Classical gravitational anomalies of Liouville theory

Pavel Haman[1,*] and Alfredo Iorio[1,†]
[1] *Faculty of Mathematics and Physics, Charles University, V Holešovičkách 2, 18000 Prague 8, Czech Republic*
(Dated: June 19, 2023)

We show that for classical Liouville field theory, diffeomorphism invariance, Weyl invariance and locality cannot hold together. This is due to a genuine Virasoro center, present in the theory, that leads to an energy-momentum tensor with non-tensorial conformal transformations, in flat space, and with a non-vanishing trace, in curved space. Our focus is on a field-independent term, proportional to the square of the Weyl gauge field, $W_\mu W^\mu$, that makes the action Weyl-invariant and was disregarded in previous investigations of Weyl and conformal symmetry. We show this term to be related to the classical center of the Virasoro.

Keywords: Gravitational anomalies; diffeomorphic invariance; conformal symmetry; classical Virasoro algebra; Liouville field theory.

Liouville field theory, with flat space action[1]

$$A_L[\Phi] = \int \mathrm{d}^2x \left(\frac{1}{2}\partial_\mu\Phi\partial^\mu\Phi - \frac{m^2}{\beta^2}e^{\beta\Phi} \right), \tag{1}$$

is an exactly solvable two-dimensional model that enjoys a prominent role in many fields of the theoretical and mathematical investigations. Among those, the geometry of surfaces [1], two-dimensional (quantum) gravity, see, e.g., [2] and [3], string theory, see, e.g., [4], conformal field theories, such as the Wess-Zumino-Witten and the Toda models, see e.g., [5], and therefore the AdS/CFT correspondence [6]. It is then of great importance to know its symmetries in all details, already at the classical level.

In particular, Liouville theory is known to enjoy both scale and full (global) conformal symmetries in flat space, hence it belongs to the cases studied in [7]. There it is assumed that Weyl and diffemorphism invariances hold together. Even though full (global) conformal symmetry is known to be in place, in a more recent work [8] it is conjectured that Liouville theory might not be made both diffeomorphic and Weyl invariant, evoking a generic “classical anomaly” as the reason for that. In this letter we prove that conjecture and provide explicit formulae for such classical gravitational anomalies. We leave to a longer forthcoming paper [9] the more detailed discussion on how such classical anomalies arise, in general and for Liouville.

Let us start by considering the Liouville action on curved background

$$\mathcal{A}_L[\Phi] = \int \mathrm{d}^2x \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{m^2}{\beta^2}e^{\beta\Phi} + \frac{1}{\beta}R\Phi \right), \tag{2}$$

routinely employed to obtain the energy-momentum tensor (EMT)

$$T_L^{\mu\nu} = -\frac{2}{\sqrt{-g}}\frac{\delta\mathcal{A}_L}{\delta g_{\mu\nu}} = \nabla^\mu\Phi\nabla^\nu\Phi - g^{\mu\nu}\left(\frac{1}{2}g^{\alpha\beta}\nabla_\alpha\Phi\nabla_\beta\Phi - \frac{m^2}{\beta^2}e^{\beta\Phi} \right) + \frac{2}{\beta}(g^{\mu\nu}\nabla_\rho\nabla^\rho - \nabla^\mu\nabla^\nu)\Phi\,, \tag{3}$$

that *on-shell* gives

$$T_L{}^\mu{}_\mu = \frac{2}{\beta^2}R\,, \tag{4}$$

* pavel.haman@mff.cuni.cz
† alfredo.iorio@mff.cuni.cz
[1] Here we use $\eta_{\mu\nu} = \mathrm{diag}(+1, -1)$.

ensuring a zero trace in the flat limit, but *not Weyl invariance*, for the curved background. To remedy it, this *ad hoc* procedure could be traded for the one of [7], based on the Weyl-gauging of the curvilinear expression for the action (1)

$$\mathcal{A}_W[\Phi, W_\mu] = \int \mathrm{d}^2x \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{m^2}{\beta^2}e^{\beta\Phi} + \frac{2}{\beta}\Phi\nabla_\mu W^\mu + \frac{2}{\beta^2}g^{\mu\nu}W_\mu W_\nu\right). \tag{5}$$

Since under Weyl transformations, $g_{\mu\nu} \to e^{2\omega}g_{\mu\nu}$ and $\Phi \to \Phi - \frac{2}{\beta}\omega$, one has $2\nabla_\mu W^\mu \xrightarrow{\text{Weyl}} e^{-2\omega}(2\nabla_\mu W^\mu - 2\nabla_\mu\nabla^\mu\omega)$, this should be compared to $R[g_{\mu\nu}] \xrightarrow{\text{Weyl}} e^{-2\omega}(R[g_{\mu\nu}] - 2\nabla_\mu\nabla^\mu\omega)$ and the (Ricci gauged, in the language of [7]) action

$$\mathcal{A}_R[\Phi] = \int \mathrm{d}^2x \sqrt{-g}\left(\frac{1}{2}g^{\mu\nu}\nabla_\mu\Phi\nabla_\nu\Phi - \frac{m^2}{\beta^2}e^{\beta\Phi} + \frac{1}{\beta}\Phi R + \frac{2}{\beta^2}g^{\mu\nu}W_\mu W_\nu\right), \tag{6}$$

enjoys Weyl invariance, ${T_R}^\mu{}_\mu = 0$, provided

$$2\nabla_\mu W^\mu = R\,, \tag{7}$$

holds. Notice that, contrary to [7], we keep here the last, Φ-independent term, that is precisely the one that ensures Weyl invariance.

