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CLASSICAL EIGHTH- AND LOWER-ORDER RUNGE-KUTTA-NYSTROM FORMULAS WITH STEPSIZE CONTROL FOR SPECIAL SECOND-ORDER DIFFERENTIAL EQUATIONS

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16. Abstract

Runge-Kutta-Nyström formulas of the eighth, seventh, sixth, fifth, and fourth orders are derived for the special second-order (vector) differential equation $\ddot{x} = f(t, x)$. These formulas include a stepsize control procedure, based on a complete coverage of the leading term of the truncation error in x. The formulas require fewer evaluations per step than other Runge-Kutta-Nyström formulas if the latter are operated by using the standard procedure for stepsize control. They also require fewer evaluations per step than this author's earlier Runge-Kutta formulas for first-order differential equations. An example is presented. With results being of the same accuracy, in this example the new Runge-Kutta-Nyström formulas save 50 percent or more computer time compared with other authors' Runge-Kutta-Nyström formulas and compared with our earlier Runge-Kutta formulas for first-order differential equations.

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CLASSICAL EIGHTH- AND LOWER-ORDER RUNGE-KUTTA-NYSTRÖM FORMULAS WITH STEPSIZE CONTROL FOR SPECIAL SECOND-ORDER DIFFERENTIAL EQUATIONS

INTRODUCTION

1. In earlier reports [1], [2], [3] the author derived and listed Runge-Kutta formulas with stepsize control for first-order differential equations. Although second-order differential equations can be converted to first-order differential equations by introducing the first derivatives as new variables, this amounts to an increased computational effort, since we then have to deal with twice as many differential equations as in the original second-order problem. Therefore, the direct Runge-Kutta integration of second-order differential equations without conversion to first-order equations might be preferable. Direct Runge-Kutta formulas for second-order differential equations were first published by E. J. NYSTRÖM [4]. We therefore refer in this report to such direct Runge-Kutta formulas for second-order differential equations as Runge-Kutta-Nyström (RKN) formulas.

2. In this report we restrict ourselves to a special class of second-order (vector) differential equations,

$$\ddot{x} = f(t,x) \quad , \tag{1}$$

which do not contain the first derivative $\dot{x}$ on the right-hand side. Such special second-order differential equations are frequently encountered in mechanics and physics.

The derivation of the equations of condition for the Runge-Kutta-Nyström coefficients is much simpler and easier for the special equations (1) than for the general equations which would also contain the first derivatives $\dot{x}$ on the right-hand side.

3. Similar to the Runge-Kutta formulas of our earlier reports [1], [2], [3], the Runge-Kutta-Nyström formulas of this report include an automatic stepsize control based on a complete coverage of the leading term of the local truncation error in x. This coverage is achieved by one additional evaluation of the differential equations. Each of our Runge-Kutta-Nyström formulas represents in fact a pair of integration formulas for x which differ

from one another by the one additional evaluation of the differential equations. The orders of these two formulas differ by 1. Therefore, the difference of the formulas represents an approximation for the leading term of the truncation error in x for the lower-order formula. By requiring that this difference remain between preset limits, an automatic stepsize control for the lower-order formula can be established.

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4. For the sake of brevity in the derivation of the formulas, let us consider in the following a system of two second-order differential equations:

$$\left.\begin{aligned} \ddot{x} &= f(t,x,y) \\ \ddot{y} &= g(t,x,y) \end{aligned}\right\} \quad . \tag{2}$$

Our results, however, will hold, in a quite obvious way, for a system of any number of second-order differential equations. We introduce

$$\left.\begin{aligned} f_0 &= f(t_0,\ x_0,\ y_0) \\ f_\kappa &= f\Big(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0\alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} f_\lambda, \\ &\qquad\qquad y_0 + \dot{y}_0\alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} g_\lambda\Big) \\ &\qquad (\kappa = 1, 2, \ldots, 11) \end{aligned}\right\} \tag{3}$$

and corresponding expressions for the second equation (2).

For any κ the sum of the coefficients $\gamma_{\kappa\lambda}$ ($\lambda = 0, 1, \ldots, \kappa - 1$) is related to α_κ by:

$$\sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} = \frac{1}{2}\,\alpha_\kappa^2 , \tag{4}$$

as can easily be seen by a Taylor expansion of the x- or y- argument of f_κ in (3).

For an eighth-order Runge-Kutta-Nyström formula with stepsize control we then require

$$\left.\begin{aligned} x &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{10} c_\kappa f_\kappa + O(h^9) \\ \hat{x} &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{11} \hat{c}_\kappa f_\kappa + O(h^{10}) \\ \dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{10} \dot{c}_\kappa f_\kappa + O(h^9) \end{aligned}\right\} \qquad (5)$$

with

$$\left.\begin{aligned} \hat{c}_\kappa &= c_\kappa \text{ for } \kappa = 0, 1, 2, \ldots, 9 \\ \hat{c}_{10} &= 0 \\ \hat{c}_{11} &= c_{10} \end{aligned}\right\} \qquad (6)$$

and corresponding formulas for the solution of the second differential equation (2). In (3) and (5) the quantities t_0, x_0, y_0, $\dot{x}_0$, $\dot{y}_0$ are the initial values for the integration step under consideration, while h stands for the integration stepsize.

As in our earlier report [2], we require that the last evaluation of the differential equation can be taken over as first evaluation for the next step, thereby reducing the number of evaluations per step by one. By this requirement, the coefficients $\gamma_{11\lambda}$ and α_{11} are determined:

$$\gamma_{110} = c_0,\ \gamma_{111} = c_1,\ \gamma_{112} = c_2,\ \ldots,\ \gamma_{1110} = c_{10},\ \alpha_{11} = 1 . \qquad (7)$$

Our problem then consists in finding the Runge-Kutta-Nyström coefficients α_κ, $\gamma_{\kappa\lambda}$, c_κ, $\dot{c}_\kappa$ so that the right-hand sides of (5) are really eighth- or ninth- order approximations of x or $\dot{x}$. By expanding in Taylor series the solution of (2) as well as the right-hand sides of (5) and equating the corresponding terms in both series, equations of condition for these coefficients can be obtained.

5. We first expand the solution x, y of (2) in a (truncated) Taylor series:

$$\left.\begin{aligned} x &= x_0 + \sum_{\nu=1}^{10} \frac{x_0^{(\nu)}}{\nu!} h^{\nu} \\ y &= y_0 + \sum_{\nu=1}^{10} \frac{y_0^{(\nu)}}{\nu!} h^{\nu} \end{aligned}\right\} \quad , \tag{8}$$

using equations (2) and their derivatives for the computation of the total derivatives $x_0^{(\nu)}$, $y_0^{(\nu)}$ in (8).

For the computation of the derivatives of (2) we introduce the differential operator:

$$D = \frac{\partial}{\partial t} + \dot{x} \frac{\partial}{\partial x} + \dot{y} \frac{\partial}{\partial y} \quad . \tag{9}$$

Obviously, the following rules hold for this operator:

$$\left.\begin{aligned} D(\varphi + \psi) &= D(\varphi) + D(\psi) \\ D(\varphi \cdot \psi) &= \varphi D(\psi) + \psi D(\varphi) \\ D[D^n(\varphi)] &= D^{n+1}(\varphi) + n\left[D^{n-1}(\varphi_x) f + D^{n-1}(\varphi_y) g\right] \end{aligned}\right\} \quad . \tag{10}$$

The operators $D^2(\varphi)$, etc., are defined as symbolic powers of D:

$$D^2(\varphi) = \left(\varphi_t + \varphi_x \cdot \dot{x} + \varphi_y \cdot \dot{y}\right)^2 = \varphi_{tt} + 2\varphi_{tx} \cdot \dot{x} + 2\varphi_{ty} \cdot \dot{y} + \varphi_{xx} \cdot \dot{x}^2 + 2\varphi_{xy} \cdot \dot{x}\dot{y} + \varphi_{yy} \cdot \dot{y}^2,$$

etc.

Observing the above rules, the total derivatives of x are obtainable from (2). The resulting lengthy expressions for these derivatives are somewhat shortened by the introduction of the following abbreviations:

$$\{\varphi(f_x)\ \psi(f)\} = \varphi(f_x)\ \psi(f) + \varphi(f_y)\ \psi(g) \tag{11}$$

$$\left.\begin{aligned} <\varphi(f_{xx})\ \psi_1(f)\ \psi_2(f)> = \varphi(f_{xx})\ \psi_1(f)\ \psi_2(f) + \varphi(f_{xy})\ [\psi_1(f)\ \psi_2(g) \\ + \psi_1(g)\ \psi_2(f)] + \varphi(f_{yy})\ \psi_1(g)\ \psi_2(g) \end{aligned}\right\} \tag{12}$$

Extending (11) to the case that $\psi(f)$ is a product of two operators, $\psi(f) = \{\psi_1(f_x)\ \psi_2(f)\}$, we define,

$$\{\varphi(f_x)\ \{\psi_1(f_x)\ \psi_2(f)\}\} = \varphi(f_x)[\psi_1(f_x)\ \psi_2(f) + \psi_1(f_y)\ \psi_2(g)]$$

$$+ \varphi(f_y)\ [\psi_1(g_x)\ \psi_2(f) + \psi_1(g_y)\ \psi_2(g)]$$

and similarly for the case that $\psi(f)$ in (11) is a product of more than two operators.

In an obvious way, (12) can also be extended to the case of third- and higher-order partial derivatives f_{xxx}, f_{xxxx}, etc. In the case of a third-order partial derivative we define:

$$<\varphi(f_{xxx})\ \psi_1(f)\ \psi_2(f)\ \psi_3(f)>$$

$$= \varphi(f_{xxx})\ \psi_1(f)\ \psi_2(f)\ \psi_3(f)$$

$$+ \varphi(f_{xxy})\ [\psi_1(f)\ \psi_2(f)\ \psi_3(g) + \psi_1(f)\ \psi_2(g)\ \psi_3(f) + \psi_1(g)\ \psi_2(f)\ \psi_3(f)]$$

$$+ \varphi(f_{xyy})\ [\psi_1(f)\ \psi_2(g)\ \psi_3(g) + \psi_1(g)\ \psi_2(f)\ \psi_3(g) + \psi_1(g)\ \psi_2(g)\ \psi_3(f)]$$

$$+ \varphi(f_{yyy})\ \psi_1(g)\ \psi_2(g)\ \psi_3(g)$$

and similarly for higher-order partial derivatives.

Using the abbreviations (11), (12), we obtain from (2) for the total derivatives of x (always taken for $t = t_0$):

$$
\left.
\begin{aligned}
x^{II} &= f \\
x^{III} &= D(f) \\
x^{IV} &= D^2(f) + \{f_x f\} \\
x^{V} &= D^3(f) + 3\{D(f_x)f\} + \{f_x\ D(f)\} \\
x^{VI} &= D^4(f) + 6\{D^2(f_x)f\} + 3\langle f_{xx} f^2\rangle + 4\{D(f_x)\ D(f)\} \\
&\quad + \{f_x\ D^2(f)\} + \{f_x\{f_x f\}\} \\
x^{VII} &= D^5(f) + 10\{D^3(f_x)f\} + 15\langle D(f_{xx})f^2\rangle + 10\{D^2(f_x)\ D(f)\} \\
&\quad + 10\langle f_{xx}\ fD(f)\rangle + 5\{D(f_x)\ D^2(f)\} + 5\{D(f_x)\{f_x f\}\} \\
&\quad + \{f_x\ D^3(f)\} + 3\{f_x\ \{D(f_x)f\}\} + \{f_x\ \{f_x D(f)\}\} \\
x^{VIII} &= D^6(f) + 15\{D^4(f_x)f\} + 45\langle D^2(f_{xx})\ f^2\rangle + 20\{D^3(f_x)\ D(f)\} \\
&\quad + 15\langle f_{xxx}\ f^3\rangle + 60\ \langle D(f_{xx})f\ D(f)\rangle + 15\{D^2(f_x)\ D^2(f)\} \\
&\quad + 15\{D^2(f_x)\{f_x f\}\} + 10\ \langle f_{xx}\ [D(f)]^2\rangle + 15\langle f_{xx} f\ D^2(f)\rangle \\
&\quad + 15\ \langle f_{xx}\ f\{f_x f\}\rangle + 6\{D(f_x)\ D^3(f)\} + 18\{D(f_x)\ \{D(f_x)f\}\} \\
&\quad + 6\{D(f_x)\ \{f_x\ D(f)\}\} + \{f_x\ D^4(f)\} + 6\{f_x\ \{D^2(f_x)f\}\} \\
&\quad + 3\{f_x\ \langle f_{xx}\ f^2\rangle\} + 4\{f_x\{D(f_x)\ D(f)\}\} + \{f_x\ \{f_x D^2(f)\}\} \\
&\quad + \{f_x\ \{f_x\ \{f_x f\}\}\}
\end{aligned}
\right\}
\qquad (13)
$$

$$
\left.
\begin{aligned}
x^{IX} = {} & D^7(f) + 21\{D^5(f_x)f\} + 105\langle D^3(f_{xx})f^2\rangle + 35\{D^4(f_x)\ D(f)\} \\
& + 105\langle D(f_{xxx})f^3\rangle + 210\langle D^2(f_{xx})f\ D(f)\rangle + 35\{D^3(f_x)\ D^2(f)\} \\
& + 35\{D^3(f_x)\ \{f_x f\}\} + 105\langle f_{xxx}\ f^2\ D(f)\rangle + 70\langle D(f_{xx})\ [D(f)]^2\rangle \\
& + 105\langle D(f_{xx})f\ D^2(f)\rangle + 105\langle D(f_{xx})\ f\ \{f_x f\}\rangle + 21\{D^2(f_x)\ D^3(f)\} \\
& + 63\{D^2(f_x)\ \{D(f_x)\ f\}\} + 21\{D^2(f_x)\ \{f_x\ D(f)\}\} + 35\langle f_{xx}\ D(f)\ D^2(f)\rangle \\
& + 35\langle f_{xx}\ D(f)\{f_x f\}\rangle + 21\langle f_{xx} f\ D^3(f)\rangle + 63\langle f_{xx} f\ \{D(f_x)f\}\rangle \\
& + 21\langle f_{xx} f\ \{f_x\ D(f)\}\rangle + 7\{D(f_x)\ D^4(f)\} + 42\{D(f_x)\ \{D^2(f_x)f\}\} \\
& + 21\{D(f_x)\langle f_{xx}f^2\rangle\} + 28\{D(f_x)\{D(f_x)D(f)\}\} + 7\{D(f_x)\ \{f_x D^2(f)\}\} \\
& + 7\{D(f_x)\ \{f_x\{f_x f\}\}\} + \{f_x\ D^5(f)\} + 10\{f_x\ \{D^3(f_x)f\}\} \\
& + 15\{f_x\langle D(f_{xx})f^2\rangle\} + 10\{f_x\ \{D^2(f_x)\ D(f)\}\} + 10\{f_x\langle f_{xx}f\ D(f)\rangle\} \\
& + 5\{f_x\ \{D(f_x)\ D^2(f)\}\} + 5\{f_x\ \{D(f_x)\ \{f_x f\}\}\} + \{f_x\ \{f_x\ D^3(f)\}\} \\
& + 3\{f_x\ \{f_x\ \{D(f_x)f\}\}\} + \{f_x\ \{f_x\ \{f_x\ D(f)\}\}\} \\[1ex]
x^{X} = {} & D^8(f) + 28\{D^6(f_x)f\} + 210\langle D^4(f_{xx})f^2\rangle + 56\{D^5(f_x)\ D(f)\} \\
& + 420\langle D^2(f_{xxx})f^3\rangle + 560\langle D^3(f_{xx})f\ D(f)\rangle + 70\{D^4(f_x)\ D^2(f)\} \\
& + 70\{D^4(f_x)\ \{f_x f\}\} + 105\langle f_{xxxx}f^4\rangle + 840\langle D(f_{xxx})f^2\ D(f)\rangle \\
& + 280\langle D^2(f_{xx})\ [D(f)]^2\rangle + 420\langle D^2(f_{xx})f\ D^2(f)\rangle + 420\langle D^2(f_{xx})f\{f_x f\}\rangle \\
& + 56\{D^3(f_x)\ D^3(f)\} + 168\{D^3(f_x)\{D\ (f_x)f\}\} + 56\{D^3(f_x)\ \{f_x\ D(f)\}\}
\end{aligned}
\right\}
\begin{gathered}(13)\\ \text{(con.)}\end{gathered}
$$

$$
\left.
\begin{aligned}
&+ 280 \langle f_{xxx} f\, [D(f)]^2 \rangle + 210 \langle f_{xxx} f^2\, D^2(f) \rangle + 210 \langle f_{xxx} f^2 \{f_x f\} \rangle \\
&+ 280 \langle D(f_{xx})\, D(f)\, D^2(f) \rangle + 280 \langle D(f_{xx})\, D(f) \{f_x f\} \rangle \\
&+ 168 \langle D(f_{xx}) f\, D^3(f) \rangle + 504 \langle D(f_{xx}) f \{ D(f_x) f\} \rangle \\
&+ 168 \langle D(f_{xx}) f \{ f_x\, D(f) \} \rangle + 28 \{ D^2(f_x)\, D^4(f) \} \\
&+ 168 \{ D^2(f_x) \{ D^2(f_x) f \} \} + 84 \{ D^2(f_x) \langle f_{xx} f^2 \rangle \} \\
&+ 112 \{ D^2(f_x) \{ D(f_x)\, D(f) \} \} + 28 \{ D^2(f_x) \{ f_x\, D^2(f) \} \} \\
&+ 28 \{ D^2(f_x) \{ f_x \{ f_x f \} \} \} + 56 \langle f_{xx}\, D(f)\, D^3(f) \rangle \\
&+ 168 \langle f_{xx}\, D(f) \{ D(f_x) f \} \rangle + 35 \langle f_{xx} [D^2(f)]^2 \rangle \\
&+ 70 \langle f_{xx}\, D^2(f) \{ f_x f \} \rangle + 56 \langle f_{xx}\, D(f) \{ f_x\, D(f) \} \rangle \\
&+ 35 \langle f_{xx} \{ f_x f \}^2 \rangle + 28 \langle f_{xx} f\, D^4(f) \rangle + 168 \langle f_{xx} f \{ D^2(f_x) f \} \rangle \\
&+ 84 \langle f_{xx} f \langle f_{xx} f^2 \rangle \rangle + 112 \langle f_{xx} f \{ D(f_x)\, D(f) \} \rangle \\
&+ 28 \langle f_{xx} f \{ f_x\, D^2(f) \} \rangle + 28 \langle f_{xx} f \{ f_x \{ f_x f \} \} \rangle + 8 \{ D(f_x)\, D^5(f) \} \\
&+ 80 \{ D(f_x) \{ D^3(f_x) f \} \} + 120 \{ D(f_x) \langle D(f_{xx}) f^2 \rangle \} \\
&+ 80 \{ D(f_x) \{ D^2(f_x)\, D(f) \} \} + 80 \{ D(f_x) \langle f_{xx} f\, D(f) \rangle \} \\
&+ 40 \{ D(f_x) \{ D(f_x)\, D^2(f) \} \} + 40 \{ D(f_x) \{ D(f_x) \{ f_x f \} \} \} \\
&+ 8 \{ D(f_x) \{ f_x\, D^3(f) \} \} + 24 \{ D(f_x) \{ f_x \{ D(f_x) f \} \} \} \\
&+ 8 \{ D(f_x) \{ f_x \{ f_x\, D(f) \} \} \} + \{ f_x D^6(f) \} + 15 \{ f_x \{ D^4(f_x) f \} \}
\end{aligned}
\right\} \quad \begin{matrix}(13)\\(\text{con.})\end{matrix}
$$

$$
\left.
\begin{aligned}
&+ 45 \{ f_x \langle D^2(f_{xx}) f^2 \rangle \} + 20 \{ f_x \{ D^3(f_x)\ D(f) \} \} \\
&+ 15 \{ f_x \langle f_{xxx} f^3 \rangle \} + 60 \{ f_x \langle D(f_{xx}) f\ D(f) \rangle \} \\
&+ 15 \{ f_x \{ D^2(f_x)\ D^2(f) \} \} + 15 \{ f_x \{ D^2(f_x)\ \{ f_x f \} \} \} \\
&+ 10 \{ f_x \langle f_{xx}\ [D(f)]^2 \rangle \} + 15 \{ f_x \langle f_{xx} f\ D^2(f) \rangle \} \\
&+ 15 \{ f_x \langle f_{xx} f \{ f_x f \} \rangle \} + 6 \{ f_x \{ D(f_x)\ D^3(f) \} \} \\
&+ 18 \{ f_x \{ D(f_x)\ \{ D(f_x) f \} \} \} + 6 \{ f_x \{ D(f_x)\ \{ f_x\ D(f) \} \} \} \\
&+ \{ f_x \{ f_x\ D^4(f) \} \} + 6 \{ f_x \{ f_x \{ D^2(f_x) f \} \} \} + 3 \{ f_x \{ f_x \langle f_{xx} f^2 \rangle \} \} \\
&+ 4 \{ f_x \{ f_x \{ D(f_x)\ D(f) \} \} \} + \{ f_x \{ f_x \{ f_x\ D^2(f) \} \} \} \\
&+ \{ f_x \{ f_x \{ f_x \{ f_x f \} \} \} \} \qquad .
\end{aligned}
\right\} \quad \begin{matrix} (13) \\ (\text{con.}) \end{matrix}
$$

Corresponding expressions hold for the total derivatives of y.

