CLASSICAL EXPANSIONS AND THEIR RELATION TO CONJUGATE HARMONIC FUNCTIONS

BY

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1. Introduction.

a. *Background.* The purpose of this paper is to study a variety of "classical" expansions in analogy to ordinary Fourier series and integrals. As our starting point we take those expansions which arise in harmonic analysis of functions on Euclidean spaces, spheres, etc., having appropriate rotational invariance. By analogy we then pass to more extensive classes of expansions and consider, among others, the ultraspherical expansions and their continuous analogues, the Fourier-Bessel (i.e., Hankel) transforms.

The common idea which unifies most of the techniques and results obtained here centers about the study of harmonic functions and their conjugates in several variables, and their variants. Ordinary Fourier series and integrals have an intimate connection with analytic functions, and this relation which they enjoy has long provided a fundamental tool in their study. Thus one of our main objectives is the development of an analogous tool for the other expansions in question.

The connection that is used between various expansions and generalized harmonic and conjugate harmonic functions seems to be of basic importance. In its formal aspects this connection has its roots in the classical literature. More recently, however, some of the ideas have been taken up in the theory of axially symmetric potential theory (see Weinstein [24]) and in the study of pseudo-analytic functions of Bers [2], [3], and Vekua [20]. We now describe in more detail the background and statements of our results(1).

We begin by recalling some of the basic properties of the ultraspherical expansions and Fourier-Bessel transforms. Fix $\lambda > 0$, and consider the ultraspherical polynomials of degree n, $P_n^\lambda(t)$, defined by the generating relation:

$$(1 - 2t\omega + \omega^2)^{-\lambda} = \sum_{k=0}^{\infty} \omega^k P_k^\lambda(t) .$$

Then the set $\{P^\lambda(\cos\theta)\}$ is orthogonal and complete over $(0,\pi)$ with respect to the measure $(\sin\theta)^{2\lambda}d\theta$. It leads to the expansion $f(\theta) \sim \sum^\infty a_k P_k^\lambda(\cos\theta)$ for an

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(1) Some of the results obtained in this paper are also described in the note [16].

"arbitrary" function f on $(0,\pi)$. In the limiting case $\lambda = 0$ we recover the usual cosine expansion because $\lim_{\lambda\to 0}\lambda^{-1}P_k^\lambda(\cos\theta)=(2/k)\cos k\theta$. This, of course, leads to the Fourier expansion for the circle, for even functions (a trivial rotational invariance, in this case). When $\lambda = 1/2$, the $P_k^\lambda(\cos\theta)$ are the usual Legendre polynomials of $\cos\theta$. As is well known, these functions arise by considering expansions on the sphere (in Euclidean 3-space) invariant under rotations of a fixed axis. The case $\lambda = 1$ should also be noted because then $P_k^\lambda(\cos\theta) = \sin(k+1)\theta/\sin\theta$. In general, when 2λ is integral, $2\lambda = n-2$, the P_k^λ arise in the Fourier analysis of functions of the surface of the sphere in Euclidean n-space, E_n, which are invariant under the rotations leaving a given axis fixed.

The Fourier-Bessel transforms have certain close analogies with the above. Let $J_\alpha(t)$ denote the usual Bessel function of order α, and set $V_\alpha(t) = t^{-\alpha}J_\alpha(t)$. Consider the (Hankel) transformation

$$f(y) \to \hat{f}(\xi) = \int_0^\infty V_{\lambda-1/2}(\xi y)f(y)y^{2\lambda}\,dy.$$

(Again $\lambda \geqq 0$.) Then, as is well known, this leads to a unitary transformation of the L^2-space over $(0,\infty)$ with measure $y^{2\lambda}dy$. The connection with the usual Fourier transform (which occurs when 2λ is integral) is as follows. Suppose we consider functions in E_n which are radial (i.e., invariant under all rotations about the origin). Then the standard (n-dimensional) Fourier transform of such a function is again radial; and when so restricted this transform becomes the above Hankel transform (with $2\lambda = n-2$).

To relate the above expansions with generalized harmonic and conjugate harmonic functions we shall be guided by the case when 2λ is integral. In the trigonometric case ($\lambda=0$), we can associate to a general function $f(\theta)\sim \sum_{k=0}^\infty a_k \cos k\theta$ its Poisson integral $u(x,y) = \sum_{k=0}^\infty a_k r^k \cos k\theta$, $(x,y) = (r\cos\theta, r\sin\theta)$ which is harmonic in the unit disc and takes on the boundary values $f(\theta)$, as $r\to 1$. From the function u we pass to the conjugate harmonic function $v(x,y) = \sum_{k=1}^\infty a_k r^k \sin k\theta$; and the properties of the analytic function $F(z) = u + iv$ are closely related to the original boundary function $f(\theta)$. In the case of E_n, when $n > 2$, there is an analogous notion of conjugacy, which while not as decisive as in the classical case still has some significant properties (see [16] for a general discussion of this point). For this one makes the definition that an n-tuple of functions on E_n, $u_1, u_2, \cdots, u_n$, are conjugate if they satisfy the equations

$$\sum_{i=1}^n \frac{\partial u_i}{\partial x_i} = 0, \qquad \frac{\partial u_i}{\partial x_j} = \frac{\partial u_j}{\partial x_i}. \tag{1.1}$$

This system implies, in particular, that each u_i is harmonic, and, moreover, that the u_i are related in a way which generalizes the usual Cauchy-Riemann equations. Now suppose we have chosen a fixed axis, say along the x_1 direction

and we consider functions u_1 which are invariant under rotations leaving this axis fixed, i.e., functions u_1 which are radial in the variables $x_2, x_3, \cdots, x_n$. If we set $x = x_1$, $y = (x_2^2 + x_3^2 + \cdots + x_n^2)^{1/2}$, and $U(x,y) = u_1(x_1, \cdots, x_n)$, then U satisfies the singular "Laplace equation"

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{2\lambda}{y}\frac{\partial U}{\partial y} = 0 \qquad (2\lambda = n-2). \tag{1.2}$$

Moreover, the $n-1$ conjugates $u_2, u_3, \cdots, u_n$ are not essentially different, and in fact can be taken so that $u_k = (-x_k/y)V(x,y)$, $k = 2, \cdots, n$; the relation between U and V, resulting from (1.1) becomes

$$\begin{aligned} U_y + V_x &= 0, \\ U_x - V_y - \frac{2\lambda}{y}V &= 0. \end{aligned} \tag{1.3}$$

Thus if Φ satisfies (1.2), and is otherwise "arbitrary," we may take $U = \Phi_x$, and $V = -\Phi_y$ to be typical solutions of (1.3). If we set $u(x,y) = U(x,y)$, and $v(x,y) = y^{2\lambda}V(x,y)$ we get the more symmetric system

$$v_x = -y^{2\lambda}u_y, \qquad v_y = y^{2\lambda}u_x.$$

u of course satisfies (1.2) and v satisfies the same equation with λ replaced by $-\lambda$.

We have just described what holds when 2λ is integral, but we take this as our model for the case of general $\lambda \geqq 0$. Let us revert to ultraspherical expansions and consider an arbitrary function

$$f(\theta) \sim \sum_{k=0}^{\infty} a_k P_k^{\lambda}(\cos\theta)$$

and its "Poisson integral"

$$U(x,y) = f(r,\theta) = \sum_{k=0}^{\infty} a_k r^k P_k^{\lambda}(\cos\theta). \tag{1.4}$$

Then U satisfies the equation (1.2) and we are, therefore, led to its conjugate related to U by (1.3). This gives

$$V(x,y) = \tilde{f}(r,\theta) = 2\lambda \sum_{k=1}^{\infty} \frac{a_k r^k \sin\theta}{k+2\lambda} P_{k-1}^{\lambda+1}(\cos\theta) \tag{1.5}$$

and leads to the generalized Hilbert transform, $f(\theta) \to \tilde{f}(\theta)$, where

$$\tilde{f}(\theta) \sim 2\lambda \sum_{k=1}^{\infty} \frac{a_k}{k+2\lambda} \sin\theta\, P_{k-1}^{\lambda+1}(\cos\theta).$$

A similar notion of conjugacy is defined for the Hankel transforms in §16 (see (16.7) and (16.8)), and for Fourier-Bessel series in §18.

b. *Summary of Chapter* I. We are now in a position to state our results. These are carried out below in detail for the ultraspherical case; their analogies for the Fourier-Bessel case are then sketched in §16.

The results of Chapter I deal mainly with the passage to the "Poisson integral" (1.4), its conjugate (1.5), and the properties of λ-harmonic functions (i.e., solutions of (1.2)). In §2 we set down various preliminaries, mostly of a formal nature. In §3 we prove a maximum principle appropriate to λ-harmonic functions. The novelty here arises because we deal with regions that contain the x-axis (the line of singularity of (1.2)) in their interior. Thus the usual Hopf maximum principle is not directly applicable. However, this is circumvented because the λ-harmonic functions that arise by (1.4) are even in y. (Similarly, the functions (1.5) have a natural extension for $y<0$ as odd functions of y.) This evenness (or oddness) is a feature that recurs repeatedly.

§4 contains the basic estimates for the kernel of the Poisson integral. It turns out that this kernel P, defined by

$$f(r,\theta)=\int_0^\pi P(r,\theta,\phi)f(\phi)(\sin\phi)^{2\lambda}\,d\phi,$$

behaves when θ is "near" ϕ in a way which is similar to the Poisson kernel in the trigonometric case. The same kind of similarity holds for the conjugate kernel Q, and the differentiated Poisson kernel, P_r (see §§7 and 12). Obtaining these estimates and applying them is a complicated matter. As such this represents one of the main technical difficulties in this paper. The estimates for P are used in §5 to relate the boundary behavior of (1.4) to an appropriate "maximal function" of the boundary values. This leads to a description of Poisson integrals (1.4) and an analogue of the Fatou theorem which are strikingly similar to the case of usual harmonic functions. These results are in §6. The purpose of §§7 and 8 is to prove the basic fact that the conjugacy mapping $f\to\tilde{f}$ is a bounded operator on L^p, $1<p<\infty$.

§§9 and 10 deal with the analogue of the H^p-spaces for ultraspherical expansions and the results center about the boundary behavior of such functions. The critical case $p=1$ leads to an extension of the classical theorem of F. and M. Riesz and states that if $\sum a_k P_k^\lambda(\cos\theta)$ and its conjugate,

$$2\lambda\ \sum(a_k/(k+2\lambda))\sin\theta\, P_{k-1}^{\lambda+1}\,(\cos\theta$$

both represent finite measures, then both these measures are absolutely continuous. Two tools are used in obtaining the results for H-spaces. The first leads to a global majorization, and follows from the fact that $|F|^p$ is a sub-solution of (1.2), where $F=U+iV$, U and V satisfy the "Cauchy-Riemann equations" (1.3) and $p\geqq 2\lambda/(2\lambda+1)$. The second tool is the known "similarity principle" for systems like (1.3) (see [2]); but since this principle is applicable

only in the case of regular coefficients, it can be used only locally (away from the line of singularity).

c. *Summary of Chapter* II. It is of interest to ask whether the above notions can be used as tools only in the study of ultraspherical and other expansions without reference being made to them in the final statements of results. This will be seen to be the case in the extension of the Littlewood-Paley theory which is given in Chapter II. The main result obtained—and we we discuss only it—is an extension of the Marcinkiewicz multiplier theorem of ordinary Fourier series. It arises from the following problem. Consider a transformation T defined by

$$Tf \sim \sum a_k \mu_k P_k^\lambda(\cos\theta),$$

where

$$f \sim \sum a_k P_k^\lambda(\cos\theta).$$

We may ask, what are the condition on the multiplier sequence $\{\mu_k\}$ so that T is a continuous transformation of L^p to itself, for appropriate p? An obvious necessary condition is that, (i) the sequence be uniformly bounded, i.e., $|\mu_k| \leqq M$, for some M. An additional condition, that of the Marcinkiewicz theorem, which is sufficient in the case of ordinary Fourier series is that, (ii) $\sum_{k=1}^n k|\mu_k - \mu_{k-1}| \leqq Mn$, for some M and all n.

It is to be noted that the sequences for which $\mu_k = 1$, for $1 \leqq k \leqq m$, and $\mu_k = 0$, for $k \geqq m$, satisfy these conditions, with a uniform M for all m. These particular multiplier sequences correspond to the operators of partial sums, and for these Pollard [14] has proved the required L^p boundedness, whenever $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$; incidentally this range of p cannot be extended. In view of this, it might be expected that in general the conditions (i) and (ii) are sufficient for the multiplier transformation T to be bounded on L^p, for the above range of p. This is indeed the case, as Theorem 10 of §14 shows.

The proof of this theorem is lengthy and follows preliminary considerations in §§11, 12, and 13. We discuss these sections briefly, so as to indicate how the notions of λ-harmonic functions and their conjugates enter into the proofs. §11 is devoted in part, to a variant of Pollard's result on partial sums, and this variant and Pollard's theorem have this in common: The formulas for partial sums involve the functions P_k^λ; but to study the partial sums effectively these formulas must be transformed so as to involve both the P_k^λ and their conjugates the $P_{k-1}^{\lambda+1}$.

In §12 we deal with the "g-function" analogue for ultraspherical series. This basic tool involves in its very definition the Poisson integral (in the sense of λ-harmonic functions) of a given boundary function f. The result of this section, is that the L^p-norm of g is comparable with the L^p-norm of f for $1 < p < \infty$, as

in the case of Fourier series. §13 may be viewed as giving a further refinement of this, leading ultimately to the desired results in §14.

d. *Summary of Chapter* III. This chapter deals with several additional results related to the above. In §15 our main concern is the convolution structure which is naturally associated to ultraspherical expansions. In the case when 2λ is integral this convolution structure is, of course, the one induced from the group of rotations in $n+1$ variables. By the aid of this structure we study the L^p properties of potential operators ("fractional integration") analogous to those obtained for Fourier series by Hardy and Littlewood. We prove that if

$$I_\alpha(f) \sim \sum_{k=1}^{\infty} k^{-\alpha} a_k P_k^\lambda(\cos\theta)$$

then the mapping $f \to I_\alpha(f)$ is continuous from L^p to L^q, where $1 < p < q < \infty$, $1/q = 1/p - \alpha/(2\lambda+1)$. This result, and others like it, are consequences of the general Theorem 13; this theorem holds for any convolution structure, and seems to have some interest in its own right.

In §16 we point out how results similar to those of ultraspherical expansions can be derived for Hankel transforms. §17 deals with variants of some of the above results, where certain weight factors are introduced in the L^p-norms. Finally, in §18, the basic definitions are given for a possible extension of our results to include the case of Fourier-Bessel (or Dini) series. From our point of view this arises by considering functions harmonic in circular cylinders (of E_{n+1}) which are invariant under rotations about the axis of the cylinder; and then passing by analogy from this case (which corresponds to 2λ being integral), to the case of general positive λ.

e. *Further problems.* We wish to discuss briefly several further problems suggested by the above.

(i) When 2λ is integral the Bessel functions and ultraspherical polynomials can be interpreted as "spherical functions" associated to symmetric spaces; in the case of Bessel functions these spaces are the Euclidean spaces, and in the case of ultraspherical expansions, these are the spheres. It would be of interest to extend the above results to other symmetric spaces; the ones for which such an extension would seem most probable are the symmetric spaces of rank one. Two illustrations should be given in this regard. First, the spherical functions for the complex projective spaces of dimension n have been determined by Élie Cartan. In the notation of Szegö [18, Chapter IV] they are the Jacobi polynomials $P_k^{(\alpha,0)}$, where $\alpha = n-1$. Thus it is strongly suggested that much of what has been done above has an appropriate variant for the more general Jacobi expansions in terms of the $P_k^{(\alpha,0)}$, where now α is any positive number. A similar situation probably exists for the noncompact symmetric spaces of rank one. This is indicated, at least on the formal level, by the known duality which allows one to

pass from compact to noncompact symmetric spaces; and more particularly by the spherical functions of the Poincaré half-plane, which have been determined explicitly(2).

(ii) As has been stressed repeatedly, the ultraspherical expansions, Hankel transforms, and Fourier-Bessel series may be considered as arising by taking harmonic functions in respectively Euclidean half-spaces, interior of spheres or interior of cylinders, which are properly invariant under rotations. However, the expansions associated with a variety of other regions would seem to be amenable to similar treatment. In this connection we should mention the "conical harmonics" associated to regions bounded by circular cones—which regions have an obvious rotational invariance. It is of interest to point out there is an intimate connection between conical harmonics and spherical functions for the Poincaré half-plane(3).

CHAPTER I

2. **Basic formulae.** We consider the ultraspherical polynomials of type $\lambda, P_n^\lambda(t)$, defined by

$$\sum_{n=0}^{\infty} \omega^n P_n^{(\lambda)}(t) = (1 - 2t\omega + \omega^2)^{-\lambda}. \tag{2.1}$$

See Szegö [18, p. 82].

For any ultraspherical expansion of the type $f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta)$, we shall associate the "harmonic" function

$$f(r,\theta) = \sum a_n r^n P_n^\lambda(\cos\theta), \qquad 0 \leqq r < 1, \tag{2.2}$$

and we shall also write it in the form $u(x,y) = f(r,\theta)$, where $x = r\cos\theta$, $y = r\sin\theta$. It can then be verified that $u(x,y)$ satisfies the differential equation

$$\Delta_\lambda(u) = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\lambda}{y}\frac{\partial u}{\partial y} = 0 \tag{2.3}$$

for (x,y) in the upper semi-disc; $x^2 + y^2 < 1$ and $0 < y$.

The formal verification of (2.3) follows easily from the differential equation:

$$f''(\theta) + 2\lambda\cot\theta f'(\theta) + n(n + 2\lambda)f(\theta) = 0$$

satisfied by $f(\theta) = P_n^\lambda(\cos\theta)$.

To the series $f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta)$ we shall also associate its "conjugate" series

(2) See the book of S. Helgason, *Differential geometry and symmetric spaces*, Academic Press, New York, 1962; Chapter X. The determination of the spherical functions of the complex projective spaces, and of the spheres, is in Cartan's paper, Rend. Circ. Mat. Palermo **53** 29), (19217–252.

(3) This connection was brought to the authors' attention by L. Ehrenpreis. For information about conical harmonics see H. Bateman, *Partial differential equations of mathematical physics*, Dover, New York, 194.; Chapter VI.

$$\tilde{f}(\theta) \sim 2\lambda \sum_{n=1}^{\infty} \frac{a_n}{n+2\lambda} \cdot \sin\theta P_{n-1}^{\lambda+1}(\cos\theta) \tag{2.4}$$

and its "conjugate harmonic" series

$$\tilde{f}(r,\theta) = 2\lambda \sum_{n=1}^{\infty} \frac{a_n}{n+2\lambda} r^n \sin\theta P_{n-1}^{\lambda+1}(\cos\theta). \tag{2.5}$$

If we now set

$$v(x,y) = y^{2\lambda}\tilde{f}(r,\theta) \tag{2.6}$$

then u and v satisfy the "Cauchy-Riemann equations":

$$v_x = -y^{2\lambda}u_y, \qquad v_y = y^{2\lambda}u_x. \tag{2.7}$$

v is harmonic in the sense conjugate to u, i.e.,

$$\Delta_{-\lambda}(v) = \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} - \frac{2\lambda}{y}\frac{\partial v}{\partial y} = 0. \tag{2.8}$$

We now pass from formal considerations to precise statements. The functions of θ, such as $P_n^\lambda(\cos\theta)$ and $f(\theta)$ will always be considered on the interval $0 \leqq \theta \leqq \pi$, unless stated to the contrary.

As is known the functions $P_n^\lambda(\cos\theta)$ are orthogonal with respect to a measure $dm_\lambda = dm_\lambda(\theta)$. More precisely we have

$$\begin{aligned} &\int_0^\pi P_n^\lambda(\cos\theta)P_m^\lambda(\cos\theta)\, dm_\lambda(\theta) \\ &\qquad = \delta_{n,m} 2^{1-2\lambda}\pi[\Gamma(\lambda)]^{-2} \frac{\Gamma(n+2\lambda)}{(n+\lambda)n!} = \delta_{nm}/\gamma_n, \end{aligned} \tag{2.9}$$

where

$$dm_\lambda(\theta) = (\sin\theta)^{2\lambda} d\theta. \tag{2.10}$$

See Szegö, [18, p. 81].

Thus we shall define the spaces $L^p = L^p(dm_\lambda)$ as those whose norm $\|f\|_p$ is given by $\|f\|_p = (\int_0^\pi |f(\theta)|^p\, dm_\lambda(\theta))^{1/p}$ if $1 \leqq p < \infty$, and $\|f\|_\infty = \operatorname{ess\,sup}|f(\theta)|$. Also B will denote the Borel measures $d\mu$ on $[0,\pi]$ which satisfy:

$$\|d\mu\| = \int_0^\pi (\sin\theta)^{2\lambda} |d\mu(\theta)| < \infty.$$

If $f(\theta)$ is a given function $(0,\pi)$ integrable with respect to $dm_\lambda(\theta)$, we form its ultraspherical expansion

$$f(\theta) \sim \sum_{n=0}^{\infty} a_n P_n^\lambda(\cos\theta),$$

where

$$a_n = \gamma_n \int_0^\pi f(\theta) P_n^\lambda(\cos\theta)\, dm_\lambda(\theta).$$

On account of (2.1) it follows easily that

$$|P_n^\lambda(\cos\theta)| \leqq An^{2\lambda-1}.$$

Remembering the definition of γ_n given in (2.4) it is then clear that the series

$$f(r,\theta) = \sum_{n=0}^{\infty} a_n r^n P_n^\lambda(\cos\theta)$$

converges absolutely when $0 \leqq r < 1$.

The function $f(r,\theta)$ will be referred to as the "Poisson integral" of $f(\theta)$. There is an explicit formula for it which follows from a theorem of Watson [22]. In fact, define the "Poisson kernel" $P(r,\theta,\phi)$ by

$$P(r,\theta,\phi) = \sum_{0}^{\infty} r^n \gamma_n P_n^\lambda(\cos\theta)\, P_n^\lambda(\cos\phi). \tag{2.11}$$

Then we have

$$\begin{aligned} P(r,\theta,\phi) &= \frac{\lambda}{\pi}(1-r^2)\int_0^\pi \frac{\sin^{2\lambda-1}t}{[1-2r(\cos\theta\cos\phi+\sin\theta\sin\phi\cos t)+r^2]^{\lambda+1}}\,dt \\ &= \frac{\lambda(1-r^2)}{\pi}\int_0^\pi \frac{\sin^{2\lambda-1}t}{[(x-\xi)^2+(y-\eta)^2+2y\eta(1-\cos t)]^{\lambda+1}}\,dt, \end{aligned} \tag{2.12}$$

where $(x,y) = (r\cos\theta,\ r\sin\theta)$, $(\xi,\eta) = (\cos\phi, \sin\phi)$. See also Huber [11].

We thus have

$$f(r,\theta) = \int_0^\pi P(r,\theta,\phi) f(\phi)\, dm_\lambda(\phi). \tag{2.13}$$

It should be noted that a similar formalism holds for Borel measures $d\mu$ which have finite norm, in place of integrable functions f.

If we now write $u(x,y) = f(r,\theta)$, it is apparent from an examination of the Poisson kernel (2.12), that $u(x,y)$ can be naturally defined in the *entire* unit disc $x^2+y^2<1$ in such a manner that $u(x,y)$ is even in y and is indefinitely differentiable (in fact real-analytic) there. It is of interest to observe that it follows from this that the function $r^n P_n^\lambda(\cos\theta)$ is real-analytic in x and y. (This also follows from the fact that $P_n^\lambda(t) = (-)^n P_n^\lambda(-t)$.) We can now verify that $u(x,y)$ satisfies the differential equation (2.3) not just when $y>0$, but also when $y<0$. Thus in the study of ultraspherical expansions we are led to study solutions of the singular "Laplace equation" (2.3), but only those solutions which have a natural continuation past the singular line $y=0$, are even, and real-analytic in the unit disc.

3. **Maximum principle.** For second-order elliptic operators like those appearing in (2.3) above there is a well-known maximum principle due to E. Hopf [9].

However, this principle is applicable only when matters are restricted to the region of regularity of the coefficients. Thus regions containing the singular line $y=0$ would be excluded from consideration. But it is exactly regions of this kind that are needed in what follows below. It is possible to bypass this difficulty by considering only even functions, as is suggested by the facts observed in the preceding section. Our result is as follows.

THEOREM 1. *Let $u(x,y)$ be of class C^2 in an open region R. If R contains segments of the line $y=0$, we shall assume that $u(x,y)=u(x,-y)$ near such segments. Assume that u satisfies $\partial^2 u/\partial x^2+\partial^2 u/\partial y^2+(2\lambda/y)(\partial u/\partial y)\geqq 0$, for $(x,y)\in R$, $y\neq 0$, and λ is a fixed non-negative number. Then u cannot attain a local maximum in R, unless u is a constant.*

Proof. Suppose, on the contrary, that $(x_0,y_0)\in R$ is a local maximum. We can first assert that $y_0=0$. For otherwise u would attain a local maximum at a point of regularity of the coefficients. This would contradict Hopf's maximum principle. We therefore can assume that $y_0=0$. To treat this case we shall show that since $\partial^2 u/\partial x^2+\partial^2 u/\partial y^2+(2\lambda/y)(\partial u/\partial y)\geqq 0$, then u satisfies the following property akin to sub-harmonicity: For all ρ sufficiently small

$$u(x_0,0)\int_0^\pi \sin^{2\lambda}\theta\, d\theta \leqq \int_0^\pi u(x_0+\rho\cos\theta,\rho\sin\theta)\sin^{2\lambda}\theta\, d\theta. \tag{3.1}$$

In fact, we have by Green's theorem, for $u,v\in C^2$ of a region D with smooth boundary curve B

$$\begin{aligned}\iint_D [v\operatorname{div}(|y|^{2\lambda}\operatorname{grad} u)-u\operatorname{div}(|y|^{2\lambda}\operatorname{grad} v)]\, dxdy \\ = \int_B |y|^{2\lambda}\left(v\frac{\partial u}{\partial n}-u\frac{\partial v}{\partial n}\right)ds.\end{aligned} \tag{3.2}$$

($\partial/\partial n$ denotes the direction of the outward normal.) We set $\Delta_\lambda(u)=u_{xx}+u_{yy}+(2\lambda/y)u_y$, and note that if $r^{-2\lambda}=((x-x_0)^2+y^2)^{-\lambda}$, the $\Delta_\lambda(r^{-2\lambda})=0$ (except when $(x,y)=(x_0,0)$). Take for D the annulus centered at $(x_0,0)$, whose outer circle has radius ρ and inner circle radius ε; set $v=r^{-2\lambda}-c$, where $c=\rho^{-2\lambda}$ (thus $v=0$ on the outer circle). Then (3.2) gives:

$$-\int_{C_\rho}|y|^{2\lambda}u\frac{\partial v}{\partial n}\,ds-\int_{C_\varepsilon}|y|^{2\lambda}v\frac{\partial u}{\partial n}\,ds+\int_{C_\varepsilon}|y|^{2\lambda}u\frac{\partial v}{\partial n}\,ds\geqq 0.$$

Here C_ρ and C_ε denote respectively the circles of radius ρ and ε taken in the positive direction. If we let $\varepsilon\to 0$ in the above, then the second integral tends to zero, and the resulting inequality is (3.1). Hence if $(x_0,0)$ is a local maximum of u then by (3.1) there is a local maximum near $(x_0,0)$ but lying

strictly in the upper half-plane. Here again we obtain a contradiction with Hopf's maximum principle and the theorem is proved.

4. **Estimate for the Poisson kernel.** The results dealing with the behavior of Poisson integrals near the boundary discussed below in §6 depend in part on the basic estimates for the Poisson kernel given below. Recall the definition and formulae (2.11) and (2.12).