A solution of (7) can be found [8] using the Green's function $K(x,y)$, such that $\nabla^2_x K(x,y) = \frac{1}{\sqrt{-g(x)}}\delta^{(2)}(x-y)$. Assuming that $W_\mu = \partial_\mu w$, with w transforming as $w \xrightarrow{\text{Weyl}} w - \omega$, the solution is $w(x) = 1/2\int \mathrm{d}^2y\,\sqrt{-g(y)}\,K(x,y)\,R(y)$. It follows that the extra term in the action proportional to $W^\mu W_\mu$ is

$$\int \mathrm{d}^2x\, W^\mu(x)W_\mu(x) = \frac{1}{4}\int \mathrm{d}^2x\,\mathrm{d}^2y\,\sqrt{-g(x)}R(x)K(x,y)\sqrt{-g(y)}R(y)\,, \tag{8}$$

which is the well-known *Polyakov string effective action* [4]. The EMT associated to the action (6) with (8), is traceless and covariantly conserved [4]. The price we pay is the evident nonlocality.

A *local solution* to (7) was found by Deser and Jackiw in [10]

$$W^\mu_{DJ} = \frac{\varepsilon^{\mu\nu}}{2\sqrt{-g}}\left[\frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}\Gamma_{\beta\alpha\nu} + (\cosh\sigma - 1)\partial_\nu\gamma + \partial_\nu r\right], \tag{9}$$

where $\varepsilon^{01} = +1$ is the Levi-Civita symbol and a "conformal" parametrization of the metric gives $g_{++}/\sqrt{-g} = e^\gamma\sinh\sigma$, $g_{+-}/\sqrt{-g} = \cosh\sigma$, $g_{--}/\sqrt{-g} = e^{-\gamma}\sinh\sigma$, and, from there, $\gamma = \ln\sqrt{g_{++}/g_{--}}$ (see Supplemental Material). The expression (9) includes the derivative of a generic Weyl scalar, r, to take into account the invariance of (7) for $W^\mu_{DJ} \to W^\mu_{DJ} + \frac{\varepsilon^{\mu\nu}}{2\sqrt{-g}}\partial_\nu r$.

W^μ_{DJ} enjoys proper Weyl transformations, $g_{\mu\nu}W^\nu_{DJ} \xrightarrow{\text{Weyl}} g_{\mu\nu}W^\nu_{DJ} - \partial_\mu\omega$, but it does not transform as a general (contravariant) vector under infinitesimal diffeomorphisms, $x^\mu \to x'^\mu = x^\mu - f^\mu(x)$,

$$W'^\mu_{DJ}(x') = \frac{\partial x'^\mu}{\partial x^\nu}W^\nu_{DJ}(x) + \frac{\varepsilon^{\mu\nu}}{2\sqrt{-g}}\partial_\nu\left[\left(\partial_- - e^{-\gamma}\tanh\frac{\sigma}{2}\,\partial_+\right)f^- - \left(\partial_+ - e^{\gamma}\tanh\frac{\sigma}{2}\,\partial_-\right)f^+\right]. \tag{10}$$

It follows that the term $g_{\mu\nu}W^\mu_{DJ}W^\nu_{DJ}$ in (6), although it keeps Weyl invariance and locality of $\mathcal{A}_R[\Phi]$, cannot be a world scalar, hence it breaks diffeomorphism invariance.

To investigate and quantify such breaking, let us start with the contribution to the EMT coming from the extra term

$$T^{\mu\nu}_{\text{extra}} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int \mathrm{d}^2x\,\frac{2}{\beta^2}\sqrt{-g}W^\mu W_\mu\,. \tag{11}$$

In the Supplemental Material it is shown that

$$\frac{\beta^2}{2}T^{\mu\nu}_{\text{extra}} = g^{\mu\nu}W_\rho W^\rho - 2W^\mu W^\nu - Rg^{\mu\nu} + \nabla^\mu W^\nu + \nabla^\nu W^\mu + 2\frac{\varepsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\beta W_\alpha[(\cosh\sigma - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\,, \tag{12}$$

where $W_\rho \equiv g_{\rho\lambda}W^\lambda$, $2\Gamma^{\mu\nu} = (g_{--}g_{++})^{-1/2}\begin{pmatrix} -\sinh\gamma & \cosh\gamma \\ \cosh\gamma & -\sinh\gamma \end{pmatrix}$ and $r^{\mu\nu} \equiv \delta r/\delta g_{\mu\nu}$. One would like to compute $\nabla_\mu T^{\mu\nu}_{\text{extra}}$ and compare it with known expressions of the *quantum* gravitational anomalies, such as [11] $\nabla_\mu T^\mu{}_\nu = \frac{1}{48\pi}\frac{\varepsilon^{\sigma\rho}}{2\sqrt{-g}}\partial_\rho\partial_\lambda\Gamma^\lambda{}_{\nu\sigma}$ or [12] $\nabla_\mu T^\mu{}_\nu = \frac{1}{48\pi}\partial_\nu R$.