6. Next, we have to expand (3) in a Taylor series. Let us list the result for $\kappa = 1$:

$$
\left.
\begin{aligned}
f_1 = {} & f + D(f)\ \alpha_1 h + \frac{1}{2} D^2(f)\ \alpha_1^2 h^2 + \frac{1}{2} \{ f_x f \}\ \alpha_1^2 h^2 + \frac{1}{6} D^3(f)\ \alpha_1^3 h^3 \\
& + \frac{1}{2} \{ D(f_x) f \}\ \alpha_1^3 h^3 + \frac{1}{24} D^4(f)\ \alpha_1^4 h^4 + \frac{1}{4} \{ D^2(f_x) f \}\ \alpha_1^4 h^4 \\
& + \frac{1}{8} \langle f_{xx} f^2 \rangle\ \alpha_1^4 h^4 + \frac{1}{120} D^5(f)\ \alpha_1^5 h^5 + \frac{1}{12} \{ D^3(f_x) f \}\ \alpha_1^5 h^5 \\
& + \frac{1}{8} \langle D(f_{xx})\ f^2 \rangle\ \alpha_1^5 h^5 + \frac{1}{720} D^6(f)\ \alpha_1^6 h^6 + \frac{1}{48} \{ D^4(f_x) f \}\ \alpha_1^6 h^6 \\
& + \frac{1}{16} \langle D^2(f_{xx}) f^2 \rangle\ \alpha_1^6 h^6 + \frac{1}{48} \langle f_{xxx} f^3 \rangle\ \alpha_1^6 h^6 + \frac{1}{5040} D^7(f)\ \alpha_1^7 h^7 \\
& + \frac{1}{240} \{ D^5(f_x) f \}\ \alpha_1^7 h^7 + \frac{1}{48} \langle D^3(f_{xx}) f^2 \rangle\ \alpha_1^7 h^7 + \frac{1}{48} \langle D(f_{xxx}) f^3 \rangle\ \alpha_1^7 h^7
\end{aligned}
\right\} (14)
$$

$$+\frac{1}{40320} D^8(f)\,\alpha_1^8 h^8 + \frac{1}{1440}\{D^6(f_x)f\}\alpha_1^8 h^8 + \frac{1}{192}\langle D^4(f_{xx})f^2\rangle \alpha_1^8 h^8$$

$$+\frac{1}{96}\langle D^2(f_{xxx})f^3\rangle \alpha_1^8 h^8 + \frac{1}{384}\langle f_{xxxx} f^4\rangle \alpha_1^8 h^8 \quad . \qquad (14)\ (\text{con.})$$

Because of (4), we replaced in (14) the coefficient γ_{10} by $\frac{1}{2}\alpha_1^2$. Using (4), we similarly eliminated $\gamma_{\nu 0}$ ($\nu = 2, 3, \ldots$) in the following expansions for f_2 f_3 .... We now list the expansion for f_2. It consists of similar terms as (14) which are obtained from (14) replacing α_1 by α_2 and of additional terms. We list only these additional terms:

$$\left.\begin{aligned}
f_2 = \ldots &+ \{f_x D(f)\}(\gamma_{21}\alpha_1) h^3 + \{D(f_x)\ D(f)\}\alpha_2(\gamma_{21}\alpha_1) h^4 \\
&+ \frac{1}{2}\{f_x D^2(f)\}(\gamma_{21}\alpha_1^2) h^4 + \frac{1}{2}\{f_x\{f_x f\}\}(\gamma_{21}\alpha_1^2) h^4 \\
&+ \frac{1}{2}\{D^2(f_x)\ D(f)\}\alpha_2^2(\gamma_{21}\alpha_1) h^5 + \frac{1}{2}\{D(f_x)\ D^2(f)\}\alpha_2(\gamma_{21}\alpha_1^2) h^5 \\
&+ \frac{1}{2}\{D(f_x)\{f_x f\}\}\alpha_2(\gamma_{21}\alpha_1^2) h^5 + \frac{1}{2}\langle f_{xx} f\ D(f)\rangle \alpha_2^2(\gamma_{21}\alpha_1) h^5 \\
&+ \frac{1}{6}\{f_x D^3(f)\}(\gamma_{21}\alpha_1^3) h^5 + \frac{1}{2}\{f_x\{D(f_x)f\}\}(\gamma_{21}\alpha_1^3) h^5 \\
&+ \frac{1}{6}\{D^3(f_x)\ D(f)\}\alpha_2^3(\gamma_{21}\alpha_1) h^6 + \frac{1}{4}\{D^2(f_x)\ D^2(f)\}\alpha_2^2(\gamma_{21}\alpha_1^2) h^6 \\
&+ \frac{1}{4}\{D^2(f_x)\{f_x f\}\}\alpha_2^2(\gamma_{21}\alpha_1^2) h^6 + \frac{1}{2}\langle D(f_{xx}) f D(f)\rangle \alpha_2^3(\gamma_{21}\alpha_1) h^6 \\
&+ \frac{1}{6}\{D(f_x)\ D^3(f)\}\alpha_2(\gamma_{21}\alpha_1^3) h^6 + \frac{1}{2}\{D(f_x)\{D(f_x)f\}\}\alpha_2(\gamma_{21}\alpha_1^3) h^6 \\
&+ \frac{1}{4}\langle f_{xx} f\ D^2(f)\rangle \alpha_2^2(\gamma_{21}\alpha_1^2) h^6 + \frac{1}{4}\langle f_{xx} f\{f_x f\}\rangle \alpha_2^2(\gamma_{21}\alpha_1^2) h^6 \\
&+ \frac{1}{2}\langle f_{xx}[D(f)]^2\rangle (\gamma_{21}\alpha_1)^2 h^6 + \frac{1}{24}\{f_x D^4(f)\}(\gamma_{21}\alpha_1^4) h^6
\end{aligned}\right\}\ (15)$$

$$
\left.
\begin{aligned}
&+\frac{1}{8}\{f_x \langle f_{xx} f^2\rangle\}(\gamma_{21}\,\alpha_1^4)\,h^6+\frac{1}{4}\{f_x\{D^2(f_x)f\}\}(\gamma_{21}\,\alpha_1^4)\,h^6\\
&+\frac{1}{24}\{D^4(f_x)\;D(f)\}\,\alpha_2^4\,(\gamma_{21}\,\alpha_1)h^7+\frac{1}{12}\{D^3(f_x)\;D^2(f)\}\,\alpha_2^3\,(\gamma_{21}\,\alpha_1^2)\,h^7\\
&+\frac{1}{12}\{D^3(f_x)\{f_x f\}\}\,\alpha_2^3\,(\gamma_{21}\,\alpha_1^2)h^7+\frac{1}{4}\langle D^2(f_{xx})f\;D(f)\rangle\,\alpha_2^4\,(\gamma_{21}\,\alpha_1)\,h^7\\
&+\frac{1}{12}\{D^2(f_x)\;D^3(f)\}\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^3)h^7+\frac{1}{4}\{D^2(f_x)\{D(f_x)f\}\}\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^3)\,h^7\\
&+\frac{1}{4}\langle D(f_{xx})f\;D^2(f)\rangle\,\alpha_2^3\,(\gamma_{21}\,\alpha_1^2)h^7+\frac{1}{4}\langle D(f_{xx})f\{f_x f\}\rangle\alpha_2^3\,(\gamma_{21}\,\alpha_1^2)h^7\\
&+\frac{1}{2}\langle D(f_{xx})\,[D(f)]^2\rangle\,\alpha_2(\gamma_{21}\,\alpha_1)^2h^7+\frac{1}{8}\langle f_{xxx}f^2\;D(f)\rangle\,\alpha_2^4\,(\gamma_{21}\,\alpha_1)h^7\\
&+\frac{1}{24}\{D(f_x)\;D^4(f)\}\,\alpha_2\,(\gamma_{21}\,\alpha_1^4)h^7+\frac{1}{8}\{D(f_x)\,\langle f_{xx}f^2\rangle\}\,\alpha_2\,(\gamma_{21}\,\alpha_1^4)\,h^7\\
&+\frac{1}{4}\{D(f_x)\,\{D^2(f_x)f\}\}\,\alpha_2(\gamma_{21}\,\alpha_1^4)h^7+\frac{1}{12}\langle f_{xx}f\;D^3(f)\rangle\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^3)h^7\\
&+\frac{1}{4}\langle f_{xx}f\{D(f_x)f\}\rangle\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^3)h^7+\frac{1}{2}\langle f_{xx}\;D(f)\;D^2(f)\rangle\,(\gamma_{21}\alpha_1)\,(\gamma_{21}\,\alpha_1^2)\,h^7\\
&+\frac{1}{2}\langle f_{xx}\;D(f)\;\{f_x f\}\rangle(\gamma_{21}\,\alpha_1)\,(\gamma_{21}\,\alpha_1^2)h^7+\frac{1}{120}\{f_x D^5(f)\}\,(\gamma_{21}\,\alpha_1^5)h^7\\
&+\frac{1}{8}\{f_x\;\langle D(f_{xx})f^2\rangle\}\,(\gamma_{21}\,\alpha_1^5)h^7+\frac{1}{12}\{f_x\{D^3(f_x)f\}\}\;(\gamma_{21}\,\alpha_1^5)\;h^7\\
&+\frac{1}{120}\{D^5(f_x)\;D(f)\}\,\alpha_2^5\,(\gamma_{21}\,\alpha_1)h^8+\frac{1}{48}\{D^4(f_x)\;D^2(f)\}\,\alpha_2^4\,(\gamma_{21}\,\alpha_1^2)\,h^8\\
&+\frac{1}{48}\{D^4(f_x)\{f_x f\}\}\,\alpha_2^4\,(\gamma_{21}\,\alpha_1^2)h^8+\frac{1}{12}\langle D^3(f_{xx})f\;D(f)\rangle\,\alpha_2^5\,(\gamma_{21}\,\alpha_1)h^8\\
&+\frac{1}{36}\{D^3\,(f_x)\;\;D^3\,(f)\}\,\alpha_2^3\,(\gamma_{21}\,\alpha_1^3)\,h^8+\frac{1}{12}\{D^3(f_x)\,\{D(f_x)f\}\}\,\alpha_2^3\,(\gamma_{21}\,\alpha_1^3)\,h^8
\end{aligned}
\right\}\;\begin{matrix}(15)\\(\text{con.})\end{matrix}
$$

$$
\left.
\begin{aligned}
&+\frac{1}{8}\left\langle D^2(f_{xx})f\,D^2(f)\right\rangle \alpha_2^4\,(\gamma_{21}\,\alpha_1^2)h^8+\frac{1}{8}\left\langle D^2(f_{xx})f\,\{f_x f\}\right\rangle \alpha_2^4\,(\gamma_{21}\,\alpha_1^2)h^8\\
&+\frac{1}{4}\left\langle D^2(f_{xx})\,[D(f)]^2\right\rangle \alpha_2^2\,(\gamma_{21}\,\alpha_1)^2h^8+\frac{1}{8}\left\langle D(f_{xxx})f^2\,D(f)\right\rangle \alpha_2^5\,(\gamma_{21}\,\alpha_1)h^8\\
&+\frac{1}{48}\{D^2(f_x)\,D^4(f)\}\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^4)\,h^8+\frac{1}{8}\{D^2(f_x)\,\{D^2(f_x)f\}\}\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^4)h^8\\
&+\frac{1}{16}\{D^2(f_x)\,\left\langle f_{xx}f^2\right\rangle\}\,\alpha_2^2\,(\gamma_{21}\,\alpha_1^4)h^8+\frac{1}{12}\left\langle D(f_{xx})f\,D^3(f)\right\rangle \alpha_2^3\,(\gamma_{21}\,\alpha_1^3)h^8\\
&+\frac{1}{4}\left\langle D(f_{xx})f\,\{D(f_x)f\}\right\rangle \alpha_2^3\,(\gamma_{21}\,\alpha_1^3)h^8\\
&+\frac{1}{2}\left\langle D(f_{xx})\,D(f)\,D^2(f)\right\rangle \alpha_2\,(\gamma_{21}\,\alpha_1)\,(\gamma_{21}\,\alpha_1^2)\,h^8\\
&+\frac{1}{2}\left\langle D(f_{xx})\,D(f)\,\{f_x f\}\right\rangle \alpha_2\,(\gamma_{21}\,\alpha_1)\,(\gamma_{21}\,\alpha_1^2)\,h^8\\
&+\frac{1}{16}\left\langle f_{xxx}\,f^2\,D^2(f)\right\rangle \alpha_2^4\,(\gamma_{21}\,\alpha_1^2)\,h^8\\
&+\frac{1}{16}\left\langle f_{xxx}\,f^2\,\{f_x f\}\right\rangle \alpha_2^4\,(\gamma_{21}\,\alpha_1^2)\,h^8+\frac{1}{4}\left\langle f_{xxx}f\,[D(f)]^2\right\rangle \alpha_2^2\,(\gamma_{21}\,\alpha_1)^2h^8\\
&+\frac{1}{120}\{D(f_x)\,D^5(f)\}\,\alpha_2\,(\gamma_{21}\,\alpha_1^5)h^8+\frac{1}{12}\{D(f_x)\,\{D^3(f_x)f\}\}\,\alpha_2\,(\gamma_{21}\,\alpha_1^5)h^8\\
&+\frac{1}{8}\{D(f_x)\,\left\langle D(f_{xx})f^2\right\rangle\}\,\alpha_2\,(\gamma_{21}\,\alpha_1^5)h^8+\frac{1}{48}\left\langle f_{xx}f\,D^4(f)\right\rangle \alpha_2^2\,(\gamma_{21}\,\alpha_1^4)h^8\\
&+\frac{1}{8}\left\langle f_{xx}f\,\{D^2(f_x)f\}\right\rangle \alpha_2^2\,(\gamma_{21}\,\alpha_1^4)h^8+\frac{1}{16}\left\langle f_{xx}f\,\left\langle f_{xx}f^2\right\rangle\right\rangle \alpha_2^2\,(\gamma_{21}\,\alpha_1^4)h^8\\
&+\frac{1}{6}\left\langle f_{xx}\,D(f)\,D^3(f)\right\rangle (\gamma_{21}\,\alpha_1)\,(\gamma_{21}\,\alpha_1^3)h^8\\
&+\frac{1}{2}\left\langle f_{xx}\,D(f)\,\{D(f_x)f\}\right\rangle (\gamma_{21}\,\alpha_1)\,(\gamma_{21}\,\alpha_1^3)\,h^8
\end{aligned}
\right\}\ \begin{matrix}(15)\\(\text{con.})\end{matrix}
$$

$$
\left.
\begin{aligned}
&+\frac{1}{8}\langle f_{xx}\,[D^2(f)]^2\rangle\,(\gamma_{21}\,\alpha_1^2)^2 h^8+\frac{1}{4}\langle f_{xx}\,D^2(f)\{f_x f\}\rangle\,(\gamma_{21}\,\alpha_1^2)^2 h^8\\
&+\frac{1}{8}\langle f_{xx}\,\{f_x f\}^2\rangle\,(\gamma_{21}\,\alpha_1^2)^2 h^8+\frac{1}{720}\{f_x\,D^6(f)\}\,(\gamma_{21}\,\alpha_1^6)h^8\\
&+\frac{1}{48}\{f_x\,\{D^4(f_x)f\}\}\,(\gamma_{21}\,\alpha_1^6)h^8+\frac{1}{16}\{f_x\,\langle D^2(f_{xx})f^2\rangle\}\,(\gamma_{21}\,\alpha_1^6)h^8\\
&+\frac{1}{48}\{f_x\,\langle f_{xxx}f^3\rangle\}\,(\gamma_{21}\,\alpha_1^6)h^8
\end{aligned}
\right\}\ \text{(15) (con.)}
$$

Next, we list the expansion of f_3. This expansion consists of terms similar to those of (14); these terms are obtained replacing α_1 by α_3 in (14). Furthermore, f_3 contains terms similar to those of (15); these terms are obtained replacing in (15) α_2^{ν} by α_3^{ν} and $(\gamma_{21}\,\alpha_1^{\nu})$ by $(\gamma_{31}\,\alpha_1^{\nu}+\gamma_{32}\,\alpha_2^{\nu})$. Finally, f_3 contains additional terms which do not appear in (14) or (15). We now list these additional terms of f_3:

$$
\left.
\begin{aligned}
f_3 = \ldots &+\{f_x\{f_x\,D(f)\}\}\,\gamma_{32}\,(\gamma_{21}\,\alpha_1)h^5+\{D(f_x)\{f_x\,D(f)\}\}\,\alpha_3\cdot\gamma_{32}\,(\gamma_{21}\,\alpha_1)\,h^6\\
&+\{f_x\,\{D(f_x)\,D(f)\}\}\,\gamma_{32}\,\alpha_2\,(\gamma_{21}\,\alpha_1)h^6+\frac{1}{2}\{f_x\,\{f_x\,D^2(f)\}\}\,\gamma_{32}\,(\gamma_{21}\,\alpha_1^2)h^6\\
&+\frac{1}{2}\{f_x\,\{f_x\{f_x f\}\}\}\,\gamma_{32}(\gamma_{21}\,\alpha_1^2)h^6+\frac{1}{2}\{D^2(f_x)\,\{f_x\,D(f)\}\}\,\alpha_3^2\cdot\gamma_{32}\,(\gamma_{21}\,\alpha_1)h^7\\
&+\{D(f_x)\,\{D(f_x)\,D(f)\}\}\,\alpha_3\cdot\gamma_{32}\,\alpha_2\,(\gamma_{21}\,\alpha_1)h^7\\
&+\frac{1}{2}\{D(f_x)\,\{f_x\,D^2(f)\}\}\,\alpha_3\cdot\gamma_{32}\,(\gamma_{21}\,\alpha_1^2)h^7\\
&+\frac{1}{2}\{D(f_x)\,\{f_x\,\{f_x f\}\}\}\,\alpha_3\cdot\gamma_{32}\,(\gamma_{21}\,\alpha_1^2)\,h^7\\
&+\frac{1}{2}\langle f_{xx}f\,\{f_x D(f)\}\rangle\,\alpha_3^2\cdot\gamma_{32}\,(\gamma_{21}\,\alpha_1)h^7\\
&+\frac{1}{2}\,\{f_x\{D^2(f_x)\,D(f)\}\}\gamma_{32}\,\alpha_2^2(\gamma_{21}\alpha_1)\,h^7+\frac{1}{2}\{f_x\{D(f_x)\,D^2(f)\}\}\,(\gamma_{32}\alpha_2)\,(\gamma_{21}\,\alpha_1^2)\,h^7
\end{aligned}
\right\}\ (16)
$$

$$
\left.
\begin{aligned}
&+\frac{1}{2}\{f_x\{D(f_x)\{f_x f\}\}\}\gamma_{32}\alpha_2(\gamma_{21}\alpha_1^2)h^7+\frac{1}{2}\{f_x\langle f_{xx}f\,D(f)\rangle\}(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1)h^7\\
&+\frac{1}{6}\{f_x\{f_x D^3(f)\}\}\gamma_{32}(\gamma_{21}\alpha_1^3)h^7+\frac{1}{2}\{f_x\{f_x\{D(f_x)f\}\}\}\gamma_{32}(\gamma_{21}\alpha_1^3)h^7\\
&+\frac{1}{6}\{f_x\{D^3(f_x)D(f)\}\}(\gamma_{32}\alpha_2^3)(\gamma_{21}\alpha_1)h^8+\frac{1}{4}\{f_x\{D^2(f_x)D^2(f)\}\}(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1^2)h^8\\
&+\frac{1}{4}\{f_x\{D^2(f_x)\{f_x f\}\}\}(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1^2)h^8+\frac{1}{2}\{f_x\langle D(f_{xx})f\,D(f)\rangle\}(\gamma_{32}\alpha_2^3)(\gamma_{21}\alpha_1)h^8\\
&+\frac{1}{6}\{f_x\{D(f_x)D^3(f)\}\}(\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1^3)h^8+\frac{1}{2}\{f_x\{D(f_x)\{D(f_x)f\}\}\}(\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1^3)h^8\\
&+\frac{1}{4}\{f_x\langle f_{xx}f\,D^2(f)\rangle\}(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1^2)h^8+\frac{1}{4}\{f_x\langle f_{xx}f\{f_x f\}\rangle\}(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1^2)h^8\\
&+\frac{1}{2}\{f_x\langle f_{xx}[D(f)]^2\rangle\}\gamma_{32}(\gamma_{21}\alpha_1)^2h^8+\frac{1}{24}\{f_x\{f_x D^4(f)\}\}\gamma_{32}(\gamma_{21}\alpha_1^4)h^8\\
&+\frac{1}{4}\{f_x\{f_x\{D^2(f_x)f\}\}\}\gamma_{32}(\gamma_{21}\alpha_1^4)h^8+\frac{1}{8}\{f_x\{f_x\langle f_{xx}f^2\rangle\}\}\gamma_{32}(\gamma_{21}\alpha_1^4)h^8\\
&+\langle f_{xx}D(f)\{f_x D(f)\}\rangle\gamma_{32}(\gamma_{31}\alpha_1+\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1)h^8\\
&+\frac{1}{2}\langle f_{xx}f\{D(f_x)D(f)\}\rangle\alpha_3^2(\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1)h^8\\
&+\frac{1}{4}\langle f_{xx}f\{f_x D^2(f)\}\rangle\alpha_3^2\cdot\gamma_{32}(\gamma_{21}\alpha_1^2)h^8+\frac{1}{4}\langle f_{xx}f\{f_x\{f_x f\}\}\rangle\alpha_3^2\cdot\gamma_{32}(\gamma_{21}\alpha_1^2)h^8\\
&+\frac{1}{2}\{D(f_x)\{D^2(f_x)D(f)\}\}\alpha_3(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1)h^8\\
&+\frac{1}{2}\{D(f_x)\{D(f_x)D^2(f)\}\}\alpha_3(\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1^2)h^8\\
&+\frac{1}{2}\{D(f_x)\{D(f_x)\{f_x f\}\}\}\alpha_3(\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1^2)h^8\\
&+\frac{1}{2}\{D(f_x)\langle f_{xx}f\,D(f)\rangle\}\alpha_3(\gamma_{32}\alpha_2^2)(\gamma_{21}\alpha_1)h^8\\
&+\frac{1}{6}\{D(f_x)\{f_x D^3(f)\}\}\alpha_3\cdot\gamma_{32}(\gamma_{21}\alpha_1^3)h^8+\frac{1}{2}\{D(f_x)\{f_x\{D(f_x)f\}\}\}\alpha_3\cdot\gamma_{32}(\gamma_{21}\alpha_1^3)h^8\\
&+\frac{1}{2}\{D^2(f_x)\{D(f_x)D(f)\}\}\alpha_3^2(\gamma_{32}\alpha_2)(\gamma_{21}\alpha_1)h^8+\frac{1}{4}\{D^2(f_x)\{f_x D^2(f)\}\}\alpha_3^2\cdot\gamma_{32}(\gamma_{21}\alpha_1^2)h^8
\end{aligned}
\right\}\quad \text{(16) (con.)}
$$

$$+\frac{1}{4}\{D^2(f_x)\{f_x\{f_x f\}\}\}\ \alpha_3^2\cdot\gamma_{32}\ (\gamma_{21}\ \alpha_1^2)h^8$$

$$+\frac{1}{2}\langle D(f_{xx})f\{f_x\ D(f)\}\rangle\ \alpha_3^3\cdot\gamma_{32}\ (\gamma_{21}\ \alpha_1)h^8 \qquad (16)\ \text{(con.)}$$

$$+\frac{1}{6}\{D^3(f_x)\{f_x\ D(f)\}\}\ \alpha_3^3\cdot\gamma_{32}\ (\gamma_{21}\ \alpha_1)h^8$$

Finally, we list the additional terms for f_4:

$$\begin{aligned} f_4 = \ldots &+ \{f_x\{f_x\{f_x\ D(f)\}\}\}\ \gamma_{43}\cdot\gamma_{32}\ (\gamma_{21}\ \alpha_1)h^7 \\ &+ \{f_x\{D(f_x)\{f_x\ D(f)\}\}\}\ (\gamma_{43}\ \alpha_3)\ \gamma_{32}\ (\gamma_{21}\ \alpha_1)h^8 \\ &+ \{f_x\{f_x\{D(f_x)\ D(f)\}\}\}\ \gamma_{43}\ (\gamma_{32}\ \alpha_2)\ (\gamma_{21}\ \alpha_1)h^8 \\ &+ \frac{1}{2}\{f_x\{f_x\{f_x\ D^2(f)\}\}\}\ \gamma_{43}\cdot\gamma_{32}\ (\gamma_{21}\ \alpha_1^2)h^8 \\ &+ \frac{1}{2}\{f_x\{f_x\{f_x\{f_x f\}\}\}\}\ \gamma_{43}\cdot\gamma_{32}\ (\gamma_{21}\ \alpha_1^2)h^8 \\ &+ \{D(f_x)\{f_x\{f_x\ D(f)\}\}\}\ \alpha_4\ \gamma_{43}\ \gamma_{32}\ (\gamma_{21}\ \alpha_1)\ h^8 \end{aligned} \qquad (17)$$

When proceeding to f_5, ... , no more additional new terms for h^8 or lower powers of h are obtained.

Formulas corresponding to (14), (15), (16), and (17) hold for g_1, g_2, g_3, g_4. In all these formulas, all functions on the right-hand side are to be taken for $t = t_0$.

7. Introducing the expressions for f_1, f_2, ... into (5) and comparing the resulting expansion (5) term by term with the expansion (8), using the values (13) for the total derivatives in (8), we obtain the equations of condition for the Runge-Kutta-Nyström coefficients as listed in Table 1.[1]

1. All tables are at the end of this report.

The left-hand part of Table 1 represents the equations of condition for x, y as obtained from the expansions of (3) and the first (or first two) equations (5); the right-hand part of Table 1 states the equations of condition for $\dot{x}$, $\dot{y}$, as obtained from the expansion of (3) and the last equation (5). For the right-hand part of Table 1, the weight factors c_κ of the table are to be replaced by the weight factors $\dot{c}_\kappa$ of the first derivatives $\dot{x}$, $\dot{y}$. In the marginal columns of Table 1, one finds listed the h-power of the corresponding term in the Taylor expansions for x, y (left-hand side) or $\dot{x}$, $\dot{y}$ (right-hand side). These terms are listed in the same order as in equations (13). Some of the equations of condition of Table 1 are duplicates or multiples of other ones and are marked by an asterisk (*). When we consider the leading term of the local truncation error, these duplicates must also be taken into account.

