Classical Information and Distillable Entanglement

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We establish a quantitative connection between the amount of lost classical information about a quantum state and the concomitant loss of entanglement. Using methods that have been developed for the optimal purification of mixed states, we find a class of mixed states with known distillable entanglement. These results can be used to determine the quantum capacity of a quantum channel which randomizes the order of transmitted signals.

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The development of quantum information processing in recent years has shown that quantum information, and, in particular, quantum entanglement, allow for the realization of applications that are not possible classically [1]. Classical information has, however, not become obsolete as a simple limiting case of the more general theory. In fact, there are interesting connections between the amount of quantum entanglement [2–4] that is held by two parties and the classical information that is available about the jointly held system [5]. An extreme example would be one where the two parties Alice and Bob are sharing an equal mixture of two Bell states. Being completely ignorant about the identity of the state, the density operator describing the system can also be described as an equal mixture of two product states, which implies that Alice and Bob share no entanglement at all. However, given the one bit of information about the identity of the state, they share one ebit of entanglement (one maximally entangled state between two qubits). While the exact relation between the amount of classical information required per gained ebit is unknown (see also [5]), this example illustrates that the retrieval of classical information can lead to an increase in the usable entanglement. Quite analogously, the loss of classical information will usually reduce the amount of entanglement held between two parties. In this paper we will consider a particularly clear way in which classical information is lost. Surprisingly, for the resulting class of mixed states the distillable entanglement can be determined. This example can also be interpreted as a noisy quantum channel which randomizes the order of transmitted signals. Using the results of this paper, we are able to determine the quantum capacity of such a quantum channel.

Imagine two spatially separated parties, Alice and Bob, who are holding two entangled pairs of particles which they would like to use later on, for example, to implement some quantum communication protocol. As Alice and Bob share two pairs of identical particles, they need a classical record about the order of the particles. This means that it is known which of Alice's particles is entangled with which particle of Bob (see left part of Fig. 1 which represents the mixed state of apparatus and system where the ancilla allows one to determine the order of the particles). Now imagine that, by some misfortune (e.g., the particles in a transmission arrive in random order), this classical record is destroyed, i.e., the ancilla is unavailable. In that case, the state of the two pairs kept by Alice and Bob is an equal mixture between two possible states: one where the first of Alice's particles is entangled with the first of Bob's particles and another where the first of Alice's particles is entangled with the second of Bob's particles. In the context of a quantum channel, one would have a situation where signals change their order randomly. The natural question for Alice and Bob is as follows: Are they still holding quantum mechanical entanglement and, if yes, how much? For example, what is the capacity of the associated quantum channel? Let us state the issue more formally.

When information ΔI about the order of a number of entangled quantum systems is lost, is the resulting state of any use for quantum communication purposes? How much entanglement $\Delta E = E_{\text{before}} - E_{\text{after}}$ has been destroyed and what can be said about the ratio $\Delta E/\Delta I$?

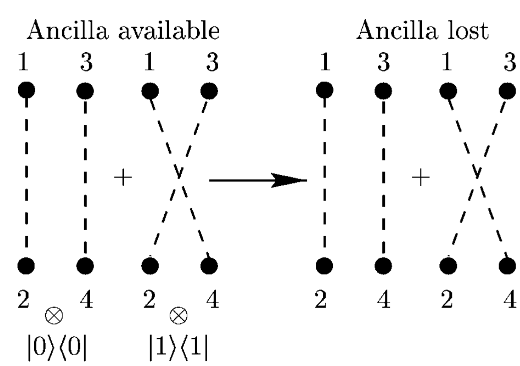

FIG. 1. In the left half of the figure, system and ancilla are in a mixed state. The ancilla allows one to determine the order of the particles. The first particle of Alice is entangled with either the first or the second particle of Bob—each with probability $p = \frac{1}{2}$. On the right-hand side, the ancilla is lost and one cannot determine the order anymore. The information that the ancilla had about the systems can in this case be quantified by the mutual information between system and ancilla [see also [6] and discussion of Eq. (8)].

It is this question that we will investigate in this Letter to further explore the connection between classical and quantum information. First, we will answer the special case in which Alice and Bob are holding two pairs of maximally entangled states. Subsequently, we will solve the case of an even number of copies of pairs of arbitrarily entangled particles. These results then give rise to a bound for the ratio $\Delta E/\Delta I$ in a more general situation. The concept of entanglement which is employed in the following is that of distillable entanglement E_D [2,4,7,8] with respect to separable operations [9]. This means that we are interested in the maximal rate with which entanglement purification can obtain maximally entangled states from a state which has arisen due to the loss of classical information.