Rather than attempting a direct computation, we take a simpler road. First, we move to isothermal light-cone coordinates, that we know to always exist in two dimensions. There one has $\hat{g}_{\pm\pm}(x) = e^{2\rho(x)}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, and we indicate with a hat all quantities evaluated there[2]. If we set $r = 0$ for a moment, we have

$$\frac{\beta^2}{4}\hat{T}^{\mu\nu}_{\text{extra}} = (\hat{g}^{\mu\alpha}\partial_\alpha\rho)(\hat{g}^{\nu\beta}\partial_\beta\rho) - \frac{1}{2}\hat{g}^{\mu\nu}(\hat{g}^{\alpha\beta}\partial_\alpha\rho\partial_\beta\rho) + \left[\hat{g}^{\mu\nu}(\hat{g}^{\alpha\beta}\partial_\alpha\partial_\beta) - \hat{g}^{\mu\alpha}\hat{g}^{\nu\beta}\partial_\alpha\partial_\beta\right]\rho\,, \tag{13}$$

and the computation becomes trivial, $\hat{\nabla}_\mu\hat{T}^{\mu\nu}_{\text{extra}} = 0$. Of course, this would not guarantee general covariance, until we have a frame-independent result (see later).

On the other hand, including r in the computation gives

$$\beta^2\hat{\nabla}_\mu\hat{T}^{\mu\nu}_{\text{extra}} = \frac{\varepsilon^{\nu\mu}}{\sqrt{-\hat{g}}}\partial_\mu\left(\hat{g}^{\alpha\beta}\partial_\alpha\partial_\beta\hat{r}\right) - 2\hat{g}^{\alpha\beta}\partial_\mu(\hat{r}^{\mu\nu}\partial_\alpha\partial_\beta\hat{r}) + 2\partial_\mu\hat{g}^{\alpha\beta}\hat{r}^{\mu\nu}\partial_\alpha\partial_\beta\hat{r}\,, \tag{14}$$

and this expression, although it differs from the recalled anomalous quantum expressions [11, 12], it is clearly nonzero, in general. In this coordinate frame, $\hat{T}^{\mu\nu}_{\text{extra}}$ not only guarantees Weyl invariance, through a traceless EMT, but for harmonic rs, $\hat{\Box}\hat{r} = 0$

$$\hat{\nabla}_\mu\hat{T}^{\mu\nu}_{\text{extra}}\Big|_{\hat{\Box}\hat{r}=0} = 0\,. \tag{15}$$

As for the previous case, for $r = 0$, this is not enough to have general covariance. We have no guarantee that (15) holds in all frames. We have to look for how much such divergence differs from a tensor, when we move away from the isothermal frame, $\Delta\hat{\nabla}_\mu\hat{T}^{\mu\nu}_{\text{extra}}(x) \equiv \nabla'_\mu T'^{\mu\nu}_{\text{extra}}(x') - (\partial x'^\nu/\partial x^\rho)\hat{\nabla}_\sigma\hat{T}^{\sigma\rho}_{\text{extra}}(x)$. This has to be, at least partially, expressible in terms of $\Delta\hat{W}^\mu(x) \equiv W'^\mu(x') - (\partial x'^\mu/\partial x^\nu)\hat{W}^\nu(x)$. For infinitesimal diffeomorphisms, W^μ transforms as (10) and, defining $\Delta r(x) \equiv r'(x') - r(x)$,

$$\Delta\hat{W}^\mu = \frac{\varepsilon^{\mu\nu}}{2\sqrt{-\hat{g}}}\partial_\nu\left[\varepsilon^{\alpha\beta}\partial_\beta\left(\frac{f_\alpha}{\sqrt{-\hat{g}}}\right) + \Delta\hat{r}\right] \equiv \frac{\varepsilon^{\mu\nu}}{2\sqrt{-\hat{g}}}\partial_\nu\xi(r,f)\,. \tag{16}$$

With these

$$\beta^2\Delta\hat{\nabla}_\mu\hat{T}^{\mu\nu}_{\text{extra}} = \frac{\varepsilon^{\nu\mu}}{\sqrt{-\hat{g}}}\partial_\mu\left[\hat{g}^{\alpha\beta}\partial_\alpha\partial_\beta\xi(r,f)\right] - 2\hat{g}^{\alpha\beta}\partial_\mu(\hat{r}^{\mu\nu}\partial_\alpha\partial_\beta\xi(r,f)) + 2\hat{r}^{\mu\nu}\partial_\mu\hat{g}^{\alpha\beta}\partial_\alpha\partial_\beta\xi(r,f)\,, \tag{17}$$

