Tohoku Math. J.
56 (2004), 467–490

CLASSICAL TRANSCENDENTAL SOLUTIONS OF THE PAINLEVÉ EQUATIONS AND THEIR DEGENERATION

TETSU MASUDA

(Received February 10, 2003, revised May 12, 2004)

Abstract. We present a determinant expression for a family of classical transcendental solutions of the Painlevé V and the Painlevé VI equation. Degeneration of these solutions along the process of coalescence for the Painlevé equations is discussed.

1. Introduction. As is well-known, the Painlevé equations (except for $P_{\rm I}$) admit two classes of classical solutions. One is classical transcendental solutions expressible in terms of special functions of hypergeometric type. Another one is algebraic or rational solutions. It is also known that the Painlevé equations admit an action of the affine Weyl groups as groups of the Bäcklund transformations. It is remarkable that classical solutions are located on special places from a viewpoint of symmetry in the parameter spaces. A rough picture is that classical transcendental solutions exist on the reflection hyperplanes of the affine Weyl group and algebraic (or rational) solutions do on the fixed points with respect to the Bäcklund transformations corresponding to automorphisms of the Dynkin diagram.

In this paper, we concentrate our attention on the classical transcendental solutions. One of the important features of these solutions is that they can be expressed in terms of 2-directional Wronskians or Casorati determinants whose entries are given by the corresponding special functions of hypergeometric type. This arises as a consequence that the Toda equation describes Bäcklund (or Schlesinger) transformations of the Painlevé equations [11, 12, 13, 14]. Indeed, such determinant expressions for $P_{\rm II}$, $P_{\rm III}$ and $P_{\rm IV}$ have been presented in [13, 14, 5, 7].

The aim of this paper is to present a determinant expression for a family of classical transcendental solutions of the Painlevé V equation

$$\begin{aligned}\frac{d^2q}{dt^2} &= \left(\frac{1}{2q}+\frac{1}{q-1}\right)\left(\frac{dq}{dt}\right)^2-\frac{1}{t}\frac{dq}{dt}\\ &\quad+\frac{(q-1)^2}{2t^2}\left(\kappa_\infty^2 q-\frac{\kappa_0^2}{q}\right)-(\theta+1)\frac{q}{t}-\frac{q(q+1)}{2(q-1)}\,,\end{aligned} \tag{1.1}$$

2000 *Mathematics Subject Classification*. Primary 34M55; Secondary 33C05, 33C15.

Partly supported by the Grant-in-Aid for Young Scientists (B), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

which is equivalent to the Hamiltonian system

$$\text{(1.2)}\qquad \mathrm{S_V}: \qquad q' = \frac{\partial H}{\partial p}\,, \quad p' = -\frac{\partial H}{\partial q}\,, \quad {}' = t\frac{d}{dt}\,,$$

with the Hamiltonian

$$\text{(1.3)}\qquad \begin{aligned} H &= q(q-1)^2p^2 - \bigl[\kappa_0(q-1)^2 + \theta q(q-1) + tq\bigr]p + \kappa(q-1)\,,\\ \kappa &= \frac{1}{4}(\kappa_0+\theta)^2 - \frac{1}{4}\kappa_\infty^2\,, \end{aligned}$$

and the Painlevé VI equation

$$\text{(1.4)}\qquad \begin{aligned} \frac{d^2q}{dt^2} =& \frac{1}{2}\left(\frac{1}{q}+\frac{1}{q-1}+\frac{1}{q-t}\right)\left(\frac{dq}{dt}\right)^2 - \left(\frac{1}{t}+\frac{1}{t-1}+\frac{1}{q-t}\right)\frac{dq}{dt}\\ &+\frac{q(q-1)(q-t)}{2t^2(t-1)^2}\left[\kappa_\infty^2 - \kappa_0^2\frac{t}{q^2} + \kappa_1^2\frac{t-1}{(q-1)^2} + (1-\theta^2)\frac{t(t-1)}{(q-t)^2}\right], \end{aligned}$$

which is equivalent to the Hamiltonian system

$$\text{(1.5)}\qquad \mathrm{S_{VI}}: \qquad q' = \frac{\partial H}{\partial p}\,, \quad p' = -\frac{\partial H}{\partial q}\,, \quad {}' = t(t-1)\frac{d}{dt}\,,$$

with the Hamiltonian

$$\text{(1.6)}\qquad \begin{aligned} H =& q(q-1)(q-t)p^2\\ &-[\kappa_0(q-1)(q-t) + \kappa_1 q(q-t) + (\theta-1)q(q-1)]p + \kappa(q-t)\,,\\ \kappa =& \frac{1}{4}(\kappa_0+\kappa_1+\theta-1)^2 - \frac{1}{4}\kappa_\infty^2\,, \end{aligned}$$

respectively.

Let us explain how one can construct a family of classical transcendental solutions of the Painlevé equations. As an example, we take $\mathrm{P_{II}}$:

$$\text{(1.7)}\qquad \frac{d^2q}{dt^2} = 2q^3 - 2tq + 2\left(\alpha+\frac{1}{2}\right),$$

which is equivalent to the Hamiltonian system

$$\text{(1.8)}\qquad \mathrm{S_{II}}: \qquad q' = \frac{\partial H}{\partial p}\,, \quad p' = -\frac{\partial H}{\partial q}\,, \quad {}' = \frac{d}{dt}\,,$$

with the Hamiltonian

$$\text{(1.9)}\qquad H = -p^2 - (q^2-t)p + \alpha q\,.$$

If $\alpha = 0$, the right-hand side of the second equation of (1.8) is divisible by p, which means that $\mathrm{S_{II}}$ admits the specialization of $p = 0$ if $\alpha = 0$. The first equation of (1.8) yields the Riccati equation $q' = -q^2 + t$. Setting $q = (\log\varphi)'$, we get a linear equation $\varphi'' = t\varphi$, which coincides with Airy's differential equation. Thus we find that $\mathrm{P_{II}}$ admits, when $\alpha = 0$, a particlar solution expressed by a rational function of the Airy function and its derivative.

We now introduce the τ-function via the Hamiltonian (1.9) as $H(\alpha) = (\log \tau(\alpha))'$. Then it is known that a sequence of τ-functions $\tau_n = \tau(\alpha + n)$ $(n \in \mathbf{Z})$ satisfies the Toda equation

$$\tau_{n+1}\tau_{n-1} = \tau_n''\tau_n - (\tau_n')^2 \,, \tag{1.10}$$

which corresponds to the Bäcklund transformation

$$\alpha \mapsto \alpha - 1\,, \quad q \mapsto -q - \frac{2\alpha}{q' + q^2 - t}\,. \tag{1.11}$$

Iteration of the Bäcklund transformation to the above Riccati solution yields a family of classical transcendental solutions. What we have to do is reduced to solving the Toda equation (1.10) with the initial conditions

$$\tau_{-1} = 0\,, \quad \tau_0 = 1\,, \quad \tau_1 = \varphi\,. \tag{1.12}$$

By using Darboux's formula, we have the following [13].

PROPOSITION 1.1 (Okamoto). *Define the functions τ_n $(n \in \mathbf{Z}_{\geq 0})$ by*

$$\tau_n = \begin{vmatrix} \varphi^{(0)} & \varphi^{(1)} & \cdots & \varphi^{(n-1)} \\ \varphi^{(1)} & \varphi^{(2)} & \cdots & \varphi^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)} & \varphi^{(n)} & \cdots & \varphi^{(2n-2)} \end{vmatrix}\,, \quad \varphi^{(k)} = \left(\frac{d}{dt}\right)^k \varphi\,, \tag{1.13}$$

where φ is the general solution of Airy's differential equation $\varphi'' = t\varphi$. Then,

$$q = \frac{d}{dt}\log\frac{\tau_{n+1}}{\tau_n}\,, \quad p = \frac{\tau_{n+1}\tau_{n-1}}{\tau_n^2}\,, \tag{1.14}$$

$$\alpha = n\,, \tag{1.15}$$

give rise to a family of classical transcendental solutions of S_{II}.

By similar procedures, it is possible to obtain a determinant expression for a family of classical transcendental solutions to P_{V} and P_{VI}, which are presented in Sections 2 and 3, respectively. As is well known, P_{VI} degenerates to $P_{\mathrm{V}}, \ldots, P_{\mathrm{I}}$ by successive limiting procedures [15, 3]. In Section 4, we discuss the degeneration of the family of classical transcendental solutions.

The author would like to thank Professors Y. Haraoka, K. Kajiwara and M. Noumi for their helpful comments.

2. Classical transcendental solutions of the Painlevé V equation. Noumi and Yamada have introduced the symmetric form of the Painlevé equations [6, 7, 8, 9, 10]. This formulation provides us with a clear description of symmetry structures of Bäcklund transformations and a systematic tool of constructing special solutions.

First, we summarize the symmetric form of the Painlevé V equation [6, 8, 9]. Introducing τ-functions via Hamiltonians, we derive a bilinear equation of Toda type. As mentioned in the example of P_{II}, one can construct a Riccati solution by restricting the system to a reflection hyperplane of the affine Weyl group. In order to get a family of classical transcendental solutions, we have to solve the bilinear equation of Toda type under the similar initial conditions to (1.12). By using Darboux's formula, we obtain the determinant expression for the family of classical transcendental solutions.

2.1. The symmetric form of the Painlevé V equation. The symmetric form of P_V is given by

$$
\begin{aligned}
f_0' &= f_0 f_2 (f_1 - f_3) + \left(\frac{1}{2} - \alpha_2\right) f_0 + \alpha_0 f_2 \,, \\
f_1' &= f_1 f_3 (f_2 - f_0) + \left(\frac{1}{2} - \alpha_3\right) f_1 + \alpha_1 f_3 \,, \\
f_2' &= f_2 f_0 (f_3 - f_1) + \left(\frac{1}{2} - \alpha_0\right) f_2 + \alpha_2 f_0 \,, \\
f_3' &= f_3 f_1 (f_0 - f_2) + \left(\frac{1}{2} - \alpha_1\right) f_3 + \alpha_3 f_1 \,,
\end{aligned}
\qquad {}' = t\frac{d}{dt} \,, \tag{2.1}
$$

with normalization conditions $\alpha_0 + \alpha_1 + \alpha_2 + \alpha_3 = 1$ and $f_0 + f_2 = f_1 + f_3 = \sqrt{t}$. The correspondences to the canonical variables and to parameters of S_V are given by

$$
q = -\frac{f_3}{f_1} \,, \quad p = \frac{1}{\sqrt{t}} f_1 (f_0 f_1 + \alpha_0) \,, \tag{2.2}
$$

and

$$
\kappa_\infty = \alpha_1 \,, \quad \kappa_0 = \alpha_3 \,, \quad \theta = \alpha_2 - \alpha_0 - 1 \,, \tag{2.3}
$$

respectively. The Bäcklund transformations of P_V are described as follows:

$$
\begin{aligned}
&s_i(\alpha_i) = -\alpha_i \,, \quad s_i(\alpha_j) = \alpha_j + \alpha_i \ (j = i \pm 1) \,, \quad s_i(\alpha_j) = \alpha_j \ (j \neq i, i \pm 1) \,, \\
&s_i(f_i) = f_i \,, \quad s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \ (j = i \pm 1) \,, \quad s_i(f_j) = f_j \ (j \neq i, i \pm 1) \,, \\
&\pi(\alpha_j) = \alpha_{j+1} \,, \quad \pi(f_j) = f_{j+1} \,,
\end{aligned} \tag{2.4}
$$

where the subscripts $i = 0, 1, 2, 3$ are understood as elements of $\mathbf{Z}/4\mathbf{Z}$. The Hamiltonians h_i of the system (2.1) are given by

$$
\begin{aligned}
h_0 = {} & f_0 f_1 f_2 f_3 + \frac{\alpha_1 + 2\alpha_2 - \alpha_3}{4} f_0 f_1 + \frac{\alpha_1 + 2\alpha_2 + 3\alpha_3}{4} f_1 f_2 \\
& - \frac{3\alpha_1 + 2\alpha_2 + \alpha_3}{4} f_2 f_3 + \frac{\alpha_1 - 2\alpha_2 - \alpha_3}{4} f_3 f_0 + \frac{(\alpha_1 + \alpha_3)^2}{4} \,,
\end{aligned} \tag{2.5}
$$

and $h_i = \pi^i(h_0)$. Then we have

$$
s_i(h_j) = h_j \ (i \neq j) \,, \quad s_i(h_i) = h_i + \sqrt{t}\, \frac{\alpha_i}{f_i} \,, \quad \pi(h_i) = h_{i+1} \,. \tag{2.6}
$$