Example.—Consider the situation where Alice and Bob share two pairs of two-level systems—i.e., qubits—each in a maximally entangled state of the form $|\psi_0\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. In this way they are sharing two ebits of entanglement. Now Bob loses the information about the order of his quantum systems. This means that Bob does not know whether his two particles are in the original order or have been permuted (see right half of Fig. 1).

Let $|\psi_1\rangle\langle\psi_1|$ be the state of the qubits labeled 1, 2, 3, and 4 in the original situation (see Fig. 1). 1 and 3 are Alice's qubits, Bob's qubits are numbered 2 and 4. In the computational basis $|\psi_1\rangle = (|0000\rangle + |0011\rangle + |1100\rangle + |1111\rangle)/2$, while in the permuted case where the role of 2 and 4 is interchanged the state is $|\psi_2\rangle = (|0000\rangle + |0110\rangle + |1001\rangle + |1111\rangle)/2$. As a result of the loss of the order of the particles on Bob's side, the composite quantum system is now described by the density operator,

$$\sigma = (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)/2\,. \tag{1}$$

It is now natural to ask how much entanglement is still accessible to Alice and Bob, i.e., how much distillable entanglement the state σ holds. To solve this question consider the spectral decomposition of σ given by

$$\sigma = \frac{1}{4}\,|\phi_1\rangle\langle\phi_1| + \frac{3}{4}\,|\phi_2\rangle\langle\phi_2|\,, \tag{2}$$

where

$$|\phi_1\rangle = (|0011\rangle - |0110\rangle - |1001\rangle + |1100\rangle)/2\,, \tag{3a}$$

$$|\phi_2\rangle = (2|0000\rangle + |0011\rangle + |0110\rangle + |1001\rangle + |1100\rangle + 2|1111\rangle)/\sqrt{12}\,. \tag{3b}$$

In the basis of angular momentum eigenstates $|j,m\rangle$ with $j = 0,1$; $m = -1,0,1$, which is given by

$$|1,-1\rangle = |00\rangle, \qquad |1,1\rangle = |11\rangle\,, \tag{4a}$$

$$|1,0\rangle = (|01\rangle + |10\rangle)/\sqrt{2}\,, \quad |0,0\rangle = (|01\rangle - |10\rangle)/\sqrt{2}\,, \tag{4b}$$

the eigenstates $|\phi_1\rangle$ and $|\phi_2\rangle$ read

$$|\phi_1\rangle = |0,0\rangle\,|0,0\rangle\,, \tag{5a}$$

$$|\phi_2\rangle = (|1,-1\rangle\,|1,-1\rangle + |1,0\rangle\,|1,0\rangle + |1,1\rangle\,|1,1\rangle)/\sqrt{3}\,. \tag{5b}$$

Here, the first ket corresponds to Alice's qubits, the second ket to Bob's.

An upper bound [4,7] for the distillable entanglement is given by the relative entropy of entanglement [2,4] $E_R(\sigma)$ of σ which in turn is smaller than or equal to the relative entropy with respect to any separable state ρ. Hence, the distillable entanglement $E_D(\sigma)$ of σ is bounded by

$$E_D(\sigma) \le S(\sigma\|\rho) = \frac{3}{4}\log 3\,, \tag{6}$$

where the disentangled state ρ is chosen as $\rho = \frac{1}{4}\sum_{j=0}^{1}\sum_{m=-j}^{j} |j,m\rangle\,|j,m\rangle\langle j,m|\langle j,m|$. Surprisingly, it turns out that the upper bound given in Eq. (6) can indeed be achieved.

In the optimal distillation protocol Alice performs a von-Neumann projective measurement with the two possible projectors $A_1 = |0,0\rangle\langle 0,0|$ and $A_2 = \sum_{m=-1}^{1} |j = 1,m\rangle\langle j = 1,m|$, while Bob remains inactive, i.e., $B_2 = \mathbb{1}_B$. With probability $p_1 = 1/4$ they obtain the normalized output state $|\phi_1\rangle\langle\phi_1|$, which is a product state and of no further use. With probability $p_2 = 3/4$ they obtain $|\phi_2\rangle\langle\phi_2|$ which has log3 ebits of entanglement. The average number of maximally entangled states that can be distilled from σ is given by