LEMMA 1. *Let us write* $\delta = 1 - r$, $1/2 \leqq r < 1$. *Then*

$$(4.1)\qquad \begin{aligned} &\text{(a)} \qquad P(r,\theta,\phi) \leqq C \cdot \delta \frac{\sin^{-\lambda}\theta \sin^{-\lambda}\phi}{\delta^2 + (\theta-\phi)^2}, \\ &\text{(b)} \qquad P(r,\theta,\phi) \leqq C \cdot \frac{\delta}{(\delta^2 + (\theta-\phi)^2)^{\lambda+1}}. \end{aligned}$$

Proof. Write $\Delta = 1 - 2r\cos(\theta - \phi) + r^2$, $\sigma = r\sin\theta\sin\phi$. By examination in (2.12) it follows that

$$P(r,\theta,\phi) = I_1 + I_2,$$

where

$$I_1 \leqq C(1-r^2)\int_0^{\pi/2} \frac{t^{2\lambda-1}}{\left[\Delta + \dfrac{t^2\sigma}{4}\right]^{\lambda+1}}\, dt,$$

and a similar estimate for I_2 as an integral taken over $\pi/2 \leqq t \leqq \pi$. If we make the change of variables $u = (\sigma/\Delta)^{1/2}t$, we get

$$I_1 \leqq C(1-r^2)\frac{\sigma^{-\lambda}}{\Delta}\int_0^{\pi(\sigma/\Delta)^{1/2}/2} \frac{u^{2\lambda-1}}{\left(1 + \dfrac{u^2}{4}\right)^{\lambda+1}}\, du.$$

Using the fact that

$$\int_0^\infty \frac{u^{2\lambda-1}}{\left(1 + \dfrac{u^2}{4}\right)^{\lambda+1}}\, du < \infty,$$

we get

$$I_1 \leqq C \cdot \delta \frac{(\sin\theta\sin\phi)^{-\lambda}}{(1-2r\cos(\theta-\phi)+r^2)} \leqq C \cdot \delta \frac{(\sin\theta\sin\phi)^{-\lambda}}{\delta^2 + (\theta-\phi)^2}.$$

Using the fact that

$$\int_0^\omega \frac{u^{2\lambda-1}}{\left(1 + \dfrac{u^2}{4}\right)^{\lambda+1}}\, du \leqq C\omega^{2\lambda},$$

gives

$$I_1 \leqq C\cdot\delta\cdot\sigma^{-\lambda}\cdot\sigma^{\lambda}\Delta^{-\lambda-1} \leqq C\ \frac{\delta}{(\delta^2+(\theta-\phi)^2)^{\lambda+1}}.$$

(We have used the easily verified estimate

$$\frac{1}{\Delta} \leqq \frac{C}{\delta^2+(\theta-\phi)^2}$$

if $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq \pi$, $\delta = 1-r$, $1/2 \leqq r < 1$.)

Similar estimates hold for I_2. This proves Lemma 1.

5. **A maximal function.** Another basic fact we shall use is the analogue of Hardy-Littlewood "max" function, but in the form appropriate for ultraspherical expansions.

For any $f \in L^1(dm_\lambda(\theta))$ we set

$$f^*(\theta) = \sup_{h\neq 0;\, 0\leqq\theta+h\leqq\pi} \frac{\displaystyle\int_\theta^{\theta+h} |f(\phi)|\ dm_\lambda(\phi)}{\displaystyle\int_\theta^{\theta+h} dm_\lambda}.$$

We then have

LEMMA 2. *$f^*(\theta)$ is finite for almost every θ. For any $\alpha > 0$, let $E_\alpha = \{\theta \mid f^*(\theta) > \alpha\}$. Then $m_\lambda(E_\alpha) \leqq (2/\alpha) \int |f|\, dm_\lambda$. If in addition $f \in L^p(dm_\lambda)$, $1 < p \leqq \infty$, then $f^* \in L^p$ and*

$$\|f^*\|_p \leqq A_p \|f\|_p.$$

Proof. It is possible to imitate the classical argument or to deduce the present lemma from its known special case. We choose the latter approach. Introduce a new variable $x = x(\theta) = \int_0^\theta dm_\lambda(\phi)$, $dx/d\theta = \sin^{2\lambda}\theta$. We then have $0 \leqq x \leqq C_\lambda$, where $C_\lambda = \int_0 dm_\lambda(\phi)$. Let now $F(x) = f(\theta)$, then

$$m_\lambda(E_\alpha) = m_0\left\{x \,\middle|\, k^{-1}\int_x^{x+k} |F(x)|\,dx > \alpha,\ \ 0 \leqq x+k \leqq C_\lambda\right\},$$

where m_0 denotes the standard Lebesgue measure. We then have by Riesz's lemma

$$m_\lambda(E_\alpha) \leqq \frac{2}{\alpha}\int_0^{C_\lambda} |F(x)|\ dx = \frac{2}{\alpha}\int_0^\pi |f(\phi)|\ dm_\lambda(\phi).$$

Similarly the inequality $\|f^*\|_p \leqq A_p\|f\|_p$ is proved.

In view of the lemma we have just proved, the basic lemma for the behavior of Poisson integrals near the boundary can now be stated:

LEMMA 3. *Let $f \in L^1(dm_\lambda)$. If $f(r,\theta)$ denotes the Poisson integral* (2.13), *and $f^*(\theta)$ is the maximal function defined above, then*

$$\sup_{0\leqq r<1} |f(r,\theta)| \leqq Cf^*(\theta).$$

Proof. To simplify matters we divide the square $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq \pi$, into four parts: (1) $0 \leqq \theta \leqq \pi/2$, $0 \leqq \phi \leqq \pi/2$, (2) $0 \leqq \theta \leqq \pi/2$, $\pi/2 \leqq \phi \leqq \pi$, (3) $\pi/2 \quad \theta \leqq \pi, 0 \leqq \phi \leqq \pi/2$, (4) $\pi/2 \leqq \theta \leqq \pi$, and $\pi/2 \leqq \phi \leqq \pi$.

Thus we have $f(r,\theta) = f^1(r,\theta) + f^2(r,\theta) + f^3(r,\theta) + f^4(r,\theta)$ where say $f^1(r,\theta)$ is the Poisson integral of the function which equals $f(\phi)$ for $0 \leqq \phi \leqq \pi/2$ and vanishes outside this interval; also in $f^1(r,\theta)$ we restrict θ by $0 \leqq \theta \leqq \pi/2$. Similarly for the other $f^j(r,\theta)$. We assume also, without loss of generality that $f \geqq 0$. It turns out that consideration of $f^1(r,\theta)$ and $f^4(r,\theta)$ are essentially identical(4), while consideration of $f^2(r,\theta)$ and $f^3(r,\theta)$ are somewhat simpler. We shall therefore discuss in detail the estimates for $f^1(r,\theta)$.

Now in the range $0 \leqq \theta \leqq \pi/2, 0 \leqq \phi \leqq \pi/2, c\theta \leqq \sin\theta \leqq \theta$, and $c\phi \leqq \sin\phi \leqq \phi$, where $c > 0$. In the estimates that follow, therefore, we shall systematically replace $\sin\theta$ by θ and $\sin\phi$ by ϕ. We now break up matters into further cases:

Case I. $2\delta = 2(1-r) \leqq \theta$.

Let E be the set of ϕ's so that $|\theta - \phi| \leqq \delta$, $0 \leqq \phi \leqq \pi/2$, then

$$\int_E P(r,\theta,\phi) f(\phi) \sin^{2\lambda}\phi \, d\phi \leqq c \int_E \delta^{-1} \frac{f(\phi)(\sin\phi)^{2\lambda}}{(\sin\theta \sin\phi)^{\lambda}} \, d\phi$$

$$\leqq \frac{c\delta^{-1}}{\theta^{\lambda}} \int_E f(\phi)\phi^{\lambda} \, d\phi$$

by estimate (a) of Lemma 1 for the Poisson kernel. However, $|\phi/\theta| \leqq c$ according to the restrictions made on θ and ϕ. Thus we have

$$\int_E P(r,\theta,\phi) f(\phi) \sin^{2\lambda}\phi \, d\phi \leqq c \frac{\delta^{-1}}{\theta^{2\lambda}} \int_E f(\phi)\phi^{2\lambda} \, d\phi.$$

However, $\int_E \sin^{2\lambda}\phi \, d\phi \geqq c \int_E \phi^{2\lambda} \, d\phi \geqq c\delta\theta^{2\lambda} \geqq c\delta(\sin\theta)^{2\lambda}$ because $|\theta - \phi| \leqq \delta$ and $2\delta \leqq \theta$. Thus finally:

$$\int_E P(r,\theta,\phi) f(\phi) \sin^{2\lambda}\phi \, d\phi \leqq C \frac{\displaystyle\int_{|\theta-\phi|<\delta} f(\phi)\sin^{2\lambda}\phi \, d\phi}{\displaystyle\int_{|\theta-\phi|<\delta} \sin^{2\lambda}\phi \, d\phi} \leqq Cf^*(\theta).$$

Next let F be the set where $\delta \leqq |\theta - \phi| \leqq \theta/2$, and F_n the set where $2^n\delta \leqq |\theta - \phi| \leqq 2^{n+1}\delta$.

Then

$$\int_F P(r,\theta,\phi) f(\phi) \sin^{2\lambda}\phi \, d\phi \leqq C\delta \int_F \frac{f(\phi)\phi^{2\lambda}}{\theta^{\lambda}\phi^{\lambda}(\theta-\phi)^2} \, d\phi$$

$$\leqq c \sum_{n=0}^{\infty} \delta \int_{F_n} \frac{f(\phi)\phi^{2\lambda}}{\theta^{2\lambda} 2^{2n} \delta^2} \, d\phi.$$

(4) This is due to the fact that $P(r,\theta,\phi) = P(r,\pi-\theta,\pi-\phi)$.

This follows by again applying estimate (a) of (4.1) on $P(r,\theta,\phi)$. Now

$$\int_{F_n} \sin^{2\lambda}\phi\, d\phi \geqq c \int_{2^n\delta \leqq |\theta-\phi| \leqq 2^{n+1}\delta} \phi^{2\lambda} d\phi \geqq c\theta^{2\lambda}\delta 2^n ,$$

since $\theta \geqq 2\delta$.

Thus we obtain

$$\int_F P(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi\, d\phi \leqq C \sum 2^{-n} \frac{\int_{F_n} f(\phi)(\phi)^{2\lambda}\, d\phi}{\int_{F_n} \sin^{2\lambda}\phi\, d\phi}$$

$$\leqq C \sum 2^{-n} f^*(\theta) = Cf^*(\theta).$$

Similarly, let G be the set where $\theta/2 < |\theta - \phi|$, and let G_n be the set where $2^n\theta < |\theta - \phi| \leqq 2^{n+1}\theta$.

Then

$$\int_G P(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi\, d\phi \leqq C\int_G \frac{\delta f(\phi)\phi^{2\lambda}}{(\theta-\phi)^{2\lambda+2}}\, d\phi$$

$$\leqq C \sum_{n=-1}^{\infty} \theta^{-(2\lambda+1)} \int_{G_n} \frac{f(\phi)\phi^{2\lambda}}{2^{2n(\lambda+1)}}\, d\phi ,$$

by estimate (b) of (4.1), and since $\theta \geqq 2\delta$.

However,

$$\int_{G_n} \sin^{2\lambda}\phi\, d\phi \geqq c \int_{2^n\theta < |\theta-\phi| \leqq 2^{n+1}\theta} \phi^{2\lambda}\, d\phi \geqq c(2^n\theta)^{2\lambda+1} .$$

Thus the sum is dominated by

$$c \sum 2^{-n} \frac{\int_{G_n} f(\phi)\phi^{2\lambda}\, d\phi}{\int_{G_n} \sin^{2\lambda}\phi\, d\phi} \leqq c \sum 2^{-n} f^*(\theta).$$

Combination of the estimates for sets E, F, and G shows that

$$f^1(r,\theta) \leqq cf^*(\theta) \quad \text{if } 2\delta \leqq \theta.$$

The remaining case for $f^1(r,\theta)$, that is the case $2\delta > \theta$, is treated similarly.

Case II. $2\delta > \theta$.

We sketch the argument:

Let E be the set where $|\theta - \phi| < \delta$. Then

$$\int_E P(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi\, d\phi \leqq c\delta^{-2\lambda-1}\int_E f(\phi)\phi^{2\lambda}\, d\phi \leqq cf^*(\theta),$$

because of estimate (b) of (4.1) and the fact that

$$\int_{|\theta-\phi|<\delta} \sin^{2\lambda}\phi \, d\phi \geqq c\delta^{2\lambda+1} \quad \text{if } 2\delta \geqq \theta.$$

Finally, let F be the set where $\delta \leqq |\theta-\phi|$, and let F_n be the set where $2^n\delta \leqq |\theta-\phi| < 2^{n+1}\delta$. Then

$$\int_F P(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi \, d\phi \leqq c\delta \int_F \frac{f(\phi)\phi^{2\lambda}}{(\theta-\phi)^{2\lambda+2}} \, d\phi$$

$$\leqq c \sum_{n=0}^{\infty} \int_{F_n} \delta^{-2\lambda-1} \frac{f(\phi)\phi^{2\lambda}d\phi}{2^{2n(\lambda+1)}} \leqq c \sum \frac{f^*(\theta)}{2^n} \leqq cf^*(\theta)$$

as in the previous arguments.

This completes the estimate $f^1(r,\theta) \leqq cf^*(\theta)$ for all r,θ. As mentioned before, f^2, f^3, and f^4 are treated similarly. This concludes the proof of the lemma.

6. **Poisson integrals and the analogue of Fatou's theorem.** The following theorem contains the basic results of Poisson integrals for L^p.

THEOREM 2. *Let $f \in L^p(dm_\lambda)$, $1 \leqq p \leqq \infty$, and let $f(r,\theta)$ denote its Poisson integral*, (2.13). *Then*

(a) $\|f(r,\theta)\|_p \leqq \|f(\theta)\|_p$, $1 \leqq p \leqq \infty$,

(b) $\|f(r,\theta)-f(\theta)\|_p \to 0$ *as* $r \to 1$, *if* $1 \leqq p < \infty$,

(c) $\lim_{r\to 1} f(r,\theta) = f(\theta)$ *almost everywhere*, $1 \leqq p \leqq \infty$,

(d) $\|\sup_{r<1}|f(r,\theta)|\|_p \leqq A_p\|f\|_p$, *if* $1 < p \leqq \infty$.

Proof. (a) We note that $P(r,\theta,\phi) \geqq 0$, $P(r,\theta,\phi) = P(r,\phi,\theta)$ and $\int_0^\pi P(r,\theta,\phi)\sin^{2\lambda}\phi \, d\phi = 1$. Set $f(r,\theta) = \int_0^\pi P(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi \, d\phi$. If $1 \leqq p < \infty$, we have by Hölder's inequality:

$$|f(r,\theta)|^p \leqq \int P(r,\theta,\phi)|f(\phi)|^p \sin^{2\lambda}\phi \, d\phi.$$

Integrating with respect to θ and interchanging the order of integration gives $\|f(r,\theta)\|_p \leqq \|f\|_p, 1 \leqq p < \infty$. The case $p = \infty$ is trivial.

(b) Let S be the linear space of all finite polynomials $\sum_{n=0}^N a_n P_n^\lambda(\cos\theta)$. Then S is dense in every $L^p(dm_\lambda)$, $1 \leqq p < \infty$. (In fact S is dense in the space of continuous functions on $[0,\pi]$ in the "sup" topology.) Since clearly $f(r,\theta) \to f(\theta)$ uniformly and hence in L^p for every $f \in S$, part (b) then follows from part (a). Notice also that if $f(\theta)$ is continuous on $[0,\pi]$ then $\|f(r,\theta)-f(\theta)\|_\infty \to 0$.

(c) If $f(\theta)$ is given and in $L^1(dm_\lambda)$ it suffices to show that given any $\varepsilon > 0$, there exists a set E_ε, so that $m_\lambda(E_\varepsilon) < \varepsilon$ and $\limsup_{r\to 1}|f(r,\theta)-f(\theta)| > \varepsilon$ only for $\theta \in E_\varepsilon$.

Thus write $f(\theta)=f_1(\theta)+f_2(\theta)$ where $f_1(\theta)\in S$ and $\|f_2\|_1<\delta$, δ to be chosen later.

Then $f(r,\theta)-f(\theta)=f_1(r,\theta)-f_1(\theta)+f_2(r,\theta)-f_2(\theta)$ and hence $|f(r,\theta)-f(\theta)| \leqq |f_1(r,\theta)-f_1(\theta)|+cf_2^*(\theta)$, since $|f_2(r,\theta)|\leqq Cf_2^*(\theta)$, by Lemma 3, and $|f_2(\theta)| \leqq f_2^*(\theta)$ trivially.

Since $\limsup_{r\to 1}|f_1(r,\theta)-f_1(\theta)|=0$ for every θ, we must show that $f_2^*(\theta)\leqq \varepsilon/c$ except in a set of measure $<\varepsilon$. However by Lemma 2,

$$m_\lambda\left\{f_2^*(\theta)\geqq\frac{\varepsilon}{c}\right\}\leqq\frac{2c}{\varepsilon}\|f_2\|_1\leqq\frac{2c\delta}{\varepsilon}.$$

Thus we need choose δ only by $\delta<\varepsilon/2c$, which proves (c). (d) is an immediate consequence of the comparison $\sup_{0\leqq r<1}|f(r,\theta)|\leqq cf^*(\theta)$ (Lemma 3) and the fact that $\|f^*\|_p\leqq A_p\|f\|_p$, $1<p\leqq\infty$ (Lemma 2). This concludes the proof of Theorem 2.

We now come to the converse problem, that of characterizing among all "harmonic" functions those which arise as Poisson integrals of L^p functions. For this purpose it is important to recall the space B of Borel measures $d\mu$ on $[0,\pi]$ whose norm

$$\|d\mu\|=\int_0^\pi(\sin\theta)^{2\lambda}|d\mu(\theta)|$$

is finite.

We define $\int_0^\pi P(r,\theta,\phi)\sin^{2\lambda}\phi\, d\mu(\phi)$ as the Poisson integral of the measure $d\mu$.

Let $u(x,y)$ be a function defined in the upper semi-disc $x^2+y^2<1$, $y>0$. We shall say $u(x,y)$ is *λ-harmonic* there (or sometimes "harmonic") if it is of class C^2 there and satisfies

$$\Delta_\lambda(u)=\frac{\partial^2u}{\partial x^2}+\frac{\partial^2u}{\partial y^2}+\frac{2\lambda}{y}\frac{\partial u}{\partial y}=0.$$

We shall say that u is *regular* and *even* if there exists a C^2 function in the entire disc $x^2+y^2<1$ which extends $u(x,y)$ (we denote the extension by $u(x,y)$ also) so that $u(x,y)=u(x,-y)$. We notice if u is λ-harmonic, regular and even, then $\Delta_\lambda(u)=0$ also if $y<0$.

We are now in a position to state our characterization of Poisson integrals.

THEOREM 3. *Let $u(x,y)$ be λ-harmonic in the upper semi-disc. Then $u(x,y)=f(r,\theta)$ is the Poisson integral of a function $f(\theta)\in L^p(dm_\lambda)$, $1<p\leqq\infty$, if and only if $u(x,y)$ is regular and even in the disc $x^2+y^2<1$ and*

$$\sup_{0\leqq r<1}\|f(r,\theta)\|_p<\infty.$$

For the case $p=1$ we have the following not-unexpected variant.

THEOREM 3′. *Let $u(x,y)$ be λ-harmonic in the upper semi-disc. Then $u(x,y)=f(r,\theta)$ is the Poisson integral of a measure $d\mu$ so that $\int_0^\pi \sin^{2\lambda}\theta\,|d\mu(\theta)|<\infty$ if and only if $u(x,y)$ is regular and even and*

$$\sup_{r<1} \|f(r,\theta)\|_1 < \infty.$$

As a corollary we obtain:

COROLLARY 1. *A λ-harmonic function $u(x,y)$ which is regular, even, and non-negative is the Poisson integral of a non-negative measure $d\mu$ so that $\int_0^\pi \sin^{2\lambda}\theta\ d\mu(\theta)<\infty$, and conversely.*

As a further consequence, taking into consideration Theorem 2, we have:

COROLLARY 2. *If $u(x,y)$ is λ-harmonic, regular, even and satisfies*

$$\sup_{0\leqq r<1} \|u(r\cos\theta, r\sin\theta)\|_p < \infty \quad \textit{for some } p,\ 1<p\leqq\infty,$$

then

$$\lim_{r\to 1} u(r\cos\theta, r\sin\theta) \textit{ exists for almost every } \theta.$$

There are two further conclusions for Poisson integrals that need be mentioned. First, that $\lim_{r\to 1} u(r\cos\theta, r\sin\theta)$ will exist almost everywhere even for Poisson integrals of measures in B. (Theorem 2-(c) covers only absolutely continuous measures.) Second, all limits at the boundary can be taken in the nontangential sense—instead of just the radial sense.

While these facts could be proved by techniques similar to those used above, we prefer postponing consideration of these matters to a later section where they shall follow in the course of the development of H^p-spaces.

Proof of Theorem 3. We have already observed that if $f(r,\theta)$ is a Poisson, integral it is λ-harmonic, regular and even. If $f(\theta)\in L^p$ then by Theorem 2, $\|f(r,\theta)\|_p \leqq \|f(\theta)\|_p$ and thus the necessity is proved. We come therefore to the converse.

For any r_0, $0<r_0<1$, write

$$g(r,\theta) = \int_0^\pi P(r,\theta,\phi) f(r_0,\phi)\ dm_\lambda(\phi).$$

Thus $g(r,\theta)$ is the Poisson integral of the continuous function $f(r_0,\theta)$. Consider now $h(r,\theta)=g(r,\theta)-f(rr_0,\theta)$. It is easy to verify directly that $f(rr_0,\theta)$ is λ-harmonic (in (r,θ)), regular and even in the disc $x^2+y^2<1$, because $f(r,\theta)$ is. Thus $h(r,\theta)$ has the same properties. Moreover, $h(r,\theta)$ is continuous in the closure $x^2+y^2\leqq 1$ since $f(rr_0,\theta)$ is, and $g(r,\theta)\to f(r_0,\theta)$ uniformly as $r\to 1$. Therefore, in addition, $h(1,\theta)=0$. We claim that $h(r,\theta)\equiv 0$. Assume the contrary, then $h(r,\theta)$ would have a local maximum (or minimum) at an *interior* point of

the disc. This contradicts the maximum principle (Theorem 1) and hence $h(r,\theta)\equiv 0$. Thus

$$f(rr_0,\theta) = \int_0^\pi P(r,\theta,\phi)f(r_0,\phi)\, dm_\lambda(\phi), \qquad 0 \leqq r_0,\ r<1. \tag{6.1}$$

Since $\|f(r,\phi)\|_p \leqq M < \infty$, we can select a sequence $r_1, r_2, \cdots, r_n, \cdots$, so that $r_n \to 1$ and $f(r_n,\theta)\to f(\theta)$ weakly as $r_n \to 1$.

It therefore follows from (6.1) that $f(r,\theta)$ is the Poisson integral of $f(\theta)$, and the theorem is proved.

Theorem 3′ is proved in the same way, except because $\int_0^\pi |f(r,\theta)|\sin^{2\lambda}\theta\, d\theta \leqq M < \infty$, we now have a sequence $f(r_n,\theta)$ which converges weakly to a measure $d\mu(\theta)$ so that $\int_0^\pi \sin^{2\lambda}\theta\, |d\mu(\theta)| < \infty$.

The proof of Corollary 1 is based on the following fact: If $u(x,y)=f(r,\theta)$ is λ-harmonic, regular, and even, then

$$u(0,0)\int_0^\pi (\sin\theta)^{2\lambda}\, d\theta = \int_0^\pi u(r\cos\theta,\, r\sin\theta)(\sin\theta)^{2\lambda} d\theta. \tag{6.2}$$

This is the mean value property, which in this context is due to Weinstein [23]. (6.2) also follows immediately from a more general inequality (3.1) for "subharmonic" functions. In any case, if $u(x,y)\geqq 0$, then (6.2) implies that $u(x,y)=f(r,\theta)$ satisfies $\sup_{r<1}\int_0^\pi |f(r,\theta)|\sin^{2\lambda}\theta\, d\theta < \infty$. Theorem 3′ is therefore applicable.

Corollary 2 is an immediate consequence of Theorems 2 and 3.

7. **The conjugate kernel.** Let us recall briefly the definitions of conjugacy introduced above.

If

$$f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta)$$

and

$$f(r,\theta) = \sum a_n r^n P_n^\lambda(\cos\theta), \qquad r<1,$$

then

$$\tilde{f}(r,\theta) = 2\lambda \sum_{n=1}^\infty \frac{a_n}{n+2\lambda} r^n \sin\theta\, P_{n-1}^{\lambda+1}(\cos\theta), \qquad 0\leqq\theta\leqq\pi. \tag{7.1}$$

If we set $u(x,y)=f(r,\theta)$ and $v(x,y)=y^{2\lambda}\tilde{f}(r,\theta)$ then u and v satisfy the Cauchy-Riemann equations $v_x = -y^{2\lambda}u_y$, and $v_y = y^{2\lambda}u_x$ and their analogues in polar coordinates: $v_r = -((r\sin\theta)^{2\lambda}/r)u_\theta$, $v_\theta = r(r\sin\theta)^{2\lambda}u_r$. That these equations are verified follows easily from the relation $(d/dt)P_n^\lambda(t) = 2\lambda P_{n-1}^{\lambda+1}(t)$, see Szegö [18, p. 83].

Let us now notice that

$$\tilde{f}(r,\theta) = \int_0^\pi Q(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi\, d\phi,$$

where

$$Q(r,\theta,\phi) = \sum_{n=1}^{\infty} \frac{2\lambda}{n+2\lambda} r^n \gamma_n \sin\theta\, P_{n-1}^{\lambda+1}(\cos\theta) P_n^\lambda(\cos\phi) \tag{7.2}$$

and where γ_n are the normalizing factors (2.9). Q will be called the conjugate Poisson kernel.

In view of the equations $v_r = -((r\sin\theta)^{2\lambda}/r)u_\theta$, $v = y^{2\lambda}\tilde{f}(r,\theta)$, and $Q(0,\theta,\phi) = 0$, we obtain

$$Q(r,\theta,\phi) = -r^{-2\lambda}\int_0^r t^{2\lambda-1}P_\theta(t,\theta,\phi)\, dt, \tag{7.3}$$

where P is the Poisson kernel (2.11). Equation (7.3) allows us to reduce the problem of analyzing the kernel Q in terms of the kernel P, for which we have an explicit formula. The result of this analysis is as follows.

LEMMA 4. *If* $0 \leqq \theta \leqq \pi$, $0 \leqq \phi \leqq \pi/2$, $0 < \lambda$, *then*

$$\begin{aligned} Q(r,\theta,\phi) &= O((\sin\phi)^{-2\lambda-1}) \quad \text{if } 2\theta < \phi, \\ &= O((\sin\theta)^{-2\lambda-1}) \quad \text{if } \tfrac{1}{2}\theta > \phi, \\ Q(r,\theta,\phi) &= c_\lambda r^\lambda (\sin\theta\sin\phi)^{-\lambda}\frac{\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2} \\ &\quad + O\left[(\sin\theta)^{-2\lambda-1}\left(1+\log^+\left(\frac{\sin\theta\sin\phi}{1-\cos(\theta-\phi)}\right)\right)\right] \quad \text{if } \frac{\theta}{2} \leqq \phi \leqq 2\theta. \end{aligned}$$

REMARKS. It follows from (7.2) that $Q(r,\theta,\phi) = Q(r,\theta',\phi')$, where $\theta' = \pi - \theta$, $\phi' = \pi - \phi$ (because $P_n(-t) = (-1)^n P_n(t)$). Thus very similar estimates hold for Q in the range $\pi/2 \leqq \phi \leqq \pi$. The lemma shows that except for the factor $c_\lambda r^\lambda(\sin\theta\sin\phi)^{-\lambda}$, the kernel Q behaves like the conjugate kernel in the trigonometric case for θ near ϕ.