Introducing τ-functions τ_i as $h_i = (\log\tau_i)'$, we find that the Bäcklund transformations for τ-functions are described as

$$s_i(\tau_j) = \tau_j \ (i \neq j)\,, \quad s_i(\tau_i) = f_i\frac{\tau_{i-1}\tau_{i+1}}{\tau_i}\,, \quad \pi(\tau_i) = \tau_{i+1}\,. \tag{2.7}$$

The canonical variables of $\mathrm{S_V}$ are recovered from τ-functions by

$$q = -\frac{\tau_3 s_3(\tau_3)}{\tau_1 s_1(\tau_1)}\,, \quad p = \frac{1}{\sqrt{t}}\frac{\tau_1 s_1(\tau_1) s_0 s_1(\tau_1)}{\tau_2^2\tau_3}\,. \tag{2.8}$$

Let us define the translation operators T_i $(i = 0, 1, 2, 3)$ by $T_1 = \pi s_3 s_2 s_1$ and $\pi T_i = T_{i+1}\pi$, which commute with each other and act on parameters α_i by

$$T_i(\alpha_{i-1}) = \alpha_{i-1} + 1\,, \quad T_i(\alpha_i) = \alpha_i - 1\,, \quad T_i(\alpha_j) = \alpha_j \ (j \neq i-1, i)\,. \tag{2.9}$$

Noting that $T_1T_2T_3T_0 = 1$, we set $\tau_{k,l,m} = T_1^k T_2^l T_3^m(\tau_0)$ $(k, l, m \in \mathbf{Z})$. Then, from (2.7) and (2.8), we have

$$\begin{aligned} T_1^kT_2^lT_3^m(f_0) &= \frac{\tau_{k,l,m}\tau_{k+2,l+1,m+1}}{\tau_{k+1,l+1,m+1}\tau_{k+1,l,m}}\,, \quad & T_1^kT_2^lT_3^m(f_1) &= \frac{\tau_{k+1,l,m}\tau_{k,l+1,m}}{\tau_{k,l,m}\tau_{k+1,l+1,m}}\,, \\ T_1^kT_2^lT_3^m(f_2) &= \frac{\tau_{k+1,l+1,m}\tau_{k+1,l,m+1}}{\tau_{k+1,l,m}\tau_{k+1,l+1,m+1}}\,, \quad & T_1^kT_2^lT_3^m(f_3) &= \frac{\tau_{k+1,l+1,m+1}\tau_{k,l,m-1}}{\tau_{k+1,l+1,m}\tau_{k,l,m}}\,, \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} T_1^kT_2^lT_3^m(q) &= -\frac{\tau_{k+1,l+1,m+1}\tau_{k,l,m-1}}{\tau_{k+1,l,m}\tau_{k,l+1,m}}\,, \\ T_1^kT_2^lT_3^m(p) &= \frac{1}{\sqrt{t}}\frac{\tau_{k+1,l,m}\tau_{k,l+1,m}\tau_{k+2,l+2,m+1}}{\tau_{k+1,l+1,m}^2\tau_{k+1,l+1,m+1}}\,, \end{aligned} \tag{2.11}$$

respectively. It is possible to derive a bilinear equation of Toda type with respect to each translation operator. For the T_1-direction, we have

$$\begin{aligned} &\tau_{k+1,l,m}\tau_{k-1,l,m} \\ &\quad = \frac{1}{\sqrt{t}}\left[(\log\tau_{k,l,m})'' + \frac{3\alpha_1 + 2\alpha_2 + \alpha_3 - 3k + l + m}{4}t\right]\tau_{k,l,m}\cdot\tau_{k,l,m}\,. \end{aligned} \tag{2.12}$$

2.2. A Riccati solution. If $\alpha_0 = 0$, the right-hand side of the first equation of (2.1) is divisible by f_0, which means that the system (2.1) admits the specialization of $f_0 = 0$ if $\alpha_0 = 0$. Then, setting $f_1 = \sqrt{t}f$, we see that f satisfies a Riccati equation $f' = tf(1-f) - (\alpha_1 + \alpha_3)f + \alpha_1$. By a dependent variable transformation $f = (d/dt)\log\varphi$, we have for φ the linear equation

$$\left[t\frac{d^2}{dt^2} + (\alpha_1 + \alpha_3 - t)\frac{d}{dt} - \alpha_1\right]\varphi = 0\,, \tag{2.13}$$

which is nothing but the confluent hypergeometric differential equation. We set $\alpha_1 = a$ and $\alpha_3 = c - a$. The general solution of (2.13) is expressed as

$$\begin{aligned}\varphi &= c_1 \frac{\Gamma(a)\Gamma(c-a)}{\Gamma(c)} F(a, c; t) \\ &\quad + c_2 \frac{1}{\sin\pi(c-a)\Gamma(2-c)} t^{1-c} F(a-c+1, 2-c; t)\,,\end{aligned} \tag{2.14}$$

where $F(a, c; t)$ denotes Kummer's confluent hypergeometric function, and c_i $(i = 1, 2)$ are arbitrary complex constants. For simplicity, we denote

$$\begin{aligned}f_{i,j} &= F(a+i, c+j; t)\,, \\ g_{i,j} &= t^{1-c-j} F(a-c+1+i-j, 2-c-j; t) \quad (i, j \in \mathbf{Z})\,.\end{aligned} \tag{2.15}$$

By making use of the contiguity relations of Kummer's function, we obtain the following.

PROPOSITION 2.1. *Define the functions* $\varphi_{i,j}$ *by*

$$\begin{aligned}\varphi_{i,j} &= c_1 \frac{\Gamma(a+i)\Gamma(c-a-i+j)}{\Gamma(c+j)} f_{i,j} \\ &\quad + c_2 \frac{1}{\sin\pi(c-a-i+j)\Gamma(2-c-j)} g_{i,j}\,.\end{aligned} \tag{2.16}$$

Then,

$$\begin{aligned}(f_0, f_1, f_2, f_3) &= \left(0, \sqrt{t}\,\frac{\varphi_{1,1}}{\varphi_{0,0}}, \sqrt{t}, \sqrt{t}\frac{\varphi_{0,1}}{\varphi_{0,0}}\right), \\ (\alpha_0, \alpha_1, \alpha_2, \alpha_3) &= (0, a, 1-c, c-a)\,,\end{aligned} \tag{2.17}$$

and

$$q = -\frac{\varphi_{0,1}}{\varphi_{1,1}}\,, \quad p = 0\,, \quad \kappa_\infty = a\,, \quad \kappa_0 = c - a\,, \quad \theta = -c\,, \tag{2.18}$$

give a Riccati solution of the symmetric form of $\mathrm{P_V}$ *and the Hamiltonian system* $\mathrm{S_V}$*, respectively.*

2.3. A Determinant formula for a family of classical transcendental solutions. First, we calculate the Hamiltonians and τ-functions for the Riccati solution in Proposition 2.1. Under the specialization (2.17), the Hamiltonians and τ-functions are calculated as

$$\begin{aligned}h_0 &= \frac{\varphi'_{0,0}}{\varphi_{0,0}} - \frac{2a-c+2}{4}t + \frac{c^2}{4}\,, \quad h_1 = -\frac{2a-c-1}{4}t + \frac{(c-1)^2}{4}\,, \\ h_2 &= -\frac{2a-c}{4}t + \frac{c^2}{4}\,, \quad h_3 = -\frac{2a-c+1}{4}t + \frac{(c-1)^2}{4}\,,\end{aligned} \tag{2.19}$$

and

$$
\begin{aligned}
&\tau_0 = \tau_{0,0,0} = \varphi_{0,0}\, t^{c^2/4} \exp\left(-\frac{2a-c+2}{4} t \right), \\
&\tau_1 = \tau_{1,0,0} = t^{(c-1)^2/4} \exp\left(-\frac{2a-c-1}{4} t \right), \\
&\tau_2 = \tau_{1,1,0} = t^{c^2/4} \exp\left(-\frac{2a-c}{4} t \right), \\
&\tau_3 = \tau_{1,1,1} = t^{(c-1)^2/4} \exp\left(-\frac{2a-c+1}{4} t \right), \\
&s_0(\tau_0) = \tau_{2,1,1} = 0\,, \\
&s_1(\tau_1) = \tau_{0,1,0} = \varphi_{1,1} t^{(c+1)^2/4} \exp\left(-\frac{2a-c+3}{4} t \right), \\
&s_2(\tau_2) = \tau_{1,0,1} = t^{(c-2)^2/4} \exp\left(-\frac{2a-c}{4} t \right), \\
&s_3(\tau_3) = \tau_{0,0,-1} = \varphi_{0,1} t^{(c+1)^2/4} \exp\left(-\frac{2a-c+1}{4} t \right),
\end{aligned}
\tag{2.20}
$$

up to multiplication by some constants, respectively. For small k, l, m, we observe that $\tau_{k,l,m}$ are expressed in the form

$$
\tau_{k,l,m} = \sigma_{k,l,m} t^{(c-k+l-m)^2/4 - k(k-1)/2} \exp\left(-\frac{2a-c+2-3k+l+m}{4} t \right), \tag{2.21}
$$

with $\sigma_{2,l,m} = 0$, $\sigma_{1,l,m} =$ const. and $\sigma_{0,l,m} = (\text{const.}) \times \varphi_{l,l-m}$. Assume that $\tau_{k,l,m}$ are expressed as (2.21) for any $k, l, m \in \mathbf{Z}$. Then the bilinear equation of Toda type (2.12) yields

$$
\sigma_{k+1,l,m}\sigma_{k-1,l,m} = \sigma''_{k,l,m}\sigma_{k,l,m} - (\sigma'_{k,l,m})^2\,. \tag{2.22}
$$