$$E_D(\sigma) = (3/4)\log 3 \approx 1.189\,. \tag{7}$$

As this realizes the bound Eq. (6), it is the maximally possible value [10]. It is worth noting that this value is greater than one. Hence, less than one ebit of entanglement is erased due to the loss of the classical information about the order. Now we need to compute the loss of classical information when the ancilla is erased. What is the information that the ancilla possesses about the order of the systems? If we have no access to the ancilla, we see an equal mixture between the nonorthogonal states $|\psi_1\rangle$ and $|\psi_2\rangle$, each corresponding to a particular piece of classical information, namely, no change or change of order. We know from Schumachers noiseless coding theorem (in the case of initially mixed states we have to employ the accessible information [11]) that the amount of classical information about the order encoded in this way equals $S(\sigma) = -\mathrm{tr}(\sigma\log\sigma)$ with σ as in Eq. (1). This quantifies our classical uncertainty about the order of the particles. If we have access to the ancilla, then all our uncertainty is removed. Therefore the ancilla possesses $S(\sigma)$ of classical information about the order of the systems. It is this information $\Delta I = S(\sigma)$ that is lost when the ancilla is discarded [6]. As a result we find

$$\frac{\Delta E_D}{\Delta I} = 1\,. \tag{8}$$

The above scenario can be generalized to the situation where Alice and Bob initially do not hold maximally entangled states but pure states of the type $\alpha|00\rangle + \beta|11\rangle$ [12] with a given degree of entanglement. This case is interesting since it leads to an operationally defined one parameter class of states for which the distillable entanglement with respect to separable operations can be analytically computed. Here we consider $\sigma = (|\psi_1\rangle\langle\psi_1| + |\psi_2\rangle\langle\psi_2|)/2$ with

$$|\psi_1\rangle = \alpha^2|0000\rangle + \alpha\beta|0011\rangle + \alpha\beta|1100\rangle + \beta^2|1111\rangle, \tag{9}$$

$$|\psi_2\rangle = \alpha^2|0000\rangle + \alpha\beta|0110\rangle + \alpha\beta|1001\rangle + \beta^2|1111\rangle, \tag{10}$$

where $\alpha \in [0,1]$, $\beta = \sqrt{1-\alpha^2}$. Following the previous calculation, we find that the distillable entanglement is

$$\begin{aligned} E_D(\sigma) &= E_R(\sigma) \\ &= (1-\alpha^2\beta^2)\log(1-\alpha^2\beta^2) \\ &\quad - (\alpha^4\log\alpha^4 + \beta^4\log\beta^4 + \alpha^2\beta^2\log\alpha^2\beta^2). \end{aligned} \tag{11}$$

Since the entanglement of the initial pure state was given by the entropy of the reduced states of Alice or Bob, that is, by $-2(\alpha^2\log\alpha^2 + \beta^2\log\beta^2)$, again, $\Delta E_D/\Delta I = 1$ for all $\alpha \in [0,1]$. Of course, Eq. (11) reduces to $E_D(\sigma) = (3/4)\log 3$ for $\alpha = \beta = 1/\sqrt{2}$.

The example we have presented here leads to a more general proposition which we are going to prove here. We restrict the argument to the case where Alice and Bob are initially sharing pairs of qubits in pure states with two-particle entanglement.

Proposition.—Let Alice and Bob share $N = 2J$ pairs of qubits each in the same state $|\phi\rangle$. The associated Hilbert space is $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B \sim (\mathbb{C}^2)^{\otimes 2J} \otimes (\mathbb{C}^2)^{\otimes 2J}$, $J = 1,2,\ldots$. Bob then loses the information ΔI about the order of the qubits completely. As a consequence, Alice and Bob are now sharing a less entangled mixed state σ. The distillable entanglement of the state σ can be calculated exactly, and the ratio between the change of distillable ΔE_D and the amount of erased information ΔI obeys for any $J = 1,2,\ldots$ the inequality

$$\frac{\Delta E_D}{\Delta I} \leq 1, \tag{12}$$

with equality for $J = 1$.

Proof: Let us first consider the case where $|\phi\rangle$ is a maximally entangled state $|\phi\rangle = (|00\rangle + |11\rangle)/\sqrt{2}$. To prove the statement of the proposition we first construct the state σ after the loss of the order of Bob's particles. Then the optimal entanglement purification protocol will be presented and its optimality proven. The loss in information can be calculated and the validity of Eq. (12) for these particular initial states is then confirmed. In the more general case of arbitrary initial states $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ the same approach can be applied, confirming Eq. (12).

