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Classical Weyl transverse gravity

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Abstract We study various classical aspects of the Weyl transverse (WTDiff) gravity in a general space-time dimension. First of all, we clarify a classical equivalence among three kinds of gravitational theories, those are, the conformally invariant scalar tensor gravity, Einstein's general relativity and the WTDiff gravity via the gauge-fixing procedure. Secondly, we show that in the WTDiff gravity the cosmological constant is a mere integration constant as in unimodular gravity, but it does not receive any radiative corrections unlike the unimodular gravity. A key point in this proof is to construct a covariantly conserved energy-momentum tensor, which is achieved on the basis of this equivalence relation. Thirdly, we demonstrate that the Noether current for the Weyl transformation is identically vanishing, thereby implying that the Weyl symmetry existing in both the conformally invariant scalar tensor gravity and the WTDiff gravity is a "fake" symmetry. We find it possible to extend this proof to all matter fields, i.e. the Weyl-invariant scalar, vector and spinor fields. Fourthly, it is explicitly shown that in the WTDiff gravity the Schwarzschild black hole metric and a charged black hole one are classical solutions to the equations of motion only when they are expressed in the Cartesian coordinate system. Finally, we consider the Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmology and provide some exact solutions.

1 Introduction

The physical importance of Weyl (local conformal) symmetry has not been clearly established in quantum gravity thus far. It is usually believed that if the energy scale under consideration goes up to the Planck mass scale, all elementary particles, which are either massive or massless at the low energy scale, could be regarded as almost massless particles where the Weyl symmetry would become a gauge symmetry and play an important role. However, it is true that a concrete implementation of the Weyl symmetry as a plausible gauge symmetry in quantum gravity encounters a lot of difficulties. For instance, if one requires an exact Weyl symmetry to be realized in gravitational theories at the classical level, only two candidate theories are deserved to be studied though they possess some defects in their own right. The one theory is the conformal gravity, for which the action is described in terms of the square term of the conformal tensor. The conformal gravity belongs to a class of higher derivative gravities so that it suffers from a serious problem, i.e. violation of the unitarity because of the emergence of massive ghosts, although it has an attractive feature as a renormalizable theory [1,2].

The other plausible candidate as a gravitational theory with the Weyl symmetry, which we consider in this article intensively, is the conformally invariant scalar–tensor gravity [3,4]. In this theory, a (ghost-like) scalar field is introduced in such a way that it couples to the scalar curvature in a conformally invariant manner. Even if this theory is a unitary theory owing to the presence of only second-order derivative terms, it suffers from a sort of triviality problem in the sense that when we take a suitable gauge condition for the Weyl symmetry (we take the scalar field to be a constant), the action of the conformally invariant scalar–tensor gravity reduces to the Einstein–Hilbert action of Einstein's general relativity. It is therefore unclear to make use of the conformally invariant scalar–tensor gravity as an alternative theory of general relativity. Of course, the conformally invariant scalar–tensor gravity is not a renormalizable theory like general relativity.

One reason why we would like to consider a gravitational theory with the Weyl symmetry stems from the cosmological constant problem [5], which is one of the most difficult problems in modern theoretical physics. The Weyl symmetry forbids the appearance of operators of dimension zero such as the cosmological constant in the action so it is expected that the Weyl symmetry might play an important role in the

Dedicated to the memory of Mario Tonin.

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cosmological constant problem [6]. In this respect, a difficulty is that the Weyl symmetry is broken by quantum effects and its violation emerges as a trace anomaly of the energy-momentum tensor [7,8]. Thus, the idea such that one utilizes the Weyl symmetry as a resolution of the cosmological constant problem makes no sense at the quantum level even if it is an intriguing idea at the classical level.

Here a naive but natural question arises: Is the Weyl symmetry always violated by radiative corrections? We think that it is not always so. What kind of the Weyl symmetry is not broken? In a pioneering work by Englert et al. [9], it has been clarified that the conformally invariant scalar–tensor gravity coupled to various matter fields is free of Weyl anomaly when the Weyl symmetry is spontaneously broken. This fact has been investigated and certified by subsequent papers [10–18]. Related to this work, in this article, we wish to put forward a new conjecture that the Weyl symmetry is not violated by radiative corrections if it is a *fake* Weyl symmetry and it is spontaneously broken. Here the word "*fake*" means that the corresponding Noether currents [19] vanish identically [6,20].

If our conjecture were really valid, we could have recourse to the Weyl symmetry as a resolution to the cosmological constant problem as follows: start with the conformally invariant scalar–tensor gravity, and gauge-fix the longitudinal diffeomorphism instead of the Weyl symmetry, by which the ghost-like scalar field can be removed from the physical spectrum, so that the unitarity issue does not occur. Consequently, we obtain the Weyl-invariant and transverse diffeomorphisms-invariant gravitational theory. We then find that the remaining Weyl symmetry is a fake symmetry, so it is not violated by quantum corrections according to our conjecture. By a detailed analysis, it turns out that this gravitational theory, which we call Weyl transverse (WTDiff) gravity [21–27], has a remarkable feature that the equations of motion can be rewritten to the same form as the standard Einstein's equations where the cosmological constant emerges as an integration constant as in unimodular gravity. In the unimodular gravity [28–44], the unimodular condition is implemented by using the Lagrange multiplier field, which plays a role as the cosmological constant and receives huge radiative corrections, so the cosmological constant problem is not solved. On the other hand, in the WTDiff gravity, there is no constraint like the unimodular condition and the unbroken Weyl symmetry severely prohibits the appearance of the cosmological constant. Hence, our conjecture would insist that in the WTDiff gravity, the cosmological constant problem is reduced to a mere problem of how to fix the initial value of the cosmological constant, which is an important first step for a resolution of the cosmological constant problem though we still have the new problem of how to fix its initial value.

This paper is organised as follows: In Sect. 2, we clarify the equivalence relation among three kinds of gravitational theories, i.e. the conformally invariant scalar tensor gravity, Einstein's general relativity and the WTDiff gravity via the gauge-fixing procedure. This equivalence makes it possible to construct a covariantly conserved energy-momentum tensor and prove that the equations of motions in the WTDiff gravity can be transformed to the Einstein equations of general relativity. The possibility of making such an energy-momentum tensor comes from the fact that the underlying theory behind the WTDiff gravity is the conformally invariant scalar–tensor gravity which is generally covariant.

In Sect. 3, we show that the Noether current for the Weyl transformation is identically vanishing, thereby implying that the Weyl symmetry existing in both the conformally invariant scalar tensor gravity and the WTDiff gravity is a "fake" symmetry. It is shown that it is possible to apply this proof for all the Weyl-invariant matter fields. It is explicitly shown in Sects. 4 and 5 that in the WTDiff gravity the Schwarzschild black hole metric and the charged black hole one are classical solutions to the equations of motion only when they are expressed in the Cartesian coordinate system. In Sect. 6, we consider the Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmology and provide an exact solution. The final section is devoted to discussions. Our notation and conventions are summarized in Appendix A. From Appendix B to D, some proof and the details of calculations are presented.

2 Equivalence among three gravitational theories

We will start by recalling the well-known recipe for obtaining the conformally invariant scalar–tensor gravity from the Einstein–Hilbert action of general relativity. The Einstein–Hilbert action is of form in a general n space-time dimension (we assume $n \neq 2$ in this article).[1]

$$\hat{S} = \frac{1}{2} \int d^n x \sqrt{-\hat{g}} \hat{R}, \tag{1}$$

where $\hat{g}_{\mu\nu}$ is a metric tensor. (The "hat" symbol is put for later convenience.) To let this action have the Weyl (local conformal) symmetry, one introduces a scalar field φ and supposes that the metric tensor $\hat{g}_{\mu\nu}$ is composed of the scalar field φ and a new metric field $g_{\mu\nu}$ as

$$\hat{g}_{\mu\nu} = \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \varphi \right)^{\frac{4}{n-2}} g_{\mu\nu}. \tag{2}$$

The key observation is that the metric tensor $\hat{g}_{\mu\nu}$ is invariant under the following Weyl transformation:

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega^2(x) g_{\mu\nu}, \quad \varphi \to \varphi' = \Omega^{-\frac{n-2}{2}}(x) \varphi, \tag{3}$$

[1] See Appendix A.1 for our notation and conventions.

where $\Omega(x)$ is a scalar parameter. Next, substituting (2) into the Einstein–Hilbert action (1) produces an action for the conformally invariant scalar–tensor gravity[2]

$$S = \int d^n x \sqrt{-g}\left[\frac{n-2}{8(n-1)}\varphi^2 R + \frac{1}{2} g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]. \tag{4}$$

Note that the scalar field φ is not normal but ghost-like owing to the positive coefficient $\frac{1}{2}$, but it is not a problem since the dynamical degree of freedom associated with φ can be nullified by taking a gauge condition.