Later in this paper, we will refer to the equations of Table 1 by a Roman numeral (I through X), indicating the h-power (left or right marginal column in Table 1) to which the equation belongs and by an Arabic numeral indicating its position in the block of the equations of the h-power under consideration. For example, the last equation in Table 1 should be referred to as (X, 72) or, if considered as an equation with the weight factors $\dot{c}_\kappa$, as (IX, 72)$^{\cdot}$.

Equations (X, 1) through (X, 72) of Table 1 are required only as equations (IX, 1)$^{\cdot}$ through (IX, 72)$^{\cdot}$ for the local truncation error in $\dot{x}$, $\dot{y}$.

8. We reduce the equations of condition of Table 1 by omitting the duplicate or multiple equations (*) and by introducing, in addition to (6) and (7), the following assumptions:

$$c_1 = \hat{c}_1 = \dot{c}_1 = 0, \ c_2 = \hat{c}_2 = \dot{c}_2 = 0, \ c_3 = \hat{c}_3 = \dot{c}_3 = 0, \ \alpha_{10} = \alpha_{11} = 1. \qquad (18)$$

$$\left.\begin{aligned}
&\gamma_{21}\,\alpha_1 = \frac{1}{6}\,\alpha_2^3 \\
&\gamma_{31}\,\alpha_1 + \gamma_{32}\,\alpha_2 = \frac{1}{6}\,\alpha_3^3 \\
&\gamma_{41}\,\alpha_1 + \gamma_{42}\,\alpha_2 + \gamma_{43}\,\alpha_3 = \frac{1}{6}\,\alpha_4^3 \\
&\quad\cdot \\
&\quad\cdot \\
&\quad\cdot \\
&\gamma_{111}\,\alpha_1 + \gamma_{112}\,\alpha_2 + \gamma_{113}\,\alpha_3 + \ldots + \gamma_{1110}\,\alpha_{10} = \frac{1}{6}\,\alpha_{11}^3
\end{aligned}\right\} \qquad (19)$$

$$
\left.\begin{aligned}
&\gamma_{21}\,\alpha_1^2 = \frac{1}{12}\,\alpha_2^4 \\
&\gamma_{31}\,\alpha_1^2 + \gamma_{32}\,\alpha_2^2 = \frac{1}{12}\,\alpha_3^4 \\
&\gamma_{41}\,\alpha_1^2 + \gamma_{42}\,\alpha_2^2 + \gamma_{43}\,\alpha_3^2 = \frac{1}{12}\,\alpha_4^4 \\
&\quad\cdot \\
&\quad\cdot \\
&\quad\cdot \\
&\gamma_{111}\,\alpha_1^2 + \gamma_{112}\,\alpha_2^2 + \gamma_{113}\,\alpha_3^2 + \ldots + \gamma_{1110}\,\alpha_{10}^2 = \frac{1}{12}\,\alpha_{11}^4
\end{aligned}\right\} \qquad (20)
$$

$$
\left.\begin{aligned}
&\gamma_{41}\,\alpha_1^3 + \gamma_{42}\,\alpha_2^3 + \gamma_{43}\,\alpha_3^3 = \frac{1}{20}\,\alpha_4^5 \\
&\gamma_{51}\,\alpha_1^3 + \gamma_{52}\,\alpha_2^3 + \gamma_{53}\,\alpha_3^3 + \gamma_{54}\,\alpha_4^3 = \frac{1}{20}\,\alpha_5^5 \\
&\quad\cdot \\
&\quad\cdot \\
&\quad\cdot \\
&\gamma_{111}\,\alpha_1^3 + \gamma_{112}\,\alpha_2^3 + \gamma_{113}\,\alpha_3^3 + \ldots + \gamma_{1110}\,\alpha_{10}^3 = \frac{1}{20}\,\alpha_{11}^5
\end{aligned}\right\} \qquad (21)
$$

$$
\left.\begin{aligned}
&c_4\,\gamma_{41} + c_5\,\gamma_{51} + \ldots + c_9\,\gamma_{91} + c_{10}\begin{Bmatrix}\gamma_{101}\\ \gamma_{111}\end{Bmatrix} = 0 \\
&\dot{c}_4\,\gamma_{41} + \dot{c}_5\,\gamma_{51} + \ldots + \dot{c}_9\,\gamma_{91} + \dot{c}_{10}\,\gamma_{101} = 0
\end{aligned}\right\} \qquad (22)
$$

$$
\left.\begin{aligned}
&c_4\,\alpha_4\,\gamma_{41} + c_5\,\alpha_5\,\gamma_{51} + \ldots + c_9\,\alpha_9\,\gamma_{91} + c_{10}\begin{Bmatrix}\gamma_{101}\\ \gamma_{111}\end{Bmatrix} = 0 \\
&\dot{c}_4\,\alpha_4\,\gamma_{41} + \dot{c}_5\,\alpha_5\,\gamma_{51} + \ldots + \dot{c}_9\,\alpha_9\,\gamma_{91} + \dot{c}_{10}\,\gamma_{101} = 0
\end{aligned}\right\} \qquad (23)
$$

$$
\left.\begin{aligned}
&c_4\,\alpha_4^2\,\gamma_{41} + c_5\,\alpha_5^2\,\gamma_{51} + \ldots + c_9\,\alpha_9^2\,\gamma_{91} + c_{10}\,\gamma_{111} = 0 \\
&\dot{c}_4\,\alpha_4^2\,\gamma_{41} + \dot{c}_5\,\alpha_5^2\,\gamma_{51} + \ldots + \dot{c}_9\,\alpha_9^2\,\gamma_{91} + \dot{c}_{10}\,\gamma_{101} = 0
\end{aligned}\right\} \qquad (24)
$$

$$\left.\begin{aligned} c_4\,\gamma_{42} + c_5\,\gamma_{52} + \ldots + c_9\,\gamma_{92} + c_{10}\,\gamma_{112} &= 0 \\ \dot{c}_4\,\gamma_{42} + \dot{c}_5\,\gamma_{52} + \ldots + \dot{c}_9\,\gamma_{92} + \dot{c}_{10}\,\gamma_{102} &= 0 \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} c_4\,\gamma_{43} + c_5\,\gamma_{53} + \ldots + c_9\,\gamma_{93} + c_{10}\,\gamma_{113} &= 0 \\ \dot{c}_4\,\gamma_{43} + \dot{c}_5\,\gamma_{53} + \ldots + \dot{c}_9\,\gamma_{93} + \dot{c}_{10}\,\gamma_{103} &= 0 \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} & c_5\,\gamma_{54}\,\gamma_{41} + c_6\,(\gamma_{64}\,\gamma_{41} + \gamma_{65}\,\gamma_{51}) + \ldots + c_9\,(\gamma_{94}\,\gamma_{41} + \ldots + \gamma_{98}\,\gamma_{81}) \\ & \qquad + c_{10}\,(\gamma_{114}\,\gamma_{41} + \ldots + \gamma_{1110}\,\gamma_{101}) = 0 \\ & \dot{c}_5\,\gamma_{54}\,\gamma_{41} + \dot{c}_6\,(\gamma_{64}\,\gamma_{41} + \gamma_{65}\,\gamma_{51}) + \ldots + \dot{c}_9\,(\gamma_{94}\,\gamma_{41} + \ldots + \gamma_{98}\,\gamma_{81}) \\ & \qquad + \dot{c}_{10}\,(\gamma_{104}\,\gamma_{41} + \ldots + \gamma_{109}\,\gamma_{91}) = 0 \end{aligned}\right\} \quad (27)$$

In the first equations (22) and (23), the upper line (γ_{101}) holds for the eighth-order formula and the lower line (γ_{111}) for the ninth-order formula. Introducing these assumptions, we convert the necessary and sufficient equations of condition of Table 1 into a system of sufficient equations of condition that can be solved in a relatively easy manner. In fact, these assumptions transform the equations of Table 1 into several separate systems of linear equations for the coefficients $\gamma_{\kappa\lambda}$.

Let us now consider the reductions of the equations of Table 1 effected by the above assumptions. Assumptions (19) — together with the first equation (18) — lead to the following identities in Table 1:

$$\left.\begin{aligned} & (V,\ 3) \equiv (V,\ 1),\ (VI,\ 4) \equiv (VI,\ 1),\ (VII,\ 4) \equiv (VII,\ 1) \\ & (VIII,\ 4) \equiv (VIII,\ 1),\ (VIII,\ 9) \equiv (VIII,\ 1),\ (IX,\ 4) \equiv (IX,\ 1) \\ & (IX,\ 10) \equiv (IX,\ 1),\ (X, 4) \equiv (X, 1),\ (X, 11) \equiv (X, 1) \\ & (X, 35) \equiv (X, 16),\ (X, 61) \equiv (X, 56) \quad . \end{aligned}\right\} \quad (28)$$

Because of the identities (28) the equations (V, 3), (VI, 4), (VII, 4), (VIII, 4), (VIII, 9), (IX, 4), (IX, 10), (X, 4), (X, 11), (X,35), (X,61), and the corresponding equations in $\dot{c}_\kappa$ can be omitted from Table 1.

Assumptions (20) — together with (19) and the first equation (18) — yield the following identities in Table 1:

$$\left.\begin{array}{l}(\mathrm{VI},\ 5) \equiv (\mathrm{VI},\ 1),\ (\mathrm{VII},\ 6) \equiv (\mathrm{VII},\ 1),\ (\mathrm{VIII},\ 7) \equiv (\mathrm{VIII},\ 1),\\ (\mathrm{VIII},\ 19) \equiv (\mathrm{VIII},\ 18),\ (\mathrm{IX},\ 7) \equiv (\mathrm{IX},\ 1),\ (\mathrm{IX},\ 16) \equiv (\mathrm{IX},\ 1)\\ (\mathrm{IX},\ 25) \equiv (\mathrm{IX},\ 24),\ (\mathrm{IX},32) \equiv (\mathrm{IX},\ 30),\ (\mathrm{X},\ 7) \equiv (\mathrm{X},1),\\ (\mathrm{X},\ 20) \equiv (\mathrm{X},\ 1),\ (\mathrm{X},\ 29) \equiv (\mathrm{X},\ 28),\ (\mathrm{X},\ 33) \equiv (\mathrm{X},\ 1)\\ (\mathrm{X},\ 48) \equiv (\mathrm{X},\ 47),\ (\mathrm{X},\ 59) \equiv (\mathrm{X},\ 58),\ (\mathrm{X},\ 71) \equiv (\mathrm{X},\ 70)\ ,\end{array}\right\} \quad (29)$$

eliminating equations (VI, 5), (VII, 6), (VIII, 7), (VIII, 19), (IX, 7), (IX, 16), (IX, 25), (IX, 32), (X, 7), (X, 20), (X, 29), (X, 33), (X, 48), (X, 59), and (X, 71) from Table 1.

Assumptions (21), together with (19) and the first equation (18), lead to the following identities:

$$\left.\begin{array}{l}(\mathrm{VII},\ 8) \equiv (\mathrm{VII},\ 1),\ (\mathrm{VIII},\ 12) \equiv (\mathrm{VIII},\ 1),\ (\mathrm{IX},\ 13) \equiv (\mathrm{IX},\ 1),\\ (\mathrm{X},\ 14) \equiv (\mathrm{X},1),\ (\mathrm{X},\ 31) \equiv (\mathrm{X},\ 1),\end{array}\right\} \quad (30)$$

thereby eliminating equations (VII, 8), (VIII, 12), (IX, 13), (X, 14) and (X, 31) from Table 1.

Because of (7) and the first equations (18) and (22):

$$\gamma_{111} = \gamma_{101} = 0 \quad (31)$$

Similarly because of (7) and (18):

$$\gamma_{112} = 0,\ \gamma_{113} = 0\ . \quad (32)$$

Therefore, we can omit the last term on the left-hand side of (22), (23), and (24) and of the first equations (25) and (26). We can also omit the last term on the left-hand side of the first equation (27) since (7) and the first equation (22) hold.

From (22) and the previous assumptions, the following identities result:

$$\left.\begin{array}{l}(\mathrm{VII},\ 10) \equiv (\mathrm{VII},\ 8),\ (\mathrm{VIII},\ 18) \equiv (\mathrm{VIII},\ 16), \\ (\mathrm{IX},\ 30) \equiv (\mathrm{IX},\ 27),\ (\mathrm{X},\ 56) \equiv (\mathrm{X},\ 53)\quad ,\end{array}\right\} \tag{33}$$

eliminating equations (VII, 10), (VIII, 18), (IX, 30), and (X, 56), from Table 1.

Assumption (23) and previous assumptions lead to:

$$(\mathrm{VIII},\ 14) \equiv (\mathrm{VIII},\ 12),\ (\mathrm{IX},\ 24) \equiv (\mathrm{IX},\ 21),\ (\mathrm{X},\ 46) \equiv (\mathrm{X},\ 43), \tag{34}$$

eliminating equations (VIII, 14), (IX, 24), and (X, 46) from Table 1.

By (24) and previous assumptions equations (IX, 15) and (X, 28) are eliminated from Table 1 since they then become identical with (IX, 1) and (X, 25), respectively.

Finally, assumptions (25) and (26), together with previous assumptions, lead to the identity:

$$(\mathrm{IX},\ 34) \equiv (\mathrm{IX},\ 27),\ (\mathrm{X},\ 64) \equiv (\mathrm{X},\ 53), \tag{35}$$

and assumption (27), together with previous assumptions, to:

$$(\mathrm{IX},\ 36) \equiv (\mathrm{IX},\ 27),\ (\mathrm{X},\ 66) \equiv (\mathrm{X},\ 53),\ (\mathrm{X},\ 70) \equiv (\mathrm{X},\ 67), \tag{36}$$

thus eliminating equations (IX, 34) (IX, 36), (X, 64), (X, 66), and (X, 70) from Table 1.

Omitting in Table 1 all equations which are duplicates or multiples of other ones or which can be eliminated by using the above assumptions, the table reduces to the following equations:(II, 1), (III, 1), (IV, 1), (V, 1), (VI, 1), (VII, 1), (VIII, 1), (IX, 1), (VIII, 15), (IX, 21), (IX, 27), which are listed in this order in Table 2. To write the last three equations in a more concise form, we introduced in Table 2 the abbreviation:

$$\gamma_{\mu 1}\,\alpha_1^{\nu} + \gamma_{\mu 2}\,\alpha_2^{\nu} + \ldots + \gamma_{\mu,\mu-1}\,\alpha_{\mu-1}^{\nu} = P_{\mu\nu}\,. \tag{37}$$

The remaining equations in Table 1 which belong to ninth-order terms for $\dot{x}$ will be considered later when dealing with the local truncation errors in x and $\dot{x}$.

In the ninth equation of Table 2, the upper line (P_{104}) holds for the eighth-order formula and the lower line ($\frac{1}{30}$) for the ninth-order formula. From the two lines of this equation, it follows that

$$P_{104} = \frac{1}{30}$$

must hold.

The equations of Table 2 have to be solved together with the assumptions (19) through (27).

9. The first eight equations of Table 2 do not contain the coefficients $\gamma_{\kappa\lambda}$. They represent linear equations for the weight factors c_κ or $\dot{c}_\kappa$. By solving these equations, we can express the weight factors by the α_κ's.

Next, we have to solve for the coefficients $\gamma_{\kappa\lambda}$. Obviously, equations (22), (23), (24), and (27) are satisfied by

$$\gamma_{41} = \gamma_{51} = \gamma_{61} = \gamma_{71} = \gamma_{81} = \gamma_{91} = \gamma_{101} = \gamma_{111} = 0. \tag{38}$$

The first equation (19) together with the first equation (20) leads to a restrictive condition for α_1:

$$\alpha_1 = \frac{1}{2}\,\alpha_2. \tag{39}$$

Similarly, the third equation (19), together with the third equation (20) and the first equation (21), yields because of $\gamma_{41} = 0$, the following restrictive condition for α_2:

$$\alpha_2 = \frac{1}{5}\,\alpha_4\,\frac{5\alpha_3 - 3\alpha_4}{2\alpha_3 - \alpha_4}\,. \tag{40}$$

The remaining α_κ's may be chosen arbitrarily. We tried to select these α_κ-values in such a way that the leading term of the truncation error becomes small for our Runge-Kutta-Nyström formula.

The remaining coefficients $\gamma_{\kappa\lambda}$ can easily be determined as follows: The first equation (19) yields γ_{21}; the second equation (19) and the second equation (20) determine γ_{31} and γ_{32}. Since $\gamma_{41} = 0$, from the third equation (19) and the third equation (20) the coefficients γ_{42} and γ_{43} can be obtained. Because of $\gamma_{51} = 0$ we can obtain the coefficients γ_{52}, γ_{53}, and γ_{54} from the fourth equation (19), the fourth equation (20), and the second equation (21).

From the ninth and the tenth equations of Table 2, we can determine P_{64}, P_{74}, P_{84}, and P_{94}, since these equations are four linear equations for these quantities (P_{44}, P_{54} being known already).

The fifth equation (19), the fifth equation (20), the third equation (21), and P_{64}, together with $\gamma_{61} = 0$, yield the coefficients γ_{62}, γ_{63}, γ_{64}, and γ_{65}.

Since we have more coefficients $\gamma_{\kappa\lambda}$ than equations of condition, we may set some of these coefficients equal to zero:

$$\gamma_{72} = \gamma_{82} = \gamma_{83} = \gamma_{104} = 0. \qquad (41)$$

The coefficients γ_{73}, γ_{74}, γ_{75}, γ_{76} can then be obtained from the sixth equation (19), the sixth equation (20), the fourth equation (21), and P_{74}. Similarly, γ_{84}, γ_{85}, γ_{86}, and γ_{87} are determined by the seventh equation (19), the seventh equation (20), the fifth equation (21), and P_{84}.

Next, we use the first equation (25) and the first equation (26) to find γ_{92} and γ_{93}. The first line of the last equation in Table 2 yields P_{95}. This value P_{95}, together with P_{94}, the eighth equation (19), the eighth equation (20), and the sixth equation (21), determines the coefficients γ_{94}, γ_{95}, γ_{96}, γ_{97}, and γ_{98}. Similarly, we use the second equations (25) and (26) to compute γ_{102} and γ_{103}. From the second line of the last equation in Table 2 we find P_{105}. Since $\gamma_{104} = 0$, we can determine the remaining coefficients γ_{105}, γ_{106}, γ_{107}, γ_{108}, and γ_{109} from P_{105}, $P_{104}\,(= \frac{1}{30})$, the ninth equation (19), the ninth equation (20), and the seventh equation (21).

This concludes the computation of the coefficients $\gamma_{\kappa\lambda}$ since $\gamma_{11\lambda}$ ($\lambda = 0, 1, 2, \ldots, 10$) is given by (7), and $\gamma_{\kappa 0}$ ($\kappa = 1, 2, \ldots, 11$) can be determined from (4).

10. Table 3 lists the coefficients of an eighth-order Runge-Kutta-Nyström formula RKN 8(9). The expression TE_x in Table 3 represents the leading term of the local truncation error in x for our eighth-order formula and is obtained as difference between the first and the second formula (5).

In Table 4, the error coefficients for the leading truncation error term and x and $\dot{x}$ are listed. If the error coefficients differ by a constant numerical factor only, the coefficient with the largest factor is listed. The product of the error coefficient and the corresponding expression in the partial derivatives, as listed in (13), times h^9 is the actual contribution to the leading truncation error term.

Because of $\mathbf{P}_{104} = \frac{1}{30}$, the tenth equation of Table 2 holds also for the eighth-order formula RKN 8, so that this equation does not contribute to the leading truncation error term in x. The only contributing equation of Table 2 is the last one [equation (IX, 27) of Table 1]. Replacing equation (IX, 27) by equation (IX, 29), because of the larger factor of the latter one, the error coefficient that determines the contribution to the leading truncation error term in x reads:

$$T_{29} = \frac{1}{8}\left(c_4 P_{45} + c_5 P_{55} + \ldots + c_9 P_{95} + c_{10} P_{105}\right) - \frac{15}{362\ 880} \quad . \qquad (42)$$

In the case of $\dot{x}$, there are seven essentially different error coefficients that contribute to the leading truncation error term: $\dot{T}_{10}$, $\dot{T}_{26}$, $\dot{T}_{45}$, $\dot{T}_{51}$, $\dot{T}_{52}$, $\dot{T}_{58}$, and $\dot{T}_{68}$.

It is essential that the error coefficients in $\dot{x}$ are of about the same order of magnitude as the error coefficients in x. If the error coefficients in $\dot{x}$ are large compared with the error coefficients in x, large truncation errors in $\dot{x}$ might be generated since in our method the integration step-size is determined by the local truncation error in x (by TE_x).

SECTION II. SEVENTH-ORDER FORMULA RKN 7(8)

11. The derivation of a seventh-order Runge-Kutta-Nyström formula RKN 7(8) closely follows the same pattern as in the case of our eighth-order formula RKN 8(9) of Section I. Naturally, a seventh-order formula requires fewer evaluations of the differential equations per step than an eighth-order formula. We shall present in the following a seventh-order formula based on nine evaluations per step, a tenth evaluation being taken over as first evaluation for the next step.

In the case of a seventh-order formula, equations (3), (5), and (6) must then be replaced by:

$$\left.\begin{aligned} f_0 &= f(t_0,\ x_0,\ y_0) \\ f_\kappa &= f\left(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0\alpha_\kappa h + h^2 . \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} f_\lambda,\right. \\ &\qquad\qquad \left. y_0 + \dot{y}_0\, \alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda}\, g_\lambda \right) \\ &\qquad\qquad (\kappa = 1,\ 2,\ \ldots,\ 9) \end{aligned}\right\} \tag{43}$$

$$\left.\begin{aligned} x &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{8} c_\kappa f_\kappa + 0(h^8) \\ \hat{x} &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{9} \hat{c}_\kappa f_\kappa + 0(h^9) \end{aligned}\right\} \tag{44}$$

$$\left.\begin{aligned} \dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{8} \dot{c}_\kappa f_\kappa + 0(h^8) \\ \hat{c}_\kappa &= c_\kappa \text{ for } \kappa = 0,\ 1,\ 2,\ \ldots,\ 7 \\ \hat{c}_8 &= 0 \\ \hat{c}_9 &= c_8 \end{aligned}\right\} \tag{45}$$

In Table 1 the last 72 equations of condition now have to be omitted, since they correspond to tenth-order terms in the Taylor expansion for x or to ninth-order terms in the expansion for $\dot{x}$.