Let $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ be the underlying Hilbert space and $S(\mathcal{H})$ the associated state space. Since $\mathcal{H}_A$ and $\mathcal{H}_B$ are $2J$-fold tensor products of Hilbert spaces isomorphic to $\mathbb{C}^2$, they can be decomposed into a direct sum of orthogonal subspaces of the form $\mathcal{H}_A = \bigoplus_{j=0}^{J}\bigoplus_{\alpha_j}\mathcal{H}^A_{j,\alpha_j}$, $\mathcal{H}_B = \bigoplus_{k=0}^{J}\bigoplus_{\beta_k}\mathcal{H}^B_{k,\beta_k}$, where $\mathcal{H}^A_{j,\alpha_j} = \mathrm{span}\{|j,m,\alpha_j\rangle\,|m = -j,-j+1,\ldots,j\}$ for $j = 0,1,\ldots,J$ and $\alpha_j = 1,2,\ldots,d_j$. The additional degeneracy is given by $d_j = \frac{2j+1}{2J+1}\binom{2J+1}{J-j}$. As in [13] we choose $|j,m,1\rangle = |j,m\rangle \otimes [(|01\rangle - |10\rangle)/\sqrt{2}]^{\otimes(J-j)}$, where $|j,m\rangle$ is the state of $2j$ qubits with a fixed value of j and m with $j-m$ qubits in $|0\rangle$. $\mathcal{H}_B$ can be decomposed into a direct sum in exactly the same fashion.

Using such a decomposition of the Hilbert space, it can easily be seen that the initial state of the N pairs shared between Alice and Bob can be written as

$$\otimes_{n=1}^{N}|\phi\rangle = \sum_{j,m,\alpha}|j,m,\alpha\rangle|j,m,\alpha\rangle/\sqrt{2^{2J}}. \tag{13}$$

The state σ after the loss of the order of Bob's particles σ is then given by $\sigma = \sum_{j=0}^{J}\sum_{\alpha_j,\beta_j=1}^{d_j} \times p_j|\psi_j(\alpha_j,\beta_j)\rangle\langle\psi_j(\alpha_j,\beta_j)|$, with $|\psi_j(\alpha_j,\beta_j)\rangle = 1/\sqrt{2j+1}\sum_{m=-j}^{j}|j,m,\alpha_j\rangle|j,m,\beta_j\rangle$ and $p_j = (2j+1)/(2^{2J}d_j)$. This particular form of the state after loss of the order of the particles can be proven using Schur's first lemma [14].

As before, the following distillation protocol is based on the fact that the subspaces of the state space corresponding to the above components of the underlying Hilbert space are locally distinguishable. Interesting enough, this protocol is related to the algorithm proposed in [13] for the optimal purification of qubits.

(i) Alice performs a local projective measurement in such a way that her reduced state is an element of $S(\mathcal{H}^A_{j,\alpha_j})$ for some $j = 0,\ldots,J$; $\alpha_j = 1,\ldots,d_j$.

(ii) If $\alpha_j \neq 1$ she applies a local unitary operation U^A_{j,α_j} such that her reduced state is included in the set $S(\mathcal{H}^A_{j,1})$. Since, in general, $|j,m,\alpha_j\rangle$ is a linear superposition of $\Pi_i|j,m,1\rangle$, where Π_i, $i = 1,2,\ldots$ are appropriate locally acting permutation operators, this is always possible.

(iii) The reduced state σ_A of Alice is at this stage of the structure $\sigma_A = \omega_A \otimes [(|01\rangle - |10\rangle)(\langle 01| - \langle 10|)/2]^{\otimes(J-j)}$. The last J-j pairs of qubits in the singlet state are neither entangled with the other qubits on her side nor entangled with any of Bob's qubits. Hence, they will be of no further use in the distillation protocol.

(iv) Bob performs a local measurement projecting his reduced state on $S(\mathcal{H}^B_{k,\beta_k})$ with some $k = 0,1,\ldots,J$ and $\beta_k = 1,2,\ldots,d_k$. Because of the particular form of the initial state $k = j$, but he may get a β_j different from the α_j obtained by Alice.

(v) In the same way as before, Bob applies a local unitary operation U^B_{j,β_j} such that his reduced state is an element of $S(\mathcal{H}^B_{j,1})$.

(vi) Alice and Bob end up with the probability $d_j^2 p_j = (2j+1)d_j/(2^{2J})$ in one of the pure states $|\psi_j\rangle\langle\psi_j|$, where $|\psi_j\rangle = 1/\sqrt{2j+1}\sum_{m=-j}^{j}|j,m,1\rangle|j,m,1\rangle$. This state contains $\log(2j+1)$ ebits of entanglement. Hence, the total average number is $\sum_j d_j^2 p_j S[\mathrm{tr}_A(|\psi_j\rangle\langle\psi_j|)] = \sum_j d_j^2 p_j \log(2j+1)$.