Moreover, we set

$$
\sigma_{k,l,m} = \omega_{k,l,m}\rho_{k,l,m}\,, \quad \omega_{k,l,m} = \omega_{k,l,m}(a, c)\,, \tag{2.23}
$$

with $\rho_{1,l,m} = 1$ and $\rho_{0,l,m} = \varphi_{l,l-m}$, and impose that the constants $\omega_{k,l,m}$ satisfy

$$
\omega_{k+1,l,m}\omega_{k-1,l,m} = \omega_{k,l,m}^2\,. \tag{2.24}
$$

Then the functions $\rho_{k,l,m}$ are determined by the recurrence relation

$$
\rho_{k+1,l,m}\rho_{k-1,l,m} = \rho''_{k,l,m}\rho_{k,l,m} - (\rho'_{k,l,m})^2\,, \tag{2.25}
$$

with the initial conditions

$$
\rho_{2,l,m} = 0\,, \quad \rho_{1,l,m} = 1\,, \quad \rho_{0,l,m} = \varphi_{l,l-m}\,. \tag{2.26}
$$

By Darboux's formula, the functions $\rho_{1-n,l,m}$ for $n \in \mathbf{Z}_{\geq 0}$ are expressed as

$$\rho_{1-n,l,m} = \begin{vmatrix} \varphi^{(0)}_{l,l-m} & \varphi^{(1)}_{l,l-m} & \cdots & \varphi^{(n-1)}_{l,l-m} \\ \varphi^{(1)}_{l,l-m} & \varphi^{(2)}_{l,l-m} & \cdots & \varphi^{(n)}_{l,l-m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)}_{l,l-m} & \varphi^{(n)}_{l,l-m} & \cdots & \varphi^{(2n-2)}_{l,l-m} \end{vmatrix}, \quad \varphi^{(i)}_{l,l-m} = \left(t\frac{d}{dt}\right)^i \varphi_{l,l-m}\,. \tag{2.27}$$

Note that the constants $\omega_{k,l,m}$ are determined by the recurrence relations (2.24) and

(2.28)

$$\omega_{1,l+1,m}\omega_{1,l-1,m} = -(a+l-1)\omega^2_{1,l,m}\,, \quad \omega_{1,l,m+1}\omega_{1,l,m-1} = (c-a-m)\omega^2_{1,l,m}\,,$$
$$\omega_{0,l+1,m}\omega_{0,l-1,m} = -(a+l-1)\omega^2_{0,l,m}\,, \quad \omega_{0,l,m+1}\omega_{0,l,m-1} = (c-a-1-m)\omega^2_{0,l,m}\,,$$

with initial conditions

$$\begin{aligned} &\omega_{1,0,0} = \omega_{1,1,0} = \omega_{1,0,1} = \omega_{1,1,1} = 1\,, \\ &\omega_{0,0,0} = \omega_{0,1,0} = \omega_{0,0,-1} = \omega_{0,1,-1} = 1\,. \end{aligned} \tag{2.29}$$

Since it is possible to set $l = m = 0$ without loss of generality, we obtain the following.

THEOREM 2.2. *Define the functions* $\tau^{i,j}_n$ *by*

$$\tau^{i,j}_n = \begin{vmatrix} \varphi^{(0)}_{i,j} & \varphi^{(1)}_{i,j} & \cdots & \varphi^{(n-1)}_{i,j} \\ \varphi^{(1)}_{i,j} & \varphi^{(2)}_{i,j} & \cdots & \varphi^{(n)}_{i,j} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)}_{i,j} & \varphi^{(n)}_{i,j} & \cdots & \varphi^{(2n-2)}_{i,j} \end{vmatrix}, \quad \varphi^{(k)}_{i,j} = \left(t\frac{d}{dt}\right)^k \varphi_{i,j}\,, \tag{2.30}$$

where $\varphi_{i,j}$ *are given by* (2.16). *Then,*

$$\begin{aligned} &f_0 = \frac{1}{\sqrt{t}}\frac{\tau^{0,0}_{n+1}\tau^{1,0}_{n-1}}{\tau^{1,0}_n\tau^{0,0}_n}\,, \quad f_1 = \sqrt{t}\,\frac{\tau^{0,0}_n\tau^{1,1}_{n+1}}{\tau^{0,0}_{n+1}\tau^{1,1}_n}\,, \\ &f_2 = \sqrt{t}\,\frac{\tau^{1,1}_n\tau^{0,-1}_n}{\tau^{0,0}_n\tau^{1,0}_n}\,, \quad f_3 = \sqrt{t}\left(\frac{c-a-1}{c-a}\right)^n\frac{\tau^{1,0}_n\tau^{0,1}_{n+1}}{\tau^{1,1}_n\tau^{0,0}_{n+1}}\,, \end{aligned} \tag{2.31}$$

$$(\alpha_0, \alpha_1, \alpha_2, \alpha_3) = (-n, a+n, 1-c, c-a)\,, \tag{2.32}$$

and

$$q = -\left(\frac{c-a-1}{c-a}\right)^n\frac{\tau^{1,0}_n\tau^{0,1}_{n+1}}{\tau^{0,0}_n\tau^{1,1}_{n+1}}\,, \quad p = -\frac{a}{c-a-1}\frac{1}{t}\frac{\tau^{0,0}_n\tau^{1,1}_{n+1}\tau^{2,1}_{n-1}}{\tau^{1,1}_n\tau^{1,1}_n\tau^{1,0}_n}\,, \tag{2.33}$$

$$\kappa_\infty = a+n\,, \quad \kappa_0 = c-a\,, \quad \theta = -c+n\,, \tag{2.34}$$

give a family of classical transcendental solutions of the symmetric form of $\mathrm{P_V}$ *and the Hamiltonian system* $\mathrm{S_V}$, *respectively.*

REMARK 2.3. Noting that (2.6) implies

$$\sqrt{t}\,\frac{\alpha_i}{f_i} = t\frac{d}{dt}\log\frac{s_i(\tau_i)}{\tau_i}\,, \tag{2.35}$$

we obtain another expression of the solutions in Theorem 2.2. For example, we have

$$(a+n)\frac{\sqrt{t}}{f_1} = t\frac{d}{dt}\log\frac{\tau_{n+1}^{1,1}}{\tau_n^{0,0}} + c - t\,. \tag{2.36}$$

REMARK 2.4. The symmetric form of P_V admits the following symmetry σ:

$$\begin{aligned}
&\sigma(t) = -t\,,\\
&\sigma(f_0) = \sqrt{-1}f_2\,,\quad \sigma(f_2) = \sqrt{-1}f_0\,,\quad \sigma(f_1) = \sqrt{-1}f_1\,,\quad \sigma(f_3) = \sqrt{-1}f_3\,,\\
&\sigma(\alpha_0) = \alpha_2\,,\quad \sigma(\alpha_2) = \alpha_0\,,\quad \sigma(\alpha_1) = \alpha_1\,,\quad \sigma(\alpha_3) = \alpha_3\,.
\end{aligned} \tag{2.37}$$

Applying σ to the family of solutions in Theorem 2.2, we get another family of solutions expressed in terms of $F(a, c; -t)$.

3. Classical transcendental solutions of the Painlevé VI equation. In this section, we construct a determinant formula for a family of classical transcendental solutions of the Painlevé VI equation by following the same recipe as in the previous section.

3.1. The symmetric form of the Painlevé VI equation. Here, we give a brief review of the symmetric form of the P_{VI} [10, 4]. We set

$$f_0 = q - t\,,\quad f_3 = q - 1\,,\quad f_4 = q\,,\quad f_2 = p\,, \tag{3.1}$$

and

$$\alpha_0 = \theta\,,\quad \alpha_1 = \kappa_\infty\,,\quad \alpha_3 = \kappa_1\,,\quad \alpha_4 = \kappa_0\,. \tag{3.2}$$

Then the Hamiltonian (1.6) is written as

$$H = f_2^2 f_0 f_3 f_4 - [(\alpha_0 - 1)f_3 f_4 + \alpha_3 f_0 f_4 + \alpha_4 f_0 f_3]f_2 + \alpha_2(\alpha_1+\alpha_2)f_0 \tag{3.3}$$

with $\alpha_0 + \alpha_1 + 2\alpha_2 + \alpha_3 + \alpha_4 = 1$ and the Hamilton equation (1.5) is written as

$$\begin{aligned}
f_4' &= 2f_2 f_0 f_3 f_4 - (\alpha_0 - 1)f_3 f_4 - \alpha_3 f_0 f_4 - \alpha_4 f_0 f_3\,,\\
f_2' &= -(f_0 f_3 + f_0 f_4 + f_3 f_4)f_2^2\\
&\quad + [(\alpha_0 - 1)(f_3 + f_4) + \alpha_3(f_0 + f_4) + \alpha_4(f_0 + f_3)]f_2 - \alpha_2(\alpha_1 + \alpha_2)\,.
\end{aligned} \tag{3.4}$$

The fundamental Bäcklund transformations of P_{VI} are given in Table 1. We define the Hamiltonians h_i $(i = 0, 1, 2, 3, 4)$ by

$$\begin{aligned}
&h_0 = H_0 + \frac{t}{4}\,,\qquad h_1 = s_5(H_0) - \frac{t-1}{4}\,,\\
&h_3 = s_6(H_0) + \frac{1}{4}\,,\quad h_4 = s_7(H_0)\,,\quad h_2 = h_1 + s_1(h_1)\,,
\end{aligned} \tag{3.5}$$

TABLE 1. Bäcklund transformations of P_{VI}.

	α_0	α_1	α_2	α_3	α_4	f_4	f_2
s_0	$-\alpha_0$	α_1	$\alpha_2+\alpha_0$	α_3	α_4	f_4	$f_2-\dfrac{\alpha_0}{f_0}$
s_1	α_0	$-\alpha_1$	$\alpha_2+\alpha_1$	α_3	α_4	f_4	f_2
s_2	$\alpha_0+\alpha_2$	$\alpha_1+\alpha_2$	$-\alpha_2$	$\alpha_3+\alpha_2$	$\alpha_4+\alpha_2$	$f_4+\dfrac{\alpha_2}{f_2}$	f_2
s_3	α_0	α_1	$\alpha_2+\alpha_3$	$-\alpha_3$	α_4	f_4	$f_2-\dfrac{\alpha_3}{f_3}$
s_4	α_0	α_1	$\alpha_2+\alpha_4$	α_3	$-\alpha_4$	f_4	$f_2-\dfrac{\alpha_4}{f_4}$
s_5	α_1	α_0	α_2	α_4	α_3	$t\dfrac{f_3}{f_0}$	$-\dfrac{f_0(f_2f_0+\alpha_2)}{t(t-1)}$
s_6	α_3	α_4	α_2	α_0	α_1	$\dfrac{t}{f_4}$	$-\dfrac{f_4(f_4f_2+\alpha_2)}{t}$
s_7	α_4	α_3	α_2	α_1	α_0	$\dfrac{f_0}{f_3}$	$\dfrac{f_3(f_3f_2+\alpha_2)}{t-1}$

where an auxiliary Hamiltonian H_0 is given by

(3.6)
$$\begin{aligned} H_0 = H &+ \frac{t}{4}[1+4\alpha_1\alpha_2+4\alpha_2^2-(\alpha_3+\alpha_4)^2] \\ &+ \frac{1}{4}[(\alpha_1+\alpha_4)^2+(\alpha_3+\alpha_4)^2+4\alpha_2\alpha_4]\,. \end{aligned}$$