Instead of equations (18) through (27), we now make the following assumptions:

$$c_1 = \hat{c}_1 = \dot{c}_1 = 0, \quad c_2 = \hat{c}_2 = \dot{c}_2 = 0, \quad \alpha_8 = \alpha_9 = 1. \tag{46}$$

$$\left.\begin{aligned} &\gamma_{21}\,\alpha_1 = \frac{1}{6}\alpha_2^3 \\ &\gamma_{31}\,\alpha_1 + \gamma_{32}\,\alpha_2 = \frac{1}{6}\,\alpha_3^3 \\ &\quad\vdots \\ &\gamma_{91}\,\alpha_1 + \gamma_{92}\,\alpha_2 + \ldots + \gamma_{98}\,\alpha_8 = \frac{1}{6}\,\alpha_9^3 \end{aligned}\right\}. \tag{47}$$

$$\left.\begin{aligned} &\gamma_{31}\,\alpha_1^2 + \gamma_{32}\,\alpha_2^2 = \frac{1}{12}\,\alpha_3^4 \\ &\gamma_{41}\,\alpha_1^2 + \gamma_{42}\,\alpha_2^2 + \gamma_{43}\,\alpha_3^2 = \frac{1}{12}\,\alpha_4^4 \\ &\quad\vdots \\ &\gamma_{91}\,\alpha_1^2 + \gamma_{92}\,\alpha_2^2 + \ldots + \gamma_{98}\,\alpha_8^2 = \frac{1}{12}\,\alpha_9^4 \end{aligned}\right\}. \tag{48}$$

$$\left.\begin{aligned} &\gamma_{31}\,\alpha_1^3 + \gamma_{32}\,\alpha_2^3 = \frac{1}{20}\,\alpha_3^5 \\ &\gamma_{41}\,\alpha_1^3 + \gamma_{42}\,\alpha_2^3 + \gamma_{43}\,\alpha_3^3 = \frac{1}{20}\,\alpha_4^5 \\ &\quad\vdots \\ &\gamma_{91}\,\alpha_1^3 + \gamma_{92}\,\alpha_2^3 + \ldots + \gamma_{98}\,\alpha_8^3 = \frac{1}{20}\,\alpha_9^5 \end{aligned}\right\}. \tag{49}$$

$$\left.\begin{aligned} &c_3\,\gamma_{31} + c_4\,\gamma_{41} + \ldots + c_7\,\gamma_{71} + c_8 \begin{Bmatrix} \gamma_{81} \\ \gamma_{91} \end{Bmatrix} = 0 \\ &\dot{c}_3\,\gamma_{31} + \dot{c}_4\,\gamma_{41} + \ldots + \dot{c}_7\,\gamma_{71} + \dot{c}_8\,\gamma_{81} = 0 \end{aligned}\right\} \quad . \tag{50}$$

$$\left.\begin{aligned} &c_3\,\alpha_3\,\gamma_{31} + c_4\,\alpha_4\,\gamma_{41} + \ldots + c_7\,\alpha_7\,\gamma_{71} + c_8\,\gamma_{91} = 0 \\ &\dot{c}_3\,\alpha_3\,\gamma_{31} + \dot{c}_4\,\alpha_4\,\gamma_{41} + \ldots + \dot{c}_7\,\alpha_7\,\gamma_{71} + \dot{c}_8\,\gamma_{81} = 0 \end{aligned}\right\} \quad . \tag{51}$$

$$\left.\begin{aligned} &c_3\,\gamma_{32} + c_4\,\gamma_{42} + \ldots + c_7\,\gamma_{72} + c_8\,\gamma_{92} = 0 \\ &\dot{c}_3\,\gamma_{32} + \dot{c}_4\,\gamma_{42} + \ldots + \dot{c}_7\,\gamma_{72} + \dot{c}_8\,\gamma_{82} = 0 \end{aligned}\right\} \quad . \tag{52}$$

Because of $\gamma_{91} = c_1 = 0$, we obtain from the first equation (50):

$$\gamma_{81} = 0 \quad . \tag{53}$$

The assumptions (46) through (52) reduce the equations of conditions of Table 1 to those of Table 5.

12. We now have to solve the equations of Table 5 together with the assumptions (47) through (52): The first seven equations of Table 5 allow us to express the weight factors by the α's. From the first equation (47) we obtain γ_{21}. The second equation (47), the first equation (48), and the first equation (49) lead to a restrictive condition for the α's:

$$\alpha_1 = \frac{1}{5}\,\alpha_3\,\frac{5\alpha_2 - 3\alpha_3}{2\alpha_2 - \alpha_3} \tag{54}$$

and to γ_{31}, γ_{32}. The third equation (47), the second equation (48), and the second equation (49) yield γ_{41}, γ_{42}, γ_{43}.

On the other hand, equations (50) and (51) allow us to express γ_{41}, γ_{51}, γ_{61}, and γ_{71} by γ_{31}, the α's and the weight factors. Therefore, we have two different expressions for γ_{41}. Equating them leads to another restrictive condition for the α's:

$$\alpha_2 = \frac{1}{5}\;\frac{2\alpha_3^2\,\dfrac{\gamma_{41}}{\gamma_{31}} - \left(\dfrac{\alpha_4}{\alpha_3}\right)^3 \alpha_4\,(5\alpha_3 - 3\alpha_4)}{\alpha_3 \cdot \dfrac{\gamma_{41}}{\gamma_{31}} - \left(\dfrac{\alpha_4}{\alpha_3}\right)^3 (2\alpha_3 - \alpha_4)} \quad . \tag{55}$$

Since γ_{51} is already known, the coefficients γ_{52}, γ_{53}, γ_{54} are obtained from the fourth equation (47), the third equation (48), and the third equation (49).

Putting $\gamma_{62} = 0$, we obtain, since γ_{61} is already known, γ_{63}, γ_{64}, γ_{65} from the fifth equation (47), the fourth equation (48), and the fourth equation (49).

Since $\gamma_{92} = c_2 = 0$, the first equation (52) yields γ_{72}. Since γ_{71} is known, we can obtain γ_{73}, γ_{74}, γ_{75}, γ_{76} from the sixth equation (47), the fifth equation (48), the fifth equation (49), and the last equation of Table 5 (written as equation with weight factors c_ν).

Similarly, the second equation (52) yields γ_{82}. Setting $\gamma_{83} = 0$, we obtain γ_{84}, γ_{85}, γ_{86}, γ_{87} from the seventh equation (47), the sixth equation (48), the sixth equation (49), and the last equation of Table 5 (written as equation with weight factors $\dot{c}_\nu$).

This concludes the computation of the coefficients $\gamma_{\kappa\lambda}$ since $\gamma_{9\lambda}$ ($\lambda = 0, 1, \ldots, 8$) are equal to the weight factors c_λ ($\lambda = 0, 1, \ldots, 8$) and $\gamma_{\kappa 0}$ ($\kappa = 1, 2, \ldots, 9$) can be determined from (4).

13. Table 6 lists the coefficients of a seventh-order Runge-Kutta-Nyström formula RKN 7 (8). The restrictive condition (55) makes it somewhat harder than in the case of the formula RKN 8 (9) to find reasonably simple α-values. This explains the somewhat unwieldy coefficients $\gamma_{\kappa\lambda}$ of Table 6.

We notice in Table 6 that α_2 and $\dot{c}_4$ are negative. It is possible to obtain positive values for all α's and all weight factors by a different choice of α_3 $\left(\text{e.g.}, \alpha_3 = \frac{1}{12}\right)$. However, the coefficients $\gamma_{\kappa\lambda}$ then turn out to be even more formidable.

In Table 7 the error coefficients of the formula of Table 6 are listed in the same way as we have listed in Table 4 the error coefficients of the formula of Table 3.

SECTION III. SIXTH-ORDER FORMULA RKN 6(7)

14. For the derivation of a sixth-order Runge-Kutta-Nyström formula RKN 6(7), we proceed in a quite similar way as in Sections I and II. In the case of a sixth-order formula RKN 6(7) we have to consider only the first 43 equations of Table 1. We base our sixth-order formulas on seven evaluations of the differential equations per step and allow for an eighth evaluation that will be taken over as first evaluation for the next step. We then have:

$$\left.\begin{aligned} f_0 &= f\,(t_0,\ x_0,\ y_0) \\ f_\kappa &= f\Bigg(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\lambda=0}^{\kappa=1} \gamma_{\kappa\lambda} f_\lambda\ , \\ &\qquad\qquad y_0 + \dot{y}_0 h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda}\, g_\lambda\Bigg) \\ &\qquad\qquad (\kappa = 1,\ 2,\ \ldots,\ 7) \end{aligned}\right\} \qquad (56)$$

,

$$\left.\begin{aligned} x &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{6} c_\kappa f_\kappa + 0\,(h^7) \\ \hat{x} &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{7} \hat{c}_\kappa f_\kappa + 0\,(h^8) \end{aligned}\right.$$

and

$$\left.\begin{aligned} \dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{6} \dot{c}_\kappa f_\kappa + 0\,(h^7) \end{aligned}\right\} \qquad (57)$$

,

and

$$\left.\begin{aligned} \hat{c}_\kappa &= c_\kappa \text{ for } \kappa=0,\ 1,\ 2,\ \ldots,\ 5 \\ \hat{c}_6 &= 0 \\ \hat{c}_7 &= c_6 \end{aligned}\right\} \qquad (58)$$

.

In the case of a sixth-order formula RKN 6(7), we make the following assumptions:

$$c_1 = \hat{c}_1 = \dot{c}_1 = 0, \quad \alpha_6 = \alpha_7 = 1. \tag{59}$$

$$\left.\begin{aligned} &\gamma_{21}\,\alpha_1 = \frac{1}{6}\,\alpha_2^3 \\ &\gamma_{31}\,\alpha_1 + \gamma_{32}\,\alpha_2 = \frac{1}{6}\,\alpha_3^3 \\ &\vdots \\ &\gamma_{71}\,\alpha_1 + \gamma_{72}\,\alpha_2 + \ldots + \gamma_{76}\,\alpha_6 = \frac{1}{6}\,\alpha_7^3 \end{aligned}\right\}, \tag{60}$$

$$\left.\begin{aligned} &\gamma_{21}\,\alpha_1^2 = \frac{1}{12}\,\alpha_2^4 \\ &\gamma_{31}\,\alpha_1^2 + \gamma_{32}\,\alpha_2^2 = \frac{1}{12}\,\alpha_3^4 \\ &\vdots \\ &\gamma_{71}\,\alpha_1^2 + \gamma_{72}\,\alpha_2^2 + \ldots + \gamma_{76}\,\alpha_6^2 = \frac{1}{12}\,\alpha_7^4 \end{aligned}\right\}, \tag{61}$$

$$\left.\begin{aligned} &c_2\,\gamma_{21} + c_3\,\gamma_{31} + c_4\,\gamma_{41} + c_5\,\gamma_{51} + c_6\,\gamma_{71} = 0 \\ &\dot{c}_2\,\gamma_{21} + \dot{c}_3\,\gamma_{31} + \dot{c}_4\,\gamma_{41} + \dot{c}_5\,\gamma_{51} + \dot{c}_6\,\gamma_{61} = 0 \end{aligned}\right\}. \tag{62}$$

Obviously, these assumptions reduce the equations of Table 1 to those of Table 8.

15. We now solve the equations of Table 8 together with the assumptions (60), (61), and (62). The first six equations of Table 8 yield the weight factors as functions of the α's. From the first equations (60) and (61), we obtain the restrictive condition:

$$\alpha_1 = \frac{1}{2}\,\alpha_2 \tag{63}$$

and the coefficient γ_{21}. The second equations (60) and (61) yield γ_{31} and γ_{32}. Setting $\gamma_{41} = 0$, we obtain γ_{42}, γ_{43} from the third equations (60) and (61). Since $\gamma_{71} = c_1 = 0$, the first equation (62) yield γ_{51}. The coefficients γ_{52}, γ_{53}, γ_{54} are then obtained from the fourth equations (60), (61) and the last equation of Table 8, written as equation with weight factors c_ν.

The coefficient γ_{61} is determined by the second equation (62). Setting $\gamma_{62} = 0$, the coefficients γ_{63}, γ_{64}, γ_{65} can be computed from the fifth equations (60) and (61) and from the last equation of Table 8, written as equation with weight factors $\dot{c}_\nu$.

This concludes the computation of the coefficients $\gamma_{\kappa\lambda}$ since $\gamma_{7\lambda}$ $(\lambda = 0, 1, \ldots, 6)$ are equal to the weight factors c_λ $(\lambda = 0, 1, \ldots, 6)$ and $\gamma_{\kappa 0}$ $(\kappa = 1, 2, \ldots 7)$ can be determined from (4).

16. In Table 9 the coefficients of a sixth-order Runge-Kutta-Nyström formula RKN 6(7) are presented. In Table 10 the error coefficients for the leading truncation error term in x and $\dot{x}$ are listed. Again, if the error coefficients differ by a constant numerical factor only, the coefficient with the largest factor is listed. As Table 10 shows, the error coefficients in $\dot{x}$ are of about the same order of magnitude as the error coefficient in x.

SECTION IV. FIFTH-ORDER FORMULA RKN 5(6)

17. To derive a fifth-order Runge-Kutta-Nyström formula RKN 5(6), we proceed in a quite similar way as in Sections I to III. We base our fifth-order formula on six evaluations of the differential equations per step and allow for a seventh evaluation that will be taken over as first evaluation for the next step. We then have.

$$\left.\begin{aligned} f_0 &= f(t_0, x_0, y_0) \\ f_\kappa &= f\left(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0 \alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} f_\lambda,\right. \\ &\qquad \left. y_0 + \dot{y}_0 \alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1} \gamma_{\kappa\lambda} g_\lambda \right) \\ &\qquad (\kappa = 1, 2, \ldots, 6) \end{aligned}\right\} . \qquad (64)$$

$$
\left.
\begin{aligned}
x &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{5} c_\kappa f_\kappa + 0(h^6) \\
\hat{x} &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{6} \hat{c}_\kappa f_\kappa + 0(h^7) \\
\dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{5} \dot{c}_\kappa f_\kappa + 0(h^6)
\end{aligned}
\right\} \quad . \tag{65}
$$

$$
\left.
\begin{aligned}
&\hat{c}_\kappa = c_\kappa \text{ for } \kappa = 0, 1, 2, 3, 4 \\
&\hat{c}_5 = 0 \\
&\hat{c}_6 = c_5
\end{aligned}
\right\} \quad . \tag{66}
$$

We make the following assumptions:

$$
c_1 = \hat{c}_1 = \dot{c}_1 = 0 \ , \quad \alpha_5 = \alpha_6 = 1 \quad , \tag{67}
$$

$$
\left.
\begin{aligned}
&\gamma_{21}\alpha_1 = \frac{1}{6}\alpha_2^3 \\
&\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2 = \frac{1}{6}\alpha_3^3 \\
&\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3 = \frac{1}{6}\alpha_4^3 \\
&\gamma_{51}\alpha_1 + \gamma_{52}\alpha_2 + \gamma_{53}\alpha_3 + \gamma_{54}\alpha_4 = \frac{1}{6}\alpha_5^3 \\
&\gamma_{61}\alpha_1 + \gamma_{62}\alpha_2 + \gamma_{63}\alpha_3 + \gamma_{64}\alpha_4 + \gamma_{65}\alpha_5 = \frac{1}{6}\alpha_6^3
\end{aligned}
\right\} \quad , \tag{68}
$$

$$
\gamma_{31} = 0 \ , \quad \gamma_{41} = 0 \ , \quad \gamma_{51} = 0 \ , \quad \gamma_{61} = 0 \quad , \tag{69}
$$

which reduce the equations of condition of Table 1 to the equations of Table 11.

18. The first five equations of Table 11 yield the weight factors c_κ or $\dot{c}_\kappa$ as functions of the α_κ's.

From the first two equations (68) we obtain γ_{21} and γ_{32}. From the third equation (68) and from the last equation of Table 11, this equation written with weight factors c_κ, the coefficients γ_{42} and γ_{43} can be determined. Using the fourth equation (68) and the last equation of Table 11, written as equation with weight factors $\dot{c}_\kappa$, the coefficients γ_{53} and γ_{54} can be expressed by the α_κ's and by γ_{52}. By still having γ_{52} available, we could select γ_{52} in such a way that some of the error coefficients of our fifth-order Runge-Kutta-Nyström formula become small. By doing so, we obtained error coefficients, some of which were considerably smaller than those obtained from setting $\gamma_{52} = 0$. The coefficients $\gamma_{6\lambda}$ of the last equation (68) are naturally equal to the weight factors c_λ, since it was intended to take over the seventh evaluation as first evaluation for the next step.

19. In Table 12 are listed the coefficients of a fifth-order Runge-Kutta-Nyström formula RKN 5(6), and in Table 13 the error coefficients in x as well as in $\dot{x}$ for the formula of Table 12. Table 13 shows that the error coefficients in $\dot{x}$ are of the same order of magnitude or smaller than the error coefficient in x.

SECTION V. FOURTH-ORDER FORMULA RKN 4(5)

20. A fourth-order Runge-Kutta-Nyström formula RKN 4(5) can be based on four evaluations per step if we allow for a fifth evaluation that will be taken over as first evaluation for the next step:

$$\left.\begin{aligned} f_0 &= f(t_0, x_0, y_0) \\ f_\kappa &= f\Big(t_0 + \alpha_\kappa h,\ x_0 + \dot{x}_0 \alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1}{}' \gamma_{\kappa\lambda} f_\kappa, \\ &\qquad y_0 + \dot{y}_0 \alpha_\kappa h + h^2 \cdot \sum_{\lambda=0}^{\kappa-1}{}' \gamma_{\kappa\lambda} g_\kappa \Big) \\ &\qquad (\kappa = 1, 2, 3, 4) \end{aligned}\right\} \tag{70}$$

$$\left.\begin{aligned} x &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{3} c_\kappa f_\kappa + 0(h^5) \\ \hat{x} &= x_0 + \dot{x}_0 h + h^2 \cdot \sum_{\kappa=0}^{4} \hat{c}_\kappa f_\kappa + 0(h^6) \\ \dot{x} &= \dot{x}_0 + h \cdot \sum_{\kappa=0}^{3} \dot{c}_\kappa f_\kappa + 0(h^5) \end{aligned}\right\} \quad . \tag{71}$$

$$\left.\begin{aligned} \hat{c}_\kappa &= c_\kappa \text{ for } \kappa = 0, 1, 2 \\ \hat{c}_3 &= 0 \\ \hat{c}_4 &= c_3 \end{aligned}\right\} \quad . \tag{72}$$

$$\alpha_3 = \alpha_4 = 1 \quad . \tag{73}$$

Table 14 shows the reduced equations of condition for a fourth-order formula RKN 4(5).

21. The first four equations of Table 14 determine the weight factors c_κ and $\dot{c}_\kappa$ as functions of the α_κ's.

Since we require that the fifth evaluation should be taken over as first evaluation for the next step, the coefficients $\gamma_{4\lambda}$ must be equal to the weight factors c_λ ($\lambda = 0, 1, 2, 3$). Setting $\gamma_{31} = 0$, the coefficients γ_{21} and γ_{32} can then be determined from the two equations in the last line of Table 14.

22. In Table 15 are listed the coefficients of a fourth-order Runge-Kutta-Nyström formula RKN 4(5) and in Table 16, the error coefficients in x as well as in $\dot{x}$ for the formula of Table 15. There is only one error coefficient in x which is different from zero. The six error coefficients in $\dot{x}$ which are all different from zero are equal in magnitude or smaller than the error coefficient in x. Again, if the error coefficients differ by a constant factor only, those with the largest factor are listed in Table 16.

SECTION VI. SOME RUNGE-KUTTA-NYSTRÖM FORMULAS OF OTHER AUTHORS

23. In Section VII we shall apply the Runge-Kutta-Nyström formulas of Sections I through V to a numerical example. We shall compare their performance with that of our earlier Runge-Kutta formulas for first-order differential equations [1], [2] and also with the performance of some Runge-Kutta-Nyström formulas by other authors.

For the convenience of the reader we list in Tables 17 through 22 the Runge-Kutta-Nyström coefficients and also the error coefficients for a fourth-, a fifth-, and a sixth-order Runge-Kutta-Nyström formula, which we have used for comparison in Section VII. As before, if the error coefficients differ only by a constant factor, we list those with the largest factor. The fourth- and the fifth-order Runge-Kutta-Nyström formulas were published by E. J. NYSTRÖM ([4], p. 24) and the sixth-order formula by J. ALBRECHT ([5], p. 103).

A comparison of Table 18 with Table 16 shows that NYSTRÖM's error coefficients are two to nine times as large as ours. Therefore, NYSTRÖM's formula of Table 17 can be expected to have a larger leading truncation error term than our formula RKN 4(5).

A comparison of Table 20 with Table 13 shows again that our formula RKN 5(6) has considerably smaller error coefficients than NYSTRÖM's formula RKN 5. For instance, our error coefficient T_5 is only about one-seventeenth of NYSTRÖM's coefficient T_5.

Again, the error coefficients of our Runge-Kutta-Nyström formula RKN 6(7), as listed in Table 10, are smaller than those of ALBRECHT's formula, listed in Table 22. For instance, our largest error coefficient in x, T_9, is only about one-fifth of ALBRECHT's coefficient T_9.

24. The formulas of NYSTRÖM and ALBRECHT of this section do not include an automatic stepsize control as our formulas of Sections I through V do. Therefore, we have to apply to the formulas of NYSTRÖM and ALBRECHT the standard stepsize control procedure that consists of recomputing two consecutive steps of stepsize h by one step of double stepsize 2h. It can easily be shown that the truncation error after one step of stepsize h is approximately

$$E = \frac{1}{2(2^n - 1)} \cdot \Delta x_2 \quad , \tag{74}$$

with n being the order of the formula under consideration and Δx_2 the difference of our two results after two steps of stepsize h or one step of stepsize 2h, respectively. If the formula under consideration requires m evaluations per step, this stepsize control procedure would increase the number of evaluations per step to 2m - 1. So NYSTRÖM's formulas RKN 4 and RKN 5 and ALBRECHT's formula RKN 6 would require five, seven, and nine evaluations per step, respectively. For NYSTRÖM's formulas, this is one evaluation more per step than our formulas RKN 4 (5), RKN 5(6) require, and for ALBRECHT's formula RKN 6, this is two evaluations more than our formula RKN 6(7) requires, since for our formulas we can take over the last evaluation as first evaluation for the next step. Only for the very first integration step our formulas would require five, seven, and eight evaluations, respectively. Compared with NYSTRÖM's or ALBRECHT's formulas, our formulas have the further advantage that they have considerably smaller leading truncation error terms and therefore permit the use of a larger integration stepsize without loss of accuracy. The numerical example of Section VII will demonstrate this advantage.

SECTION VII: APPLICATION TO A NUMERICAL PROBLEM

25. In this section we apply the Runge-Kutta-Nyström formulas of this report and some of the Runge-Kutta formulas of our earlier reports [1], [2] to a numerical problem. For comparison, we also apply the Runge-Kutta-Nyström formulas of Section VI to the same problem.

In Table 23 the problem is stated, and the results of the numerical integration are presented for the various formulas.

All calculations were executed on an IBM-7094 computer in double precision (16 decimal places). The computer was equipped with an electronic clock to measure the running time for the various formulas.

26. The stepsize control for our Runge-Kutta-Nyström formulas RKN 4(5), ..., RKN 8(9) was set up in the following way. For a preset tolerance TOL (in Table 23 we used TOL = $0.1 \cdot 10^{-16}$), we computed for each step the products TOLX=TOL $\cdot$ $|x_0|$, TOLY=TOL $\cdot$ $|y_0|$ which we consider as the tolerable errors in x and y for the step under consideration. Having computed the approximate truncation errors TE_x, TE_y

according to Tables 3, 6, 9, 12, or 15, respectively, we then determined the maximum of the ratios $|TE_x|/TOLX$, $|TE_y|/TOLY$ and then required that for this maximum (max) the following inequalities hold:

$$\left(\frac{1}{2}\right)^{n+1} \leqq \max \leqq 1 \quad , \tag{75}$$

n being the order of our formula.

If necessary, we halved or doubled the stepsize until (75) held. However, since the values TE_x, TE_y are only approximations of the true truncation error, it can happen that (75) never holds: If a certain stepsize h is too small $\max < \left(\frac{1}{2}\right)^{n+1}$, the stepsize 2h might be too large: $\max > 1$. In such a case, we accepted the smaller stepsize h as final. The stepsize control for our Runge-Kutta formulas RK 4(5) , . . . , RK 8(9) was set up quite similarly, but here we also tested the truncation errors in $\dot{x}$ and $\dot{y}$, since for these formulas we had to rewrite the differential equations of Table 23 as a system of four first-order differential equations in x, y, $\dot{x}$, $\dot{y}$.

