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CLASSICAL FIFTH-, SIXTH-, SEVENTH-, AND EIGHTH-ORDER RUNGE-KUTTA FORMULAS WITH STEPSIZE CONTROL

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CLASSICAL FIFTH-, SIXTH-, SEVENTH-, AND EIGHTH-ORDER RUNGE-KUTTA FORMULAS WITH STEPSIZE CONTROL

INTRODUCTION

1. In two earlier papers [1], [2], the author has described a Runge-Kutta procedure which provides a stepsize control by one or two additional evaluations of the differential equations. This earlier procedure, requiring an m-fold differentiation and a suitable transformation of the differential equations, yielded (m+4)-th order Runge-Kutta formulas - as well as (m+5)-th order formulas for the purpose of stepsize control. The stepsize control was based on a complete coverage of the leading local truncation error term. The procedure required altogether six evaluations of the differential equations, regardless of m.

   Since for m=0 no differentiations have to be performed, our earlier formulas represent, in this special case, classical Runge-Kutta formulas of the fourth order, requiring six evaluations of the differential equations and including a complete coverage of the leading truncation error term for the purpose of stepsize control.

2. In this paper we shall derive classical Runge-Kutta formulas of the fifth, sixth, seventh, and eighth order including a stepsize control procedure which is again based on a complete coverage of the leading local truncation error term. Naturally, these new formulas require more evaluations per step of the differential equations than the known classical Runge-Kutta formulas without stepsize control procedure.

However, they require fewer evaluations per step than the known classical formulas if Richardson's extrapolation to the limit is used for such formulas as a stepsize control device. Since the application of Richardson's extrapolation to the limit means practically a doubling of the computational effort for the benefit of the stepsize control only, it is worthwhile to look for a less expensive stepsize control procedure.

Less expensive procedures have been suggested by different authors. However, these procedures generally do not make any effort to cover the truncation error, but rather try to estimate the truncation error from the last or the last few considered terms of the Taylor series. Since such a procedure has no mathematical base, these estimates are rather unreliable. Generally, since the terms in a convergent Taylor series are decreasing with increasing order, the last considered term will be larger than the leading truncation error term. Therefore, a stepsize control procedure based on the last or last few considered terms of the Taylor series will, in general, largely underestimate the permissible stepsize, thereby wasting computer time and building up unnecessarily large round-off errors.

3. The new formulas of our paper contain one or more free parameters. By a proper choice of these parameters the leading term of the local truncation error reduces substantially. This, in general, results in an increase of the permissible stepsize. This increase, together with the smaller number of evaluations per step, accounts for the superiority of our new formulas compared with the known Runge-Kutta formulas operated with Richardson's principle as stepsize control procedure.

4. Naturally, the new classical Runge-Kutta formulas of this paper, being of the eighth or lower order, are in general less economical than our earlier Runge-Kutta transformation formulas [1], [2] which represent Runge-Kutta formulas of any desired order. However, our new formulas have certain advantages, since they require no preparatory work (like differentiation of the differential equations) by the programmer. The new formulas including the stepsize control procedure can easily be written as a subroutine.

One might try to further raise the order of our new classical Runge-Kutta formulas hoping to make them still more economical. However, we believe that we have more or less reached the optimum with our eighth-order formula. The examples that we ran show that the gain of our eighth-order formula RK8(9) if compared with our seventh-order formula RK7(8) is not very substantial any more.

We made the same experience with a ninth- and a tenth-order Runge-Kutta formula that E.B. SHANKS has developed recently. These (not yet published) new formulas of SHANKS do not bring a substantial gain any more compared with SHANKS's eighth-order Runge-Kutta formula.

PART I. FIFTH-ORDER FORMULAS

SECTION I. THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

5. Although all formulas of this paper hold for systems of differential equations, we state them — for the sake of brevity — for a single differential equation

$$y' = f(x, y) \tag{1}$$

only.

In the customary way, we set:

$$\left.\begin{aligned} f_0 &= f(x_0,\ y_0) \\ f_\kappa &= f\left(x_0 + \alpha_\kappa h,\ y_0 + h \sum_{\lambda=0}^{\kappa-1} \beta_{\kappa\lambda} f_\lambda\right) \qquad (\kappa = 1, 2, 3, \ldots 7) \end{aligned}\right\} \tag{2}$$

and we require:

$$\left.\begin{aligned} y &= y_0 + h \sum_{\kappa=0}^{5} c_\kappa f_\kappa + 0(h^6) \\ \hat{y} &= y_0 + h \sum_{\kappa=0}^{7} \hat{c}_\kappa f_\kappa + 0(h^7) \end{aligned}\right\} \tag{3}$$

with h as integration step size.

Equations (3) mean that we try to determine the coefficients α_κ, $\beta_{\kappa\lambda}$, c_κ, $\hat{c}_\kappa$ in such a way, that the first formula (3) represents a fifth-order and the second formula (3) a sixth-order Runge-Kutta formula. According to (2) and (3), this means that the coefficients α_κ and $\beta_{\kappa\lambda}$ have to be the same in both formulas for the first six evaluations of f. This restriction explains why we shall need eight evaluations (instead of seven) for our sixth-order formula.

6. Since the equations of condition for the Runge-Kutta coefficients, resulting from Taylor expansions of (2) and (3), are well known in the literature, we restrict ourselves to stating these equations. For Runge-Kutta formulas up to the eighth order these equations — in condensed form — are listed in a paper by J.C. BUTCHER ([3], Table 1). For the convenience of the reader, we list these equations for a sixth-order formula such as the second formula (3) in the customary summation form. However, since these equations then become somewhat lengthy, we introduce the following abbreviation:

$$\beta_{\kappa 1}\alpha_1^\lambda + \beta_{\kappa 2}\alpha_2^\lambda + \ldots . + \beta_{\kappa\kappa-1}\alpha_{\kappa-1}^\lambda = P_{\kappa\lambda} \quad (\kappa = 2, 3, \ldots, 7; \lambda = 1, 2, 3) \tag{4}$$

The 37 equations of condition for our sixth-order formula, listed in the same order as in BUTCHER's paper, then read:

TABLE I. EQUATIONS OF CONDITION FOR SIXTH-ORDER FORMULA

(I, 1) $\sum_{\kappa=0}^{7} \hat{c}_\kappa - 1 = 0$

(II, 1) $\sum_{\kappa=1}^{7} \hat{c}_\kappa \alpha_\kappa - \frac{1}{2} = 0$

(III, 1) $\sum_{\kappa=2}^{7} \hat{c}_\kappa P_{\kappa 1} - \frac{1}{6} = 0$

(III, 2) $\frac{1}{2} \sum_{\kappa=1}^{7} \hat{c}_\kappa \alpha_\kappa^2 - \frac{1}{6} = 0$

(IV, 1) $\sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right) - \frac{1}{24} = 0$

(IV, 2) $\frac{1}{2} \sum_{\kappa=2}^{7} \hat{c}_\kappa P_{\kappa 2} - \frac{1}{24} = 0$

TABLE I. (Continued)

$$(\mathrm{IV},3)\quad \sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa P_{\kappa 1} - \frac{1}{8} = 0$$

$$(\mathrm{IV},4)\quad \frac{1}{6} \sum_{\kappa=1}^{7} \hat{c}_\kappa \alpha_\kappa^3 - \frac{1}{24} = 0$$

$$(\mathrm{V},1)\quad \sum_{\kappa=4}^{7} \hat{c}_\kappa \left[\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{\mu=2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1} \right) \right] - \frac{1}{120} = 0$$

$$(\mathrm{V},2)\quad \frac{1}{2} \sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2} \right) - \frac{1}{120} = 0$$

$$(\mathrm{V},3)\quad \sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} \alpha_\lambda P_{\lambda 1} \right) - \frac{1}{40} = 0$$

$$(\mathrm{V},4)\quad \frac{1}{6} \sum_{\kappa=2}^{7} \hat{c}_\kappa P_{\kappa 3} - \frac{1}{120} = 0$$

$$(\mathrm{V},5)\quad \sum_{\kappa=3}^{7} \hat{c}_\kappa \alpha_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1} \right) - \frac{1}{30} = 0$$

$$(\mathrm{V},6)\quad \frac{1}{2} \sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa P_{\kappa 2} - \frac{1}{30} = 0$$

$$(\mathrm{V},7)\quad \frac{1}{2} \sum_{\kappa=2}^{7} \hat{c}_\kappa P_{\kappa 1}^2 - \frac{1}{40} = 0$$

$$(\mathrm{V},8)\quad \frac{1}{2} \sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa^2 P_{\kappa 1} - \frac{1}{20} = 0$$

$$(\mathrm{V},9)\quad \frac{1}{24} \sum_{\kappa=1}^{7} \hat{c}_\kappa \alpha_\kappa^4 - \frac{1}{120} = 0$$

$$(\mathrm{VI},1)\quad \sum_{\kappa=5}^{7} \hat{c}_\kappa \left\{ \sum_{\lambda=4}^{\kappa-1} \beta_{\kappa\lambda} \left[\sum_{\mu=3}^{\lambda-1} \beta_{\lambda\mu} \left(\sum_{\nu=2}^{\mu-1} \beta_{\mu\nu} P_{\nu 1} \right) \right] \right\} - \frac{1}{720} = 0$$

$$(\mathrm{VI},2)\quad \frac{1}{2} \sum_{\kappa=4}^{7} \hat{c}_\kappa \left[\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{\mu=2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2} \right) \right] - \frac{1}{720} = 0$$

TABLE I. (Continued)

$$(\mathrm{VI},3)\quad \sum_{\kappa=4}^{7} \hat{c}_\kappa \left[\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_{\mu=2}^{\lambda-1} \beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] - \frac{1}{240} = 0$$

$$(\mathrm{VI},4)\quad \frac{1}{6}\sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3}\right) - \frac{1}{720} = 0$$

$$(\mathrm{VI},5)\quad \sum_{\kappa=4}^{7} \hat{c}_\kappa \left[\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda\left(\sum_{\mu=2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] - \frac{1}{180} = 0$$

$$(\mathrm{VI},6)\quad \frac{1}{2}\sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda P_{\lambda 2}\right) - \frac{1}{180} = 0$$

$$(\mathrm{VI},7)\quad \frac{1}{2}\sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P^2_{\lambda 1}\right) - \frac{1}{240} = 0$$

$$(\mathrm{VI},8)\quad \frac{1}{2}\sum_{\kappa=3}^{7} \hat{c}_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda}\alpha^2_\lambda P_{\lambda 1}\right) - \frac{1}{120} = 0$$

$$(\mathrm{VI},9)\quad \frac{1}{24}\sum_{\kappa=2}^{7} \hat{c}_\kappa P_{\kappa 4} - \frac{1}{720} = 0$$

$$(\mathrm{VI},10)\quad \sum_{\kappa=4}^{7} \hat{c}_\kappa \alpha_\kappa \left[\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda}\left(\sum_{\mu=2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 1}\right)\right] - \frac{1}{144} = 0$$

$$(\mathrm{VI},11)\quad \frac{1}{2}\sum_{\kappa=3}^{7} \hat{c}_\kappa \alpha_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 2}\right) - \frac{1}{144} = 0$$

$$(\mathrm{VI},12)\quad \sum_{\kappa=3}^{7} \hat{c}_\kappa \alpha_\kappa \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda}\alpha_\lambda P_{\lambda 1}\right) - \frac{1}{48} = 0$$

$$(\mathrm{VI},13)\quad \frac{1}{6}\sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa P_{\kappa 3} - \frac{1}{144} = 0$$

$$(\mathrm{VI},14)\quad \sum_{\kappa=3}^{7} \hat{c}_\kappa P_{\kappa 1} \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right) - \frac{1}{72} = 0$$

$$(\mathrm{VI},15)\quad \frac{1}{2}\sum_{\kappa=2}^{7} \hat{c}_\kappa P_{\kappa 1} P_{\kappa 2} - \frac{1}{72} = 0$$

TABLE I. (Concluded)

$$\text{(VI, 16)} \quad \frac{1}{2}\sum_{\kappa=3}^{7} \hat{c}_\kappa \alpha_\kappa^2 \left(\sum_{\lambda=2}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 1}\right) - \frac{1}{72} = 0$$

$$\text{(VI, 17)} \quad \frac{1}{4}\sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa^2 P_{\kappa 2} - \frac{1}{72} = 0$$

$$\text{(VI, 18)} \quad \frac{1}{2}\sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa P_{\kappa 1}^2 - \frac{1}{48} = 0$$

$$\text{(VI, 19)} \quad \frac{1}{6}\sum_{\kappa=2}^{7} \hat{c}_\kappa \alpha_\kappa^3 P_{\kappa 1} - \frac{1}{72} = 0$$

$$\text{(VI, 20)} \quad \frac{1}{120}\sum_{\kappa=1}^{7} \hat{c}_\kappa \alpha_\kappa^5 - \frac{1}{720} = 0$$

The Roman numerals in front of the equations in Table I indicate the order of the terms in the Taylor expansion.

A similar table holds for a fifth-order formula such as the first formula (3). We obtain this table from Table I by omitting the sixth-order equations (VI, 1) through (VI, 20), and replacing, in the remaining equations, $\hat{c}_\kappa$ with c_κ and the upper limit 7 of the κ-sums with 5.

Naturally, all equations of this new table for a fifth-order formula as well as those of Table I have to be satisfied simultaneously.

7. The equations of Table I which represent necessary and sufficient conditions for a sixth-order Runge-Kutta formula, can, however, be replaced by a much simpler system of sufficient condition.

First, we make the assumptions:

$$\alpha_5 = \alpha_7 = 1, \ \alpha_6 = 0; \ \hat{c}_1 = c_1 = 0, \ \hat{c}_2 = c_2, \ \hat{c}_3 = c_3, \ \hat{c}_4 = c_4, \ \hat{c}_5 = 0, \ \hat{c}_6 = \hat{c}_7 = c_5 \qquad (5)$$

which greatly facilitate the simultaneous solution of both systems of equations (for fifth-order and sixth-order Runge-Kutta formulas). Then, we introduce the following assumptions (A), (B), (C) which are well-known to reduce our equations of condition to a great extent:

$$
\text{(A)}\left\{\begin{array}{llll}
(*)\; P_{21} = \frac{1}{2}\alpha_2^2 & \text{(A2)} & (*)\; P_{51} = \frac{1}{2}\alpha_5^2 = \frac{1}{2} & \text{(A5)} \\
(*)\; P_{31} = \frac{1}{2}\alpha_3^2 & \text{(A3)} & P_{61} = \frac{1}{2}\alpha_6^2 = 0 & \text{(A6)} \\
(*)\; P_{41} = \frac{1}{2}\alpha_4^2 & \text{(A4)} & P_{71} = \frac{1}{2}\alpha_7^2 = \frac{1}{2} & \text{(A7)}
\end{array}\right.
$$

$$
\text{(B)}\left\{\begin{array}{lrl}
(*) & c_2\beta_{21} + c_3\beta_{31} + c_4\beta_{41} + c_5\left\{\begin{matrix}\beta_{51} \\ \beta_{61} + \beta_{71}\end{matrix}\right\} = c_1(1 - \alpha_1) = 0 & \text{(B1)} \\
(*) & c_3\beta_{32} + c_4\beta_{42} + c_5\left\{\begin{matrix}\beta_{52} \\ \beta_{62} + \beta_{72}\end{matrix}\right\} = c_2(1 - \alpha_2) & \text{(B2)} \\
(*) & c_4\beta_{43} + c_5\left\{\begin{matrix}\beta_{53} \\ \beta_{63} + \beta_{73}\end{matrix}\right\} = c_3(1 - \alpha_3) & \text{(B3)} \\
(*) & c_5\left\{\begin{matrix}\beta_{54} \\ \beta_{64} + \beta_{74}\end{matrix}\right\} = c_4(1 - \alpha_4) & \text{(B4)} \\
 & c_5\,(\beta_{65} + \beta_{75}) = c_5(1 - \alpha_5) = 0 & \text{(B5)} \\
 & c_5\beta_{76} = c_5 & \text{(B6)}
\end{array}\right.
$$

$$
\text{(C)}\left\{\begin{array}{lll}
(*) & c_2\alpha_2\beta_{21} + c_3\alpha_3\beta_{31} + c_4\alpha_4\beta_{41} + c_5\left\{\begin{matrix}\beta_{51} \\ \beta_{71}\end{matrix}\right\} = 0 & \text{(C1)} \\
 & c_2\alpha_2^2\beta_{21} + c_3\alpha_3^2\beta_{31} + c_4\alpha_4^2\beta_{41} + c_5\beta_{71} = 0 & \text{(C2)} \\
 & c_3\alpha_3\beta_{32}\beta_{21} + c_4\alpha_4(\beta_{42}\beta_{21} + \beta_{43}\beta_{31}) + c_5(\beta_{72}\beta_{21} + \beta_{73}\beta_{31} + \beta_{74}\beta_{41} & \\
 & \qquad + \beta_{75}\beta_{51} + \beta_{76}\beta_{61}) = 0 \; . & \text{(C3)}
\end{array}\right.
$$

The asterisk (*) in front of some of the equations indicates that these equations hold for the fifth-order formulas as well as for the sixth-order formulas. If the equations marked by an asterisk are split in two lines, the top line holds for the fifth-order formulas and the bottom line for the sixth-order formulas. The equations without an asterisk are required for the sixth-order formulas only.

8. Inserting the assumptions (A) into Table I, one immediately finds the following identities:

$$\left.\begin{array}{l}(III, 1) \equiv (III, 2);\ (IV, 3) \equiv 3(IV, 4);\ (V, 7) \equiv 3(V, 9); \\ (V, 8) \equiv 6(V, 9);\ (VI, 8) \equiv 2(VI, 7);\ (VI, 14) \equiv (VI, 16); \\ (VI, 15) \equiv (VI, 17);\ (VI, 18) \equiv 15(VI, 20);\ (VI, 19) \equiv 10(VI, 20).\end{array}\right\} \quad (6)$$

Therefore, the equations on the left-hand sides of the identities (6) can be omitted from Table I.

Using also assumption (B1), three more identies are obtained:

$$(IV, 1) \equiv (IV, 2);\quad (V, 3) \equiv 3(V, 4);\quad (VI, 7) \equiv 3(VI, 9); \qquad (7)$$

thus eliminating equations (VI, 1), (V, 3), (VI, 7) from Table I.

The assumptions (B) lead to the following identities:

$$\left.\begin{array}{ll}(IV, 2) \equiv (III, 2) - 3(IV, 4); & (V, 1) \equiv (IV, 1) - (V, 5); \\ (V, 2) \equiv (IV, 2) - (V, 6); & (V, 4) \equiv (IV, 4) - 4(V, 9); \\ (VI, 1) \equiv (V, 1) - (VI, 10); & (VI, 2) \equiv (V, 2) - (VI, 11); \\ (VI, 4) \equiv (V, 4) - (VI, 13); & (VI, 5) \equiv (V, 5) - 2(VI, 16); \\ (VI, 6) \equiv (V, 6) - 2(VI, 17); & (VI, 9) \equiv (V, 9) - 5(VI, 20);\end{array}\right\} \quad (8)$$

thus eliminating ten more equations from Table I.