Introducing τ-functions τ_i as $h_i = (\log\tau_i)'$, we find that the Bäcklund transformations for τ-functions are described as follows:

(3.7)
$$\begin{gathered} s_i(\tau_j) = \tau_j \quad (i \neq j,\ i, j = 0, 1, 2, 3, 4)\,, \\ s_i(\tau_i) = f_i\frac{\tau_2}{\tau_i} \quad (i = 0, 3, 4)\,, \quad s_1(\tau_1) = \frac{\tau_2}{\tau_1}\,, \quad s_2(\tau_2) = \frac{f_2}{\sqrt{t}}\frac{\tau_0\tau_1\tau_3\tau_4}{\tau_2}\,, \end{gathered}$$

(3.8)
$$\begin{aligned} s_5:\quad &\tau_0 \mapsto [t(t-1)]^{1/4}\tau_1\,, \qquad \tau_1 \mapsto [t(t-1)]^{-1/4}\tau_0\,, \\ &\tau_3 \mapsto t^{-1/4}(t-1)^{1/4}\tau_4\,, \qquad \tau_4 \mapsto t^{1/4}(t-1)^{-1/4}\tau_3\,, \\ &\tau_2 \mapsto [t(t-1)]^{-1/2}f_0\tau_2\,, \end{aligned}$$

$$\begin{aligned} s_6:\quad &\tau_0 \mapsto i t^{1/4}\tau_3\,, \qquad \tau_3 \mapsto -i t^{-1/4}\tau_0\,,\\ &\tau_1 \mapsto t^{-1/4}\tau_4\,, \qquad \tau_4 \mapsto t^{1/4}\tau_1\,,\\ &\tau_2 \mapsto t^{-1/2} f_4 \tau_2\,, \end{aligned} \tag{3.9}$$

$$\begin{aligned} s_7:\quad &\tau_0 \mapsto (-1)^{-3/4}(t-1)^{1/4}\tau_4\,, \quad \tau_4 \mapsto (-1)^{3/4}(t-1)^{-1/4}\tau_0\,,\\ &\tau_1 \mapsto (-1)^{3/4}(t-1)^{-1/4}\tau_3\,, \quad \tau_3 \mapsto (-1)^{-3/4}(t-1)^{1/4}\tau_1\,,\\ &\tau_2 \mapsto -i(t-1)^{-1/2} f_3\tau_2\,. \end{aligned} \tag{3.10}$$

Let us define the following translation operators

$$\begin{aligned} \widehat{T}_{13} &= s_1 s_2 s_0 s_4 s_2 s_1 s_7\,, \quad \widehat{T}_{40} = s_4 s_2 s_1 s_3 s_2 s_4 s_7\,,\\ \widehat{T}_{34} &= s_3 s_2 s_0 s_1 s_2 s_3 s_5\,, \quad T_{14} = s_1 s_4 s_2 s_0 s_3 s_2 s_6\,, \end{aligned} \tag{3.11}$$

which act on parameters α_i as

$$\begin{aligned} \widehat{T}_{13}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (0,1,0,-1,0)\,,\\ \widehat{T}_{40}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (-1,0,0,0,1)\,,\\ \widehat{T}_{34}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (0,0,0,1,-1)\,,\\ T_{14}(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) &= (\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) + (0,1,-1,0,1)\,. \end{aligned} \tag{3.12}$$

Note that the action of these operators on τ-functions is not commutative. For example, we have

$$\begin{aligned} \widehat{T}_{13}\widehat{T}_{40}(\tau_0) &= -\widehat{T}_{40}\widehat{T}_{13}(\tau_0)\,, \quad \widehat{T}_{13}\widehat{T}_{34}(\tau_0) = -i\widehat{T}_{34}\widehat{T}_{13}(\tau_0)\,,\\ \widehat{T}_{13}T_{14}(\tau_0) &= iT_{14}\widehat{T}_{13}(\tau_0)\,, \quad \widehat{T}_{40}\widehat{T}_{34}(\tau_0) = i\widehat{T}_{34}\widehat{T}_{40}(\tau_0)\,,\\ \widehat{T}_{40}T_{14}(\tau_0) &= -iT_{14}\widehat{T}_{40}(\tau_0)\,, \quad \widehat{T}_{34}T_{14}(\tau_0) = iT_{14}\widehat{T}_{34}(\tau_0)\,. \end{aligned} \tag{3.13}$$

Setting $\tau_{k,l,m,n} = T_{14}^n \widehat{T}_{34}^m \widehat{T}_{40}^l \widehat{T}_{13}^k(\tau_0)$ $(k,l,m,n \in \mathbf{Z})$, we have

$$\begin{aligned} T_{14}^n \widehat{T}_{34}^m \widehat{T}_{40}^l \widehat{T}_{13}^k(f_0) &= -it^{1/2}(t-1)^{1/2}\frac{\tau_{k,l,m,n}\tau_{k-1,l-2,m-1,n+1}}{\tau_{k-1,l-1,m-1,n}\tau_{k,l-1,m,n+1}}\,,\\ T_{14}^n \widehat{T}_{34}^m \widehat{T}_{40}^l \widehat{T}_{13}^k(f_3) &= i(t-1)^{1/2}\frac{\tau_{k,l-1,m-1,n}\tau_{k-1,l-1,m,n+1}}{\tau_{k-1,l-1,m-1,n}\tau_{k,l-1,m,n+1}}\,,\\ T_{14}^n \widehat{T}_{34}^m \widehat{T}_{40}^l \widehat{T}_{13}^k(f_4) &= t^{1/2}\frac{\tau_{k,l-1,m,n}\tau_{k-1,l-1,m-1,n+1}}{\tau_{k-1,l-1,m-1,n}\tau_{k,l-1,m,n+1}}\,,\\ T_{14}^n \widehat{T}_{34}^m \widehat{T}_{40}^l \widehat{T}_{13}^k(f_2) &= -(t-1)^{-1/2}\frac{\tau_{k-1,l-1,m-1,n}\tau_{k,l-1,m,n+1}\tau_{k+1,l,m,n-1}}{\tau_{k,l,m,n}\tau_{k,l-1,m-1,n}\tau_{k,l-1,m,n}}\,. \end{aligned} \tag{3.14}$$

It is possible to derive a bilinear equation of Toda type with respect to each translation operator. For the T_{14}-direction, we get

$$\tau_{k,l,m,n+1}\tau_{k,l,m,n-1} = -t^{-1/2}\left[(t-1)\frac{d}{dt}(\log\tau_{k,l,m,n})' - (\log\tau_{k,l,m,n})' + \frac{(\alpha_1+\alpha_4+k+l-m+2n)^2}{4} + \frac{1}{2}\right]\tau_{k,l,m,n}^2 . \tag{3.15}$$

3.2. A Riccati solution. If $\alpha_2 = 0$, it is possible to specialize $f_2 = 0$. Then the Hamilton equation (3.4) yields a Riccati equation $q' = \alpha_1 q^2 + [(\alpha_3+\alpha_4)t + (\alpha_0 + \alpha_4 - 1)]q - \alpha_4 t$. We set $\alpha_0 = -b$, $\alpha_1 = a$, $\alpha_3 = c - a$ and $\alpha_4 = b - c + 1$. By a dependent variable transformation $aq = -(\log\varphi)' - (b+1)t + c$, we have for φ the linear equation

$$t(t-1)\frac{d^2\varphi}{dt^2} + [(a+b+3)t - (c+1)]\frac{d\varphi}{dt} + (a+1)(b+1)\varphi = 0 , \tag{3.16}$$

which is nothing but the hypergeometric differential equation. The general solution of (3.16) is expressed as

$$\begin{aligned}\varphi = {} & c_1\frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(c+1)}F(a+1, b+1, c+1; t)\\ & + c_2\frac{\Gamma(a-c+1)\Gamma(b-c+1)}{\Gamma(1-c)}t^{-c}F(a-c+1, b-c+1, 1-c; t) ,\end{aligned} \tag{3.17}$$

where $F(a, b, c; t)$ denotes Gauss's hypergeometric function and c_i $(i = 1, 2)$ are arbitrary complex constants. For simplicity, we denote

$$\begin{aligned} f_m^{kl} &= F(a+k, b+l, c+m; t) ,\\ g_m^{kl} &= t^{1-c-m}F(a-c+1+k-m, b-c+1+l-m, 2-c-m; t) ,\end{aligned} \tag{3.18}$$

for $k, l, m \in \mathbf{Z}$. By using the contiguity relations of Gauss's hypergeometric function, we obtain the following.

PROPOSITION 3.1. *Define the function* $\varphi_{k,l,m}$ *by*

$$\begin{aligned}\varphi_{k,l,m} = {} & c_1\frac{\Gamma(a+k+1)\Gamma(b+l+2)}{\Gamma(c+m+1)} f_{m+1}^{k+1,l+2}\\ & + c_2\frac{\Gamma(a-c+1+k-m)\Gamma(b-c+2+l-m)}{\Gamma(1-c-m)} g_{m+1}^{k+1,l+2} .\end{aligned} \tag{3.19}$$

Then,

$$\begin{aligned} & f_0 = b\frac{\varphi_{-1,-2,-1}}{\varphi_{0,-1,0}} , \quad f_3 = (c-a)\frac{\varphi_{-1,-1,0}}{\varphi_{0,-1,0}} , \quad f_4 = \frac{\varphi_{-1,-1,-1}}{\varphi_{0,-1,0}} , \quad f_2 = 0 ,\\ & \alpha_0 = -b , \quad \alpha_1 = a , \quad \alpha_3 = c-a , \quad \alpha_4 = b-c+1 ,\end{aligned} \tag{3.20}$$

give a Riccati solution of the symmetric form of $\mathrm{P_{VI}}$.