---

[2] For this choice, $\varepsilon^{\alpha\beta}\partial_\beta\hat{W}_\alpha = 0$.

that, for the choice (9), and for $r = 0$, eventually gives a compact expression

$$\Delta\hat{\nabla}_\mu \hat{T}^{\mu\nu}_{\text{extra}} = \frac{1}{\beta^2}\frac{\varepsilon^{\nu\mu}}{\sqrt{-\hat{g}}}\partial_\mu\Big[\hat{g}^{\alpha\beta}\partial_\alpha\partial_\beta\Big(\partial_- f^- - \partial_+ f^+\Big)\Big]\,. \tag{18}$$

This expression does not vanish for a general f^μ. *This proves the loss of diffeomorphism invariance in the Weyl invariant formulation of Liouville theory* (6)*, with local solution* (9).

Quadratic transformations, $f^\mu = a^\mu{}_{\alpha\beta}x^\alpha x^\beta + b^\mu{}_\alpha x^\alpha + c^\mu$, which include Poincaré transformations, $f^\mu = \omega^\mu{}_\nu x^\nu + c^\mu$, preserve the tensorial nature of $\hat{\nabla}_\mu \hat{T}^{\mu\nu}_{\text{extra}}$ and so do conformal transformations, obeying $\Box f^\mu = 0$. Therefore, for infinitesimal conformal and Poincaré transformations in flat space, the extra term, $T^{\mu\nu}_{\text{extra}}$, is covariantly conserved, regardless of the choice of r. In other words, $T^{\mu\nu}_{\text{extra}}$ does not violate the symmetries of $T^{\mu\nu}$ in the flat limit, that is the same conclusion of [7].

To complete the proof that this lack of diffeomorphic invariance is indeed the classical version of the quantum anomaly, we need to relate it to a nontrivial center of the Virasoro algebra. To do so, let us first consider the flat limit of the EMT (3)

$$\Theta_{\mu\nu} = \partial_\mu\Phi\partial_\nu\Phi - \eta_{\mu\nu}\left(\frac{1}{2}\partial_\alpha\Phi\partial^\alpha\Phi - \frac{m^2}{\beta^2}e^{\beta\Phi}\right) + \frac{2}{\beta}(\eta_{\mu\nu}\Box - \partial_\mu\partial_\nu)\Phi\,, \tag{19}$$

that is traceless on-shell. The associated Noether charges, written in the light-cone frame[3]

$$Q^\pm[f] = \int \mathrm{d}x^\pm\,\Theta_{\pm\pm}f^\pm = \int \mathrm{d}x^\pm\left((\partial_\pm\Phi)^2 - \frac{2}{\beta}\partial_\pm^2\Phi\right)f^\pm\,, \tag{20}$$

through the Poisson brackets $\{\Phi(x),\Phi(y)\}|_{x^+=y^+} = -\frac{1}{4}\,\mathrm{sgn}(x^- - y^-)$ and $\{\Phi(x),\Phi(y)\}|_{x^-=y^-} = -\frac{1}{4}\,\mathrm{sgn}(x^+ - y^+)$, generate the right transformations

$$\delta_f\Phi \equiv \Big\{\Phi(x^+,x^-), Q^\pm[f]\Big\} = f^\pm(x^\pm)\partial_\pm\Phi(x^+,x^-) + \frac{1}{\beta}\partial_\pm f^\pm(x^\pm)\,. \tag{21}$$

They are made of two terms, both necessary for the invariance of the flat action (1): the usual Lie derivative of the scalar field, $f^\alpha\partial_\alpha\Phi$, and an affine term. It is easy to verify that these charges obey an algebra with a genuine central extension

$$\{Q^\pm[f], Q^\pm[g]\} = Q^\pm[k] + \frac{1}{\beta^2}\Delta^\pm[f,g]\,, \tag{22}$$

where $k^\mu = f^\nu\partial_\nu g^\mu - g^\nu\partial_\nu f^\mu$ and $\Delta^\pm[f,g] = \int \mathrm{d}x^\pm\,(g^\pm\partial_\pm^3 f^\pm - f^\pm\partial_\pm^3 g^\pm)$. By restricting to a periodic manifold, with a periodicity P, $x^\pm \propto x^\pm + P$, generators can be decomposed into

$$Q_n^\pm \equiv \frac{P}{2\pi}\int \mathrm{d}x^\pm\,\Theta_{\pm\pm}e^{i\frac{2\pi}{P}nx^\pm} = \frac{P}{2\pi}Q^\pm[e^{i\frac{2\pi}{P}nx^\pm}]\,, \tag{23}$$

and the algebra (22) can be recast into the following form

$$i\{Q_n^\pm, Q_m^\pm\} = (n-m)Q_{n+m}^\pm + \frac{4\pi}{\beta^2}n^3\delta_{n+m,0} \tag{24}$$

that is just the Virasoro algebra with genuine central charge

$$c = \frac{48\pi}{\beta^2}\,. \tag{25}$$

[3] Light-cone coordinates are defined in Supplemental Material. Further details on the expression of these charges in this frame can be found in [9]