Finally, we make use also of the assumptions (C) and obtain the following identities:

$$\left.\begin{array}{l} (V,5) \equiv (V,6);\ \ (VI,3) \equiv 3(V,4) - 3(VI,13); \\ (VI,10) \equiv (VI,11);\ \ (VI,12) \equiv 3(VI,13); \\ (VI,16) \equiv (VI,17)\ . \end{array}\right\} \qquad (9)$$

9. Cancelling all equations listed on the left-hand sides of the identities (6), (7), (8), (9), Table I reduces to the following ten equations:

$$\left.\begin{array}{l} (I,1),\ (II,1)\ (III,2),\ (IV,4),\ (V,6),\ (V,9),\ (VI,11),\ (VI,13), \\ (VI,17),\ (VI,20)\ . \end{array}\right\} \qquad (10)$$

We arrange these remaining equations in two groups,(D) and (E), the former one not containing any β-coefficients. We obtain:

$$(D)\ \left\{\begin{array}{llr} (*) & \left\{\begin{matrix} c_o \\ \hat{c}_o \end{matrix}\right\} + c_2 + c_3 + c_4 + \left\{\begin{matrix} c_5 \\ 2c_5 \end{matrix}\right\} = 1 & (D0) \\ (*) & c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4 + c_5 = \frac{1}{2} & (D1) \\ (*) & c_2\alpha_2^2 + c_3\alpha_3^2 + c_4\alpha_4^2 + c_5 = \frac{1}{3} & (D2) \\ (*) & c_2\alpha_2^3 + c_3\alpha_3^3 + c_4\alpha_4^3 + c_5 = \frac{1}{4} & (D3) \\ (*) & c_2\alpha_2^4 + c_3\alpha_3^4 + c_4\alpha_4^4 + c_5 = \frac{1}{5} & (D4) \\ & c_2\alpha_2^5 + c_3\alpha_3^5 + c_4\alpha_4^5 + c_5 = \frac{1}{6} & (D5) \end{array}\right.$$

and:

$$
\left\{
\begin{aligned}
&(*)\quad c_2\alpha_2P_{22} + c_3\alpha_3P_{32} + c_4\alpha_4P_{42} + c_5 \begin{Bmatrix} P_{52} \\ P_{72} \end{Bmatrix} = \frac{1}{15} \\
&c_2\alpha_2P_{23} + c_3\alpha_3P_{33} + c_4\alpha_4P_{43} + c_5P_{73} = \frac{1}{24} \\
&c_2\alpha_2^2P_{22} + c_3\alpha_3^2P_{32} + c_4\alpha_4^2P_{42} + c_5P_{72} = \frac{1}{18} \\
&c_3\alpha_3\ \beta_{32}\ P_{22} + c_4\alpha_4(\beta_{42}P_{22} + \beta_{43}P_{32}) + \\
&c_5(\beta_{72}P_{22} + \beta_{73}P_{32} + \beta_{74}P_{42} + \beta_{75}P_{52} + \beta_{76}P_{62}) = \frac{1}{72} \quad .
\end{aligned}
\right.
$$

Using (C) and introducing the abbreviations:

$$
p_{\kappa\lambda} = P_{\kappa\lambda} - \beta_{\kappa 1}\alpha_1^{\lambda} \qquad (\kappa = 2, 3, 4, 5, 6, 7;\ \lambda = 1, 2, 3) \tag{11}
$$

the last group of equations reduces to:

$$
(E)\left\{
\begin{aligned}
&(*)\quad c_3\alpha_3p_{32} + c_4\alpha_4p_{42} + c_5 \begin{Bmatrix} p_{52} \\ p_{72} \end{Bmatrix} = \frac{1}{15} && (E1) \\
&c_3\alpha_3p_{33} + c_4\alpha_4p_{43} + c_5p_{73} = \frac{1}{24} && (E2) \\
&c_3\alpha_3^2p_{32} + c_4\alpha_4^2p_{42} + c_5p_{72} = \frac{1}{18} && (E3) \\
&c_4\alpha_4\beta_{43}p_{32} + c_5(\beta_{73}p_{32} + \beta_{74}p_{42} + \beta_{75}p_{52} + \beta_{76}p_{62}) = \frac{1}{72} \quad . && (E4)
\end{aligned}
\right.
$$

Finally, the coefficients $\beta_{\kappa 0}(\kappa = 1, 2, 3, \ldots, 7)$ are obtained from the well-known equations:

$$
(\mathrm{F})\left\{\begin{array}{lll}
(*) & \beta_{10} = \alpha_1 & \text{(F1)} \\
(*) & \beta_{20} + \beta_{21} = \alpha_2 & \text{(F2)} \\
(*) & \beta_{30} + \beta_{31} + \beta_{32} = \alpha_3 & \text{(F3)} \\
(*) & \beta_{40} + \beta_{41} + \beta_{42} + \beta_{43} = \alpha_4 & \text{(F4)} \\
(*) & \beta_{50} + \beta_{51} + \beta_{52} + \beta_{53} + \beta_{54} = \alpha_5 = 1 & \text{(F5)} \\
 & \beta_{60} + \beta_{61} + \beta_{62} + \beta_{63} + \beta_{64} + \beta_{65} = \alpha_6 = 0 & \text{(F6)} \\
 & \beta_{70} + \beta_{71} + \beta_{72} + \beta_{73} + \beta_{74} + \beta_{75} + \beta_{76} = \alpha_7 = 1 & \text{(F7)}
\end{array}\right.
$$

The asterisks (*) in (D), (E), (F) have the same meaning as in equations (A), (B), (C). In the following section we shall show how the sufficient equations of condition (D) and (E) can be solved, taking into account the assumptions (A), (B), (C).

SECTION II. A SOLUTION OF THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

10. When solving our equations of condition, we shall try to express the β- and c- coefficients as functions of the α-coefficients. However, not all four coefficients α_1, α_2, α_3, α_4 are independent of one another. In the following two numbers we shall show that α_4 and α_3 can be expressed by α_2. Therefore, we have only two independent α-coefficients; namely α_1 and α_2. Our β- and c- coefficients will then be expressible by α_1 and α_2 only.

11. To find the relation between α_4 and α_2, we form:

$$c_3(1-\alpha_3)\,P_{31} + c_4(1-\alpha_4)P_{41}\ . \tag{12}$$

Using the assumptions (A), (B), (C), (D), we obtain from (12):

$$c_3(1-\alpha_3)\beta_{32}\,\alpha_2 + c_4(1-\alpha_4)\,(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) = \frac{1}{24}\ . \tag{13}$$

We further form the expression:

$$\alpha_2^2\cdot(\mathrm{B2}) + \alpha_3^2\cdot(\mathrm{B3}) + \alpha_4^2\cdot(\mathrm{B4}) - (\mathrm{E1})\ , \tag{14}$$

thereby eliminating the term $c_5(\beta_{52}\alpha_2^2 + \beta_{53}\alpha_3^2 + \beta_{54}\alpha_4^2)$ in (E1) and obtaining:

$$c_3(1-\alpha_3)\beta_{32}\alpha_2^2 + c_4(1-\alpha_4)\,(\beta_{42}\alpha_2^2 + \beta_{43}\alpha_3^2) = \frac{1}{60}\ . \tag{15}$$

We now eliminate the terms with c_3 in (13) and (15) and find:

$$c_4(1-\alpha_4)\,\beta_{43}\alpha_3\,(\alpha_3 - \alpha_2) = \frac{1}{60} - \frac{1}{24}\,\alpha_2\ . \tag{16}$$

We establish a second relation between c_4, β_{43} and the α- coefficients. We start by forming:

$$c_3\,\alpha_3\,P_{31} + c_4\,\alpha_4\,P_{41} + c_5 P_{51} \tag{17}$$

Using the assumptions (A), (B), (D), we find from (17):

$$c_3\alpha_3\beta_{32}\,\alpha_2 + c_4\alpha_4(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) + c_5(\beta_{52}\alpha_2 + \beta_{53}\alpha_3 + \beta_{54}\alpha_4) = \frac{1}{8} \tag{18}$$

By forming:

$$c_3\,\alpha_3^2\,P_{31} + c_4\,\alpha_4^2\,P_{41} + c_5\,P_{51} \tag{19}$$

and using (A), (B), (D), we find in the same way:

$$c_3\alpha_3^2\,\beta_{32}\alpha_2 + c_4\alpha_4^2(\beta_{42}\alpha_2 + \beta_{43}\alpha_3) + c_5(\beta_{52}\alpha_2 + \beta_{53}\alpha_3 + \beta_{54}\alpha_4) = \frac{1}{10}\,. \tag{20}$$

We finally form:

$$(\mathrm{E}3) - \alpha_2 \cdot (20) + \alpha_2 \cdot (18) - (\mathrm{E}1), \tag{21}$$

thereby eliminating the terms with c_3 and with c_5. As second relation between c_4, β_{43} and the α-coefficients, we then obtain:

$$c_4\beta_{43}\,\alpha_3\,\alpha_4\,(\alpha_3 - \alpha_2)\,(1 - \alpha_4) = \frac{1}{90} - \frac{1}{40}\,\alpha_2\,. \tag{22}$$

From the two relations (16) and (22), we find as a relation between α_4 and α_2:

$$\alpha_4 = \frac{1}{3} \cdot \frac{9\alpha_2 - 4}{5\alpha_2 - 2}\,. \tag{23}$$

12. A further relation between α_2, α_3, and α_4 can be obtained from (D1), (D2), (D3), (D4), and (D5) as condition of compatibility. Eliminating from these equations the terms with c_5, c_2, c_4, c_3, in this order, leads to:

$$\alpha_3 = \frac{5\alpha_2\alpha_4 - 3(\alpha_2 + \alpha_4) + 2}{10\alpha_2\alpha_4 - 5(\alpha_2 + \alpha_4) + 3} \quad . \tag{24}$$

Introducing (23) in (24) we find as a relation between α_3 and α_2:

$$\alpha_3 = \frac{\alpha_2}{15\alpha_2^2 - 10\alpha_2 + 2} \quad . \tag{25}$$

13. The weight factors c_κ (or $\hat{c}_\kappa$) are now obtained from equations (D1), (D2), (D3), (D4):

$$\left.\begin{aligned}
c_2 &= \hat{c}_2 = \frac{1}{180} \cdot \frac{45\alpha_2^2 - 30\alpha_2 + 4}{(15\alpha_2^2 - 10\alpha_2 + 2)\ (5\alpha_2 - 2)\ \alpha_2\ (\alpha_3 - \alpha_2)\ (\alpha_4 - \alpha_2)\ (1 - \alpha_2)} \\
c_3 &= \hat{c}_3 = \frac{1}{180} \cdot \frac{15\alpha_2^2 - 10\alpha_2 + 2}{(5\alpha_2 - 2)\ \alpha_3\ (\alpha_2 - \alpha_3)\ (\alpha_4 - \alpha_3)\ (1 - \alpha_3)} \\
c_4 &= \hat{c}_4 = -\frac{1}{20} \cdot \frac{(5\alpha_2^2 - 5\alpha_2 + 1)\ (5\alpha_2 - 2)}{(15\alpha_2^2 - 10\alpha_2 + 2)\ \alpha_4\ (\alpha_2 - \alpha_4)\ (\alpha_3 - \alpha_4)\ (1 - \alpha_4)} \\
c_5 &= \hat{c}_6 = \hat{c}_7 = \frac{1}{2} - (c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4)
\end{aligned}\right\} \tag{26}$$

14. From the two equations (E1), it follows:

$$p_{72} = p_{52} \tag{27}$$

Multiplying equations (B2), (B3), (B4) by α_2^2, α_3^2, α_4^2, respectively, adding these three equations, and using (D2) and (D3), leads to:

$$c_3 p_{32} + c_4 p_{42} + c_5 p_{52} = \frac{1}{12} \quad . \tag{28}$$

The three equations (28), (E1), (E3) represent a system of three linear equations for the three unknowns p_{32}, p_{42}, p_{52}. Its solution reads:

$$\left.\begin{aligned} p_{32} &= \frac{1}{180} \cdot \frac{3\alpha_4 - 2}{c_3(\alpha_4 - \alpha_3)\ (1 - \alpha_3)} \\ p_{42} &= \frac{1}{180} \cdot \frac{3\alpha_3 - 2}{c_4(\alpha_3 - \alpha_4)\ (1 - \alpha_4)} \\ p_{52} &= \frac{1}{180} \cdot \frac{15\alpha_3\alpha_4 - 12(\alpha_3 + \alpha_4) + 10}{c_5(1 - \alpha_3)\ (1 - \alpha_4)} \quad . \end{aligned}\right\} \tag{29}$$

15. The four equations (A2), (A3), (A4), (A5) yield:

$$\left.\begin{aligned} \beta_{21}\alpha_1 &= \frac{1}{2}\ \alpha_2^2 \\ \beta_{31}\alpha_1 &= \frac{1}{2}\ \alpha_3^2 - p_{31} \\ \beta_{41}\alpha_1 &= \frac{1}{2}\ \alpha_4^2 - p_{41} \\ \beta_{51}\alpha_1 &= \frac{1}{2} - \frac{1}{c_5}\left(\frac{1}{6} - c_3 p_{31} - c_4 p_{41}\right) \quad . \end{aligned}\right\} \tag{30}$$

The last equation (30) is obtained using equations (B2), (B3), (B4) and equations (D1), (D2).

Introducing (30) into equation (C1) results in the following expression for p_{41}:

$$p_{41} = -\frac{c_3 p_{31}(1-\alpha_3) - \frac{1}{24}}{c_4(1-\alpha_4)} \tag{31}$$

Since β_{32} is already known from the first equation (29), equation (31) and the second equation (29) represent two equations for the two unknowns β_{42}, β_{43}. Their solution reads:

$$\left.\begin{aligned} c_4\beta_{42} &= \frac{15\alpha_2\alpha_3(\alpha_4-\alpha_3) + 4(\alpha_3-\alpha_2) - 6\alpha_3(\alpha_4-\alpha_2)}{360\alpha_2^2(\alpha_3-\alpha_2)(\alpha_4-\alpha_3)(1-\alpha_4)} \\ c_4\beta_{43} &= -\frac{1}{120}\cdot\frac{5\alpha_2 - 2}{\alpha_3(\alpha_3-\alpha_2)(1-\alpha_4)} \end{aligned}\right\} \tag{32}$$

From equations (B2), (B3), (B4), it now follows:

$$\left.\begin{aligned} c_5\beta_{52} &= c_2(1-\alpha_2) - c_3\beta_{32} - c_4\beta_{42} \\ c_5\beta_{53} &= c_3(1-\alpha_3) - c_4\beta_{43} \\ c_5\beta_{54} &= c_4(1-\alpha_4) \end{aligned}\right\} \tag{33}$$

It can easily be shown that the coefficients β_{52}, β_{53}, β_{54}, as obtained from (33), satisfy the last equation (29).

Since the coefficients $\beta_{21}\alpha_1$, $\beta_{31}\alpha_1$. $\beta_{41}\alpha_1$, $\beta_{51}\alpha_1$ are obtainable from (30), all Runge-Kutta coefficients for our fifth order formula are now determined as functions of the parameters α_1 and α_2.

16. We still have to find the coefficients $\beta_{6\lambda}$ and $\beta_{7\lambda}$ for our sixth-order formula which is required for the stepsize control.

From the two equations (C1) it follows:

$$\beta_{71} = \beta_{51} \ . \tag{34}$$

From (B6) we obtain:

$$\beta_{76} = 1 \ . \tag{35}$$

Equation (B5) suggests:

$$\beta_{65} = \beta_{75} = 0 \ . \tag{36}$$

From equations (A7), (E1), (E2), the following expressions for p_{71}, p_{72}, p_{73} are obtained:

$$\left.\begin{aligned}
c_5 p_{71} &= \frac{1}{360} \cdot \frac{60\alpha_2(1-\alpha_3)(1-\alpha_4) + 2(3\alpha_4 - 2) - 15\alpha_2(1-\alpha_3)}{\alpha_2(1-\alpha_3)(1-\alpha_4)} \\
c_5 p_{72} &= \frac{1}{180} \cdot \frac{15\alpha_3\alpha_4 - 12(\alpha_3+\alpha_4) + 10}{(1-\alpha_3)(1-\alpha_4)} \\
c_5 p_{73} &= \frac{1}{360} \cdot \frac{15(1-\alpha_3)(1-\alpha_4) + 15\alpha_2\alpha_3\alpha_4(1-\alpha_3) - 6\alpha_3\alpha_4(1-\alpha_2-\alpha_3) - 4\alpha_2}{(1-\alpha_3)(1-\alpha_4)}
\end{aligned}\right\} \tag{37}$$

Equations (37) yield the following solution for β_{74}, β_{73}, β_{72}:

$$
\left.
\begin{aligned}
c_5\beta_{74} &= -\frac{1}{60}\cdot\frac{(12\alpha_2-5)(5\alpha_2^2-5\alpha_2+1)}{(15\alpha_2^2-10\alpha_2+2)\alpha_4(\alpha_2-\alpha_4)(\alpha_3-\alpha_4)} \\
c_5\beta_{73} &= \frac{1}{\alpha_3(\alpha_3-\alpha_2)}\left[\frac{60\alpha_2\alpha_4-45\alpha_2-30\alpha_4+24}{360(1-\alpha_4)} - c_5\beta_{74}\alpha_4(\alpha_4-\alpha_2)\right] \\
c_5\beta_{72} &= \frac{1}{\alpha_2^2}\left[\frac{15\alpha_3\alpha_4-12(\alpha_3+\alpha_4)+10}{180(1-\alpha_3)(1-\alpha_4)} - c_5(\beta_{73}\alpha_3^2-\beta_{74}\alpha_4^2)\right]
\end{aligned}
\right\} \quad (38)
$$

Finally, equations (B2), (B3), (B4) yield for the coefficients β_{62}, β_{63}, β_{64}:

$$
\left.
\begin{aligned}
c_5\beta_{62} &= c_2(1-\alpha_2) - c_3\beta_{32} - c_4\beta_{42} - c_5\beta_{72} \\
c_5\beta_{63} &= c_3(1-\alpha_3) - c_4\beta_{43} - c_5\beta_{73} \\
c_5\beta_{64} &= c_4(1-\alpha_4) - c_5\beta_{74}
\end{aligned}
\right\} \quad (39)
$$

17. It can be verified that the Runge-Kutta coefficients, as obtained in this section, satisfy indeed all our equations of condition (A) through (F), if c_0, $\hat{c}_0$, β_{10}, β_{20}, ..., β_{70} are determined from (D0) and (F). From the derivation of our Runge-Kutta coefficients it is not obvious that they satisfy equations (C3) and (E4), since these two equations have not been used in the derivation. However, when expressing all coefficients by α_2 explicitly, it can be shown that the two equations (C3) and (E4) become identities in α_2.

SECTION III. THE LEADING TERM OF THE LOCAL TRUNCATION ERROR

18. The leading truncation error term of our fifth-order Runge-Kutta formula is a sum of twenty members, each being of the form:

$$T_\nu \cdot [\ldots.]_\nu \cdot h^5 \tag{40}$$

The T_ν are numerical constants that are characteristic for the individual Runge-Kutta formula under consideration; they are, however, independent of the differential equation (1). The bracket in (40) contains certain combinations of partial derivatives of the right-hand side of (1). The twenty T_ν- values, expressed by the Runge-Kutta coefficients of Section II, can be obtained from the left-hand sides of equations (VI,1) through (VI,20) of Table I, if, in these equations, we replace $\hat{c}_\kappa$ by c_κ and the upper limit 7 of the κ- sums by 5.

If we list these T_ν - values in the same order as in Table I, the first T_ν read:

$$\left.\begin{aligned}
T_1 &= c_5\beta_{54}\beta_{43}\beta_{32}\beta_{21}\alpha_1 - \frac{1}{720}\\
T_2 &= \frac{1}{2}\sum_{\kappa=4}^{5} c_\kappa\left[\sum_{\lambda=3}^{\kappa-1}\beta_{\kappa\lambda}\left(\sum_{\mu=2}^{\lambda-1}\beta_{\lambda\mu}P_{\mu 2}\right)\right] - \frac{1}{720}\\
T_3 &= \sum_{\kappa=4}^{5} c_\kappa\left[\sum_{\lambda=3}^{\kappa-1}\beta_{\kappa\lambda}\left(\sum_{\mu=2}^{\lambda-1}\beta_{\lambda\mu}\alpha_\mu P_{\mu 1}\right)\right] - \frac{1}{240}
\end{aligned}\right\} \tag{41}$$

$$\vdots$$

19. Naturally, it will be desirable to establish Runge-Kutta formulas with small T_ν - values. If we succeed in making all T_ν - values sufficiently small, the sum of the terms (40) representing the leading term of the local truncation error can also be expected to be reasonably small.

Due to the special structure of our Runge-Kutta formulas (assumptions (5)), the majority of our T_ν-values are zero, as we shall see immediately.

Because of (5), (27), (34), the equations of condition (C2), (D5), (E3) hold for fifth-order Runge-Kutta formulas also. Since (D5) and (E3) correspond to equations (VI, 20) and (VI, 17) of Table I, we have $T_{20} = 0$ and $T_{17} = 0$. From $T_{20} = 0$ we obtain, using (6), (7), (8): $T_{19} = 0$, $T_{18} = 0$, $T_9 = 0$, $T_7 = 0$, $T_8 = 0$. Correspondingly, from $T_{17} = 0$ we find, using (6), (8), (9): $T_{15} = 0$, $T_6 = 0$, $T_{16} = 0$, $T_5 = 0$, $T_{14} = 0$.

By a proper choice of the parameter α_1 we can also make $T_{11} = 0$. The straightforward computation, using the results of Section I and II, shows that $T_{11} = 0$ requires

$$\alpha_1 = \frac{2}{15} \cdot \frac{5\alpha_2 - 1}{\alpha_2} \quad . \tag{42}$$

From the identity $(VI, 2) \equiv (V, 2) - (VI, 11)$ in (8), it follows that $T_{11} = 0$ leads to $T_2 = 0$ also. Therefore, altogether, fourteen T_ν-values of the leading truncation error term are zero.

20. We now compute T_{10} using the results of Sections I and II. Expressing T_{10} by the only free parameter α_2, the computation yields

$$T_{10} = \frac{1}{360} \cdot \frac{(3\alpha_2 - 1)\,(5\alpha_2 - 1)}{15\alpha_2^2 - 10\alpha_2 + 2} \quad , \tag{43}$$

and from the identity (VI, 1) $\equiv$ (V, 1) - (VI, 10) in (8), it follows:

$$T_1 = - T_{10} \ . \tag{44}$$

21. We still have to determine T_3, T_4, T_{12}, T_{13}. Again, a straightforward computation, using the results of Sections I and II, leads to:

$$T_{13} = - \frac{1}{360} \cdot \frac{(3\alpha_2 - 1)\,(5\alpha_2^2 - 5\alpha_2 + 1)}{15\alpha_2^2 - 10\alpha_2 + 2} \ . \tag{45}$$

Finally, from (8) and (9) we obtain:

$$T_4 = - T_{13} \, , \ T_3 - 3T_{13} \, , \ T_{12} = 3T_{13} \ . \tag{46}$$

SECTION IV. EXAMPLE FOR A FIFTH-ORDER RUNGE-KUTTA FORMULA

22. From (43) through (46), it follows that all T_ν-values become zero for $\alpha_2 = 1/3$. However, this value for α_2 leads to $\alpha_3 = 1$, $\alpha_4 = 1$, thereby making equations (D1) through (D5) incompatible. Therefore, we have to discard the value $\alpha_2 = 1/3$. This means that we cannot make all T_ν-values zero. This was to be expected, however, because otherwise we would obtain a sixth-order Runge-Kutta formula with only six evaluations.

One might try to keep the T_ν-values small by choosing α_2 close to 1/3 (this means α_3 and α_4 close to 1). However, such a choice of α_2 is not advisable, since this would lead to large values for some of the weight coefficients (26).