To show that the above protocol is actually optimal, we consider the relative entropy functional of the state σ after permutation with respect to an appropriate separable state ρ. The separable state ρ is taken to be $\rho = \sum_j p_j\rho_j$, where

$$\rho_j = \sum_{\alpha_j,\beta_j=1}^{d_j}\sum_{m=-j}^{j}\frac{(|j,m,\alpha_j\rangle\langle j,m,\alpha_j| \otimes |j,m,\beta_j\rangle\langle j,m,\beta_j|)}{2j+1}. \tag{14}$$

Since all subspaces associated with different values of j, m, α_j, and β_j are orthogonal and with $p_j = \frac{2j+1}{d_j 2^{2J}}$ this expression is given by $S(\sigma\|\rho) = \sum_{j=0}^{J} d_j^2 p_j \log(2j+1)$. This is identical to the value given for the average number of maximally entangled states obtained when employing the above procedure, and, therefore, also identical to the distillable entanglement $E_D(\sigma)$ with respect to separable operations. As we are again dealing with pure states as in the example, the information that the ancilla holds about the system is again $\Delta I = S(\sigma)$ [see Eq. (8)]. It follows that $\Delta E_D/\Delta I \leq 1$ for all N for this particular initial state.

For any pure state we can set $|\phi\rangle = \alpha|00\rangle + \beta|11\rangle$ with $\alpha \in [0,1]$, $\beta = \sqrt{1-\alpha^2}$ [12]. The same argument as before holds, and the same protocol is optimal for the distillation with respect to separable operations. The state σ after permutation is found to be $\sigma = \sum_{j=0}^{J} p_j|\psi_j(\alpha_j,\beta_j)\rangle\langle\psi_j(\alpha_j,\beta_j)|$, where now $p_j = \sum_{m=-j}^{j}\alpha^{2(J-m)}\beta^{2(J+m)}/d_j$; the (unnormalized) states $|\psi_j(\alpha_j,\beta_j)\rangle\langle\psi_j(\alpha_j,\beta_j)|$ are defined as $|\psi_j(\alpha_j,\beta_j)\rangle = \sum_{m=-j}^{j}\alpha^{j-m}\beta^{j+m}|j,m,\alpha_j\rangle|j,m,\beta_j\rangle$. Again, $E_D(\sigma) = \sum_{j=0}^{J} d_j^2 p_j S[\mathrm{tr}_A|\psi_j(1,1)\rangle\langle\psi_j(1,1)|]$ and $\Delta I = S(\sigma)$. From this and the fact that, initially, $E_D(|\phi\rangle\langle\phi|^N) = -N(\alpha^2\log\alpha^2 + \beta^2\log\beta^2)$ ebits of entanglement are present, it follows that also in this case $\Delta E_D/\Delta I \leq 1$ for all N. Finally, the same result holds for the case where $N = 2J+1$ with $J = 1,2,\ldots$, which can be solved in an analogous way.

Hence, in scenarios of the type discussed in the proposition, concomitant with the loss of a number of bits of classical information, not more than one ebit of distillable entanglement is destroyed per bit of classical information. These findings (see also [5]) lead us to the following.

Conjecture.—Let ω be a state of a bipartite quantum system taken from a set of (possibly entangled) states $\omega_1,\ldots,\omega_N$, each of which is assigned a classical probability $p_1,\ldots,p_N$. After the loss of the classical information about the identity of the state ω the state of the quantum system is taken to be $\sigma = \sum_{n=1}^{N} p_n\omega_n$. The change in distillable entanglement $\Delta E_D = E_D(\omega) - E_D(\sigma)$ and the loss of classical information ΔI then obeys the inequality $\Delta E_D/\Delta I \leq 1$.

In summary, we have investigated a practically relevant situation in which classical information about the order of particles can be lost, e.g., during the transmission via a quantum channel. Surprisingly, the general class of mixed states obtained from this procedure have known distillable entanglement and therefore the corresponding quantum channel has known quantum capacity [15]. It turns out that the ratio between the loss of entanglement and the amount of classical information lost in such a situation can be related by an inequality and we conjecture the general validity of this inequality. These results shed new light on the relationship between entanglement purification and channel capacity on the one hand, and classical information on the other.

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