Proof. Let us define D by

$$D = 1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2.$$

We then have

$$P(r,\theta,\phi) = (\lambda/\pi)(1-r^2)\int_0^\pi \frac{\sin^{2\lambda-1}t}{D^{\lambda+1}}\, dt.$$

Differentiation of both sides establishes the elementary identity

$$\int_0^r \frac{t^{2\lambda}(1-t^2)}{(1-2at+t^2)^{\lambda+2}}\, dt = \frac{r^{2\lambda+1}}{(\lambda+1)(1-2ar+r^2)^{\lambda+2}} - \frac{\lambda}{\lambda+1}\int_0^r \frac{t^{2\lambda}}{(1-2at+t^2)^{\lambda+1}}\, dt\,.$$

As we saw before

$$Q(r,\theta,\phi) = -r^{-2\lambda}\int_0^r t^{2\lambda-1}P_\theta(t,\theta,\phi)\,dt.$$

But

$$P_\theta(t,\theta,\phi) = -\frac{(\lambda+1)\lambda}{\pi}(1-t^2)t\int_0^\pi \frac{b(\sin\tau)^{2\lambda-1}}{(1-2at+t^2)^{\lambda+1}}\,d\tau,$$

where

$$a = \cos\theta\cos\phi + \sin\theta\sin\phi\cos\tau,$$

$$b = 2\sin\theta\cos\phi - 2\cos\theta\sin\phi\cos\tau.$$

Substituting in the above identity gives

$$Q(r,\theta,\phi) = -r^{-2\lambda+1}R(r,\theta,\phi) + \lambda r^{-2\lambda}\int_0^r R(t,\theta,\phi)\,dt,$$

with

$$R(r,\theta,\phi) = \frac{-\lambda}{\pi}r^{2\lambda-1}\int_0^r D_\theta\cdot D^{-\lambda-1}\sin^{2\lambda-1}t\,dt. \tag{7.4}$$

Thus it suffices to obtain similar estimates for the kernel $R(r,\theta,\phi)$.

Consider first the case, $\phi > 2\theta$. Now

$$D^{\lambda+1} \geqq ((1-r)^2 + 2r(1-\cos(\theta-\phi)))^{\lambda+1} \geqq c(1-\cos(\theta-\phi))^{\lambda+1}$$
$$\geqq c|\sin(\theta-\phi)|^{2\lambda+2} \geqq c(\sin\phi)^{2\lambda+2},$$

since $\phi > 2\theta$.

However, $|D_\theta| = 2r|\sin\theta\cos\phi - \cos\theta\sin\phi\cos t| \leqq c\sin\phi$ (since $\phi > 2\theta$), we therefore have by substituting these estimates in (7.4)

$$|R(r,\theta,\phi)| \leqq c(\sin\phi)^{-2\lambda-1} \quad \text{for } \phi > 2\theta,\ 0 \leqq \theta \leqq \pi. \tag{7.5}$$

By an almost identical argument

$$|R(r,\theta,\phi)| \leqq c(\sin\theta)^{-2\lambda-1}, \quad \frac{1}{2}\theta > \phi,\ 0 \leqq \theta \leqq \pi. \tag{7.6}$$

We now consider the more critical range, $\theta/2 \leqq \phi \leqq 2\theta$.

Write,

$$R(r,\theta,\phi) = \frac{-\lambda r^{2\lambda-1}}{\pi}\int_0^\pi D_\theta D^{-\lambda-1}\sin^{2\lambda-1}t\,dt$$

$$= \frac{-\lambda r^{2\lambda-1}}{\pi}\int_0^{\pi/2} D_\theta D^{-\lambda-1}\sin^{2\lambda-1}t\,dt + \frac{-\lambda r^{2\lambda-1}}{\pi}\int_{\pi/2}^\pi D_\theta D^{-\lambda-1}\sin^{2\lambda-1}t\,dt \tag{7.7}$$

$$= R_1(r,\theta,\phi) + R_2(r,\theta,\phi).$$

In R_1 the behavior near $t=0$ will be decisive and in R_2 the behavior near $t=\pi$ will be critical. Thus in R_1 we shall systematically use the fact that $\cos t = 1 - t^2/2 + O(t^4)$, and $\sin^{2\lambda-1} t = t^{2\lambda-1} + O(t^{2\lambda+1})$, as $t \to 0$; similar facts are used when dealing with R_2, whose contribution is solely to the error term.

Now

$$\begin{aligned} D^{\lambda+1} &= [1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2]^{\lambda+1} \\ &= [1 - 2r\cos(\theta-\phi) + r^2 + rt^2\sin\theta\sin\phi + O(t^4)r\sin\theta\sin\phi]^{\lambda+1}. \end{aligned}$$

Since $1 - 2r\cos(\theta-\phi) + r^2 \geqq 0$, and $r\sin\theta\sin\phi \geqq 0$, we get by the mean value theorem

$$\begin{aligned} D^{-\lambda-1} &= [1 - 2r\cos(\theta-\phi) + r^2 + t^2 r\sin\theta\sin\phi]^{-\lambda-1} \\ &\quad + O(t^4)r\sin\theta\sin\phi \cdot D^{-\lambda-2}. \end{aligned} \tag{7.8}$$

Also

$$D_\theta = 2r\sin(\theta-\phi) + O(t^2)\sin\phi \tag{7.9}$$

since $\theta/2 \leqq \phi \leqq 2\theta$, $0 \leqq \theta \leqq \pi$. Inserting the estimates (7.8) and (7.9) in (7.7), simplifying, and using the fact that $t^2 r\sin\theta\sin\phi \leqq cD$, we get

$$\begin{aligned} R_1(r,\theta,\phi) &= \frac{-\lambda}{\pi} r^{2\lambda-1} \int_0^{\pi/2} \frac{t^{2\lambda+1}\sin(\theta-\phi)}{[1 - 2r\cos(\theta-\phi) + r^2 + rt^2\sin\theta\sin\phi]^{\lambda+1}}\, dt \\ &\quad + O(\sin\phi)\int_0^{\pi/2} t^{2\lambda+1}[1 - 2r\cos(\theta-\phi)tr^2 + rt^2\sin\theta\sin\phi]^{-\lambda-1}\, dt. \end{aligned} \tag{7.10}$$

We handle these integrals by making the substitution

$$u = \left(\frac{r\sin\theta\sin\phi}{1 - 2r\cos(\theta - \phi + r^2}\right)^{1/2} t = \left(\frac{\sigma}{\Delta}\right)^{1/2} t,$$

as in the proof of Lemma 1 of §4. (Here, $\sigma = r\sin\theta\sin\phi$, and $\Delta = 1 - 2r\cos(\theta-\phi) + r^2$.)

Thus the first integral in (7.10) becomes

$$\frac{-\lambda}{\pi} r^{2\lambda-1} \frac{\sin(\theta-\phi)}{\Delta} \sigma^{-\lambda} \int_0^{(\sigma/\Delta)^{1/2}(\pi/2)} \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+1}}\, du.$$

However

$$\int_0^{\xi} \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+1}}\, du = \int_0^{\infty} \frac{u^{2\lambda-1}}{(1+u^2)^{\lambda+1}}\, du + O(1/\xi^2) \qquad \xi \to \infty.$$

Therefore the first integral in (7.10) becomes

$$c_\lambda r^{\lambda-1}(\sin\theta\sin\phi)^{-\lambda}\,\frac{\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}+O(|\sin(\theta-\phi)|\,\sigma^{-\lambda-1}),$$

and this equals

$$c_\lambda r^{\lambda-1}(\sin\theta\sin\phi)^{-\lambda}\,\frac{\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}+O(\sin\theta)^{-2\lambda-1}, \tag{7.11}$$

since $\theta/2 \leqq \phi \leqq 2\theta$.

The second integral in (7.10) is dealt with similarly, and leads to the estimate

$$O\left(\sin\phi\cdot\sigma^{-\lambda-1}\left[1+\log^+\left(\frac{\sigma}{\Delta}\right)\right]\right).$$

Since $\sin\phi \geqq c\sin\theta$ and $\Delta \geqq c[1-\cos(\theta-\phi)]$, in this range the above O term is

$$O(\sin\theta)^{-2\lambda-1}\left(1+\log^+\left(\frac{\sin\theta\sin\phi}{1-\cos(\theta-\phi)}\right)\right). \tag{7.12}$$

Hence (7.11) and (7.12) combined give us the required estimates for R_1 and thus for R, and finally for Q in the range $\theta/2 \leqq \phi \leqq 2\theta$. This concludes the proof of the lemma.

8. **L^p theorems for conjugate functions.** Let us recall the definition (7.1) of the conjugate harmonic function $\tilde{f}(r,\theta)$:

$$\tilde{f}(r,\theta) = 2\lambda\sum_{n=1}^{\infty}\frac{a_n}{n+2\lambda}r^n\sin\theta\,P_{n-1}^{\lambda+1}(\cos\theta), \qquad 0 \leqq r < 1,$$

where

$$f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta).$$

We shall need also the L^p classes with norm $\|f\|_p = (\int_0^\pi |f(\phi)|^p\,dm_\lambda(\phi))^{1/p}$, where $dm_\lambda(\phi) = (\sin\phi)^{2\lambda}d\phi$.

We come now to one of our main results—an extension of the M. Riesz theorem for conjugate functions for the L^p norm, and of Kolmogoroff's inequality for L^1.

THEOREM 4. (a) *If $f(\theta)\in L^p(dm_\lambda)$, $1<p<\infty$, then $\|\tilde{f}(r,\theta)\|_p \leqq A_p\|f\|_p$.*
(b) *If $f\in L^1(dm_\lambda)$, $s>0$, and $E_s = \{\theta \mid |f(r,\theta)|>s\}$. Then*

$$m_\lambda(E_s) \leqq (A/s)\,\|f\|_1.$$

Proof. We consider, for fixed r, $0\leqq r<1$, simultaneously two operators, T_r and T_r^* defined by

$$(T_r f)(\theta) = \int_0^\pi Q(r,\theta,\phi)f(\phi)\,dm_\lambda(\phi) \tag{8.1}$$

and

$$(8.2)\qquad (T_r^*f)(\theta) = \int_0^\pi (r,\phi Q,\theta)f(\phi)\,dm_\lambda(\phi).$$

Thus $T_r(f)(\theta) = \tilde{f}(r,\theta)$, and T_r^* is the adjoint of T_r. We prove first

$$(8.3)\qquad \| T_r f\|_2 \leqq \|f\|_2.$$

In fact,

$$\| T_r f\|_2^2 = (2\lambda)^2 \sum_{n=1}^{\infty} \frac{|a_n|^2 r^{2n}}{(n+2\lambda)^2}(\gamma_n^*)^{-1},$$

where

$$(\gamma_n^*)^{-1} = \int_0^\pi [P_{n-1}^{\lambda+1}(\cos\theta)]^2\,dm_{\lambda+1}(\theta) = 2^{-1-2\lambda}\pi(\Gamma(\lambda+1))^{-2}\frac{\Gamma(n+2\lambda+1)}{(n+\lambda)(n-1)!}.$$

However,

$$\|f\|_2^2 = \sum |a_n|^2\gamma_n^{-1},$$

where

$$\gamma_n^{-1} = \int_0^\pi [P_n^\lambda(\cos\theta)]^2\,dm_\lambda(\theta) = 2^{1-2\lambda}\pi(\Gamma(\lambda))^{-2}\frac{\Gamma(n+2\lambda)}{(n+\lambda)n!}$$

(see equation (2.9)).

Thus (8.3) is equivalent with $\sup_n (2\lambda)^2(n+2\lambda)^{-2}(\gamma_n/\gamma_n^*) \leqq 1$, which after reduction becomes $\sup_n (n/(n+2\lambda)) \leqq 1$, and this proves (8.3).

By a standard duality argument we then obtain

$$(8.4)\qquad \| T_r^* f\|_2 \leqq \|f\|_2.$$

We have therefore proved part (a) of the theorem in the special case $p=2$. In order to make further progress we must now prove the "weak-type" estimate, that is part (b) of the theorem. We shall prove this for both T_r and T_r^*. In so doing we shall systematically use the following remark: If finitely many functions satisfy the weak-type estimates arising in (b) for all $s>0$, then so do their linear combinations with constant coefficients, with possibly larger constant.

We now make a few simplifying assumptions. First we take $f \geqq 0$; next we assume that $f(\phi)=0$, if $\pi/2 \leqq \phi \leqq \pi$. The case when the support of f lies in $(\pi/2,\pi)$ is dealt with similarly.

Then

$$\begin{aligned}(T_r f)(\theta) &= \int_0^{\pi/2} Q(r,\theta,\phi)f(\phi)\,dm_\lambda(\phi)\\ &= \int_0^{\theta/2} Q(r,\theta,\phi)f(\phi)\,dm_\lambda(\phi) + \int_{2\theta}^{\pi/2} Q(r,\theta,\phi)f(\phi)\,dm_\lambda(\phi)\\ &\quad + \int_{\theta/2}^{2\theta} Q(r,\theta,\phi)f(\phi)\,dm_\lambda(\phi).\end{aligned}$$

Let us set

$$M(f)(\theta) = (\sin\theta)^{-2\lambda-1} \int_0^{\pi/2} f(\phi)\, dm_\lambda(\phi), \tag{8.5}$$

$$N(f)(\theta) = (\sin\theta)^{-2\lambda-1} \int_{\theta/2}^{2\theta} \log^+ \frac{\sin\theta\sin\phi}{1-\cos(\theta-\phi)} f(\phi)\, dm_\lambda(\phi) \tag{8.6}$$

and

$$Q_0(r,\theta,\phi) = c_\lambda r^\lambda \frac{\sin^{-\lambda}\theta \sin^{-\lambda}\phi \sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2}. \tag{8.7}$$

Then by virtue of Lemma 4 we have, after a simple reduction,

$$\left| T_r(f)(\theta) - \int_{\theta/2}^{2\theta} Q_0(r,\theta,\phi) f(\phi)\, dm_\lambda(\phi) \right| \leqq c[M(f)(\theta) + N(f)(\theta)].$$

It is easily seen that

$$m_\lambda(\theta \mid Mf(\theta) > s) \leqq \frac{c}{s} \int_0^{\pi/2} f\, dm_\lambda(\phi). \tag{8.8}$$

Next we claim that

$$m_\lambda(\theta \mid Nf(\theta) > s) \leqq \frac{c}{s} \int_0^{\pi/2} f\, dm_\lambda. \tag{8.9}$$

This will follow if we show that $\int_0^\pi Nf(\theta)\, dm_\lambda(\theta) \leqq c \int_0^\pi f(\phi)\, dm_\lambda(\phi)$. In fact

$$\int_0^\pi Nf(\theta)\, dm_\lambda(\theta) = \int_0^\pi (\sin\theta)^{-1} \left\{ \int_{\theta/2}^{2\theta} f(\phi) \log^+ \frac{\sin\theta\sin\phi}{1-\cos(\theta-\phi)}\, dm_\lambda(\phi) \right\} d\theta$$

$$= \int_0^{\pi/2} f(\phi) K(\phi)\, dm_\lambda(\phi),$$

where

$$K(\phi) = \int_{\phi/2}^{2\phi} \log^+ \left(\frac{\sin\theta\sin\phi}{1-\cos(\theta-\phi)} \right) \frac{d\theta}{\sin\theta}, \qquad 0 \leqq \phi \leqq \pi/2.$$

However, a simple reduction shows, that in this range

$$K(\phi) \leqq A \int_{\phi/2}^{2\phi} \log^+ \left(\frac{\theta\phi}{(\theta-\phi)^2} \right) \frac{d\theta}{\theta} + B = A \int_{1/2}^{2} \log^+ \left(\frac{\theta}{(\theta-1)^2} \right) \frac{d\theta}{\theta} + B.$$

Thus $K(\phi)$ is bounded and therefore we obtain (8.9).

We now come to the main contribution—that given by the kernel $Q_0(r,\theta,\phi)$—

which is the principal term of $Q(r,\theta,\phi)$. For this purpose consider the sets $E_n = (2^{-n-1}\pi, 2^{-n}\pi)$ and the sets $E_n' = (2^{-n-2}\pi,\ 2^{-n+1}\pi)$. In the integral

$$\int_{\theta/2}^{2\theta} Q_0(r,\theta,\phi) f(\phi)\, dm_\lambda(\phi) \tag{8.10}$$

if θ lies in the set E_n, the integration in ϕ ranges only over part of E_n'. Thus if $\chi_{E_n'}$ is the characteristic function of E_n', the integral of (8.10) is identical with the sum

$$\sum_{n=0}^{\infty} \chi_{E_n}(\theta) \int_{\theta/2}^{2\theta} Q_0(r,\theta,\phi) f_n(\phi)\, dm_\lambda(\phi)\,, \tag{8.11}$$

where

$$f_n(\phi) = f(\phi)\chi_{E_n'}(\phi).$$

We would like, however, to compare the sum (8.11) with the sum

$$\sum_{n=0}^{\infty} \chi_{E_n}(\theta) \int_{E_n'} Q_0(r,\theta,\phi) f(\phi)\ dm_\lambda(\phi). \tag{8.12}$$

The difference of the nth terms of these two sums is dominated by

$$\chi_{E_n}(\theta) \int_{F_n} \left| Q_0(r,\theta,\phi)\right| f(\phi)\, dm_\lambda(\phi),$$

where

$$F_n = (2^{-n-2}\pi, \theta/2) \cup (2\theta, 2^{-n+1}\pi), \qquad \theta \in E_n.$$

However, in this range it is easily verified that $\left|Q_0(r,\theta,\phi)\right| \leqq c(\sin\theta)^{-2\lambda-1}$. Thus it follows that the difference of the sums (8.11) and (8.12) is dominated by

$$c(\sin\theta)^{-2\lambda-1} \sum_{n=0}^{\infty} \int_{E_n'} f(\phi)\, dm_\lambda \leqq c'(\sin\theta)^{-2\lambda-1} \int_0^{\pi} f(\phi)\, dm_\lambda(\phi).$$

This is of the form Mf (see (8.5)), and leads to the desired estimates. We have therefore finally reduced the problem to that of the sum (8.12).

Let then E^s denote the set of θ's for which the absolute value of the sum (8.12) exceeds s. Then

$$E^s = \bigcup_{n=0}^{\infty} (E^s \cap E_n), \quad \text{and} \quad m_\lambda(E^s) = \sum_{n=0}^{\infty} m_\lambda(E^s \cap E_n).$$

We deal with the set $E^s \cap E_0$ separately. Therefore we take $n \geqq 1$. It is clear that on E_n, $(\sin\theta)^{-\lambda}$ is of the order $2^{+n\lambda}$; thus

$$E^s \cap E_n \subset \left\{\theta \,\middle|\, \left| \int_{E_n'} \frac{(\sin\phi)^{-\lambda}\sin(\theta-\phi)}{1-2r\cos(\theta-\phi)+r^2} f(\phi)\, dm_\lambda(\phi) \right| > c2^{-n\lambda}s \right\}.$$

Thus by the classical theorem on conjugate functions for trigionometric series

$$\int_{E^s\cap E_n} d\theta \leqq c'\frac{2^{+n\lambda}}{s}\int_{E_n'} f(\phi)\sin^\lambda d\phi$$

$$\leqq C'\frac{2^{+n\lambda}}{s}\cdot 2^{+n\lambda}\int_{E_n'} f(\phi)\sin^{2\lambda}\phi\, d\phi.$$

However,

$$m_\lambda(E^s\cap E_n) = \int_{E^s\cap E_n} \sin^{2\lambda} d\theta \leqq C2^{-2n\lambda}\int_{E^s\cap E_n} d\theta.$$

Thus,

$$\sum_{n=1}^{\infty} m_\lambda(E^s\cap E_n)\leqq C'/s\sum_{n=1}^{\infty}\int_{E_n'} f(\phi)\, dm_\lambda \leqq C/s\int_0^\pi f(\phi)\, dm_\lambda(\phi).$$

It remains to consider $E^s\cap E_0$. Now $E_0 = (\pi/2,\pi) = (\pi/2,(3/4)\pi)\cup[(3/4)\pi,\pi) = E_0^{(1)}\cup E_0^{(2)}$. The measure $m_\lambda(E^s\cap E_0^{(1)})\leqq (C/s)\int_0^\pi f(\phi)\,dm_\lambda$ by the same arguments used to consider $m_\lambda(E^s\cap E_n)$, $n\geqq 1$. Finally, when $\theta\in[(3/4)\pi,\pi]$, $\phi\in[0,\pi/2]$ we use the trivial estimate $|Q_0(r,\theta,\phi)|\leqq C(\sin\theta)^{-2\lambda-1}$, which reduces the estimate of the measure $m_\lambda(E^s\cap E_0^{(2)})$ to the integral $M(f)(\theta)$ considered earlier.

Combining these estimates, and taking into consideration the remarks made earlier, we obtain

$$m_\lambda(\theta\,|\,(T_rf(\theta)|>s)\leqq A/s\,\|f\|_1. \tag{8.13}$$

This proves part (b) of the theorem.

Since we have essentially identical estimates for the kernels $Q(r,\phi,\theta)$ and $Q(r,\theta,\phi)$ we also obtain

$$m_\lambda(\theta\,|\,|T_r^*f(\theta)|>s)\leqq A/s\cdot\|f\|_1. \tag{8.14}$$

A combination of (8.13) and (8.3) via the Marcinkiewicz interpolation theorem ([5]) proves

$$\|T_r(f)\|_p\leqq A_p\|f\|_p,\quad 1<p\leqq 2. \tag{8.15}$$

Similarly, (8.14) and (8.4) lead to

$$\|T_r^*(f)\|_p\leqq A_p\|f\|_p,\quad 1<p\leqq 2. \tag{8.16}$$

However, by a standard duality argument (8.16) gives

$$\|T_r(f)\|_q\leqq A_q\|f\|_q,\quad 2\leqq q<\infty. \tag{8.17}$$

Therefore the theorem is completely proved.

([5]) See Zygmund [26, Vol. II, p. 111].

COROLLARY 1. *Let $f \in L^p(dm_\lambda)$, $1 < p < \infty$; then $\tilde{f}(r,\theta)$ converges in L^p = norm to $\tilde{f}(\theta)$, as $r \to 1$, and*

$$\|\tilde{f}\|_p \leqq A_p \|f\|_p .$$

Moreover,

$$\tilde{f}(\theta) \sim 2\lambda \sum \frac{a_n}{n+2\lambda}(\sin\theta)P_{n-1}^{\lambda+1}(\cos\theta).$$

COROLLARY 2. *If $d\mu$ is a measure belonging to B, $d\mu \sim \sum a_n P_n^\lambda(\cos\theta)$ and*

$$\tilde{f}(r,\theta) = 2\lambda \sum_{n=1}^{\infty} \frac{a_n}{n+2\lambda} r^n \sin\theta\, P_{n-1}^{\lambda+1}(\cos\theta),$$

then

$$\left(\int_0^\pi |\tilde{f}(r,\theta)|^p \, dm_\lambda(\theta)\right)^{1/p} \leqq A_p \|d\mu\|, \quad 0 < p < 1.$$

Corollary 1 is proved in a familiar way from part (a) of the theorem. To prove Corollary 2, let $0 < r_0 < 1$, and let $g(\theta) = f(r_0,\theta)$, where $f(r_0,\theta)$ is the Poisson integral of the measure $d\mu$. Then $\|f(r_0,\theta)\|_1 \leqq \|d\mu\|$. Now apply part (b) of the theorem to g and let $r_0 \to 1$.

We see from the theorem and its corollary that the conjugate mapping, $f \to \tilde{f}$ is bounded on L^p, $1 < p < \infty$, and satisfies an appropriate substitute relation for L^1. This brings us naturally to the problem of the inverse mapping. Let us consider, for the moment, certain facts about the conjugate mapping, $f \to \tilde{f}$, in the classical trigonometric case. The adjoint of the mapping $f \to \tilde{f}$ is the mapping $f \to -\tilde{f}$; while the inverse of the mapping $f \to \tilde{f}$ (defined on those f without constant term) is again the mapping $f \to -\tilde{f}$. However, in the general case we consider, this is no longer true. Thus we had to consider side-by-side with the mapping T_r the mapping T_r^*, in our proof of Theorem 4.

To study the inverse mapping, define the operator S_r by

$$S_r(f)(\theta) = (2\lambda)^{-1} \sum_{n=1}^{\infty} a_n(n+2\lambda) r^n P_n^\lambda(\cos\theta) \tag{8.18}$$

whenever

$$f(\theta) \sim \sum_{n=1}^{\infty} a_n(\sin\theta) P_{n-1}^{\lambda+1}(\cos\theta).$$

It follows that

$$S_r(f)(\theta) = \int_0^\pi Q_-(r,\theta,\phi) f(\phi)\, dm_\lambda(\phi),$$

where

$$Q_-(r,\theta,\phi) = (2\lambda)^{-1} \sum_{n=1}^{\infty} (n+2\lambda)\gamma_n^* r^n P_n^\lambda(\cos\theta) \sin\phi\, P_{n-1}^{\lambda+1}(\cos\phi).$$

(For the definition of the constants γ_n^* see the formulae following (8.3).)

If we recall the formula for $Q(r,\theta,\phi)$ (see (7.2)) and the fact that

$$(2\lambda)^{-2}(n+2\lambda)^2\frac{\gamma_n^*}{\gamma_n}=\frac{n+2\lambda}{n}=1+\frac{2\lambda}{n},$$

we get

$$Q_-(r,\theta,\phi)=Q(r,\phi,\theta)+2\lambda\int_0^r t^{-1}Q(t,\phi,\theta)\,dt.$$

Thus,

$$S_r=T_r^*+2\lambda\int_0^r t^{-1}T_t^*\,dt. \tag{8.19}$$

However, we have seen that $\|T_rf\|_p\leqq A_p\|f\|_p$, $1<p<\infty$, and thus $\|T_r^*f\|_p\leqq A_p\|f\|_p$, $1<p<\infty$. From (8.19) it then follows that

$$\|S_r(f)\|_p\leqq A_p\|f\|_p,\quad 1<p<\infty,\quad 0\leqq r<1. \tag{8.20}$$

Incidentally, we have used the semitrivial estimate $\|T_r^*f\|_p\leqq Ar\|f\|_p$ if $0\leqq r\leqq 1/2$.

When f is, say, a polynomial,

$$f(\theta)=\sum_{n=1}^N a_nP_n^\lambda(\cos\theta),\qquad \int_0^\pi f(\theta)\,dm_\lambda(\theta)=0,$$

then it follows immediately from the definitions of f and S_r that $\lim_{r\to 1}S_r(f)(\theta)=f(\theta)$ in L^p-norm.

Combining this with (8.20) gives the following further corollary.

COROLLARY 3. *If $f(\theta)\in L^p(dm_\lambda)$, $1<p<\infty$, and*

$$\int_0^\pi f(\theta)\,dm_\lambda(\theta)=0,$$

then

$$B_p\|f\|_p\leqq\|\tilde f\|_p\leqq A_p\|f\|_p.$$

9. Theory of H^p-spaces: Harmonic majorization. Let u and v be a pair of conjugate harmonic functions on the upper semi-disc, that is

$$v_x=-y^{2\lambda}u_y\quad\text{and}\quad v_y=y^{2\lambda}u_x$$

there.

We have studied in §6 the conditions under which $u(x,y)$ can be represented as a Poisson integral of boundary values taken on the perimeter of the upper semi-disc. Whenever this is possible we have

$$u(x,y)=f(r,\theta)=\sum_{n=0}^\infty a_nr^nP_n^\lambda(\cos\theta),\quad (x,y)=(r\cos\theta,r\sin\theta). \tag{9.1}$$

$u(x, y)$ then has the additional property that it can be extended past the singular line $y = 0$ to the whole disc so that it is even in y and regular there. For u's represented as in (9.1) we can write down a similar series representing the conjugate v,

$$v(x, y) = y^{2\lambda}\tilde{f}(r, \theta) = y^{2\lambda} \cdot 2\lambda \sum_{n=1}^{\infty} \frac{a_n r^n}{n + 2\lambda} \sin\theta P_{n-1}^{\lambda+1}(\cos\theta). \tag{9.2}$$

It is to be noted that $\tilde{f}(r, \theta)$ can also be extended past the line $y = 0$, so that as a function of x and y it is *odd* in y, and regular in the entire disc.