This recipe for introducing the Weyl symmetry to a theory suggests that the Weyl symmetry obtained in this way might be a *fake* symmetry and the scalar field φ be a spurion field [20]. Indeed, as shown later, the Noether current for the Weyl symmetry is identically vanishing for both local and global Weyl transformations [6,20]. The physical property and the importance of this *fake* Weyl symmetry will also be discussed later in dealing with the Noether currents.

Now let us invert the order of the above argument and this time start with the conformally invariant scalar–tensor gravity (4). It is easy to see that the gauge condition for the Weyl symmetry

$$\varphi = 2\sqrt{\frac{n-1}{n-2}} \tag{5}$$

transforms the action (4) of the conformally invariant scalar–tensor gravity into the Einstein–Hilbert action (1) of general relativity (without the hat symbol).

An interesting gauge condition not for the Weyl symmetry but for the longitudinal diffeomorphism is given by

$$\varphi = 2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}}, \tag{6}$$

where we have defined $|g| = -g$ because of $g < 0$. Here let us examine this gauge condition (6) more closely. Under the Weyl transformation (3), the RHS of Eq. (6) is transformed as

$$2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}} \to \Omega^{-\frac{n-2}{2}}\, 2\sqrt{\frac{n-1}{n-2}}|g|^{-\frac{n-2}{4n}}, \tag{7}$$

which is the same transformation property as φ under the Weyl transformation as seen in (3). Thus, the gauge condition (6) does not break the Weyl symmetry. Instead, the gauge condition (6) *does* break the longitudinal diffeomorphism as explained in what follows: First, notice that with the gauge condition (6) the metric tensor (2) reads

$$\hat{g}_{\mu\nu} = |g|^{-\frac{1}{n}} g_{\mu\nu}. \tag{8}$$

Taking the determinant of this metric reveals us that $\hat{g}_{\mu\nu}$ is the unimodular metric satisfying the unimodular condition

$$\hat{g}(x) = -1. \tag{9}$$

[2] See Refs. [45–48] for various applications of this action.

Given the unimodular condition (9), any variation of the unimodular metric gives rise to an equation

$$\hat{g}^{\mu\nu}\delta\hat{g}_{\mu\nu} = 0. \tag{10}$$

When one restricts the variation to be diffeomorphisms

$$\delta\hat{g}_{\mu\nu} = \hat{\nabla}_\mu\xi_\nu + \hat{\nabla}_\nu\xi_\mu = \hat{g}_{\mu\rho}\partial_\nu\xi^\rho + \hat{g}_{\nu\rho}\partial_\mu\xi^\rho + \xi^\rho\partial_\rho\hat{g}_{\mu\nu}, \tag{11}$$

with $\hat{\nabla}_\mu$ and ξ_μ being the covariant derivative with respect to the metric tensor $\hat{g}_{\mu\nu}$ and an infinitesimal parameter, respectively, Eq. (10) yields

$$\partial_\mu\xi^\mu = 0, \tag{12}$$

where we have used

$$\hat{g}^{\mu\nu}\partial_\rho\hat{g}_{\mu\nu} = 2\partial_\rho\left(\log\sqrt{-\hat{g}}\right) = 0, \tag{13}$$

which comes from the unimodular condition (9). Equation (12) implies that the full group of diffeomorphisms (Diff) is broken down to the transverse diffeomorphisms (TDiff),[3] thereby showing that the gauge condition (6) certainly breaks the longitudinal diffeomorphism.

Inserting the gauge condition (6) to the action of the conformally invariant scalar–tensor gravity (4), one arrives at an action of the Weyl transverse (WTDiff) gravity

$$\begin{aligned} S &= \int d^n x \mathcal{L} \\ &= \frac{1}{2}\int d^n x |g|^{\frac{1}{n}}\left[R + \frac{(n-1)(n-2)}{4n^2}\frac{1}{|g|^2} g^{\mu\nu}\partial_\mu|g|\partial_\nu|g|\right]. \end{aligned} \tag{14}$$

It is straightforward to derive the equations of motion from this action. The detailed calculation is presented in Appendix C by means of two different methods. Then the equations of motion read

$$R_{\mu\nu} - \frac{1}{n}g_{\mu\nu}R = T_{(g)\mu\nu} - \frac{1}{n}g_{\mu\nu}T_{(g)}, \tag{15}$$

where the energy-momentum tensor $T_{(g)\mu\nu}$ is defined as

$$\begin{aligned} T_{(g)\mu\nu} = &\frac{(n-2)(2n-1)}{4n^2}\frac{1}{|g|^2}\partial_\mu|g|\partial_\nu|g| \\ &- \frac{n-2}{2n}\frac{1}{|g|}\nabla_\mu\nabla_\nu|g|, \end{aligned} \tag{16}$$

with $\nabla_\mu\nabla_\nu|g| = \partial_\mu\partial_\nu|g| - \Gamma^\rho_{\mu\nu}\partial_\rho|g|$. Note that Eq. (15) is purely the traceless part of the standard Einstein equations. By an explicit calculation, it is possible to verify that the action (14) and the equations of motion (15) are invariant under the Weyl transformation (3) and the transverse group of diffeomorphisms. The proof is given in Appendix B.

[3] See Appendix B for more details of TDiff.

The most important point associated with this energy-momentum tensor (16) is that it is not covariantly conserved

$$\nabla^\mu T_{(g)\mu\nu} \neq 0. \tag{17}$$

This is because the WTDiff gravity action (14) is not invariant under the full group of diffeomorphisms but only its subgroup, that is, the transverse diffeomorphisms (TDiff). However, our starting action (4) of the conformally invariant scalar–tensor gravity is generally covariant, so it should be possible to find an alternative energy-momentum tensor which is covariantly conserved (see Appendix D).

To find the desired energy-momentum tensor, let us begin by deriving the equations of motion of the conformally invariant scalar–tensor gravity (4). The equations of motion for the metric tensor $g_{\mu\nu}$ and the scalar field φ are, respectively, given by

$$\frac{n-2}{8(n-1)}\left[\varphi^2 G_{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)(\varphi^2)\right] = \frac{1}{4}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi \tag{18}$$

and

$$\frac{n-2}{4(n-1)}\varphi R = \Box\varphi, \tag{19}$$

where $G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R$ is the Einstein tensor and $\Box\varphi = g^{\mu\nu}\nabla_\mu\nabla_\nu\varphi$. Equation (19) is the equation of motion for the spurion field φ so it is not an independent equation. Actually, taking the trace part of Eq. (18) naturally leads to Eq. (19). Thus, it is sufficient to take only the equations of motion (18) into consideration.

Next, we will rewrite (18) as

$$G_{\mu\nu} = \frac{1}{\varphi^2}(\nabla_\mu\nabla_\nu - g_{\mu\nu}\Box)(\varphi^2) + \frac{8(n-1)}{n-2}\frac{1}{\varphi^2}\left[\frac{1}{4}g_{\mu\nu}\partial_\rho\varphi\partial^\rho\varphi - \frac{1}{2}\partial_\mu\varphi\partial_\nu\varphi\right] = T_{\mu\nu}, \tag{20}$$

where we have defined a new energy-momentum tensor $T_{\mu\nu}$. Since the Einstein tensor $G_{\mu\nu}$ satisfies the Bianchi, identity

$$\nabla^\mu G_{\mu\nu} = 0, \tag{21}$$

the new energy-momentum $T_{\mu\nu}$ should satisfy the covariant conservation law

$$\nabla^\mu T_{\mu\nu} = 0. \tag{22}$$

Finally, substituting the gauge condition (6) into $T_{\mu\nu}$, one has

$$T_{\mu\nu} = T_{(g)\mu\nu} + \frac{n-2}{2n}g_{\mu\nu} \times \left[-\frac{5n-3}{4n}\frac{1}{|g|^2}(\partial_\rho|g|)^2 + \frac{1}{|g|}\nabla_\rho\nabla^\rho|g|\right]. \tag{23}$$

Note that the existence of the extra terms except $T_{(g)\mu\nu}$ makes it possible to hold the covariant conservation law (22). Indeed, it is straightforward to check that this energy-momentum tensor (23) satisfies the covariant conservation law (22) by a direct calculation. Another indirect but easy proof is to consider the conservation law (22) in the local Lorentz frame where $g_{\mu\nu} = \eta_{\mu\nu}$ and $\partial_\rho g_{\mu\nu} = 0$. Then we can explicitly check that

$$\partial^\mu T_{\mu\nu} = -\frac{n-2}{2n}\frac{1}{|g|}\partial^\mu\partial_\mu\partial_\nu|g| + \frac{n-2}{2n}\frac{1}{|g|}\partial^\mu\partial_\mu\partial_\nu|g| = 0, \tag{24}$$

which implies the conservation law (22) in a curved space-time.