3.3. A Determinant formula for a family of classical transcendental solutions. First, we calculate the Hamiltonians and τ-functions for the Riccati solution in Proposition 3.1. Under the specialization (3.20), the Hamiltonians are calculated as

$$h_i = A_i(t-1) + B_i t\,, \quad (i = 0, 1, 3, 4)\,, \tag{3.21}$$

with

$$\begin{aligned} A_0 &= -\frac{1}{4}(a+b-c+1)^2 - \frac{1}{4}(a-b-1)^2\,, & B_0 &= \frac{1}{4}(a+b-c+1)^2 + \frac{1}{2}\,,\\ A_1 &= -\frac{1}{4}(a+b-c)^2 - \frac{1}{4}(a-b-1)^2 - \frac{1}{4}\,, & B_1 &= \frac{1}{4}(a+b-c)^2 + \frac{1}{4}\,,\\ A_3 &= -\frac{1}{4}(a+b-c+1)^2 - \frac{1}{4}(a-b)^2 - \frac{1}{4}\,, & B_3 &= \frac{1}{4}(a+b-c+1)^2 + \frac{1}{2}\,,\\ A_4 &= -\frac{1}{4}(a+b-c)^2 - \frac{1}{4}(a-b)^2\,, & B_4 &= \frac{1}{4}(a+b-c)^2 + \frac{1}{4}\,. \end{aligned} \tag{3.22}$$

Then we have

$$\tau_i = t^{A_i}(t-1)^{B_i}\,, \quad s_2 s_i(\tau_i) = 0 \quad (i = 0, 1, 3, 4)\,, \tag{3.23}$$

$$\begin{aligned} s_0(\tau_0) &= b\varphi_{-1,-2,-1} t^{A_0+b}(t-1)^{B_0}\,,\\ s_1(\tau_1) &= \varphi_{0,-1,0} t^{A_1+(c-a)}(t-1)^{B_1+(a+b-c+1)}\,,\\ s_3(\tau_3) &= (c-a)\varphi_{-1,-1,0} t^{A_3+a}(t-1)^{B_3}\,,\\ s_4(\tau_4) &= \varphi_{-1,-1,-1} t^{A_4-(b-c+1)}(t-1)^{B_4+(a+b-c+1)}\,, \end{aligned} \tag{3.24}$$

up to multiplication by some constants. For small k, l, m, n, we observe that $\tau_{k,l,m,n}$ are expressed in the form

(3.25)
$$\tau_{k,l,m,n} = \sigma_{k,l,m,n} t^{-(\hat{a}+\hat{b}-\hat{c}+2n)^2/4-(\hat{a}-\hat{b}-n)^2/4+n(\hat{b}+n)-n(n-1)/2}(t-1)^{(\hat{a}+\hat{b}-\hat{c}+2n)^2/4+1/2}\,,$$

with $\sigma_{k,l,m,-1} = 0$, $\sigma_{k,l,m,0} =$ const. and $\sigma_{k,l,m,1} =$ (const.) $\times\ \varphi_{k,l,m}$, where we denote $\hat{a} = a + k$, $\hat{b} = b + l + 1$ and $\hat{c} = c + m$. Assume that $\tau_{k,l,m,n}$ are expressed as (3.25) for any $k, l, m, n \in \mathbf{Z}$. Then the bilinear equation of Toda type (3.15) yields

$$\sigma_{k,l,m,n+1}\sigma_{k,l,m,n-1} = -[(\delta^2\sigma_{k,l,m,n})\sigma_{k,l,m,n} - (\delta\sigma_{k,l,m,n})^2]\,, \quad \delta = t\frac{d}{dt}\,. \tag{3.26}$$

Moreover, we set

$$\sigma_{k,l,m,n} = \omega_{k,l,m,n}\rho_{k,l,m,n}\,, \quad \omega_{k,l,m,n} = \omega_{k,l,m,n}(a,b,c)\,, \tag{3.27}$$

with $\rho_{k,l,m,0} = 1$ and $\rho_{k,l,m,1} = \varphi_{k,l,m}$, and impose that the constants $\omega_{k,l,m,n}$ satisfy

$$\omega_{k,l,m,n+1}\omega_{k,l,m,n-1} = -\omega_{k,l,m,n}^2\,. \tag{3.28}$$

Then the function $\rho_{k,l,m,n}$ are determined by the recurrence relation

$$\rho_{k,l,m,n+1}\rho_{k,l,m,n-1} = (\delta^2\rho_{k,l,m,n})\rho_{k,l,m,n} - (\delta\rho_{k,l,m,n})^2 \tag{3.29}$$

with initial conditions

$$\rho_{k,l,m,-1} = 0\,, \quad \rho_{k,l,m,0} = 1\,, \quad \rho_{k,l,m,1} = \varphi_{k,l,m}\,. \tag{3.30}$$

By Darboux's formula, the functions $\rho_{k,l,m,n}$ for $n \in \mathbf{Z}_{\geq 0}$ are expressed as

$$\rho_{k,l,m,n} = \begin{vmatrix} \varphi^{(0)}_{k,l,m} & \varphi^{(1)}_{k,l,m} & \cdots & \varphi^{(n-1)}_{k,l,m} \\ \varphi^{(1)}_{k,l,m} & \varphi^{(2)}_{k,l,m} & \cdots & \varphi^{(n)}_{k,l,m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)}_{k,l,m} & \varphi^{(n)}_{k,l,m} & \cdots & \varphi^{(2n-2)}_{k,l,m} \end{vmatrix}\,, \quad \varphi^{(i)}_{k,l,m} = \left(t\frac{d}{dt}\right)^i \varphi_{k,l,m}\,. \tag{3.31}$$

Note that the constants $\omega_{k,l,m,n}$ are determined by recurrence relations (3.28) and

$$\begin{aligned} \omega_{k+1,l,m,i}\omega_{k-1,l,m,i} &= i\hat{a}(\hat{c}-\hat{a})\omega^2_{k,l,m,i}\,, \\ \omega_{k,l+1,m,i}\omega_{k,l-1,m,i} &= -i\hat{b}(\hat{c}-\hat{b})\omega^2_{k,l,m,i}\,, \qquad (i=0,1) \\ \omega_{k,l,m+1,i}\omega_{k,l,m-1,i} &= (\hat{c}-\hat{a})(\hat{c}-\hat{b})\omega^2_{k,l,m,i}\,, \end{aligned} \tag{3.32}$$

with initial conditions

$$\begin{aligned} &\omega_{-1,-2,-1,1} = (-1)^{-1/4}b\,, && \omega_{0,-2,-1,1} = b\,, \\ &\omega_{-1,-1,-1,1} = 1\,, && \omega_{0,-1,-1,1} = (-1)^{-1/4}\,, \\ &\omega_{-1,0,0,1} = -(-1)^{-3/4}(c-a)\,, && \omega_{0,0,0,1} = -i\,, \\ &\omega_{-1,-1,0,1} = -i(c-a)\,, && \omega_{0,-1,0,1} = (-1)^{-3/4}\,, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned} &\omega_{-1,-2,-1,0} = (-1)^{-3/4}b\,, && \omega_{0,-2,-1,0} = -b\,, \\ &\omega_{-1,-1,-1,0} = 1\,, && \omega_{0,-1,-1,0} = (-1)^{-3/4}\,, \\ &\omega_{-1,0,0,0} = (-1)^{-3/4}(c-a)\,, && \omega_{0,0,0,0} = 1\,, \\ &\omega_{-1,-1,0,0} = c-a\,, && \omega_{0,-1,0,0} = (-1)^{-3/4}\,. \end{aligned} \tag{3.34}$$

Since it is possible to set $k = l = m = 0$ without loss of generality, we obtain the following.

THEOREM 3.2. *Define the functions* $\tau^{k,l,m}_n$ *by*

$$\tau^{k,l,m}_n = \begin{vmatrix} \varphi^{(0)}_{k,l,m} & \varphi^{(1)}_{k,l,m} & \cdots & \varphi^{(n-1)}_{k,l,m} \\ \varphi^{(1)}_{k,l,m} & \varphi^{(2)}_{k,l,m} & \cdots & \varphi^{(n)}_{k,l,m} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi^{(n-1)}_{k,l,m} & \varphi^{(n)}_{k,l,m} & \cdots & \varphi^{(2n-2)}_{k,l,m} \end{vmatrix}\,, \quad \varphi^{(i)}_{k,l,m} = \left(t\frac{d}{dt}\right)^i \varphi_{k,l,m}\,, \tag{3.35}$$

where $\varphi_{k,l,m}$ are given by (3.19). *Then,*

$$f_0 = b\,\frac{\tau_n^{0,0,0}\tau_{n+1}^{-1,-2,-1}}{\tau_n^{-1,-1,-1}\tau_{n+1}^{0,-1,0}}\,, \quad f_3 = (c-a)\,\frac{\tau_n^{0,-1,-1}\tau_{n+1}^{-1,-1,0}}{\tau_n^{-1,-1,-1}\tau_{n+1}^{0,-1,0}}\,,$$
$$f_4 = \frac{\tau_n^{0,-1,0}\tau_{n+1}^{-1,-1,-1}}{\tau_n^{-1,-1,-1}\tau_{n+1}^{0,-1,0}}\,, \quad f_2 = at^{-1}\,\frac{\tau_n^{-1,-1,-1}\tau_{n+1}^{0,-1,0}\tau_{n-1}^{1,0,0}}{\tau_n^{0,0,0}\tau_n^{0,-1,-1}\tau_n^{0,-1,0}}\,, \tag{3.36}$$

$$(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (-b, a+n, -n, c-a, b-c+1+n)\,, \tag{3.37}$$

give a family of classical transcendental solutions of the symmetric form of $\mathrm{P_{VI}}$.

4. Degeneration of classical transcendental solutions. It is well-known that, starting from $\mathrm{P_{VI}}$, one can obtain $\mathrm{P_V}, \dots, \mathrm{P_I}$ by successive limiting procedures in the following diagram [15, 3],

$$\begin{array}{ccccc} \mathrm{P_{VI}} \longrightarrow & \mathrm{P_V} & \longrightarrow & \mathrm{P_{III}} & \\ & \downarrow & & \downarrow & \\ & \mathrm{P_{IV}} & \longrightarrow & \mathrm{P_{II}} & \longrightarrow \mathrm{P_I}\,, \end{array} \tag{4.1}$$

which corresponds to the degeneration diagram of the special functions of hypergeometric type

$$\begin{array}{ccc} \text{Gauss} \longrightarrow \text{Kummer} & \longrightarrow & \text{Bessel} \\ \downarrow & & \downarrow \\ \text{Hermite-Weber} & \longrightarrow & \text{Airy}\,. \end{array} \tag{4.2}$$

In this section, we show that, starting from the family of special function solutions of $\mathrm{P_{VI}}$ given in Theorem 3.2, we obtain classical transcendental solutions to other Painlevé equations by degeneration.

4.1. From $\mathrm{P_{VI}}$ to $\mathrm{P_V}$. As is known, the Hamiltonian system $\mathrm{S_{VI}}$ is reduced to $\mathrm{S_V}$ by putting

$$t \mapsto 1-\varepsilon t\,, \quad \kappa_1 \mapsto \varepsilon^{-1}+\theta+1\,, \quad \theta \mapsto -\varepsilon^{-1}\,, \tag{4.3}$$

and taking the limit $\varepsilon \to 0$. We consider the degeneration of the family of classical transcendental solutions given in Theorem 3.2. It is known that $\mathrm{P_{VI}}$ admits an outer symmetry as

$$\sigma_{34}: \quad \alpha_3 \leftrightarrow \alpha_4\,, \quad t \mapsto 1-t\,, \quad f_4 \mapsto -f_3, \quad f_2 \mapsto -f_2\,. \tag{4.4}$$

Applying σ_{34} to the family of solutions in Theorem 3.2, we get another family of solutions for parameters $(\alpha_0,\alpha_1,\alpha_2,\alpha_3,\alpha_4) = (-b, a+n, -n, b-c+1+n, c-a)$ or

$$\kappa_\infty = a+n\,, \quad \kappa_0 = c-a\,, \quad \kappa_1 = b-c+1+n\,, \quad \theta = -b\,. \tag{4.5}$$

Then it is easy to see that by putting $t \mapsto \varepsilon t$ and $b = \varepsilon^{-1}$, the Hamiltonian system $\mathrm{S_{VI}}$ with (4.5) is reduced to $\mathrm{S_V}$ with

$$\kappa_\infty = a+n\,, \quad \kappa_0 = c-a\,, \quad \theta = -c+n\,, \tag{4.6}$$

in the limit $\varepsilon \to 0$. It is obvious that Gauss's function $F(a, b, c; t)$ is reduced to Kummer's function $F(a, c; t)$ by this process. Thus we get

$$\varphi_{k,l,m} \to \varepsilon^{-l}\Gamma^{-1}(c-a-k+m)\,\varphi_{k+1,m+1}\,, \tag{4.7}$$

where $\varphi_{i,j}$ are given by (2.16). Note that we redefine the constants c_1 and c_2 appropriately. It is easy to see that we have

$$\tau_n^{k,l,m} \to \varepsilon^{-ln}\Gamma^{-n}(c-a-k+m)\,\tau_n^{k+1,m+1}\,. \tag{4.8}$$

Therefore, we obtain the family of classical transcendental solutions of $\mathrm{S_V}$ in Theorem 2.2.