In view of the L^p boundedness results obtained in §8 it is natural to consider $f(r, \theta)$ and $\tilde{f}(r, \theta)$ on the same footing—as opposed to u and v. For this reason it is convenient to use adjusted functions U and V defined by

$$\begin{aligned} U(x, y) &= u(x, y) = f(r, \theta), \\ V(x, y) &= y^{-2\lambda} v(x, y) = \tilde{f}(r, \theta). \end{aligned} \tag{9.2$'$}$$

We shall write also in complex notation

$$F(z) = F(x, y) = U + iV, \qquad z = x + iy.$$

It is to be noted that U and V satisfy the equations

$$\begin{aligned} &V_x + U_y = 0, \\ &\frac{2\lambda}{y} V + V_y - U_x = 0, \qquad 0 \leqq \lambda. \end{aligned} \tag{9.3}$$

Recall also that if $f(r, \theta)$ is the Poisson integral of a function, say in $L^p(dm_\lambda)$, $1 < p < \infty$, then by Theorems 2 and 4 it follows that

$$\sup_{r<1} \int_0^\pi \left| F(re^{i\theta}) \right|^p \, dm_\lambda(\theta) < \infty. \tag{9.4}$$

We are now in a position to make our basic definition. Let $F = U + iV$ be a function regular in the unit disc. U is even in y and V is odd in y. Suppose that F satisfies the system (9.3) in the upper semi-disc, and for some $p, p > 0$, F satisfies the boundedness condition (9.4). Then we say that F belongs to the class H^p.

By the remarks made above it can be shown that the H^p theory, when $1 < p < \infty$, is essentially equivalent with the L^p theory already studied. Our main interest, therefore, will be with $p \leqq 1$. Here matters stand in close analogy with the H^p theory of harmonic functions in several variables, developed earlier; see [17]. Thus it will be possible to develop an H^p theory for a range of p's less than 1, but not all p's, $0 \leqq p \leqq 1$. More precisely we shall deal with those p's which satisfy $p \geqq 2\lambda/(2\lambda + 1)$; for only those we have the following basic lemma concerning "sub-harmonicity."

LEMMA 5. *Let $F = U + iV$ satisfy* (9.3) *in a region in the upper half-plane. Suppose $|F| > 0$ there. Then if $p \geqq 2\lambda/(2\lambda+1)$, and $\Delta_\lambda = \partial^2/\partial x^2 + \partial^2/\partial y^2 + (2\lambda/y)\partial/\partial y$ (where $\lambda \geqq 0$), then*

$$\Delta_\lambda |F|^p \geqq 0. \tag{9.5}$$

Proof. Write $F = (U, V)$, $F_x = (U_x, U_x)$ and $F \cdot F_x = UU_x + VV_x \cdots$, etc. Then

$$\frac{\partial}{\partial x} |F|^p = p|F|^{p-2} F_x \cdot F,$$

$$\frac{\partial}{\partial y} |F|^p = p|F|^{p-2} F_y \cdot F,$$

$$\frac{\partial^2}{\partial x^2} |F|^p = p[(p-2)|F|^{p-4}(F_x \cdot F)^2 + |F|^{p-2}\{|F_x|^2 + F \cdot F_{xx}\}].$$

Similarly with $(\partial^2/\partial y^2)|F|^p$.

Adding we obtain

$$\begin{aligned} &\Delta_\lambda |F|^p \\ &= p|F|^{p-4}[(p-2)((F_x \cdot F)^2 + (F_y \cdot F)^2) + |F|^2\{|F_x|^2 + |F_y|^2 + F \cdot \Delta_\lambda F\}]. \end{aligned} \tag{9.6}$$

Notice, however, that $\Delta_\lambda F = \Delta_\lambda U + i\Delta_\lambda V$. From (9.3) it follows that $\Delta_\lambda U = 0$, while

$$V_{xx} + V_{yy} + \left(\frac{2\lambda}{y} V\right)_y = 0;$$

thus $\Delta_\lambda V = (2\lambda/y^2)V$, and hence $F \cdot \Delta_\lambda F = 2\lambda(V^2/y^2)$. Substituting in (9.6) we then see that $\Delta_\lambda |F|^p \geqq 0$ if $p \geqq 2$, since everything is positive in this case. Having disposed of this simpler case, let us now suppose that $p < 2$. Then (9.5) is equivalent with

$$(F_x \cdot F)^2 + (F_y \cdot F)^2 \leqq \frac{1}{2-p}|F|^2\left[|F_x|^2 + |F_y|^2 + \frac{2\lambda V^2}{y^2}\right]. \tag{9.7}$$

Let M denote the matrix

$$\begin{bmatrix} U_x & V_x \\ -U_y & -V_y \end{bmatrix}.$$

Notice that the equations (9.3) can be translated into the two statements (i) M is symmetric and (ii) trace of $M = 2\lambda V/y$. The inequality (9.7) can be translated to

$$|M[F]|^2 \leqq \frac{1}{2-p}|F|^2\left[\|M\|^2 + \frac{(\mathrm{tr}(M))^2}{2\lambda}\right], \tag{9.8}$$

where $\|M\|^2$ denotes the square of the 'Hilbert-Schmidt' norm of the matrix.

If we consider F to be an arbitrary two-component vector, then (9.8) becomes

$$|M|^2 \leqq \left(\frac{1}{2-p}\right)\left[\|M\|^2 + \frac{(\mathrm{tr}(M))^2}{2\lambda}\right], \tag{9.9}$$

where $|M|$ denotes the usual norm of the matrix M—considered as an operator. We shall show that the inequality holds for any symmetric 2×2 matrix M, as long as $p \geqq 2\lambda/(2\lambda+1)$.

In fact, $|M|$, $\|M\|$, and $\mathrm{tr}(M)$ are invariant under orthogonal changes of coordinates; so we may assume that M is already in diagonal form,

$$M = \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{pmatrix}.$$

Then (9.9) becomes

$$\max_k \mu_k^2 \leqq \frac{1}{(2-p)}[\mu_1^2 + \mu_2^2 + (2\lambda)^{-1}(\mu_1+\mu_2)^2]. \tag{9.10}$$

We have

$$\mu_1 = -\mu_2 + (\mu_1 + \mu_2).$$

Therefore, $\mu_1^2 \leqq \alpha\mu_2^2 + \beta(\mu_1+\mu_2)^2$, whenever $\alpha > 0$, $\beta > 0$, $1/\alpha + 1/\beta = 1$. Add $\alpha\mu_1^2$ to both sides; this gives

$$\mu_1^2 \leqq \frac{\alpha}{1+\alpha}\left([\mu_1^2 + \mu_2^2] + \frac{\beta}{\alpha}[\mu_1+\mu_2]^2\right).$$

Choose $\beta = (2\lambda+1)/2\lambda$, $\alpha = 2\lambda+1$; then $\beta/\alpha = 1/2\lambda$, and $\alpha/(1+\alpha) = (2\lambda+1)/(2\lambda+2)$. Thus (9.10) is valid as long as $1/(2-p) \geqq (2\lambda+1)/(2\lambda+2)$, which is $2 > p \geqq 2\lambda/(2\lambda+1)$. This proves (9.10) and, therefore, the lemma.

REMARK. The lower bound $2\lambda/(2\lambda+1)$ which occurs for p in Lemma 5 cannot be improved. One only need consider $U = x(x^2+y^2)^{-\lambda-1}$ and $V = -y(x^2+y^2)^{-\lambda-1}$. Then $|F|^p = (r^{-2\lambda-1})^p = r^{-\mu}$. But $\Delta_\lambda r^{-\mu} < 0$ unless $\mu \geqq 2\lambda$.

We now come to the majorization theorem which is the key result in the development of the H^p results.

THEOREM 5. *Let $\lambda > 0$, and $\nu = 2\lambda/(2\lambda+1)$. Suppose that $F \in H^p$, $\nu \leqq p$.*

(a) *If $p = \nu$, there exists finite positive measure $d\mu$ ($d\mu \in B$), so that if $h(r,\theta)$ is its Poisson integral, then*

$$|F(re^{i\theta})|^\nu \leqq h(r,\theta), \qquad 0 \leqq r < 1, \quad \textit{for all } \theta, \tag{9.11}$$

and

$$\sup_{r<1} \int_0^\pi |F(re^{i\theta})|^\nu \, dm_\lambda(\theta) = \int_0^\pi \sin^{2\lambda}\theta \, d\mu(\theta) \; (= \|d\mu\|). \tag{9.12}$$

(b) *If* $p > \nu$, *then there exists a function* $g(\theta) \in L^q$ *where* $q = p/\nu$, *so that if* $h(r,\theta)$ *is the Poisson integral of* g, *then* (9.11) *still holds.* (9.12) *is then to be replaced by*

$$\sup_{r<1} \| F(re^{i\theta}) \|_p^\nu = \| g \|_q. \tag{9.13}$$

Proof. Let us consider first part (a). Fix ρ, $0 < \rho < 1$, and set $g_\rho(\theta) = |F(\rho e^{i\theta})|^\nu$. Let $h_\rho(r,\theta)$ be the Poisson integral of $g_\rho(\theta)$. The main step in the proof consists in proving the inequality

$$|F(\rho r e^{i\theta})| \leqq h_\rho(r,\theta), \qquad 0 \leqq r \leqq 1. \tag{9.13}$$

For this purpose consider the difference $\delta(r,\theta) = |F(\rho r e^{i\theta})| - h_\rho(r,\theta)$. We observe that $h_\rho(r,\theta)$ is regular, and even in the unit disc and satisfies $\Delta_\lambda(h_\rho) = 0$ there, since h_ρ is a Poisson integral. Moreover, recalling the properties of U,V and F, we see that $|F(\rho r e^{i\theta})|$ is even and regular near any point where $|F| > 0$, and satisfies $\Delta_\lambda |F| \geqq 0$, there, by Lemma 5. Notice also that $\delta(1,\theta) = 0$, by the definition of h_ρ. Suppose now, that contrary to (9.13) we had $\delta(r,\theta) > 0$ for some point (r,θ). Since δ is continuous in the closed disc and vanishes on the boundary, its maximum would be attained at some point (r_0,θ_0) and $\delta(r_0,\theta_0) > 0$. However $h(r,\theta) \geqq 0$, since it is a Poisson integral of a non-negative function. Therefore $|F(\rho r_0 e^{i\theta})| > 0$, and hence $\Delta_\lambda(\delta) \geqq 0$ near the point (r_0,θ_0). This would violate the maximum principle (Theorem 1 of §3), and hence (9.13) is proved.

Now let ρ_n be a sequence tending to 1 from below. Let $g_n(\theta) = |F(\rho_n e^{i\theta})|$ and let $h_n(r,\theta)$ be the Poisson integral of $g_n(\theta)$. Then by (9.13) we have

$$|F(\rho_n r e^{i\theta})|^\nu \leqq h_n(r,\theta). \tag{9.14}$$

Moreover,

$$\int_0^\pi g_n(\theta) \sin^{2\lambda}\theta \, d\theta \leqq \sup_{\rho<1} \int_0^\pi |F(\rho e^{i\theta})|^\nu \sin^{2\lambda}\theta \, d\theta < \infty. \tag{9.15}$$

Thus, as is well known, we can find a subsequence $\{g_{n_k}(\theta)\}$ and a measure $d\mu$, so that, $g_{n_k} \to d\mu$ weakly. Let $h(r,\theta)$ be the Poisson integral of $d\mu$. From (9.14) it follows that

$$|F(re^{i\theta})| \leqq h(r,\theta), \qquad 0 \leqq r < 1, \tag{9.16}$$

and from (9.15) that

$$\begin{gathered} \int_0^\pi \sin^{2\lambda}(\theta) \, d\mu(\theta) \leqq \sup_{r<1} \int_0^\pi |F(re^{i\theta})|^\nu \sin^{2\lambda}\theta \, d\theta. \\ \int_0^\pi h(r,\theta) \sin^{2\lambda}\theta \, d\theta = \int_0^\pi \sin^{2\lambda}(\theta) \, d\mu(\theta). \end{gathered} \tag{9.17}$$

Combining this with (9.16) and (9.17) gives the identity (9.12). Thus part (a) of the theorem is completely proved. Part (b) is proved in a very similar way and needs no further comment.

10. **Theory of H^p-space: Basic results.** We are now in a position to state the basic results regarding the theory of H^p-spaces for ultraspherical expansions. The proofs given below depend on two ideas: the theory of majorization developed in the preceding section, which gives certain *global* results, and the "similarity principle" for pseudo-analytic functions, which will lead to certain *local* (i.e., almost everywhere) conclusions.

Let us recall the definition of the H^p-spaces given at the beginning of §9. We then have

THEOREM 6. (a) *Let $F \in H^p$, $2\lambda/(2\lambda+1) \leqq p$. Then $\lim_{r\to 1} F(re^{i\theta}) = F(e^{i\theta})$ exists for almost every θ. More generally, for almost every θ, $\lim F(\rho e^{i\phi})$ exists as $\rho e^{i\phi}$ approaches the point $e^{i\theta}$ nontangentially.*

(b) *If $F(e^{i\theta}) = 0$ in a set of θ of positive measure, then $F \equiv 0$.*

(c) *If $2\lambda/(2\lambda+1) < p$, then $\int_0^\pi |F(re^{i\theta}) - F(e^{i\theta})|^p \, dm_\lambda(\theta) \to 0$, as $r \to 1$.*

As a simple consequence of part (c) of the theorem, when $p = 1$, we obtain the following analogue of the theorem of F. and M. Riesz.

COROLLARY 1. *Let $d\mu_1$ and $d\mu_2$ be two finite measures (that is, $\int_0^\pi (\sin\theta)^{2\lambda} |d\mu_i| < \infty$). Suppose that*

$$d\mu_1 \sim \sum a_n P_n^\lambda(\cos\theta)$$

and

$$d\mu_2 \sim 2\lambda \sum \frac{a_n}{n+2\lambda} \sin P_{n-1}^{\lambda+1}(\cos\theta).$$

Then both $d\mu_1$ and $d\mu_2$ are absolutely continuous.

We also can complete the generalization of Fatou's theorem given Corollary 2 of Theorem 3, §6.

COROLLARY 2. *Let $u(x,y)$ be λ-harmonic, regular and even and satisfy*

$$\sup_{0 \leqq r < 1} \int_0^\pi |u(r\cos, r\sin\theta)| \, dm_\lambda(\theta) < \infty.$$

Then $\lim_{r\to 1} u(r\cos\theta, r\sin\theta)$ exists for a.e. θ, and in fact nontangentially.

We come now to the proof of the theorem and its corollaries. Let $d\mu$ be a positive measure which is finite, i.e., $\int_0^\pi \sin^{2\lambda}\theta \, d\mu(\theta) < \infty$. Let $h(r,\theta)$ denote its Poisson integral

$$h(r,\theta) = \int_0^\pi P(r,\theta,\phi) \sin^{2\lambda}\phi \, d\mu(\phi).$$

We denote by $\mu^*(\theta)$ the maximal function associated to $d\mu$, that is

$$\mu^*(\theta) = \sup_h \frac{\int_\theta^{\theta+h} \sin^{2\lambda}\phi \; d\mu(\phi)}{\int_\theta^{\theta+1} \sin^{2\lambda}\phi \; d\phi}$$

The well-known theorem of differentiation of measures tells us that $\mu^*(\theta) < \infty$ for a.e. θ.

Let us observe that

$$\sup_{0 \leqq r < 1} h(r,\theta) \leqq c\mu^*(\theta). \tag{10.1}$$

In fact, in the case when $d\mu$ is absolutely continuous (10.1) is merely a restatement of Lemma 3 of §5. The proof for a not-necessarily absolutely continuous measure is exactly the same.

(10.1) gives us an appropriate estimate for the radial approach to the boundary. In what follows we shall need "a nontangential" analogue of (10.1) This may be formulated as follows. Let Ω be a fixed triangle, lying strictly inside the unit disc, except that one of its vertices is at the point $r = 1$, $\theta = 0$. For any θ_0, define $\Omega(\theta_0)$ to be the triangle rotated (about the origin) so that this vertex is the point $r = 1$, $\theta = \theta_0$. The chosen triangle $\Omega = \Omega(0)$ is kept fixed in the rest of our discussion. Its particular shape is irrelevant.

We can then state

LEMMA 6.

$$\sup_{re^{i\psi} \in \Omega(\theta)} h(r, e^{i\psi}) \leqq c\mu^*(\theta).$$

Proof. In view of the definition of the Poisson kernel quoted above, it suffices to show that

$$P(r,\psi,\phi) \leqq cP(r,\theta,\phi), \qquad re^{i\psi} \in \Omega(\theta). \tag{10.2}$$

Let us recall, however, that

$$P(r,\theta,\phi) = \left(\frac{\lambda}{\pi}\right)(1-r^2) \int_0^\pi \frac{\sin^{2\lambda-1} t}{D^{\lambda+1}} dt \,,$$

where

$$\begin{aligned} D &= 1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2 \\ &= (x-\xi)^2 + (y-\eta)^2 + 2y\eta[1-\cos t], \end{aligned}$$

where $(\xi,\eta) = (\cos\phi, \sin\phi)$ and $(x,y) = (r\cos\theta, r\sin\theta)$, (see (2.12)).

Now the expression D has an interpretation in terms of a distance measured in three-dimensional Euclidean space. Thus consider the (xy)-plane embedded

in three-dimensional Euclidean space, and let Δ denote the disc perpendicular to the (xy)-plane and be determined by the fact that $(\xi,\eta,0)$ and $(\xi,-\eta,0)$ are a pair of diametrically opposite points on its perimeter. Then if $P=(x,y,0)$ and Q is an appropriate point in the perimeter of this disc, then D is the square of the distance from P to Q. The point Q may be fixed on the perimeter of Δ as follows: t is the angle made by the ray joining $(\xi,0,0)$ and Q with the y direction.

From this it can be seen that

$$D(r,\theta,\phi,t) \leqq cD(r,\psi,\phi,t), \qquad re^{i\psi} \in \Omega(\theta),$$

and therefore (10.2) follows. This proves the lemma.

We revert now to the proof of Theorem 6 and consider first part (a). Let $\delta > 0$, and let R_δ be the open region consisting of that part of the open unit disc where $y > \delta$, and let $\bar{R}_\delta$ denote its closure. Restrict $F(z) = U(x,y) + iV(x,y)$ to R_δ. Then F satisfies the system of "Cauchy-Riemann" equation (9.3) where the coefficients are regular in $\bar{R}_\delta$. Therefore by the similarity principle for pseudo-analytic functions (see, e.g., Bers [2]) there exists a function $S(z)$ continuous in $\bar{R}_\delta$, and an analytic function $G(z)$ in R_δ, so that

$$F(z) = e^{S(z)}G(z), \qquad z \in R_\delta. \tag{10.3}$$

However, by the majorization theorem (Theorem 5), $|F(z)|^\nu \leqq h(r,\theta)$, where $h(r,\theta)$ is the Poisson integral of a finite positive measure $d\mu(\theta)$. Thus in view of Lemma 6, $F(re^{i\theta})$ is bounded for $re^{i\theta} \in \Omega(\theta_0)$ and for almost every θ_0. By (10.3) $G(re^{i\theta})$ is bounded in the same triangles, for almost all θ_0, so that $\sin\theta_0 \geqq \delta$. Then by the theorem of Privalov-Plessener (see [26, Vol. II, p. 200]), it follows that $G(z)$ has nontangential limits on the perimeter of unit disc for a.e. θ_0, so that $\sin\theta_0 \geqq \delta > 0$. By (10.3) (since $S(z)$ is continuous in $\bar{R}_\delta$), the same is true for $F(z)$. Finally, since $\delta > 0$, but is otherwise arbitrary, part (a) of the theorem is completely proved. Part (b) is proved similarly, using the fact that G can vanish only on a set of zero measure on the perimeter of the disc unless it vanishes identically.

We come now to the proof of part (c).

Let $p > \nu = 2\lambda/(\lambda+1)$, and $q = p/\nu$. Then by part (b) of Theorem 5, there exists a $g \in L^q$, so that if $h(r,\theta)$ is its Poisson integral

$$|F(re^{i\theta})|^\nu \leqq h(r,\theta).$$

However, by Lemmas 2 and 3,

$$\sup_{r<1} h(r,\theta) \leqq cg^*(\theta) \in L^q.$$

Therefore

$$|F(re^{i\theta}) - F(e^{i\theta})|^p \leqq cg^*(\theta),$$

and part (c) of the theorem is proved by the use of the Lebesgue dominated convergence theorem, in conjunction with part (a), which is already known.

We now come to the proof of Corollary 1. Suppose that $d\mu_1 \sim \sum a_n P_n^\lambda(\cos\theta)$ and $d\mu_2 \sim 2\lambda \sum (a_n/(n+2\lambda)) P_{n-1}^{\lambda+1}(\cos\theta)$.

Let

$$U(x,y) = \sum a_n r^n P_n^\lambda(\cos\theta), \quad r<1, \quad (x,y) = (r\cos\theta, r\sin\theta),$$

$$V(x,y) = 2\lambda \sum \frac{a_n}{n+2\lambda} r^n \sin\theta\, P_{n-1}^{\lambda+1}(\cos\theta).$$

Then, as we know, $F = U + iV$ satisfies the system (9.3), and because U and V are Poisson integrals, then

$$\int_0^\pi |F(re^{i\theta})|\, dm_\lambda(\theta) \leqq \|d\mu_1\| + \|d\mu_2\|.$$

Thus F is in H^1, and therefore there exists $F(e^{i\theta}) = f(\theta) + i\tilde{f}(\theta) \in L^1(dm_\lambda)$, so that

$$\int_0^\pi |F(re^{i\theta}) - F(e^{i\theta})|\, dm_\lambda(\theta) \to 0.$$

Then it is easily verified that $f(\theta)d\theta = d\mu_1(\theta)$ and $\tilde{f}(\theta)d\theta = d\mu_2(\theta)$, since both $f(\theta)d\theta$ and $d\mu_1$ have the same ultraspherical expansions of index λ, and $\tilde{f}(\theta)d\theta$, $d\mu_2$ have the same expansions of index $\lambda+1$.

To prove Corollary 2 recall that the assumptions on u imply that it is the Poisson integral of a finite measure. (See Theorem 3′ in §6.) If we now define $F(z) = U(x,y) + iV(x,y)$ as above it then follows by Theorem 4, Corollary 2 that $F \in H^p$, for all $p<1$. The result then follows from part (a) of the present theorem.

We should point out conclusion (a) and (b) of the theorem could be proved for all p, $0<p$, without the extra restriction $2\lambda/(2\lambda+1) \leqq p$. This could be done as follows. Define, as in the proof of the theorem, the function $G(z)$, analytic in R_σ, so that (10.3) holds. Then $\sup_{r<1} \int_0^\pi |F(re^{i\theta})|^p dm_\lambda(\theta) < \infty$ implies easily $\sup_{r<1, \sin\theta \geqq \delta} \int |G(re^{i\theta})|^p\, d\theta < \infty$.

Then a modification of the classical arguments shows that G has nontangential limits for almost all points on the boundary of the disc. The same then follows for $F(z)$. Similar arguments hold for the nonvanishing on the boundary.

Chapter II

11. **Partial sums of ultraspherical expansions.** The purpose of the next few sections is to develop an analogue of the Littlewood-Paley theory for ultraspherical

expansions, culminating in the "multiplier theorem" of §14.

In the trigonometric case there are two basic tools used in the development of the Littlewood-Paley theory. The first are certain inequalities for partial sums, and the second deals with the auxiliary "g-function." In this section we shall extend the partial sum inequalities to the ultraspherical case.

Thus let

$$f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta),$$

and define S_n by

$$S_n(f) = S_n(\theta;\, f) = \sum_{k=0}^{n} a_k P_k^\lambda(\cos\theta).$$

A basic result for the partial sums is the inequality

$$\|S_n(f)\|_p \leqq A_p \|f\|_p, \qquad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}. \tag{11.1}$$

The case corresponding to $\lambda = 0$ is of course the classical inequality of M. Riesz, and the case for positive λ is due to Pollard [14].

We shall need an extension of (11.1) which can be stated as follows:

Let $f_1, f_2, \cdots, f_k$ be a given k-tuple of functions (k arbitrary) and write $F = (f_1, \cdots, f_k)$, $|F| = (|f_1|^2 + |f_2|^2 + \cdots + |f_k|^2)^{1/2}$.

Similarly let $n_1, \cdots, n_k$ be an arbitrary k-tuple of positive integers, and write $S(F)$ for the k-tuple

$$S(F) = (S_{n_1}(f_1),\ S_{n_2}(f_2), \cdots, S_{n_k}(f_k)).$$

Using this notation our result can then be stated as

THEOREM 7.

$$\| |S(F)| \|_p \leqq A_p \| |F| \|_p, \quad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}$$

with A_p independent of k.

The proof of the theorem makes use of the basic reduction of Pollard which occurs in the proof of (11.1) together with the following known lemma, see [26, Vol. II, p. 224].

LEMMA 7. *Let T be a linear transformation which satisfies*

$$\|Tf\|_p \leqq A\|f\|_p \quad \textit{for some } p. \tag{11.2}$$

Write $|T(F)| = (|Tf_1|^2 + |Tf_2|^2 + \cdots + |Tf_k|^2)^{1/2}$, *with* $|F| = (|f_1|^2 + \cdots + |f_k|^2)^{1/2}$. *Then* $\| |T(F)| \|_p \leqq A \| |F| \|_p$. *$A$ is the same constant as in* (11.2).

One should observe the difference between this lemma and Theorem 7. In the lemma we have one fixed operator, T, acting simultaneously on a k-tuple of functions. In the theorem we deal with k *different* partial sums, acting on the k-tuple of functions.

We now come to the proof of the theorem, which we have indicated is based on Pollard's work. Thus to avoid duplication we shall repeat only the main steps of his reduction omitting details. Following his notation, x and y are variables in the interval $(-1,1)$, which are related by our variables by the change $x=\cos\theta$, $y=\cos\phi$.

We have $\omega(x)dx=(\sin\theta)^{2\lambda}d\theta$ as the basic measure with $\omega(x)=(1-x^2)^{\lambda-1/2}$. We write $p_n(x)=\gamma_n^{1/2}P_n^{\lambda}(\cos\theta)$, which are the normalized polynomials for the weight $\omega(x)$. (γ_n are the factors given in (2.9).) We also have the conjugate polynomials $q_n(x)$, defined by $(1-x^2)^{-1}q_n(x)=(\gamma_n^*)^{1/2}P_{n-1}^{\lambda+1}(\cos\theta)$, which are the orthonormal polynomials with respect to $(1-x^2)\omega(x)$. (See Pollard [14, p. 356].)

Two basic estimates satisfied by the p_n and q_n are

$$\begin{aligned} &|(1-x^2)^{1/4}\omega(x)^{1/2}p_n(x)| \leqq A, \\ &|(1-x^2)^{-1/4}\omega(x)^{1/2}q_n(x)| \leqq A. \end{aligned} \tag{11.3}$$

(See Pollard [14, p. 362]; also Szegö [18, 7.33.6, p. 167].)

Now consider the Dirichlet kernel $k_n(x,y)$ given by

$$k_n(x,y) = \sum_{j=0}^{n} p_j(x)p_j(y).$$

According to Christoffel's formula (see Szegö [18, p. 42]),

$$k_n(x,y) = c_n \frac{p_{n+1}(x)p_n(y)-p_n(x)p_{n+1}(y)}{x-y}$$

for appropriate constants c_n.