In the case of the Runge-Kutta-Nyström formulas of Section VI we used equation (74) as approximation for the truncation error.

27. Table 23 shows the result of the various formulas for $t = 10$. Since our problem has a solution in closed form, the total errors Δx, Δy, $\Delta\dot{x}$, $\Delta\dot{y}$ are easily available and are listed in Table 23 for $t = 10$. In each of the five groups of formulas, these errors do not differ much from one another. This means that all fourth-order formulas, etc., are of about the same accuracy.

However, the formulas of each group differ considerably from one another with respect to the number of steps required to cover the interval from $t_i = \sqrt{\pi/2}$ to $t = 10$ and with respect to the execution time on the computer. Our new Runge-Kutta-Nyström formulas RKN 4(5), . . . , RKN 8(9) require half or less than half the execution time of our earlier Runge-Kutta formulas RK 4(5), . . . , RK 8(9) or of the formulas of Section VI.

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TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9)

	x ←	c_K, $\dot{c}_K$	→ $\dot{x}$	
1. h^2	$\frac{1}{2} =$	$c_0 + c_1 + c_2 + c_3 + c_4 + \ldots$	$= 1$	h
1. h^3	$\frac{1}{6}$	$c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + \ldots$	$= \frac{1}{2}$	h^2
1. h^4	$\frac{1}{24} =$	$\frac{1}{2} c_1\alpha_1^2 + \frac{1}{2} c_2\alpha_2^2 + \frac{1}{2} c_3\alpha_3^2 + \frac{1}{2} c_4\alpha_4^2 + \ldots$	$= \frac{1}{6}$	h^3
2. h^4(*)	$\frac{1}{24} =$	$\frac{1}{2} c_1\alpha_1^2 + \frac{1}{2} c_2\alpha_2^2 + \frac{1}{2} c_3\alpha_3^2 + \frac{1}{2} c_4\alpha_4^2 + \ldots$	$= \frac{1}{6}$	(*) h^3
1. h^5	$\frac{1}{120} =$	$\frac{1}{6} c_1\alpha_1^3 + \frac{1}{6} c_2\alpha_2^3 + \frac{1}{6} c_3\alpha_3^3 + \frac{1}{6} c_4\alpha_4^3 + \ldots$	$= \frac{1}{24}$	h^4
2. h^5(*)	$\frac{3}{120}$	$\frac{1}{2} c_1\alpha_1^3 + \frac{1}{2} c_2\alpha_2^3 + \frac{1}{2} c_3\alpha_3^3 + \frac{1}{2} c_4\alpha_4^3 + \ldots$	$= \frac{3}{24}$	(*) h^4
3. h^5	$\frac{1}{120}$	$c_2 \cdot \gamma_{21}\alpha_1 + c_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + c_4(\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + \ldots$	$= \frac{1}{24}$	h^4
1. h^6	$\frac{1}{720} =$	$\frac{1}{24} c_1\alpha_1^4 + \frac{1}{24} c_2\alpha_2^4 + \frac{1}{24} c_3\alpha_3^4 + \frac{1}{24} c_4\alpha_4^4 + \ldots$	$= \frac{1}{120}$	h^5
2. h^6(*)	$\frac{6}{720}$	$\frac{1}{4} c_1\alpha_1^4 + \frac{1}{4} c_2\alpha_2^4 + \frac{1}{4} c_3\alpha_3^4 + \frac{1}{4} c_4\alpha_4^4 + \ldots$	$= \frac{6}{120}$	(*) h^5
3. h^6(*)	$\frac{3}{720}$	$\frac{1}{8} c_1\alpha_1^4 + \frac{1}{8} c_2\alpha_2^4 + \frac{1}{8} c_3\alpha_3^4 + \frac{1}{8} c_4\alpha_4^4 + \ldots$	$= \frac{3}{120}$	(*) h^5
4. h^6	$\frac{4}{720}$	$c_2\alpha_2 \cdot \gamma_{21}\alpha_1 + c_3\alpha_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + c_4\alpha_4(\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + \ldots$	$= \frac{4}{120}$	h^5
5. h^6	$\frac{1}{720}$	$\frac{1}{2} c_2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{2} c_3(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{2} c_4(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$= \frac{1}{120}$	h^5
6. h^6(*)	$\frac{1}{720} =$	$\frac{1}{2} c_2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{2} c_3(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{2} c_4(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$= \frac{1}{120}$	(*) h^5
1. h^7	$\frac{1}{5\,040} =$	$\frac{1}{120} c_1\alpha_1^5 + \frac{1}{120} c_2\alpha_2^5 + \frac{1}{120} c_3\alpha_3^5 + \frac{1}{120} c_4\alpha_4^5 + \ldots$	$= \frac{1}{720}$	h^6
2. h^7(*)	$\frac{10}{5\,040} =$	$\frac{1}{12} c_1\alpha_1^5 + \frac{1}{12} c_2\alpha_2^5 + \frac{1}{12} c_3\alpha_3^5 + \frac{1}{12} c_4\alpha_4^5 + \ldots$	$= \frac{10}{720}$	(*) h^6
3. h^7(*)	$\frac{15}{5\,040} =$	$\frac{1}{8} c_1\alpha_1^5 + \frac{1}{8} c_2\alpha_2^5 + \frac{1}{8} c_3\alpha_3^5 + \frac{1}{8} c_4\alpha_4^5 + \ldots$	$= \frac{15}{720}$	(*) h^6
4. h^7	$\frac{10}{5\,040} =$	$\frac{1}{2} c_2\alpha_2^2 \cdot \gamma_{21}\alpha_1 + \frac{1}{2} c_3\alpha_3^2 (\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + \frac{1}{2} c_4\alpha_4^2 (\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + \ldots$	$= \frac{10}{720}$	h^6
5. h^7(*)	$\frac{10}{5\,040} =$	$\frac{1}{2} c_2\alpha_2^2 \cdot \gamma_{21}\alpha_1 + \frac{1}{2} c_3\alpha_3^2 (\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + \frac{1}{2} c_4\alpha_4^2 (\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + \ldots$	$= \frac{10}{720}$	(*) h^6
6. h^7	$\frac{5}{5\,040} =$	$\frac{1}{2} c_2\alpha_2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{2} c_3\alpha_3(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{2} c_4\alpha_4(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$= \frac{5}{720}$	h^6
7. h^7(*)	$\frac{5}{5\,040} =$	$\frac{1}{2} c_2\alpha_2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{2} c_3\alpha_3(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{2} c_4\alpha_4(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$= \frac{5}{720}$	(*) h^6
8. h^7	$\frac{1}{5\,040} =$	$\frac{1}{6} c_2 \cdot \gamma_{21}\alpha_1^3 + \frac{1}{6} c_3(\gamma_{31}\alpha_1^3 + \gamma_{32}\alpha_2^3) + \frac{1}{6} c_4(\gamma_{41}\alpha_1^3 + \gamma_{42}\alpha_2^3 + \gamma_{43}\alpha_3^3) + \ldots$	$= \frac{1}{720}$	h^6
9. h^7(*)	$\frac{3}{5\,040} =$	$\frac{1}{2} c_2 \cdot \gamma_{21}\alpha_1^3 + \frac{1}{2} c_3(\gamma_{31}\alpha_1^3 + \gamma_{32}\alpha_2^3) + \frac{1}{2} c_4(\gamma_{41}\alpha_1^3 + \gamma_{42}\alpha_2^3 + \gamma_{43}\alpha_3^3) + \ldots$	$= \frac{3}{720}$	(*) h^6
10. h^7	$\frac{1}{5\,040} =$	$c_3 \cdot \gamma_{32} \cdot \gamma_{21}\alpha_1 + c_4[\gamma_{42} \cdot \gamma_{21}\alpha_1 + \gamma_{43}(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2)] + \ldots$	$= \frac{1}{720}$	h^6

TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9) (Continued)

	$x \leftarrow$	c_κ , $\dot{c}_\kappa$	$\rightarrow \dot{x}$	
1. h^8	$\frac{1}{40\ 320}$	$\frac{1}{720} c_1\alpha_1^6 + \frac{1}{720} c_2\alpha_2^6 + \frac{1}{720} c_3\alpha_3^6 + \frac{1}{720} c_4\alpha_4^6 + \ldots$	$\frac{1}{5\ 040}$	h^7
2. h^8 (*)	$\frac{15}{40\ 320}$	$\frac{1}{48} c_1\alpha_1^6 + \frac{1}{48} c_2\alpha_2^6 + \frac{1}{48} c_3\alpha_3^6 + \frac{1}{48} c_4\alpha_4^6 + \ldots$	$\frac{15}{5\ 040}$	(*) h^7
3. h^8 (*)	$\frac{45}{40\ 320}$	$\frac{1}{16} c_1\alpha_1^6 + \frac{1}{16} c_2\alpha_2^6 + \frac{1}{16} c_3\alpha_3^6 + \frac{1}{16} c_4\alpha_4^6 + \ldots$	$\frac{45}{5\ 040}$	(*) h^7
4. h^8	$\frac{20}{40\ 320}$	$\frac{1}{6} c_2\alpha_2^3 \cdot \gamma_{21}\alpha_1 + \frac{1}{6} c_3\alpha_3^3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + \frac{1}{6} c_4\alpha_4^3(\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + \ldots$	$\frac{20}{5\ 040}$	h^7
5. h^8 (*)	$\frac{15}{40\ 320}$	$\frac{1}{48} c_1\alpha_1^6 + \frac{1}{48} c_2\alpha_2^6 + \frac{1}{48} c_3\alpha_3^6 + \frac{1}{48} c_4\alpha_4^6 + \ldots$	$\frac{15}{5\ 040}$	(*) h^7
6. h^8 (*)	$\frac{60}{40\ 320}$	$\frac{1}{2} c_2\alpha_2^3 \cdot \gamma_{21}\alpha_1 + \frac{1}{2} c_3\alpha_3^3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2) + \frac{1}{2} c_4\alpha_4^3(\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3) + \ldots$	$\frac{60}{5\ 040}$	(*) h^7
7. h^8	$\frac{15}{40\ 320}$	$\frac{1}{4} c_2\alpha_2^2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{4} c_3\alpha_3^2(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{4} c_4\alpha_4^2(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$\frac{15}{5\ 040}$	h^7
8. h^8 (*)	$\frac{15}{40\ 320}$	$\frac{1}{4} c_2\alpha_2^2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{4} c_3\alpha_3^2(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{4} c_4\alpha_4^2(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$\frac{15}{5\ 040}$	(*) h^7
9. h^8	$\frac{10}{40\ 320}$	$\frac{1}{2} c_2(\gamma_{21}\alpha_1)^2 + \frac{1}{2} c_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2)^2 + \frac{1}{2} c_4(\gamma_{41}\alpha_1 + \gamma_{42}\alpha_2 + \gamma_{43}\alpha_3)^2 + \ldots$	$\frac{10}{5\ 040}$	h^7
10. h^8 (*)	$\frac{15}{40\ 320}$	$\frac{1}{4} c_2\alpha_2^2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{4} c_3\alpha_3^2(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{4} c_4\alpha_4^2(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$\frac{15}{5\ 040}$	(*) h^7
11. h^8 (*)	$\frac{15}{40\ 320}$	$\frac{1}{4} c_2\alpha_2^2 \cdot \gamma_{21}\alpha_1^2 + \frac{1}{4} c_3\alpha_3^2(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2) + \frac{1}{4} c_4\alpha_4^2(\gamma_{41}\alpha_1^2 + \gamma_{42}\alpha_2^2 + \gamma_{43}\alpha_3^2) + \ldots$	$\frac{15}{5\ 040}$	(*) h^7
12. h^8	$\frac{6}{40\ 320}$	$\frac{1}{6} c_2\alpha_2 \cdot \gamma_{21}\alpha_1^3 + \frac{1}{6} c_3\alpha_3(\gamma_{31}\alpha_1^3 + \gamma_{32}\alpha_2^3) + \frac{1}{6} c_4\alpha_4(\gamma_{41}\alpha_1^3 + \gamma_{42}\alpha_2^3 + \gamma_{43}\alpha_3^3) + \ldots$	$\frac{6}{5\ 040}$	h^7
13. h^8 (*)	$\frac{18}{40\ 320}$	$\frac{1}{2} c_2\alpha_2 \cdot \gamma_{21}\alpha_1^3 + \frac{1}{2} c_3\alpha_3(\gamma_{31}\alpha_1^3 + \gamma_{32}\alpha_2^3) + \frac{1}{2} c_4\alpha_4(\gamma_{41}\alpha_1^3 + \gamma_{42}\alpha_2^3 + \gamma_{43}\alpha_3^3) + \ldots$	$\frac{18}{5\ 040}$	(*) h^7
14. h^8	$\frac{6}{40\ 320}$	$c_3\alpha_3 \cdot \gamma_{32} \cdot \gamma_{21}\alpha_1 + c_4\alpha_4[\gamma_{42} \cdot \gamma_{21}\alpha_1 + \gamma_{43}(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2)] + \ldots$	$\frac{6}{5\ 040}$	h^7
15. h^8	$\frac{1}{40\ 320}$	$\frac{1}{24} c_2 \cdot \gamma_{21}\alpha_1^4 + \frac{1}{24} c_3(\gamma_{31}\alpha_1^4 + \gamma_{32}\alpha_2^4) + \frac{1}{24} c_4(\gamma_{41}\alpha_1^4 + \gamma_{42}\alpha_2^4 + \gamma_{43}\alpha_3^4) + \ldots$	$\frac{1}{5\ 040}$	h^7
16. h^8 (*)	$\frac{6}{40\ 320}$	$\frac{1}{4} c_2 \cdot \gamma_{21}\alpha_1^4 + \frac{1}{4} c_3(\gamma_{31}\alpha_1^4 + \gamma_{32}\alpha_2^4) + \frac{1}{4} c_4(\gamma_{41}\alpha_1^4 + \gamma_{42}\alpha_2^4 + \gamma_{43}\alpha_3^4) + \ldots$	$\frac{6}{5\ 040}$	(*) h^7
17. h^8 (*)	$\frac{3}{40\ 320}$	$\frac{1}{8} c_2 \cdot \gamma_{21}\alpha_1^4 + \frac{1}{8} c_3(\gamma_{31}\alpha_1^4 + \gamma_{32}\alpha_2^4) + \frac{1}{8} c_4(\gamma_{41}\alpha_1^4 + \gamma_{42}\alpha_2^4 + \gamma_{43}\alpha_3^4) + \ldots$	$\frac{3}{5\ 040}$	(*) h^7
18. h^8	$\frac{4}{40\ 320}$	$c_3 \cdot \gamma_{32}\alpha_2 \cdot \gamma_{21}\alpha_1 + c_4[\gamma_{42}\alpha_2 \cdot \gamma_{21}\alpha_1 + \gamma_{43}\alpha_3(\gamma_{31}\alpha_1 + \gamma_{32}\alpha_2)] + \ldots$	$\frac{4}{5\ 040}$	h^7
19. h^8	$\frac{1}{40\ 320}$	$\frac{1}{2} c_3 \cdot \gamma_{32} \cdot \gamma_{21}\alpha_1^2 + \frac{1}{2} c_4[\gamma_{42} \cdot \gamma_{21}\alpha_1^2 + \gamma_{43}(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2)] + \ldots$	$\frac{1}{5\ 040}$	h^7
20. h^8 (*)	$\frac{1}{40\ 320}$	$\frac{1}{2} c_3 \cdot \gamma_{32} \cdot \gamma_{21}\alpha_1^2 + \frac{1}{2} c_4[\gamma_{42} \cdot \gamma_{21}\alpha_1^2 + \gamma_{43}(\gamma_{31}\alpha_1^2 + \gamma_{32}\alpha_2^2)] + \ldots$	$\frac{1}{5\ 040}$	(*) h^7

TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9) (Continued)

	$x \leftarrow$	c_k, $\dot{c}_k$	$\rightarrow \dot{x}$	
1. h^9	$\frac{1}{362\,880} =$	$\frac{1}{5\,040} c_1 \alpha_1^7 + \frac{1}{5\,040} c_2 \alpha_2^7 + \frac{1}{5\,040} c_3 \alpha_3^7 + \frac{1}{5\,040} c_4 \alpha_4^7 + \ldots$	$= \frac{1}{40\,320}$	h^8
2. h^9 (*)	$\frac{21}{362\,880} =$	$\frac{1}{240} c_1 \alpha_1^7 + \frac{1}{240} c_2 \alpha_2^7 + \frac{1}{240} c_3 \alpha_3^7 + \frac{1}{240} c_4 \alpha_4^7 + \ldots$	$= \frac{21}{40\,320}$	(*) h^8
3. h^9 (*)	$\frac{105}{362\,880} =$	$\frac{1}{48} c_1 \alpha_1^7 + \frac{1}{48} c_2 \alpha_2^7 + \frac{1}{48} c_3 \alpha_3^7 + \frac{1}{48} c_4 \alpha_4^7 + \ldots$	$= \frac{105}{40\,320}$	(*) h^8
4. h^9	$\frac{35}{362\,880} =$	$\frac{1}{24} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1 + \frac{1}{24} c_3 \alpha_3^4 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) + \frac{1}{24} c_4 \alpha_4^4 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3) + \ldots$	$= \frac{35}{40\,320}$	h^8
5. h^9 (*)	$\frac{105}{362\,880} =$	$\frac{1}{48} c_1 \alpha_1^7 + \frac{1}{48} c_2 \alpha_2^7 + \frac{1}{48} c_3 \alpha_3^7 + \frac{1}{48} c_4 \alpha_4^7 + \ldots$	$= \frac{105}{40\,320}$	(*) h^8
6. h^9 (*)	$\frac{210}{362\,880} =$	$\frac{1}{4} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1 + \frac{1}{4} c_3 \alpha_3^4 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) + \frac{1}{4} c_4 \alpha_4^4 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3) + \ldots$	$= \frac{210}{40\,320}$	(*) h^8
7. h^9	$\frac{35}{362\,880} =$	$\frac{1}{12} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{12} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{12} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{35}{40\,320}$	h^8
8. h^9 (*)	$\frac{35}{362\,880} =$	$\frac{1}{12} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{12} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{12} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{35}{40\,320}$	(*) h^8
9. h^9 (*)	$\frac{105}{362\,880} =$	$\frac{1}{8} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1 + \frac{1}{8} c_3 \alpha_3^4 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) + \frac{1}{8} c_4 \alpha_4^4 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3) + \ldots$	$= \frac{105}{40\,320}$	(*) h^8
10. h^9	$\frac{70}{362\,880} =$	$\frac{1}{2} c_2 \alpha_2 (\gamma_{21} \alpha_1)^2 + \frac{1}{2} c_3 \alpha_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)^2 + \frac{1}{2} c_4 \alpha_4 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3)^2 + \ldots$	$= \frac{70}{40\,320}$	h^8
11. h^9 (*)	$\frac{105}{362\,880} =$	$\frac{1}{4} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{4} c_3 \alpha_3^3 (\gamma_{21} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{4} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{105}{40\,320}$	(*) h^8
12. h^9 (*)	$\frac{105}{362\,880} =$	$\frac{1}{4} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{4} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{4} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{105}{40\,320}$	(*) h^8
13. h^9	$\frac{21}{362\,880} =$	$\frac{1}{12} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{12} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{12} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{21}{40\,320}$	h^8
14. h^9 (*)	$\frac{63}{362\,880} =$	$\frac{1}{4} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{4} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{4} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{63}{40\,320}$	(*) h^8
15. h^9	$\frac{21}{362\,880} =$	$\frac{1}{2} c_3 \alpha_3^2 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 \alpha_4^2 [\gamma_{42} \cdot \gamma_{21} \alpha_1 + \gamma_{43} (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$= \frac{21}{40\,320}$	h^8
16. h^9	$\frac{35}{362\,880} =$	$\frac{1}{2} c_2 \cdot \gamma_{21} \alpha_1 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \ldots$	$= \frac{35}{40\,320}$	h^8
17. h^9 (*)	$\frac{35}{362\,880} =$	$\frac{1}{2} c_2 \cdot \gamma_{21} \alpha_1 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \ldots$	$= \frac{35}{40\,320}$	(*) h^8
18. h^9 (*)	$\frac{21}{362\,880} =$	$\frac{1}{12} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{12} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{12} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{21}{40\,320}$	(*) h^8
19. h^9 (*)	$\frac{63}{362\,880} =$	$\frac{1}{4} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{4} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{4} c_4 \alpha_4^2 (\gamma_{11} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{63}{40\,320}$	(*) h^8
20. h^9 (*)	$\frac{21}{362\,880} =$	$\frac{1}{2} c_3 \alpha_3^2 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 \alpha_4^2 [\gamma_{42} \cdot \gamma_{21} \alpha_1 + \gamma_{43} (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$= \frac{21}{40\,320}$	(*) h^8

TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9) (Continued)

	$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
21. h^9	$\frac{7}{362\ 880} = \frac{1}{24} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{24} c_3 \alpha_3 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{24} c_4 \alpha_4 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \dots$	$= \frac{7}{40\ 320}$	h^8
22. h^9 (*)	$\frac{42}{362\ 880} = \frac{1}{4} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{4} c_3 \alpha_3 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{4} c_4 \alpha_4 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \dots$	$= \frac{42}{40\ 320}$	(*) h^8
23. h^9 (*)	$\frac{21}{362\ 880} = \frac{1}{8} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{8} c_3 \alpha_3 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{8} c_4 \alpha_4 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \dots$	$= \frac{21}{40\ 320}$	(*) h^8
24. h^9	$\frac{28}{362\ 880} = c_3 \alpha_3 \cdot \gamma_{32} \alpha_2 \cdot \gamma_{21} \alpha_1 + c_4 \alpha_4 [\gamma_{42} \alpha_2 \cdot \gamma_{21} \alpha_1 + \gamma_{43} \alpha_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \dots$	$= \frac{28}{40\ 320}$	h^8
25. h^9	$\frac{7}{362\ 880} = \frac{1}{2} c_3 \alpha_3 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_4 \alpha_4 [(\gamma_{42} \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \dots$	$= \frac{7}{40\ 320}$	h^8
26. h^9 (*)	$\frac{7}{362\ 880} = \frac{1}{2} c_3 \alpha_3 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_4 \alpha_4 [\gamma_{42} \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \dots$	$= \frac{7}{40\ 320}$	(*) h^8
27. h^9	$\frac{1}{362\ 880} = \frac{1}{120} c_2 \cdot \gamma_{21} \alpha_1^5 + \frac{1}{120} c_3 (\gamma_{31} \alpha_1^5 + \gamma_{32} \alpha_2^5) + \frac{1}{120} c_4 (\gamma_{41} \alpha_1^5 + \gamma_{42} \alpha_2^5 + \gamma_{43} \alpha_3^5) + \dots$	$= \frac{1}{40\ 320}$	h^8
28. h^9 (*)	$\frac{10}{362\ 880} = \frac{1}{12} c_2 \cdot \gamma_{21} \alpha_1^5 + \frac{1}{12} c_3 (\gamma_{31} \alpha_1^5 + \gamma_{32} \alpha_2^5) + \frac{1}{12} c_4 (\gamma_{41} \alpha_1^5 + \gamma_{42} \alpha_2^5 + \gamma_{43} \alpha_3^5) + \dots$	$= \frac{10}{40\ 320}$	(*) h^4
29. h^9 (*)	$\frac{15}{362\ 880} = \frac{1}{8} c_2 \cdot \gamma_{21} \alpha_1^5 + \frac{1}{8} c_3 (\gamma_{31} \alpha_1^5 + \gamma_{32} \alpha_2^5) + \frac{1}{8} c_4 (\gamma_{41} \alpha_1^5 + \gamma_{42} \alpha_2^5 + \gamma_{43} \alpha_3^5) + \dots$	$= \frac{15}{40\ 320}$	(*) h^8
30. h^9	$\frac{10}{362\ 880} = \frac{1}{2} c_3 \cdot \gamma_{32} \alpha_2^2 \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 [\gamma_{42} \alpha_2^2 \cdot \gamma_{21} \alpha_1 + \gamma_{43} \alpha_3^2 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \dots$	$= \frac{10}{40\ 320}$	h^8
31. h^9 (*)	$\frac{10}{362\ 880} = \frac{1}{2} c_3 \cdot \gamma_{32} \alpha_2^2 \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 [\gamma_{42} \alpha_2^2 \cdot \gamma_{21} \alpha_1 + \gamma_{43} \alpha_3^2 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \dots$	$= \frac{10}{40\ 320}$	(*) h^8
32. h^9	$\frac{5}{362\ 880} = \frac{1}{2} c_3 \cdot \gamma_{32} \alpha_2 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_4 [\gamma_{42} \alpha_2 \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} \alpha_3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \dots$	$= \frac{5}{40\ 320}$	h^8
33. h^9 (*)	$\frac{5}{362\ 880} = \frac{1}{2} c_3 \cdot \gamma_{32} \alpha_2 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_4 [\gamma_{42} \alpha_2 \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} \alpha_3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \dots$	$= \frac{5}{40\ 320}$	(*) h^8
34. h^9	$\frac{1}{362\ 880} = \frac{1}{6} c_3 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^3 + \frac{1}{6} c_4 [\gamma_{42} \cdot \gamma_{21} \alpha_1^3 + \gamma_{43} (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3)] + \dots$	$= \frac{1}{40\ 320}$	h^8
35. h^9 (*)	$\frac{3}{362\ 880} = \frac{1}{2} c_3 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^3 + \frac{1}{2} c_4 [\gamma_{42} \cdot \gamma_{21} \alpha_1^3 + \gamma_{43} (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3)] + \dots$	$= \frac{3}{40\ 320}$	(*) h^8
36. h^9	$\frac{1}{362\ 880} = c_4 \cdot \gamma_{43} \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1 + c_5 \{\gamma_{53} \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1 + \gamma_{54} [\gamma_{42} \cdot \gamma_{21} \alpha_1 + \gamma_{43} (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)]\} + \dots$	$= \frac{1}{40\ 320}$	h^8

TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9) (Continued)

	$x \longleftarrow c_\kappa,\ \dot{c}_\kappa \longrightarrow \dot{x}$		
1. h^{10}	$\frac{1}{3\ 628\ 800} = \frac{1}{40\ 320} c_1 \alpha_1^8 + \frac{1}{40\ 320} c_2 \alpha_2^8 + \frac{1}{40\ 320} c_3 \alpha_3^8 + \frac{1}{40\ 320} c_4 \alpha_4^8 + \ldots$	$= \frac{1}{362\ 880}$	h^9
2. h^{10} (*)	$\frac{28}{3\ 628\ 800} = \frac{1}{1\ 440} c_1 \alpha_1^8 + \frac{1}{1\ 440} c_2 \alpha_2^8 + \frac{1}{1\ 440} c_3 \alpha_3^8 + \frac{1}{1\ 440} c_4 \alpha_4^8 + \ldots$	$= \frac{28}{362\ 880}$	(*) h^9
3. h^{10} (*)	$\frac{210}{3\ 628\ 800} = \frac{1}{192} c_1 \alpha_1^8 + \frac{1}{192} c_2 \alpha_2^8 + \frac{1}{192} c_3 \alpha_3^8 + \frac{1}{192} c_4 \alpha_4^8 + \ldots$	$= \frac{210}{362\ 880}$	(*) h^9
4. h^{10}	$\frac{56}{3\ 628\ 800} = \frac{1}{120} c_2 \alpha_2^5 \cdot \gamma_{21} \alpha_1 + \frac{1}{120} c_3 \alpha_3^5 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) + \frac{1}{120} c_4 \alpha_4^5 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3) + \ldots$	$= \frac{56}{362\ 880}$	h^9
5. h^{10} (*)	$\frac{420}{3\ 628\ 800} = \frac{1}{96} c_1 \alpha_1^8 + \frac{1}{96} c_2 \alpha_2^8 + \frac{1}{96} c_3 \alpha_3^8 + \frac{1}{96} c_4 \alpha_4^8 + \ldots$	$= \frac{420}{362\ 800}$	(*) h^9
6. h^{10} (*)	$\frac{560}{3\ 628\ 800} = \frac{1}{12} c_2 \alpha_2^5 \cdot \gamma_{21} \alpha_1 + \frac{1}{12} c_3 \alpha_3^5 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) + \frac{1}{12} c_4 \alpha_4^5 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3) + \ldots$	$= \frac{560}{362\ 880}$	(*) h^9
7. h^{10}	$\frac{70}{3\ 628\ 800} = \frac{1}{48} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{48} c_3 \alpha_3^4 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{48} c_4 \alpha_4^4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$\frac{70}{362\ 880}$	h^9
8. h^{10} (*)	$\frac{70}{628\ 800} = \frac{1}{48} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{48} c_3 \alpha_3^4 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{48} c_4 \alpha_4^4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$\frac{70}{362\ 880}$	(*) h^9
9. h^{10} (*)	$\frac{105}{3\ 628\ 800} = \frac{1}{384} c_1 \alpha_1^8 + \frac{1}{384} c_2 \alpha_2^8 + \frac{1}{384} c_3 \alpha_3^8 + \frac{1}{384} c_4 \alpha_4^8 + \ldots$	$= \frac{105}{362\ 880}$	(*) h^9
10. h^{10} (*)	$\frac{840}{3\ 628\ 800} = \frac{1}{8} c_2 \alpha_2^5 \cdot \gamma_{21} \alpha_1 + \frac{1}{8} c_3 \alpha_3^5 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) + \frac{1}{8} c_4 \alpha_4^5 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3) + \ldots$	$\frac{840}{362\ 880}$	(*) h^9
11. h^{10}	$\frac{280}{3\ 628\ 800} = \frac{1}{4} c_2 \alpha_2^2 (\gamma_{21} \alpha_1)^2 + \frac{1}{4} c_3 \alpha_3^2 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)^2 + \frac{1}{4} c_4 \alpha_4^2 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3)^2 + \ldots$	$= \frac{280}{362\ 880}$	h^9
12. h^{10} (*)	$\frac{420}{3\ 628\ 800} = \frac{1}{8} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{8} c_3 \alpha_3^4 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{8} c_4 \alpha_4^4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{420}{362\ 880}$	(*) h^9
13. h^{10} (*)	$\frac{420}{3\ 628\ 800} = \frac{1}{8} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{8} c_3 \alpha_3^4 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{8} c_4 \alpha_4^4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{420}{362\ 880}$	(*) h^9
14. h^{10}	$\frac{56}{3\ 628\ 800} = \frac{1}{36} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{36} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{36} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{56}{362\ 880}$	h^9
15. h^{10} (*)	$\frac{168}{3\ 628\ 800} = \frac{1}{12} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{12} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{12} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{168}{362\ 880}$	(*) h^9
16. h^{10}	$\frac{56}{3\ 628\ 800} = \frac{1}{6} c_3 \alpha_3^3 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1 + \frac{1}{6} c_4 \alpha_4^3 [\gamma_{42} \cdot \gamma_{21} \alpha_1 + \gamma_{43} (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$= \frac{56}{362\ 880}$	h^9
17. h^{10} (*)	$\frac{280}{3\ 628\ 800} = \frac{1}{4} c_2 \alpha_2^2 (\gamma_{21} \alpha_1)^2 + \frac{1}{4} c_3 \alpha_3^2 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)^2 + \frac{1}{4} c_4 \alpha_4^2 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3)^2 + \ldots$	$= \frac{280}{362\ 880}$	(*) h^9
18. h^{10} (*)	$\frac{210}{3\ 628\ 800} = \frac{1}{16} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{16} c_3 \alpha_3^4 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{16} c_4 \alpha_4^4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{210}{362\ 880}$	(*) h^9
19. h^{10} (*)	$\frac{210}{3\ 628\ 880} = \frac{1}{16} c_2 \alpha_2^4 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{16} c_3 \alpha_3^4 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \frac{1}{16} c_4 \alpha_4^4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2) + \ldots$	$= \frac{210}{362\ 880}$	(*) h^9
20. h^{10}	$\frac{280}{3\ 628\ 800} = \frac{1}{2} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_3 \alpha_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \ldots$	$= \frac{280}{362\ 880}$	h^9
21. h^{10} (*)	$\frac{280}{3\ 628\ 800} = \frac{1}{2} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1 \cdot \gamma_{21} \alpha_1^2 + \frac{1}{2} c_3 \alpha_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2) + \ldots$	$= \frac{280}{362\ 880}$	(*) h^9
22. h^{10} (*)	$\frac{168}{3\ 628\ 800} = \frac{1}{12} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{12} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{12} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{168}{362\ 880}$	(*) h^9
23. h^{10} (*)	$\frac{504}{3\ 628\ 800} = \frac{1}{4} c_2 \alpha_2^3 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{4} c_3 \alpha_3^3 (\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \frac{1}{4} c_4 \alpha_4^3 (\gamma_{41} \alpha_1^3 + \gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \ldots$	$= \frac{504}{362\ 880}$	(*) h^9
24. h^{10} (*)	$\frac{168}{3\ 628\ 800} = \frac{1}{2} c_3 \alpha_3^3 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 \alpha_4^3 [\gamma_{42} \cdot \gamma_{21} \alpha_1 + \gamma_{43} (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$= \frac{168}{362\ 880}$	(*) h^9
25. h^{10}	$\frac{28}{3\ 628\ 800} = \frac{1}{48} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{48} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{48} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \ldots$	$= \frac{28}{362\ 880}$	h^9
26. h^{10} (*)	$\frac{168}{3\ 628\ 800} = \frac{1}{8} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{8} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{8} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \ldots$	$= \frac{168}{362\ 880}$	(*) h^9

TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9) (Continued)

	$x \leftarrow$ c_κ, $\dot{c}_\kappa$ $\rightarrow \dot{x}$		
27. h^{10} (*)	$\frac{84}{3\ 628\ 800} \cdot \frac{1}{16} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{16} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{16} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \ldots$	$\frac{84}{362\ 880}$	(*) h^9
28. h^{10}	$\frac{112}{3\ 628\ 800}\ \frac{1}{2} c_3 \alpha_3^2 \cdot \gamma_{32} \alpha_2 \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 \alpha_4^2 [\gamma_{42} \alpha_2 \cdot \gamma_{21} \alpha_1 + \gamma_{43} \alpha_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$\frac{112}{362\ 880}$	h^9
29. h^{10}	$\frac{28}{3\ 628\ 800} \cdot \frac{1}{4} c_3 \alpha_3^2 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^2 + \frac{1}{4} c_4 \alpha_4^2 [\gamma_{42} \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \ldots$	$\frac{28}{362\ 880}$	h^9
30. h^{10} (*)	$\frac{28}{3\ 628\ 800}\ \frac{1}{4} c_3 \alpha_3^2 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^2 + \frac{1}{4} c_4 \alpha_4^2 [\gamma_{42} \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \ldots$	$\frac{28}{362\ 880}$	(*) h^9
31. h^{10}	$\frac{56}{3\ 628\ 800} = \frac{1}{6} c_2 \cdot \gamma_{21} \alpha_1 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{6} c_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)(\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \ldots$	$\frac{56}{362\ 880}$	h^9
32. h^{10} (*)	$\frac{168}{3\ 628\ 800}\ \frac{1}{2} c_2 \cdot \gamma_{21} \alpha_1 \cdot \gamma_{21} \alpha_1^3 + \frac{1}{2} c_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)(\gamma_{31} \alpha_1^3 + \gamma_{32} \alpha_2^3) + \ldots$	$\frac{168}{362\ 880}$	(*) h^9
33. h^{10}	$\frac{35}{3\ 628\ 800} = \frac{1}{8} c_2 (\gamma_{21} \alpha_1^2)^2 + \frac{1}{8} c_3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)^2 + \frac{1}{8} c_4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2)^2 +$	$\frac{35}{362\ 886}$	h^9
34. h^{10} (*)	$\frac{70}{3\ 628\ 800}\ \frac{1}{4} c_2 (\gamma_{21} \alpha_1^2)^2 + \frac{1}{4} c_3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)^2 + \frac{1}{4} c_4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2)^2 +.$	$\frac{70}{362\ 880}$	(*) h^9
35. h^{10}	$\frac{56}{3\ 628\ 800}\ c_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2) \gamma_{32} \cdot \gamma_{21} \alpha_1 + c_4 (\gamma_{41} \alpha_1 + \gamma_{42} \alpha_2 + \gamma_{43} \alpha_3)[\gamma_{42} \cdot \gamma_{21} \alpha_1 + \gamma_{43} (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$\frac{56}{362\ 880}$	h^7
36. h^{10} (*)	$\frac{35}{3\ 628\ 800} = \frac{1}{8} c_2 (\gamma_{21} \alpha_1^2)^2 + \frac{1}{8} c_3 (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)^2 + \frac{1}{8} c_4 (\gamma_{41} \alpha_1^2 + \gamma_{42} \alpha_2^2 + \gamma_{43} \alpha_3^2)^2 + \ldots$	$\frac{35}{362\ 880}$	(*) h^9
37. h^{10} (*)	$\frac{28}{3\ 628\ 800} = \frac{1}{48} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{48} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{48} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \ldots$	$\frac{28}{362\ 880}$	(*) h^9
38. h^{10} (*)	$\frac{168}{3\ 628\ 800} \cdot \frac{1}{8} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{8} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{8} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \ldots$	$\frac{168}{362\ 880}$	(*) h^9
39. h^{10} (*)	$\frac{84}{3\ 628\ 800} = \frac{1}{16} c_2 \alpha_2^2 \cdot \gamma_{21} \alpha_1^4 + \frac{1}{16} c_3 \alpha_3^2 (\gamma_{31} \alpha_1^4 + \gamma_{32} \alpha_2^4) + \frac{1}{16} c_4 \alpha_4^2 (\gamma_{41} \alpha_1^4 + \gamma_{42} \alpha_2^4 + \gamma_{43} \alpha_3^4) + \ldots$	$\frac{84}{362\ 880}$	(*) h^9
40. h^{10} (*)	$\frac{112}{3\ 628\ 800} \cdot \frac{1}{2} c_3 \alpha_3^2 \cdot \gamma_{32} \alpha_2 \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 \alpha_4^2 [\gamma_{42} \alpha_2 \cdot \gamma_{21} \alpha_1 + \gamma_{43} \alpha_3 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$\frac{112}{362\ 880}$	(*) h^9
41. h^{10} (*)	$\frac{28}{3\ 628\ 800} = \frac{1}{4} c_3 \alpha_3^2 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^2 + \frac{1}{4} c_4 \alpha_4^2 [\gamma_{42} \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \ldots$	$\frac{28}{362\ 880}$	(*) h^9
42. h^{10} (*)	$\frac{28}{3\ 628\ 800}\ \frac{1}{4} c_3 \alpha_3^2 \cdot \gamma_{32} \cdot \gamma_{21} \alpha_1^2 + \frac{1}{4} c_4 \alpha_4^2 [\gamma_{42} \cdot \gamma_{21} \alpha_1^2 + \gamma_{43} (\gamma_{31} \alpha_1^2 + \gamma_{32} \alpha_2^2)] + \ldots$	$= \frac{28}{362\ 880}$	(*) h^9
43. h^{10}	$\frac{8}{3\ 628\ 800}\ \frac{1}{120} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1^5 + \frac{1}{120} c_3 \alpha_3 (\gamma_{31} \alpha_1^5 + \gamma_{32} \alpha_2^5) + \frac{1}{120} c_4 \alpha_4 (\gamma_{41} \alpha_1^5 + \gamma_{42} \alpha_2^5 + \gamma_{43} \alpha_3^5) + \ldots$	$\frac{8}{362\ 880}$	h^9
44. h^{10} (*)	$\frac{80}{3\ 628\ 800} \cdot \frac{1}{12} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1^5 + \frac{1}{12} c_3 \alpha_3 (\gamma_{31} \alpha_1^5 + \gamma_{32} \alpha_2^5) + \frac{1}{12} c_4 \alpha_4 (\gamma_{41} \alpha_1^5 + \gamma_{42} \alpha_2^5 + \gamma_{43} \alpha_3^5) + \ldots$	$\frac{80}{362\ 880}$	(*) h^9
45. h^{10} (*)	$\frac{120}{3\ 628\ 800}\ \frac{1}{8} c_2 \alpha_2 \cdot \gamma_{21} \alpha_1^5 + \frac{1}{8} c_3 \alpha_3 (\gamma_{31} \alpha_1^5 + \gamma_{32} \alpha_2^5) + \frac{1}{8} c_4 \alpha_4 (\gamma_{41} \alpha_1^5 + \gamma_{42} \alpha_2^5 + \gamma_{43} \alpha_3^5) + \ldots$	$= \frac{120}{362\ 880}$	(*) h^9
46. h^{10}	$\frac{80}{3\ 628\ 800} \cdot \frac{1}{2} c_3 \alpha_3 \cdot \gamma_{32} \alpha_2^2 \cdot \gamma_{21} \alpha_1 + \frac{1}{2} c_4 \alpha_4 [\gamma_{42} \alpha_2^2 \cdot \gamma_{21} \alpha_1 + \gamma_{43} \alpha_3^2 (\gamma_{31} \alpha_1 + \gamma_{32} \alpha_2)] + \ldots$	$= \frac{80}{362\ 880}$	h^9

TABLE 1. COMPLETE EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9) (Concluded)

	$x \leftarrow c_\kappa,\ \dot{c}_\kappa \rightarrow \dot{x}$		
47. h^{10} (*)	$\frac{80}{3\ 628\ 800} = \frac{1}{2} c_3\, \alpha_3 \cdot \gamma_{32}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1 + \frac{1}{2} c_4\, \alpha_4\, [\gamma_{42}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1 + \gamma_{43}\, \alpha_3^2\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)] + \ldots$	$= \frac{80}{362\ 880}$	(*) h^9
48. h^{10}	$\frac{40}{3\ 628\ 800} = \frac{1}{2} c_3\, \alpha_3 \cdot \gamma_{32}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{2} c_4\, \alpha_4\, [\gamma_{42}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, \alpha_3\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)] + \ldots$	$= \frac{40}{362\ 880}$	h^9
49. h^{10} (*)	$\frac{40}{3\ 628\ 800} = \frac{1}{2} c_3\, \alpha_3 \cdot \gamma_{32}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{2} c_4\, \alpha_4\, [\gamma_{42}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, \alpha_3\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)] + \ldots$	$= \frac{40}{362\ 880}$	(*) h^9
50. h^{10}	$\frac{8}{3\ 628\ 800} = \frac{1}{6} c_3\, \alpha_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^3 + \frac{1}{6} c_4\, \alpha_4\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^3 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^3 + \gamma_{32}\, \alpha_2^3)] + \ldots$	$= \frac{8}{362\ 880}$	h^9
51. h^{10} (*)	$\frac{24}{3\ 628\ 800} = \frac{1}{2} c_3\, \alpha_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^3 + \frac{1}{2} c_4\, \alpha_4\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^3 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^3 + \gamma_{32}\, \alpha_2^3)] + \ldots$	$= \frac{24}{362\ 880}$	(*) h^9
52. h^{10}	$\frac{8}{3\ 628\ 800} = c_4\, \alpha_4 \cdot \gamma_{43} \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1 + c_5\, \alpha_5\, \{\gamma_{53} \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1 + \gamma_{54}\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1 + \gamma_{43}\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)]\} + \ldots$	$= \frac{8}{362\ 880}$	h^9
53. h^{10}	$\frac{1}{3\ 628\ 800} = \frac{1}{720} c_2 \cdot \gamma_{21}\, \alpha_1^6 + \frac{1}{720} c_3\, (\gamma_{31}\, \alpha_1^6 + \gamma_{32}\, \alpha_2^6) + \frac{1}{720} c_4\, (\gamma_{41}\, \alpha_1^6 + \gamma_{42}\, \alpha_2^6 + \gamma_{43}\, \alpha_3^6) + \ldots$	$= \frac{1}{362\ 880}$	h^9
54. h^{10} (*)	$\frac{15}{3\ 628\ 800} = \frac{1}{48} c_2 \cdot \gamma_{21}\, \alpha_1^6 + \frac{1}{48} c_3\, (\gamma_{31}\, \alpha_1^6 + \gamma_{32}\, \alpha_2^6) + \frac{1}{48} c_4\, (\gamma_{41}\, \alpha_1^6 + \gamma_{42}\, \alpha_2^6 + \gamma_{43}\, \alpha_3^6) + \ldots$	$= \frac{15}{362\ 880}$	(*) h^9
55. h^{10} (*)	$\frac{45}{3\ 628\ 800} = \frac{1}{16} c_2 \cdot \gamma_{21}\, \alpha_1^6 + \frac{1}{16} c_3\, (\gamma_{31}\, \alpha_1^6 + \gamma_{32}\, \alpha_2^6) + \frac{1}{16} c_4\, (\gamma_{41}\, \alpha_1^6 + \gamma_{42}\, \alpha_2^6 + \gamma_{43}\, \alpha_3^6) + \ldots$	$= \frac{45}{362\ 880}$	(*) h^9
56. h^{10}	$\frac{20}{3\ 628\ 800} = \frac{1}{6} c_3 \cdot \gamma_{32}\, \alpha_2^3 \cdot \gamma_{21}\, \alpha_1 + \frac{1}{6} c_4\, [\gamma_{42}\, \alpha_2^3 \cdot \gamma_{21}\, \alpha_1 + \gamma_{43}\, \alpha_3^3\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)] + \ldots$	$= \frac{20}{362\ 880}$	h^9
57. h^{10} (*)	$\frac{15}{3\ 628\ 800} = \frac{1}{48} c_2 \cdot \gamma_{21}\, \alpha_1^6 + \frac{1}{48} c_3\, (\gamma_{31}\, \alpha_1^6 + \gamma_{32}\, \alpha_2^6) + \frac{1}{48} c_4\, (\gamma_{41}\, \alpha_1^6 + \gamma_{42}\, \alpha_2^6 + \gamma_{43}\, \alpha_3^6) + \ldots$	$= \frac{15}{362\ 880}$	(*) h^9
58. h^{10} (*)	$\frac{60}{3\ 628\ 800} = \frac{1}{2} c_3 \cdot \gamma_{32}\, \alpha_2^3 \cdot \gamma_{21}\, \alpha_1 + \frac{1}{2} c_4\, [\gamma_{42}\, \alpha_2^3 \cdot \gamma_{21}\, \alpha_1 + \gamma_{43}\, \alpha_3^3\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)] + \ldots$	$= \frac{60}{362\ 880}$	(*) h^9
59. h^{10}	$\frac{15}{3\ 628\ 800} = \frac{1}{4} c_3 \cdot \gamma_{32}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{4} c_4\, [\gamma_{42}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, \alpha_3^2\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)] + \ldots$	$= \frac{15}{362\ 880}$	h^9
60. h^{10} (*)	$\frac{15}{3\ 628\ 800} = \frac{1}{4} c_3 \cdot \gamma_{32}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{4} c_4\, [\gamma_{42}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, \alpha_3^2\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)] + \ldots$	$= \frac{15}{362\ 880}$	(*) h^9
61. h^{10}	$\frac{10}{3\ 628\ 800} = \frac{1}{2} c_3 \cdot \gamma_{32}\, (\gamma_{21}\, \alpha_1)^2 + \frac{1}{2} c_4\, [\gamma_{42}\, (\gamma_{21}\, \alpha_1)^2 + \gamma_{43}\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)^2] + \ldots$	$= \frac{10}{362\ 880}$	h^9
62. h^{10} (*)	$\frac{15}{3\ 628\ 800} = \frac{1}{4} c_3 \cdot \gamma_{32}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{4} c_4\, [\gamma_{42}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, \alpha_3^2\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)] + \ldots$	$= \frac{15}{362\ 880}$	(*) h^9
63. h^{10} (*)	$\frac{15}{3\ 628\ 800} = \frac{1}{4} c_3 \cdot \gamma_{32}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{4} c_4\, [\gamma_{42}\, \alpha_2^2 \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, \alpha_3^2\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)] + \ldots$	$= \frac{15}{362\ 880}$	(*) h^9
64. h^{10}	$\frac{6}{3\ 628\ 800} = \frac{1}{6} c_3 \cdot \gamma_{32}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^3 + \frac{1}{6} c_4\, [\gamma_{42}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^3 + \gamma_{43}\, \alpha_3\, (\gamma_{31}\, \alpha_1^3 + \gamma_{32}\, \alpha_2^3)] + \ldots$	$= \frac{6}{362\ 880}$	h^9
65. h^{10} (*)	$\frac{18}{3\ 628\ 800} = \frac{1}{2} c_3 \cdot \gamma_{32}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^3 + \frac{1}{2} c_4\, [\gamma_{42}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1^3 + \gamma_{43}\, \alpha_3\, (\gamma_{31}\, \alpha_1^3 + \gamma_{32}\, \alpha_2^3)] + \ldots$	$= \frac{18}{362\ 880}$	(*) h^9
66. h^{10}	$\frac{6}{3\ 628\ 800} = c_4 \cdot \gamma_{43}\, \alpha_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1 + c_5\, \{\gamma_{53}\, \alpha_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1 + \gamma_{54}\, \alpha_4\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1 + \gamma_{43}\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)]\} + \ldots$	$= \frac{6}{362\ 880}$	h^9
67. h^{10}	$\frac{1}{3\ 628\ 800} = \frac{1}{24} c_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^4 + \frac{1}{24} c_4\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^4 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^4 + \gamma_{32}\, \alpha_2^4)] + \ldots$	$= \frac{1}{362\ 880}$	h^9
68. h^{10} (*)	$\frac{6}{3\ 628\ 800} = \frac{1}{4} c_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^4 + \frac{1}{4} c_4\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^4 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^4 + \gamma_{32}\, \alpha_2^4)] + \ldots$	$= \frac{6}{362\ 880}$	(*) h^9
69. h^{10} (*)	$\frac{3}{3\ 628\ 800} = \frac{1}{8} c_3 \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^4 + \frac{1}{8} c_4\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^4 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^4 + \gamma_{32}\, \alpha_2^4)] + \ldots$	$= \frac{3}{362\ 880}$	(*) h^9
70. h^{10}	$\frac{4}{3\ 628\ 800} = c_4 \cdot \gamma_{43} \cdot \gamma_{32}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1 + c_5\, \{\gamma_{53} \cdot \gamma_{32}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1 + \gamma_{54}\, [\gamma_{42}\, \alpha_2 \cdot \gamma_{21}\, \alpha_1 + \gamma_{43}\, \alpha_3\, (\gamma_{31}\, \alpha_1 + \gamma_{32}\, \alpha_2)]\} + \ldots$	$= \frac{4}{362\ 880}$	h^9
71. h^{10}	$\frac{1}{3\ 628\ 800} = \frac{1}{2} c_4 \cdot \gamma_{43} \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{2} c_5\, \{\gamma_{53} \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{54}\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)]\} + \ldots$	$= \frac{1}{362\ 880}$	h^9
72. h^{10} (*)	$\frac{1}{3\ 628\ 800} = \frac{1}{2} c_4 \cdot \gamma_{43} \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^2 + \frac{1}{2} c_5\, \{\gamma_{53} \cdot \gamma_{32} \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{54}\, [\gamma_{42} \cdot \gamma_{21}\, \alpha_1^2 + \gamma_{43}\, (\gamma_{31}\, \alpha_1^2 + \gamma_{32}\, \alpha_2^2)]\} + \ldots$	$= \frac{1}{362\ 880}$	(*) h^9