23. Since we cannot make all T_ν-values zero, we might try to make the T_ν-values (45) and (46) zero by assuming:

$$5\alpha_2^2 - 5\alpha_2 + 1 = 0 \tag{47}$$

This would leave us with only two T_ν-values different from zero (T_1 and T_{10}). From (47) we would obtain:

$$\alpha_2 = \frac{1}{10}\,(5 \pm \sqrt{5}) \tag{48}$$

and from (43), (44):

$$T_{10} = -\,T_1 = \frac{5 \pm 3\sqrt{5}}{3600}\quad . \tag{49}$$

Obviously, the minus sign in (48) would be preferable, since it leads to a smaller value T_{10}.

However, the values (48) do not lead to a solution of our problem: for $\alpha_2 = \frac{1}{10}(5 - \sqrt{5})$ we would obtain from (47): $\alpha_3 = \frac{1}{10}(5 + \sqrt{5})$. From equations (D1) through (D5) we derive, by eliminating c_5, c_2, and c_3, the two relations:

$$\left.\begin{aligned} c_4\alpha_4(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_4)(1 - \alpha_4) &= \frac{1}{60}\,[10\alpha_2\alpha_3 - 5(\alpha_2 + \alpha_3) + 3] \\ c_4\alpha_4^2(\alpha_3 - \alpha_4)(\alpha_2 - \alpha_4)(1 - \alpha_4) &= \frac{1}{60}\,[5\alpha_2\alpha_3 - 3(\alpha_2 + \alpha_3) + 2] \end{aligned}\right\} \quad (50)$$

For our values α_2, α_3 the right-hand sides of (50) would both become zero. Therefore, one of the following conditions must hold: $\alpha_4 = 0$, $\alpha_4 = 1$, $\alpha_4 = \alpha_2$, $\alpha_4 = \alpha_3$ or $c_4 = 0$. It can be shown, however, that for any of these conditions our equations of condition (A) through (E) lead to contradictions.

24. However, by choosing for α_2 a value in the vicinity of $\frac{1}{10}(5 - \sqrt{5})$ the terms (45) and (46) can be kept small. On the other hand, the terms (43) and (44) vary only slightly in the vicinity of $\alpha_2 = \frac{1}{10}(5 - \sqrt{5})$ for which value they have an extremum. Therefore, instead of choosing $\alpha_2 = \frac{1}{10}(5 - \sqrt{5}) \approx 0.276$ we might, for example, take $\alpha_2 = \frac{4}{15} \approx 0.267$. This value is relatively close to $\frac{1}{10}(5 - \sqrt{5})$; it yields relatively simple Runge-Kutta coefficients — all weight factors (26) being non-negative — and the error coefficients T_{10} and T_{13} become relatively small. Table II shows the Runge-Kutta coefficients for $\alpha_2 = 4/15$, as obtained from Section II. In this case, the leading term (the sixth-order term) of the truncation error obtained as the difference of the two formulas (3), would read:

$$TE = \frac{5}{66}\,(f_0 + f_5 - f_6 - f_7)\,h \qquad (51)$$

From (43) and (45) we then obtain for our error coefficients:

$$T_{10} = -\frac{1}{2160} \approx -0.463.10^{-3};\; T_{13} = \frac{1}{32400} \approx 0.309\cdot 10^{-4}\,. \qquad (52)$$

TABLE II. RK5(6)

<table>
<tr><td rowspan="2">$\kappa \diagdown \lambda$</td><td rowspan="2">α_κ</td><td colspan="7">$\beta_{\kappa\lambda}$</td><td rowspan="2">c_κ</td><td rowspan="2">$\hat{c}_\kappa$</td></tr>
<tr><td>0</td><td>1</td><td>2</td><td>3</td><td>4</td><td>5</td><td>6</td></tr>
<tr><td>0</td><td>0</td><td>0</td><td></td><td></td><td></td><td></td><td></td><td></td><td>$\frac{31}{384}$</td><td>$\frac{7}{1408}$</td></tr>
<tr><td>1</td><td>$\frac{1}{6}$</td><td>$\frac{1}{6}$</td><td></td><td></td><td></td><td></td><td></td><td></td><td colspan="2">0</td></tr>
<tr><td>2</td><td>$\frac{4}{15}$</td><td>$\frac{4}{75}$</td><td>$\frac{16}{75}$</td><td></td><td></td><td></td><td></td><td></td><td colspan="2">$\frac{1125}{2816}$</td></tr>
<tr><td>3</td><td>$\frac{2}{3}$</td><td>$\frac{5}{6}$</td><td>$-\frac{8}{3}$</td><td>$\frac{5}{2}$</td><td></td><td></td><td></td><td></td><td colspan="2">$\frac{9}{32}$</td></tr>
<tr><td>4</td><td>$\frac{4}{5}$</td><td>$-\frac{8}{5}$</td><td>$\frac{144}{25}$</td><td>-4</td><td>$\frac{16}{25}$</td><td></td><td></td><td></td><td colspan="2">$\frac{125}{768}$</td></tr>
<tr><td>5</td><td>1</td><td>$\frac{361}{320}$</td><td>$-\frac{18}{5}$</td><td>$\frac{407}{128}$</td><td>$-\frac{11}{80}$</td><td>$\frac{55}{128}$</td><td></td><td></td><td>$\frac{5}{66}$</td><td>0</td></tr>
<tr><td>6</td><td>0</td><td>$-\frac{11}{640}$</td><td>0</td><td>$\frac{11}{256}$</td><td>$-\frac{11}{160}$</td><td>$\frac{11}{256}$</td><td>0</td><td></td><td></td><td>$\frac{5}{66}$</td></tr>
<tr><td>7</td><td>1</td><td>$\frac{93}{640}$</td><td>$-\frac{18}{5}$</td><td>$\frac{803}{256}$</td><td>$-\frac{11}{160}$</td><td>$\frac{99}{256}$</td><td>0</td><td>1</td><td></td><td>$\frac{5}{66}$</td></tr>
</table>

SECTION V. NUMERICAL COMPARISON WITH OTHER FIFTH-ORDER RUNGE-KUTTA FORMULAS

25. For comparison, we consider some other fifth-order Runge-Kutta formulas. The first two fifth-order Runge-Kutta formulas ever developed are due to W. KUTTA ([4], p. 446/447) . Since both formulas of W. KUTTA contained minor arithmetical errors which later were corrected by this author ([5], p. 424), and by E. J. NYSTRÖM ([6], p. 5), respectively, we state in Tables III and V KUTTA's formulas in their correct form. We also present in Tables IV and VI the error coefficients T_1 through T_{20} for KUTTA's formulas. These coefficients have been rounded to three significant digits.

TABLE III. KUTTA 1

$\kappa \diagdown \lambda$	α_κ	$\beta_{\kappa\lambda}$ 0	1	2	3	4	c_κ
0	0	0					$\frac{17}{144}$
1	$\frac{1}{5}$	$\frac{1}{5}$					0
2	$\frac{2}{5}$	0	$\frac{2}{5}$				$\frac{25}{36}$
3	1	$\frac{9}{4}$	-5	$\frac{15}{4}$			$\frac{1}{72}$
4	$\frac{3}{5}$	$-\frac{63}{100}$	$\frac{9}{5}$	$-\frac{13}{20}$	$\frac{2}{25}$		$-\frac{25}{72}$
5	$\frac{4}{5}$	$-\frac{6}{25}$	$\frac{4}{5}$	$\frac{2}{15}$	$\frac{8}{75}$	0	$\frac{25}{48}$

TABLE IV. ERROR COEFFICIENTS FOR KUTTA 1

n	T_{5n+1}	T_{5n+2}	T_{5n+3}	T_{5n+4}	T_{5n+5}
0	$-0.139\cdot 10^{-2}$	$-0.556\cdot 10^{-3}$	$-0.833\cdot 10^{-3}$	$-0.278\cdot 10^{-3}$	$0.278\cdot 10^{-2}$
1	$0.111\cdot 10^{-2}$	$0.417\cdot 10^{-3}$	$0.833\cdot 10^{-3}$	$0.139\cdot 10^{-3}$	$0.139\cdot 10^{-2}$
2	$0.556\cdot 10^{-3}$	$0.833\cdot 10^{-3}$	$0.278\cdot 10^{-3}$	$-0.389\cdot 10^{-3}$	$-0.222\cdot 10^{-3}$
3	$-0.389\cdot 10^{-3}$	$-0.222\cdot 10^{-3}$	$-0.250\cdot 10^{-3}$	$-0.167\cdot 10^{-3}$	$-0.167\cdot 10^{-4}$

TABLE V. KUTTA 2

$\kappa \backslash \lambda$	α_κ	$\beta_{\kappa\lambda}$ 0	1	2	3	4	c_κ
0	0	0					$\frac{23}{192}$
1	$\frac{1}{3}$	$\frac{1}{3}$					0
2	$\frac{2}{5}$	$\frac{4}{25}$	$\frac{6}{25}$				$\frac{125}{192}$
3	1	$\frac{1}{4}$	-3	$\frac{15}{4}$			0
4	$\frac{2}{3}$	$\frac{2}{27}$	$\frac{10}{9}$	$-\frac{50}{81}$	$\frac{8}{81}$		$-\frac{27}{64}$
5	$\frac{4}{5}$	$\frac{2}{25}$	$\frac{12}{25}$	$\frac{2}{15}$	$\frac{8}{75}$	0	$\frac{125}{192}$

TABLE VI. ERROR COEFFICIENTS FOR KUTTA 2

n	T_{5n+1}	T_{5n+2}	T_{5n+3}	T_{5n+4}	T_{5n+5}
0	$-0.139 \cdot 10^{-2}$	0	$-0.833 \cdot 10^{-3}$	$-0.278 \cdot 10^{-3}$	$0.278 \cdot 10^{-2}$
1	0	$0.417 \cdot 10^{-3}$	$0.833 \cdot 10^{-3}$	$0.139 \cdot 10^{-3}$	$0.139 \cdot 10^{-2}$
2	0	$0.833 \cdot 10^{-3}$	$0.278 \cdot 10^{-3}$	$-0.556 \cdot 10^{-3}$	$-0.926 \cdot 10^{-4}$
3	$-0.555 \cdot 10^{-3}$	$-0.926 \cdot 10^{-4}$	$-0.278 \cdot 10^{-3}$	$-0.185 \cdot 10^{-3}$	$-0.185 \cdot 10^{-4}$

Comparing Tables IV and VI with the error coefficients (52) of our Runge-Kutta formula, we notice that in both formulas of KUTTA the coefficient T_{10} is three times as large as in our formula, and the coefficient T_{13} even nine times as large as our coefficient T_{13}. Furthermore, in KUTTA's formulas the term T_5, which is zero in our formula, is six times as large as our largest term $T_1 = -T_{10}$. Finally, KUTTA's first formula has twenty non-vanishing error coefficients T_ν, and his second formula has still seventeen such terms, whereas our formula carries only six non-vanishing error coefficients T_ν.

For these reasons we may expect that our formula (Table II), having a smaller truncation error, shall integrate a problem in fewer steps than KUTTA's formulas (Table III and Table V).

26. More recently, several other authors (H. P. KONEN, H. A. LUTHER, J. A. ZONNEVELD, etc.) have derived more fifth-order Runge-Kutta formulas. However, we cannot consider all these fifth-order formulas, especially since most of them are really not much of an improvement with respect to the error coefficients if compared with KUTTA's formulas.

27. For the numerical comparison of our formula RK5(6) with KUTTA's formulas, we apply these formulas to an example of a system of two differential equations, for which the exact solution is known. Therefore, in our example, we can easily check on the accumulated errors of these formulas.

Example:

$$\left.\begin{array}{l} y' = -2xy\cdot \log z, \quad z' = 2xz\cdot \log y \\ \text{Initial values: } x_o = 0,\ y_o = e,\ z_o = 1 \\ \text{Exact solution: } y = e^{\cos(x^2)},\ z = e^{\sin(x^2)} \end{array}\right\} \tag{53}$$

Since no other step-size control procedure seems to be known for KUTTA's formulas, we have applied RICHARDSON's extrapolation to the limit as stepsize control for these formulas.

All computations were executed on an IBM-7094 computer (16 decimal places). The same tolerance (10^{-16}) for the local truncation error was allowed for all formulas.

Table VII shows the results of our computer runs for example (53); all runs cover the interval from $x = 0$ to $x = 5$.

TABLE VII. COMPARISON OF FIFTH-ORDER METHODS FOR EXAMPLE (53)

		Results for $x=5$ and Tolerance 10^{-16}				
Method	Number of Substitutions per Step	Number of Steps	Total Number of Evaluations	Running Time on IBM-7094 (min)	Accumulated Errors in y and z: Δy	Δz
KUTTA 1	11	6335	69685	7.10	$-0.3071\cdot 10^{-12}$	$-0.2772\cdot 10^{-12}$
KUTTA 2	11	6290	69190	6.95	$-0.2383\cdot 10^{-12}$	$-0.2802\cdot 10^{-12}$
RK5(6)	8	4779	38232	3.84	$+0.1072\cdot 10^{-12}$	$-0.2190\cdot 10^{-12}$

The table shows that in this example our formula RK5(6) requires approximately 55 percent of the evaluations or 55 percent of the computer running time compared with KUTTA's formulas, all formulas yielding about the same accuracy.

Such a saving could be expected since the computational effort per step — including stepsize control — is smaller for our method than for KUTTA's formulas. Furthermore, we save steps since our local truncation error is smaller.

PART II. SIXTH-ORDER FORMULAS

SECTION VI. THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

28. For sixth-order Runge-Kutta formulas with stepsize control, we allow for ten evaluations of the differential equations per step.

Therefore, we have instead of (2) and (3):

$$\left.\begin{aligned} f_0 &= f(x_0, y_0) \\ f_\kappa &= f(x_0 + \alpha_\kappa h,\ y_0 + h \sum_{\lambda=0}^{\kappa-1} \beta_{\kappa\lambda} f_\lambda) \quad (\kappa = 1, 2, 3, \ldots, 9) \end{aligned}\right\} \quad (54)$$

and

$$\left.\begin{aligned} y &= y_0 + h \sum_{\kappa=0}^{7} c_\kappa f_\kappa + 0(h^7) \\ \hat{y} &= y_0 + h \sum_{\kappa=0}^{9} \hat{c}_\kappa f_\kappa + 0(h^8) \end{aligned}\right\} \quad (55)$$

29. Since Table I contains the Taylor terms up to the sixth-order only, we need a supplementary table for the seventh-order terms. Such a table can again be obtained from J. C. BUTCHER's paper ([3], Table I). It would contain 48 equations. Because of the length of this supplementary table, we do not reproduce it here, but refer directly to BUTCHER's paper. We denote the equations of this new table — in the same order as in BUTCHER's table — by (VII, 1) through (VII, 48).

This supplementary table in combination with Table I contains all equations of condition for a seventh-order Runge-Kutta formula. Naturally, in Table I the κ-sums now run up to 9 (instead of 7). Since we are looking for a pair of Runge-Kutta formulas (55), the original Table I (with $\hat{c}_\kappa$ replaced by c_κ) must also be satisfied for our sixth-order formula.

30. We proceed in a very similar way as in Part I and make the following assumptions:

$$\left.\begin{array}{l} \alpha_7 = \alpha_9 = 1, \ \alpha_8 = 0, \ \hat{c}_1 = c_1 = 0, \ \hat{c}_2 = c_2 = 0, \ \hat{c}_3 = c_3, \ \hat{c}_4 = c_4, \ \hat{c}_5 = c_5, \\ \hat{c}_6 = c_6, \ \hat{c}_7 = 0, \ \hat{c}_8 = \hat{c}_9 = c_7 \\ \\ \beta_{31} = \beta_{41} = \beta_{51} = \beta_{61} = \beta_{71} = \beta_{81} = \beta_{91} = 0 \end{array}\right\} \quad (56)$$

Further, analogous to Part I, the following assumptions shall hold:

$$\text{(A1)} \left\{ \begin{array}{lll|lll} (*) & P_{21} = \frac{1}{2}\alpha_2^2 & \text{(A1-2)} & (*) & P_{61} = \frac{1}{2}\alpha_6^2 & \text{(A1-6)} \\ (*) & P_{31} = \frac{1}{2}\alpha_3^2 & \text{(A1-3)} & (*) & P_{71} = \frac{1}{2}\alpha_7^2 = \frac{1}{2} & \text{(A1-7)} \\ (*) & P_{41} = \frac{1}{2}\alpha_4^2 & \text{(A1-4)} & & P_{81} = \frac{1}{2}\alpha_8^2 = 0 & \text{(A1-8)} \\ (*) & P_{51} = \frac{1}{2}\alpha_5^2 & \text{(A1-5)} & & P_{91} = \frac{1}{2}\alpha_9^2 = \frac{1}{2} & \text{(A1-9)} \end{array} \right.$$

$$\text{(A2)} \left\{ \begin{array}{lll|lll} (*) & P_{32} = \frac{1}{3}\alpha_3^3 & \text{(A2-3)} & (*) & P_{72} = \frac{1}{3}\alpha_7^3 = \frac{1}{3} & \text{(A2-7)} \\ (*) & P_{42} = \frac{1}{3}\alpha_4^3 & \text{(A2-4)} & & P_{82} = \frac{1}{3}\alpha_8^3 = 0 & \text{(A2-8)} \\ (*) & P_{52} = \frac{1}{3}\alpha_5^3 & \text{(A2-5)} & & P_{92} = \frac{1}{3}\alpha_9^3 = \frac{1}{3} & \text{(A2-9)} \\ (*) & P_{62} = \frac{1}{3}\alpha_6^3 & \text{(A2-6)} & & & \end{array} \right.$$

$$(\mathrm{B})\left\{\begin{array}{lr}
(*)\quad c_2\beta_{21} + c_3\beta_{31} + c_4\beta_{41} + c_5\beta_{51} + c_6\beta_{61} + c_7 \left\{\begin{matrix}\beta_{71} \\ \beta_{81} + \beta_{91}\end{matrix}\right\} = c_1(1-\alpha_1) = 0 & (\mathrm{B1}) \\[2ex]
(*)\quad c_3\beta_{32} + c_4\beta_{42} + c_5\beta_{52} + c_6\beta_{62} + c_7 \left\{\begin{matrix}\beta_{72} \\ \beta_{82} + \beta_{92}\end{matrix}\right\} = c_2(1-\alpha_2) = 0 & (\mathrm{B2}) \\[2ex]
(*)\quad c_4\beta_{43} + c_5\beta_{53} + c_6\beta_{63} + c_7 \left\{\begin{matrix}\beta_{73} \\ \beta_{83} + \beta_{93}\end{matrix}\right\} = c_3(1-\alpha_3) & (\mathrm{B3}) \\[2ex]
(*)\quad c_5\beta_{54} + c_6\beta_{64} + c_7 \left\{\begin{matrix}\beta_{74} \\ \beta_{84} + \beta_{94}\end{matrix}\right\} = c_4(1-\alpha_4) & (\mathrm{B4}) \\[2ex]
(*)\quad c_6\beta_{65} + c_7 \left\{\begin{matrix}\beta_{75} \\ \beta_{85} + \beta_{95}\end{matrix}\right\} = c_5(1-\alpha_5) & (\mathrm{B5}) \\[2ex]
(*)\quad c_7 \left\{\begin{matrix}\beta_{76} \\ \beta_{86} + \beta_{96}\end{matrix}\right\} = c_6(1-\alpha_6) & (\mathrm{B6}) \\[2ex]
c_7\ (\beta_{87} + \beta_{97}) = c_7(1-\alpha_7) = 0 & (\mathrm{B7}) \\[2ex]
c_7\ \beta_{98} = c_7(1-\alpha_8) = c_7 & (\mathrm{B8})
\end{array}\right.$$

$$(\mathrm{C})\left\{\begin{array}{lr}
(*)\quad c_3\alpha_3\beta_{32} + c_4\alpha_4\beta_{42} + c_5\alpha_5\beta_{52} + c_6\alpha_6\beta_{62} + c_7 \left\{\begin{matrix}\beta_{72} \\ \beta_{92}\end{matrix}\right\} = 0 & (\mathrm{C1}) \\[2ex]
c_3\alpha_3^2\beta_{32} + c_4\alpha_4^2\beta_{42} + c_5\alpha_5^2\beta_{52} + c_6\alpha_6^2\beta_{62} + c_7\ \beta_{92} = 0 & (\mathrm{C2}) \\[2ex]
c_4\alpha_4\beta_{43}\beta_{32} + c_5\alpha_5(\beta_{53}\beta_{32} + \beta_{54}\beta_{42}) + c_6\alpha_6(\beta_{63}\beta_{32} + \beta_{64}\beta_{42} + \beta_{65}\beta_{52}) & \\
\quad + c_7(\beta_{93}\beta_{32} + \beta_{94}\beta_{42} + \beta_{95}\beta_{52} + \beta_{96}\ \beta_{62} + \beta_{97}\beta_{72} + \beta_{98}\beta_{82}) = 0 & (\mathrm{C3})
\end{array}\right.$$

The asterisk (*) in front of some of these equations means now, naturally, that these equations must hold for the seventh-order formula as well as for the sixth-order formula.

31. We now reduce the necessary and sufficient equations of condition of our supplementary table to fewer and simpler sufficient equations of condition by inserting the assumptions (A1), (A2), (B), and (C) into the table.

Since the reduction is completely analogous to the procedure of Number 8, we shall present here the results of the reduction only, without going into all details. The reduction leads to equations (VII, 24), (VII, 29), (VII, 37), and (VII, 48) as the only four independent equations of the table. All other equations of the table can be expressed by these four equations and by some sixth-order equations of Table I.