It is the algebra of the flat Liouville EMT components that inevitably includes the genuine center (25)

$$\{\Theta_{\pm\pm}(x), \Theta_{\pm\pm}(y)\}\Big|_{x^{\mp}=y^{\mp}} = \Theta'_{\pm\pm}(x)\delta(x^{\pm}-y^{\pm}) + 2\Theta_{\pm\pm}(x)\delta'(x^{\pm}-y^{\pm}) - \frac{c}{24\pi}\delta'''(x^{\pm}-y^{\pm})\,, \quad (26)$$

and so do its transformations

$$\delta_f \Theta_{\pm\pm} = f^{\pm}\partial_{\pm}\Theta_{\pm\pm} + 2\Theta_{\pm\pm}\partial_{\pm}f^{\pm} - \frac{c}{24\pi}\partial^3_{\pm}f^{\pm}\,. \quad (27)$$

This center is not there in the trace of flat EMT (19), but it is proportional to the trace (4) of the curved space EMT (3).

A deeper study of this, including the general framework for classically anomalous transformations, we have done it in [9]. Here we want to show that the above is indeed related to the extra term $T^{\mu\nu}_{\rm extra}$ that, in curved space, preserves Weyl invariance but breaks diffeomorphic invariance. To do so, let us first rewrite (27) as the difference between the full transformation and its tensorial part

$$\Delta\Theta_{\pm\pm} \equiv \delta_f \Theta_{\pm\pm} - f^{\pm}\partial_{\pm}\Theta_{\pm\pm} - 2\Theta_{\pm\pm}\partial_{\pm}f^{\pm} = -\frac{c}{24\pi}\partial^3_{\pm}f^{\pm}\,, \quad (28)$$

as we did earlier in the curved context. We then simply notice that, for the infinitesimal diffeomorphism, $x^\mu \to x^\mu - f^\mu(x)$, the non-tensorial transformation of $T^{\mu\nu}_{\rm extra}$ is

$$\beta^2 \Delta\hat{T}^{\mu\nu}_{\rm extra}(x) = \partial_\alpha\partial_\beta\xi(r,f)\left(\hat{g}^{\mu\alpha}\frac{\varepsilon^{\nu\beta}}{\sqrt{-\hat{g}}} + \hat{g}^{\nu\alpha}\frac{\varepsilon^{\mu\beta}}{\sqrt{-\hat{g}}}\right) + 2\hat{g}^{\mu\nu}\frac{\varepsilon^{\alpha\beta}}{\sqrt{-\hat{g}}}\partial_\alpha\rho\partial_\beta\xi(r,f)\,, \quad (29)$$

where the same notation of (16) and (17) has been used. Assuming conformal diffeomorphisms and taking the flat limit we have

$$\Delta\hat{T}^{\pm\pm}_{\rm extra}(x)\Big|_{\rho\to 0} = -\frac{2}{\beta^2}\,\partial^3_{\mp}f^{\mp} = \Delta\Theta_{\mp\mp}\,, \quad (30)$$

which is exactly[4] (28) with c given by (25).

This center was removed from the trace (4) but re-emerged in (30). We have then proved that, also for *classical* Liouville theory, lack of Weyl invariance or of diffeomorphism invariance is related to the Virasoro center, like in the quantum case. This gives a precise mathematical meaning to what we are now entitled to call “classical gravitational anomalies”. Whether this is possible for more general classical systems, it is an important open question. Another direction for further research that we are considering is the connection of such anomalous transformation of the EMT with “classical Unruh and Hawking effects”.

Acknowledgments. We gladly acknowledge support from Charles University Research Center (UNCE/SCI/013).

[1] J. Liouville, J. Math. Pures Appl. , 71 (1853).
[2] R. Jackiw, in *Quantum Theory of Gravity*, edited by S. Christensen (Adam-Hilger Ltd., 1984) Chap. A Two-Dimensional Model of Gravity, pp. 403–420.
[3] D. Grumiller, R. Ruzziconi, and C. Zwikel, SciPost Phys. **12**, 032 (2022).

[4] Due to the signature of the light-cone metric, $\Theta_{\mp\mp} = \Theta^{\pm\pm}$. See Supplemental Material.

