln \omega^{-1}$ from the first two diagrams in figure 24, from the region where the momentum of the internal graviton is smaller than ω.

The net logarithmic contribution from the region of integration where the loop momentum is larger than ω is given by

$$i\,\lambda\,\widehat{S}_{\text{em}}^{(1)}\,K_{\text{gr}}^{\text{reg}}\,. \tag{7.8}$$

On the other hand the contribution to $\Gamma_{\gamma\gamma g}^{(n,1)}$ from the small loop momentum region is given by:

$$i\,\lambda\,(\ln\omega^{-1}+\ln R^{-1})\left[\frac{i}{4\pi}\sum_{\substack{b\\\eta_b=-1}}k.p_b\sum_a\frac{\varepsilon_\mu p_a^\mu}{p_a.k}\,q_a-\frac{1}{8\pi^2}\sum_{a=1}^n\frac{q_a\varepsilon_\mu p_a^\mu}{p_a.k}\sum_{b=1}^n(p_b.k)\ln\left(\frac{-p_b^2}{(p_b.\hat{k})^2}\right)\right]. \tag{7.9}$$

One difference from the previous diagrams of this type, e.g. the ones shown in figure 11, is that the divergent contribution comes only from the region where the internal graviton momentum becomes small, and not when the internal photon momentum becomes small. This reflects the fact that while photons feel the long range gravitational force due to other particles, the graviton, being charge neutral, does not feel any long range Coulomb force. After removing the $i\,\lambda$ factors from (7.8) and (7.9), we have to add them to (5.20) to get the total soft factor $S_{\text{em}}^{(1)}$. This gives

$$S_{\text{em}}^{(1)}=\widehat{S}_{\text{em}}^{(1)}\left(K_{\text{em}}^{\text{reg}}+K_{\text{gr}}^{\text{reg}}\right)$$

$$+(\ln\omega^{-1}+\ln R^{-1})\left[\frac{i}{4\pi}\sum_{\substack{b\\\eta_b=-1}}k.p_b\sum_a\frac{\varepsilon_\mu p_a^\mu}{p_a.k}\,q_a-\frac{1}{8\pi^2}\sum_{a=1}^n\frac{q_a\varepsilon_\mu p_a^\mu}{p_a.k}\sum_{b=1}^n(p_b.k)\ln\left(\frac{-p_b^2}{(p_b.\hat{k})^2}\right)\right]. \tag{7.10}$$

This reproduces terms proportional to $\ln\omega^{-1}$ in the sum of (2.4) and (2.5) after using the explicit forms of $K_{\text{em}}^{\text{reg}}$ and $K_{\text{gr}}^{\text{reg}}$ given in (5.28) and (6.25).

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