For the benefit of the truncation error terms, which we shall consider later, we now list those equations which can be expressed by (VII, 24) and (VII, 29):

$$\left.\begin{array}{lll}
(\mathrm{VII},1) \equiv & (\mathrm{VI},1) & -(\mathrm{VII},21) \\
(\mathrm{VII},2) \equiv & (\mathrm{VI},2) & -(\mathrm{VII},22) \\
(\mathrm{VII},3) \equiv & (\mathrm{VI},3) & -3(\mathrm{VII},24) \\
(\mathrm{VII},4) \equiv & (\mathrm{VI},4) & -(\mathrm{VII},24) \\
(\mathrm{VII},21) \equiv & (\mathrm{VII},24) & \\
(\mathrm{VII},22) \equiv & (\mathrm{VII},24) & \\
(\mathrm{VII},23) \equiv & 3(\mathrm{VII},24) &
\end{array}\right\} \quad (57)$$

and

$$\left.\begin{array}{lll}
(\mathrm{VII},5) \equiv & (\mathrm{VI},5) & -4(\mathrm{VII},29) \\
(\mathrm{VII},6) \equiv & (\mathrm{VI},6) & -4(\mathrm{VII},29) \\
2(\mathrm{VII},7) \equiv & (\mathrm{VI},8) & -6(\mathrm{VII},29) \\
(\mathrm{VII},8) \equiv & (\mathrm{VI},8) & -6(\mathrm{VII},29) \\
(\mathrm{VII},9) \equiv & (\mathrm{VI},9) & -\ (\mathrm{VII},29 \\
(\mathrm{VII},25) \equiv & 4(\mathrm{VII},29) & \\
(\mathrm{VII},26) \equiv & 4(\mathrm{VII},29) & \\
(\mathrm{VII},27) \equiv & 3(\mathrm{VII},29) & \\
(\mathrm{VII},28) \equiv & 6(\mathrm{VII},29) &
\end{array}\right\} \quad (58)$$

The remaining equations of our supplementary table can be expressed by equations (VII, 37) and (VII, 48). However, because of our assumptions (56), these two equations do not contribute to the truncation error of our sixth-order formula.

It should be noticed that the fifth and sixth identities in (57) are correct only if assumption (C3) is assumed to hold. Since condition (C3) does not hold in the case of a sixth-order formula, it can not be followed from (VII, 21) ≡ (VII, 24) and (VII, 22) ≡ (VII, 24) that the truncation error coefficients T_{21} and T_{22} are identical with T_{24}.

32. We now list for our pair of sixth- and seventh-order Runge-Kutta formulas the sufficient equations of condition that result after we have performed all the reductions in Table I and in the supplementary table.

We arrange them again in two groups (D) and (E) in the same way as in Part I.

$$(\mathrm{D})\left\{\begin{array}{lll}
(*) & \left\{\begin{matrix} c_o \\ \hat{c}_o \end{matrix}\right\} + c_3 + c_4 + c_5 + c_6 + \left\{\begin{matrix} c_7 \\ 2c_7 \end{matrix}\right\} = 1 & (\mathrm{D0}) \\
(*) & c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 + c_6\alpha_6 + c_7 = \frac{1}{2} & (\mathrm{D1}) \\
(*) & c_3\alpha_3^2 + c_4\alpha_4^2 + c_5\alpha_5^2 + c_6\alpha_6^2 + c_7 = \frac{1}{3} & (\mathrm{D2}) \\
(*) & c_3\alpha_3^3 + c_4\alpha_4^3 + c_5\alpha_5^3 + c_6\alpha_6^3 + c_7 = \frac{1}{4} & (\mathrm{D3}) \\
(*) & c_3\alpha_3^4 + c_4\alpha_4^4 + c_5\alpha_5^4 + c_6\alpha_6^4 + c_7 = \frac{1}{5} & (\mathrm{D4}) \\
(*) & c_3\alpha_3^5 + c_4\alpha_4^5 + c_5\alpha_5^5 + c_6\alpha_6^5 + c_7 = \frac{1}{6} & (\mathrm{D5}) \\
 & c_3\alpha_3^6 + c_4\alpha_4^6 + c_5\alpha_5^6 + c_6\alpha_6^6 + c_7 = \frac{1}{7} & (\mathrm{D6})
\end{array}\right.$$

and

$$
(\mathrm{E})\left\{\begin{array}{ll}
(*)\ c_3\alpha_3P_{33} + c_4\alpha_4P_{43} + c_5\alpha_5P_{53} + c_6\alpha_6P_{63} + c_7\left\{\begin{matrix}P_{73}\\ P_{93}\end{matrix}\right\} = \dfrac{1}{24} & (\mathrm{E1})\\[2ex]
c_3\alpha_3P_{34} + c_4\alpha_4P_{44} + c_5\alpha_5P_{54} + c_6\alpha_6P_{64} + c_7\,P_{94} = \dfrac{1}{35} & (\mathrm{E2})\\[2ex]
c_3\alpha_3^2P_{33} + c_4\alpha_4^2P_{43} + c_5\alpha_5^2P_{53} + c_6\alpha_6^2P_{63} + c_7\,P_{93} = \dfrac{1}{28} & (\mathrm{E3})\\[2ex]
c_4\alpha_4\beta_{43}P_{33} + c_5\alpha_5(\beta_{53}P_{33} + \beta_{54}P_{43}) + c_6\alpha_6(\beta_{63}P_{33} + \beta_{64}P_{43} + \beta_{65}P_{53}) & \\
\quad + c_7\,(\beta_{93}P_{33} + \beta_{94}P_{43} + \beta_{95}P_{53} + \beta_{96}P_{63} + \beta_{97}P_{73} + \beta_{98}P_{83}) = \dfrac{1}{140} & (\mathrm{E4})
\end{array}\right.
$$

Group D is an obvious extension of the former group (D) of Number 9, the last equation (D6) representing equation (VII, 48) of the supplementary table. In group (E) the first equation represents equation (E2) of the former group (E) of Number 9. Equations (E2), (E3), (E4) of group (E) correspond to equations (VII, 29), (VII, 37), (VII, 24) of the supplementary table.

Because of our assumptions (A2) the equations (E1), (E3) of our former group (E) in Number 9 become identical to the equations (D4), (D5) and can, therefore, be omitted from our new group (E). Introducing our new assumptions (A2) and (C1) into equation (E4) of our former group (E), it follows immediately that this equation becomes identical with equation (E1) of our new group (E).

Since the extension of our former group (F) for sixth-order Runge-Kutta formulas is quite obvious, we omit this extension here.

SECTION VII. A SOLUTION OF THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

33. In the case of our sixth-order Runge-Kutta formula, six α-coefficients, α_1, α_2, α_3, α_4, α_5, α_6, are at our disposal. We try again to express the β- and c-coefficients as functions of these α-coefficients. Again, we shall show that not all six α-coefficients are independent of one another. Obviously, from (A1-3) and (A2-3), it follows

$$\alpha_2 = \frac{2}{3}\alpha_3 \ . \tag{59}$$

34. A further relation between the α-coefficients is obtained from equations (D1) through (D6) as condition of compatibility. Eliminating from these equations the terms with c_7, c_6, c_3, c_4, c_5, in this order, leads to:

$$\alpha_5 = \frac{1}{7} \cdot \frac{35\alpha_3\alpha_4\alpha_6 - 21(\alpha_3\alpha_4 + \alpha_3\alpha_6 + \alpha_4\alpha_6) + 14(\alpha_3 + \alpha_4 + \alpha_6) - 10}{10\alpha_3\alpha_4\alpha_6 - 5(\alpha_3\alpha_4 + \alpha_3\alpha_6 + \alpha_4\alpha_6) + 3(\alpha_3 + \alpha_4 + \alpha_6) - 2} \tag{60}$$

35. We now establish a relation between α_3, α_4, α_6, which will allow us to express α_2, α_3, α_5 by α_4 and α_6 so that α_1, α_4, α_6 will be the only free α-coefficients. To find such a relation, we proceed in the following manner:

From (A1) we find by multiplying these equations with $c_3(1 - \alpha_3)$, $c_4(1 - \alpha_4)$, ..., adding them and using (D2) and (D3):

$$c_3(1 - \alpha_3)\ P_{31} + c_4(1 - \alpha_4)\ P_{41} + c_5(1 - \alpha_5)\ P_{51} + c_6(1 - \alpha_6)\ P_{61} = \frac{1}{24} \tag{61}$$

and in the same way from (A2), using (D3) and (D4):

$$c_3(1 - \alpha_3)\ P_{32} + c_4(1 - \alpha_4)\ P_{42} + c_5(1 - \alpha_5)\ P_{52} + c_6(1 - \alpha_6)\ P_{62} = \frac{1}{60} \tag{62}$$

A third equation is found by forming:

$$\alpha_2^3(B2) + \alpha_3^3(B3) + \alpha_4^3(B4) + \alpha_5^3(B5) + \alpha_6^3(B6) - (E1) \; , \tag{63}$$

thereby eliminating in (63) the terms with c_7. We obtain:

$$c_3(1 - \alpha_3)\; P_{33} + c_4(1 - \alpha_4)\; P_{43} + c_5(1 - \alpha_5)\; P_{53} + c_6(1 - \alpha_6)\; P_{63} = \frac{1}{120} \tag{64}$$

From equations (61), (62), (64), we can easily eliminate the terms with c_3 and c_4. We obtain:

$$\left.\begin{aligned} &c_5(1 - \alpha_5)\; \beta_{54}\alpha_4(\alpha_4 - \alpha_3)\; (\alpha_4 - \alpha_2) + c_6(1 - \alpha_6)\; [\beta_{64}\alpha_4(\alpha_4 - \alpha_3)\; (\alpha_4 - \alpha_2) \\ &+ \beta_{65}\alpha_5(\alpha_5 - \alpha_3)\; (\alpha_5 - \alpha_2)\;] = \frac{1}{120}\; [5\alpha_2\alpha_3 - 2(\alpha_2 + \alpha_3) + 1] \end{aligned}\right\} \tag{65}$$

If we had multiplied (A1) with $c_3(1 - \alpha_3^2)$, $c_4(1 - \alpha_4^2)$, ..., we would have obtained instead of (61):

$$c_3(1 - \alpha_3^2)\; P_{31} + c_4(1 - \alpha_4^2)\; P_{41} + c_5(1 - \alpha_5^2)\; P_{51} + c_6(1 - \alpha_6^2)\; P_{61} = \frac{1}{15} \tag{66}$$

and by multiplying (A2) with $c_3(1 - \alpha_3^2)$, $c_4(1 - \alpha_4^2)$, ...,

$$c_3(1 - \alpha_3^2)\; P_{32} + c_4(1 - \alpha_4^2)\; P_{42} + c_5(1 - \alpha_5^2)\; P_{52} + c_6(1 - \alpha_6^2)\; P_{62} = \frac{1}{36} \tag{67}$$

Analogous to (63) we now form:

$$\alpha_2^3(B2) + \alpha_3^3(B3) + \alpha_4^3(B4) + \alpha_5^3(B5) + \alpha_6^3(B6) - (E3) \tag{68}$$

and find:

$$c_3(1-\alpha_3^2)\,P_{33} + c_4(1-\alpha_4^2)\,P_{43} + c_5(1-\alpha_5^2)\,P_{53} + c_6(1-\alpha_6^2)\,P_{63} = \frac{1}{70}\,. \tag{69}$$

From equations (66), (67), (69), we can again eliminate the terms with c_3 and c_4 and find:

$$\left.\begin{aligned} &c_5(1-\alpha_5^2)\,\beta_{54}\alpha_4(\alpha_4-\alpha_3)(\alpha_4-\alpha_2) + c_6(1-\alpha_6^2)\,[\beta_{64}\alpha_4(\alpha_4-\alpha_3)(\alpha_4-\alpha_2) \\ &+ \beta_{65}\alpha_5(\alpha_5-\alpha_3)(\alpha_5-\alpha_2)] = \frac{1}{1260}\,[84\alpha_2\alpha_3 - 35(\alpha_2+\alpha_3) + 18]\,. \end{aligned}\right\} \tag{70}$$

Obviously, equations (65) and (70) permit the elimination of the terms with c_6 and yield:

$$\left.\begin{aligned} &c_5\beta_{54}(\alpha_6-\alpha_5)(1-\alpha_5)\;\alpha_4(\alpha_4-\alpha_3)(\alpha_4-\alpha_2) = \frac{1}{2520}\,\{21(1+\alpha_6)[5\alpha_2\alpha_3 \\ &\qquad - 2(\alpha_2+\alpha_3) + 1] - 2[84\alpha_2\alpha_3 - 35(\alpha_2+\alpha_3) + 18]\}\,. \end{aligned}\right\} \tag{71}$$

To eliminate $c_5\beta_{54}$, we need a second relation between $c_5\beta_{54}$ and the α-coefficients. We start from (A1-5) and (A2-5). Multiplying these equations with c_5 and eliminating the terms with β_{53} yields:

$$c_5\beta_{54}\alpha_4(\alpha_4-\alpha_3) = c_5\beta_{52}\alpha_2(\alpha_3-\alpha_2) - \frac{1}{6}\,c_5\alpha_5^2(3\alpha_3-2\alpha_5)\,. \tag{72}$$

From (B2), (C1), (C2) we find by eliminating the terms with β_{62} and β_{72}:

$$c_5\beta_{52}(\alpha_6-\alpha_5)(1-\alpha_5) = -c_3\beta_{32}(\alpha_6-\alpha_3)(1-\alpha_3) - c_4\beta_{42}(\alpha_6-\alpha_4)(1-\alpha_4) \tag{73}$$

Combining (72) and (73) yields:

$$
\left.\begin{aligned}
c_5\beta_{54}(\alpha_6 - \alpha_5)(1 - \alpha_5)\ \alpha_4(\alpha_4 - \alpha_3) &= - c_3\beta_{32}\alpha_2(\alpha_3 - \alpha_2)(\alpha_6 - \alpha_3)(1 - \alpha_3) \\
- c_4\beta_{42}\alpha_2(\alpha_3 - \alpha_2)(\alpha_6 - \alpha_4)(1 - \alpha_4) &- \frac{1}{6} c_5\alpha_5^2(3\alpha_3 - 2\alpha_5)(\alpha_6 - \alpha_5)(1 - \alpha_5)
\end{aligned}\right\} \quad (74)
$$

From (A1-4) and (A2-4) we find by eliminating the terms with β_{43}:

$$
\beta_{42}\alpha_2(\alpha_3 - \alpha_2) = \frac{1}{6}\alpha_4^2(3\alpha_3 - 2\alpha_4) \quad (75)
$$

and from (A1-3) and (59)

$$
\beta_{32}\alpha_2(\alpha_3 - \alpha_2) = \frac{1}{6}\alpha_3^2(3\alpha_3 - 2\alpha_3) \quad (76)
$$

Inserting (75) and (76) into (74), we obtain:

$$
\left.\begin{aligned}
c_5\beta_{54}(\alpha_6 - \alpha_5)(1 - \alpha_5)\ \alpha_4(\alpha_4 - \alpha_3) = &- \frac{1}{2}\alpha_3[c_3\alpha_3^2(\alpha_6 - \alpha_3)(1 - \alpha_3) \\
&+ c_4\alpha_4^2(\alpha_6 - \alpha_4)(1 - \alpha_4) + c_5\alpha_5^2(\alpha_6 - \alpha_5)(1 - \alpha_5)] \\
&+ \frac{1}{3}[c_3\alpha_3^3(\alpha_6 - \alpha_3)(1 - \alpha_3) + c_4\alpha_4^3(\alpha_6 - \alpha_4)(1 - \alpha_4) \\
&+ c_5\alpha_5^3(\alpha_6 - \alpha_5)(1 - \alpha_5)]
\end{aligned}\right\} \quad (77)
$$

The values of the brackets in (77) can easily be obtained from equations (D1) through (D6) by eliminating c_7 and c_6. For the first bracket we obtain $\frac{1}{60}(5\alpha_6 - 3)$ and for the second one $\frac{1}{60}(3\alpha_6 - 2)$. Inserting these values in (77), we find:

$$
c_5\beta_{54}(\alpha_6 - \alpha_5)(1 - \alpha_5)\ \alpha_4(\alpha_4 - \alpha_3) = \frac{1}{360}[2(3\alpha_6 - 2) - 3\alpha_3(5\alpha_6 - 3)] \quad (78)
$$

From equations (71) and (78) we can now easily find the wanted relation between the α-coefficients by forming:

$$(78) \cdot (\alpha_4 - \alpha_2) - (71) = 0 \tag{79}$$

We obtain:

$$\alpha_3 = \frac{1}{7} \cdot \frac{42\alpha_4\alpha_6 - 28\alpha_4 - 21\alpha_6 + 15}{15\alpha_4\alpha_6 - 9\alpha_4 - 6\alpha_6 + 4} \tag{80}$$

36. Having subjected the α-coefficients to the restrictive conditions (59), (60), and (80), we can now easily find the c- and β-coefficients in terms of the α-coefficients. The procedure is quite similar to that of Part I.

From equations (D1) through (D6) we obtain for the c-coefficients:

$$\left.\begin{aligned}
c_3 &= \hat{c}_3 = \frac{1}{60} \cdot \frac{10\alpha_4\alpha_5\alpha_6 - 5(\alpha_4\alpha_5 + \alpha_4\alpha_6 + \alpha_5\alpha_6) + 3(\alpha_4 + \alpha_5 + \alpha_6) - 2}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)(\alpha_6 - \alpha_3)(1 - \alpha_3)} \\
c_4 &= \hat{c}_4 = \frac{1}{60} \cdot \frac{10\alpha_3\alpha_5\alpha_6 - 5(\alpha_3\alpha_5 + \alpha_3\alpha_6 + \alpha_5\alpha_6) + 3(\alpha_3 + \alpha_5 + \alpha_6) - 2}{\alpha_4(\alpha_3 - \alpha_4)(\alpha_5 - \alpha_4)(\alpha_6 - \alpha_4)(1 - \alpha_4)} \\
c_5 &= \hat{c}_5 = \frac{1}{60} \cdot \frac{10\alpha_3\alpha_4\alpha_6 - 5(\alpha_3\alpha_4 + \alpha_3\alpha_6 + \alpha_4\alpha_6) + 3(\alpha_3 + \alpha_4 + \alpha_6) - 2}{\alpha_5(\alpha_3 - \alpha_5)(\alpha_4 - \alpha_5)(\alpha_6 - \alpha_5)(1 - \alpha_5)} \\
c_6 &= \hat{c}_6 = \frac{1}{60} \cdot \frac{10\alpha_3\alpha_4\alpha_5 - 5(\alpha_3\alpha_4 + \alpha_3\alpha_5 + \alpha_4\alpha_5) + 3(\alpha_3 + \alpha_4 + \alpha_5) - 2}{\alpha_6(\alpha_3 - \alpha_6)(\alpha_4 - \alpha_6)(\alpha_5 - \alpha_6)(1 - \alpha_6)} \\
c_7 &= \hat{c}_8 = \hat{c}_9 = \frac{1}{2} - (c_3\alpha_3 + c_4\alpha_4 + c_5\alpha_5 + c_6\alpha_6)
\end{aligned}\right\} \tag{81}$$

37. From (A1-2) and (59), it follows:

$$\beta_{21}\,\alpha_1 = \frac{2}{9}\,\alpha_3^2\,; \tag{82}$$

and from (A1-3) and (59):

$$\beta_{32} = \frac{3}{4}\,\alpha_3\,. \tag{83}$$

From (A1-4) and (A2-4), we find:

$$\left.\begin{aligned} \beta_{42} &= \frac{1}{6}\cdot\frac{\alpha_4^2\,(3\alpha_3 - 2\alpha_4)}{\alpha_2\,(\alpha_3 - \alpha_2)} \\ \beta_{43} &= \frac{1}{6}\cdot\frac{\alpha_4^2\,(3\alpha_2 - 2\alpha_4)}{\alpha_3\,(\alpha_2 - \alpha_3)} \end{aligned}\right\} \tag{84}$$

Equations (B2), (C1), (C2) can now be considered as three equations for β_{52}, β_{62}, β_{72}, since β_{32} and β_{42} are already known. From these equations, we obtain:

$$\left.\begin{aligned} \beta_{52} &= -\frac{c_3(\alpha_6 - \alpha_3)(1 - \alpha_3)\,\beta_{32} + c_4(\alpha_6 - \alpha_4)(1 - \alpha_4)\,\beta_{42}}{c_5(\alpha_6 - \alpha_5)(1 - \alpha_5)} \\ \beta_{62} &= -\frac{c_3(\alpha_5 - \alpha_3)(1 - \alpha_3)\,\beta_{32} + c_4(\alpha_5 - \alpha_4)(1 - \alpha_4)\,\beta_{42}}{c_6(\alpha_5 - \alpha_6)(1 - \alpha_6)} \\ \beta_{72} &= -\frac{1}{c_7}\,(c_3\beta_{32} + c_4\beta_{42} + c_5\beta_{52} + c_6\beta_{62}) \end{aligned}\right\} \tag{85}$$

From equations (A1-5) and (A2-5), we now obtain β_{53} and β_{54}:

$$\left.\begin{aligned}
\beta_{53} &= \frac{\frac{1}{6}\,\alpha_5^2(3\alpha_4 - 2\alpha_5) - \beta_{52}\alpha_2(\alpha_4 - \alpha_2)}{\alpha_3(\alpha_4 - \alpha_3)} \\
\beta_{54} &= \frac{\frac{1}{6}\,\alpha_5^2(3\alpha_3 - 2\alpha_5) - \beta_{52}\alpha_2(\alpha_3 - \alpha_2)}{\alpha_4(\alpha_3 - \alpha_4)}
\end{aligned}\right\} \quad (86)$$

We eliminate P_{73} from equations (E1) and (E3) and find:

$$\left.\begin{aligned}
P_{63} = \frac{1}{c_6\alpha_6(1 - \alpha_6)} \Bigg[\frac{1}{168} &- c_3\alpha_3(1 - \alpha_3)\, P_{33} - c_4\alpha_4(1 - \alpha_4)\, P_{43} \\
&- c_5\alpha_5(1 - \alpha_5)\, P_{53}\Bigg]
\end{aligned}\right\} \quad (87)$$

Since P_{33}, P_{43}, P_{53} are already known from (83), (84), (85), (86), P_{63} is a known quantity now.