TABLE 2. REDUCED EQUATIONS OF CONDITION FOR EIGHTH-ORDER FORMULA RKN 8(9)

	$x \leftarrow$	$c_K,\ \dot{c}_K$	$\rightarrow \dot{x}$	
h^2	$\frac{1}{2}$	$= c_0 + c_4 + c_5 + c_6 + c_7 + c_8 + c_9 + c_{10}$	$= 1$	h
h^3	$\frac{1}{6}$	$= c_4\ \alpha_4 + c_5\ \alpha_5 + c_6\ \alpha_6 + c_7\ \alpha_7 + c_8\ \alpha_8 + c_9\ \alpha_9 + c_{10}$	$= \frac{1}{2}$	h^2
h^4	$\frac{1}{12}$	$= c_4\ \alpha_4^2 + c_5\ \alpha_5^2 + c_6\ \alpha_6^2 + c_7\ \alpha_7^2 + c_8\ \alpha_8^2 + c_9\ \alpha_9^2 + c_{10}$	$= \frac{1}{3}$	h^3
h^5	$\frac{1}{20}$	$= c_4\ \alpha_4^3 + c_5\ \alpha_5^3 + c_6\ \alpha_6^3 + c_7\ \alpha_7^3 + c_8\ \alpha_8^3 + c_9\ \alpha_9^3 + c_{10}$	$= \frac{1}{4}$	h^4
h^6	$\frac{1}{30}$	$= c_4\ \alpha_4^4 + c_5\ \alpha_5^4 + c_6\ \alpha_6^4 + c_7\ \alpha_7^4 + c_8\ \alpha_8^4 + c_9\ \alpha_9^4 + c_{10}$	$= \frac{1}{5}$	h^5
h^7	$\frac{1}{42}$	$= c_4\ \alpha_4^5 + c_5\ \alpha_5^5 + c_6\ \alpha_6^5 + c_7\ \alpha_7^5 + c_8\ \alpha_8^5 + c_9\ \alpha_9^5 + c_{10}$	$= \frac{1}{6}$	h^6
h^8	$\frac{1}{56}$	$= c_4\ \alpha_4^6 + c_5\ \alpha_5^6 + c_6\ \alpha_6^6 + c_7\ \alpha_7^6 + c_8\ \alpha_8^6 + c_9\ \alpha_9^6 + c_{10}$	$= \frac{1}{7}$	h^7
h^9	$\frac{1}{72}$	$= c_4\ \alpha_4^7 + c_5\ \alpha_5^7 + c_6\ \alpha_6^7 + c_7\ \alpha_7^7 + c_8\ \alpha_8^7 + c_9\ \alpha_9^7 + c_{10}$	$= \frac{1}{8}$	h^8
h^8	$\frac{1}{1\ 680}$	$= c_4\ P_{44} + c_5\ P_{54} + c_6\ P_{64} + c_7\ P_{74} + c_8\ P_{84} + c_9\ P_{94} + c_{10} \left\{ \begin{matrix} P_{104} \\ \frac{1}{30} \end{matrix} \right\}$		
		$\dot{c}_4\ P_{44} + \dot{c}_5\ P_{54} + \dot{c}_6\ P_{64} + \dot{c}_7\ P_{74} + \dot{c}_8\ P_{84} + \dot{c}_9\ P_{94} + \dot{c}_{10} + P_{104}$	$= \frac{1}{210}$	h^7
h^9	$\frac{1}{2\ 160}$	$= c_4\ \alpha_4\ P_{44} + c_5\ \alpha_5\ P_{54} + c_6\ \alpha_6\ P_{64} + c_7\ \alpha_7\ P_{64} + c_8\ \alpha_8\ P_{84} + c_9\ \alpha_9\ P_{94} + c_{10} \cdot \frac{1}{30}$		
		$\dot{c}_4\ \alpha_4\ P_{44} + \dot{c}_5\ \alpha_5\ P_{54} + \dot{c}_6\ \alpha_6\ P_{64} + \dot{c}_7\ \alpha_7\ P_{74} + \dot{c}_8\ \alpha_8\ P_{84} + \dot{c}_9\ \alpha_9\ P_{94} + \dot{c}_{10}\ P_{104}$	$= \frac{1}{240}$	h^8
h^9	$\frac{1}{3\ 024}$	$= c_4\ P_{45} + c_5\ P_{55} + c_6\ P_{65} + c_7\ P_{75} + c_8\ P_{85} + c_9\ P_{95} + c_{10} \cdot \frac{1}{42}$		
		$\dot{c}_4\ P_{45} + \dot{c}_5\ P_{55} + \dot{c}_6\ P_{65} + \dot{c}_7\ P_{75} + \dot{c}_8\ P_{85} + \dot{c}_9\ P_{95} + \dot{c}_{10}\ P_{106}$	$= \frac{1}{336}$	h^8

TABLE 3. COEFFICIENTS FOR AN EIGHTH-ORDER FORMULA RKN 8(9)

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$: 0	1	2	3	4	5	6	7	8	9	10	c_κ	$\dot{c}_\kappa$
0	0												$\frac{121}{4\,200}$	$\frac{41}{840}$
1	$\frac{7}{80}$	$\frac{49}{12\,800}$											0	0
2	$\frac{7}{40}$	$\frac{49}{9\,600}$	$\frac{49}{4\,800}$										0	0
3	$\frac{5}{12}$	$\frac{16\,825}{381\,024}$	$-\frac{625}{11\,907}$	$\frac{18\,125}{190\,512}$									0	0
4	$\frac{1}{2}$	$\frac{23}{840}$	0	$\frac{50}{609}$	$\frac{9}{580}$								$\frac{43}{525}$	$\frac{34}{105}$
5	$\frac{1}{6}$	$\frac{533}{68\,040}$	0	$\frac{5\,050}{641\,277}$	$-\frac{19}{5\,220}$	$\frac{23}{12\,636}$							$\frac{33}{350}$	$\frac{9}{35}$
6	$\frac{1}{3}$	$-\frac{4\,469}{85\,050}$	0	$-\frac{2\,384\,000}{641\,277}$	$\frac{3\,896}{19\,575}$	$-\frac{1\,451}{15\,795}$	$\frac{502}{135}$						$\frac{17}{140}$	$\frac{9}{280}$
7	$\frac{2}{3}$	$\frac{694}{10\,125}$	0	0	$-\frac{5\,504}{10\,125}$	$\frac{424}{2\,025}$	$-\frac{104}{2\,025}$	$\frac{364}{675}$					$\frac{3}{56}$	$\frac{9}{280}$
8	$\frac{5}{6}$	$\frac{30\,203}{691\,200}$	0	0	0	$\frac{5\,797}{172\,800}$	$\frac{79\,391}{518\,400}$	$\frac{20\,609}{345\,600}$	$\frac{70\,609}{2\,073\,600}$				$\frac{31}{1\,050}$	$\frac{9}{35}$
9	$\frac{1}{12}$	$\frac{1\,040\,381\,917}{14\,863\,564\,800}$	0	$\frac{548\,042\,275}{109\,444\,608}$	$\frac{242\,737}{5\,345\,280}$	$\frac{569\,927\,617}{6\,900\,940\,800}$	$-\frac{2\,559\,686\,731}{530\,841\,600}$	$-\frac{127\,250\,389}{353\,894\,400}$	$-\frac{53\,056\,229}{2\,123\,366\,400}$	$\frac{23}{5\,120}$			$\frac{512}{5\,775}$	0
10	1	$-\frac{33\,213\,637}{179\,088\,000}$	0	$\frac{604\,400}{324\,597}$	$\frac{63\,826}{445\,875}$	0	$-\frac{6\,399\,863}{2\,558\,400}$	$\frac{110\,723}{511\,680}$	$\frac{559\,511}{35\,817\,600}$	$\frac{372\,449}{7\,675\,200}$	$\frac{756\,604}{839\,475}$		$\frac{1}{550}$	$\frac{41}{840}$
11	1	$\frac{121}{4\,200}$	0	0	0	$\frac{43}{525}$	$\frac{33}{350}$	$\frac{17}{140}$	$\frac{3}{56}$	$\frac{31}{1\,050}$	$\frac{512}{5\,775}$	$\frac{1}{550}$	$TE_x = \frac{1}{550}(f_{10} - f_{11})\,h^2$	

TABLE 4. ERROR COEFFICIENTS FOR RKN 8(9) OF TABLE 3

$$T_{29} = \frac{1}{8} \left(c_4 P_{45} + c_5 P_{55} + \ldots\right) - \frac{1}{24\ 192} \approx 0.000\ 000\ 096\ 588$$

$$\dot{T}_{10} = \frac{1}{48} \left(\dot{c}_4 \alpha_4^8 + \dot{c}_5 \alpha_5^8 + \ldots\right) - \frac{1}{432} \approx 0.000\ 000\ 535\ 837$$

$$\dot{T}_{26} = \frac{1}{8} \left(\dot{c}_4 \alpha_4^2 P_{44} + \dot{c}_5 \alpha_5^2 P_{54} + \ldots\right) - \frac{1}{2\ 160} \approx 0.000\ 000\ 004\ 019$$

$$\dot{T}_{45} = \frac{1}{8} \left(\dot{c}_4 \alpha_4 P_{45} + \dot{c}_5 \alpha_5 P_{55} + \ldots\right) - \frac{1}{3\ 024} \approx 0.000\ 000\ 616\ 351$$

$$\dot{T}_{51} = \frac{1}{2} \left[\dot{c}_4 \alpha_4 \left(\gamma_{42} P_{23} + \gamma_{43} P_{33}\right) + \dot{c}_5 \alpha_5 \left(\gamma_{52} P_{23} + \gamma_{53} P_{33} + \gamma_{54} P_{43}\right) + \ldots\right] - \frac{1}{15\ 120} \approx 0.000\ 000\ 092\ 283$$

$$\dot{T}_{52} = \frac{1}{6} \left\{\dot{c}_4 \alpha_4 \gamma_{43} \cdot \gamma_{32} \alpha_2^3 + \dot{c}_5 \alpha_5 \left[\gamma_{53} \cdot \gamma_{32} \alpha_2^3 + \gamma_{54} \left(\gamma_{52} \alpha_2^3 + \gamma_{53} \alpha_3^3\right)\right] + \ldots\right\} - \frac{1}{45\ 360} \approx -\ 0.000\ 000\ 043\ 977$$

$$\dot{T}_{58} = \frac{1}{12} \left(\dot{c}_4 P_{46} + \dot{c}_5 P_{56} + \ldots\right) - \frac{1}{6\ 048} \approx -\ 0.000\ 000\ 088\ 515$$

$$\dot{T}_{68} = \frac{1}{4} \left[\dot{c}_5 \gamma_{54} P_{44} + \dot{c}_6 \left(\gamma_{64} P_{44} + \gamma_{65} P_{54}\right) + \ldots\right] - \frac{1}{60\ 480} \approx -\ 0.000\ 000\ 000\ 332$$

TABLE 5. REDUCED EQUATIONS OF CONDITION FOR SEVENTH-ORDER FORMULA RKN 7(8)

	$x \longleftarrow$	c_κ, $\dot{c}_\kappa$	$\longrightarrow \dot{x}$	
h^2	$\frac{1}{2}$	$= c_0 + c_3 + c_4 + c_5 + c_6 + c_7 + c_8$	$= 1$	h
h^3	$\frac{1}{6}$	$= c_3\ \alpha_3 + c_4\ \alpha_4 + c_5\ \alpha_5 + c_6\ \alpha_6 + c_7\ \alpha_7 + c_8$	$= \frac{1}{2}$	h^2
h^4	$\frac{1}{12}$	$= c_3\ \alpha_3^2 + c_4\ \alpha_4^2 + c_5\ \alpha_5^2 + c_6\ \alpha_6^2 + c_7\ \alpha_7^2 + c_8$	$= \frac{1}{3}$	h^3
h^5	$\frac{1}{20}$	$= c_3\ \alpha_3^3 + c_4\ \alpha_4^3 + c_5\ \alpha_5^3 + c_6\ \alpha_6^3 + c_7\ \alpha_7^3 + c_8$	$= \frac{1}{4}$	h^4
h^6	$\frac{1}{30}$	$= c_3\ \alpha_3^4 + c_4\ \alpha_4^4 + c_5\ \alpha_5^4 + c_6\ \alpha_6^4 + c_7\ \alpha_7^4 + c_8$	$= \frac{1}{5}$	h^5
h^7	$\frac{1}{42}$	$= c_3\ \alpha_3^5 + c_4\ \alpha_4^5 + c_5\ \alpha_5^5 + c_6\ \alpha_6^5 + c_7\ \alpha_7^5 + c_8$	$= \frac{1}{6}$	h^6
h^8	$\frac{1}{56}$	$= c_3\ \alpha_3^6 + c_4\ \alpha_4^6 + c_5\ \alpha_5^6 + c_6\ \alpha_6^6 + c_7\ \alpha_7^6 + c_8$	$= \frac{1}{7}$	h^7
h^8	$\frac{1}{1\,680}$	$= c_3\ P_{34} + c_4\ P_{44} + c_5\ P_{54} + c_6\ P_{64} + c_7\ P_{74} + c_8\ \frac{1}{30}$		
		$\dot{c}_3\ P_{34} + \dot{c}_4\ P_{44} + \dot{c}_5\ P_{54} + c_6\ P_{64} + \dot{c}_7\ P_{74} + \dot{c}_8\ P_{84}$	$= \frac{1}{210}$	h^7

TABLE 6. COEFFICIENTS FOR A SEVENTH-ORDER FORMULA RKN 7(8)

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$: 0	1	2	3	4	5	6	7	8	c_κ	$\hat{c}_\kappa$
0	0										$\frac{67}{2\ 016}$	$\frac{23}{2\ 016}$
1	$\frac{19}{375}$	$\frac{361}{281\ 250}$									0	0
2	$-\frac{7}{10}$	$\frac{10\ 437}{7\ 600}$	$-\frac{343}{304}$								0	0
3	$\frac{1}{10}$	$\frac{547}{319\ 200}$	$\frac{1\ 125}{342\ 304}$	$-\frac{1}{4\ 729\ 200}$							$\frac{440}{3\ 969}$	$\frac{880}{3\ 969}$
4	$\frac{1}{5}$	$\frac{74}{9\ 975}$	$-\frac{1\ 125}{791\ 578}$	$-\frac{1}{157\ 640}$	$\frac{311}{22\ 200}$						$\frac{25}{252}$	$-\frac{25}{2\ 016}$
5	$\frac{2}{5}$	$\frac{1\ 028}{29\ 925}$	$-\frac{6\ 375}{1\ 583\ 156}$	$-\frac{55}{319\ 221}$	$-\frac{13}{1\ 665}$	$\frac{467}{8\ 100}$					$\frac{425}{3\ 024}$	$\frac{1\ 075}{3\ 024}$
6	$\frac{3}{5}$	$\frac{148\ 349}{19\ 254\ 600}$	$\frac{6\ 375}{1\ 583\ 156}$	0	$\frac{1\ 299\ 964}{14\ 060\ 925}$	$\frac{4\ 783}{253\ 350}$	$\frac{173\ 101}{3\ 040\ 200}$				$\frac{5}{72}$	$\frac{65}{1\ 008}$
7	$\frac{4}{5}$	$\frac{116\ 719\ 112}{18\ 953\ 746\ 875}$	$\frac{1\ 125}{791\ 578}$	$\frac{1\ 680\ 359}{2\ 992\ 696\ 875}$	$\frac{51\ 962\ 281}{585\ 871\ 875}$	$\frac{104\ 130\ 509}{855\ 056\ 250}$	$\frac{1\ 995\ 658}{47\ 503\ 125}$	$\frac{15\ 029}{253\ 125}$			$\frac{625}{14\ 112}$	$\frac{4\ 225}{14\ 112}$
8	1	$\frac{604\ 055\ 892\ 451}{4\ 935\ 014\ 784\ 000}$	0	$-\frac{206\ 360\ 699}{115\ 664\ 409\ 000}$	0	$\frac{32\ 963\ 694\ 031}{528\ 751\ 584\ 000}$	$\frac{9\ 676\ 095\ 011}{39\ 166\ 784\ 000}$	$\frac{1\ 641\ 775\ 937}{176\ 250\ 528\ 000}$	$\frac{2\ 851\ 784\ 579}{47\ 000\ 140\ 800}$		$\frac{11}{4\ 536}$	$\frac{1\ 087}{18\ 144}$
9	1	$\frac{67}{2\ 016}$	0	0	$\frac{440}{3\ 969}$	$\frac{25}{252}$	$\frac{425}{3\ 024}$	$\frac{5}{72}$	$\frac{625}{14\ 112}$	$\frac{11}{4\ 536}$	$TE_x - \frac{11}{4\ 536}(f_8 - f_9)h^2$	

TABLE 7. ERROR COEFFICIENTS FOR RKN 7(8) OF TABLE 6.

$$T_{15} = \frac{1}{4} \left(c_3 P_{34} + c_4 P_{44} + \ldots \right) - \frac{1}{6\ 720} \approx -0.000\ 000\ 774$$

$$\dot{T}_6 = \frac{1}{24} \left(\dot{c}_3 \alpha_3^7 + \dot{c}_4 \alpha_4^7 + \ldots \right) - \frac{1}{192} \approx 0.000\ 003\ 492$$

$$\dot{T}_{15} = \frac{1}{12} \left[\dot{c}_3 \alpha_3^2 \cdot \gamma_{32} \alpha_2^3 + \dot{c}_4 \alpha_4^2 \left(\gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3 \right) + \ldots \right] - \frac{1}{1\ 920} \approx 0.000\ 000\ 348$$

$$\dot{T}_{22} = \frac{1}{4} \left(\dot{c}_3 \alpha_3 P_{34} + \dot{c}_4 \alpha_4 P_{44} + \ldots \right) - \frac{1}{960} \approx 0.000\ 000\ 131$$

$$\dot{T}_{25} = \frac{1}{2} \left[\dot{c}_3 \alpha_3 \gamma_{32} P_{22} + \dot{c}_4 \alpha_4 \left(\gamma_{42} P_{22} + \gamma_{43} P_{32} \right) + \ldots \right] - \frac{1}{5\ 760} \approx -0.000\ 000\ 014$$

$$\dot{T}_{29} = \frac{1}{8} \left(\dot{c}_3 P_{35} + \dot{c}_4 P_{45} + \ldots \right) - \frac{1}{2\ 688} \approx -0.000\ 002\ 667$$

$$\dot{T}_{36} = \frac{1}{6} \left\{ \dot{c}_4 \gamma_{43} \cdot \gamma_{32} \alpha_2^3 + \dot{c}_5 \left[\gamma_{53} \cdot \gamma_{32} \alpha_2^3 + \gamma_{54} \left(\gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3 \right) \right] + \ldots \right\} - \frac{1}{40\ 320} \approx -0.000\ 000\ 177$$

TABLE 8. REDUCED EQUATIONS OF CONDITION FOR SIXTH-ORDER FORMULA RKN 6(7)