[4] A. M. Polyakov, Phys. Lett. B **103**, 207 (1981).
[5] P. Di Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer-Verlag, 1997).
[6] J. Penedones, in *New Frontiers in Fields and Strings*, edited by J. Polchinski, P. Vieira, and O. DeWolfe (World Scientific, 2017) Chap. TASI Lectures on AdS/CFT, pp. 75–136.
[7] A. Iorio, L. O'Raifeartaigh, I. Sachs, and C. Wiesendanger, Nucl. Phys. B **495**, 433 (1997).
[8] R. Jackiw, Theor. Math. Phys. **148**, 941 (2006).
[9] P. Haman and A. Iorio, *Trace and diffeomorphism anomalies of the classical Liouville theory, Virasoro algebras, Weyl-gauging and all that* (forthcoming, 2023).
[10] S. Deser and R. Jackiw, Int. J. Mod. Phys. B **495**, 1499 (1996).
[11] S. P. Robinson and F. Wilczek, Phys. Rev. Lett. **95**, 011303 (2005).
[12] R. Jackiw 10.48550/arxiv.hep-th/9501016 (1995).

Supplemental Material

I. LIGHT-CONE COORDINATES

We define the light-cone coordinates in two dimensions as

$$x^{\pm} = \frac{1}{\sqrt{2}}(x^0 \pm x^1)\,. \tag{1}$$

The derivatives in ligh-cone coordinates are

$$\partial_+ = \frac{1}{\sqrt{2}}(\partial_0 + \partial_1), \quad \partial_- = \frac{1}{\sqrt{2}}(\partial_0 - \partial_1)\,. \tag{2}$$

The coordinate transformation $(x^0, x^1) \to (x^+, x^-)$ can be obtained by acting with the matrix

$$S = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}\,. \tag{3}$$

Light-cone components of the metric are, thus, obtained as

$$g_{++} = \frac{1}{2}(g_{00} + 2g_{01} + g_{11}) \tag{4}$$

$$g_{+-} = \frac{1}{2}(g_{00} - g_{11}) \tag{5}$$

$$g_{--} = \frac{1}{2}(g_{00} - 2g_{01} + g_{11})\,, \tag{6}$$

and the components of inverse metric $g^{\mu\nu}$ are

$$g^{++} = \frac{g_{--}}{g}, \quad g^{+-} = -\frac{g_{+-}}{g}, \quad g^{--} = \frac{g_{++}}{g}\,. \tag{7}$$

The light-cone Minkowski metric, with the signature $\eta_{\mu\nu} = \mathrm{diag}(+1, -1)$, is

$$\eta_{\mu\nu} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\,, \tag{8}$$

with $\mu, \nu \in \{+, -\}$. Thus, the scalar product is $x^2 = -2x^+x^-$. Raising and lowering indices changes the $+$ to $-$ sign and vice versa, e.g., $\partial_{\pm} = \partial^{\mp}$, $\Theta_{\mp\mp} = \Theta^{\pm\pm}$, etc..

II. IMPROVEMENT OF THE ENERGY-MOMENTUM TENSOR

Here we compute the extra improvement term of the EMT

$$T^{\mu\nu}_{\mathrm{extra}} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta}{\delta g_{\mu\nu}}\int \mathrm{d}^2x\,\frac{2}{\beta^2}\sqrt{-g}W^{\mu}W_{\mu} \equiv -\frac{2}{\sqrt{-g}}\frac{\delta \Delta A}{\delta g_{\mu\nu}}\,.$$

To facilitate calculations it is easier to introduce the "conformal" metric

$$\gamma_{\mu\nu} \equiv \frac{g_{\mu\nu}}{\sqrt{-g}}, \qquad \gamma^{\mu\nu} \equiv \sqrt{-g}g^{\mu\nu}, \qquad \sqrt{-g} \equiv \rho\,, \tag{9}$$

which, in light-cone coordinates, can be parametrized as

$$\gamma_{++} = e^{\gamma}\sinh\sigma, \quad \gamma_{+-} = \cosh\sigma, \quad \gamma_{--} = e^{-\gamma}\sinh\sigma\,. \tag{10}$$

In this parametrization a new quantity R^μ can be defined

$$R^\mu \equiv 2\sqrt{-g}W^\mu\,. \tag{11}$$

Using the following identity

$$\epsilon^{\mu\nu}\epsilon^{\alpha\beta} = \gamma^{\mu\beta}\gamma^{\nu\alpha} - \gamma^{\mu\alpha}\gamma^{\nu\beta}\,, \tag{12}$$

it can be seen that

$$R^\mu = -\gamma^{\mu\nu}\partial_\nu\rho - \partial_\nu\gamma^{\mu\nu} + \epsilon^{\mu\nu}(\cosh\sigma - 1)\partial_\nu\gamma + \epsilon^{\mu\nu}\partial_\nu r\,. \tag{13}$$

The natural way to compute the EMT for ΔA is by varying the latter with respect to $\gamma^{\mu\nu}$

$$T_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\Delta A}{\delta g^{\mu\nu}} = 2\frac{\delta\Delta A}{\delta\gamma^{\mu\nu}} - \gamma_{\mu\nu}\gamma^{\alpha\beta}\frac{\delta\Delta A}{\delta\gamma^{\alpha\beta}}\,. \tag{14}$$