However, the q_n's can be expressed in terms of the p_n's (see Pollard [14, p. 357]). Making the substitution and combining terms gives

$$k_n(x,y)=\alpha_n\frac{p_{n+1}(x)q_n(y)}{x-y}+\beta_n\frac{q_n(x)p_{n+1}(y)}{x-y}+\delta_n p_{n+1}(x)p_{n+1}(y), \tag{11.4}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, and $\{\delta_n\}$ are bounded sequences. (See [14, p. 358].)

We quote also the following lemma of Pollard:

LEMMA 8. *Let* $(Tf)(x)=\int_{-1}^{1}K(x,y)f(y)\,dy$, *where*

$$K(x,y) = \left|\frac{\left(\frac{1-y^2}{1-x^2}\right)^c-1}{x-y}\right|,$$

with $-1<c<1$ *and* $c<1/p<c+1$, $1<p$. *Then*

$$\int_{-1}^{1} |Tf(x)|^p \, dx \leqq A^p \int_{-1}^{1} |f(x)|^p \, dx.$$

Now

$$S_n(f) = \int_{-1}^{1} k_n(x,y) f(y) \omega(y) \, dy = S_n^1(f) + S_n^2(f) + S_n^3(f).$$

Here S_n^1 is the transformation whose kernel is $\alpha_n p_{n+1}(x) q_n(y)/(x-y)$; S_n^2 has the kernel $\beta_n q_n(x) p_{n+1}(y)/(x-y)$ and S_n^3 has the kernel $\gamma_n p_{n+1}(x) p_{n+1}(y)$.

Let $f_1, \cdots, f_k$ be k given functions and $n_1, \cdots, n_k$ be given positive integers. For $1 \leqq j \leqq 3$, set

$$S^{(j)}(F) = (S_{n_1}^j(f_1), \cdots, S_{n_k}^j(f_k))$$

and

$$|S^j(F)| = (|S_{n_1}^j(f_1)|^2 + \cdots + (|S_{n_k}^j(f_k)|^2)^{1/2}.$$

It will then suffice to show

$$\int_{-1}^{1} |S^j(F)(x)|^p \omega(x) \, dx \leqq A_p^p \int_{-1}^{1} |F(x)|^p \omega(x) \, dx, \quad j = 1,2,3. \tag{11.5}$$

Consider $S^1(F)$ first.

In view of the inequalities (11.3) and the boundedness of α_n we can write

$$\begin{aligned} &\omega^{1/p}(x) S_{n_k}^1(f_k)(x) \\ &= B_{n_k}(x) \int_{-1}^{1} \frac{\omega^{1/p}(y) f_k(y) A_{n_k}(y) \, dy}{x-y} + C_{n_k}(x) \int_{-1}^{1} K(x,y) \omega^{1/p}(y) f_k(y) A_{n_k}(y) \, dy, \end{aligned} \tag{11.6}$$

where $A_n(x)$, $B_n(x)$, and $C_n(x)$ are uniformly bounded and $K(x,y)$ is the kernel of Lemma 8, with $c = 1/4 + (\lambda - 1/2)(1/2 - 1/p)$.

Apply now Lemmas 7 and 8, and the well-known fact that

$$\int_{-1}^{1} \left| \int_{-1}^{1} \frac{f(y) \, dy}{x-y} \right|^p dx \leqq A_p^p \int_{-1}^{1} |f(x)|^p \, dx.$$

This gives (11.5) for $j = 1$,

$$\frac{2\lambda + 1}{\lambda + 1} < p < \frac{2\lambda + 1}{\lambda}.$$

A very similar argument holds for S^2.

To consider S^3 we recall that $|p_{n+1}(x)| \leqq A(1-x^2)^{-\lambda/2}$ (see (11.3)). Hence $|S_{n_k}(f_k)| \leqq A(1-x^2)^{-\lambda/2} \int_{-1}^{1} f(y)(1-y^2)^{-\lambda/2} \omega(y) \, dy$ $(\omega(y) = (1-y^2)^{\lambda - 1/2})$.

But the transformation T

$$T: f(x) \to (1-x^2)^{-\lambda/2} \int_{-1}^{1} f(y)(1-y^2)^{-\lambda/2}(1-y^2)^{\lambda-1/2}\, dy$$

satisfies

$$\int_{-1}^{1} |Tf(x)|^p (1-x^2)^{\lambda-1/2}\, dx \leqq A_p \int_{-1}^{1} |f(x)|^p (1-x^2)^{\lambda-1/2}\, dx$$

(by Hölder's inequality) when $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$. Therefore, another application of Lemma 7 shows (11.5) when $j = 3$, and concludes the proof of the theorem.

REMARK. It should be noted that a basic step in the proof is the passage from the Christoffel formula, which involves only the p_n's, to the formula (11.4), which involves in addition the conjugate polynomials q_n. Here we have another instance of the significance of the notion of conjugacy studied in Chapter I of this paper.

We shall now prove a rather technical corollary to the theorem, which, however, is important for its applications in §12.

Besides the k-tuple $f_1, f_2, \cdots, f_k$, considered above, we shall also deal with the respective Poisson integrals $f_1(r,\theta)$, $f_2(r,\theta), \cdots, f_k(r,\theta)$. We select an arbitrary k-tuple of r's, $r_1, r_2, \cdots, r_k$, with $0 \leqq r_j < 1$, and we let δ_j denote an arbitrary sub-interval of $(r_j, 1)$, with $|\delta_j|$ standing for the length of δ_j.

COROLLARY.

$$\begin{aligned} &\left\| \left(\sum_{j=1}^{k} |S_{n_j}(\theta, f_j(r_j,\theta))|^2 \right)^{1/2} \right\|_p \\ &\leqq A_p' \left\| \left(\sum_{j=1}^{k} |\delta_j|^{-1} \int_{\delta_j} |f_j(\rho,\theta)|^2 d\rho \right)^{1/2} \right\|_p, \quad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}. \end{aligned} \tag{11.7}$$

NOTE. The constant A_p' may be taken to be $(A_p)^2$, where A_p is the constant arising in the theorem.

Let us set $F = (f_1, \cdots, f_k)$, $|F| = (|f_1|^2 + \cdots + |f_k|^2)^{1/2}$ and $F(r,\theta) = (f_1(r_1,\theta), f_2(r_2,\theta), \cdots, f_k(r_k,\theta))$, with

$$|F(r,\theta)| = (|f_1(r_1,\theta)|^2 + \cdots + |f_k(r_k,\theta)|^2)^{1/2}.$$

We prove first that if $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$

$$\| \, |F(r,\theta)| \, \|_p \leqq A_p \| \, |F(\theta)| \, \|_p. \tag{11.8}$$

In fact

$$(r,\theta) = \sum_{l=0}^{\infty} a_l r^l P_l(\cos\theta) = (1-r) \sum S_l(\theta, f) r^l.$$

Therefore,

$$|F(r,\theta)|^2 = \sum_{j=1}^{k} |f_j(r_j,\theta)|^2 \leqq \sum_{j=1}^{k} (1-r_j) \sum_{l=0}^{\infty} |S_l(\theta,f_j)|^2 r_j^l,$$

by Schwarz's inequality. Thus if we apply the theorem, we obtain

$$\| |F(r,\theta)| \|_p \leqq A_p \left\| \left(\sum_{j=1}^{k} (1-r_j) \sum_{l=0}^{\infty} |f_j(\theta)|^2 r_j^l \right)^{1/2} \right\|_p$$

$$= A_p \left\| \left(\sum_{j=1}^{k} |f_j(\theta)|^2 \right)^{1/2} \right\|_p = A_p \| |F(\theta)| \|_p.$$

Next consider r_j' restricted by $r_j' > r_j$, $j = 1,\cdots,k$. Define

$$F(r',\theta) = (f_1(r_1',\theta), f_2(r_2',\theta),\cdots,f_k(r_k',\theta)).$$

An immediate consequence of (11.8) is the the generalization

$$\| \, |F(r,\theta)| \, \|_p \leqq A_p \| \, |F(r',\theta)| \, \|_p, \qquad r < r', \tag{11.9}$$

where $r < r'$ stands for $r_j < r_j'$, $j=1,\cdots,k$. (Again $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$.) (11.9) can then be given the following variant. Let δ_j denote a subinterval of $(r_j,1)$; we have

$$\| \, |F(r,\theta)| \, \|_p < A_p \left\| \left[\sum_{j=1}^{k} (1/|\delta_j|) \int_{\delta_j} |f_j(\rho,\theta)|^2 \, d\rho \right]^{1/2} \right\|_p. \tag{11.10}$$

In fact (11.10) may be proved by breaking the δ_j up into sufficiently small intervals, applying (11.9), and then by passing to the limit.

We now come to the proof of (11.7).

The left side of (11.7) is, by the theorem, bounded by

$$A_p \left\| \left(\sum_{j=1}^{k} |f_j(r_j,\theta)|^2 \right)^{1/2} \right\|_p = A_p \| \, |F(r,\theta)| \, \|_p.$$

Apply now (11.10) and we get (11.7) with $A_p' = (A_p)^2$.

12. **The g-function.** We come now to the second auxilliary tool used in the development of the analogue of the Littlewood-Paley theory. The results that we shall obtain are L^p inequalities for the g-function, which is defined as follows.

Let $f(\theta) \sim \sum a_n P_n^\lambda(\cos\theta)$, and let $f(r,\theta)$ be its Poisson integral,

$$f(r,\theta) = \sum a_n r^n P_n^\lambda(\cos\theta).$$

Then we define

$$g(\theta) = g(\theta; f) = \left(\int_0^1 (1-r) |f_r(r,\theta)|^2 \, dr \right)^{1/2}. \tag{12.1}$$

The usefulness of this function will become clearer in the subsequent section. Suffice it to say that we may regard g as an Abelian analogue of another function which has more immediate application. Our theorem then is

THEOREM 8.

$$\| g(\theta; f) \|_p \leqq A_p \| f \|_p, \qquad 1 < p < \infty. \tag{12.2}$$

$$\textit{If} \int_0^\infty f(\theta) \sin^{2\lambda}\theta \, d\theta = 0, \textit{ then } \| f \|_p \geqq A_p \| g(\theta; f) \|_p, \quad 1 < p < \infty. \tag{12.3}$$

Before we come to the proof of the theorem we shall give an outline of its ideas.

One proves first the direct inequality (12.2), and then the converse (12.3) follows from this by a rather standard duality argument. The main difficulty, therefore, lies in the proof of (12.2). Now

$$f_r(r, \theta) = \int_0^\pi P_r(r, \theta, \phi) f(\phi) (\sin \phi)^{2\lambda} \, d\phi.$$

It is clear that what is needed, among other things, are estimates on the behavior of the differentiated Poisson kernel, $P_r(r, \theta, \phi)$. These are given in Lemma 9 below and are in the same spirit as the estimates for the Poisson kernel in Lemma 1 of §4, and the conjugate kernel in Lemma 4 of §7.

Let us recall the abbreviations used earlier

$$\Delta = 1 - 2r\cos(\theta - \phi) + r^2 \quad \text{and} \quad \sigma = r \sin\theta \sin\phi.$$

Then our estimates for $P_r(r, \theta, \phi)$ become, if $1/2 < r < 1$,

$$\begin{aligned} |P_r(r, \theta, \phi)| &\leqq C/\Delta^{\lambda+1}, \\ P_r(r, \theta, \phi) &= P_r^+(r, \theta, \phi) + P_r^+(r, \theta, -\phi), \end{aligned} \tag{12.4}$$

where

$$\begin{aligned} P_r^+(r, \theta, \phi) = \frac{\sigma^{-\lambda}}{\pi} \frac{\partial}{\partial r} \left(\frac{1 - r^2}{\Delta} \right) &+ O\left(\frac{\sigma^{-\lambda}(1 - r)}{\Delta} \right) \\ &+ O\left[\sigma^{-\lambda}/(\sigma + \Delta)(1 + \log^+(\sigma/\Delta)) \right]. \end{aligned} \tag{12.5}$$

It is to be noted that the main term of the asymptotic estimate (12.5),

$$\frac{\sigma^{-\lambda}}{\pi} \frac{\partial}{\partial r} \left(\frac{1 - r^2}{\Delta} \right),$$

is, except for the factor $\sigma^{-\lambda}$, essentially the $P_r(r, \theta, \phi)$ in the trigonometric case ($\lambda = 0$). The asymptotic formula (12.5) will be used when θ is near ϕ; and its main contribution, due to the first theorem, will then be handled by using the classical result for the g-function.

When θ is not near ϕ, the simpler estimate (12.4) will be sufficient.

LEMMA 9. *The estimates* (12.4) *and* (12.5) *hold for* $P_r(r,\theta,\phi)$ *when* $1/2 \leqq r < 1$ (6).

Proof of lemma. Recall that

$$P(r,\theta,\phi) = (\lambda/\pi)(1-r^2)\int_0^\pi \frac{\sin^{2\lambda-1}t}{D^{\lambda+1}}\,dt,$$

where

$$D = 1 - 2r(\cos\theta\cos\phi + \sin\theta\sin\phi\cos t) + r^2$$

We notice two useful facts about D

$$D \geqq \Delta \qquad (\Delta = 1 - 2r\cos(\theta-\phi) + r^2). \tag{12.6}$$

and

$$(1-r)\,|D_r| \leqq \frac{2}{r}D. \tag{12.7}$$

In fact $rD_r - r^2 + 1 = D$, and $D \geqq (1-r)^2$, as can be seen by rewriting $D = (1-r)^2 + 2r[1-\cos(\theta-\phi)] + 2r\sin\theta\sin\phi[1-\cos t]$. Combining these two gives (12.7).

Differentiating in the integral formula for $P(r,\theta,\phi)$ then gives

$$|P_r(r,\theta,\phi)| \leqq \frac{\lambda 2r}{\pi}\int_0^\pi \frac{\sin^{2\lambda-1}}{D^{\lambda+1}}t\;dt + \frac{\lambda(\lambda+1)}{\pi}(1-r^2)\int_0^\pi \frac{|D_r|}{D^{\lambda+2}}\sin^{2\lambda-1}t\;dt.$$

When we substitute the estimates (12.6) and (12.7) (taking into account $1/2 \leqq r < 1$), we obtain the first conclusion of the lemma, i.e., (12.4).

We write

$$P(r,\theta,\phi) = \frac{\lambda(1-r^2)}{\pi}\int_0^\pi \frac{\sin^{2\lambda-1}t}{D^{\lambda+1}}\,dt = \frac{\lambda(1-r^2)}{\pi}\int_0^{\pi/2} + \frac{\lambda(1-r^2)}{\pi}\int_{\pi/2}^\pi.$$

Call

$$P^+(r,\theta,\phi) = \frac{\lambda(1-r^2)}{\pi}\int_0^{\pi/2} \frac{\sin^{2\lambda-1}t}{D^{\lambda+1}}\,dt.$$

Then

$$P(r,\theta,\phi) = P^+(r,\theta,\phi) + P^+(r,\theta,-\phi).$$

We shall now prove the estimate (12.5) for $P_r^+(r,\theta,\phi)$. In fact

$$P_r^+(r,\theta,\phi) = \frac{-2\lambda r}{\pi}\int_0^{\pi/2} \frac{\sin^{2\lambda-1}t}{D^{\lambda+1}}dt - \frac{\lambda(\lambda+1)}{\pi}(1-r^2)\int_0^{\pi/2}\sin^{2\lambda-1}t\frac{D_r}{D^{\lambda+2}}\,dt. \tag{12.8}$$

(6) When $0 \leqq r \leqq 1/2$ the trivial estimate $|P_r(r,\theta,\phi)| \leqq A$ will suffice.

To study the situation further we shall concentrate on the main singularity, which occurs when θ is near ϕ (and t is near zero) in the integrals in (12.8).

Thus we write $\Delta^* = \Delta + t^2 r \sin\theta \sin\phi = \Delta + t^2\sigma$. Since $D = \Delta + 2\sigma(1 - \cos t)$, it follows that $D = \Delta^* + O(\sigma t^4)$. Notice that $D_r - \Delta_r = O(\sigma t^4)$.

In this way, we are led to replace systematically D by Δ^*, $\sin^{2\lambda-1} t$ by $t^{2\lambda-1}$, and D_r by Δ_r. We therefore define $P_r^*(r,\theta,\phi)$ by

$$P_r^*(r,\theta,\phi) = \frac{-2\lambda r}{\pi} \int_0^{\pi/2} \frac{t^{2\lambda-1}}{(\Delta^*)^{\lambda+1}}\, dt - \frac{\lambda(\lambda+1)(1-r^2)\Delta_r}{\pi} \int_0^{\pi/2} \frac{t^{2\lambda-1}}{(\Delta^*)^{\lambda+2}}\, dt. \tag{12.9}$$

Let us try to determine by how much $P_r^*(r,\theta,\phi)$ actually differs from $P_r^+(r,\theta,\phi)$ (see (12.8)). Now by the mean value theorem

$$[\Delta + 2\sigma(1 - \cos t)]^{-\lambda-1} - [\Delta + \sigma t^2]^{-\lambda-1} = O[\Delta + \sigma t^2]^{-\lambda-2}\sigma t^4,$$

since $\Delta \geqq 0$, $\sigma \geqq 0$ and $t \geqq 0$. In other words

$$D^{-\lambda-1} - (\Delta^*)^{-\lambda-1} = O(\Delta^*)^{-\lambda-2}\sigma t^4.$$

Similarly

$$D^{-\lambda-2} - (\Delta^*)^{-\lambda-2} = O(\Delta^*)^{-\lambda-3}\sigma t^4.$$

Also we should point out that since $\Delta = (1-r)^2 + 2r(1 - \cos(\theta - \phi))$ and $1/2 \leqq r < 1$, then $|(1-r)\Delta_r| \leqq 4\Delta$.

Let us now define

$$I_\mu^\nu = \int_0^{\pi/2} \frac{t^\nu}{(\Delta^*)^\mu}\, dt. \tag{12.10}$$

Considering what has been said above, and after some simple reductions, we obtain that the difference $P_r^+(r,\theta,\phi) - P_r^*(r,\theta,\phi)$ is bounded by a constant multiple of

$$\begin{aligned}(1-r)\Delta I_{\lambda+2}^{2\lambda-1} + (1-r)\sigma I_{\lambda+2}^{2\lambda+1} + (1-r)\sigma^2 I_{\lambda+3}^{2\lambda+3} \\ + \Delta I_{\lambda+2}^{2\lambda+1} + \Delta\sigma I_{\lambda+3}^{2\lambda+3} + I_{\lambda+1}^{2\lambda+1} + \sigma I_{\lambda+2}^{2\lambda+3}.\end{aligned} \tag{12.11}$$

By a change of variables

$$I_\mu^\nu = \sigma^{-\nu-1/2}\Delta^{\nu/2-\mu+1/2} \int_0^\xi \frac{t^\nu}{(1+t^2)^\mu}\, dt, \text{ where } \xi = \frac{\pi}{2}\left(\frac{\sigma}{\Delta}\right)^{1/2}.$$

However,

$$\int_0^\xi \frac{t^\nu}{(1+t^2)^\mu}\, dt \leqq A\left(\frac{\xi}{1+\xi}\right)^{\nu+1} \text{ if } \nu > -1,\ \nu - 2\mu < -1;$$

also

$$\int_0^\xi \frac{t^\nu}{(1+t^2)^\mu}\,dt \leqq A\left(\frac{\xi}{1+\xi}\right)^{\nu+1}[1+\log^+\xi] \text{ if } \nu>-1,\ \nu-2\mu=-1.$$

From this we get by straightforward calculation the following estimates:

$$\begin{aligned}
I_{\lambda+2}^{2\lambda-1} &= O(\sigma^{-\lambda}\Delta^{-2})\,(\lambda>0)\,(\text{a more precise form is given below}),\\
I_{\lambda+2}^{2\lambda+1} &= O(\sigma^{-\lambda}\Delta^{-1}(\sigma+\Delta)^{-1}) = O(\sigma^{-\lambda-1}\Delta^{-1}),\\
I_{\lambda+3}^{2\lambda+3} &= O(\sigma^{-\lambda-1}\Delta^{-1}(\sigma+\Delta)^{-1}) = O(\sigma^{-\lambda-2}\Delta^{-1}),\\
I_{\lambda+2}^{2\lambda+3} &= O(\sigma^{-\lambda-1}\Delta^{-1}),\\
I_{\lambda+1}^{2\lambda+1} &= O(\sigma^{-\lambda}(\sigma+\Delta)^{-1}(1+\log^+(\sigma/\Delta))).
\end{aligned}$$

Therefore we see the total contribution in (12.11) is bounded by a constant multiple of

$$\sigma^{-\lambda}(1-r)\Delta^{-1}+\frac{\sigma^{-\lambda}}{\sigma+\Delta}(1+\log^+(\sigma/\Delta)). \tag{12.12}$$

To analyze $P_r^*(r,\theta,\phi)$ we need to look at the integrals $I_{\lambda+1}^{2\lambda-1}$ and $I_{\lambda+2}^{2\lambda-1}$ more closely (see (12.9)).

Recall that

$$I_\mu^\nu = \sigma^{-\nu-1/2}\Delta^{\nu/2-\mu+1/2}\int_0^\xi \frac{t^\nu}{(1+t^2)^\mu}\,dt, \text{ where } \xi=\frac{\pi}{2}\left(\frac{\sigma}{\Delta}\right)^{1/2}.$$

But

$$\int_0^\xi \frac{t^\nu}{(1+t^2)^\mu}\,dt=\int_0^\infty \frac{t^\nu}{(1+t^2)^\mu}\,dt-\int_\xi^\infty \frac{t^\nu}{(1+t^2)^\mu} \text{ if } \nu-2\mu<-1.$$

From this, and a well-known formula in the theory of the gamma function, it follows that

$$I_{\lambda+1}^{2\lambda-1} = (2\lambda)^{-1}\sigma^{-\lambda}\Delta^{-1}+O\left(\frac{\sigma^{-\lambda}}{(\sigma+\Delta)}\right)$$

and

$$I_{\lambda+2}^{2\lambda-1} = (\lambda(\lambda+1))^{-1}\sigma^{-\lambda}\Delta^{-2}+O\left(\sigma^{-\lambda}\frac{\Delta^{-1}}{(\sigma+\Delta)}\right).$$

Combining all the above via the estimate in (12.12) gives us (12.5) and the lemma is proved.

We come now to the proof of Theorem 8.

Proof of (12.2). Recall that

$$g(\theta,f) = \left(\int_0^1 (1-r)|f_r(r,\theta)|^2\,dr\right)^{1/2},$$

where

$$f_r(r,\theta) = \int_0^\pi P_r(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi \; d\phi.$$

Thus we have $g(\theta,f) \leqq g_1(\theta,f) + g_2(\theta,f) + g_3(\theta,f) + g_4(\theta,f)$. $g_k(\theta,f)$ is defined like $g(\theta,f)$, where $P_r(r,\theta,\phi)$ is replaced by $P_r^{(k)}(r,\theta,\phi)$, and

$$\begin{aligned} P_r^{(1)}(r,\theta,\phi) &= P_r(r,\theta,\phi) && \text{if } 0 \leqq r \leqq 1/2, \\ &= 0 && \text{otherwise;} \\ P_r^{(2)}(r,\theta,\phi) &= P_r(r,\theta,\phi) && \text{if } 1/2 \leqq r < 0 \text{ and } 0 \leqq \phi \leqq (3/4)\theta, \\ &= 0 && \text{otherwise;} \\ P_r^{(3)}(r,\theta,\phi) &= P_r(r,\theta,\phi) && \text{if } 1/2 \leqq r < 0 \text{ and } (4/3)\theta \leqq \phi \leqq \pi, \\ &= 0 && \text{otherwise;} \\ P_r^{(4)}(r,\theta,\phi) &= P_r(r,\theta,\phi) && \text{if } 1/2 \leqq r < 0 \text{ and } (3/4)\theta < \phi < (4/3)\theta. \end{aligned}$$

We need to prove, therefore, that $\| g_k(\theta,f) \|_p \leqq A_p \| f \|_p$, $k = 1,2,3,4$.

The inequality $\| g_1(\theta,f) \|_p \leqq A \| f \|_p$ is trivial since $P_r^1(r,\theta,\phi)$ is uniformly bounded.

To prove the other three inequalities, we notice that it suffices to assume that $f(\phi)=0$ when $\phi \in [\pi/2, \pi]$, because $P_r(r,\theta,\phi) = P_r(r,\pi-\theta, \pi-\phi)$ makes the consideration of the intervals $[0,\pi/2]$ and $[\pi/2,\pi]$ completely symmetric.

Now by estimate (12.4) of $P_r(r,\theta,\phi)$, we see that

$$|P_r^{(2)}(r,\theta,\phi)| \leqq C[(1-r)^2 + 2r(1-(\cos\theta-\phi))]^{-\lambda-1} \leqq C((1-r^2)+\theta)^{2/-\lambda-1},$$

since $0 \leqq \phi \leqq (3/4)\theta$. Hence

$$\begin{aligned} g_2(\theta,f) &= \int_{1/2}^1 (1-r) \left| \int_0^{(3/4)\theta} P_r^{(2)}(r,\theta,\phi)f(\phi)\sin^{2\lambda}\phi \; d\phi \right|^2 dr \\ &\leqq C \int_{1/2}^1 (1-r)[(1-r)^2+\theta^2]^{-2\lambda-2} \left[\int_0^{(3/4)\theta} |f(\phi)|^p \sin^{2\lambda}\phi \, d\phi \right]^{p/2} \theta^{2/q+1} \, dr \end{aligned}$$

by Hölder's inequality. If we carry out the integration with respect to r we obtain

$$g^2(\theta,f) \leqq C\theta^{2/q-2(2\lambda+1)} \left[\int_0^{(3/4)\theta} |f(\phi)|^p \sin^{2\lambda p}\phi \; d\phi \right]^{p/2}.$$

Thus

$$\begin{aligned} \int_0^\pi (g_2(\theta,f))^p \sin^{2\lambda}\theta \, d\theta &\leqq C \int_0^\pi \theta^{-1-2\lambda p+2\lambda} d\theta \int_0^{(3/4)\theta} |f(\phi)|^p \sin^{2\lambda p}\phi \; d\phi \\ &\leqq C \int_0^{\pi/2} |f(\phi)|^p \sin^{2\lambda p}\phi \, d\phi \int_{(4/3)\theta}^\pi \phi^{-1-2\lambda p+2\lambda} \; d\phi \\ &\leqq C \int_0^\pi |f(\phi)|^p \sin^{2\lambda}\phi \; d\phi, \end{aligned}$$

since $f(\phi) = 0$ on $[\pi/2, \pi]$.

By very similar arguments one can also prove that

$$\| g_3(\theta, f) \|_p \leqq A \| f \|_p .$$

We now come to the case of $g_4(\theta, f)$. For its study we shall use the asymptotic estimate (12.5) for $P_r(r, \theta, \phi)$. We notice that this leads to two terms, one for P_r^+ and the other for $P_r^+(r, \theta, -\phi)$. Again considerations of symmetry reduce the problem to that of $P_r^+(r, \theta, \phi)$.