Equations (A1-6), (A2-6) and the definition of P_{63} can now be considered as three equations for β_{63}, β_{64}, β_{65}, since β_{62} is already known. Their solution reads:

$$\left.\begin{aligned}
\beta_{63} &= \frac{P_{61}\alpha_4\alpha_5 - P_{62}(\alpha_4 + \alpha_5) + P_{63} - \beta_{62}\alpha_2(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2)}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)} \\
\beta_{64} &= \frac{P_{61}\alpha_3\alpha_5 - P_{62}(\alpha_3 + \alpha_5) + P_{63} - \beta_{62}\alpha_2(\alpha_3 - \alpha_2)(\alpha_5 - \alpha_2)}{\alpha_4(\alpha_3 - \alpha_4)(\alpha_5 - \alpha_4)} \\
\beta_{65} &= \frac{P_{61}\alpha_3\alpha_4 - P_{62}(\alpha_3 + \alpha_4) + P_{63} - \beta_{62}\alpha_2(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)}{\alpha_5(\alpha_3 - \alpha_5)(\alpha_4 - \alpha_5)}
\end{aligned}\right\} \quad (88)$$

the quantities P_{61}, P_{62}, P_{63} being represented by the right-hand sides of equations (A1-6), (A2-6), (87).

From equations (B3) through (B6), we obtain the coefficients β_{73}, β_{74}, β_{75}, β_{76} in the form:

$$\left.\begin{aligned}
c_7\beta_{73} &= c_3(1 - \alpha_3) - (c_4\beta_{43} + c_5\beta_{53} + c_6\beta_{63}) \\
c_7\beta_{74} &= c_4(1 - \alpha_4) - (c_5\beta_{54} + c_6\beta_{64}) \\
c_7\beta_{75} &= c_5(1 - \alpha_5) - c_6\beta_{65} \\
c_7\beta_{76} &= c_6(1 - \alpha_6) \; .
\end{aligned}\right\} \tag{89}$$

38. We still have to determine the coefficients $\beta_{8\lambda}$ and $\beta_{9\lambda}$, required for our seventh-order formula.

We determine first the coefficients $\beta_{9\lambda}$. From (B8), it follows:

$$\beta_{98} = 1 \, , \tag{90}$$

while (B7) suggests:

$$\beta_{97} = \beta_{87} = 0 \; . \tag{91}$$

From the two equations (C1), we obtain:

$$\beta_{92} = \beta_{72} \, , \tag{92}$$

and from the two equations (E1):

$$P_{93} = P_{73} \;. \tag{93}$$

The still missing coefficients β_{93}, β_{94}, β_{95}, β_{96} can be obtained from equations (A1-9), (A2-9), the definition of P_{93}, and the definition of P_{94}. We find:

$$\left.\begin{aligned}
\beta_{93} &= \frac{P_{91}\alpha_4\alpha_5\alpha_6 - P_{92}(\alpha_4\alpha_5 + \alpha_4\alpha_6 + \alpha_5\alpha_6) + P_{93}(\alpha_4 + \alpha_5 + \alpha_6) - P_{94} - \beta_{92}\alpha_2(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2)(\alpha_6 - \alpha_2)}{\alpha_3(\alpha_4 - \alpha_3)(\alpha_5 - \alpha_3)(\alpha_6 - \alpha_3)} \\
\beta_{94} &= \frac{P_{91}\alpha_3\alpha_5\alpha_6 - P_{92}(\alpha_3\alpha_5 + \alpha_3\alpha_6 + \alpha_5\alpha_6) + P_{93}(\alpha_3 + \alpha_5 + \alpha_6) - P_{94} - \beta_{92}\alpha_2(\alpha_3 - \alpha_2)(\alpha_5 - \alpha_2)(\alpha_6 - \alpha_2)}{\alpha_4(\alpha_3 - \alpha_4)(\alpha_5 - \alpha_4)(\alpha_6 - \alpha_4)} \\
\beta_{95} &= \frac{P_{91}\alpha_3\alpha_4\alpha_6 - P_{92}(\alpha_3\alpha_4 + \alpha_3\alpha_6 + \alpha_4\alpha_6) + P_{93}(\alpha_3 + \alpha_4 + \alpha_6) - P_{94} - \beta_{92}\alpha_2(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)(\alpha_6 - \alpha_2)}{\alpha_5(\alpha_3 - \alpha_5)(\alpha_4 - \alpha_5)(\alpha_6 - \alpha_5)} \\
\beta_{96} &= \frac{P_{91}\alpha_3\alpha_4\alpha_5 - P_{92}(\alpha_3\alpha_4 + \alpha_3\alpha_5 + \alpha_4\alpha_5) + P_{93}(\alpha_3 + \alpha_4 + \alpha_5) - P_{94} - \beta_{92}\alpha_2(\alpha_3 - \alpha_2)(\alpha_4 - \alpha_2)(\alpha_5 - \alpha_2)}{\alpha_6(\alpha_3 - \alpha_6)(\alpha_4 - \alpha_6)(\alpha_5 - \alpha_6)}
\end{aligned}\right\} \tag{94}$$

The quantity P_{94} is obtainable from (E2) as:

$$P_{94} = \frac{1}{c_7}\left(\frac{1}{35} - c_3\,\alpha_3\,P_{34} - c_4\,\alpha_4\,P_{44} - c_5\,\alpha_5\,P_{54} - c_6\,\alpha_6\,P_{64}\right) . \tag{95}$$

The coefficients $\beta_{8\lambda}$ can now be found from equations (B):

$$\left.\begin{array}{l|l}
\beta_{82} = \beta_{72} - \beta_{92} = 0 & \beta_{85} = \beta_{75} - \beta_{95} \\
\beta_{83} = \beta_{73} - \beta_{93} & \beta_{86} = \beta_{76} - \beta_{96} \\
\beta_{84} = \beta_{74} - \beta_{94} & \beta_{87} = 0 \;.
\end{array}\right\} \tag{96}$$

All Runge-Kutta coefficients for our pair of sixth- and seventh-order formulas are now known. Under the restrictive conditions (59), (60), and (80) our coefficients satisfy all equations of condition (A1), (A2), (B), (C), (D), and (E).

SECTION VIII. THE LEADING TERM OF THE LOCAL TRUNCATION ERROR

39. According to our supplementary table we have 48 terms T_ν ($\nu = 1, 2, 3, .. 48$) that contribute to the leading truncation error term of a sixth-order Runge-Kutta formula. We obtain these terms from the left-hand sides of the equations of our table by replacing $\hat{c}_\kappa$ with c_κ and the upper limit 9 of the κ-sums by 7. Fortunately, because of our assumptions (56) and our conditions (A1), (A2), (B), and (C), the majority of the terms T_ν are zero. For those T_ν, that are not zero, we obtain from (57) and (58):

$$T_1 = - T_{21}, \; T_2 = - T_{22}, \; T_3 = - 3T_{24}, \; T_4 = - T_{24}, \; T_{23} = 3T_{24} \tag{97}$$

$$\left.\begin{aligned} &T_5 = - 4T_{29}, \; T_6 = - 4T_{29}, \; T_7 = - 3T_{29}, \; T_8 = - 6T_{29}, \; T_9 = - T_{29} \\ &T_{25} = 4T_{29}, \; T_{26} = 4T_{29}, \; T_{27} = 3T_{29}, \; T_{28} = 6T_{29} \end{aligned}\right\} \tag{98}$$

with:

$$\left.\begin{aligned} T_{21} &= \frac{1}{6} \sum_{\kappa=5}^{7} c_\kappa \alpha_\kappa \left[\sum_{\lambda=4}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{\mu=3}^{\lambda-1} \beta_{\lambda\mu} \alpha_\mu^3 \right) \right] - \frac{1}{840} \\ T_{22} &= \frac{1}{2} \sum_{\kappa=4}^{7} c_\kappa \alpha_\kappa \left[\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum_{\mu=2}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 2} \right) \right] - \frac{1}{840} \\ T_{24} &= \frac{1}{6} \sum_{\kappa=4}^{7} c_\kappa \alpha_\kappa \left(\sum_{\lambda=3}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 3} \right) - \frac{1}{840} \\ T_{29} &= \frac{1}{24} \sum_{\kappa=3}^{7} c_\kappa \alpha_\kappa P_{\kappa 4} - \frac{1}{840} \end{aligned}\right\} \tag{99}$$

40. The expressions (99) can be simplified by further use of the equations of condition. We finally obtain:

$$\left.\begin{aligned}
T_{29} &= \frac{1}{24}\, c_7 (P_{74} - P_{94}) \\
T_{24} &= \frac{1}{6}\, c_7 (\beta_{83}P_{33} + \beta_{84}P_{43} + \beta_{85}P_{53} + \beta_{86}P_{63}) \\
T_{21} &= T_{24} - \frac{1}{6}\, c_7 (\beta_{83}\beta_{32} + \beta_{84}\beta_{42} + \beta_{85}\beta_{52} + \beta_{86}\beta_{62})\, \alpha_2^3 \\
T_{22} &= \frac{3}{2} \cdot \frac{\alpha_1}{\alpha_2} (T_{24} - T_{21}) + T_{21} \quad .
\end{aligned}\right\} \qquad (100)$$

We observe that only T_{22} depends on α_1. By a proper choice of α_1 we might be able to keep T_{22} small.

SECTION IX. EXAMPLE FOR A SIXTH-ORDER RUNGE-KUTTA FORMULA

41. It would present an extremely laborious and unpleasant problem to express all c- and β-coefficients by the free coefficients α_1, α_4, α_6 and to determine these α-coefficients in such a way that the error terms (100) become reasonably small.

However, since we have to consider these error terms to obtain an effective sixth-order Runge-Kutta formula, we used an electronic computer to determine the error terms T_{29}, T_{24}, T_{21}, T_{22}/α_1 for a variety of pairs of coefficients α_4 α_6.

We selected α_4 and α_6 from the following set: 1/2, 1/3, 2/3, 1/4, 3/4, ... using all positive fractions with denominators 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 14, 15. Inspecting the print-out of the above-mentioned error terms, we choose for our Runge-Kutta formula a combination of α_4 and α_6 that leads to reasonably small error terms.

We choose:

$$\alpha_4 = \frac{1}{2}, \quad \alpha_6 = \frac{6}{7} \quad (\alpha_1 = \frac{1}{2}\alpha_2). \tag{101}$$

This combination leads to:

$$\left.\begin{array}{l} T_{29} \approx +0.376 \cdot 10^{-5}, \quad T_{24} \approx -0.919 \cdot 10^{-4}, \quad T_{21} \approx -0.273 \cdot 10^{-4}, \\ T_{22} \approx -0.758 \cdot 10^{-4} \end{array}\right\} \tag{102}$$

The other non-zero error terms of our formula can then be computed from (97) and (98).

The Runge-Kutta coefficients of our formula were determined from the equations of Section VII and are listed in Table VIII.

TABLE VIII. RK 6 (7)

$\kappa \backslash \lambda$	α_κ	$\beta_{\kappa\lambda}$: 0	1	2	3	4	5	6	7	8	c_κ	$\hat{c}_\kappa$
0	0	0									$\frac{77}{1440}$	$\frac{11}{864}$
1	$\frac{2}{33}$	$\frac{2}{33}$									0	
2	$\frac{4}{33}$	0	$\frac{4}{33}$								0	
3	$\frac{2}{11}$	$\frac{1}{22}$	0	$\frac{3}{22}$							$\frac{1771561}{6289920}$	
4	$\frac{1}{2}$	$\frac{43}{64}$	0	$-\frac{165}{64}$	$\frac{77}{32}$						$\frac{32}{105}$	
5	$\frac{2}{3}$	$-\frac{2383}{486}$	0	$\frac{1067}{54}$	$-\frac{26312}{1701}$	$\frac{2176}{1701}$					$\frac{243}{2560}$	
6	$\frac{6}{7}$	$\frac{10077}{4802}$	0	$-\frac{5643}{686}$	$\frac{116259}{16807}$	$-\frac{6240}{16807}$	$\frac{1053}{2401}$				$\frac{16807}{74880}$	
7	1	$-\frac{733}{176}$	0	$\frac{141}{8}$	$-\frac{335763}{23296}$	$\frac{216}{77}$	$-\frac{4617}{2816}$	$\frac{7203}{9152}$			$\frac{11}{270}$	0
8	0	$\frac{15}{352}$	0	0	$-\frac{5445}{46592}$	$\frac{18}{77}$	$-\frac{1215}{5632}$	$\frac{1029}{18304}$	0			$\frac{11}{270}$
9	1	$-\frac{1833}{352}$	0	$\frac{141}{8}$	$-\frac{51237}{3584}$	$\frac{18}{7}$	$-\frac{729}{512}$	$\frac{1029}{1408}$	0	1		$\frac{11}{270}$

Truncation Error Term: $TE = \frac{11}{270}(f_0 + f_7 - f_8 - f_9)h$ (103)

SECTION X. NUMERICAL COMPARISON WITH OTHER SIXTH-ORDER RUNGE-KUTTA FORMULAS

42. The first sixth-order Runge-Kutta formulas were developed by A. HUŤA [7], [8]. We use his second formula ([8], p. 23), since it has somewhat simpler coefficients. HUŤA's formulas are based on eight evaluations of the differential equations per step.

Later, J. C. BUTCHER [9] and other authors found sixth-order Runge-Kutta formulas that require seven evaluations per step only. We consider one of BUTCHER's formulas ([9], p. 193) as of special interest, since this formula has favorably small truncation error coefficients.

We computed the error coefficients T_ν ($\nu = 1, 2, 3, \ldots, 48$) for HUŤA's formula as well as for BUTCHER's formula. Comparing them with our error coefficients (102), we found that HUŤA's largest coefficient T_{38} is more than 32 times as large as our largest coefficients $T_3 = -3T_{24}$ and $T_{23} = 3T_{24}$, while BUTCHER's largest coefficients T_6 and T_{26} are less than three times as large as our coefficients T_3 and T_{23}. While our formula RK6(7) of Table VIII has only 18 non-vanishing error coefficients T_ν, HUŤA's formula has 36 such coefficients and in BUTCHER's formula all 48 coefficients are different from zero.

43. For the numerical comparison of our formula RK6(7) with HUŤA's and BUTCHER's formulas, we again apply these formulas to our problem (53). We ran our formulas on the computer in exactly the same way and under exactly the same conditions as in Number 27, again using RICHARDSON's principle as stepsize control procedure for HUŤA's and BUTCHER's formulas. Table IX shows our results.

In this example our formula RK6(7) requires only approximately 58 percent of the evaluations and of the computer running time compared with BUTCHER's formula. If we compare our formula with the earlier HUŤA formula, this percentage decreases even more to approximately 40-42 percent.

TABLE IX. COMPARISON OF SIXTH-ORDER METHODS FOR EXAMPLE (53)

Method	Number of Substitutions per Step	Results for $x = 5$ and Tolerance 10^{-16}				
		Number of Steps	Total Number of Evaluations	Running Time on IBM-7094 (min)	Accumulated Errors in y and z	
					Δy	Δz
HUŤA	15	3596	53940	5.65	$-0.4863 \cdot 10^{-13}$	$-0.1731 \cdot 10^{-12}$
BUTCHER	13	2958	38454	3.87	$-0.5951 \cdot 10^{-13}$	$-0.1495 \cdot 10^{-12}$
RK6(7)	10	2245	22450	2.25	$-0.4641 \cdot 10^{-13}$	$-0.1079 \cdot 10^{-12}$

PART III. SEVENTH-ORDER FORMULAS

SECTION XI. THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

44. For a seventh-order Runge-Kutta formula RK7(8) with stepsize control we allow for thirteen evaluations of the differential equations per step:

$$\left.\begin{aligned} f_0 &= f(x_0,\ y_0) \\ f_\kappa &= f\left(x_0 + \alpha_\kappa h,\ y_0 + h \sum_{\lambda=0}^{\kappa-1} \beta_{\kappa\lambda} f_\lambda\right) \quad (\kappa = 1,2,3,\ldots,12) \end{aligned}\right\} \quad (104)$$

$$\left.\begin{aligned} y &= y_0 + h \sum_{\kappa=0}^{10} c_\kappa f_\kappa + 0(h^8) \\ \hat{y} &= y_0 + h \sum_{\kappa=0}^{12} \hat{c}_\kappa f_\kappa + 0(h^9) \end{aligned}\right\} \quad (105)$$

45. We now need the equations of condition for the eighth-order terms. These are also listed in BUTCHER's paper ([3], Table 1). There are 115 such equations, and we shall refer to them as equations (VIII, 1) through (VIII, 115) - in the same order as in BUTCHER's paper.

46. To reduce these 115 necessary and sufficient conditions to a system of simpler and fewer sufficient conditions, we make - very similar as in Part I and in Part II - the following assumptions:

$$\left.\begin{aligned} &\alpha_{10} = \alpha_{12} = 1,\quad \alpha_{11} = 0;\quad \hat{c}_1 = c_1 = 0,\quad \hat{c}_2 = c_2 = 0,\quad \hat{c}_3 = c_3 = 0, \\ &\hat{c}_4 = c_4 = 0,\quad c_5 = \hat{c}_5,\quad c_6 = \hat{c}_6,\quad c_7 = \hat{c}_7,\quad c_8 = \hat{c}_8,\quad c_9 = \hat{c}_9, \\ &\hat{c}_{10} = 0,\quad \hat{c}_{11} = \hat{c}_{12} = c_{10} \\ &\beta_{31} = \beta_{41} = \beta_{51} = \beta_{61} = \beta_{71} = \beta_{81} = \beta_{91} = \beta_{101} = \beta_{111} = \beta_{121} = 0 \\ &\beta_{52} = \beta_{62} = \beta_{72} = \beta_{82} = \beta_{92} = \beta_{102} = \beta_{112} = \beta_{122} = 0 \end{aligned}\right\} \quad (106)$$

and:

$$\text{(A1)} \qquad P_{\nu 1} = \frac{1}{2}\,\alpha_\nu^{2} \quad (\nu = 2, 3, 4, \ldots, 12) \qquad \text{(A1-}\nu\text{)}$$

$$\text{(A2)} \qquad P_{\nu 2} = \frac{1}{3}\,\alpha_\nu^{3} \quad (\nu = 2, 3, 4, \ldots, 12) \qquad \text{(A2-}\nu\text{)}$$

$$\text{(A3)} \qquad P_{\nu 3} = \frac{1}{4}\,\alpha_\nu^{4} \quad (\nu = 5, 6, 7, \ldots, 12) \qquad \text{(A3-}\nu\text{)}$$

$$\text{(B)} \left\{ \begin{array}{llll}
\sum_{\nu=5}^{9} c_\nu \beta_{\nu 3} + c_{10} & \left\{\begin{matrix} \beta_{103} \\ \beta_{113} + \beta_{123} \end{matrix}\right\} & = c_3 (1-\alpha_3) = 0 & \text{(B3)} \\
\sum_{\nu=5}^{9} c_\nu \beta_{\nu 4} + c_{10} & \left\{\begin{matrix} \beta_{104} \\ \beta_{114} + \beta_{124} \end{matrix}\right\} & = c_4 (1-\alpha_4) = 0 & \text{(B4)} \\
\sum_{\nu=6}^{9} c_\nu \beta_{\nu 5} + c_{10} & \left\{\begin{matrix} \beta_{105} \\ \beta_{115} + \beta_{125} \end{matrix}\right\} & = c_5 (1-\alpha_5) & \text{(B5)} \\
\sum_{\nu=7}^{9} c_\nu \beta_{\nu 6} + c_{10} & \left\{\begin{matrix} \beta_{106} \\ \beta_{116} + \beta_{126} \end{matrix}\right\} & = c_6 (1-\alpha_6) & \text{(B6)} \\
\sum_{\nu=8}^{9} c_\nu \beta_{\nu 7} + c_{10} & \left\{\begin{matrix} \beta_{107} \\ \beta_{117} + \beta_{127} \end{matrix}\right\} & = c_7 (1-\alpha_7) & \text{(B7)} \\
c_9 \beta_{98} + c_{10} & \left\{\begin{matrix} \beta_{108} \\ \beta_{118} + \beta_{128} \end{matrix}\right\} & = c_8 (1-\alpha_8) & \text{(B8)} \\
c_{10} & \left\{\begin{matrix} \beta_{109} \\ \beta_{119} + \beta_{129} \end{matrix}\right\} & = c_9 (1-\alpha_9) & \text{(B9)} \\
c_{10} & (\beta_{1110} + \beta_{1210}) & = c_{10} (1-\alpha_{10}) = 0 & \text{(B10)} \\
c_{10} & \beta_{1211} & = c_{10} (1-\alpha_{11}) = c_{10} & \text{(B11)}
\end{array} \right.$$

$$
(C)\quad\left\{\begin{array}{ll}
\sum_{\mu=5}^{9} c_\mu\,\alpha_\mu\,\beta_{\mu 3} + c_{10}\left\{\begin{matrix}\beta_{103}\\ \beta_{123}\end{matrix}\right\} = 0 & \text{(C1)}\\[2ex]
\sum_{\mu=5}^{9} c_\mu\,\alpha_\mu\,\beta_{\mu 4} + c_{10}\,\beta_{124} = 0 & \text{(C2)}\\[2ex]
\sum_{\mu=5}^{9} c_\mu\,\alpha_\mu^{\,2}\beta_{\mu 3} + c_{10}\,\beta_{123} = 0 & \text{(C3)}\\[2ex]
\sum_{\mu=5}^{9} c_\mu\,\alpha_\mu^{\,2}\beta_{\mu 4} + c_{10}\,\beta_{124} = 0 & \text{(C4)}\\[2ex]
\sum_{\mu=6}^{9} c_\mu\,\alpha_\mu\left(\sum_{\nu=5}^{\mu-1}\beta_{\mu\nu}\,\beta_{\nu 3}\right) + c_{10}\sum_{\nu=5}^{11}\beta_{12\nu}\,\beta_{\nu 3} = 0 & \text{(C5)}\\[2ex]
\sum_{\mu=6}^{9} c_\mu\,\alpha_\mu\left(\sum_{\nu=5}^{\mu-1}\beta_{\mu\nu}\,\beta_{\nu 4}\right) + c_{10}\sum_{\nu=6}^{11}\beta_{12\nu}\,\beta_{\nu 4} = 0 & \text{(C6)}
\end{array}\right.
$$