One remarkable feature of this energy-momentum tensor $T_{\mu\nu}$ is that there exists a nontrivial relation between $T_{\mu\nu}$ and $T_{(g)\mu\nu}$

$$T_{\mu\nu} - \frac{1}{n}g_{\mu\nu}T = T_{(g)\mu\nu} - \frac{1}{n}g_{\mu\nu}T_{(g)}, \tag{25}$$

which stems from the fact that the actions of both the conformally invariant scalar–tensor gravity and the WTDiff gravity are invariant under the Weyl transformation. It is worthwhile to stress that our findings (23) critically depend on the classical equivalence between the conformally invariant scalar–tensor gravity and the WTDiff gravity. In other words, without this equivalence, it would be difficult, if not impossible, to construct the covariantly conserved energy-momentum tensor (23) which also satisfies the important relation (25).

Now we are ready to show that the equations of motion of the WTDiff gravity, Eq. (15), reproduce the standard Einstein equations. To this aim, using Eq. (25), let us first replace the RHS in Eq. (15) with its covariantly conserved counterpart

$$R_{\mu\nu} - \frac{1}{n}g_{\mu\nu}R = T_{\mu\nu} - \frac{1}{n}g_{\mu\nu}T. \tag{26}$$

Taking the covariant derivative of this equation, and using the Bianchi identity (21) and the covariant conservation law (22), one obtains

$$\frac{n-2}{2n}\nabla_\mu R = -\frac{1}{n}\nabla_\mu T. \tag{27}$$

This equation says that $R + \frac{2}{n-2}T$ is a constant, which we will call $\frac{2n}{n-2}\Lambda$,

$$R + \frac{2}{n-2}T = \frac{2n}{n-2}\Lambda. \tag{28}$$

Eliminating T from Eq. (26) in terms of Eq. (28), one can reach the standard Einstein equations

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R + \Lambda g_{\mu\nu} = T_{\mu\nu}. \tag{29}$$

Although we have obtained the Einstein equations from the equations of motion of the WTDiff gravity in this way, the cosmological constant Λ emerges as a mere integration constant and has nothing to do with any terms in the action or vacuum fluctuations. To put it differently, Eq. (26) does not include the cosmological constant and the contribution from radiative corrections to the cosmological constant cancels in the RHS of Eq. (26), thereby guaranteeing the stability of the cosmological constant against quantum corrections.

This feature of the emergence of the cosmological constant as an integration constant is a common feature of the WTDiff gravity and the unimodular gravity [28–44]. However, there is an important difference between the two theories. In the unimodular gravity, from the viewpoint of quantum field theories, the unimodular condition must be properly implemented via the Lagrange multiplier field, which turns out to correspond to the cosmological constant in the unimodular gravity, thereby rendering its initial value radiatively unstable. In this sense, the cosmological constant problem cannot be solved within the framework of the unimodular gravity.

On the other hand, in the WTDiff gravity, there is no constraint like the unimodular condition and the *fake* Weyl symmetry is expected to forbid operators of dimension zero such as the cosmological constant. Moreover, we have a plausible conjecture such that the *fake* Weyl symmetry might never be violated by quantum effects, that is, no Weyl anomaly, owing to its "fakeness". In other words, the *fake* Weyl symmetry could survive even at the quantum level, by which suppressing the radiative corrections to the cosmological constant. If our conjecture were true, the cosmological constant problem would amount to be a mere problem of how to fix the integration constant Λ. From this point of view, we should clarify quantum aspects of the WTDiff gravity in the future.

3 Fake Weyl symmetry

In our previous article [6], motivated with the article [20] we have studied Weyl symmetry (local conformal symmetry) in the WTDiff gravity and the conformally invariant scalar–tensor gravity in four space-time dimensions, and we have shown that the Noether currents for both the local and the global Weyl symmetries are identically vanishing. In this sense, the Weyl symmetry existing in the WTDiff gravity and the conformally invariant scalar–tensor gravity is called a "fake" Weyl symmetry. In this section, we will generalize this study to not only an arbitrary space-time dimension but also all matter fields involving scalar, vector and spinor fields.

3.1 Gravity

Let us start with a gravitational sector and consider the action of the WTDiff gravity, Eq. (14). Here it is more convenient to work with the Lagrangian density than the action itself,

$$\mathcal{L} = \frac{1}{2}|g|^{\frac{1}{n}} \left[R + \frac{(n-1)(n-2)}{4n^2} \frac{1}{|g|^2} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| \right], \quad (30)$$

which is invariant under the Weyl transformation up to a surface term as shown in Appendix B.

Now we wish to calculate the Noether current for Weyl symmetry by using the Noether procedure [19]. We will closely follow the line of arguments in Ref. [20]. The general variation of the Lagrangian density (30) reads

$$\delta\mathcal{L} = \frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} \delta g_{\mu\nu} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu g_{\nu\rho})} \delta(\partial_\mu g_{\nu\rho}) + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu g_{\rho\sigma})} \delta(\partial_\mu \partial_\nu g_{\rho\sigma}). \quad (31)$$

In this expression, let us note that the Lagrangian density under consideration includes second-order derivatives of $g_{\mu\nu}$ in the scalar curvature R. Setting $\Omega(x) = e^{-\Lambda(x)}$, the infinitesimal variation $\delta\mathcal{L}$ under the Weyl transformation (3) is given by

$$\delta\mathcal{L} = \partial_\mu X_1^\mu, \quad (32)$$

where X_1^μ is defined as

$$X_1^\mu = (n-1)|g|^{\frac{1}{n}} g^{\mu\nu} \partial_\nu \Lambda. \quad (33)$$

Next, using the equations of motion

$$\frac{\partial \mathcal{L}}{\partial g_{\mu\nu}} = \partial_\rho \frac{\partial \mathcal{L}}{\partial(\partial_\rho g_{\mu\nu})} - \partial_\rho \partial_\sigma \frac{\partial \mathcal{L}}{\partial(\partial_\rho \partial_\sigma g_{\mu\nu})}, \quad (34)$$

the variation $\delta\mathcal{L}$ in (31) can be cast into the form

$$\delta\mathcal{L} = \partial_\mu K_1^\mu, \quad (35)$$

where K_1^μ is defined as

$$K_1^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu g_{\nu\rho})} \delta g_{\nu\rho} + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu g_{\rho\sigma})} \partial_\nu \delta g_{\rho\sigma} - \partial_\nu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \partial_\nu g_{\rho\sigma})} \delta g_{\rho\sigma}. \quad (36)$$

Using this formula, an explicit calculation yields

$$K_1^\mu = X_1^\mu, \quad (37)$$

thereby giving us the result that the Noether current for the Weyl symmetry vanishes identically

$$J_1^\mu = K_1^\mu - X_1^\mu = 0. \quad (38)$$

Incidentally, let us note that both the expressions X_1^μ and K_1^μ are gauge invariant under the Weyl transformation as seen in Eq. (33). This fact will be utilized later.