So we are led to consider the three functions, $l(\theta)$, $m(\theta)$, and $n(\theta)$ defined respectively by

$$[l(\theta)]^2 = \left[\int_{1/2}^{1} (1-r) \int_{(3/4)\theta}^{(4/3)\theta} \sigma^{-\lambda} \frac{\partial}{\partial r}\left(\frac{1-r^2}{\Delta}\right) f(\phi) \sin^{2\lambda}\phi \, d\phi \right]^2 dr,$$

$$[m(\theta)]^2 = \int_{1/2}^{1} (1-r) \left[\int_{(3/4)\theta}^{(4/3)\theta} (1-r) \sigma^{-\lambda\Delta^{-1}} |f(\phi)| \sin^{2\lambda}\phi \, d\phi \right]^2 dr$$

and

$$[n(\theta)]^2 = \int_{1/2}^{2} (1-r) \left[\int_{(3/4)\theta}^{(4/3)\theta} (1-r) \left(\frac{\sigma^{-\lambda}}{(\sigma+\Delta)}\right) \left(1 + \log^+\left(\frac{\sigma}{\Delta}\right)\right) |f(\phi)| \sin^{2\lambda}\phi \, d\phi \right]^2 dr.$$

We need to prove that

$$\| n(\theta) \|_p \leqq A_p \| f \|_p$$

with similar inequalities for m and l. Consider $n(\theta)$ first. When $(3/4)\theta \leqq \phi \leqq (4/3)\theta$, $0 \leqq \phi \leqq \pi/2$, we have the simple estimates $\sigma^{-\lambda} \leqq C\theta^{-2\lambda}$, $\sigma + \Delta \geqq C[\theta^2 + (1-r)^2]$, $\Delta \geqq C(\theta - \phi)^2$, and $\sigma \geqq C\theta\phi$. Substitute these estimates, integrate with respect to r and use the fact that

$$\int_{1/2}^{1} \frac{(1-r)}{(\theta^2 + (1-r)^2)^2} \, dr = O(\theta^{-2}).$$

Then we get

$$n(\theta) \leqq C\theta^{-1} \int_{(3/4)\theta}^{(4/3)\theta} \left[1 + \log^+\left(\frac{\phi}{\theta - \phi}\right)^2 \right] |f(\phi)| \, d\phi . \tag{12.13}$$

But then by Hölder's inequality

$$\int_0^{\pi} (n(\theta))^p \sin^{2\lambda}\theta \, d\theta \leqq \int_0^{\pi} \theta^{2\lambda - p} \int_{(3/4)\theta}^{(4/3)\theta} |f(\phi)|^p \, d\phi \left\{ \int_{(3/4)\theta}^{(4/3)\theta} \left[1 + \log^+\left(\frac{\phi}{\theta - \phi}\right)^2 \right]^q d\phi \right\}^{p/q} d\theta,$$

where $1/p + 1/q = 1$. However,

$$\int_{(3/4)\theta}^{(4/3)\theta} \left[1 + \log^+\left(\frac{\phi}{\theta - \phi}\right)^2 \right]^q d\phi = \theta \cdot \int_{3/4}^{4/3} \left[1 + \log^+\left(\frac{\phi}{1 - \phi}\right)^2 \right]^q d\phi .$$

Thus

$$\int_0^\pi (n(\theta))^p \sin^{2\lambda}\theta \, d\theta \leqq C \int_0^\pi \theta^{2\lambda-1} \int_{(3/4)\theta}^{(4/3)\theta} |f(\phi)|^p \, d\phi \leqq C \int_0^\pi |f(\phi)|^p \sin^{2\lambda}\phi \, d\phi,$$

since

$$f(\phi) = 0 \quad \text{when } \phi \in [\pi/2, \pi].$$

We now consider $m(\theta)$, and therefore we define $F_r(\theta)$ by

$$F_r(\theta) = (1-r) \int_{(3/4)\theta}^{(4/3)\theta} \sigma^{-\lambda} \Delta^{-1} |f(\phi)| \sin^{2\lambda}\phi \, d\phi.$$

Since $\sigma = r \sin\theta \sin\phi$, and $f(\phi) = 0$ if $\phi > \pi/2$, we get

$$F_r(\theta)(\sin\theta)^{2\lambda/p} \leqq C(1-r^2) \int_{(3/4)\theta}^{(4/3)\theta} \Delta^{-1} |f(\phi)| (\sin\phi)^{2\lambda/p} \, d\phi$$

$$\leqq C(1-r^2) \int_0^\pi \Delta^{-1} |f(\phi)| (\sin\phi)^{2\lambda/p} \, d\phi.$$

Thus, by a well-known property of the ordinary Poisson kernel, $(1-r^2)/\Delta$ we have

$$\int_0^\pi [F_r(\theta)]^p \sin^{2\lambda}(\theta) \, d\theta \leqq C \int_0^\pi |f(\phi)^p| \sin^{2\lambda}\phi \, d\phi,$$

with C independent of r, $0 \leqq r < 1$.

But

$$m(\theta) = \left(\int_{1/2}^1 (1-r)[F_r(\theta)]^2 \, dr \right)^{1/2}.$$

Therefore Minkowski's inequality for integrals and the above shows that

$$\int_0^\pi (m(\theta))^p \sin^{2\lambda}\theta \, d\theta \leqq C \int_0^\pi |f(\phi)|^p \sin^{2\lambda}\phi \, d\phi.$$

Finally we come to $l(\theta)$, and here the argument is somewhat intricate.

Since in our case we always have $(3/4)\theta \leqq \phi \leqq (4/3)\theta$, and $0 \leqq \phi \leqq \pi/2$ (i.e. $f(\phi)=0$ if $\pi/2 < \phi \leqq \pi$), then we may limit ourselves to $\theta \in [0, (4/3)\cdot\pi/2]$. Thus we define the intervals $E_n = [(4/3)\cdot\pi/2^{n-1}, (4/3)\cdot\pi/2^n]$; hence $\bigcup_{n=1}^\infty E_n = [0, (4/3)\cdot\pi/2]$. The E_n's are the appropriate intervals for the θ variable. The corresponding intervals for the ϕ variable are gotten by multiplying the right-hand end point by $4/3$ and the left-hand end point by $3/4$. In this way we define $E_n^* = [\pi/2^{n-1}, (4/3)^2 \cdot \pi/2^n]$. Notice that while the sets E_n are essentially non-overlapping, the sets E_n^* overlap at most twice.

Now we have

$$(l(\theta))^2 = \int_{1/2}^{1} (1-r)\,|G_r(\theta)|^2\,dr,$$

where

$$G_r(\theta) = \int_{(3/4)\theta}^{(4/3)\theta} \sigma^{-\lambda} \frac{\partial}{\partial r}\left(\frac{(1-r^2)}{\Delta}\right) f(\phi)\sin^{2\lambda}\phi\ d\phi = \int_{(3/4)\theta}^{(4/3)\theta} I(\phi)\ d\phi\,.$$

In view of the above we may write this as

$$G_r(\theta) = \sum_{n=1}^{\infty} \chi_{E_n}(\theta) \int_{(3/4)\theta}^{(4/3)\theta} \chi_{E_n^*}(\phi) I(\phi)\ d\phi,$$

where χ_E denotes the characteristic function of the set E. We should like, however, to replace the above sum by

$$G_r^*(\theta) = \sum_{n=1}^{\infty} \chi_{E_n}(\theta) \int_{E_n^*} I(\phi)\ d\phi\,. \tag{12.14}$$

But the difference between $G_r(\theta)$ and $G_r^*(\theta)$ is bounded by

$$\sum_{n=1}^{\infty} \chi_{E_n}(\theta) \int_{E_n'} |I(\phi)|\ d\phi,$$

where

$$E_n' = [\pi/2^{n-1},\ (3/4)\theta] \cup [(4/3)\theta, (4/3)^2 \cdot \pi/2^n].$$

From the definitions of the sets E_n, E_n^* and their overlapping character, it follows that

$$|G_r(\theta) - G_r^*(\theta)| \leqq 2\int_{(3/8)\theta}^{(3/4)\theta} |I(\phi)|\ d\phi + 2\int_{(4/3)\theta}^{(8/3)\theta} |I(\phi)|\ d\phi\,.$$

For the range of ϕ's appearing in the above it is easy to see that

$$\frac{\partial}{\partial r}\left(\frac{(1-r^2)}{\Delta}\right) = O((\theta^2 + (1-r)^2)^{-1}).$$

In effect we are led back to the argument used to estimate $n(\theta)$ above. By that argument, we get

$$\left(\int_{1/2}^{1} (1-r)\,|G_r(\theta) - G_r^*(\theta)|^2\,dr\right)^{1/2} \leqq C\theta^{-1}\left[\int_{(3/8)\theta}^{(3/4)\theta} |f(\phi)|\,d\phi + \int_{(4/3)\theta}^{(8/3)\theta} |f(\phi)|\,d\phi\right].$$

(See the argument leading to (12.13).)

Following through as before, we obtain

$$\left\|\left(\int_{1/2}^{1}(1-r)\,|G_r(\theta)-G_r^*(\theta)|^2\,dr\right)^{1/2}\right\|_p \leqq A\,\|f\|_p.$$

Hence we have reduced everything to proving that

$$\left\|\left(\int_{1/2}^{1}(1-r)\,|G_r^*(\theta)|^2\,dr\right)^{1/2}\right\|_p \leqq A_p\,\|f\|_p. \tag{12.15}$$

However, by definition (12.14),

$$\left\|\left(\int_{1/2}^{1}(1-r)\,|G_r^*(\theta)|^2\,dr\right)^{1/2}\right\|_p^p$$

$$= \sum_{n=1}^{\infty}\int_{E_n}\left[\int_{1/2}^{1}(1-r)\left|\int_{E_n^*}\sigma^{-\lambda}\frac{\partial}{\partial r}\left(\frac{(1-r^2)}{\Delta}\right)f(\phi)\sin^{2\lambda}\phi\,d\phi\right|^2 dr\right]^{p/2}\sin^{2\lambda}\theta\,d\theta$$

$$\leqq C\sum_{n=1}^{\infty}2^{-n(2\lambda-p\lambda)}\int_{E_n}\left[\int_0^1(1-r)\left|\int_0^{\pi}f(\phi)\sin^{2\lambda}\phi\,\chi_{E_n^*}(\phi)\,\frac{\partial}{\partial r}\left(\frac{(1-r^2)}{\Delta}\right)d\phi\right|^2 dr\right]^{p/2}d\theta$$

since $\sigma = r\sin\theta\sin\phi$.

Let us now use the classical inequality for the g-function(7), which states in particular

$$\int_0^{\pi}\left[\int_0^1(1-r)\left|\int_0^{\pi}f(\phi)\,\sin^{\lambda}\phi\,\chi_{E_n^*}(\phi)\frac{\partial}{\partial r}\left(\frac{(1-r^2)}{\Delta}\right)d\phi\right|^2 dr\right]^{p/2}d\theta$$

$$\leqq C\int_0^{\pi}|f(\phi)(\sin\phi)^{\lambda}|^p\chi_{E_n^*}(\phi)\,d\phi = C\int_{E_n^*}|f(\phi)|^p\sin^{\lambda p}\phi\,d\phi.$$

Combining this with the above gives us

$$\left\|\left(\int_{1/2}^{1}(1-r)\,|G_r^*(\theta)|^2\,dr\right)^{1/2}\right\|_p^p \leqq C\sum_{n=1}^{\infty}2^{-n(2\lambda-p\lambda)}\int_{E_n^*}|f(\phi)|^p\sin^{\lambda p}\phi\,d\phi$$

$$\leqq C\int_0^{\pi}|f(\phi)|^p\sin^{2\lambda}\phi\,d\phi.$$

This proves (12.5), and therefore the proof of the direct part of the theorem (12.2) is complete.

Proof of (12.3). The whole matter here is rather simple and is based on the following "duality": if $\int_0^{\pi}f_1(\theta)\sin^{2\lambda}\theta\,d\theta = 0$,

$$\left|\int_0^{\pi}f_1(\theta)f_2(\theta)\sin^{2\lambda}\theta\,d\theta\right| \leqq 4\int_0^{\pi}g(\theta,f_1)\,g(\theta,f_2)\sin^{2\lambda}\theta\,d\theta. \tag{12.16}$$

(7) See [26, Chapter 14].

This holds whenever $f_1(\theta)$ and $f_2(\theta)$ are regular enough, and in particular when f_1 and f_2 have finite expansions.

In fact, let $f_1(\theta) \sim \sum_{n=1} a_n P_n(\cos\theta)$, and $f_2(\theta) \sim \sum_{n=0} b_n P_n(\cos\theta)$.

Set $\langle f_1, f_2\rangle = \int_0^\pi f_1(\theta) f_2(\theta) \sin^{2\lambda}\theta \, d\theta$.

We then have $\langle f_1, f_2\rangle = \sum_{n=1}^\infty a_n b_n \gamma_n^{-1}$, where γ_n are the normalizing constants. However,

$$\sum_{n=1} a_n b_n \gamma_n^{-1} = 4 \int_0^1 \frac{1}{\rho} \left\{ \int_0^r \left(\sum_{n=1} a_n b_n \gamma_n^{-1} n^2 r^{2n-1} \right) dr \right\} d\rho \ .$$

This means that

$$\langle f_1, f_2\rangle = 4 \int_0^1 \frac{1}{\rho} \left\{ \int_0^r \left\langle \frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r} \right\rangle r \, dr \right\} d\rho .$$

We have set $f_1(r,\theta) = \sum a_n r^n P_n^\lambda(\cos\theta)$, and $\partial f_1/\partial r = \sum n a_n r^{n-1} P_n^\lambda(\cos\theta)$, with a similar notation for f_2.

We therefore obtain that

$$|\langle f_1, f_2\rangle| \leqq 4 \int_0^1 (1-r) \left\langle \frac{\partial f_1}{\partial r}, \frac{\partial f_2}{\partial r} \right\rangle dr .$$

Applying Fubini's theorem and Schwarz's inequality gives (12.16).

Next, (12.16) extends to the case when $f_1 \in L^p, 1 < p < \infty$, $\int_0^\pi f_1(\theta) \sin^{2\lambda}\theta \, d\theta = 0$, and $f_2 \in L^q$, $1/p + 1/q = 1$.

In fact, we already know that the sub-linear mapping $f \to g(f)$ is bounded in L^p (and L^q); moreover its sub-linearity implies that

$$|g(f_1^{(n)}) - g(f_1^{(m)})| \leqq g(f_1^n - f_1^m) .$$

Thus if $\{f_1^n\}$ is a convergent sequence in the L^p norm, so is $\{g(f_1^n)\}$. Hence a passage to the limit shows that (12.16) holds whenever $f_1 \in L^p$ and $f_2 \in L^q$, $1/p + 1/q - 1$, $\int_0^\pi f_1(\theta) \sin^{2\lambda}\theta \, d\theta = 0$.

Finally

$$\|f_1\|_p = \sup_{\|f_2\|_q \leqq 1} \left| \int_0^\pi f_1(\theta) f_2(\theta) \sin^{2\lambda}\theta \, d\theta \right| \leqq 4 \| g(\theta, f_1) \|_p \| g(\theta, f_2) \|_q \leqq 4 A_q \| g(\theta, f_1) \|_p ,$$

by (12.2). This proves (12.3) and the proof of our theorem is complete.

13. **Partial sums analogue of the g-function.** The g-function which was studied in the previous section is intimately connected with the Poisson integral

$$\begin{aligned} f(r,\theta) &= \sum_{k=0}^\infty a_k r^k P_k^\lambda(\cos\theta), \\ f(\theta) &\sim \sum_{k=0}^\infty a_k P_k^\lambda(\cos\theta). \end{aligned} \tag{13.1}$$

It is, therefore, closely related to the Abel-summability of the expansion in question. What we need, however, are the analogous facts for ordinary convergence instead of Abel-summability. This is the result obtained in the theorem of this section; thus what follows below may be considered as an argument of Tauberian character. Let us look at the situation more closely. Instead of the Abel means (13.1), we consider the ordinary partial sums

$$S_n(\theta) = S_n(\theta,f) = \sum_{k=0}^{n} a_k P_k^\lambda(\cos\theta). \tag{13.2}$$

Similarly in place of $f_r(r,\theta) = \sum k a_k r^{k-1} P_k^\lambda(\cos\theta)$, we consider

$$S_n'(\theta) = S_n'(\theta,f) = \sum_{k=1}^{n} k a_k P_k^\lambda(\cos\theta). \tag{13.3}$$

Moreover, as is familiar, n and r are to be connected by $1/n = 1 - r$. Therefore, instead of

$$g(\theta) = \left(\int_0^1 (1-r)\left|f_r(r,\theta)\right|^2 dr\right)^{1/2}$$

we are led to consider its analogue

$$g^*(\theta) = g^*(\theta,f) = \left(\sum_{n=1}^{\infty} \frac{\left|S_n'(\theta,f)\right|^2}{n^3}\right)^{1/2}. \tag{13.4}$$

If we recall the argument of the classical Abelian theorem, we are led to expect an inequality of the form

$$g(\theta) \leqq c g^*(\theta). \tag{13.5}$$

Indeed, we shall see that this is the case. In the converse direction, we cannot hope for anything as simple as (13.5). However, it will turn out that (13.5) may be reversed not for individual θ's, but as a whole, in the sense of an L^p inequality. We must add one further reservation to this, which arises because the partial sums inequalities of §10 are limited to the case when the exponent p satisfies $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$. For this reason we may expect the result

$$\|g^*(\theta)\|_p \leqq A_p \|g(\theta)\|_p, \qquad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}. \tag{13.6}$$

We formulate the above as a theorem.

THEOREM 9. *The inequalities* (13.5) *and* (13.6) *hold.*

We prove (13.5) first.

Since $f_r(r,\theta) = \sum k a_k r^{k-1} P_k^\lambda(\cos\theta)$ and $S_n'(\theta) = \sum_{k=1}^n k a_k P_k^\lambda(\cos\theta)$, it follows that

$$f_r(r,\theta) = (1-r)\sum_{k=1}^{\infty} S_k'(\theta) r^{k-1}.$$

By Schwarz's inequality

$$|f_r(r,\theta)|^2 \leqq (1-r)^2\left(\sum_{k=1}^{\infty} |S_k'(\theta)|^2 r^{k-1}\right)\sum_{k=1}^{\infty} r^{k-1}$$

$$= (1-r)\sum_{k=1}^{\infty} |S_k'(\theta)|^2 r^{k-1}.$$

Hence

$$\int_0^1 (1-r)|f_r(r,\theta)|^2\,dr \leqq c\sum_{k=1}^{\infty}\frac{|S_k'(\theta)|^2}{k^3}$$

because

$$\int_0^1 (1-r)^2 r^{k-1}\,dr \approx 1/k^3.$$

This proves (13.5).

The inequality (13.6) lies deeper, because unlike (13.5) it makes use of Theorem 7 on partial sums, more particularly its corollary. In proving (13.6) we follow the argument in [26, Chapter 15] closely.

We have

$$S_n'(\theta) = \sum_{k=1}^{n} k a_k P_k^\lambda(\cos\theta) = \sum_{k=1}^{n} k a_k r^{k-1} P_k^\lambda(\cos\theta) r^{1-k}.$$

Sum the last expression by parts, set $r = r_n = 1 - 1/n$, and use Schwarz's inequality. This gives

$$|S_n'(\theta)|^2 \leqq A\left\{n^{-1}\sum_{k=1}^{n-1}|S_k'(\theta,f_n)|^2 + |S_n'(\theta,f_n)|^2\right\},$$

where $f_n = f(r_n,\theta)$, $r_n = 1 - 1/n$.

But

$$g^*(\theta) = \sum_{k=1}^{\infty}\frac{|S_n'(\theta)|^2}{n^3}$$

$$\leqq A\left\{\sum_{n=1}^{\infty} n^{-4}\sum_{k=1}^{n-1}|S_k'(\theta,f_n)|^2 + \sum_{n=1}^{\infty}\frac{|S_n'(\theta,f_n)|^2}{n^3}\right\} = G_1(\theta) + G_2(\theta)$$

If we apply the corollary to Theorem 7 (see §11) we obtain

$$\|G_1(\theta)\|_p \leqq A_p\left\|\left\{\sum_{n=1}^{\infty} n^{-4}\sum_{k=1}^{n-1}|\delta_n^{-1}|\int_{\delta_n}|f_r(r,\theta)|^2\,dr\right\}^{1/2}\right\|_p,$$

where $\delta_n = (r_n, r_{n+1})$, and $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$.

But $|\delta_n| = r_{n+1} - r_n = 1/(n+1) - 1/n = 1/n(n+1) \geqq 1/(n+1)^2$. Hence,

$$\| G_1(\theta) \|_p \leqq A_p \left\| \left\{ \sum_{n=1}^{\infty} n^{-1} \int_{\delta_n} |f_r(r,\theta)|^2 \, dr \right\}^{1/2} \right\|_p$$

$$\leqq A_p \left\| \left(\int_0^1 (1-r) |f_r(r,\theta)|^2 \, dr \right)^{1/2} \right\|_p = A_p \| g \|_p .$$

Similarly for $G_2(\theta)$.

This proves (13.6) and hence the theorem.

The following corollary is an immediate consequence of the above theorem and Theorem 8 in the previous section.

COROLLARY 1. *If* $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$, *then* $\| f(\theta) \|_p$ *and* $\|_\pi g^*(\theta,f) \|_p$ *are equivalent norms for the class of functions normalized by* $\int_0 f(\theta) \sin^{2\lambda}\theta \, d\theta = 0$.

Another corollary that can be deduced is in effect a generalization of the theorem. Let $v_1, v_2, \cdots, v_n, \cdots$ be a series of positive constants satisfying

$$\sum_{k=1}^{n} v_k \leqq Mn \quad \text{for all } n \text{ and some constant } M. \tag{13.7}$$

Instead of $(g^*(\theta))^2$, as defined by (13.4), form the series

$$\sum_{n=1}^{\infty} \frac{|S_n'(\theta)|^2}{n^3} v_n . \tag{13.8}$$

COROLLARY 2.

$$\left\| \left\{ \sum \frac{|S_n'(\theta)|^2}{n^3} v_n \right\}^{1/2} \right\|_p \leqq A_p M \| f \|_p , \quad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda} .$$

The constant A_p *does not depend on the sequence* $\{v_k\}$ *or* f.

The proof is sufficiently similar to that of the theorem so that we shall omit it. See also [26, Chapter 15].

As a special case we can take

$$v_n = \frac{n^3}{(n+1)^2} \qquad \text{if } n = 2^k,$$

$$= 0 \qquad \text{otherwise.}$$

Then since $(n+1)[S_n(\theta) - \sigma_n(\theta)] = S_n'(\theta)$, where $\sigma_n(\theta)$ are the Cesàro means (13.8) becomes

$$\sum_{k=0}^{\infty} |S_2^k(\theta) - \sigma_2^k(\theta)|^2 .$$

14. **Multiplier theorem and behavior of lacunary partial sums.** We come now to the main results of this part, which we formulate as two separate theorems. The first is an extension of the Marcinkiewicz multiplier theorem and includes the Littlewood-Paley theorem of decomposition into dyadic blocks. It may be formulated as follows:

Let $\mu_0, \mu_1, \cdots, \mu_n, \cdots$ be a sequence of constants (multipliers) and consider the linear transformation T, defined at least formally by

$$(14.1) \qquad Tf \sim \sum_{n=1}^{\infty} a_n \mu_n P_n^{\lambda}(\cos\theta) \quad \text{whenever } f \sim \sum_{n=1}^{\infty} a_n P_n^{\lambda}(\cos\theta).$$

What is required are conditions on the sequence $\{\mu_n\}$ which will guarantee that the transformation T is well defined and is continuous on L^p, that is

$$\| Tf \|_p \leqq A \| f \|_p.$$

It is easy to see that when $p=2$, a necessary and sufficient condition of the sequence is the existence of a constant M, so that

$$\text{(i)} \qquad |\mu_n| \leqq M \quad \text{for all } n.$$

We consider, in addition to this, the condition

$$\text{(ii)} \qquad \sum_{k=1}^{n} k \left| \mu_k - \mu_{k+1} \right| \leqq M_n \quad \text{for all } n.$$

The result is then

THEOREM 10. *Suppose the sequence* $\{\mu_n\}$ *satisfies the conditions* (i) *and* (ii) *above. Then the transformation* T, *defined by* (14.1) *on all polynomials satisfies*

$$(14.2) \qquad \| Tf \|_p \leqq A_p M \| f \|_p, \qquad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}.$$

Thus T *has a unique extension to all of* L^p, *which again satisfies* (14.1) *and* (14.2).

REMARK. It should be noticed that the condition (ii) is equivalent with the condition

$$\text{(ii}'\text{)} \qquad \sum_{k=2^n}^{2^{n+1}} \left| u_k - u_{k+1} \right| \leqq M' \quad \text{for all } n.$$

This second form is the way such conditions are usually stated.

Proof. The main difficulty is the proof of inequality (14.2) for polynomials. The rest follows with easily managed technicalities. We begin by making the trivial simplifying assumption that $a_0 = 0$.

Let us set $F = Tf$. Then in view of Corollary 1 of Theorem 9, it is enough to prove

$$\|g^*(\theta,F)\|_p \leqq A_p' M \|f\|_p, \qquad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}. \tag{14.3}$$

Now

$$S_n'(\theta,F) = \sum_{k=1}^{n} a_k k \mu_k P_k^\lambda(\cos\theta) = \sum_{k=1}^{n-1} S_k'(\theta,f)[\mu_k - \mu_{k+1}] + S_n'(\theta,f)\mu_n.$$

Therefore by Schwarz's inequality, and conditions (i) and (ii)

$$|S_n'(\theta,F)|^2 \leqq 2Mn \sum_{k=1}^{n-1} |S_k'(\theta,f)|^2 \frac{|\mu_k - \mu_{k+1}|}{k} + 2M|S_n'(\theta,f)|^2.$$

Finally then,

$$(g^*(\theta,F))^2 = \sum_{n=1}^{\infty} \frac{|S_n'(\theta,F)|^2}{n^3} \leqq AM\left(\sum_{n=1}^{\infty} \frac{|S_n'(\theta,f)|^2 v_n}{n^3} + \sum_{n=1}^{\infty} \frac{|S_n'(\theta,f)|^2}{n^3}\right),$$

where A is an absolute constant and $v_n = n|\mu_n - \mu_{n+1}|$.

Thus $\{v_n\}$ satisfies the condition (13.7) and Theorem 9 and its second corollary then show that

$$\|g^*(\theta,F)\|_2 \leqq A_p M \|f\|_p.$$

The extension to the case where f is an arbitrary L^p function follows by straightforward limiting arguments.

That the usual decomposition theorem into dyadic blocks is contained in Theorem 10 can be seen as follows. For this we apply standard arguments related to the Rademacher functions $\phi_0(t), \phi_1(t), \cdots, \phi_n(t), \cdots$ defined on (0,1). These functions are orthonormal and independent, and take on only the values ± 1. For each $t \in (0,1)$ define the linear operator T_t by

$$T_t f \sim \sum_{n=0}^{\infty} \phi_n(t) \sum_{k=2^n}^{2^{n+1}} a_k P_k^\lambda(\cos\theta).$$

Then the sequence of multipliers μ_n, so obtained, satisfy the conditions (i) and (ii) with $M = 1$.

Hence we have $\|T_t f\|_p \leqq A_p \|f\|_p$. If we assume that $a_0 = 0$, and notice that $T_t^2 = I$, we then have

$$A_p^{-1}\|f\|_p \leqq \|T_t f\|_p \leqq A_p \|f\|_p, \qquad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}.$$

If we integrate this with respect to t, we get (see [26, Chapter 15], for similar arguments)

$$A_p'^{-1}\|f\|_p \leqq \|\Delta(f)\|_p \leqq A_p'\|f\|_p,$$

where

$$\Delta(f)^2 = \sum_{n=0}^{\infty}\left|\sum_{k=2^n}^{2^{n+1}-1} a_k P_k^\lambda(\cos\theta)\right|^2.$$

We come now to the second result which deals with lacunary partial sums. In fact let $n_1, n_2, \cdots, n_k, \cdots$ be a lacunary sequence of positive integers, corresponding to Hadamard gaps, and described by the following condition: There exists a $\sigma > 1$, so that $n_{k+1}/n_k \geqq \sigma > 1$. The extension of the Littlewood-Paley theorem for the trigonometric case is then as follows.

THEOREM 11. *Let $f(\theta) \in L^p$, $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$. Then*
(i) $\|\sup_k |S_{n_k}(\theta,f)|\,\|_p \leqq A_p\|f\|_p$,
(ii) $\lim_{k\to\infty} S_{n_k}(\theta,f)) = f(\theta)$, *almost everywhere and in the L^p-norm.*

Proof. The key conclusion is (i); (ii) follows from it by straightforward arguments. In proving (i) we first deduce a maximal inequality for Cesàro means.