4.2. From $\mathrm{P_V}$ to $\mathrm{P_{III}}$. From $\mathrm{P_V}$, we can obtain two coalescence limits. First, we consider the Painlevé III equation

$$\frac{d^2q}{dt^2} = \frac{1}{q}\left(\frac{dq}{dt}\right)^2 - \frac{1}{t}\frac{dq}{dt} - \frac{4}{t}[\eta_\infty\theta_\infty q^2 + \eta_0(\theta_0+1)] + 4\eta_\infty^2 q^3 - \frac{4\eta_0^2}{q}\,, \tag{4.9}$$

which is equivalent to the Hamiltonian system

$$\mathrm{S_{III}}: \qquad q' = \frac{\partial H}{\partial p}\,, \quad p' = -\frac{\partial H}{\partial q}\,, \quad {}' = t\frac{d}{dt}\,, \tag{4.10}$$

with the Hamiltonian

$$H = 2q^2p^2 - [2\eta_\infty tq^2 + (2\theta_0+1)q + 2\eta_0 t]p + \eta_\infty(\theta_0+\theta_\infty)tq\,. \tag{4.11}$$

This system can be derived from $\mathrm{S_V}$ by putting

$$\begin{gathered} q \mapsto 1+\varepsilon tq\,, \quad p \mapsto \varepsilon^{-1}t^{-1}p\,, \quad t \mapsto \eta_0\varepsilon t^2\,, \quad H \mapsto \frac{1}{2}(H+qp) \\ \kappa_\infty \mapsto -\eta_\infty\varepsilon^{-1} + \theta_\infty\,, \quad \kappa_0 \mapsto \eta_\infty\varepsilon^{-1}\,, \quad \theta \mapsto \theta_0\,, \end{gathered} \tag{4.12}$$

and taking the limit $\varepsilon \to 0$.

Let us apply the limiting procedure to the family of classical transcendental solutions of $\mathrm{S_V}$ given in Theorem 2.2. It is easy to see that by putting $a \mapsto -\eta_\infty\varepsilon^{-1} + c$, the Hamiltonian system $\mathrm{S_V}$ with (2.34) is reduced to $\mathrm{S_{III}}$ with

$$\theta_\infty = \nu + n + 1\,, \quad \theta_0 = -\nu + n - 1\,, \tag{4.13}$$

in the limit $\varepsilon \to 0$, where we denote $c = \nu + 1$. Without loss of generality, it is possible to set $\sigma := 4\eta_\infty\eta_0 = \pm 1$. Then we find that

$$F(a, c; t) \to \Gamma(\nu+1)\left(\frac{t}{2}\right)^{-\nu} Z_\nu(t)\,, \tag{4.14}$$

where $Z_\nu = Z_\nu(t)$ denotes

$$Z_\nu = \begin{cases} J_\nu: & \text{Bessel} \quad \sigma = +1\,, \\ I_\nu: & \text{modified Bessel} \quad \sigma = -1\,. \end{cases} \tag{4.15}$$

This means that Kummer's function is reduced to the (modified) Bessel function in this limit. Thus we get

$$\varphi_{i,j} \to (-1)^i (\eta_\infty \varepsilon^{-1})^j \left(\frac{t}{2}\right)^{-\nu-j} \varphi_{\nu+j} \,, \tag{4.16}$$

with

$$\varphi_{\nu+j} = c_1 Z_{\nu+j} + c_2 (-\sigma)^{-j} Z_{-\nu-j} \,. \tag{4.17}$$

This leads to

$$\tau_n^{i,j} \to \left(\frac{1}{2}\right)^{(n-1)n} (-1)^{in} (\eta_\infty \varepsilon^{-1})^{jn} \left(\frac{t}{2}\right)^{-(\nu+j)n} \tau_n^{\nu+j} \,, \tag{4.18}$$

where $\tau_n^{\nu+j}$ are defined by

$$\tau_n^{\nu+j} = \begin{vmatrix} \varphi_{\nu+j}^{(0)} & \varphi_{\nu+j}^{(1)} & \cdots & \varphi_{\nu+j}^{(n-1)} \\ \varphi_{\nu+j}^{(1)} & \varphi_{\nu+j}^{(2)} & \cdots & \varphi_{\nu+j}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \varphi_{\nu+j}^{(n-1)} & \varphi_{\nu+j}^{(n)} & \cdots & \varphi_{\nu+j}^{(2n-2)} \end{vmatrix} , \quad \varphi_{\nu+j}^{(i)} = \left(t\frac{d}{dt}\right)^i \varphi_{\nu+j} \,. \tag{4.19}$$

Therefore, from Theorem 2.2 and Remark 2.3, we obtain the following [14].

PROPOSITION 4.1 (Okamoto). *Define the functions* τ_n^ν *by* (4.19). *Then,*

$$q = \frac{1}{2\eta_\infty} \frac{\tau_{n+1}^\nu \tau_n^{\nu+1}}{\tau_n^\nu \tau_{n+1}^{\nu+1}} = \frac{1}{2\eta_\infty}\left(\frac{d}{dt}\log\frac{\tau_{n+1}^{\nu+1}}{\tau_n^\nu} + \frac{\nu+1-n}{t}\right), \quad p = -\frac{1}{4\eta_0 t}\frac{\tau_{n+1}^{\nu+1}\tau_{n-1}^{\nu+1}}{\tau_n^{\nu+1}\tau_n^{\nu+1}} \,, \tag{4.20}$$

$$\theta_\infty = \nu + n + 1 \,, \quad \theta_0 = -\nu + n - 1 \,, \tag{4.21}$$

with $4\eta_\infty\eta_0 = \pm 1$ *give a family of classical transcendental solutions of* S_{III}.

4.3. From P_{V} to P_{IV}. Next, we consider the Painlevé IV equation

$$\frac{d^2q}{dt^2} = \frac{1}{2q}\left(\frac{dq}{dt}\right)^2 + \frac{3}{2}q^3 + 2tq^2 + \frac{1}{2}t^2 q - (-\kappa_0 + 2\theta_\infty + 1)q - \frac{\kappa_0^2}{2q} \,, \tag{4.22}$$

which is equivalent to the Hamiltonian system

$$S_{\mathrm{IV}}: \qquad q' = \frac{\partial H}{\partial p} \,, \quad p' = -\frac{\partial H}{\partial q} \,, \quad ' = \frac{d}{dt} \,, \tag{4.23}$$

with the Hamiltonian

$$H = qp^2 - (q^2 + tq + \kappa_0)p + \theta_\infty q \,. \tag{4.24}$$

This system can also be derived from S_{V} by coalescence, whose process is achieved by putting

$$\begin{gathered} q \mapsto \varepsilon q \,, \quad p \mapsto \varepsilon^{-1} p \,, \quad t \mapsto \varepsilon^{-2}(1+\varepsilon t) \,, \quad H + \kappa \mapsto \varepsilon^{-1} H \,, \\ \theta \mapsto \varepsilon^{-2} + 2\theta_\infty - \kappa_0 \,, \quad \kappa_\infty \mapsto \varepsilon^{-2} \,, \end{gathered} \tag{4.25}$$

and taking the limit $\varepsilon \to 0$.

Let us consider the degeneration of the classical transcendental solutions of S_{V}. Applying the Bäcklund transformation π^2 to the solutions in Theorem 2.2, we obtain the following.

COROLLARY 4.2. *Define the functions* $\tau_n^{i,j}$ *by* (2.30). *Then,*

$$q = -\left(\frac{c-a}{c-a-1}\right)^n \frac{\tau_n^{0,0}\tau_{n+1}^{1,1}}{\tau_n^{1,0}\tau_{n+1}^{0,1}}, \quad p = (1-a)\left(\frac{c-a-1}{c-a}\right)^n \frac{\tau_n^{1,0}\tau_{n+1}^{0,1}\tau_{n+1}^{-1,-1}}{\tau_{n+1}^{0,0}\tau_{n+1}^{0,0}\tau_n^{0,0}}, \tag{4.26}$$

$$\kappa_\infty = c-a, \quad \kappa_0 = a+n, \quad \theta = c-n-2, \tag{4.27}$$

give a family of classical transcendental solutions of S_{V}.

It is easy to see that by putting $c \mapsto \varepsilon^{-2} + a$, the Hamiltonian system S_{V} with (4.27) is reduced to S_{IV} with

$$\kappa_0 = -\nu + n, \quad \theta_\infty = -\nu - 1, \tag{4.28}$$

in the limit $\varepsilon \to 0$, where we denote $a = -\nu$. We find that

$$\frac{\Gamma(c-a)}{\Gamma(c)} F(a,c;t) \to (-\varepsilon)^{-\nu} H_\nu(t), \tag{4.29}$$

where $H_\nu(t)$ denotes the Hermite-Weber function. By a Kummer transformation

$$F(a-c+1, 2-c; t) = e^t F(1-a, 2-c; -t), \tag{4.30}$$

we get

$$\frac{\Gamma(c-a)\Gamma(a-c+1)}{\Gamma(2-c)} t^{1-c} F(a-c+1, 2-c; t) \to (-i\varepsilon)^{-\nu-1} e^{t^2/2} H_{-\nu-1}(it). \tag{4.31}$$

Thus we have

$$\varphi_{k,j} \to (-\varepsilon)^{-\nu+k}[c_1\Gamma(-\nu+k)H_{\nu-k}(t) + c_2 e^{i\pi(-\nu+k-1)/2} e^{t^2/2} H_{-\nu+k-1}(it)]. \tag{4.32}$$

Let us rewrite this expression in terms of the hyperbolic cylinder function $D_\nu(t)$. Noting the relations $H_\nu(t) = e^{t^2/4} D_\nu(t)$ and

$$D_{-\nu-1}(it) = \frac{\Gamma(-\nu)}{\sqrt{2\pi}}[e^{i\pi(\nu+1)/2} D_\nu(t) - e^{-i\pi(\nu-1)/2} D_\nu(-t)], \tag{4.33}$$

we get $\varphi_{k,j} \to \varepsilon^k e^{t^2/4} \varphi_{\nu-k}$ with

$$\varphi_{\nu-k} = c_1 \frac{D_{\nu-k}(t)}{\Gamma(\nu-k+1)} + c_2 \Gamma(-\nu+k) D_{\nu-k}(-t). \tag{4.34}$$

This leads to $\tau_n^{k,j} \to \varepsilon^{-n(n-1)} \varepsilon^{kn} \tau_n^{\nu-k}$, where $\tau_n^{\nu-k}$ are defined by

$$\tau_n^{\nu-k} = \begin{vmatrix} \phi_{\nu-k}^{(0)} & \phi_{\nu-k}^{(1)} & \cdots & \phi_{\nu-k}^{(n-1)} \\ \phi_{\nu-k}^{(1)} & \phi_{\nu-k}^{(2)} & \cdots & \phi_{\nu-k}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \phi_{\nu-k}^{(n-1)} & \phi_{\nu-k}^{(n)} & \cdots & \phi_{\nu-k}^{(2n-2)} \end{vmatrix}, \quad \phi_{\nu-k}^{(m)} = \left(\frac{d}{dt}\right)^m (e^{t^2/4}\varphi_{\nu-k}). \tag{4.35}$$

Therefore, we obtain the following.