	$x \longleftarrow$	$c_\kappa, \; \dot{c}_\kappa$	$\longrightarrow \dot{x}$	
h^2	$\frac{1}{2}$	$= c_0 + c_2 + c_3 + c_4 + c_5 + c_6$	$= 1$	h
h^3	$\frac{1}{6}$	$= c_2\,\alpha_2 + c_3\,\alpha_3 + c_4\,\alpha_4 + c_5\,\alpha_5 + c_6$	$= \frac{1}{2}$	h^2
h^4	$\frac{1}{12}$	$= c_2\,\alpha_2^2 + c_3\,\alpha_3^2 + c_4\,\alpha_4^2 + c_5\,\alpha_5^2 + c_6$	$= \frac{1}{3}$	h^3
h^5	$\frac{1}{20}$	$= c_2\,\alpha_2^3 + c_3\,\alpha_3^3 + c_4\,\alpha_4^3 + c_5\,\alpha_5^3 + c_6$	$= \frac{1}{4}$	h^4
h^6	$\frac{1}{30}$	$= c_2\,\alpha_2^4 + c_3\,\alpha_3^4 + c_4\,\alpha_4^4 + c_5\,\alpha_5^4 + c_6$	$= \frac{1}{5}$	h^5
h^7	$\frac{1}{42}$	$= c_2\,\alpha_2^5 + c_3\,\alpha_3^5 + c_4\,\alpha_4^5 + c_5\,\alpha_5^5 + c_6$	$= \frac{1}{6}$	h^6
h^7	$\frac{1}{840}$	$= c_2\,P_{23} + c_3\,P_{33} + c_4\,P_{43} + c_5\,P_{53} + c_6 \cdot \frac{1}{20}$ $\dot{c}_2\,P_{23} + \dot{c}_3\,P_{33} + \dot{c}_4\,P_{43} + \dot{c}_5\,P_{53} + \dot{c}_6\,P_{63}$	$= \frac{1}{120}$	h^6

TABLE 9. COEFFICIENTS FOR A SIXTH-ORDER FORMULA RKN 6(7)

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$ 0	1	2	3	4	5	6	c_κ	$\dot{c}_\kappa$
0	0								$\frac{61}{1\,008}$	$\frac{19}{288}$
1	$\frac{1}{10}$	$\frac{1}{200}$							0	0
2	$\frac{1}{5}$	$\frac{1}{150}$	$\frac{1}{75}$						$\frac{475}{2\,016}$	$\frac{25}{96}$
3	$\frac{2}{5}$	$\frac{2}{75}$	0	$\frac{4}{75}$					$\frac{25}{504}$	$\frac{25}{144}$
4	$\frac{3}{5}$	$\frac{9}{200}$	0	$\frac{9}{100}$	$\frac{9}{200}$				$\frac{125}{1\,008}$	$\frac{25}{144}$
5	$\frac{4}{5}$	$\frac{199}{3\,600}$	$-\frac{19}{150}$	$\frac{47}{120}$	$-\frac{119}{1\,200}$	$\frac{89}{900}$			$\frac{25}{1\,008}$	$\frac{25}{96}$
6	1	$-\frac{179}{1\,824}$	$\frac{17}{38}$	0	$-\frac{37}{152}$	$\frac{219}{456}$	$-\frac{157}{1\,824}$		$\frac{11}{2\,016}$	$\frac{19}{288}$
7	1	$\frac{61}{1\,008}$	0	$\frac{475}{2\,016}$	$\frac{25}{504}$	$\frac{125}{1\,008}$	$\frac{25}{1\,008}$	$\frac{11}{2\,016}$	$TE_x = \frac{11}{2\,016}(f_6 - f_7)h^2$	

TABLE 10. ERROR COEFFICIENTS FOR RKN 6(7) OF TABLE 9

$$T_9 = \frac{1}{2}(c_2 P_{23} + c_3 P_{33} + c_4 P_{43} + c_5 P_{53} + c_6 P_{63}) - \frac{1}{1\ 680} \approx -0.000\ 014\ 909$$

$$\dot{T}_6 = \frac{1}{12}(\dot{c}_2 \alpha_2^6 + \dot{c}_3 \alpha_3^6 + \dot{c}_4 \alpha_4^6 + \dot{c}_5 \alpha_5^6 + \dot{c}_6 \alpha_6^6) - \frac{1}{84} \approx 0.000\ 017\ 460$$

$$\dot{T}_{13} = \frac{1}{2}(\dot{c}_2 \alpha_2 P_{23} + \dot{c}_3 \alpha_3 P_{33} + \dot{c}_4 \alpha_4 P_{43} + \dot{c}_5 \alpha_5 P_{53} + \dot{c}_6 \alpha_6 P_{63}) - \frac{1}{280}$$

$$\approx -0.000\ 022\ 644$$

$$\dot{T}_{14} = \frac{1}{6}[\dot{c}_3 \alpha_3 \cdot \gamma_{32} \alpha_2^3 + \dot{c}_4 \alpha_4 (\gamma_{42} \alpha_2^3 + \gamma_{43} \alpha_3^3) + \dot{c}_5 \alpha_5 (\gamma_{52} \alpha_2^3 + \gamma_{53} \alpha_3^3 + \gamma_{54} \alpha_4^3)$$

$$+ \dot{c}_6 \alpha_6 (\gamma_{62} \alpha_2^3 + \gamma_{63} \alpha_3^3 + \gamma_{64} \alpha_4^3 + \gamma_{65} \alpha_5^3)] - \frac{1}{840} \approx -0.000\ 008\ 185$$

$$\dot{T}_{16} = \frac{1}{4}(\dot{c}_2 P_{24} + \dot{c}_3 P_{34} + \dot{c}_4 P_{44} + \dot{c}_5 P_{54} + \dot{c}_6 P_{64}) - \frac{1}{840} \approx -0.000\ 078\ 323$$

TABLE 11. REDUCED EQUATIONS OF CONDITION FOR FIFTH-ORDER FORMULA RKN 5(6)

	x ⟵ c_κ $\dot{c}_\kappa$ ⟶ $\dot{x}$	
h^2	$\frac{1}{2} = c_0 + c_2 + c_3 + c_4 + c_5 = 1$	h
h^3	$\frac{1}{6} = c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5 = \frac{1}{2}$	h^2
h^4	$\frac{1}{12} = c_2\alpha_2^2 + c_3\alpha_3^2 + c_4\alpha_4^2 + c_5 = \frac{1}{3}$	h^3
h^5	$\frac{1}{20} = c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 + c_5 = \frac{1}{4}$	h^4
h^6	$\frac{1}{30} = c_2\alpha_2^4 + c_3\alpha_3^4 + c_4\alpha_4^4 + c_5 = \frac{1}{5}$	h^5
h^6	$\frac{1}{360} = c_2 P_{22} + c_3 P_{32} + c_4 P_{42} + c_5 \cdot \frac{1}{12}$ $\dot{c}_2 P_{22} + \dot{c}_3 P_{32} + \dot{c}_4 P_{42} + \dot{c}_5 P_{52} = \frac{1}{60}$	h^5

TABLE 12. COEFFICIENTS FOR A FIFTH-ORDER FORMULA RKN 5(6)

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$ 0	1	2	3	4	5	c_κ	$\dot{c}_\kappa$
0	0							$\frac{11}{240}$	$\frac{1}{24}$
1	$\frac{1}{12}$	$\frac{1}{288}$						0	0
2	$\frac{1}{6}$	$\frac{1}{216}$	$\frac{1}{108}$					$\frac{108}{475}$	$\frac{27}{95}$
3	$\frac{1}{2}$	0	0	$\frac{1}{8}$				$\frac{8}{45}$	$\frac{1}{3}$
4	$\frac{4}{5}$	$\frac{16}{125}$	0	$\frac{4}{125}$	$\frac{4}{25}$			$\frac{125}{2\,736}$	$\frac{125}{456}$
5	1	$-\frac{247}{1\,152}$	0	$\frac{12}{19}$	$\frac{7}{432}$	$\frac{4\,375}{65\,664}$		$\frac{1}{300}$	$\frac{1}{15}$
6	1	$\frac{11}{240}$	0	$\frac{108}{475}$	$\frac{8}{45}$	$\frac{125}{2\,736}$	$\frac{1}{300}$	$TE_x = \frac{1}{300}(f_5 - f_6)h^2$	

TABLE 13. ERROR COEFFICIENTS FOR RKN 5(6) OF TABLE 12

$T_5 = \frac{1}{2}(c_2 P_{22} + c_3 P_{32} + c_4 P_{42} + c_5 P_{52}) - \frac{1}{720} \approx -0.000\ 031\ 829$
$\dot{T}_3 = \frac{1}{8}(\dot{c}_2 \alpha_2^5 + \dot{c}_3 \alpha_3^5 + \dot{c}_4 \alpha_4^5 + \dot{c}_5 \alpha_5^5) - \frac{1}{48} \approx 0.000\ 034\ 722$
$\dot{T}_6 = \frac{1}{2}(\dot{c}_2 \alpha_2 P_{22} + \dot{c}_3 \alpha_3 P_{32} + \dot{c}_4 \alpha_4 P_{42} + \dot{c}_5 \alpha_5 P_{52}) - \frac{1}{144} \approx -0.000\ 028\ 935$
$\dot{T}_9 = \frac{1}{2}(\dot{c}_2 P_{23} + \dot{c}_3 P_{33} + \dot{c}_4 P_{43} + \dot{c}_5 P_{53}) - \frac{1}{240} \approx -0.000\ 005\ 838$
$\dot{T}_{10} = \frac{1}{6}(\dot{c}_3 P_{33} + \dot{c}_4 P_{43} + \dot{c}_5 P_{53}) - \frac{1}{720} \approx -0.000\ 002\ 200$

TABLE 14. REDUCED EQUATIONS OF CONDITION FOR FOURTH-ORDER FORMULA RKN 4(5)

	$x \leftarrow c_K$	$\dot{c}_K \rightarrow \dot{x}$	
h^2	$\frac{1}{2} = c_0 + c_1 + c_2 + c_3$	$= 1$	h
h^3	$\frac{1}{6} = c_1\alpha_1 + c_2\alpha_2 + c_3$	$= \frac{1}{2}$	h^2
h^4	$\frac{1}{12} = c_1\alpha_1^2 + c_2\alpha_2^2 + c_3$	$= \frac{1}{3}$	h^3
h^5	$\frac{1}{20} = c_1\alpha_1^3 + c_2\alpha_2^3 + c_3$	$= \frac{1}{4}$	h^4
h^5	$\frac{1}{120} = c_2 \cdot \gamma_{21}\alpha_1 + c_3 \cdot \frac{1}{6}$		
	$\dot{c}_2\ \gamma_{21}\ \alpha_1 + \dot{c}_3(\gamma_{31}\ \alpha_1 + \gamma_{32}\ \alpha_2)$	$= \frac{1}{24}$	h^4

TABLE 15. COEFFICIENTS FOR A FOURTH-ORDER FORMULA RKN 4(5)

$\kappa \backslash \lambda$	α_κ	$\gamma_{\kappa\lambda}$ 0	1	2	3	c_κ	$\dot{c}_\kappa$
0	0					$\frac{13}{120}$	$\frac{1}{8}$
1	$\frac{1}{3}$	$\frac{1}{18}$				$\frac{3}{10}$	$\frac{3}{8}$
2	$\frac{2}{3}$	0	$\frac{2}{9}$			$\frac{3}{40}$	$\frac{3}{8}$
3	1	$\frac{1}{3}$	0	$\frac{1}{6}$		$\frac{1}{60}$	$\frac{1}{8}$
4	1	$\frac{13}{120}$	$\frac{3}{10}$	$\frac{3}{40}$	$\frac{1}{60}$	$TE_x = \frac{1}{60}(f_3 - f_4)h^2$	

TABLE 16. ERROR COEFFICIENTS FOR RKN 4(5) OF TABLE 15

$$T_3 = c_2 P_{21} + c_3 P_{31} - \frac{1}{120} \approx -0.000\ 925\ 926$$

$$\dot{T}_2 = \frac{1}{4}(\dot{c}_1 \alpha_1^4 + \dot{c}_2 \alpha_2^4 + \dot{c}_3 \alpha_3^4) - \frac{1}{20} \approx 0.000\ 925\ 926$$

$$\dot{T}_4 = \dot{c}_2 \alpha_2 \cdot P_{21} + \dot{c}_3 \alpha_3 P_{31} - \frac{1}{30} \approx -0.000\ 925\ 926$$

$$\dot{T}_5 = \frac{1}{2}(\dot{c}_2 P_{22} + \dot{c}_3 P_{32}) - \frac{1}{120} \approx 0.000\ 925\ 926$$

TABLE 17. COEFFICIENTS FOR NYSTRÖM'S FOURTH-ORDER FORMULA RKN 4

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$		c_κ	$\dot{c}_\kappa$
		0	1		
0	0			$\frac{1}{6}$	$\frac{1}{6}$
1	$\frac{1}{2}$	$\frac{1}{8}$		$\frac{1}{3}$	$\frac{2}{3}$
2	1	0	$\frac{1}{2}$	0	$\frac{1}{6}$

TABLE 18. ERROR COEFFICIENTS FOR NYSTRÖM'S FOURTH-ORDER FORMULA RKN 4 OF TABLE 17

$$T_2 = \frac{1}{2} c_1 \alpha_1^3 - \frac{1}{40} \approx - 0.004\ 166\ 667$$

$$T_3 = - \frac{1}{120} \approx - 0.008\ 333\ 333$$

$$\dot{T}_2 = \frac{1}{4} (\dot{c}_1 \alpha_1^4 + \dot{c}_2 \alpha_2^4) - \frac{1}{20} \approx 0.002\ 083\ 333$$

$$\dot{T}_4 = \dot{c}_2 \alpha_2 P_{21} - \frac{1}{30} \approx 0.008\ 333\ 333$$

$$\dot{T}_5 = \frac{1}{2} \dot{c}_2 P_{22} - \frac{1}{120} \approx 0.002\ 083\ 333$$

TABLE 19. COEFFICIENTS FOR NYSTRÖM'S FIFTH-ORDER FORMULA RKN 5

$\kappa \backslash \lambda$	α_κ	$\gamma_{\kappa\lambda}$ 0	$\gamma_{\kappa\lambda}$ 1	$\gamma_{\kappa\lambda}$ 2	c_κ	$\dot{c}_\kappa$
0	0				$\frac{1}{24}$	$\frac{1}{24}$
1	$\frac{1}{5}$	$\frac{1}{50}$			$\frac{25}{84}$	$\frac{125}{336}$
2	$\frac{2}{3}$	$-\frac{1}{27}$	$\frac{7}{27}$		$\frac{9}{56}$	$\frac{27}{56}$
3	1	$\frac{3}{10}$	$-\frac{2}{35}$	$\frac{9}{35}$	0	$\frac{5}{48}$

TABLE 20. ERROR COEFFICIENTS FOR NYSTRÖM'S FIFTH-ORDER FORMULA RKN 5 OF TABLE 19

$$T_2 = \frac{1}{4}(c_1 \alpha_1^4 + c_2 \alpha_2^4) - \frac{1}{120} \approx -0.000\ 277\ 778$$

$$T_5 = \frac{1}{2} c_2 P_{22} - \frac{1}{720} \approx -0.000\ 555\ 556$$

$$\dot{T}_3 = \frac{1}{8}(\dot{c}_1 \alpha_1^5 + \dot{c}_2 \alpha_2^5 + \dot{c}_3 \alpha_3^5) - \frac{1}{48} \approx 0.000\ 138\ 889$$

$$\dot{T}_6 = \frac{1}{2}(\dot{c}_2 \alpha_2 P_{22} + \dot{c}_3 \alpha_3 P_{32}) - \frac{1}{144} \approx 0.000\ 555\ 556$$

$$\dot{T}_9 = \frac{1}{2}(\dot{c}_2 P_{23} + \dot{c}_3 P_{33}) - \frac{1}{240} \approx 0.000\ 277\ 778$$

TABLE 21. COEFFICIENTS FOR ALBRECHT'S SIXTH-ORDER FORMULA RKN 6

κ \ λ	α_κ	$\gamma_{\kappa\lambda}$ 0	1	2	3	c_κ	$\dot{c}_\kappa$
0	0					$\frac{7}{90}$	$\frac{7}{90}$
1	$\frac{1}{4}$	$\frac{1}{32}$				$\frac{4}{15}$	$\frac{16}{45}$
2	$\frac{1}{2}$	$-\frac{1}{24}$	$\frac{1}{6}$			$\frac{1}{15}$	$\frac{2}{15}$
3	$\frac{3}{4}$	$\frac{3}{32}$	$\frac{1}{8}$	$-\frac{1}{16}$		$\frac{4}{45}$	$\frac{16}{45}$
4	1	0	$\frac{3}{7}$	$-\frac{1}{14}$	$\frac{1}{7}$	0	$\frac{7}{90}$

TABLE 22. ERROR COEFFICIENTS FOR ALBRECHT'S SIXTH-ORDER FORMULA RKN 6 OF TABLE 21

$$T_3 = \frac{1}{8}(c_1\,\alpha_1^5 + c_2\,\alpha_2^5 + c_3\,\alpha_3^5) - \frac{1}{336} \approx -0.000\ 046\ 503$$

$$T_4 = \frac{1}{2}(c_2\,\alpha_2^2\,P_{21} + c_3\,\alpha_3^2\,P_{31}) - \frac{1}{504} \approx -0.000\ 074\ 405$$

$$T_6 = \frac{1}{2}(c_2\,\alpha_2\,P_{22} + c_3\,\alpha_3\,P_{32}) - \frac{1}{1\ 008} \approx -0.000\ 037\ 202$$

$$T_9 = \frac{1}{2}(c_2\,P_{23} + c_3\,P_{33}) - \frac{1}{1\ 680} \approx -0.000\ 074\ 405$$

$$T_{10} = c_3\,\gamma_{32}\,P_{21} - \frac{1}{5\ 040} \approx 0.000\ 032\ 873$$

$$\dot{T}_3 = \frac{1}{16}(\dot{c}_1\,\alpha_1^6 + \dot{c}_2\,\alpha_2^6 + \dot{c}_3\,\alpha_3^6 + \dot{c}_4\,\alpha_4^6) - \frac{1}{112} \approx 0.000\ 023\ 251$$

$$\dot{T}_6 = \frac{1}{2}(\dot{c}_2\,\alpha_2^3\,P_{21} + \dot{c}_3\,\alpha_3^3\,P_{31} + \dot{c}_4\,\alpha_4^3\,P_{41}) - \frac{1}{84} \approx 0.000\ 074\ 405$$

$$\dot{T}_7 = \frac{1}{4}(\dot{c}_2\,\alpha_2^2\,P_{22} + \dot{c}_3\,\alpha_3^2\,P_{32} + \dot{c}_4\,\alpha_4^2\,P_{42}) - \frac{1}{336} \approx 0.000\ 018\ 601$$

$$\dot{T}_9 = \frac{1}{2}(\dot{c}_2\,P_{21}^2 + \dot{c}_3\,P_{31}^2 + \dot{c}_4\,P_{41}^2) - \frac{1}{504} \approx 0.000\ 066\ 138$$

$$\dot{T}_{13} = \frac{1}{2}(\dot{c}_2\,\alpha_2\,P_{23} + \dot{c}_3\,\alpha_3\,P_{33} + \dot{c}_4\,\alpha_4\,P_{43}) - \frac{1}{280} \approx 0.000\ 074\ 405$$

$$\dot{T}_{14} = \dot{c}_3\,\alpha_3\,\gamma_{32}\,P_{21} + \dot{c}_4\,\alpha_4\,(\gamma_{42}\,P_{21} + \gamma_{43}\,P_{31}) - \frac{1}{840} \approx -0.000\ 033\ 069$$

$$\dot{T}_{16} = \frac{1}{4}(\dot{c}_2\,P_{24} + \dot{c}_3\,P_{34} + \dot{c}_4\,P_{44}) - \frac{1}{840} \approx 0.000\ 046\ 503$$

$$\dot{T}_{18} = \dot{c}_3 \cdot \gamma_{32}\,\alpha_2\,P_{21} + \dot{c}_4\,(\gamma_{42}\,\alpha_2\,P_{21} + \gamma_{43}\,\alpha_3\,P_{31}) - \frac{1}{1\ 260} \approx 0.000\ 074\ 405$$

$$\dot{T}_{19} = \frac{1}{2}\,[\dot{c}_3\,\gamma_{32}\,P_{22} + \dot{c}_4\,(\gamma_{42}\,P_{22} + \gamma_{43}\,P_{32})\,] - \frac{1}{5\ 040} \approx 0.000\ 018\ 601$$

TABLE 23. APPLICATION OF THE VARIOUS FORMULAS TO A NUMERICAL PROBLEM

Problem: $\begin{cases} \ddot{x} = -4t^2x - \dfrac{2y}{\sqrt{x^2+y^2}} \\ \ddot{y} = -4t^2y + \dfrac{2x}{\sqrt{x^2+y^2}} \end{cases}$ Initial Values: $t_i = \sqrt{\pi/2}$ $\begin{cases} x_i = 0, & \dot{x}_i = -\sqrt{2\pi} \\ y_i = 1, & \dot{y}_i = 0 \end{cases}$ Solution: $\begin{cases} x = \cos(t^2) \\ y = \sin(t^2) \end{cases}$

Formula	t	Steps	Δx	Δy	$\Delta \dot{x}$	$\Delta \dot{y}$	7094 Time (min)
NYSTRÖM : RKN-4	10	172 011	$-0.2099 \cdot 10^{-11}$	$-0.3437 \cdot 10^{-11}$	$+0.6858 \cdot 10^{-10}$	$-0.4174 \cdot 10^{-10}$	14.38
RK 4(5)	10	124 073	$-0.1300 \cdot 10^{-11}$	$-0.2169 \cdot 10^{-11}$	$+0.4346 \cdot 10^{-10}$	$-0.2615 \cdot 10^{-10}$	13.95
RKN 4(5)	10	112 529	$-0.1293 \cdot 10^{-11}$	$-0.2114 \cdot 10^{-11}$	$+0.4231 \cdot 10^{-10}$	$-0.2577 \cdot 10^{-10}$	6.35
NYSTRÖM : RKN-5	10	27 584	$-0.3156 \cdot 10^{-12}$	$-0.5825 \cdot 10^{-12}$	$+0.1158 \cdot 10^{-10}$	$-0.6269 \cdot 10^{-11}$	3.03
RK 5(6)	10	27 177	$-0.3609 \cdot 10^{-12}$	$-0.5067 \cdot 10^{-12}$	$+0.1015 \cdot 10^{-10}$	$-0.7176 \cdot 10^{-11}$	3.99
RKN 5(6)	10	18 465	$-0.2273 \cdot 10^{-12}$	$-0.3933 \cdot 10^{-12}$	$+0.7808 \cdot 10^{-11}$	$-0.4555 \cdot 10^{-11}$	1.53
ALBRECHT : RKN-6	10	10 465	$-0.1242 \cdot 10^{-12}$	$-0.2273 \cdot 10^{-12}$	$+0.4539 \cdot 10^{-11}$	$-0.2412 \cdot 10^{-11}$	1.60
RK 6(7)	10	10 873	$-0.1040 \cdot 10^{-12}$	$-0.1806 \cdot 10^{-12}$	$+0.3626 \cdot 10^{-11}$	$-0.2069 \cdot 10^{-11}$	2.05
RKN 6(7)	10	7 841	$-0.7783 \cdot 10^{-13}$	$-0.1376 \cdot 10^{-12}$	$+0.2739 \cdot 10^{-11}$	$-0.1593 \cdot 10^{-11}$	0.78
RK 7(8)	10	4 541	$-0.6745 \cdot 10^{-13}$	$-0.1242 \cdot 10^{-12}$	$-0.2471 \cdot 10^{-11}$	$-0.1343 \cdot 10^{-11}$	1.18
RKN 7(8)	10	2 752	$-0.2531 \cdot 10^{-13}$	$-0.3833 \cdot 10^{-13}$	$+0.7905 \cdot 10^{-12}$	$-0.5063 \cdot 10^{-12}$	0.43
RK 8(9)	10	2 869	$-0.4130 \cdot 10^{-13}$	$-0.8210 \cdot 10^{-12}$	$+0.1630 \cdot 10^{-11}$	$-0.8669 \cdot 10^{-12}$	1.0[illegible]
RKN 8(9)	10	1 452	$-0.1626 \cdot 10^{-13}$	$-0.3095 \cdot 10^{-13}$	$+0.6093 \cdot 10^{-12}$	$-0.3251 \cdot 10^{-12}$	0.25

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