As an alternative derivation of the same result, one can also appeal to a more conventional method where the Lagrangian density in (30) does not explicitly involve second-order derivatives of $g_{\mu\nu}$ in the curvature scalar R. To do so, one makes use of the following well-known formula, which holds in general space-time dimensions: When one writes the scalar curvature

$$R = R_1 + R_2, \tag{39}$$

the formula takes the form [49]

$$R_1 = -2R_2 + \frac{1}{\sqrt{-g}}\partial_\mu(\sqrt{-g}A^\mu), \tag{40}$$

where one has defined the following quantities:

$$\begin{aligned} R_1 &= g^{\mu\nu}\left(\partial_\rho \Gamma^\rho_{\mu\nu} - \partial_\nu \Gamma^\rho_{\mu\rho}\right), \\ R_2 &= g^{\mu\nu}\left(\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} - \Gamma^\sigma_{\rho\nu}\Gamma^\rho_{\mu\sigma}\right) \\ &= g^{\mu\nu}\Gamma^\sigma_{\rho\sigma}\Gamma^\rho_{\mu\nu} + \frac{1}{2}\Gamma^\rho_{\mu\nu}\partial_\rho g^{\mu\nu}, \\ A^\mu &= g^{\nu\rho}\Gamma^\mu_{\nu\rho} - g^{\mu\nu}\Gamma^\rho_{\nu\rho}. \end{aligned} \tag{41}$$

Here let us note that R_2 is free of second-order derivatives of $g_{\mu\nu}$, which are now involved in the term including A^μ. Using this formula, we can rewrite the Lagrangian density (30) to the form

$$\mathcal{L} = \mathcal{L}_0 + \frac{1}{2}\partial_\mu\left(|g|^{\frac{1}{n}}A^\mu\right), \tag{42}$$

where $\mathcal{L}_0$ is defined as

$$\mathcal{L}_0 = \frac{1}{2}|g|^{\frac{1}{n}}\left[-R_2 + \frac{n-2}{2n}\frac{1}{|g|}A^\mu\partial_\mu|g| + \frac{(n-1)(n-2)}{4n^2}\frac{1}{|g|^2}g^{\mu\nu}\partial_\mu|g|\partial_\nu|g|\right]. \tag{43}$$

We are now ready to show that the Noether current for the Weyl symmetry is also zero by the more conventional method. First of all, let us observe that the variation of $\mathcal{L}$ under the Weyl transformation (3) comes from only the total derivative term

$$\delta\mathcal{L} = \partial_\mu\left[(n-1)|g|^{\frac{1}{n}}g^{\mu\nu}\partial_\nu\Lambda\right] = \frac{1}{2}\partial_\mu\left[\delta(|g|^{\frac{1}{n}}A^\mu)\right]. \tag{44}$$

The total derivative terms are irrelevant to dynamics so in what follows let us focus our attention only on the Lagrangian $\mathcal{L}_0$, which is free of second-order derivatives of $g_{\mu\nu}$.

Second, by an explicit calculation we find that the Lagrangian $\mathcal{L}_0$ is invariant under the Weyl transformation without any surface term

$$X_2^\mu = 0. \tag{45}$$

Finally, applying the Noether theorem [19] for $\mathcal{L}_0$, we can derive the following result:

$$K_2^\mu = \frac{\partial\mathcal{L}_0}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}) = 0. \tag{46}$$

Hence, the Noether current for the Weyl symmetry identically vanishes as before,

$$J_2^\mu = K_2^\mu - X_2^\mu = 0. \tag{47}$$

At this stage, we should refer to an ambiguity associated with the Noether currents for local Weyl symmetry. Our calculation in this section is based on the first Noether theorem, which is applicable for global symmetries, and the second theorem, which can be applied to local (gauge) symmetries. Of course, the latter case includes the former one as a special case, and the two Noether theorems give the same result such that the Noether currents are identically vanishing. However, we should recall the well-known fact that the Noether currents for local (gauge) symmetries always reduce to superpotentials, which give us some ambiguity. Thus, the more reliable statement, which is obtained from our calculation at hand, is that the global Weyl symmetry has a vanishing Noether current, and hence neither charge nor symmetry generator.

Next, we shall provide a simpler proof that the Noether current for the Weyl symmetry in both the conformally invariant scalar–tensor gravity and the WTDiff gravity vanishes. This proof is based on the observation that via the metric (2) and the gauge condition (6), the two theories become equivalent, and the Noether currents are gauge-invariant quantities. For simplicity, we will consider the action which includes only first-order derivatives of the metric tensor $g_{\mu\nu}$.

As the starting action, we will take the action (4) of the conformally invariant scalar–tensor gravity. As in the case of the WTDiff gravity, this action can be rewritten in the first-order derivative form

$$S = \int d^n x\left[\mathcal{L}_3 + \frac{n-2}{8(n-1)}\partial_\mu(\sqrt{-g}\varphi^2 A^\mu)\right], \tag{48}$$

where $\mathcal{L}_3$ is defined by

$$\mathcal{L}_3 = \sqrt{-g}\left[-\frac{n-2}{8(n-1)}\varphi^2 R_2 - \frac{n-2}{8(n-1)}A^\mu\partial_\mu(\varphi^2) + \frac{1}{2}g^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\right]. \tag{49}$$

The total derivative term in S plays no role in bulk dynamics, so we will henceforth pay our attention to $\mathcal{L}_3$. It is easy to show that $\mathcal{L}_3$ is invariant under the Weyl transformation without a surface term, which gives us

$$X_3^\mu = 0. \tag{50}$$

Then the Noether theorem [19] provides us with

$$K_3^\mu = \frac{\partial\mathcal{L}_3}{\partial(\partial_\mu\varphi)}\frac{n-2}{2}\varphi + \frac{\partial\mathcal{L}_3}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}). \tag{51}$$

Here we would like to give a simpler proof of $K_3^\mu = 0$ without much calculations. The key observation for this proof is to recall that three kinds of gravitational theories are related to each other by a Weyl-invariant metric (2), from which, taking the differentiation, we can derive an equation

$$\partial_\mu \hat{g}_{\nu\rho} = \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{4}{n-2}} \left(\frac{4}{n-2}\frac{1}{\varphi}\partial_\mu\varphi g_{\nu\rho} + \partial_\mu g_{\nu\rho}\right). \tag{52}$$

Using this equation, one finds that

$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}_3}{\partial(\partial_\mu \hat{g}_{\nu\rho})}\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{4}{n-2}} \frac{4}{n-2}\frac{1}{\varphi}g_{\nu\rho},$$
$$\frac{\partial \mathcal{L}_3}{\partial(\partial_\mu g_{\nu\rho})} = \frac{\partial \mathcal{L}_3}{\partial(\partial_\mu \hat{g}_{\nu\rho})}\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{4}{n-2}}. \tag{53}$$

From Eq. (53), Eq. (51) produces the expected result,

$$K_3^\mu = 0. \tag{54}$$

As a result, the Noether current for the Weyl symmetry is vanishing

$$J_3^\mu = K_3^\mu - X_3^\mu = 0. \tag{55}$$

This is our simpler proof of the vanishing Noether current for the Weyl symmetry in the conformally invariant scalar–tensor gravity. Since the current is gauge invariant, our proof can be directly applied to any conformally invariant gravitational theories such as the WTDiff gravity obtained via the trick (2) and the gauge condition (6). From our simple proof, we can also explain why the Weyl symmetry existing in both the conformally invariant scalar–tensor gravity and the WTDiff gravity is identically vanishing. It has been already shown in the previous section that these two gravitational theories are equivalent to general relativity, and the Noether currents for the Weyl symmetry are Weyl-invariant quantities. Since there is no Weyl symmetry in general relativity, the Noether current for the Weyl symmetry should be trivially zero in general relativity. The equivalence among the three theories and gauge invariance of the Noether currents naturally lead to a conclusion that the Weyl currents in the conformally invariant scalar–tensor gravity and the WTDiff gravity should be vanishing as well.

So far, we have confined our attention to only the gravitational sector. Since there are plenty of matters around us, it is natural to ask if effects of matter fields could change our conclusion or not. In the following subsections, we will show that the introduction of conformal matters does not modify the fact that the Weyl current vanishes.