PROPOSITION 4.3. *Define the functions* $\tau_n^{\nu-k}$ *by* (4.35). *Then,*

$$q = -\frac{\tau_n^{\nu}\tau_{n+1}^{\nu-1}}{\tau_n^{\nu-1}\tau_{n+1}^{\nu}}\,, \quad p = (\nu+1)\frac{\tau_n^{\nu-1}\tau_{n+1}^{\nu+1}}{\tau_{n+1}^{\nu}\tau_n^{\nu}}\,, \tag{4.36}$$

$$\kappa_0 = -\nu + n\,, \quad \theta_\infty = -\nu - 1\,, \tag{4.37}$$

give a family of classical transcendental solutions of S_{IV}.

A direct derivation of this proposition is given in Appendix.

REMARK 4.4. A special case ($c_2 = 0$) of Proposition 4.3 is stated in [5]. In [7], the case of $\nu \in \mathbf{Z}$, where the τ-functions are reduced to some polynomials, is discussed.

4.4. From $\mathrm{P_{III}}$ to $\mathrm{P_{II}}$. Both $\mathrm{P_{III}}$ and $\mathrm{P_{IV}}$ go to $\mathrm{P_{II}}$ by coalescence. First, we consider the degeneration from S_{III} to S_{II}, which is achieved by putting

$$\begin{gathered} q \mapsto 1 + \varepsilon q\,, \quad p \mapsto \varepsilon^{-1} p\,, \quad \theta_0 \mapsto 2\varepsilon^{-3} + \alpha^{(1)}\,, \quad \theta_\infty \mapsto -2\varepsilon^{-3} + \alpha^{(2)}\,, \\ t \mapsto -2\varepsilon^{-3}\left(1 - \frac{1}{2}\varepsilon^2 t\right), \quad H \mapsto -2\varepsilon^{-2} H - 2\varepsilon^{-3}\alpha\,, \end{gathered} \tag{4.38}$$

and taking the limit $\varepsilon \to 0$, where we set $\alpha = (\alpha^{(1)} + \alpha^{(2)})/2$.

Let us consider the degeneration of the family of classical transcendental solutions. From (4.38), it is the case of $\sigma = 1$ (the Bessel function) that we can take the degeneration limit. By the relation $J_{-\nu} = \cos(\nu\pi) J_\nu - \sin(\nu\pi) Y_\nu$, we rewrite (4.17) as $\varphi_{\nu+j} = c_1 J_{\nu+j} + c_2 Y_{\nu+j}$. Then, from (4.38), we see that by putting $\nu = -2\varepsilon^{-3}$, the Hamiltonian system $\mathrm{S_{III}}$ with (4.21) is reduced to $\mathrm{S_{II}}$ with $\alpha = n$. It is known that we have [1]

$$\begin{aligned} J_\nu(\nu + z\nu^{1/3}) &= 2^{1/3}\nu^{-1/3}\mathrm{Ai}(-2^{1/3}z) + O(\nu^{-1})\,, \\ Y_\nu(\nu + z\nu^{1/3}) &= -2^{1/3}\nu^{-1/3}\mathrm{Bi}(-2^{1/3}z) + O(\nu^{-1})\,, \end{aligned} \tag{4.39}$$

which lead to

$$J_\nu(t) \to -\varepsilon \mathrm{Ai}(t) + O(\varepsilon^3)\,, \quad Y_\nu(t) \to \varepsilon \mathrm{Bi}(t) + O(\varepsilon^3)\,. \tag{4.40}$$

Thus we get $\tau_n^{\nu+j} \to \varepsilon^n(-2\varepsilon^{-2})^{(n-1)n}\tau_n$, where the functions τ_n are defined by (1.13). Therefore, we obtain Proposition 1.1.

4.5. From $\mathrm{P_{IV}}$ to $\mathrm{P_{II}}$. It is well-known that the Hamiltonian system $\mathrm{S_{II}}$ is also derived from $\mathrm{S_{IV}}$ by degeneration. This process is achieved by putting

$$\begin{gathered} q \mapsto \varepsilon^{-3}(1 - \varepsilon^2 q)\,, \quad p \mapsto -\varepsilon p\,, \quad t \mapsto -2\varepsilon^{-3}\left(1 + \frac{1}{2}\varepsilon^4 t\right), \\ \kappa_0 \mapsto \varepsilon^{-6}\,, \quad \theta_\infty \mapsto \alpha\,, \quad H \mapsto -\varepsilon^{-1} H + \varepsilon^{-3}\alpha\,, \end{gathered} \tag{4.41}$$

and taking the limit $\varepsilon \to 0$.

Let us consider the degeneration of classical transcendental solutions. Applying the Bäcklund transformation π to the solutions in Theorem A.2, we obtain the following.

COROLLARY 4.5. *Define the functions* τ_n^ν *by* (4.35). *Then,*

$$q = \frac{d}{dt}\log\frac{\tau_{n+1}^{\nu}}{\tau_n^{\nu}} - t\,, \quad p = \frac{\tau_{n+1}^{\nu}\tau_{n-1}^{\nu-1}}{\tau_n^{\nu-1}\tau_n^{\nu}}\,, \tag{4.42}$$

$$\kappa_0 = \nu + 1\,, \quad \theta_\infty = n\,, \tag{4.43}$$

give a family of classical transcendental solutions of S_{IV}.

It is easy to see that by putting $\nu \mapsto \varepsilon^{-6} - 1$, the Hamiltonian system S_{IV} with (4.43) is reduced to S_{II} with $\alpha = n$ in the limit $\varepsilon \to 0$.

Let us consider the degeneration of classical transcendental solutions. According to [2], we find that the parabolic cylinder function is reduced to the Airy function as

$$\frac{D_{\nu+j}(t)}{\Gamma(\nu+j+1)} \to (-\varepsilon)^{3j}\mathrm{Ai}(t)\,, \quad \Gamma(-\nu-j)D_{\nu+j}(-t) \to (-\varepsilon)^{3j}\mathrm{Ai}(\omega t)\,, \tag{4.44}$$

with $\omega = e^{2\pi i/3}$. Thus we have $\varphi_{\nu+j} \to (-\varepsilon)^{3j}\varphi$, where φ denote the general solution of Airy's differential equation. Normalizing the τ-functions (4.35) as $\tau_n^{\nu+j} = e^{nt^2/4}\,\tilde{\tau}_n^{\nu+j}$, we get $\tilde{\tau}_n^{\nu+j} \to (-\varepsilon)^{-n(n-1)+3jn}\tau_n$, where the functions τ_n are defined by (1.13). Therefore, we obtain Proposition 1.1.

A. Classical transcendental solutions of the Painlevé IV equation.

A.1. The symmetric form of the Painlevé IV equation. The symmetric form of P_{IV} is given by [6,7]

$$\begin{aligned} f_0' &= f_0(f_1 - f_2) + \alpha_0\,,\\ f_1' &= f_1(f_2 - f_0) + \alpha_1\,, \qquad {}' = \frac{d}{dt}\,,\\ f_2' &= f_2(f_0 - f_1) + \alpha_2\,, \end{aligned} \tag{A.1}$$

with normalization conditions $\alpha_0 + \alpha_1 + \alpha_2 = 1$ and $f_0 + f_1 + f_2 = t$. The correspondences to the canonical variables and to parameters of S_{IV} are given by

$$q = -f_1\,, \quad p = f_2\,, \qquad \kappa_0 = \alpha_1\,, \quad \theta_\infty = -\alpha_2\,. \tag{A.2}$$

The Bäcklund transformations of P_{IV} are described as follows:

$$\begin{aligned} &s_i(\alpha_i) = -\alpha_i\,, \quad s_i(\alpha_j) = \alpha_j + \alpha_i \ (j = i \pm 1)\,, \quad \pi(\alpha_j) = \alpha_{j+1}\,,\\ &s_i(f_i) = f_i\,, \qquad s_i(f_j) = f_j \pm \frac{\alpha_i}{f_i} \ (j = i \pm 1)\,, \quad \pi(f_j) = f_{j+1}\,, \end{aligned} \tag{A.3}$$

where the subscripts $i = 0, 1, 2$ are understood as elements of $\mathbf{Z}/3\mathbf{Z}$. The Hamiltonians h_i of the system (A.1) are given by

$$h_0 = f_0 f_1 f_2 + \frac{\alpha_1 - \alpha_2}{3} f_0 + \frac{\alpha_1 + 2\alpha_2}{3} f_1 - \frac{2\alpha_1 + \alpha_2}{3} f_2\,, \tag{A.4}$$

and $h_i = \pi^i(h_0)$. Introducing τ-functions τ_i as $h_i = (\log \tau_i)'$, we find that the Bäcklund transformations for τ-functions are described by

$$s_i(\tau_j) = \tau_j \ (i \neq j)\,, \quad s_i(\tau_i) = f_i \frac{\tau_{i-1}\tau_{i+1}}{\tau_i}\,, \quad \pi(\tau_i) = \tau_{i+1}\,. \tag{A.5}$$

The f-variables are recovered from the τ-functions by

$$f_i = \frac{\tau_i s_i(\tau_i)}{\tau_{i-1}\tau_{i+1}} = \frac{d}{dt}\log\frac{\tau_{i-1}}{\tau_{i+1}} + \frac{t}{3}\,. \tag{A.6}$$

Let us define the translation operators T_i $(i = 0, 1, 2)$ by $T_1 = \pi s_2 s_1$ and $\pi T_i = T_{i+1}\pi$, which commute with each other and act on the parameters α_i as

$$T_i(\alpha_{i-1}) = \alpha_{i-1} + 1\,, \quad T_i(\alpha_i) = \alpha_i - 1\,, \quad T_i(\alpha_j) = \alpha_j \ (j \neq i-1, i)\,. \tag{A.7}$$

Noting that $T_1T_2T_0 = 1$, we set $\tau_{k,l} = T_1^k T_2^l(\tau_0)$ $(k, l \in \mathbf{Z})$. Then we have from (A.6)

$$\begin{aligned} T_1^k T_2^l(f_0) &= \frac{\tau_{k,l}\tau_{k+2,l+1}}{\tau_{k+1,l+1}\tau_{k+1,l}} = \frac{d}{dt}\log\frac{\tau_{k+1,l+1}}{\tau_{k+1,l}} + \frac{t}{3}\,,\\ T_1^k T_2^l(f_1) &= \frac{\tau_{k+1,l}\tau_{k,l+1}}{\tau_{k,l}\tau_{k+1,l+1}} = \frac{d}{dt}\log\frac{\tau_{k,l}}{\tau_{k+1,l+1}} + \frac{t}{3}\,,\\ T_1^k T_2^l(f_2) &= \frac{\tau_{k+1,l+1}\tau_{k,l-1}}{\tau_{k+1,l}\tau_{k,l}} = \frac{d}{dt}\log\frac{\tau_{k+1,l}}{\tau_{k,l}} + \frac{t}{3}\,. \end{aligned} \tag{A.8}$$

It is possible to derive a bilinear equation of Toda type with respect to each translation operator. For the T_1-direction, we have

$$\tau_{k+1,l}\tau_{k-1,l} = \left[(\log\tau_{k,l})'' + \frac{2\alpha_1 + \alpha_2 - 2k + l}{3}\right]\tau_{k,l}\cdot\tau_{k,l}\,. \tag{A.9}$$

A.2. A Riccati solution. We derive a Riccati solution of (A.1). First, we set $\alpha_0 = 0$ and $f_0 = 0$. Then f_1 satisfies a Riccati equation $f_1' = f_1(t - f_1) + \alpha_1$. By a dependent variable transformation $f_1 = (\log\varphi)' + t/2$, we have for φ the linear equation

$$\left(\frac{d^2}{dt^2} - \alpha_1 + \frac{1}{2} - \frac{t^2}{4}\right)\varphi = 0\,, \tag{A.10}$$

which is nothing but Weber's differential equation. We set $\alpha_1 = -\nu$. Then the general solution of (A.10) is expressed by

$$\varphi = c_1\frac{D_\nu(t)}{\Gamma(\nu+1)} + c_2\Gamma(-\nu)D_\nu(-t)\,, \tag{A.11}$$

where c_i $(i = 1, 2)$ are arbitrary complex constants. By using the contiguity relations of the hyperbolic cylinder function, we obtain the following.