47. The reduction leads to equations (VIII, 57), (VIII, 68), (VIII, 86), and (VIII, 115) as the only four independent equations for the 8-th order terms. After the reduction we again arrange the remaining independent equations of condition in two groups (D) and (E) in the same way as before:

$$
(D)\quad\left\{\begin{array}{ll}
\left\{\begin{matrix}c_0\\ \hat{c}_0\end{matrix}\right\} + \sum_{\nu=5}^{9} c_\nu + \left\{\begin{matrix}c_{10}\\ 2c_{10}\end{matrix}\right\} = 1 & \text{(D0)}\\[2ex]
\sum_{\nu=5}^{10} c_\nu\,\alpha_\nu^{\,\mu} = \frac{1}{\mu+1}\quad (\mu = 1, 2, 3, \ldots, 7) & \text{(D}\mu\text{)}
\end{array}\right.
$$

$$
\text{(E)}\left\{
\begin{aligned}
&\sum_{\mu=5}^{9} c_\mu \alpha_\mu P_{\mu 4} + c_{10} \begin{Bmatrix} P_{104} \\ P_{124} \end{Bmatrix} = \frac{1}{35} && \text{(E1)} \\
&\sum_{\mu=5}^{9} c_\mu \alpha_\mu \left(\sum_{\nu=4}^{\mu-1} \beta_{\mu\nu} P_{\nu 3} \right) + c_{10} \begin{Bmatrix} \sum\limits_{\nu=4}^{9} \beta_{10\,\nu} P_{\nu 3} \\ \sum\limits_{\nu=4}^{11} \beta_{12\,\nu} P_{\nu 3} \end{Bmatrix} = \frac{1}{140} && \text{(E2)} \\
&\sum_{\mu=5}^{9} c_\mu \alpha_\mu P_{\mu 5} + c_{10} P_{125} = \frac{1}{48} && \text{(E3)} \\
&\sum_{\mu=5}^{9} c_\mu \alpha_\mu^2 P_{\mu 4} + c_{10} P_{124} = \frac{1}{40} && \text{(E4)} \\
&\sum_{\mu=6}^{9} c_\mu \alpha_\mu \left(\sum_{\nu=5}^{\mu-1} \beta_{\mu\nu} P_{\nu 4} \right) + c_{10} \sum_{\nu=5}^{11} \beta_{12\,\nu} P_{\nu 4} = \frac{1}{240} && \text{(E5)}
\end{aligned}
\right.
$$

Obviously, equation (D7) represents equation (VIII, 115) of BUTCHER's table, while equations (E3), (E4), (E5) are identical with BUTCHER's equations (VIII, 68), (VIII, 86), and (VIII, 57). Equations (E1) and (E2) originate from seventh-order terms and, under our new assumptions, are identical with equations (E2) and (E4) of Part II. Equations (E1) and (E3) of Part II may be omitted now, since, because of our new assumptions (A3), they now become identical with (D5) and (D6).

SECTION XII. A SOLUTION OF THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

48. From equations (D1) through (D7) we obtain a relation between α_5, α_6, α_7, α_8, α_9 as condition of compatibility. The relation can be written in the form:

$$\alpha_5 = \frac{1}{2}\,\frac{70\alpha_6\alpha_7\alpha_8\alpha_9 - 42(\alpha_6\alpha_7\alpha_8 + \ldots + \alpha_7\alpha_8\alpha_9) + 28(\alpha_6\alpha_7 + \ldots + \alpha_8\alpha_9) - 20(\alpha_6 + \ldots + \alpha_9) + 15}{70\alpha_6\alpha_7\alpha_8\alpha_9 - 35(\alpha_6\alpha_7\alpha_8 + \ldots + \alpha_7\alpha_8\alpha_9) + 21(\alpha_6\alpha_7 + \ldots + \alpha_8\alpha_9) - 14(\alpha_6 + \ldots + \alpha_9) + 10} \qquad (107)$$

This rather lengthy expression simplifies to $\alpha_5 = 1/2$, if we require $\alpha_6 + \alpha_7 = 1$ and $\alpha_8 + \alpha_9 = 1$. Therefore, we assume for the following:

$$\alpha_5 = \frac{1}{2}, \qquad \alpha_6 + \alpha_7 = 1, \qquad \alpha_8 + \alpha_9 = 1, \qquad (108)$$

since such a choice of α_5, α_6, α_7, α_8, α_9 also leads to rather simple expressions for the weight coefficients c_5, c_6, ..., c_{10}. We obtain:

$$\left.\begin{aligned}
c_5 &= \frac{16}{105}\cdot\frac{14(5\alpha_6^2 - 5\alpha_6 + 1)\,\alpha_8^2 - 14\,(5\alpha_6^2 - 5\alpha_6 + 1)\,\alpha_8 + (14\alpha_6^2 - 14\alpha_6 + 3)}{(1-2\alpha_6)^2(1-2\alpha_8)^2} \\
c_6 = c_7 &= \frac{1}{420}\cdot\frac{7\alpha_8(1-\alpha_8)-1}{\alpha_6(1-\alpha_6)(1-2\alpha_6)^2(\alpha_8-\alpha_6)(1-\alpha_6-\alpha_8)} \\
c_8 = c_9 &= -\frac{1}{420}\cdot\frac{7\alpha_6(1-\alpha_6)-1}{\alpha_8(1-\alpha_8)(1-2\alpha_8)^2(\alpha_8-\alpha_6)(1-\alpha_6-\alpha_8)} \\
c_{10} &= \frac{1}{420}\cdot\frac{70\alpha_6(1-\alpha_6)\alpha_8(1-\alpha_8) - 7[\alpha_6(1-\alpha_6) + \alpha_8(1-\alpha_8)] + 1}{\alpha_6(1-\alpha_6)\alpha_8(1-\alpha_8)}
\end{aligned}\right\} \qquad (109)$$

Naturally, we have assumed in (109) that α_6, α_7, α_8, α_9 are different from one another and different from 0, 1/2, 1.

49. Further restrictive conditions between the α-coefficients shall now be established.

From (A1-2) and (A2-2), it follows:

$$\alpha_1 = \frac{2}{3} \alpha_2 \tag{110}$$

and from (A1-3) and (A2-3):

$$\alpha_2 = \frac{2}{3} \alpha_3 \tag{111}$$

From (A1-5), (A2-5), and (A3-5), we obtain the following relation between α_3 and α_4:

$$\alpha_3 = \frac{1}{2} \alpha_5 \frac{4\alpha_4 - 3\alpha_5}{3\alpha_4 - 2\alpha_5} \tag{112}$$

or because of $\alpha_5 = 1/2$:

$$\alpha_3 = \frac{1}{8} \cdot \frac{8\alpha_4 - 3}{3\alpha_4 - 1} \tag{112a}$$

From (108), (110), (111), (112a), it follows that all α-coefficients can be expressed by α_4, α_6, α_8.

Later in No.52, however, we shall show that α_8 is already determined by our equations of condition and by our assumptions. Therefore, α_4 and α_6 shall be the only free α-coefficients.

50. We now determine the β-coefficients. From (A1-2) we obtain:

$$\beta_{21} = \frac{3}{4} \alpha_2 \tag{113}$$

and from (A1-3) and (111):

$$\beta_{32} = \frac{3}{4} \alpha_3 \tag{114}$$

Equations (A1-4) and (A2-4) lead to:

$$\left.\begin{aligned} \beta_{42} &= \frac{1}{6}\,\frac{\alpha_4^2}{\alpha_3-\alpha_2}\cdot\frac{3\alpha_3-2\alpha_4}{\alpha_2} \\ \beta_{43} &= -\frac{1}{6}\,\frac{\alpha_4^2}{\alpha_3-\alpha_2}\cdot\frac{3\alpha_2-2\alpha_4}{\alpha_3} \end{aligned}\right\} \quad (115)$$

and equations (A1-5) and (A2-5) lead to:

$$\left.\begin{aligned} \beta_{53} &= \frac{1}{6}\,\frac{\alpha_5^2}{\alpha_4-\alpha_3}\cdot\frac{3\alpha_4-2\alpha_5}{\alpha_3} \\ \beta_{54} &= -\frac{1}{6}\,\frac{\alpha_5^2}{\alpha_4-\alpha_3}\cdot\frac{3\alpha_3-2\alpha_5}{\alpha_4} \end{aligned}\right\} \quad (116)$$

Equations (A1-6), (A2-6) and (A3-6) represent three linear equations for the three unknowns β_{63}, β_{64}, and β_{65}. Their solution reads:

$$\left.\begin{aligned} \beta_{63} &= \frac{1}{12}\,\alpha_6^2\,\frac{6\alpha_4\alpha_5-4(\alpha_4+\alpha_5)\alpha_6+3\alpha_6^2}{\alpha_3(\alpha_4-\alpha_3)(\alpha_5-\alpha_3)} \\ \beta_{64} &= \frac{1}{12}\,\alpha_6^2\,\frac{6\alpha_3\alpha_5-4(\alpha_3+\alpha_5)\alpha_6+3\alpha_6^2}{\alpha_4(\alpha_3-\alpha_4)(\alpha_5-\alpha_4)} \\ \beta_{65} &= \frac{1}{12}\,\alpha_6^2\,\frac{6\alpha_3\alpha_4-4(\alpha_3+\alpha_4)\alpha_6+3\alpha_6^2}{\alpha_5(\alpha_4-\alpha_5)(\alpha_3-\alpha_5)} \end{aligned}\right\} \quad (117)$$

Assuming:

$$\beta_{73} = 0 \quad (118)$$

we can determine, in the same way, β_{74}, β_{75}, β_{76} from equations (A1-7), (A2-7), (A3-7):

$$\left.\begin{aligned} \beta_{74} &= \frac{1}{12}\,\alpha_7^2\,\frac{6\alpha_5\alpha_6-4(\alpha_5+\alpha_6)\alpha_7+3\alpha_7^2}{\alpha_4(\alpha_5-\alpha_4)(\alpha_6-\alpha_4)} \\ \beta_{75} &= \frac{1}{12}\,\alpha_7^2\,\frac{6\alpha_4\alpha_6-4(\alpha_4+\alpha_6)\alpha_7+3\alpha_7^2}{\alpha_5(\alpha_4-\alpha_5)(\alpha_6-\alpha_5)} \\ \beta_{76} &= \frac{1}{12}\,\alpha_7^2\,\frac{6\alpha_4\alpha_5-4(\alpha_4+\alpha_5)\alpha_7+3\alpha_7^2}{\alpha_6(\alpha_4-\alpha_6)(\alpha_5-\alpha_6)} \end{aligned}\right\} \quad (119)$$

From equations (B3), (C1) and (C3), we can now determine β_{83}, β_{93}, and $\beta_{103} = \beta_{123}$:

$$\left.\begin{aligned}
\beta_{83} &= -\frac{c_5(\alpha_9-\alpha_5)(1-\alpha_5)\beta_{53} + c_6(\alpha_9-\alpha_6)(1-\alpha_6)\beta_{63}}{c_8(\alpha_9-\alpha_8)(1-\alpha_8)} \\
\beta_{93} &= -\frac{c_5(\alpha_8-\alpha_5)(1-\alpha_5)\beta_{53} + c_6(\alpha_8-\alpha_6)(1-\alpha_6)\beta_{63}}{c_9(\alpha_8-\alpha_9)(1-\alpha_9)} \\
\beta_{103} = \beta_{123} &= -\frac{c_5(\alpha_8-\alpha_5)(\alpha_9-\alpha_5)\beta_{53} + c_6(\alpha_8-\alpha_6)(\alpha_9-\alpha_6)\beta_{63}}{c_{10}(1-\alpha_8)(1-\alpha_9)}
\end{aligned}\right\} \quad (120)$$

Assuming:

$$\beta_{104} = \beta_{124} \ , \qquad (121)$$

we obtain, in the same way, β_{84}, β_{94}, and $\beta_{104} = \beta_{124}$ from equations (B4), (C2), and (C4):

$$\left.\begin{aligned}
\beta_{84} &= -\frac{c_5(\alpha_9-\alpha_5)(1-\alpha_5)\beta_{54} + c_6(\alpha_9-\alpha_6)(1-\alpha_6)\beta_{64} + c_7(\alpha_9-\alpha_7)(1-\alpha_7)\beta_{74}}{c_8(\alpha_9-\alpha_8)(1-\alpha_8)} \\
\beta_{94} &= -\frac{c_5(\alpha_8-\alpha_5)(1-\alpha_5)\beta_{54} + c_6(\alpha_8-\alpha_6)(1-\alpha_6)\beta_{64} + c_7(\alpha_8-\alpha_7)(1-\alpha_7)\beta_{74}}{c_9(\alpha_8-\alpha_9)(1-\alpha_9)} \\
\beta_{104} = \beta_{124} &= -\frac{c_5(\alpha_8-\alpha_5)(\alpha_9-\alpha_5)\beta_{54} + c_6(\alpha_8-\alpha_6)(\alpha_9-\alpha_6)\beta_{64} + c_7(\alpha_8-\alpha_7)(\alpha_9-\alpha_7)\beta_{74}}{c_{10}(1-\alpha_8)(1-\alpha_9)}
\end{aligned}\right\} \quad (122)$$

51. From equations (B) we obtain by multiplying equation (B3) with α_3^4, equation (B4) with α_4^4, etc., adding all these equations and using equations (D4) and (D5):

$$c_5 P_{54} + c_6 P_{64} + c_7 P_{74} + c_8 P_{84} + c_9 P_{94} + c_{10} P_{104} = \frac{1}{30} \qquad (123)$$

This equation together with equations (E1) and (E4) can be considered as a system of three equations for the three quantities P_{84}, P_{94}, P_{104}, since P_{54}, P_{64}, P_{74} are already known from No.50.

The system has the following solution:

$$
\left.\begin{aligned}
c_8(\alpha_9-\alpha_8)(1-\alpha_8)P_{84} &= \frac{1}{840}(4\alpha_9-3)-c_5(\alpha_9-\alpha_5)(1-\alpha_5)P_{54}-c_3(\alpha_9-\alpha_6)(1-\alpha_6)P_{64}-c_7(\alpha_9-\alpha_7)(1-\alpha_7)P_{74}\\
c_9(\alpha_9-\alpha_9)(1-\alpha_9)P_{94} &= \frac{1}{840}(4\alpha_8-3)-c_5(\alpha_8-\alpha_5)(1-\alpha_5)P_{54}-c_6(\alpha_8-\alpha_6)(1-\alpha_8)P_{64}-c_7(\alpha_8-\alpha_7)(1-\alpha_7)P_{74}\\
c_{10}(1-\alpha_8)(1-\alpha_9)P_{104} &= \frac{1}{840}[28\alpha_8\alpha_9-24(\alpha_8+\alpha_9)+21]-c_5(\alpha_9-\alpha_5)(\alpha_8-\alpha_5)P_{54}-c_3(\alpha_9-\alpha_6)(\alpha_8-\alpha_6)P_{64}\\
&\qquad -c_7(\alpha_9-\alpha_7)(\alpha_8-\alpha_7)P_{74}
\end{aligned}\right\}\quad(124)
$$

The quantities P_{54}, P_{64}, P_{74} which occur on the right-hand sides of equations (124) can be expressed by α_4 and α_6. Using the results of No.49 and No.50 we find:

$$
\left.\begin{aligned}
P_{54} &= \frac{1}{1536}\cdot\frac{1}{3\alpha_4-1}\,(8\alpha_4^2+24\alpha_4-9)\\
P_{64} &= \frac{\alpha_6^2}{96}\cdot\frac{1}{3\alpha_4-1}\,[8\alpha_4^2(9\alpha_6^2-10\alpha_6+3)+3\alpha_4(12\alpha_6^2+4\alpha_6-3)-3\alpha_6(7\alpha_6-2)]\\
P_{74} &= \frac{1}{24}\,(1-\alpha_6)^2[(6\alpha_6^3-5\alpha_6^2-4\alpha_6+3)+\alpha_4(14\alpha_6^2-10\alpha_6+2)]
\end{aligned}\right\}(125)
$$

52. Equations (A1-8), (A2-8), (A3-8) and the definition of P_{84} may be written in the form:

$$
\left.\begin{aligned}
\beta_{85}\alpha_5+\beta_{86}\alpha_6+\beta_{87}\alpha_7 &= \frac{1}{2}\alpha_8^2-\beta_{83}\alpha_3-\beta_{84}\alpha_4\\
\beta_{85}\alpha_5^2+\beta_{86}\alpha_6^2+\beta_{87}\alpha_7^2 &= \frac{1}{3}\alpha_8^3-\beta_{83}\alpha_3^2-\beta_{84}\alpha_4^2\\
\beta_{85}\alpha_5^3+\beta_{86}\alpha_6^3+\beta_{87}\alpha_7^3 &= \frac{1}{4}\alpha_8^4-\beta_{83}\alpha_3^3-\beta_{84}\alpha_4^3\\
\beta_{85}\alpha_5^4+\beta_{86}\alpha_6^4+\beta_{87}\alpha_7^4 &= P_{84}-\beta_{83}\alpha_3^4-\beta_{84}\alpha_4^4
\end{aligned}\right\}(126)
$$

Since we know β_{83}, β_{84}, P_{84} from equations (120), (122), (124), we can consider equations (126) as a system of four linear equations for the three unknowns β_{85}, β_{86}, β_{87}.

As condition of compatibility, we obtain from (126) the following relation:

$$\left.\begin{aligned}&\frac{1}{12}\alpha_8^2[6\alpha_5\alpha_6\alpha_7 - 4\alpha_8(\alpha_5\alpha_6 + \alpha_5\alpha_7 + \alpha_6\alpha_7) + 3\alpha_8^2(\alpha_5 + \alpha_6 + \alpha_7)] - P_{84}\\ &\qquad - \beta_{83}\alpha_3(\alpha_5 - \alpha_3)(\alpha_6 - \alpha_3)(\alpha_7 - \alpha_3) - \beta_{84}\alpha_4(\alpha_5 - \alpha_4)(\alpha_6 - \alpha_4)(\alpha_7 - \alpha_4) = 0\end{aligned}\right\}\quad (127)$$

In (127) we now have to insert our expressions for P_{84}, β_{83}, and β_{84} and express all β- and c-coefficients by the α-coefficients. Finally, we have to insert our restrictive conditions (108) and(112a) for the α-coefficients, thereby converting (127) into a condition between α_6 and α_8 since the terms with α_4 drop out.

A tedious but straightforward computation finally reduces the left-hand side of (127) to a product of several factors. One factor is a function of α_8 only and can be made zero by a proper choice of α_8. The other factors contain α_6 also. They can not be made zero for rational values α_6, α_8, if we assume that α_6, α_7, α_8, α_9 be different from another and different from 0, 1/2, 1, in order to prevent the denominators of (109) from becoming zero.

If we omit all factors that are different from zero, we are finally left with:

$$6\alpha_8^2 - 7\alpha_8 + 2 = 0 \; . \qquad (128)$$

From (128) follows $\alpha_8 = 1/2$ or $\alpha_8 = 2/3$. Since $\alpha_8 = 1/2$ has again to be discarded, we obtain:

$$\alpha_8 = \frac{2}{3} \qquad (129)$$

as resulting from condition (127).

The restrictive conditions (108), (110), (111), (112), (129) reduce the independent α-coefficients to two, namely α_4 and α_6.

53. Since the computation of the remaining β-coefficients and auxiliary expressions is straightforward, we just indicate from which equations they are obtained without stating the explicit expressions for them:

β_{85}, β_{86}, β_{87} from (A1-8), (A2-8), (A3-8)

P_{94}, $P_{104} = P_{124}$ from (E1), (E4)

β_{95}, β_{96}, β_{97}, β_{98} from (A1-9), (A2-9), (A3-9), P_{94}

β_{105}, β_{106}, β_{107}, β_{108}, β_{109} from (B5), (B6), (B7), (B8), (B9)

β_{125}, β_{126}, β_{127}, β_{128}, β_{129}. from (A1-12), (A2-12), (A3-12), (C5), (C6)

$\beta_{1210} = 0$, $\beta_{1211} = 1$ from (B10), (B11)

$$\left.\begin{array}{l} \beta_{113} = 0,\ \beta_{114} = 0,\ \beta_{115},\ \beta_{116} \\ \beta_{117},\ \beta_{118},\ \beta_{119},\ \beta_{1110} = 0 \end{array}\right\}\text{from}\left\{\begin{array}{l} \text{(B3), (B4), (B5), (B6)} \\ \text{(B7), (B8), (B9), (B10)} \end{array}\right.$$

All Runge-Kutta coefficients for our pair of seventh- and eighth-order formulas are now known if we still determine c_0, $\hat{c}_0$ from equation (D0) and the coefficients $\beta_{\nu 0}$ ($\nu = 1, 2, 3, \ldots, 12$) from the extended equations (F). If we submit the α-coefficients to the restrictive conditions (108), (110), (111), (112), (129), the Runge-Kutta coefficients satisfy all our equations of condition.