3.2 Scalar field

First, let us turn our attention to a real scalar field ϕ in an n-dimensional curved space-time. The action is consisted of a kinetic term and a potential $V(\phi)$

$$S_\phi = \int \mathrm{d}^n x |g|^{\frac{1}{2}} \left[-\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - V(\phi)\right]. \tag{56}$$

Note that this action is manifestly invariant under the full group of diffeomorphisms (Diff). Under the Weyl transformation, the scalar field ϕ has the same transformation law as the spurion field φ,

$$\phi \to \phi' = \Omega^{-\frac{n-2}{2}}(x)\phi. \tag{57}$$

The trick to enlarge gauge symmetries from Diff to WDiff is now to make a Weyl-invariant scalar field $\hat{\phi} = \varphi^{-1}\phi$ in addition to the Weyl-invariant metric (2), and then replace the metric and the scalar field in the action (56) by the corresponding Weyl-invariant objects. As a result, a WDiff-invariant scalar action takes the form

$$\begin{aligned}\hat{S}_\phi &= \int \mathrm{d}^n x\, \hat{\mathcal{L}}_\phi \\ &= \int \mathrm{d}^n x |\hat{g}|^{\frac{1}{2}}\left[-\frac{1}{2}\hat{g}^{\mu\nu}\partial_\mu\hat{\phi}\partial_\nu\hat{\phi} - V(\hat{\phi})\right] \\ &= \int \mathrm{d}^n x |g|^{\frac{1}{2}}\left[-\frac{n-2}{8(n-1)}\varphi^2 g^{\mu\nu}\partial_\mu\left(\frac{\phi}{\varphi}\right)\partial_\nu\left(\frac{\phi}{\varphi}\right)\right. \\ &\quad \left. -\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{2n}{n-2}} V\left(\frac{\phi}{\varphi}\right)\right].\end{aligned} \tag{58}$$

We shall calculate the Noether current for Weyl symmetry by the two different methods. One method, which is called the WDiff method, is to calculate the current in the WDiff-invariant action without gauge-fixing the Weyl symmetry like the conformally invariant scalar–tensor gravity. The other method, which is called the WTDiff method, is to gauge-fix the longitudinal diffeomorphism by the gauge condition (6), by which the WDiff-invariant action is reduced to the WTDiff-invariant one, and then calculate the Noether current for the Weyl symmetry like the WTDiff gravity. The Weyl current is a gauge-invariant quantity, so the two methods should provide the same result.

First, let us calculate the Noether current for the Weyl symmetry on the basis of the WDiff matter action (58). It is easy to see that the action (58) is invariant under the Weyl transformation without a surface term, so we have

$$X_\phi^\mu = 0. \tag{59}$$

Again, the Noether theorem [19] yields

$$\begin{aligned}K_\phi^\mu &= \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu\phi)}\frac{n-2}{2}\phi + \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu\varphi)}\frac{n-2}{2}\varphi \\ &\quad + \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}).\end{aligned} \tag{60}$$

Next, the Weyl-invariant combinations (2) and $\hat{\phi} = \varphi^{-1}\phi$ give us the relations

$$\frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu\phi)} = \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu\hat{\phi})}\frac{1}{\varphi},$$

$$\frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu \varphi)} = \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu \hat{g}_{\nu\rho})} \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{4}{n-2}} \frac{4}{n-2}\frac{1}{\varphi} g_{\nu\rho} - \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu \hat{\phi})}\frac{\phi}{\varphi^2},$$
$$\frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu g_{\nu\rho})} = \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu \hat{g}_{\nu\rho})} \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{4}{n-2}}. \tag{61}$$

Using these relations (61), K^μ_ϕ in (60) becomes zero,

$$K^\mu_\phi = 0. \tag{62}$$

The Noether current for the Weyl symmetry is therefore vanishing

$$J^\mu_\phi = K^\mu_\phi - X^\mu_\phi = 0. \tag{63}$$

This is a general result and even after gauge-fixing the longitudinal diffeomorphism this result should be valid since the Weyl current is gauge invariant under the Weyl transformation. Indeed, this is so by calculating the Weyl current in WTDiff scalar action below.

Now let us take the gauge condition (6) for the longitudinal diffeomorphism, which does not break the local Weyl symmetry. Inserting the gauge condition (6) to the WDiff-invariant scalar action (58) leads to the WTDiff-invariant scalar action

$$\begin{aligned}\hat{S}_\phi &= \int \mathrm{d}^n x \ \hat{\mathcal{L}}_\phi \\ &= \int \mathrm{d}^n x \left\{ -\frac{n-2}{8(n-1)}|g|^{\frac{1}{2}} g^{\mu\nu} \left[\partial_\mu \phi \partial_\nu \phi + \frac{n-2}{2n}\frac{\phi}{|g|}\partial_\mu |g| \partial_\nu \phi + \frac{(n-2)^2}{16n^2}\frac{\phi^2}{|g|^2}\partial_\mu |g| \partial_\nu |g|\right] - V\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}|g|^{\frac{n-2}{4n}}\phi\right)\right\}.\end{aligned} \tag{64}$$

Since the action (64) is invariant under the Weyl transformation without a surface term, we have

$$X^\mu_\phi = 0. \tag{65}$$

The Noether theorem [19] gives us the formula

$$K^\mu_\phi = \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu \phi)}\frac{n-2}{2}\phi + \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}). \tag{66}$$

It is useful to evaluate each term in (66) separately to see its gauge invariance. In fact, the result is given by

$$\begin{aligned}\frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu \phi)}\frac{n-2}{2}\phi &= -\frac{n-2}{4}\hat{\phi}^2 \hat{g}^{\mu\nu}\partial_\nu \log\left(\hat{\phi}^2\right), \\ \frac{\partial \hat{\mathcal{L}}_\phi}{\partial(\partial_\mu g_{\nu\rho})}(-2g_{\nu\rho}) &= \frac{n-2}{4}\hat{\phi}^2 \hat{g}^{\mu\nu}\partial_\nu \log\left(\hat{\phi}^2\right).\end{aligned} \tag{67}$$

As promised, each term is manifestly gauge invariant under the Weyl transformation, since it is expressed in terms of only gauge-invariant quantities. Adding the two terms in (67), we have

$$K^\mu_\phi = 0. \tag{68}$$

Thus, the Noether current for the Weyl symmetry in the WTDiff method is certainly vanishing,

$$J^\mu_\phi = K^\mu_\phi - X^\mu_\phi = 0. \tag{69}$$

The two results in (63) and (69) clearly account for that the Noether current for the Weyl symmetry is vanishing in both the WDiff-invariant scalar action and the WTDiff-invariant one.

3.3 Vector field

Next, we will move on to spin 1 abelian gauge field, that is, the electro-magnetic field. It is well known that the Maxwell action for the electro-magnetic field is invariant in only four space-time dimensions, but not so in an arbitrary space-time dimension. It is therefore necessary to extend the Maxwell action in four dimensions in such a way it is also invariant under the Weyl transformation in general dimensions. We are now accustomed to the recipe for accomplishing this work: Start with a Diff-invariant action and then replace all fields with the corresponding Weyl-invariant fields, by which we have the WDiff-invariant action. Furthermore, the WTDiff-invariant action is obtained by selecting the gauge condition (6) for the longitudinal diffeomorphism. According to this recipe, let us start with the conventional Maxwell action which is invariant under Diff in n space-time dimensions:

$$S_A = -\frac{1}{4}\int \mathrm{d}^n x |g|^{\frac{1}{2}} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}, \tag{70}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. The Weyl transformation for the vector field is defined as usual

$$A_\mu \to A'_\mu = A_\mu. \tag{71}$$

Then the WDiff-invariant action reads

$$\begin{aligned}\hat{S}_A &= \int \mathrm{d}^n x \hat{\mathcal{L}}_A \\ &= -\frac{1}{4}\int \mathrm{d}^n x |\hat{g}|^{\frac{1}{2}} \hat{g}^{\mu\nu} \hat{g}^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}, \\ &= -\frac{1}{4}\int \mathrm{d}^n x |g|^{\frac{1}{2}} \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi\right)^{\frac{2(n-4)}{n-2}} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma},\end{aligned} \tag{72}$$

and the WTDiff-invariant action takes the form

$$\hat{S}_A = \int d^n x \hat{\mathcal{L}}_A = -\frac{1}{4}\int d^n x |g|^{\frac{2}{n}} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma}. \tag{73}$$

Based on these actions, it is again easy to evaluate the Noether current associated with the Weyl symmetry by the two methods. For instance, in the WTDiff method, since the WTDiff-invariant action is invariant without a surface term, we have

$$X_A^\mu = 0. \tag{74}$$

The Noether theorem [19] again produces the formula

$$\Lambda K_A^\mu = \frac{\partial \hat{\mathcal{L}}_A}{\partial(\partial_\mu A_\nu)} \delta A_\nu + \frac{\partial \hat{\mathcal{L}}_A}{\partial(\partial_\mu g_{\nu\rho})} \delta g_{\nu\rho}, \tag{75}$$

where Λ is the infinitesimal parameter for the Weyl transformation. Since $\delta A_\nu = \frac{\partial \hat{\mathcal{L}}_A}{\partial(\partial_\mu g_{\nu\rho})} = 0$, we soon reach the result

$$K_A^\mu = 0. \tag{76}$$

Hence, we have the vanishing Noether current

$$J_A^\mu = K_A^\mu - X_A^\mu = 0. \tag{77}$$

It is straightforward to derive the same result on the basis of the WDiff-invariant action (72).