PROPOSITION A.1. *Define the function* $\varphi_{\nu-k}$ *by* (4.34). *Then,*

$$(f_0, f_1, f_2) = \left(0, \frac{\varphi_{\nu-1}}{\varphi_\nu}, (\nu+1)\frac{\varphi_{\nu+1}}{\varphi_\nu}\right), \quad (\alpha_0, \alpha_1, \alpha_2) = (0, -\nu, \nu+1)\,, \tag{A.12}$$

give a Riccati solution of the symmetric form of $\mathrm{P_{IV}}$.

A.3. A Determinant formula for a family of classical transcendental solutions. First, we calculate the Hamiltonians and τ-functions for the Riccati solution in Proposition A.1.

Under the specialization (A.12), the Hamiltonians and τ-functions are calculated as

$$h_0 = \frac{\varphi_\nu'}{\varphi_\nu} + \frac{\nu+1/2}{3}t\,, \quad h_1 = \frac{\nu+1}{3}t\,, \quad h_2 = \frac{\nu}{3}t\,, \tag{A.13}$$

and

$$\begin{aligned}
&\tau_0 = \tau_{0,0} = \varphi_\nu \exp\left(\frac{\nu+1/2}{6}\,t^2\right),\\
&\tau_1 = \tau_{1,0} = \exp\left(\frac{\nu+1}{6}\,t^2\right),\\
&\tau_2 = \tau_{1,1} = \exp\left(\frac{\nu}{6}\,t^2\right),\\
&s_0(\tau_0) = \tau_{2,1} = 0\,,\\
&s_1(\tau_1) = \tau_{0,1} = \varphi_{\nu-1} \exp\left(\frac{\nu-1/2}{6}\,t^2\right),\\
&s_2(\tau_2) = \tau_{0,-1} = (\nu+1)\varphi_{\nu+1} \exp\left(\frac{\nu+3/2}{6}\,t^2\right),
\end{aligned} \tag{A.14}$$

up to multiplication by some constants, respectively. Introducing functions $\sigma_{k,l}$ by

$$\tau_{k,l} = \sigma_{k,l} \exp\left(\frac{\nu+2k-l-1}{6}\,t^2\right), \tag{A.15}$$

we see that $\sigma_{2,l} = 0$, $\sigma_{1,l} = \text{const.}$ and $\sigma_{0,l} = (\text{const.}) \times e^{t^2/4}\varphi_{\nu-l}$. Moreover, we set

$$\sigma_{k,l} = \omega_{k,l}\rho_{k,l}\,, \quad \omega_{k,l} = \omega_{k,l}(\nu)\,, \tag{A.16}$$

with $\rho_{1,l} = 1$ and $\rho_{0,l} = e^{t^2/4}\varphi_{\nu-l}$, and impose that the constants $\omega_{k,l}$ satisfy

$$\omega_{k+1,l}\omega_{k-1,l} = \omega_{k,l}^2\,. \tag{A.17}$$

From the bilinear equation of Toda type (A.9), the function $\rho_{k,l}$ are determined by

$$\rho_{k+1,l}\rho_{k-1,l} = \rho_{k,l}''\rho_{k,l} - (\rho_{k,l}')^2\,, \tag{A.18}$$

with initial conditions

$$\rho_{2,l} = 0\,, \quad \rho_{1,l} = 1\,, \quad \rho_{0,l} = e^{t^2/4}\varphi_{\nu-l}\,. \tag{A.19}$$

By Darboux's formula, the functions $\rho_{1-n,l}$ for $n \in \mathbf{Z}_{\geq 0}$ are expressed as

$$\rho_{1-n,l} = \begin{vmatrix} \rho_{0,l}^{(0)} & \rho_{0,l}^{(1)} & \cdots & \rho_{0,l}^{(n-1)} \\ \rho_{0,l}^{(1)} & \rho_{0,l}^{(2)} & \cdots & \rho_{0,l}^{(n)} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{0,l}^{(n-1)} & \rho_{0,l}^{(n)} & \cdots & \rho_{0,l}^{(2n-2)} \end{vmatrix}, \quad \rho_{0,l}^{(i)} = \left(\frac{d}{dt}\right)^i \rho_{0,l}\,. \tag{A.20}$$

Note that the constants $\omega_{k,l}$ are determined by recurrence relations (A.17) and $\omega_{i,l+1}\omega_{i,l-1} = (\nu+1-l)\,\omega_{i,l}^2$ with initial conditions $\omega_{i,0} = \omega_{i,1} = 1$ $(i = 0, 1)$. Since it is possible to set $l = 0$ without loss of generality, we obtain the following.

THEOREM A.2. *Define the functions* $\tau_n^{\nu-k}$ *by* (4.35). *Then,*

$$f_0 = \frac{\tau_{n+1}^{\nu}\tau_{n-1}^{\nu-1}}{\tau_n^{\nu-1}\tau_n^{\nu}} = \frac{d}{dt}\log\frac{\tau_n^{\nu-1}}{\tau_n^{\nu}}\,,$$

$$f_1 = \frac{\tau_n^{\nu}\tau_{n+1}^{\nu-1}}{\tau_{n+1}^{\nu}\tau_n^{\nu-1}} = \frac{d}{dt}\log\frac{\tau_{n+1}^{\nu}}{\tau_n^{\nu-1}}\,, \tag{A.21}$$

$$f_2 = (\nu+1)\frac{\tau_n^{\nu-1}\tau_{n+1}^{\nu+1}}{\tau_n^{\nu}\tau_{n+1}^{\nu}} = \frac{d}{dt}\log\frac{\tau_n^{\nu}}{\tau_{n+1}^{\nu}} + t\,,$$

$$(\alpha_0, \alpha_1, \alpha_2) = (-n, -\nu+n, \nu+1)\,, \tag{A.22}$$

and

$$q = -\frac{\tau_n^{\nu}\tau_{n+1}^{\nu-1}}{\tau_{n+1}^{\nu}\tau_n^{\nu-1}} = -\frac{d}{dt}\log\frac{\tau_{n+1}^{\nu}}{\tau_n^{\nu-1}}\,, \quad p = (\nu+1)\frac{\tau_n^{\nu-1}\tau_{n+1}^{\nu+1}}{\tau_n^{\nu}\tau_{n+1}^{\nu}} = \frac{d}{dt}\log\frac{\tau_n^{\nu}}{\tau_{n+1}^{\nu}} + t\,, \tag{A.23}$$

$$\kappa_0 = -\nu+n\,, \quad \theta_\infty = -\nu-1\,, \tag{A.24}$$

give a family of classical transcendental solutions of the symmetric form of $\mathrm{P_{IV}}$ *and the Hamiltonian system* $\mathrm{S_{IV}}$, *respectively.*

REFERENCES

[1] M. ABRAMOWITZ AND I. A. STEGUN, Handbook of Mathematical Functions, Dover Publications, New York, 1964.

[2] Y. HARAOKA, Confluence of cycles for hypergeometric functions on $Z_{2,n+1}$, Trans. Amer. Math. Soc. 349 (1997), 675–712.

[3] K. IWASAKI, H. KIMURA, S. SHIMOMURA AND M. YOSHIDA, From Gauss to Painlevé – A Modern Theory of Special Functions, Aspects of Mathematics E16, Vieweg, 1991.

[4] K. KAJIWARA, T. MASUDA, M. NOUMI, Y. OHTA AND Y. YAMADA, Determinant formulas for the Toda and discrete Toda equations, Funkcial. Ekvac. 44 (2001), 291–307.

[5] K. KAJIWARA AND Y. OHTA, Determinant structure of the rational solutions for the Painlevé IV equation, J. Phys. A 31 (1998), 2431–2446.

[6] M. NOUMI, Painlevé equations through symmetry, American Mathematical Society, 2004.

[7] M. NOUMI AND Y. YAMADA, Symmetries in the fourth Painlevé equation and Okamoto polynomials, Nagoya Math. J. 153 (1999), 53–86.

[8] M. NOUMI AND Y. YAMADA, Higher order Painlevé equations of type $A_l^{(1)}$, Funkcial. Ekvac. 41 (1998), 483–503.

[9] M. NOUMI AND Y. YAMADA, Affine Weyl groups, discrete dynamical systems and Painlevé equations, Comm. Math. Phys. 199 (1998), 281–295.

[10] M. NOUMI AND Y. YAMADA, A new Lax pair for the sixth Painlevé equation associated with $\widehat{\mathfrak{so}}(8)$, In: Microlocal Analysis and Complex Fourier Analysis (Eds. T. Kawai and K. Fujita), 238–252, World Scientific, 2002.

[11] K. OKAMOTO, Studies on the Painlevé equations I, sixth Painlevé equation P_{VI}, Ann. Mat. Pura Appl. (4) 146 (1987), 337–381.

[12] K. OKAMOTO, Studies on the Painlevé equations II, fifth Painlevé equation P_V, Japan J. Math. 13 (1987), 47–76.

[13] K. OKAMOTO, Studies on the Painlevé equations III, second and fourth Painlevé equations, P_{II} and P_{IV}, Math. Ann. 275 (1986), 222–254.

[14] K. OKAMOTO, Studies on the Painlevé equations IV, third Painlevé equation P_{III}, Funkcial. Ekvac. 30 (1987), 305–332.

[15] P. PAINLEVÉ, Sur les équations différentielles du second ordre à points critiques fixes, C. R. Acad. Sci. Paris 143 (1906), 1111–1117.

DEPARTMENT OF MATHEMATICS
KOBE UNIVERSITY
ROKKO, KOBE, 657–8501
JAPAN

E-mail: masuda@math.kobe-u.ac.jp