To begin, we see that

$$\delta\Delta A = \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\delta\gamma_{\mu\nu}R^\mu R^\nu + \frac{1}{\beta^2}\int \mathrm{d}^2x\,\gamma_{\mu\nu}R^\nu\delta R^\mu\,, \tag{15}$$

where

$$\delta R^\mu = -\delta\gamma^{\mu\nu}\partial_\nu\sigma - \gamma^{\mu\nu}\partial_\nu\delta\sigma - \partial_\nu\delta\gamma^{\mu\nu} + \partial_\nu[\epsilon^{\mu\nu}(\cosh\sigma - 1)\delta\gamma] - \bar\Gamma^\mu_{\alpha\beta}\delta\gamma^{\alpha\beta} + \epsilon^{\mu\nu}\partial_\nu\delta r\,. \tag{16}$$

The last two terms are the result of the following computation [10]

$$\delta[\epsilon^{\mu\nu}(\cosh\sigma - 1)\partial_\nu\gamma] - \partial_\nu[\epsilon^{\mu\nu}(\cosh\sigma - 1)\delta\gamma] = -\bar\Gamma^\mu_{\alpha\beta}\delta\gamma^{\alpha\beta}\delta\gamma^{\alpha\beta}\,, \tag{17}$$

where

$$\bar\Gamma^\mu_{\alpha\beta} = \frac{1}{2}\gamma^{\mu\nu}\left(\partial_\alpha\gamma_{\nu\beta} + \partial_\beta\gamma_{\nu\alpha} - \partial_\nu\gamma_{\alpha\beta}\right)\,. \tag{18}$$

Let us now split $\delta\Delta A$ into four terms

$$\delta\Delta A = \delta\Delta A^1 + \delta\Delta A^2 + \delta\Delta A^3 + \delta\Delta A^4\,, \tag{19}$$

where

$$\begin{aligned}
\delta\Delta A^1 =& \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\delta\gamma_{\mu\nu}R^\nu R^\mu\,,\\
\delta\Delta A^2 =& \frac{1}{\beta^2}\int \mathrm{d}^2x\,\gamma_{\mu\nu}R^\nu(-\delta\gamma^{\mu\lambda}\partial_\lambda\rho - \partial_\lambda\delta\gamma^{\mu\lambda} - \bar\Gamma^\mu_{\alpha\beta}\delta\gamma^{\alpha\beta})\\
=& \frac{1}{2\beta^2}\int \mathrm{d}^2x\,\delta\gamma^{\alpha\beta}\left[g_{\beta\lambda}\nabla_\alpha\left(\frac{R^\lambda}{\sqrt{-g}}\right) + g_{\alpha\lambda}\nabla_\beta\left(\frac{R^\lambda}{\sqrt{-g}}\right)\right]\,,\\
\delta\Delta A^3 =& -\frac{1}{\beta^2}\int \mathrm{d}^2x\,R^\mu\partial_\mu\delta\rho = -\frac{1}{2\beta^2}\int \mathrm{d}^2x\,\sqrt{-g}Rg_{\alpha\beta}\delta g^{\alpha\beta}\,,\\
\delta\Delta A^4 =& \frac{1}{\beta^2}\int \mathrm{d}^2x\,R^\mu\gamma_{\mu\nu}\{\partial_\lambda[\epsilon^{\nu\lambda}(\cosh\omega - 1)\delta\gamma] + \epsilon^{\nu\lambda}\partial_\lambda\delta r\}\\
=& -\frac{1}{\beta^2}\int \mathrm{d}^2x\,\partial_\lambda(R^\mu\gamma_{\mu\nu})\epsilon^{\nu\lambda}\delta g_{\alpha\beta}[(\cosh\omega - 1)\Gamma^{\alpha\beta} + r^{\alpha\beta}]\,.
\end{aligned}$$

TO derive the second expression for $\delta\Delta A^2$ we used

$$\begin{aligned}
\partial_\mu T^{\alpha\dots}_{\beta\dots} + \partial_\mu \rho T^{\alpha\dots}_{\beta\dots} &= \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}T^{\alpha\dots}_{\beta\dots})\,,\\
g_{\alpha\lambda}\nabla_\beta V^\lambda + g_{\beta\lambda}\nabla_\alpha V^\lambda &= g_{\alpha\lambda}\partial_\beta V^\lambda + g_{\beta\lambda}\partial_\alpha V^\lambda + V^\lambda\partial_\lambda g_{\alpha\beta}\,,\\
\gamma_{\alpha\beta}\delta\gamma^{\alpha\beta} &= 0\,.
\end{aligned}$$

To derive $\delta\Delta A^3$ we used the fact that R^μ is a solution of

$$\sqrt{-g}R = \partial_\mu R^\mu\,.$$

Finally, rewriting $\delta\Delta A^4$, we defined

$$\begin{aligned}
\delta\gamma &= \Gamma^{\mu\nu}\delta g_{\mu\nu}\,,\\
\delta r &= r^{\mu\nu}\delta g_{\mu\nu}\,.
\end{aligned}$$