Thus let $\sigma_n(\theta) = \sigma_n(\theta,f) = (n+1)^{-1}(S_0(\theta,f) + \cdots + S_n(\theta,f))$. We shall prove that

$$\left\|\sup_n |\sigma_n(\theta,f)|\,\right\|_p \leqq A_p\|f\|_p, \qquad \frac{2\lambda+1}{\lambda+1} < p < \frac{2\lambda+1}{\lambda}. \tag{14.4}$$

In fact, we shall deduce (14.4) from its Abelian analogue, Theorem 2(d), in §6; the Tauberian condition will turn out to be, in effect, (13.6).

Let us temporarily use the following notation: $\sum_{k=0}^{\infty} u_k$ will be a numerical series, and we shall set

$$S_n = \sum_{k=0}^{n} u_k, \qquad \sigma_n = (n+1)^{-1}\sum_{k=0}^{n} S_k, \qquad f(r) = \sum_{k=0}^{\infty} u_k r^k.$$

Ultimately we shall make $u_k = a_k P_k^\lambda(\cos\theta)$, and therefore S_n will be $S_n(\theta,f)$, $f(r) = f(r,\theta), \cdots$, etc.

We shall also have need of the expression

$$\tau = \left\{\sum_{k=0}^{\infty}(k+1)|\sigma_k - \sigma_{k-1}|^2\right\}^{1/2}, \quad \sigma_{-1} = 0. \tag{14.5}$$

By summation by parts we have

$$f(r) = \sum u_k r^k = (1-r)\sum S_k r^k = (1-r)^2 \sum \sigma_k(k+1)r^k.$$

If we apply differences to the σ_k and sum by parts again we get

$$f(r) = (1-r)\sum_{k=0}^{\infty}(\sigma_k - \sigma_{k+1})(k+1)r^k + \sum_{k=0}^{\infty}(\sigma_k - \sigma_{k-1})r^{k+1}.$$

Therefore

$$f(r)-\sigma_n=(1-r)\sum_{k=0}^{\infty}(\sigma_k-\sigma_{k-1})(k+1)r^k+\sum_{k=0}^{n}(\sigma_k-\sigma_{k-1})(r^{k+1}-1)$$
$$+\sum_{n-1}^{\infty}(\sigma_k-\sigma_{k-1})r^{k+1}.$$

By Schwarz's inequality, the fact that $1-r^{k+1}\leqq(k+2)(1-r)$ $(0\leqq r\leqq 1)$, it follows that

$$|f(r)-\sigma_n|\leqq\tau\left\{(1-r)\left(\sum_{k=0}^{\infty}(k+1)r^{2k}\right)^{1/2}+(1-r)\left(\sum_{k=0}^{n}\frac{(k+2)^2}{k+1}\right)^{1/2}+\sum_{k=n+1}^{\infty}\frac{r^{2k+2}}{k+1}\Big)^{1/2}\right\}.$$

If we set $r=1-1/n$, the terms within the brackets are all bounded. Hence

$$\sup|\sigma_n|\leqq\sup_{0\leqq r<1}|f(r)|+A\tau.$$

If we particularize to $u_k=a_kP_k^{\lambda}(\cos\theta)$, we get

$$\sup_n|\sigma_n(\theta,f)|\leqq\sup_{0\leqq r<1}|f(r,\theta)|+A\left\{\sum_{k=0}^{\infty}(k+1)|\sigma_k(\theta,f)-\sigma_{k-1}(\theta,f)|^2\right\}^{1/2},$$

where $f(r,\theta)$ is the Poisson integral of $f(\theta)$,

$$f(r,\theta)=\sum_{k=0}^{\infty}a_kr^kP_k^{\lambda}(\cos\theta).$$

Next we notice that $\sigma_n(\theta,f)-\sigma_{n-1}(\theta,f)=(n(n+1))^{-1}S_n'(\theta,f)$. Therefore

$$\sum_{k=1}^{\infty}(k+1)|\sigma_k(\theta,f)-\sigma_{k-1}(\theta,f)|^2=\sum_{k=1}^{\infty}k^{-2}(k^{-1}+1)^{-1}|S_k'(\theta,f)|^2\leqq(g^*(\theta,f))^2.$$

Since we have taken $a_0=0$, then $\sigma_0=0$, and therefore

$$\sup_n|\sigma_n(\theta,f)|\leqq\sup_{0\leqq r<1}|f(r,\theta)|+Ag^*(\theta,f).$$

A combination of Theorem 2(d) of(§6) and Corollary 1 to Theorem 9 (in §13 then gives (14.4).

Next

$$\sup_{n_k}|S_{n_k}(\theta,f)|\leqq\sup_{n_k}|\sigma_{n_k}(\theta,f)|+\sup_{n_k}|S_{n_k}(\theta,f)-\sigma_{n_k}(\theta,f)|.$$

Thus

$$\sup_{n_k}|S_{n_k}(\theta,f)|\leqq\sup_{n_k}|\sigma_{n_k}(\theta,f)|+\left(\sum_{n_k}|S_{n_k}(\theta,f)-\sigma_{n_k}(\theta,f)|^2\right)^{1/2}. \tag{14.6}$$

However once the fixed lacunary sequence $\{n_k\}$ has been chosen then we have

$$\left\| \left(\sum_{n_k} |S_{n_k}(\theta,f) - \sigma_{n_k}(\theta,f)|^2 \right)^{1/2} \right\|_p \leqq A_p \| f(\theta) \|_p. \tag{14.7}$$

In fact the special case of (14.7) is then $n_k = 2^k$, obtained in the remarks following Corollary 2 to Theorem 9. The general case of $\{n_k\}$ considered is then treated in exactly the same way.

(14.4), (14.5) and (14.7) immediately imply conclusion (i) of the theorem. As we have stated before, (ii) then follows from (i) by well-known arguments.

CHAPTER III

15. **Convolution structure and fractional integration.** Up to the present our main emphasis has been to associate to each development its appropriate "harmonic" (and "conjugate harmonic") functions. This is seen most explicitly in Chapter I (§§2–9); to a large degree this is also the case for Chapter II, in view of the definition of the g-function which plays a key role there.

We now shift our point of view and study a convolution structure naturally associated with ultraspherical (and many other) expansions. Such convolution structures have been studied for some time by several authors, but the more precise knowledge in the case of ultraspherical developments appears first in Bochner [4].

We begin first by giving a definition of a *convolution structure*, and then proving a general theorem of "fractional integration" for such a structure, and finally we shall apply this in the case of ultraspherical expansions.

The resulting theorem can be stated as follows:

THEOREM 12. *Define the operator* $I_\alpha, 0 < \alpha$, *as follows*: *If*

$$f \sim \sum_{n=0}^{\infty} a_n P_n^\lambda(\cos\theta),$$

then

$$I_\alpha(f) \sim \sum_{n=1}^{\infty} n^{-\alpha} a_n P_n^\lambda(\cos\theta).$$

Then

$$\| I_\alpha(f) \|_r \leqq A_{r,p} \| f \|_p$$

whenever $1 < p < \infty$, $1 < r < \infty$, *and* $1/r = 1/p - \alpha/(2\lambda + 1)$.

We must study first, however, some general properties of convolution structures.

Let M be a measure space with measure dm. We define the L^p-spaces and their norms, $\| \cdot \|_p$, in the usual fashion. We say we have a convolution structure

on M if we have a bilinear mapping of $L^1 \times L^1$ to L^1 given by $(f,g) \to f*g$, which satisfies the following properties:

$$\|f*g\|_1 \leqq \|f\|_1 \|g\|_1, \tag{1}$$

$$\|f*g\|_\infty \leqq \|f\|_1 \|g\|_\infty, \text{ if } g \in L^1 \cap L^\infty, \tag{2}$$

$$\|f*g\|_\infty \leqq \|f\|_\infty \|g\|_1, \text{ if } f \in L^1 \cap L^\infty. \tag{3}$$

It should be noted that we have not assumed that $f*g = g*f$—that is, we shall not need commutativity of the convolution product. An example of the above structure is the usual convolution on any locally compact (unimodular) group—and more generally on various homogeneous spaces of such groups.

We observe first Young's inequality for convolution structures, which we state as a lemma.

LEMMA 10. *Let* $1 \leqq p, q, r \leqq \infty$, *with* $1/r = 1/q + 1/p - 1$. *Then*

$$\|f*g\|_r \leqq \|f\|_p \|g\|_q \tag{15.1}$$

if

$$f \in L^1 \cap L^p \quad \textit{and} \quad g \in L^1 \cap L^g.$$

Proof. It suffices to prove (15.1) when both f and g are *simple* functions (that is, finite linear combinations of characteristic functions of sets of finite measure). From this special case the more general one stated then follows by a simple limiting argument. In fact, once (15.1) has been proved a similar argument shows that the convolution has a natural (unique) extension to a bilinear mapping on all of $L^p \times L^q$ to L^r, satisfying (15.1), whenever p, q, and r satisfy the conditions of the lemma.

However (15.1), for simple functions, follows in a straightforward fashion from a two-fold application of the bilinear Riesz-Thorin convexity theorem in conjunction with conditions (1), (2) and (3) above. (See Zygmund [26, Vol. II, p. 106].)

We come now to fractional integration. It was observed by R. O'Neil that the classical (Euclidean n-space) fractional integration theorem can be restated as an inequality like (15.1), but where the L^q-norm of g is replaced by a weaker norm—the so-called "weak-type" norm.

For our purposes, the weak-type norm of exponent q, $\|\cdot\|_q^*$, can be defined as follows

$$\|g\|_q^* = \sup_E \frac{\int |g| \chi_E \, dm}{\|\chi_E\|_{q'}}. \tag{15.2}$$

The sup is taken over all sets E of finite positive measure, χ_E denoting the characteristic function of the set E, and q' denoting the conjugate exponent to q, $1/q' + 1/q = 1$.

We observe that

$$\| g \|_q^* \leqq \| g \|_q,$$

and $\| \cdot \|_q^*$ has the properties of a norm: $\| c_1 g_1 + c_2 g_2 \|_q^* \leqq | c_1 | \, \| g_1 \|_q^* + | c_2 | \, \| g_2 \|_q^*$, for constants c_1 and c_2; and $\| g \|_q^* = 0$, only if $g = 0$. It should also be noted that $\| g \|_q^* = \| g \|_q$, when $q = 1$ or ∞; and for all $q, 1 \leqq q \leqq \infty$, when g is the characteristic function of a set. In general, however, $\| g \|_q^*$ is smaller than $\| g \|_q$.

With those definitions we can now state the theorem of fractional integration for convolution structures(8).

THEOREM 13(9).

$$\| f * g \|_r \leqq A_{p,q} \| f \|_p \| g \|_q^* \tag{15.3}$$

if $1 < p, q, r < \infty$ *and* $1/r = 1/q + 1/p - 1$.

Proof. Let us fix g with $\| g \|_q^* = 1$, and consider the linear transformation $T: f \to f * g$. We remark first that $T(f)$ is a well-defined function (in the space $L^p + L^\infty$). In fact write $g = g^1 + g_1$, where $g^1 = g$ if $|g| > 1$ and $g^1 = 0$ otherwise; $g_1 = g$ if $|g| \leqq 1$, $g_1 = 0$ otherwise. Notice first that $g^1 \in L^1$; also the conditions on p, q, r imply that $q < p'$, $(1/p + 1/p' = 1)$. From this it is easily follows that $g_1 \in L^{p'}$. Now $Tf = f * g$ will be *defined* $Tf = T_1 f + T^1 f$ where $T_1 f * g_1 = f * g_1$, $T^1 f = f * g^1$. Then $T_1 f \in L^\infty$, and $T^1 f \in L^p$(by Lemma 10). It is to be noted that the definition of Tf actually does not depend on the particular splitting used of g into a function of L^1 and $L^{p'}$.

In proving (15.3) we shall prove first the weaker statement

$$\| Tf \|_r^* \leqq A_{p,q} \| f \|_p \qquad (Tf = f * g) \tag{15.4}$$

with $1 < p, q, r < \infty$ and $1/r = 1/p + 1/q - 1$.

To do this we shall repeat in a more precise way the argument which showed that Tf was well defined, paying attention to the all-important details.

Thus we write $Tf = T^\alpha f + T_\alpha f = f * g^\alpha + f * g_\alpha$, where $g^\alpha = g$ if $|g| > \alpha$, while $g_\alpha = g$ if $|g| \leqq \alpha$. α is positive and will be fixed momentarily.

We begin by computing $\| g^\alpha \|_1$ and $\| g_\alpha \|_{p'}$. However, $\| g_\alpha \|_1 = \int_E |g| \, dx \leqq (m(E^\alpha))^{1-1/q}$, where $E^\alpha = \{x \mid |g(x)| > \alpha\}$, since $\| g \|_q^* \leqq 1$. But $\alpha < |g(x)|$ on E^α; and thus

$$\alpha m(E^\alpha) < \int_E |g| \, dx \leqq m(E^\alpha)^{1-1/q}.$$

(8) In the case of ordinary convolution on a group the theorem was first proved by O'Neil. See his paper [12].

(9) The precise meaning of $f * g$ is discussed in the proof of the theorem.

Combining the extreme terms given $m(E^\alpha) \leqq \alpha^{-q}$, and therefore

$$\|g^\alpha\|_1 \leqq \alpha^{-q+1} \quad \text{if } \|g\|_q^* \leqq 1. \tag{15.5}$$

Next, if $\lambda(\beta) = m(E^\beta)$, then

$$\|g_\alpha\|_{p'}^{p'} = -\int_0^\alpha \beta^{p'} d\lambda(\beta) = p' \int_0^\alpha \beta^{p'-1} \lambda(\beta)\, d\beta.$$

We have already seen that $m(E^\alpha) \leqq \alpha^{-q}$ for all $\alpha > 0$; hence $\lambda(\beta) \leqq \beta^{-q}$ This gives

$$\|g_\alpha\|_{p'}^{p'} \leqq p' \int_0^\alpha \beta^{p'-1-q}\, d\beta = \frac{p'}{p'-q} \alpha^{p'-q}.$$

Finally

$$\|g_\alpha\|_{p'} \leqq \left(\frac{p'}{p'-q}\right)^{1/p'} \alpha^{1-q/p'} = c\alpha^{1-q/p'}. \tag{15.6}$$

Now

$$\int_E |T^\alpha f|\, dx \leqq \|T^\alpha f\|_p (m(E))^{1-1/p} = \|f * g^\alpha\|_p (m(E))^{1-1/p}$$

$$\leqq \|f\|_p \|g^\alpha\|_1 m(E)^{1-1/p} \quad \|f\|_p\, \alpha^{-q} m(E)^{1-1/p},$$

by (15.5) and Lemma 10 again.

Now choose α by setting

$$\alpha^{-q+1} m(E)^{1-1/p} = \alpha^{1-q/p'} m(E).$$

This determines α as $\alpha^{-q} = m(E)$ and gives

$$\alpha^{-q+1} m(E)^{1-1/p} = \alpha^{1-q/p'} m(E) = m(E)^{1-1/r},$$

where $1/r = 1/p + 1/q - 1$.

Hence we have

$$\int_E |Tf|\, dx \leqq \int_E |T^\alpha f|\, dx + \int_E |T_\alpha f|\, dx$$

$$\leqq (c+1) m(E)^{1-1/r} \|f\|_p.$$

If we take the sup over the sets E we obtain

$$\|Tf\|_r^* \leqq (c+1)\|f\|_p,$$

and this is (15.3).

Let us now interpret the condition $\|g\|_r^* \leqq A$, for a general function g. In order to do this consider the distribution function λ of $|g|$. That is, $\lambda(\alpha) = m(E^\alpha) = m\{x : |g(x)| > \alpha\}$.

Now

$$\alpha\lambda(\alpha) \leqq \int_{E_\alpha} |g|\,dm \leqq Am(E^\alpha)^{1-1/r} = A(\lambda(\alpha))^{1-1/r}.$$

Therefore, $\lambda(\alpha) \leqq A^r\alpha^{-r}$, $0<\alpha<\infty$.

This shows that the inequality (15.3) implies

$$(15.7)\qquad m\{x:|Tf|>\alpha\} \leqq \left(\frac{A_{p,q}\|f\|_p}{\alpha}\right)^r,\ 1<p,q,\ r<\infty,\ 1/r=1/p+1/q-1.$$

For fixed p,q,r, (15.4) is exactly the statement that the mapping $f\to Tf$ is of *weak type* (p,r), in the sense of the Marcinkiewicz interpolation theorem; see [26, Vol. II, p. 111]. Applying this theorem then gives

$$\|Tf\|_r = \|f*g\|_r \leqq A'_{p,q}\|f\|_p,\quad 1<p,q,r<\infty,\ 1/r=1/p+1/q-1.$$

Finally, if we drop the normalization $\|g\|_q^* = 1$, we get (15.2) and the theorem.

We now come to the convolution structure associated with ultraspherical expansions.

It will be convenient to change our notation slightly by renormalizing the ultraspherical polynomials and the underlying measure of our space. We fix $\lambda>0$, throughout the discussion.

Thus set $P_n^*(\cos\theta) = P_n^\lambda(\cos\theta)/P_n^\lambda(1)$, and hence $P_n^*(1) = 1$. We shall also set $\|f\|_p = (c_\lambda^{-1}\int_0^\pi |f(\theta)|^p \sin^{2\lambda-1}\theta\, d\theta)^{1/p}$, where $c_\lambda = \int_0^\pi \sin^{2\lambda-1}\theta\, d\theta$. The normalizing constants ρ_n are defined by

$$\rho_n^{-1} = \|P_n^*\|_2.$$

Thus

$$\rho_n = \frac{\lambda+n}{\lambda}\,\frac{\Gamma(n+2\lambda)}{n!\Gamma(2\lambda)}$$

(see [4, p. 25]). If $f(\theta)\sim \sum a_n P_n^\lambda(\cos\theta)$ we also set

$$f(\theta)\sim \sum a_n^*\rho_n P_n^*(\cos\theta).$$

We now define $f*g$ by

$$(15.8)\qquad (f*g)(\theta) = c_\lambda^{-2}\int_0^\pi\int_0^\pi f(\phi)g(\psi)\sin^{2\lambda-1}\phi\sin^{2\lambda-1}t\, d\phi dt,$$

where

$$\cos\psi = \cos\theta\cos\phi + \sin\theta\sin\phi\cos t.$$

The crucial properties of this convolution product are two-fold.

$$(15.9)\qquad |f*g| \leqq |f|*|g|.$$

$$(15.10)\qquad P_n^* * P_m^* = \rho_n^{-1}\delta_{n,m}P_n^*.$$

The first is obvious, and the second is in [4, p. 29].

From (15.10) it follows that if

$$f \sim \sum a_n^* \rho_n P_n^* \quad \text{and} \quad g \sim \sum b_n^* \rho P_n^* \tag{15.11}$$

then

$$f * g \sim \sum a_n^* b_n^* \rho_n P_n^*$$

which is the multiplicative property of convolutions. From (15.9), (15.10), and (15.11) it follows easily that $f*g = g*f$, and since $f*1 = \|f\|_1 \leqq \|f\|_\infty$ we get $\|f*g\|_\infty = \|g*f\|_\infty \leqq \|f\|_1 \|g\|_\infty$.

Also if f and g are non-negative

$$\|f*g\|_1 = c_\lambda^{-1} \int_0^\pi (f*g)(\theta) \sin^{2\lambda-1}\theta \, d\theta = \|f\|_1 \|g\|_1 .$$

Thus, in general,

$$\|f*g\|_1 \leqq \|f\|_1 \|g\|_1 .$$

We have therefore verified the basic properties of the convolution structure.

We now come to the proof of Theorem 12 stated earlier in this section. It can be restated as follows:

If

$$f \sim \sum_{n=0}^{\infty} a_n^* \rho_n P_n^*(\cos\theta)$$

and

$$I_\alpha(f) \sim \sum_{n=1}^{\infty} a_n^* \rho_n n^{-\alpha} P_n^*(\cos\theta)$$

then

$$\| I_\alpha(f) \|_r \leqq A_{r,p} \| f \|_p ,$$

whenever

$$1 < p < \infty, \; 1 < r < \infty \quad \text{and} \quad 1/r = 1/p - \alpha/(2\lambda + 1).$$

However,

$$I_\alpha(f) = f * K_\alpha ,$$

where

$$K_\alpha(\theta) \sim \sum_{n=1}^{\infty} n^{-\alpha} \rho_n P_n^*(\cos\theta).$$

Therefore, in view of Theorem 13, it suffices to prove that

$$\| K_\alpha \|_q^* \leqq A_{q,\alpha} < \infty \quad \text{for } 1/q - 1 = -\alpha/(2\lambda + 1),$$

where $\| \cdot \|_q^*$ is the weak-type norm.

Let us set $x = \cos\theta$, and $K_\alpha(\theta) = k_\alpha(x)$. Then as is known, (see [4, p. 25]),

$$\frac{1-r^2}{(1-2rx+r^2)^{\lambda+1}} = \sum_{n=0}^{\infty} \rho_n r^n P_n^*(x), \qquad 0 \leqq r < 1.$$

Thus

$$k_\alpha(x) = \frac{1}{\Gamma(\alpha)} \int_0^1 \left[\frac{1-r^2}{(1-2rx+r^2)^{\lambda+1}} - 1\right](\log 1/r)^{\alpha-1} \frac{dr}{r}, \qquad \alpha > 0.$$

We break up this integral into two parts: from 0 to $1/2$ and $1/2$ to 1. We notice without difficulty that the first integral is uniformly bounded in x. We now consider the second integral, dropping first the factor $\Gamma(\alpha)^{-1}$, and then the constant $\int_{1/2}^1 (\log 1/r)^{\alpha-1} r^{-1}\, dr$. What remains is bounded by a constant multiple of

$$\int_{1/2}^1 \frac{1-r^2}{(1-2rx+r^2)^{\lambda+1}} (1-r)^{\alpha-1}\, dr. \tag{15.12}$$

We remark that here we may assume that say $3/4 \leqq x < 1$, since in the interval $-1 \leqq x \leqq 3/4$ this integral is uniformly bounded. We now decompose the integral in (15.12) as a sum of an integral where $(1-r)^2 \geqq 1-x$, and another where $(1-r)^2 < 1-x$.

Now,

$$1 - 2rx + r^2 = (1-r)^2 + 2r(1-x).$$

Thus

$$(1-2rx+r^2)^{\lambda+1} \geqq (1-r)^{2\lambda+2} \quad \text{when } (1-r)^2 \geqq (1-x)$$

and

$$(1-2rx+r^2)^{\lambda+1} \geqq (1-x)^{\lambda+1} \quad \text{when } (1-r)^2 < (1-x) \ (r \geqq 1/2).$$

Combining the above we see that (15.12) is majorized by twice

$$\int_{1/2}^{1-(1-x)^{1/2}} (1-r)^{\alpha-2\lambda-2}\, dr + (1-x)^{-\lambda-1} \int_{1-(1-x)^{1/2}}^1 (1-r)^\alpha\, dr.$$

Altogether, this is bounded by $A(1-x)^{\alpha/2-\lambda-1/2}$. Thus

$$|k_\alpha(x)| = |K_\alpha(\cos\theta)| \leqq A(1-x)^{\alpha/2-\lambda-1/2} + A \leqq A'(1-x)^{\alpha/2-\lambda-1/2} = A(\sin\theta/2)^{\alpha-2\lambda-1},$$

since $\alpha < 2\lambda + 1$, $x = \cos\theta$.

Finally it suffices to show that

$$\|(\sin\theta/2)^{\alpha-2\lambda-1}\|_q^* < \infty, \quad 1 - 1/q = \frac{\alpha}{2\lambda+1},$$

or what is the same

$$\sup_E \frac{\int_E (\sin\theta/2)^{\alpha-2\lambda-1}\sin^{2\lambda}\theta\, d\theta}{\left(\int_E \sin^{2\lambda}\theta\, d\theta\right)^{1-1/q}} < \infty.$$

Since $\alpha - 2\lambda - 1 < 0$, the function $(\sin\theta/2)^{\alpha-2\lambda-1}$ is decreasing on $(0,\pi)$. Thus the above supremum is equal to

$$\sup_{0<l\leqq\pi} \frac{\int_0^l (\sin\theta/2)^{\alpha-2\lambda-1}\sin^{2\lambda}\theta\, d\theta}{\left(\int_0^l \sin^{2\lambda}\theta\, d\theta\right)^{1-1/q}}.$$

But the last quotient is bounded by constant $\times \sup l^{\alpha} / l^{(2\lambda+1)(1-1/q)}$.

However, $\alpha/(2\lambda+1) = 1 - 1/q$, which proves the assertion, and hence the theorem.

16. **Fourier-Bessel transforms (Hankel transforms).** We shall now discuss the analogue of our study of ultraspherical expansions which leads to Bessel function and the Fourier-Bessel (or Hankel) transforms.

The wide-ranging analogy between (discrete) ultraspherical expansions and (continuous) Fourier-Bessel expansions will be used in what follows below. The analogy holds not only for the statements of the theorems but carries over also to the details of their proofs. Two slight qualifications must be added to this assertion. First, various formulae and estimates take a somewhat simpler form in the Fourier-Bessel case than the ultraspherical case. Second, the ultraspherical case, as opposed to the Fourier-Bessel transform case, deals with a compact domain as contrasted to a noncompact domain. Thus certain additional arguments are needed in problems of "harmonic majorization" (see Lemma 11 below).

However, in the main, the analogy between these two cases is so close that it is not necessary to duplicate most of the proofs in the second case. For this reason, and to save space, our discussion will be brief, omitting proofs in every case (except for Lemma 11).

Our starting point is, as indicated in the Introduction, functions in Euclidean n-space which are radial. Thus let $\bar{y} = (y_1, \cdots, y_n) \in E_n$, and let $\bar{f}(\bar{y}) = f(y_1, \cdots, y_n)$ be a radial function, i.e., $\bar{f}(\bar{y}) = f(y)$, where $y = |\bar{y}| = (y_1^2 + \cdots + y_n^2)^{1/2}$.

Consider the Fourier transform of such a function

$$\bar{f}^{\wedge}(\bar{\xi}) = (2\pi)^{-n/2} \int_{E_n} \bar{f}(\bar{y}) e^{i\bar{\xi}\cdot\bar{y}}\, d\bar{y}.$$

Then

$$\hat{f}(\xi) = \int_0^{\infty} V_{(n-2)/2}(\xi y) f(y) y^{n-1}\, dy,$$

with $\xi = |\bar{\xi}|$ and

$$V_a(t) = t^{-a} J_a(t).$$

(See [5, Chapter II].)

If $\bar{u}(x, \bar{y})$ is the harmonic function in $x > 0$, with boundary values $\bar{f}(\bar{y})$ (when $x = 0$), then this Poisson integral is given by

$$\bar{u}(x, \bar{y}) = u(x, y) = \int_0^\infty e^{-x\xi} V_{(n-2)/2}(\xi y) f(\xi) \xi^{n-1} \, d\xi$$

$$= \int_0^\infty P(x, y, z) f(z) z^{n-1} \, dz,$$

with

$$P(x, y, z) = (yz)^{-(n-2)/2} \int_0^\infty e^{-x\xi} J_{(n-2)/2}(\xi y) J_{(n-2)/2}(\xi z) \xi \, d\xi.$$

Also $u(x, y)$ satisfies the radial "Laplace equation"

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{n-1}{y} \frac{\partial u}{\partial y} = 0.$$

Incidentally, the Plancherel formula in this case becomes

$$\int_0^\infty |\hat{f}(\xi)|^2 \xi^{n-1} \, d\xi = \int_0^\infty |f(y)|^2 y^{n-1} \, dy.$$

We now set $n - 1 = 2\lambda$, and we pass by analogy from half-integral λ to general positive λ. In general, we have

$$u(x, y) = \int_0^\infty P(x, y, z) f(z) z^{2\lambda} \, dz, \tag{16.1}$$

where

$$P(x, y, z) = (yz)^{-\lambda - 1/2} \int_0^\infty e^{-x\xi} J_{\lambda - 1/2}(y\xi) J_{\lambda - 1/2}(z\xi) \xi \, d\xi. \tag{16.1'}$$

$u(x, y)$ now satisfies the equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{2\lambda}{y} \frac{\partial u}{\partial y} = 0. \tag{16.2}$$

All our L^p-spaces will be taken with respect to the measure $y^{2\lambda} dy$, so that

$$\|f\|_p = \left(\int_0^\infty |f(y)|^p y^{2\lambda} \, dy \right)^{1/p}.$$

In this way the mapping

$$f(y) \to \hat{f}(\xi) = \int_0^\infty V_{\lambda-1/2}(\xi y) f(y) y^{2\lambda}\, dy \tag{16.3}$$

becomes a unitary transformation on L^2. (See [19, Chapter 8].)