3.4 Spinor field

Finally, as one of matter fields, let us consider the Dirac spinor field. It is well known that in general n space-time dimensions, the action for massless Dirac spinor fields is invariant under the Weyl transformation [9]. We find it useful to recall the symmetry properties of the Dirac action whose Lagrangian density is[4]

$$\mathcal{L}_\psi = -\frac{1}{2} e \bar{\psi} \left(\not{D} - \overleftarrow{\not{D}} \right) \psi - e m \bar{\psi}\psi = e \left(-\bar{\psi} e_a^\mu \gamma^a D_\mu \psi - m \bar{\psi}\psi \right), \tag{78}$$

where $e = \det e_{a\mu}$, $\not{D} = \gamma^\mu D_\mu$, and in the last equality we have used the integration by parts. In case of the massless Dirac field ($m = 0$), in general n space-time dimensions, the action $\int d^n x \mathcal{L}_\psi$ is invariant under the following Weyl transformation:

$$e_\mu^a \to e'^a_\mu = \Omega(x) e_\mu^a, \quad e^{a\mu} \to e'^{a\mu} = \Omega^{-1}(x) e^{a\mu}, \quad \psi \to \psi' = \Omega^{-\frac{n-1}{2}}(x)\psi. \tag{79}$$

In the presence of the scalar field φ, we can make even the mass term invariant under the Weyl transformation. To do so, as before, it is sufficient to introduce the Weyl-invariant fields and then replace each field in the Lagrangian (78) by the corresponding Weyl-invariant field. The Weyl-invariant fields are given by

$$\hat{e}_a^\mu = \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi \right)^{-\frac{2}{n-2}} e_a^\mu, \quad \hat{\psi} = \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi \right)^{-\frac{n-1}{n-2}} \psi. \tag{80}$$

By replacing each field with the corresponding Weyl-invariant one in Eq. (78), we have the Weyl-invariant massive Dirac Lagrangian density $\hat{\mathcal{L}}_\psi$,

$$\hat{\mathcal{L}}_\psi = \hat{e}\left(-\hat{\bar{\psi}} \hat{e}_a^\mu \gamma^a \hat{D}_\mu \hat{\psi} - m \hat{\bar{\psi}}\hat{\psi} \right), = e\left[-\bar{\psi} e_a^\mu \gamma^a D_\mu \psi - \left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\varphi \right)^{\frac{2}{n-2}} m\bar{\psi}\psi \right]. \tag{81}$$

Let us note that the first term has the same expression as before since the massless Dirac Lagrangian density is invariant under the Weyl transformation. The action $\int d^n x \hat{\mathcal{L}}_\psi$ is invariant under both the Weyl transformation and the full group of Diff. To reduce the symmetries from WDiff to WTDiff, we will take the gauge condition (6) for the longitudinal diffeomorphism. The resulting Lagrangian density reads

$$\hat{\mathcal{L}}_\psi = e\left(-\bar{\psi} e_a^\mu \gamma^a D_\mu \psi - e^{-\frac{1}{n}} m \bar{\psi}\psi \right). \tag{82}$$

The Noether current for the Weyl symmetry should be calculated by using either action since the current is a gauge-invariant quantity. We will use the Lagrangian density (82), which is invariant under the Weyl transformation without surface terms, i.e.

$$X_\psi^\mu = 0. \tag{83}$$

The Noether theorem gives us the expression

$$\Lambda K_\psi^\mu = \frac{\partial^R \hat{\mathcal{L}}_\psi}{\partial(\partial_\mu \psi)} \delta\psi + \frac{\partial \hat{\mathcal{L}}_\psi}{\partial(\partial_\mu e_\nu^a)} \delta e_\nu^a, \tag{84}$$

where we have used the right-derivative notation with respect to the spinor field and the second-order formalism of gravity, that is, the spin connection has been regarded as a function of the vielbein. A straightforward calculation of each term in (84) yields

$$\frac{\partial^R \hat{\mathcal{L}}_\psi}{\partial(\partial_\mu \psi)} \delta\psi = -\Lambda \frac{n-1}{2} e \bar{\psi} e_a^\mu \gamma^a \psi, \quad \frac{\partial \hat{\mathcal{L}}_\psi}{\partial(\partial_\mu e_\nu^a)} \delta e_\nu^a = \Lambda \frac{n-1}{2} e \bar{\psi} e_a^\mu \gamma^a \psi, \tag{85}$$

both of which are gauge invariant as expected. Therefore, we have

$$K_\psi^\mu = 0. \tag{86}$$

[4] See Appendix A.2 for notation and some definitions related to spinors.

Accordingly, even in this case, we have the identically vanishing Noether current

$$J_\psi^\mu = K_\psi^\mu - X_\psi^\mu = 0. \tag{87}$$

To close this section, we should comment on the trace or Weyl (conformal) anomaly. It is well known that in a curved space-time, certain matter fields, such as the electro-magnetic field in four dimensions and massless Dirac fields in any dimensions, exhibit Weyl (local conformal) invariance at the classical level as mentioned above. Weyl invariance of the action implies that the trace of the energy-momentum tensor is zero. We are also familiar with the fact that a theory based on a classical action which is Weyl invariant in general loses its Weyl invariance in the quantum theory as a result of renormalization, i.e. owing to the existence of the renormalization scale. The energy-momentum tensor therefore acquires a non-zero trace, known as the trace or Weyl (conformal) anomaly [7,8].

However, this well-known result does not generally hold in the present formalism where there is the spurion field φ. In our formalism, we keep the situation in mind such that the conformally invariant scalar–tensor gravity coexists with the various conformally invariant matter fields. In this situation, the spurion field φ is assumed to be broken spontaneously $\varphi = \langle\varphi\rangle + \sigma$ where the massless "meson" σ is the Goldstone boson restoring conformal symmetry, even if there is no potential for triggering the spontaneous symmetry breakdown. Note that $\sigma = 0$ corresponds to the "unitary gauge" leading to general relativity or the WTDiff gravity depending on the choice of $\langle\varphi\rangle$. The key idea is that we can use the vacuum expectation value of the spurion field, $\langle\varphi\rangle$, as the renormalization scale instead of the conventional fixed renormalization scale μ [9–18]. With this idea, we have a conformally invariant effective potential without trace anomaly and the coupling constants still run with the momentum scale [15].

4 Schwarzschild solution

In this section, we wish to show that the Schwarzschild metric is a classical solution to the equations of motion of the WTDiff gravity, Eq. (15), or equivalently Eq. (26). Before doing so, we soon realize that a notable feature of Eq. (15) is that the traceless Einstein tensor defined as $G_{\mu\nu}^T = R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$ in the LHS has a beautiful geometrical structure, whereas the traceless energy-momentum tensor $T_{(g)\mu\nu}^T = T_{(g)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(g)}$ in the RHS has a complicated expression, and the presence of the metric determinant g and its derivative $\nabla_\mu \nabla_\nu |g|$ reflects the fact that the equations of motion are not invariant under Diff, but only TDiff. It is therefore natural to fix the Weyl symmetry first by the gauge condition

$$g = -1, \tag{88}$$

which is nothing but the unimodular condition (9). Since the traceless energy-momentum tensor in the RHS of Eq. (15) trivially vanishes, the resultant equations of motion read

$$G_{\mu\nu}^T \equiv R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = 0. \tag{89}$$

The space-time defined by Eq. (89) is called an Einstein space in four dimensions. The Riemannian spaces which are conformally related to Einstein spaces have been addressed for a long time [50].