By a direct calculations it follows that

$$\Gamma^{\mu\nu} = \frac{1}{2\sqrt{g_{--}g_{++}}}\begin{pmatrix} -\sinh\gamma & \cosh\gamma \\ \cosh\gamma & -\sinh\gamma \end{pmatrix}\,. \tag{20}$$

With this preliminaries, we are now ready for the computation of the EMT in four steps

$$T^i_{\mu\nu} = \frac{2}{\sqrt{-g}}\frac{\delta\Delta A^i}{\delta g^{\mu\nu}}\,,$$

with $i = 1,2,3,4$, and the results are

$$\begin{aligned}
T^1_{\mu\nu} &= \frac{1}{\beta^2}\left(\frac{1}{2}\gamma_{\mu\nu}\gamma_{\alpha\beta}R^\alpha R^\beta - \gamma_{\mu\alpha}\gamma_{\nu\beta}R^\alpha R^\beta\right)\,,\\
T^2_{\mu\nu} &= \frac{1}{\beta^2}\left(g_{\mu\lambda}\nabla_\nu\left(\frac{R^\lambda}{\sqrt{-g}}\right) + g_{\nu\lambda}\nabla_\mu\left(\frac{R^\lambda}{\sqrt{-g}}\right)\right) - \frac{1}{\beta^2}Rg_{\mu\nu}\,,\\
T^3_{\mu\nu} &= -\frac{1}{\beta^2}Rg_{\mu\nu}\,,\\
T^4_{\mu\nu} &= \frac{2}{\beta^2\sqrt{-g}}\partial_\beta\left(\frac{R^\lambda g_{\alpha\lambda}}{\sqrt{-g}}\right)\epsilon^{\alpha\beta}[(\cosh\sigma - 1)\Gamma_{\mu\nu} + r_{\mu\nu}]\,.
\end{aligned} \tag{21}$$

Adding these together we have

$$\begin{aligned}
\beta^2 T^{\mu\nu}_{\text{extra}} &= \frac{1}{g}\left(R^\mu R^\nu - \frac{1}{2}g^{\mu\nu}R\cdot R\right) - 2Rg^{\mu\nu}\\
&\quad + g^{\mu\alpha}\nabla_\alpha\left(\frac{R^\nu}{\sqrt{-g}}\right) + g^{\nu\alpha}\nabla_\alpha\left(\frac{R^\mu}{\sqrt{-g}}\right)\\
&\quad + \frac{2}{\sqrt{-g}}\partial_\beta\left(\frac{R^\lambda g_{\alpha\lambda}}{\sqrt{-g}}\right)\epsilon^{\alpha\beta}[(\cosh\sigma - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\\
&= 2g^{\mu\nu}W^\alpha W^\beta g_{\alpha\beta} - 4W^\mu W^\nu - 2Rg^{\mu\nu}\\
&\quad + 2g^{\mu\alpha}\nabla_\alpha W^\nu + 2g^{\nu\alpha}\nabla_\alpha W^\mu\\
&\quad + \frac{4}{\sqrt{-g}}\partial_\beta\left(W^\lambda g_{\alpha\lambda}\right)\epsilon^{\alpha\beta}[(\cosh\sigma - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]\,.
\end{aligned} \tag{22}$$

Since this improvement term should cancel the trace of the EMT, let us compute its trace

$$\frac{\beta^2}{2}{T_{\text{extra}}}^{\mu}{}_{\mu} = -R + 2\frac{\epsilon^{\alpha\beta}}{\sqrt{-g}}\partial_\beta\left(W^\lambda g_{\alpha\lambda}\right)[(\cosh\sigma - 1)\Gamma^{\mu\nu} + r^{\mu\nu}]g_{\mu\nu}\,. \tag{23}$$

Recalling that

$$\Gamma^{\mu\nu} = \frac{2}{(g_{00}+g_{11})^2 - 4g_{01}^2}\left[\frac{1}{2}(\delta_0^\mu\delta_1^\nu + \delta_1^\mu\delta_0^\nu)(g_{00}+g_{11}) - g_{01}(\delta_0^\mu\delta_0^\nu + \delta_1^\nu\delta_1^\mu)\right]\,,$$

we see that

$$g_{\mu\nu}\Gamma^{\mu\nu} = 0\,, \tag{24}$$

therefore

$${T_{\text{extra}}}^{\mu}{}_{\mu} = -\frac{2}{\beta^2}R + \frac{4}{\sqrt{-g}}\partial_\beta\left(W^\lambda g_{\alpha\lambda}\right)\epsilon^{\alpha\beta}r^{\mu\nu}g_{\mu\nu}\,. \tag{25}$$

From here we see another condition for r

$$g_{\mu\nu}r^{\mu\nu} = 0\,, \tag{26}$$

with which

$${T_{\text{extra}}}^{\mu}{}_{\mu} = -\frac{2}{\beta^2}R\,. \tag{27}$$

Hence the trace of the improvement cancels the anomalous trace of the canonical EMT

$$T^\mu_\mu = \frac{2}{\beta^2}R\,.$$

This proves the Weyl invariance of the improved Liouville action.