$u(x,y)$ can be re-expressed in terms of this transformation:

$$u(x,y) = \int_0^\infty e^{-x\xi} V_{\lambda-1/2}(\xi y) \hat{f}(\xi) \xi^{2\lambda}\, d\xi .$$

We return to the Poisson kernel $P(x,y,z)$, and its conjugate $Q(x,y,z)$ (to be defined later), and find explicit expressions for these.

For this purpose we consider the formula of Weinstein [23]

$$\begin{aligned} \phi_b(x,y) &= S_{p-1} \int_0^\pi (b^2 + x^2 + y^2 - 2by \cos a)^{-p/2} (\sin a)^{p-1}\, da \\ &= C_q b^{-q} y^{-q} \int_0^\infty e^{-xt} J_q(yt) J_q(bt)\, dt \end{aligned}$$

with

$$S_{p-1}^{-1} = \pi^{1/2} \Gamma(p/2) \left(\Gamma\left(\frac{p+1}{2} \right) \right)^{-1}, \quad C_q = 2^{2q} \Gamma^2(q+1) (\Gamma(2q+1))^{-1},$$

and $p = 2q + 1$. $\phi_b(x,y)$ also satisfies the "Laplace equation" (16.2) with $p = 2\lambda$. By differentiating with respect to x we get

$$P(x,y,b) = \frac{2\lambda x}{\pi} \int_0^\pi (b^2 + x^2 + y^2 - 2by \cos a)^{-\lambda-1} (\sin a)^{2\lambda-1}\, da . \tag{16.4}$$

Since we have $P(x,y,b) = -C_q^{-1}\, \partial(\phi_b(x,y))/\partial x$, and ϕ satisfies (16.2), its conjugate should be given by $C_q^{-1}\, \partial(\phi_b(x,y))/\partial y$.

We thus define $Q(x,y,b)$ by $Q(x,y,b) = C_q^{-1}\, \partial(\phi_b)/\partial y$, and we get

$$Q(x,y,b) = -(by)^{\lambda-1/2} \int_0^\infty e^{-xt} J_{\lambda+1/2}(yt) J_{\lambda-1/2}(bt)\, t\, dt \tag{16.5}$$

and

$$Q(x,y,b) = -\frac{2\lambda}{\pi} \int_0^\pi \frac{(y - b\cos\alpha) \cdot (\sin a)^{2\lambda-1}}{(b^2 + x^2 + y^2 - 2by \cos a)^{\lambda+1}}\, da . \tag{16.5'}$$

For an arbitrary f (say $f \in L^p(y^{2\lambda} dy)$), we have defined its Poisson integral $u(x,y)$ by (16.1) and u satisfies (16.2). Define the conjugate $v(x,y)$ to $u(x,y)$ by

$$v(x,y) = y^{2\lambda} V(x,y) = y^{2\lambda} \int_0^\infty Q(x,y,z) f(z) z^{2\lambda}\, dz .$$

Then it follows that u and v satisfy the "Cauchy-Riemann" equations:

$$v_y = y^{2\lambda} u_x, \qquad v_x = -y^{2\lambda} u_y, \tag{16.6}$$

or if $U(x,y) = u(x,y)$, $V(x,y) = y^{-2\lambda} v(x,y)$,

$$V_x + U_y = 0, \qquad U_x - V_y - \frac{2\lambda}{y} V = 0. \tag{16.6'}$$

We therefore see that the notion of conjugacy given by (16.6) and naturally associated to the Fourier-Bessel transform (16.3) is the same as the notion of conjugacy studied earlier and associated to ultraspherical expansions.

What distinguishes these cases are the domains of the (xy)-plane under consideration. In the ultraspherical case we had the upper semi-disc $x^2 + y^2 < 1, y > 0$ and since $U(x,y)$ was even in y and $V(x,y)$ was odd in y, we ultimately considered the whole disc.

In the present case the domain is initially the quarter-plane $x > 0$, $y > 0$; again because the $U(x,y)$ and $V(x,y)$ which arise are respectively even and odd in y we extend consideration to the whole half-plane $x > 0$. In the case of Fourier-Bessel series considered briefly in §18 below the domain becomes the half-strip $0 \leqq x, 0 \leqq y \leqq 1$; and then because of evenness the doubled strip $0 \leqq x, -1 \leqq y \leqq 1$.

We shall now briefly indicate how the results of §§3 to 15 can be reformulated and proved in variants appropriate for this setting([10]).

(a) Coming first to the maximum principle (Theorem 1 in §3), we see that in view of (16.2) and what has been said above, this theorem is equally applicable to the present case. However, in the present case the domains in question are not bounded. It will be necessary, therefore, to supplement Theorem 1 with an additional lemma and its corollary.

LEMMA 11. *Suppose* (1) *$u(x,y)$ is continuous in $0 \leqq x < \infty$, $-\infty < y < \infty$, and even in y.*

(2) *In the region where $u(x,y) > 0$, u is of class C^2 and satisfies $\partial^2 u/\partial x^2 + \partial^2 u/\partial y^2 + 2\lambda y^{-1} \partial u/\partial y \geqq 0$, there.*

(3) $u(0,y) = 0$.

(4) *For some p, $1 \leqq p < \infty$,*

$$\sup_{0<x<\infty} \int_0^\infty |u(x,y)|^p y^{2\lambda}\, dy \leqq M < \infty.$$

Conclusion: $u(x,y) \leqq 0$.

COROLLARY. *If in addition $\partial^2 y/\partial x^2 + \partial^2 y/\partial y^2 + 2\lambda y^{-1} \partial u/\partial y = 0$ everywhere then $u(x,y) \equiv 0$.*

([10]) Recall that we always have $\lambda \geqq 0$.

Proof. Fix $0<\rho<\infty$, and define $f_\rho(\theta)$ by $f_\rho(\theta)=|u(\rho\cos\theta,\ \rho\sin\theta)|$ for $|\theta|\leqq\pi/2$, and $f_\rho(\theta)=-f_\rho(\pi-\theta)$, for $(\pi/2)\leqq\theta\leqq 3\pi/2$. Then $f_\rho(\theta)$ is continuous, since $f(\pm\pi/2)=0$. Now

$$\int_0^R\int_0^\pi |f(\theta)|^p(\sin\theta)^{2\lambda}\,d\theta\rho^{2\lambda+1}d\rho=\frac{1}{2}\iint_{x^2+y^2\leqq R^2}|u(x,y)|^p y^{2\lambda}\,dydx\leqq MR,$$

because of condition (4).

Thus if $J(\rho)=\int_0^\pi|f_l(\theta)|^p(\sin\theta)^{2\lambda}\,d\theta$, then $\int_0^R J(\rho)\rho^{2\lambda+1}\,d\rho\leqq MR$, and hence $\liminf_{\rho\to\infty}J(\rho)\rho^{2\lambda+1}\leqq M$. In particular, there exists a subsequence $\rho_1,\rho_2,\cdots,\rho_n,\cdots,\rho_n\to\infty$, so that $J(\rho_n)=\int_0^\pi|f_{\rho_n}(\theta)|^p(\sin\theta)^{2\lambda}\,d\theta\to 0$.

Let now $h_\rho(x.y)$ be the Poisson integral (for the disc $x^2+y^2\leqq\rho^2$) of $f_\rho(\theta)$ in the sense of §§2–4. That is,

$$h_\rho(x,y)=\int_0^\pi P(r/\rho,\theta,\phi)f_\rho(\phi)(\sin\phi)^{2\lambda}d\phi\qquad(x=r\cos\theta,y=r\sin\theta).$$

Then we observe that $h_\rho(x,y)$ is continuous in $x^2+y^2\leqq\rho^2$, even in y, and odd in x, the latter because of the "oddness" of f. Thus $h_\rho(0,y)=0$. Moreover $h_\rho(x,y)$ satisfies the equation (16.2) and majorizes $u(x,y)$ on the half-circle $x^2+y^2=\rho^2$, $x\geqq 0$, and also when $x=0$. Thus by the maximum principle (Theorem 1)

$$u(x,y)\leqq h_\rho(x,y)\quad\text{whenever } x^2+y^2\leqq\rho^2,\quad x\geqq 0.$$

We now restrict the ρ's to our sequence $\rho_n\to\infty$. Since $\int_0^\pi|f_{\rho_n}(\theta)|^p(\sin\theta)^{2\lambda}d\theta\to 0$, it follows easily that for each fixed (x,y), $h_{\rho_n}(x,y)\to 0$. This proves the lemma. The corollary is an immediate consequence.

(b) The analogue of Lemma 1 are the estimates

$$P(x,t,z)\leqq\frac{cx(yz)^{-\lambda}}{x^2+(y-z)^2},$$

$$P(x,y,z)\leqq\frac{cx}{(x^2+(y-z)^2)^{\lambda+1}},$$

where P is given by (16.1′) (or (16.4)).

(c) For $f\in L^p(y^{2\lambda}dy)$, the maximal function f^* is defined by

$$f^*(y)=\sup_{0\leqq y+h}\frac{\int_y^{y+h}|f(z)|z^{2\lambda}\,dz}{\int_y^{y+h}z^{2\lambda}\,dz}.$$

The obvious form of Lemma 2 holds for this variant of f^*, and we also have $\sup_x|u(x,y)|\leqq cf^*(y)$, where u is given by (16.1).

(d) Theorems 2, 3, 3′, and Corollary 2 (in §6) have their obvious analogues in this context. As an example of this we state

ANALOGUE OF THEOREM 3. *Let $u(x,y)$ satisfy* (16.2) *in the quarter-plane $x>0, y>0$; then $u(x,y)$ is the Poisson integral* (16.1) *of an $f \in L^p(y^{2\lambda}dy)$, $1<p<\infty$, if and only if $u(x,y)$ is regular and even in y in the half-plane $x>0$, and $\sup_{x<0}\int_0^\infty |u(x,y)|^p y^{2\lambda}\,dy<\infty$.*

The proof of this theorem is almost identical with that of Theorem 3, except that we need to appeal to the corollary of Lemma 11 of this section.

(e) For the conjugate kernel $Q(x,y,z)$ (given by (16.5)) we have the following estimates:

$$\begin{aligned} Q(x,y,z) &= O(y^{-2\lambda-1}), \quad \text{if} \quad y \leqq z/2, \\ &= O(z^{-2\lambda-1}), \quad \text{if} \quad z \leqq y/2, \end{aligned}$$

$$Q(x,y,z) = c_\lambda (yz)^{-\lambda} \cdot \frac{y-z}{x^2+(y-z)^2} + O\left(y^{-2\lambda-1}\left(1+\log^+\left(\frac{yz}{y-z}\right)\right)\right), \quad \text{if } y/2 \leqq z \leqq 2y.$$

(f) The analogue of Theorem 4 (part (a)) is the statement

$$\sup_{0<x}\left(\int_0^\infty |V(x,y)|^p y^{2\lambda}\,dy\right)^{1/p} \leqq A_p \int_0^\infty |f(y)^p|\, y^{2\lambda}\,dy^{1/p}, \quad 1<p<\infty,$$

with V defined by $V(x,y)=\int_0^\infty Q(x,y,z)f(z)z^{2\lambda}\,dz$.

(g) Going back to formula (16.5) for Q and letting $x \to 0$, we can reinterpret (f) above as follows:

Suppose F is the Fourier-Bessel transform of f, i.e.,

$$f(y) = \int_0^\infty V_{\lambda-1/2}(yt)F(t)t^{2\lambda}\,dt. \tag{16.7}$$

Then

$$\lim_{x\to 0} V(x,y) = \tilde{f}(y) = -x\int_0^\infty V_{\lambda+1/2}(yt)F(t)t^{2\lambda+1}\,dt. \tag{16.8}$$

The mapping $f \to \tilde{f}$ defined thus (initially only for $L^2(y^{2\lambda}dy)$) is the analogue of the Hilbert transform in this case. It is clear from the above that

$$\int_0^\infty |\tilde{f}(y)|^2 y^{2\lambda}\,dy = \int_0^\infty |f(y)|^2 y^{2\lambda}\,dy.$$

The result in (e) shows that the mapping $f \to \tilde{f}$ extends to $L^p(y^{2\lambda}dy)$, $1<p<\infty$, and is boundded there.

(h) The results for H^p-spaces (as in §§9 and 10) can be developed in this

case also. The analogue of the harmonic majorization lemma (Theorem 5) requires now the additional Lemma 11 of this section for its proof. Otherwise the arguments are essentially identical. Theorem 6 has its analogue which may be stated as follows.

ANALOGUE OF THEOREM 6. *Suppose* $F(z) = U(x, y) + iV(x, y)$ *satisfies* (16.6′) *with* U *even and* V *odd in* y. *Suppose also that for* $p \geqq 2\lambda/(2\lambda+1)$, $\sup_{x>0} \int_0^\infty |F(x+iy)|^p y^{2\lambda}\, dy < \infty$. *Then* (a) $\lim_{x\to 0} F(x+iy) = F(iy)$ *exists for almost every* y, *even in the nontangential sense.* (b) *If* $F(iy) = 0$ *on a set of positive measure, then* $F(z) \equiv 0$. (c) *If* $p > 2\lambda/(2\lambda+1)$, *then* $\lim_{x\to 0} \int_0^\infty |F(x+iy) - F(iy)|^p y^{2\lambda}\, dy = 0$.

This theorem (in particular (c) when $p = 1$) has then the following

COROLLARY. *Suppose we have a pair of measures* $d\mu_1$, $d\mu_2$, *so that* $\int_0^\infty y^{2\lambda} |d\mu_i(y)| < \infty$. *If* $\int_0^\infty V_{\lambda-1/2}(xy) y^{2\lambda}\, d\mu_1(y) = -x \int_0^\infty V_{\lambda+1/2}(xy) y^{2\lambda+1}\, d\mu_2(y)$, *then both* $d\mu_i$ *are absolutely continuous with respect to Lebesgue measure.*

This, of course, is another extension of the classical F. and M. Riesz theorem on absolute continuity of analytic measures.

(i) Coming now to the results of Chapter II, and in particular those of §14, we shall limit ourselves to the two main statements in the present version.

(1) Suppose we consider the multiplier transformation T, intially defined on $L^2(y^{2\lambda}dy)$ by

$$(Tf)(y) = \int_0^\infty F(t)\mu(t) V_{\lambda-1/2}(yt) t^{2\lambda}\, dt,$$

where

$$F(t) = \int_0^\infty V_{\lambda-1/2}(yt) f(y)^{2\lambda}\, dy$$

with the multiplier $\mu(t)$ of bounded variation in every finite closed subinterval of $(0, \infty)$ and satisfying $|\mu(s)| \leqq M$ and $\int_0^s t|d\mu(t)| \leqq Ms$, $0 < s < \infty$. Then the transformation $f \to Tf$ has a unique bounded extension from $L^p(y^{2\lambda}dy)$ to itself, with $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$ ([11]).

(2) Suppose we define the partial integrals $S_a(f)$, by

$$S_a(f) = \int_0^a F(t) V_{\lambda-1/2}(yt) t^{2\lambda}\, dt.$$

If $f \in L^p(y^{2\lambda}dy)$, then $S_{a_k}(f) \to f$ almost everywhere and dominately (in L^p), for $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$, whenever a_k is lacunary in the sense that $\liminf_{k\to\infty}(a_{k+1}/a_k) > 1$.

([11]) The result is extendable to other L^p-spaces involving weight functions (see the remarks in §17). In this way one obtains anew the "Main Theorem" of [6].

(j) Finally we come to the topic of (fractional) potentials dealt with in §15. Let us define the operators I_α by

$$I_\alpha(f) = \int_0^\infty F(t)V_{\lambda-1/2}(ty)t^{-\alpha}t^{2\lambda}\,dt$$

with

$$F(t) = \int_0^\infty V_{\lambda-1/2}(ty)f(y)y^{2\lambda}\,dy,$$

at least when f is continuous with compact support and $0<\alpha<2\lambda+1$. Now it is known that $\lim_{\varepsilon\to 0}\int_0^\infty e^{-\varepsilon t}V_{\lambda-1/2}(ty)t^{-\alpha}t^{2\lambda}\,dt = c_{\alpha,\lambda}y^{-2\lambda-1+\alpha}$. (See [21, p. 391].)

Then if terms of the known convolution structure for the Fourier-Bessel transforms (see Bochner [4]) we can write

$$I_\alpha(f) = f * K_\alpha, \quad \text{where } K_\alpha(y) = c'_{\alpha,\lambda}y^{-2\lambda-1+\alpha}.$$

The general Theorem 13 of §15 then applies and shows that $\|I_\alpha(f)\|_r \leqq A_{r,p}\|f\|_p$, with $1<p<r<\infty$ and $1/r = 1/p - \alpha/(2\lambda+1)$.

17. **Remarks on weighted inequalities.** In some problems it is of interest to extend the usual statements of L^p boundedness by modifying the measures with suitable weight factors([12]).

Thus in the ultraspherical case one would look for inequalities of the form

$$\|T(f)(\theta)(\sin\theta)^\alpha\|_p \leqq A_{p,\alpha}\|f(\theta)(\sin\theta)^\alpha\|_p \tag{17.1}$$

with $\|f\| = (\int_0^\pi |f(\theta)|^p(\sin\theta)^{2\lambda}\,d\theta)^{1/p}$.

Similarly in the Fourier-Bessel transform case one would have

$$\|T(f)(y)y^\alpha\|_p \leqq A_{p,\alpha}\|f(y)y^\alpha\|_p \tag{17.2}$$

with $\|f\|_p = (\int_0^\infty |f(y)|^p y^{2\lambda}\,dy)^{1/p}$.

The validity of such inequalities falls in two ranges. The broader range

$$\frac{-(2\lambda+1)}{p} < \alpha < \frac{2\lambda+1}{p'}, \qquad \frac{1}{p}+\frac{1}{p'}=1 \tag{R_1}$$

and the narrower range

$$\frac{-(2\lambda+1)}{p} + \lambda < \alpha < \frac{2\lambda+1}{p'} - \lambda \qquad (\text{recall } \lambda \geqq 0). \tag{R_2}$$

Thus in the case of ultraspherical expansions Theorems 2((b)–(d)), 3, 4(a), Corollaries 1 and 3 and Theorem 8 are all extensible in this sense, with the L^p-norm taken in the broader sense (R_1). This extension can be carried out without any difficulty, and goes back to the fact that those theorems in question are based

([12]) See in particular [1], [6], and [8], where further references can be found.

on estimates for the Poisson and conjugate kernels which allow such variants. This situation is also typified by similar inequalities in Euclidean n-space, (where $n = 2\lambda + 1$), treated in [15].

However, results which essentially go back to partial sums, Theorems 7, 9, 10 and 11, are valid only in the narrower range (R_2). This situation is already typified, to some extent, by some of the results in [13] and [6].

A similar situation in terms of the inequalities (17.2) holds throughout for the Fourier-Bessel transforms considered in §16.

18. **Concluding remarks: Fourier-Bessel series.** The general setting which we used above seems applicable to a variety of other classical expansions. We shall illustrate this by describing the formal setting in the case of Fourier-Bessel series. While we have not pursued this matter any further, it seems highly likely that the analogy with the above can be pushed quite far.

From our point of view the Fourier-Bessel series (also referred to as Dini series) arise by studying functions $u(x, y_1, \cdots, y_n)$ harmonic in the cylinder $x > 0$, $y_1^2 + y_2^2 + \cdots + y_n^2 < 1$, of the $(n + 1)$-dimensional Euclidean space; these functions are given by their boundary values at $x = 0$, $u(0, y_1, y_2, \cdots, y_n) = f(y_1, \cdots, y_n)$, and have vanishing normal derivative along the rest of the boundary(13). By restricting consideration to radial functions f, we obtain Fourier-Bessel series expansions of order $(n-2)/2$; and then by analogy we pass to Fourier-Bessel series of real order greater than $-1/2$.

Thus if $u(x, y_1, \cdots, y_n) = u(x, y)$, $(y = (y_1^2 + \cdots + y_n^2)^{1/2})$, is harmonic in $x > 0$, $0 \leqq y < 1$, and satisfies $u(0, y) = f(y)$, with $u_y(x, 1) = 0$, then

$$u(x, y) = \sum_{m=0}^{\infty} a_m e^{-\gamma_m x} V_{\lambda - 1/2}(\gamma_m y) \qquad (2\lambda = n - 1), \tag{18.1}$$

where $V_\lambda(t) = t^{-\lambda} T_\lambda(t)$, and $\gamma_0 = 0$, $\gamma_1, \gamma_2, \cdots, \gamma_m \cdots$ are the successive zeros of $T'_{\lambda - 1/2}(t)$. We may now drop the assumption that 2λ is integral, and assume only $\lambda \geqq 0$.

The natural L^p-norms in this case are taken (as before) with the measure $y^{2\lambda} dy$; thus $\|f\|_p = (\int_0^1 |f(y)|^p y^{2\lambda} dy)^{1/p}$. For $p = 2$ we know that the collection $\{V_{\lambda - 1/2}(\gamma_m y)\}$ is mutually orthogonal and as a matter of fact complete. (See Watson [21, Chapter 18].)

We should observe that from (18.1) we obtain the expansion

$$f(y) \sim \sum_{m=0}^{\infty} a_m V_{\lambda - 1/2}(\gamma_m y). \tag{18.2}$$

$u(x, y)$ given by (18.1) satisfies the singular Laplace equation (16.2) and thus we should expect its conjugate V to be given by (16.6). From this it follows that

(13) This boundary condition is typical of a more general class that can be treated similarly.

$$V(x,y) = -y \sum_{m=1}^{\infty} a_m e^{-\gamma_m x} \gamma_m V_{\lambda+1/2}(\gamma_m y). \tag{18.3}$$

If we now let $x \to 0$, we obtain the function $\tilde{f}(y)$ conjugate to $f(y)$, given by

$$\tilde{f}(y) \sim -y \sum_{m=1}^{\infty} a_m \gamma_m V_{\lambda+1/2}(\gamma_m y). \tag{18.4}$$

(Compare this notion of conjugacy with (2.4) for ultraspherical series and (16.7)–(16.8) for Fourier-Bessel transforms.)

We state now two results which can be expected to hold in view of the close analogy with the above, and which would be typical. The first would be another extension of the M. Riesz conjugate function inequality, and would take the form that the mapping $f(y) \to \tilde{f}(y)$ (given by (18.4) is bounded in the L^p-norm (as defined above) with $1 < p < \infty$.

Secondly, one could reasonably expect an analogue of the multiplier theorem of §14. It would state:

Let the transformation $f \to Tf$ be given by

$$Tf \sim \sum_{m=0}^{\infty} a_m \mu_m V_{\lambda-1/2}(\gamma_m y)$$

with f given by (18.2) *and μ_m satisfying*: (i) $|\mu_k| \leqq M$, (ii) $\sum_{m=1}^{k} m\,|\mu_m - \mu_{m-1}| \leqq Mk$, *all k. Then the transformation is bounded in the L^p-norm, with* $(2\lambda+1)/(\lambda+1) < p < (2\lambda+1)/\lambda$.

References

1. R. Askey and I. I. Hirschman, *Weighted quadratic norms and ultraspherical polynomials*. I, Trans. Amer. Math. Soc. **91** (1959), 294–313.

2. L. Bers, *Function-theoretical properties of solutions of partial differential equations of elliptic type*, Contributions to the theory of partial differential equations, pp. 69–94, Annals of Mathematics Studies No. 33, Princeton Univ. Press, Princeton, N. J., 1954.

3. L. Bers and A. Gelbart, *On a class of differential equations in mechanics of continua*, Quart. Appl. Math. **1** (1943), 168–188.

4. S. Bochner, *Sturm-Liouville and heat equations whose eigenfunctionsareultraspherical polynomials or associated Bessel functions*, Proceedings of the conference on differential equations, pp. 23–48, Univ. of Maryland, College Park, Md., 1955.

5. S. Bochner and K.Chandrasekharan, *Fourier transforms*, Annals of Mathematics Studies No. 19, Princeton Univ. Press, Princeton, N. J., 1949.

6. D. L. Guy, *Hankel multiplier transformations and weighted p-norms*, Trans. Amer. Math. Soc. **95** (1960), 137–189.

7. C. S. Herz, *On the mean inversion of Fourier and Hankel transforms*, Proc. Nat. Acad. Sci. U.S.A. **40** (1954), 996–999.

8. I. I. Hirschman, *The decomposition of Walsh and Fourier series*, Mem. Amer. Math. Soc. No. 15 (1955), 65 pp.

9. E. Hopf, *Elementare Bemerkungen über die Lösungen partieller Differentialgleichungen von elliptischen Typus*, S.-B. Berlin Math. Ges. (1927), 147–152.

10. J. Horvath, *Sur les fonctions conjuguées à plusieurs variables*, Nederl. Akad. Wetensch. Proc. Ser. A. **16** (1953), 17–29.

11. A. Huber, *On the uniqueness of generalized axially symmetric potentials*, Ann. of Math. (2) **60** (1954), 351–358.

12. R. O'Neil, *Convolution operators and $L(p, q)$ spaces*, Duke Math. J. **30** (1963), 129–142.

13. H. Pollard, *The mean convergence of orthogonal series of polynomials*, Proc. Nat. Acad Sci. U.S.A. **32** (1946), 8–10.

14 ———, *The mean convergence of orthogonal series*. II, Trans. Amer. Math. Soc. **63** 1948), 355–367.

15. E. M. Stein, *Note on singular integrals*, Proc. Amer. Math. Soc. **8** (1957), 250–254.

16 ———, *Conjugate harmonic functions in several variables*, pp. 414–420, Internat. Congress of Mathematicians, Stockholm, 1962.

17. E. M. Stein and G. Weiss, *On the theory of harmonic functions of several variables*. I, Acta Math. **106** (1961), 137–174.

18. G. Szegö, *Orthogonal polynomials*, Amer. Math. Soc. Colloq. Publ. Vol. 23, Amer. Math. Soc., Providence, R. I., 1939.

19. E. C. Titchmarsh, *Introduction to the theory of Fourier integrals*, Clarendon Press, Oxford, 1937.

20. I. N. Vekua, *Generalized analytic functions*, Pergamon Press, London, 1962 (translated from the Russian edition, Fizmatgiz, Moscow, 1959).

21. G. N. Watson, *Theory of Bessel functions*, Cambridge Univ. Press, Cambridge, 1922.

22. ———, *Notes on generating functions of polynomials*. III. *Polynomials of Legendre and Gegenbauer*, J. London Math. Soc. **8** (1933), 289–292.

23. A. Weinstein, *Discontinuous integrals and generalized potential theory*, Trans. Amer. Math. Soc. **63** (1948), 342–354.

24. ———, *Generalized axially symmetric potential theory*, Bull. Amer. Math. Soc. **59** (1953), 20–38.

25. G. M. Wing, *The mean convergence of orthogonal series*, Amer. J. Math. **72** (1950), 792–808.

26. A. Zygmund, *Trigonometric series*, 2nd. ed., Vols. I, II, Cambridge Univ. Press, New York, 1959.

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