SECTION XIII. THE LEADING TERM OF THE LOCAL TRUNCATION ERROR

54. There are 115 terms T_ν ($\nu = 1, 2, 3, \ldots, 115$) that contribute to the leading truncation error term of a seventh-order Runge-Kutta formula. However, because of our assumptions and our conditions (A1), (A2), (A3), (B), and (C), most terms T_ν are zero. Indeed, there are only 40 error coefficients T_ν that are different from zero. All those 40 coefficients T_ν are multiples of four coefficients T_{49}, T_{52}, T_{57}, and T_{68}:

$$\left.\begin{array}{llllll}
T_1 = -T_{49}, & T_2 = -T_{52}, & T_3 = -3T_{52}, & T_4 = -T_{52}, & T_{50} = T_{52}, & T_{51} = 3T_{52} \\
T_5 = -4T_{57}, & T_6 = -4T_{57}, & T_7 = -3T_{57}, & T_8 = -6T_{57}, & T_9 = -T_{57}, & T_{53} = 4T_{57} \\
T_{54} = 4T_{57}, & T_{55} = 3T_{57}, & T_{56} = 6T_{57}, & T_{10} = -5T_{68}, & T_{11} = -5T_{68}, & T_{12} = -15T_{68} \\
T_{13} = -5T_{68}, & T_{14} = -10T_{68}, & T_{15} = -10T_{68}, & T_{16} = -10T_{68}, & T_{17} = -10T_{68}, & T_{18} = -15T_{68} \\
T_{19} = -10T_{68}, & T_{20} = -T_{68}, & T_{58} = 5T_{68}, & T_{59} = 5T_{68}, & T_{60} = 15T_{68}, & T_{61} = 5T_{68} \\
T_{62} = 10T_{68}, & T_{63} = 10T_{68}, & T_{64} = 10T_{68}, & T_{65} = 10T_{68}, & T_{66} = 15T_{68}, & T_{67} = 10T_{68}
\end{array}\right\} \quad (130)$$

55. The four basic error coefficients T_{49}, T_{52}, T_{57}, T_{68} can be written in the following form:

$$\left.\begin{array}{l}
T_{49} = \sum\limits_{\kappa=6}^{10} c_\kappa \alpha_\kappa \left\langle \sum\limits_{\lambda=5}^{\kappa-1} \beta_{\kappa\lambda} \left\{ \sum\limits_{\mu=4}^{\lambda-1} \beta_{\lambda\mu} \left[\sum\limits_{\nu=3}^{\mu-1} \beta_{\mu\nu} \left(\sum\limits_{\rho=2}^{\nu-1} \beta_{\nu\rho} P_{\rho 1} \right) \right] \right\} \right\rangle - \dfrac{1}{5760} \\[2ex]
T_{52} = \dfrac{1}{6} \sum\limits_{\kappa=5}^{10} c_\kappa \alpha_\kappa \left[\sum\limits_{\lambda=4}^{\kappa-1} \beta_{\kappa\lambda} \left(\sum\limits_{\mu=3}^{\lambda-1} \beta_{\lambda\mu} P_{\mu 3} \right) \right] - \dfrac{1}{5760} \\[2ex]
T_{57} = \dfrac{1}{24} \sum\limits_{\kappa=5}^{10} c_\kappa \alpha_\kappa \left(\sum\limits_{\lambda=4}^{\kappa-1} \beta_{\kappa\lambda} P_{\lambda 4} \right) - \dfrac{1}{5760} \\[2ex]
T_{68} = \dfrac{1}{120} \sum\limits_{\kappa=5}^{10} c_\kappa \alpha_\kappa P_{\kappa 5} - \dfrac{1}{5760}
\end{array}\right\} \quad (131)$$

SECTION XIV. EXAMPLE FOR A SEVENTH-ORDER RUNGE-KUTTA FORMULA

56. For our seventh-order Runge -Kutta formula we select for the independent parameters α_4, α_6 such a pair of values for which the four error terms (131) become small.

Since it seems hardly possible to find such a pair of values α_4, α_6 analytically, we again ran a variety of combinations α_4, α_6 on an electronic computer and printed out the error terms (131) for each combination. Inspecting these printouts, we chose for our Runge-Kutta formula the following combination:

$$\alpha_4 = 5/12, \quad \alpha_6 = 5/6 . \tag{132}$$

This combination results in rather small values for our error terms (131):

$$T_{49} \approx 0.164 \cdot 10^{-5}, \; T_{52} \approx -0.567 \cdot 10^{-6}, \; T_{57} \approx 0.765 \cdot 10^{-6}, \; T_{68} \approx -0.919 \cdot 10^{-7} \tag{133}$$

It also leads to relatively simple Runge-Kutta coefficients, which are listed in Table X.

TABLE X. RK 7(8)

$\kappa \backslash \lambda$	α_κ	$\beta_{\kappa\lambda}$ 0	1	2	3	4	5	6	7	8	9	10	11	c_κ	$\hat{c}_\kappa$
0	0	0												$\frac{41}{840}$	0
1	$\frac{2}{27}$	$\frac{2}{27}$												0	
2	$\frac{1}{9}$	$\frac{1}{36}$	$\frac{1}{12}$											0	
3	$\frac{1}{6}$	$\frac{1}{24}$	0	$\frac{1}{8}$										0	
4	$\frac{5}{12}$	$\frac{5}{12}$	0	$-\frac{25}{16}$	$\frac{25}{16}$									0	
5	$\frac{1}{2}$	$\frac{1}{20}$	0	0	$\frac{1}{4}$	$\frac{1}{5}$								$\frac{34}{105}$	
6	$\frac{5}{6}$	$-\frac{25}{108}$	0	0	$\frac{125}{108}$	$-\frac{65}{27}$	$\frac{125}{54}$							$\frac{9}{35}$	
7	$\frac{1}{6}$	$\frac{31}{300}$	0	0	0	$\frac{61}{225}$	$-\frac{2}{9}$	$\frac{13}{900}$						$\frac{9}{35}$	
8	$\frac{2}{3}$	2	0	0	$-\frac{53}{6}$	$\frac{704}{45}$	$-\frac{107}{9}$	$\frac{67}{90}$	3					$\frac{9}{280}$	
9	$\frac{1}{3}$	$-\frac{91}{108}$	0	0	$\frac{23}{108}$	$-\frac{976}{135}$	$\frac{311}{54}$	$-\frac{19}{60}$	$\frac{17}{6}$	$-\frac{1}{12}$				$\frac{9}{280}$	
10	1	$\frac{2383}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{301}{82}$	$\frac{2133}{4100}$	$\frac{45}{82}$	$\frac{45}{164}$	$\frac{18}{41}$			$\frac{41}{840}$	0
11	0	$\frac{3}{205}$	0	0	0	0	$-\frac{6}{41}$	$-\frac{3}{205}$	$-\frac{3}{41}$	$\frac{3}{41}$	$\frac{6}{41}$	0			$\frac{41}{840}$
12	1	$-\frac{1777}{4100}$	0	0	$-\frac{341}{164}$	$\frac{4496}{1025}$	$-\frac{289}{82}$	$\frac{2193}{4100}$	$\frac{51}{82}$	$\frac{33}{164}$	$\frac{12}{41}$	0	1		$\frac{41}{840}$

Truncation Error Term: $TE = \frac{41}{840}(f_0 + f_{10} - f_{11} - f_{12})h$ (134)

SECTION XV. NUMERICAL COMPARISON WITH OTHER SEVENTH-ORDER RUNGE-KUTTA FORMULAS

57. The only other 7-th order Runge-Kutta formula, known in the literature, is due to E.B. SHANKS ([10], p.34). It is based on nine evaluations of the differential equations per step.

Computing the error coefficients $T_\nu (\nu = 1, 2, 3, \ldots, 115)$ for SHANKS's formula, we found that these coefficients are rather large compared with the coefficients (130), (133), of our formula RK7(8).

While our formula RK7(8) contains only 40 non-zero error coefficients T_ν, SHANKS's 7-th order formula has 102 non-zero terms T_ν. The largest coefficient T_ν in SHANKS's formula is $-0.353 \cdot 10^{-2}$, and there are 15 terms T_ν in SHANKS's formula which are larger in absolute value than 10^{-3}. The largest coefficient T_ν of our formula RK7(8) is $-0.459 \cdot 10^{-5}$ and in our formula there are 16 coefficients T_ν which are larger in absolute value than 10^{-6} and 24 such coefficients which are smaller in absolute value than 10^{-6}.

58. For the numerical comparison of our formula RK7(8) with SHANKS's formula we applied these formulas again to our problem (53) in exactly the same way as in Part I and Part II. We again used RICHARDSON's principle as stepsize control procedure for SHANKS's formula since no other satisfactory stepsize control procedure seems to be known for SHANKS's formulas. Table XI shows the results of our comparison.

TABLE XI. COMPARISON OF SEVENTH-ORDER METHODS FOR EXAMPLE (53)

Method	Number of Substitutions per Step	Results for $x = 5$ and Tolerance 10^{-16}				
		Number of Steps	Total Number of Evaluations	Running Time on IBM-7094 (min)	Accumulated Errors in y and z	
					Δy	Δz
SHANKS	17	1423	24 191	2,49	$-0,1332 \cdot 10^{-13}$	$-0,7377 \cdot 10^{-13}$
RK7(8)	13	818	10 634	1,12	$-0,2509 \cdot 10^{-13}$	$-0,5135 \cdot 10^{-13}$

The table shows that, in this example, the computer running time for our formula RK7(8) is only about 45% of the running time of SHANKS's 7-th order formula. The relatively large number of integration steps (1423), required by SHANKS's formula, reflects the unfavorable magnitude of the error coefficients of SHANKS's formula.

PART IV. EIGHTH-ORDER FORMULAS

SECTION XVI. THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

59. An eighth-order Runge-Kutta formula RK8(9) with stepsize control can be established if we allow for seventeen evaluations of the differential equations per step:

$$\left.\begin{aligned} f_0 &= f(x_0\,,\ y_0) \\ f_\kappa &= f(x_0 + \alpha_\kappa h,\ y_0 + h\sum_{\lambda=0}^{\kappa-1} \beta_{\kappa\lambda} f_\lambda) \qquad (\kappa = 1, 2, 3, \ldots, 16) \end{aligned}\right\} \quad (135)$$

$$\left.\begin{aligned} y &= y_0 + h\sum_{\kappa=0}^{14} c_\kappa f_\kappa + 0(h^9) \\ \hat{y} &= y_0 + h\sum_{\kappa=0}^{16} \hat{c}_\kappa f_\kappa + 0(h^{10}) \end{aligned}\right\} \quad (136)$$

60. Since BUTCHER's paper ([3], Table I) lists the equations of condition up to the eighth-order only, we have to find the ninth-order equations that we need for our stepsize control by either extending BUTCHER's table or by using a scheme suggested by SHANKS [10].

We like to mention, however, that more recently another approach, based on quadrature formulas and also applicable to Runge-Kutta formulas of any order, was described by C. STIMBERG ([11], pp.1/9). For fourth-order Runge-Kutta formulas, C. STIMBERG has presented his method in more detail in another paper [12]. For a similar approach based on quadrature formulas see also J.S. ROSEN [13].

61. There are 286 ninth-order equations of condition. However, this number of equations is greatly reduced if we make - very similar as in Parts I, II, and III - the following assumptions:

$$\left.\begin{array}{l}
\alpha_{14}=\alpha_{16}=1, \quad \alpha_{15}=0, \quad \hat{c}_1=c_1=0, \quad \hat{c}_2=c_2=0, \quad \ldots, \quad \hat{c}_7=c_7=0, \\
\hat{c}_8=c_8, \quad \hat{c}_9=c_9, \quad \ldots, \quad \hat{c}_{13}=c_{13}, \quad \hat{c}_{14}=0, \quad \hat{c}_{15}=\hat{c}_{16}=c_{14} \\
\beta_{31}=\beta_{41}=\beta_{51}=\ldots=\beta_{161}=0, \quad \beta_{52}=\beta_{62}=\beta_{72}=\ldots=\beta_{162}=0 \\
\beta_{73}=\beta_{83}=\beta_{93}=\ldots=\beta_{163}=0, \quad \beta_{84}=\beta_{94}=\beta_{104}=\ldots=\beta_{164}=0
\end{array}\right\} \quad (137)$$

and:

(A1) $$P_{\nu 1}=\frac{1}{2}\alpha_\nu^2 \quad (\nu=2,3,4,\ldots,16)$$ (A1-ν)

(A2) $$P_{\nu 2}=\frac{1}{3}\alpha_\nu^3 \quad (\nu=2,3,4,\ldots,16)$$ (A2-ν)

(A3) $$P_{\nu 3}=\frac{1}{4}\alpha_\nu^4 \quad (\nu=5,6,7,\ldots,16)$$ (A3-ν)

(A4) $$P_{\nu 4}=\frac{1}{5}\alpha_\nu^5 \quad (\nu=8,9,10,\ldots,16)$$ (A4-ν)

(B)

$$\sum_{\nu=8}^{13} c_\nu\beta_{\nu 5}+c_{14}\left\{\begin{matrix}\beta_{145}\\ \beta_{155}+\beta_{165}\end{matrix}\right\}=c_5(1-\alpha_5)=0 \quad \text{(B5)}$$

$$\sum_{\nu=8}^{13} c_\nu\beta_{\nu 6}+c_{14}\left\{\begin{matrix}\beta_{146}\\ \beta_{156}+\beta_{166}\end{matrix}\right\}=c_6(1-\alpha_6)=0 \quad \text{(B6)}$$

$$\sum_{\nu=8}^{13} c_\nu\beta_{\nu 7}+c_{14}\left\{\begin{matrix}\beta_{147}\\ \beta_{157}+\beta_{167}\end{matrix}\right\}=c_7(1-\alpha_7)=0 \quad \text{(B7)}$$

$$\sum_{\nu=9}^{13} c_\nu\beta_{\nu 8}+c_{14}\left\{\begin{matrix}\beta_{148}\\ \beta_{158}+\beta_{168}\end{matrix}\right\}=c_8(1-\alpha_8) \quad \text{(B8)}$$

$$\sum_{\nu=10}^{13} c_\nu\beta_{\nu 9}+c_{14}\left\{\begin{matrix}\beta_{149}\\ \beta_{159}+\beta_{169}\end{matrix}\right\}=c_9(1-\alpha_9) \quad \text{(B9)}$$

$$\sum_{\nu=11}^{13} c_\nu\beta_{\nu 10}+c_{14}\left\{\begin{matrix}\beta_{1410}\\ \beta_{1510}+\beta_{1610}\end{matrix}\right\}=c_{10}(1-\alpha_{10}) \quad \text{(B10)}$$

$$\sum_{\nu=12}^{13} c_\nu\beta_{\nu 11}+c_{14}\left\{\begin{matrix}\beta_{1411}\\ \beta_{1511}+\beta_{1611}\end{matrix}\right\}=c_{11}(1-\alpha_{11}) \quad \text{(B11)}$$

$$c_{13}\beta_{1312}+c_{14}\left\{\begin{matrix}\beta_{1412}\\ \beta_{1512}+\beta_{1612}\end{matrix}\right\}=c_{12}(1-\alpha_{12}) \quad \text{(B12)}$$

$$c_{14}\left\{\begin{matrix}\beta_{1413}\\ \beta_{1513}+\beta_{1613}\end{matrix}\right\}=c_{13}(1-\alpha_{13}) \quad \text{(B13)}$$

$$c_{14}(\beta_{1514}+\beta_{1614})=c_{14}(1-\alpha_{14})=0 \quad \text{(B14)}$$

$$c_{14}\beta_{1615}=c_{14}(1-\alpha_{15})=c_{14} \quad \text{(B15)}$$

$$
(C)\quad \left\{
\begin{array}{ll}
\sum_{\mu=8}^{13} c_\mu \alpha_\mu \beta_{\mu 5} + c_{14}\beta_{165} = 0 & \qquad (C1) \\
\sum_{\mu=8}^{13} c_\mu \alpha_\mu \beta_{\mu 6} + c_{14}\beta_{166} = 0 & \qquad (C2) \\
\sum_{\mu=8}^{13} c_\mu \alpha_\mu \beta_{\mu 7} + c_{14}\beta_{167} = 0 & \qquad (C3) \\
\sum_{\mu=8}^{13} c_\mu \alpha_\mu^2 \beta_{\mu 5} + c_{14}\beta_{165} = 0 & \qquad (C4) \\
\sum_{\mu=8}^{13} c_\mu \alpha_\mu^2 \beta_{\mu 6} + c_{14}\beta_{166} = 0 & \qquad (C5) \\
\sum_{\mu=8}^{13} c_\mu \alpha_\mu^2 \beta_{\mu 7} + c_{14}\beta_{167} = 0 & \qquad (C6) \\
\sum_{\mu=9}^{13} c_\mu \alpha_\mu \left(\sum_{\nu=8}^{\mu-1} \beta_{\mu\nu}\beta_{\nu 5} \right) + c_{14} \sum_{\nu=8}^{13} \beta_{16\nu}\beta_{\nu 5} = 0 & \qquad (C7) \\
\sum_{\mu=9}^{13} c_\mu \alpha_\mu \left(\sum_{\nu=8}^{\mu-1} \beta_{\mu\nu}\beta_{\nu 6} \right) + c_{14} \sum_{\nu=8}^{13} \beta_{16\nu}\beta_{\nu 6} = 0 & \qquad (C8) \\
\sum_{\mu=9}^{13} c_\mu \alpha_\mu \left(\sum_{\nu=8}^{\mu-1} \beta_{\mu\nu}\beta_{\nu 7} \right) + c_{14} \sum_{\nu=8}^{13} \beta_{16\nu}\beta_{\nu 7} = 0 & \qquad (C9)
\end{array}
\right.
$$

62. The assumptions of No. 61 reduce the equations of condition for our pair of eighth- and ninth-order Runge-Kutta formulas to the following two groups (D) and (E) in very much the same way as before in Parts I, II, or III:

$$
(D)\quad \left\{
\begin{array}{ll}
\begin{Bmatrix} c_0 \\ \hat{c}_0 \end{Bmatrix} + \sum_{\nu=8}^{13} c_\nu + \begin{Bmatrix} c_{14} \\ 2c_{14} \end{Bmatrix} = 1 & \qquad (D0) \\
\sum_{\nu=8}^{14} c_\nu \alpha_\nu^\mu = \frac{1}{\mu+1} \quad (\mu = 1, 2, 3, \ldots, 8) & \qquad (D\mu)
\end{array}
\right.
$$

$$
(E)\quad \begin{cases}
\sum_{\mu=8}^{13} c_\mu \alpha_\mu P_{\mu 5} + c_{14} \begin{Bmatrix} P_{14\,5} \\ P_{16\,5} \end{Bmatrix} = \frac{1}{48} & \text{(E1)} \\
\sum_{\mu=8}^{13} c_\mu \alpha_\mu P_{\mu 6} + c_{14} P_{16 6} = \frac{1}{63} & \text{(E2)} \\
\sum_{\mu=8}^{14} c_\mu \alpha_\mu^2 P_{\mu 5} = \frac{1}{54} & \text{(E3)} \\
\sum_{\mu=9}^{13} c_\mu \alpha_\mu \left(\sum_{\nu=8}^{\mu-1} \beta_{\mu\nu} P_{\nu 5} \right) + c_{14} \sum_{\nu=8}^{13} \beta_{16\,\nu} P_{\nu 5} = \frac{1}{378} & \text{(E4)}
\end{cases}
$$

SECTION XVII. A SOLUTION OF THE EQUATIONS OF CONDITION FOR THE RUNGE-KUTTA COEFFICIENTS

63. We only briefly indicate how a solution of the equations of condition (A1), (A2), (A3), (A4), (B), (C), (D), and (E) can be obtained, since we follow a very similar procedure as in the earlier parts of our paper.

The eight equations (Dμ) for the seven weight factors c_8, c_9, ..., c_{14} lead to a condition of compatibility for the coefficients α_ν. This condition and some other relations simplify considerably if we assume:

$$\alpha_6 + \alpha_7 = 1, \quad \alpha_{10} + \alpha_{11} = 1, \quad \alpha_{12} + \alpha_{13} = 1 \tag{138}$$

Our condition of compatibility can then be written as:

$$\alpha_8 = \frac{1}{3} \cdot \frac{1}{2\alpha_9 - 1} \cdot \frac{42(5\alpha_9 - 3)\alpha_{10}\,\alpha_{11}\,\alpha_{12}\,\alpha_{13} + 6(4-7\alpha_9)(\alpha_{10}\,\alpha_{11} + \alpha_{12}\alpha_{13}) - (5-9\alpha_9)}{70\alpha_{10}\,\alpha_{11}\,\alpha_{12}\,\alpha_{13} - 14(\alpha_{10}\,\alpha_{11} + \alpha_{12}\,\alpha_{13}) + 3} \tag{139}$$

If this relationship between α_8, α_9, α_{10}, α_{11}, α_{12}, α_{13} holds, the weight factors c_8, c_9, ..., c_{14} can be determined from the first seven equations (Dμ).