Now we wish to show that the Schwarzschild metric in the Cartesian coordinate system is a classical solution to the equations of motion (89). For this purpose, we will look for a gravitational field outside an isolated, static, spherically symmetric object with mass M. In the far region from the isolated object, we assume that the metric tensor is in an asymptotically Lorentzian form,

$$g_{\mu\nu} \to \eta_{\mu\nu} + \mathcal{O}\left(\frac{1}{r^{n-3}}\right), \tag{90}$$

where $\eta_{\mu\nu}$ is the Minkowski metric, and the radial coordinate r is defined as

$$r = \sqrt{(x^1)^2 + (x^2)^2 + \cdots + (x^{n-1})^2} = \sqrt{(x^i)^2}, \tag{91}$$

with i running over spatial coordinates ($i = 1, 2, \ldots, n-1$).

Let us recall that the most spherically symmetric line element in n space-time dimensions reads

$$\begin{aligned} ds^2 = &-A(r)dt^2 + B(r)(x^i dx^i)^2 + C(r)(dx^i)^2 \\ &+ D(r)dt\, x^i dx^i, \end{aligned} \tag{92}$$

where $A(r)$ and $C(r)$ are positive functions depending on only r. Requiring the invariance under the time reversal $t \to -t$ leads to $D = 0$. As is well known, we can set $C(r) = 1$ by redefining the radial coordinate r [51]. Thus, the line element under consideration takes the form in the Cartesian coordinate system

$$ds^2 = -A(r)dt^2 + (dx^i)^2 + B(r)(x^i dx^i)^2. \tag{93}$$

From this line element (93), the non-vanishing components of the metric tensor are given by

$$g_{tt} = -A, \quad g_{ij} = \delta_{ij} + B x^i x^j, \tag{94}$$

and the components of its inverse matrix are

$$g^{tt} = -\frac{1}{A}, \quad g^{ij} = \delta^{ij} - \frac{B}{1+Br^2} x^i x^j. \tag{95}$$

Moreover, using these components of the metric tensor, the affine connection is calculated to be

$$\Gamma^t_{ti} = \frac{A'}{2A}\frac{x^i}{r}, \quad \Gamma^i_{tt} = \frac{A'}{2(1+Br^2)}\frac{x^i}{r},$$

$$\Gamma^i_{jk} = \frac{1}{2(1+Br^2)} \frac{x^i}{r}(2Br\delta_{jk} + B' x^j x^k), \tag{96}$$

where the prime denotes the differentiation with respect to r, for instance, $A' = \frac{dA}{dr}$.

Here let us take the gauge condition (88) for the Weyl transformation into consideration. By means of the metric tensor (94), the gauge condition (88) is cast into the form

$$A(1+Br^2) = 1. \tag{97}$$

Using this gauge condition (97) and Eqs. (94)–(96), the Ricci tensor and the scalar curvature can easily be calculated to be

$$\begin{aligned}
R_{tt} &= \frac{1}{2} A \left(A'' + \frac{n-2}{r} A' \right), \\
R_{ij} &= \left[\frac{n-3}{r^2}(1-A) - \frac{A'}{r} \right] \delta_{ij} \\
&\quad + \frac{1}{r^2} \left[\frac{n-3}{r^2}(A-1) + \frac{1}{r}\frac{A'}{A}(1-\frac{n}{2}+A) - \frac{1}{2}\frac{A''}{A} \right] x^i x^j, \\
R &= -A'' - \frac{2(n-2)}{r} A' - \frac{(n-2)(n-3)}{r^2}(A-1).
\end{aligned} \tag{98}$$

These results produce the concrete expressions for the non-vanishing components of the traceless Einstein tensor $G^T_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$,

$$\begin{aligned}
G^T_{tt} &= \left(\frac{1}{2} - \frac{1}{n} \right) A \left[A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) \right], \\
G^T_{ij} &= \left\{ \frac{1}{n}\delta_{ij} + \frac{1}{r^2}\frac{1}{A} \left[-\frac{1}{2} + \frac{1}{n}(1-A) \right] x^i x^j \right\} \\
&\quad \left[A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) \right].
\end{aligned} \tag{99}$$

Consequently, Eq. (89) reduces to the equation

$$A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) = 0. \tag{100}$$

Noticing that the LHS of Eq. (100) can be rewritten as

$$\begin{aligned}
&A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) \\
&= \frac{1}{r^{n-3}}\frac{d^2}{dr^2}[r^{n-3}(A-1)] \\
&\quad - (n-2)\frac{1}{r^{n-2}}\frac{d}{dr}[r^{n-3}(A-1)],
\end{aligned} \tag{101}$$

Eq. (100) is easily solved to be

$$A(r) = 1 - \frac{2M}{r^{n-3}} + ar^2, \tag{102}$$

where M and a are integration constants. From the boundary condition (90), we have to choose $a = 0$, and we can obtain the expression for $B(r)$ in terms of the gauge condition (97). Accordingly, we arrive at the expressions for $A(r)$ and $B(r)$:

$$A(r) = 1 - \frac{2M}{r^{n-3}}, \quad B(r) = \frac{2M}{r^2(r^{n-3} - 2M)}. \tag{103}$$

Then the line element is of form

$$\begin{aligned}
ds^2 &= -\left(1 - \frac{2M}{r^{n-3}}\right) dt^2 + (dx^i)^2 \\
&\quad + \frac{2M}{r^2(r^{n-3} - 2M)}(x^i dx^i)^2.
\end{aligned} \tag{104}$$

In this way, we have succeeded in showing that the Schwarzschild metric in the Cartesian coordinate system is a classical solution in the WTDiff gravity as in general relativity.

However, there is a caveat. The Schwarzschild metric in the Cartesian coordinate system, (104), can be rewritten in the spherical coordinate system as

$$ds^2 = -\left(1 - \frac{2M}{r^{n-3}}\right) dt^2 + \frac{1}{1 - \frac{2M}{r^{n-3}}} dr^2 + r^2 d\Omega^2_{n-2}, \tag{105}$$

where

$$d\Omega^2_{n-2} = d\theta_2^2 + \sin^2\theta_2 d\theta_3^2 + \cdots + \prod_{i=2}^{n-2} \sin^2\theta_i d\theta^2_{n-1}. \tag{106}$$

This form of the Schwarzschild metric is very familiar to physicists, but this is not a classical solution to the equations of motion of the WTDiff gravity, (89). The reason is that when transforming from the Cartesian coordinates to the spherical coordinates, we have a non-vanishing Jacobian factor which is against TDiff. To put it differently, while the determinant of the metric tensor in Eq. (104) is -1, the one in Eq. (105) is not so, which is against the gauge condition (88). In order to show that Eq. (105) is also a classical solution, one has to solve the equations of motion under the condition $g \neq -1$, which is at present a difficult task due to the complicated structure of the energy-momentum tensor.

5 Charged black hole solution

In the previous section, we have investigated classical solutions in the WTDiff gravity and found that the Schwarzschild metric is indeed a classical solution to the equations of motion of the WTDiff gravity. A study of the Schwarzschild solution is of physical importance since the Schwarzschild solution corresponds to the basic one-body problem of classical astronomy, and the reliable experimental verifications of the Einstein equations are almost all based on the Schwarzschild line element. Then it is natural to ask ourselves whether a charged black hole metric is also a classical solution to the equations of motion of the WTDiff gravity coupled to an electro-magnetic field or not. In this section, we will prove that it is indeed the case in general n space-time dimensions.

Our starting action is the sum of the WTDiff gravity action (14) and the WTDiff-invariant Maxwell action (73)

$$S = \int d^n x \left\{ \frac{1}{2}|g|^{\frac{1}{n}} \left[R + \frac{(n-1)(n-2)}{4n^2} \frac{1}{|g|^2} g^{\mu\nu} \partial_\mu |g| \partial_\nu |g| \right] - \frac{1}{4}|g|^{\frac{2}{n}} g^{\mu\nu} g^{\rho\sigma} F_{\mu\rho} F_{\nu\sigma} \right\}. \tag{107}$$

It is worthwhile to point out that although the WTDiff gravity has been already shown to be equivalent to general relativity, the WTDiff-invariant Maxwell action for the vector field A_μ is not equivalent to the conventional Maxwell action except in four dimensions and is its Weyl-invariant generalization. Thus, it is a nontrivial task to examine whether the action (107) possesses a charged black hole solution as a classical solution.