64. Besides (139) there are further restrictive conditions for the coefficients α_ν. Obviously the conditions (110), (111), (112) still hold in the case of our eighth-order formula. Furthermore, from equations (A1-8), (A2-8), (A3-8), (A4-8) the following condition results:

$$\alpha_5 = \frac{1}{5}\,\alpha_8\,\frac{20\alpha_6\,\alpha_7 - 15\alpha_8 + 12\alpha_8^2}{6\alpha_6\,\alpha_7 - 4\alpha_8 + 3\alpha_8^2} \tag{140}$$

From (110), (111), (112), (138), (139), (140), it follows that all α-coefficients can be expressed by α_4, α_6, α_9, α_{10}, α_{12}.

We shall show, however, in No. 63, that the coefficient α_9 is already determined as soon as α_6, α_{10}, α_{12} are fixed. Therefore, α_4, α_6, α_{10}, α_{12} are the only free α_ν-coefficients.

65. The β-coefficients are computed in very much the same way as in Part III. The expressions for β_{21}, β_{32}, β_{42}, β_{43}, β_{53}, β_{54}, β_{63}, β_{64}, β_{65}, β_{74}, β_{75}, β_{76} are identical with those in Part III.

$$\text{Equations} \left\{ \begin{array}{l} (A1\text{-}8),\ (A2\text{-}8),\ (A3\text{-}8) \\ (A1\text{-}9),\ (A2\text{-}9),\ (A3\text{-}9),\ (A4\text{-}9) \\ (A1\text{-}10), (A2\text{-}10), (A3\text{-}10), (A4\text{-}10) \\ (A1\text{-}11), (A2\text{-}11), (A3\text{-}11), (A4\text{-}11) \end{array} \right\} \text{yield} \left\{ \begin{array}{l} \beta_{85},\ \beta_{86},\ \beta_{87} \\ \beta_{95},\ \beta_{96},\ \beta_{97},\ \beta_{98} \\ \beta_{106},\ \beta_{107},\ \beta_{108},\ \beta_{109} \\ \beta_{117},\ \beta_{118},\ \beta_{119},\ \beta_{1110} \end{array} \right\},$$

if we assume

$$\beta_{105} = 0, \ \beta_{115} = \beta_{116} = 0 \quad , \tag{141}$$

Assuming:

$$\beta_{165} = \beta_{145}, \quad \beta_{166} = \beta_{146}, \quad \beta_{167} = \beta_{147} \tag{142}$$

$$\text{equations} \left\{ \begin{array}{l} (B5),\ (C1),\ (C4) \\ (B6),\ (C2),\ (C5) \\ (B7),\ (C3),\ (C6) \end{array} \right\} \text{ yield } \left\{ \begin{array}{l} \beta_{125},\ \beta_{135},\ \beta_{145} \\ \beta_{126},\ \beta_{136},\ \beta_{146} \\ \beta_{127},\ \beta_{137},\ \beta_{147} \end{array} \right\}$$

From equations (A1-12), (A2-12), (A3-12), (A4-12), we can now obtain the coefficients β_{128}, β_{129}, β_{1210}, β_{1211}.

66. Using the expressions for $\beta_{\kappa\lambda}$ as obtained from No. 65, the auxiliary quantities P_{85}, P_{95}, P_{105}, P_{115}, P_{125} are determined if α_6, α_8, α_9, α_{10}, α_{12} are given.

From (B) it follows easily:

$$c_8 P_{85} + c_9 P_{95} + c_{10} P_{105} + c_{11} P_{115} + c_{12} P_{125} + c_{13} P_{135} + c_{14} P_{145} = \frac{1}{42} \tag{143}$$

Equations (143), (E1), (E3) can now be written in the form:

$$\left.\begin{aligned} c_{13} P_{135} + c_{14} P_{145} &= \frac{1}{42} - \sum_{\nu=8}^{12} c_\nu P_{\nu 5} \\ c_{13}\, \alpha_{13} P_{135} + c_{14} P_{145} &= \frac{1}{48} - \sum_{\nu=8}^{12} c_\nu \alpha_\nu P_{\nu 5} \\ c_{13}\, \alpha_{13}^2 P_{135} + c_{14} P_{145} &= \frac{1}{54} - \sum_{\nu=8}^{12} c_\nu \alpha_\nu^2 P_{\nu 5} \end{aligned}\right\}, \tag{144}$$

representing three equations for the two unknowns P_{135}, P_{145}.

From (144) the following condition of compatibility can be obtained:

$$\sum_{\nu=8}^{12} c_\nu (\alpha_{13} - \alpha_\nu)(1 - \alpha_\nu)\, P_{\nu 5} = \frac{1}{3024}\,(9\alpha_{13} - 7) \tag{145}$$

Equations (145) can be considered as a restrictive condition for α_8. Therefore, only α_4, α_6, α_{10}, α_{12} can be considered as free parameters.

67. The computation of the remaining β-coefficients is again straightforward and need hardly any explanation.

From (A1-13), (A2-13), (A3-13), (A4-13), and the definition of P_{135} - the quantity P_{135} is already obtained from (144) - we obtain β_{138}, β_{139}, β_{1310}, β_{1311}, β_{1312}.

The upper equations (B8) through (B13) yield β_{148}, β_{149}, β_{1410}, β_{1411}, β_{1412}, β_{1413}. This concludes the determination of the β-coefficients for our eighth-order formula.

68. We still have to determine the coefficients $\beta_{15\lambda}$ and $\beta_{16\lambda}$ for the ninth-order formula.

From (E2) we can determine P_{166}. Since $P_{165} = P_{145}$, equations (A1-16), (A2-16), (A3-16), (A4-16), the definition of P_{165}, and the definition of P_{166} represent six equations for the six unknowns β_{168}, β_{169}, β_{1610}, β_{1611}, β_{1612}, β_{1613}, since the coefficients β_{165}, β_{166}, β_{167} are already determined by (142).

Equation (B14) suggests:

$$\beta_{1614} = \beta_{1514} = 0 \ , \tag{146}$$

and equation (B15) yields:

$$\beta_{1615} = 1 \ . \tag{147}$$

Finally, the coefficients $\beta_{15\lambda}$ can be determined by comparing the upper and the lower parts of equations (B).

All Runge-Kutta coefficients for our pair of eighth- and ninth-order formulas are now known if we still determine c_0, $\hat{c}_0$ from equation (D0) and the coefficients $\beta_{\nu 0}$ ($\nu = 1, 2, 3, \ldots, 16$) from the extended equations (F). If we submit the α-coefficients to the above restrictive conditions, the Runge-Kutta coefficients satisfy all our equations of condition.

SECTION XVIII. EXAMPLE FOR AN EIGHTH-ORDER RUNGE-KUTTA FORMULA

69. In Table XII we present an example for an eighth-order Runge-Kutta formula RK8(9). Since α_8, as obtained from equation (145), is in general no simple rational number anymore, we give in Table XII the decimal representation of the Runge-Kutta coefficients. The β- and c-coefficients not listed in Table XII are zero. The free parameters α_4, α_6, α_{10}, α_{12} were selected in such a way as to make the error coefficients of the leading error term of our Runge-Kutta formula as small as possible.

The computation of the coefficients was performed on an IBM-7094 electronic computer in 40-digit arithmetic*.

*The author is indebted to Mr. F.R. Calhoun, R-COMP-CSC, for developing the 40-digit package and making it available for these computations.

TABLE XII. RK 8 (9)

α_1 = 0.4436 8940 3764 9818 3109 5994 0428 1370
α_2 = 0.6655 3410 5647 4727 4664 3991 0642 2055
α_3 = 0.9983 0115 8471 2091 1996 5986 5963 3083
α_4 = 0.3155 0000 0000 0000 0000 0000 0000 0000
α_5 = 0.5054 4100 9481 6906 8626 5161 2673 7384
α_6 = 0.1714 2857 1428 5714 2857 1428 5714 2857
α_7 = 0.8285 7142 8571 4285 7142 8571 4285 7143
α_8 = 0.6654 3966 1210 1156 2534 9537 6925 5586
α_9 = 0.2487 8317 9680 6265 2069 7222 7456 0771
α_{10} = 0.1090 0000 0000 0000 0000 0000 0000 0000
α_{11} = 0.8910 0000 0000 0000 0000 0000 0000 0000
α_{12} = 0.3995 0000 0000 0000 0000 0000 0000 0000
α_{13} = 0.6005 0000 0000 0000 0000 0000 0000 0000
α_{14} = 1
α_{15} = 0
α_{16} = 1

β_{10} = 0.4436 8940 3764 9818 3109 5994 0428 1370
β_{20} = 0.1663 8352 6411 8681 8666 0997 7660 5514
β_{21} = 0.4991 5057 9235 6045 5998 2993 2981 6541
β_{30} = 0.2495 7528 9617 8022 7999 1496 6490 8271
β_{32} = 0.7487 2586 8853 4068 3997 4489 9472 4812
β_{40} = 0.2066 1891 1634 0060 2426 5567 1039 3185
β_{42} = 0.1770 7880 3779 8634 7040 3809 9728 8319
β_{43} = -0.6819 7715 4138 6949 4669 3770 7681 5048 $\cdot 10^{-1}$
β_{50} = 0.1092 7823 1526 6640 8227 9038 9092 6157
β_{53} = 0.4021 5962 6423 6799 5421 9905 6369 0087 $\cdot 10^{-2}$
β_{54} = 0.3921 4118 1690 7898 0444 3923 3017 4325
β_{60} = 0.9889 9281 4091 6466 5304 8447 6543 4355 $\cdot 10^{-1}$
β_{63} = 0.3513 8370 2279 6396 6951 2044 8735 6703 $\cdot 10^{-2}$
β_{64} = 0.1247 6099 9831 6001 6621 5206 2587 2489
β_{65} = -0.5574 5546 8349 8979 9643 7429 0146 6348 $\cdot 10^{-1}$
β_{70} = -0.3680 6865 2862 4220 3724 1531 0108 0691
β_{74} = -0.2227 3897 4694 7600 7645 0240 2094 4166 $\cdot 10^{+1}$
β_{75} = 0.1374 2908 2567 0291 0729 5656 9124 5744 $\cdot 10^{+1}$
β_{76} = 0.2049 7390 0271 1160 3002 1593 5409 2206 $\cdot 10^{+1}$
β_{80} = 0.4546 7962 6413 4715 0077 3519 5060 3349 $\cdot 10^{-1}$

TABLE XII. RK 8 (9) (Continued)

$\beta_{85} = 0.3254\ 2131\ 7015\ 8914\ 7114\ 6774\ 6964\ 8853$

$\beta_{86} = 0.2847\ 6660\ 1385\ 2790\ 8888\ 1824\ 2057\ 3687$

$\beta_{87} = 0.9783\ 7801\ 6759\ 7915\ 2435\ 8683\ 9727\ 1099 \cdot 10^{-2}$

$\beta_{90} = 0.6084\ 2071\ 0626\ 2205\ 7051\ 0941\ 4520\ 5182 \cdot 10^{-1}$

$\beta_{95} = -0.2118\ 4565\ 7440\ 3700\ 7526\ 3252\ 7525\ 1206 \cdot 10^{-1}$

$\beta_{96} = 0.1959\ 6557\ 2661\ 7083\ 1957\ 4644\ 9066\ 2983$

$\beta_{97} = -0.4274\ 2640\ 3648\ 1760\ 3675\ 1448\ 3534\ 2899 \cdot 10^{-2}$

$\beta_{98} = 0.1743\ 4365\ 7368\ 1491\ 1965\ 3234\ 5255\ 8189 \cdot 10^{-1}$

$\beta_{100} = 0.5405\ 9783\ 2969\ 3191\ 7365\ 7857\ 2411\ 1182 \cdot 10^{-1}$

$\beta_{106} = 0.1102\ 9325\ 5978\ 2892\ 6539\ 2831\ 2764\ 8228$

$\beta_{107} = -0.1256\ 5008\ 5200\ 7255\ 6414\ 1477\ 6378\ 2250 \cdot 10^{-2}$

$\beta_{108} = 0.3679\ 0043\ 4775\ 8146\ 0136\ 3840\ 4356\ 6339 \cdot 10^{-2}$

$\beta_{109} = -0.5778\ 0542\ 7709\ 7207\ 3940\ 8406\ 2857\ 1866 \cdot 10^{-1}$

$\beta_{110} = 0.1273\ 2477\ 0686\ 6711\ 4646\ 6451\ 8179\ 9160$

$\beta_{117} = 0.1144\ 8395\ 0063\ 9510\ 5323\ 6588\ 7572\ 1817$

$\beta_{118} = 0.2877\ 3020\ 7096\ 9799\ 2776\ 2022\ 0184\ 9198$

$\beta_{119} = 0.5094\ 5379\ 4596\ 1136\ 3153\ 7358\ 8507\ 9465$

$\beta_{1110} = -0.1479\ 9682\ 2443\ 7257\ 5900\ 2421\ 4444\ 9640$

$\beta_{120} = -0.3652\ 6793\ 8766\ 1674\ 0535\ 8485\ 4439\ 4333 \cdot 10^{-2}$

$\beta_{125} = 0.8162\ 9896\ 0123\ 1891\ 9777\ 8194\ 2124\ 7030 \cdot 10^{-1}$

$\beta_{126} = -0.3860\ 7735\ 6356\ 9350\ 6490\ 5176\ 9434\ 3215$

$\beta_{127} = 0.3086\ 2242\ 9246\ 0510\ 6450\ 4741\ 6602\ 5206 \cdot 10^{-1}$

$\beta_{128} = -0.5807\ 7254\ 5283\ 2060\ 2815\ 8293\ 7473\ 3518 \cdot 10^{-1}$

$\beta_{129} = 0.3359\ 8659\ 3288\ 8497\ 1493\ 1434\ 5136\ 2322$

$\beta_{1210} = 0.4106\ 6880\ 4019\ 4995\ 8613\ 5496\ 2278\ 6417$

$\beta_{1211} = -0.1184\ 0245\ 9723\ 5598\ 5520\ 6331\ 5615\ 4536 \cdot 10^{-1}$

$\beta_{130} = -0.1237\ 5357\ 9212\ 4514\ 3254\ 9790\ 9613\ 5669 \cdot 10^{+1}$

$\beta_{135} = -0.2443\ 0768\ 5513\ 5478\ 5358\ 7348\ 6136\ 6763 \cdot 10^{+2}$

$\beta_{136} = 0.5477\ 9568\ 9327\ 7865\ 6050\ 4365\ 2899\ 1173$

$\beta_{137} = -0.4441\ 3863\ 5334\ 1324\ 6374\ 9598\ 9656\ 9346 \cdot 10^{+1}$

$\beta_{138} = 0.1001\ 3104\ 8137\ 1326\ 6094\ 7926\ 1785\ 1022 \cdot 10^{+2}$

$\beta_{139} = -0.1499\ 5773\ 1020\ 5175\ 8447\ 1709\ 8507\ 3142 \cdot 10^{+2}$

$\beta_{1310} = 0.5894\ 6948\ 5232\ 1701\ 3620\ 8245\ 3965\ 1427 \cdot 10^{+1}$

$\beta_{1311} = 0.1738\ 0377\ 5034\ 2898\ 4877\ 6168\ 5744\ 0542 \cdot 10^{+1}$

$\beta_{1312} = 0.2751\ 2330\ 6931\ 6673\ 0263\ 7586\ 2286\ 0276 \cdot 10^{+2}$

$\beta_{140} = -0.3526\ 0859\ 3883\ 3452\ 2700\ 5029\ 5887\ 5588$

$\beta_{145} = -0.1839\ 6103\ 1448\ 4827\ 0375\ 0441\ 9898\ 8231$

TABLE XII. RK 8 (9) (Continued)

$\beta_{146} = -0.6557\ 0189\ 4497\ 4164\ 5138\ 0068\ 7998\ 5251$

$\beta_{147} = -0.3908\ 6144\ 8804\ 3986\ 3435\ 0255\ 2024\ 1310$

$\beta_{148} = 0.2679\ 4646\ 7128\ 5002\ 2936\ 5844\ 2327\ 1209$

$\beta_{149} = -0.1038\ 3022\ 9913\ 8249\ 0865\ 7698\ 5850\ 7427 \cdot 10^{+1}$

$\beta_{1410} = 0.1667\ 2327\ 3242\ 5867\ 1664\ 7273\ 4616\ 8501 \cdot 10^{+1}$

$\beta_{1411} = 0.4955\ 1925\ 8553\ 1597\ 7067\ 7329\ 6707\ 1441$

$\beta_{1412} = 0.1139\ 4001\ 1323\ 9706\ 3228\ 5867\ 3814\ 1784 \cdot 10^{+1}$

$\beta_{1413} = 0.5133\ 6696\ 4246\ 5861\ 3688\ 1990\ 9719\ 1534 \cdot 10^{-1}$

$\beta_{150} = 0.1046\ 4847\ 3406\ 1481\ 0391\ 8730\ 0240\ 6755 \cdot 10^{-2}$

$\beta_{158} = -0.6716\ 3886\ 8449\ 9028\ 2237\ 7784\ 4617\ 8020 \cdot 10^{-2}$

$\beta_{159} = 0.8182\ 8762\ 1894\ 2502\ 1265\ 3300\ 6524\ 8999 \cdot 10^{-2}$

$\beta_{1510} = -0.4264\ 0342\ 8644\ 8334\ 7277\ 1421\ 3808\ 7561 \cdot 10^{-2}$

$\beta_{1511} = 0.2800\ 9029\ 4741\ 6893\ 6545\ 9763\ 3115\ 3703 \cdot 10^{-3}$

$\beta_{1512} = -0.8783\ 5333\ 8762\ 3867\ 6639\ 0578\ 1314\ 5633 \cdot 10^{-3}$

$\beta_{1513} = 0.1025\ 4505\ 1108\ 2555\ 8084\ 2177\ 6966\ 4009 \cdot 10^{-1}$

$\beta_{160} = -0.1353\ 6550\ 7861\ 7406\ 7080\ 4421\ 6888\ 9966 \cdot 10^{+1}$

$\beta_{165} = -0.1839\ 6103\ 1448\ 4827\ 0375\ 0441\ 9898\ 8231$

$\beta_{166} = -0.6557\ 0189\ 4497\ 4164\ 5138\ 0068\ 7998\ 5251$

$\beta_{167} = -0.3908\ 6144\ 8804\ 3986\ 3435\ 0255\ 2024\ 1310$

$\beta_{168} = 0.2746\ 6285\ 5812\ 9992\ 5758\ 9622\ 0773\ 2989$

$\beta_{169} = -0.1046\ 4851\ 7535\ 7191\ 5887\ 0351\ 8857\ 2676 \cdot 10^{+1}$

$\beta_{1610} = 0.1671\ 4967\ 6671\ 2315\ 5012\ 0044\ 8830\ 6588 \cdot 10^{+1}$

$\beta_{1611} = 0.4952\ 3916\ 8258\ 4180\ 8131\ 1869\ 9074\ 0287$

$\beta_{1612} = 0.1148\ 1836\ 4662\ 7330\ 1905\ 2257\ 9595\ 4930 \cdot 10^{+1}$

$\beta_{1613} = 0.4108\ 2191\ 3138\ 3305\ 5603\ 9813\ 2752\ 7525 \cdot 10^{-1}$

$\beta_{1615} = 1$

$c_0 = 0.3225\ 6083\ 5002\ 1624\ 9913\ 6129\ 0096\ 0247 \cdot 10^{-1}$

$c_8 = 0.2598\ 3725\ 2837\ 1540\ 3018\ 8870\ 2317\ 1963$

$c_9 = 0.9284\ 7805\ 9965\ 7702\ 7788\ 0637\ 1430\ 2190 \cdot 10^{-1}$

$c_{10} = 0.1645\ 2339\ 5147\ 6434\ 2891\ 6477\ 3184\ 2800$

$c_{11} = 0.1766\ 5951\ 6378\ 6007\ 4367\ 0842\ 9839\ 7547$

$c_{12} = 0.2392\ 0102\ 3203\ 5275\ 9374\ 1089\ 3332\ 0941$

$c_{13} = 0.3948\ 4274\ 6042\ 0285\ 3746\ 7521\ 1882\ 9325 \cdot 10^{-2}$

$c_{14} = 0.3072\ 6495\ 4758\ 6064\ 0406\ 3683\ 0552\ 2124 \cdot 10^{-1}$

$$TE = c_{14}(f_0 + f_{14} - f_{15} - f_{16})h \qquad (148)$$

SECTION XIX. NUMERICAL COMPARISON WITH OTHER EIGHTH-ORDER RUNGE-KUTTA FORMULAS

70. We compare our formula RK8(9) with SHANKS's eighth-order formula (8-12) ([13], p.34) which is the only other eighth-order Runge-Kutta formula known to us. We apply both formulas to our problem (53) in exactly the same way as in Parts I, II, and III using RICHARDSON's principle as stepsize control procedure for SHANKS's formula. Table XIII shows our results.

TABLE XIII. COMPARISON OF EIGHTH-ORDER METHODS FOR EXAMPLE (53)

Method	Number of Substitutions per Step	Results for $x = 5$ and Tolerance 10^{-12}				
		Number of Steps	Total Number of Evaluations	Running Time on IBM-7094 (min)	Accumulated Errors in y and z	
					Δy	Δz
SHANKS	23	694	15 962	1.65	$-0.1710 \cdot 10^{-13}$	$-0.4646 \cdot 10^{-13}$
RK8(9)	17	510	8 670	0.91	$-0.1776 \cdot 10^{-14}$	$-0.3553 \cdot 10^{-13}$

The table shows that, in this example, the computer running time for our formula RK8(9) is about 55% of the running time of SHANKS's eighth-order formula.

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