It is straightforward to derive the equations of motion for the gauge field A_μ and the metric tensor $g_{\mu\nu}$. The result is given by

$$\partial_\mu (|g|^{\frac{2}{n}} F^{\mu\nu}) = 0 \tag{108}$$

and

$$R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R = T_{(g,A)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(g,A)}, \tag{109}$$

where the energy-momentum tensor $T_{(g,A)\mu\nu}$ is defined as

$$T_{(g,A)\mu\nu} = \frac{(n-2)(2n-1)}{4n^2} \frac{1}{|g|^2} \partial_\mu |g| \partial_\nu |g| - \frac{n-2}{2n} \frac{1}{|g|} \nabla_\mu \nabla_\nu |g| + |g|^{\frac{1}{n}} F_{\mu\alpha} F_\nu{}^\alpha. \tag{110}$$

This energy-momentum tensor $T_{(g,A)\mu\nu}$ is not covariantly conserved as $T_{(g)\mu\nu}$ in Eq. (16), but it is possible to construct a covariantly conserved energy-momentum tensor as before. Along the same line of arguments as in Sect. 2, the covariantly conserved energy-momentum tensor is found to be

$$T_{\mu\nu} = T_{(g,A)\mu\nu} + \frac{n-2}{2n} g_{\mu\nu} \left[-\frac{5n-3}{4n} \frac{1}{|g|^2} (\partial_\rho |g|)^2 + \frac{1}{|g|} \nabla_\rho \nabla^\rho |g| \right] - \frac{1}{4} |g|^{\frac{1}{n}} g_{\mu\nu} F_{\rho\sigma} F^{\rho\sigma}. \tag{111}$$

Moreover, as expected from Weyl invariance of the WTDiff-invariant Maxwell action, this covariantly conserved energy-momentum tensor $T_{\mu\nu}$ satisfies the relation

$$T_{\mu\nu} - \frac{1}{n} g_{\mu\nu} T = T_{(g,A)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(g,A)}. \tag{112}$$

Hence, as in the case of the absence of the electro-magnetic field, even if we add the electro-magnetic field to the WTDiff gravity, we can derive the standard Einstein equations (29) where the cosmological constant appears as an integration constant.

Since we want to find a charged black hole solution, we look for a gravitational field outside an isolated, static, spherically symmetric object with mass M and electric charge Q. We again take the asymptotically Lorentzian space-time, Eq. (90), as a boundary condition for the metric tensor. We also work with the line element (93), and take the unimodular condition (88) as a gauge condition for the Weyl symmetry, so in this case we have perfectly the same equations as Eqs. (93)–(99). With the unimodular condition (88) for the Weyl symmetry, the Maxwell equation and the energy-momentum tensor are, respectively, reduced to the form

$$\partial_\mu F^{\mu\nu} = 0, \tag{113}$$

$$T_{(g,A)\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha. \tag{114}$$

As for the electro-magnetic field $A_\mu(x)$, we assume that it has a static, spherically symmetric electric potential

$$A_t = -\phi(r), \quad A_i = 0, \tag{115}$$

where $\phi(r)$ is a function of r. First, let us solve the Maxwell equation (113). With the ansatzes (93) and (115), the Maxwell equation (113) is cast into a single equation,

$$\frac{d}{dr}(r^{n-2}\phi') = 0, \tag{116}$$

which is easily integrated to be

$$\phi(r) = \sqrt{\frac{n-2}{n-3}} \frac{Q}{r^{n-3}} + c, \tag{117}$$

where Q, which corresponds to an electric charge, and c are integration constants. To fix the constant c, we will impose the boundary condition

$$\lim_{r \to \infty} \phi(r) = 0, \tag{118}$$

which uniquely determines $c = 0$. Thus, we obtain the final expression for $\phi(r)$

$$\phi(r) = \sqrt{\frac{n-2}{n-3}} \frac{Q}{r^{n-3}}. \tag{119}$$

Next, let us try to solve the *traceless* Einstein equations (109) with the unimodular gauge condition (88). For this purpose, we will calculate the *traceless* energy-momentum tensor defined as $T^T_{(g,A)\mu\nu} \equiv T_{(g,A)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(g,A)}$ whose result is summarized as

$$T^T_{(g,A)tt} = A \frac{(n-2)^2(n-3)}{n} \frac{Q^2}{r^{2(n-2)}},$$

$$T^T_{(g,A)ij} = \frac{2(n-2)(n-3)}{n} \times \left(\delta_{ij} - \frac{A + \frac{n}{2} - 1}{A} \frac{x^i x^j}{r^2} \right) \frac{Q^2}{r^{2(n-2)}}. \tag{120}$$

Consequently, the *traceless* Einstein equations (109) reduce to the equation

$$A'' + (n-4)\frac{1}{r}A' - 2(n-3)\frac{1}{r^2}(A-1) - 2(n-2)(n-3)\frac{Q^2}{r^{2(n-2)}} = 0. \tag{121}$$

This equation can be rewritten as

$$\frac{1}{r^{n-3}}\frac{d^2}{dr^2}\left[r^{n-3}\left(A - 1 - \frac{Q^2}{r^{2(n-3)}}\right)\right] - (n-2)\frac{1}{r^{n-2}}\frac{d}{dr}\left[r^{n-3}\left(A - 1 - \frac{Q^2}{r^{2(n-3)}}\right)\right] = 0. \tag{122}$$

By performing an integration, $A(r)$ turns out to be

$$A(r) = 1 - \frac{2M}{r^{n-3}} + \frac{Q^2}{r^{2(n-3)}} + ar^2, \tag{123}$$

where M and a are integration constants. From the boundary condition (90), we have to choose $a = 0$, and we can obtain the expression for $B(r)$ in terms of the gauge condition (97). As a result, we obtain the expressions for $A(r)$ and $B(r)$,

$$A(r) = 1 - \frac{2M}{r^{n-3}} + \frac{Q^2}{r^{2(n-3)}}, \quad B(r) = \frac{2Mr^{n-3} - Q^2}{r^2\left[r^{2(n-3)} - 2Mr^{n-3} + Q^2\right]}. \tag{124}$$

Then the line element is of the form

$$ds^2 = -\left[1 - \frac{2M}{r^{n-3}} + \frac{Q^2}{r^{2(n-3)}}\right]dt^2 + (dx^i)^2 + \frac{2Mr^{n-3} - Q^2}{r^2\left[r^{2(n-3)} - 2Mr^{n-3} + Q^2\right]}(x^i dx^i)^2. \tag{125}$$

Hence, we have shown that the charged black hole metric in the Cartesian coordinate system is indeed a classical solution in the WTDiff gravity coupled to the WTDiff-invariant Maxwell theory in an arbitrary space-time dimension.

Again we should make an important remark. The charged black hole metric (125) in the Cartesian coordinate system can be rewritten in a more familiar form in the spherical coordinate system,

$$ds^2 = -\left[1 - \frac{2M}{r^{n-3}} + \frac{Q^2}{r^{2(n-3)}}\right]dt^2 + \frac{1}{1 - \frac{2M}{r^{n-3}} + \frac{Q^2}{r^{2(n-3)}}}dr^2 + r^2 d\Omega_{n-2}^2. \tag{126}$$

However, Eq. (126) is *not* a classical solution in the WTDiff gravity plus the WTDiff-invariant Maxwell theory. This situation is very similar to the Schwarzschild black hole metric. Namely, the dependence of classical solutions on the chosen coordinate system is a notable feature of the WTDiff gravity where there is no the full group of diffeomorphisms, but only TDiff.

6 Cosmology

As a final application of the classical WTDiff gravity, we would like to consider cosmology in the WTDiff gravity coupled with the WTDiff-invariant scalar matter. Before attempting to solve the *traceless* Einstein equations, following the same method as before we can construct the energy-momentum tensor satisfying the covariant conservation law in this case as well. An interesting point here is that such a covariantly conserved energy-momentum tensor plays a critical role in the construction of classical solutions, which should be contrasted to the cases treated thus far where the covariantly conserved energy-momentum tensors are only needed to have a connection with the standard Einstein equations.

The action with which we begin is the sum of the WTDiff gravity action (14) and the WTDiff-invariant scalar action (64),

$$S = \int d^n x \left\{ \frac{1}{2}|g|^{\frac{1}{n}}\left[R + \frac{(n-1)(n-2)}{4n^2}\frac{1}{|g|^2}g^{\mu\nu}\partial_\mu|g|\partial_\nu|g|\right] - \frac{n-2}{8(n-1)}|g|^{\frac{1}{2}}g^{\mu\nu}\left[\partial_\mu\phi\partial_\nu\phi + \frac{n-2}{2n}\frac{\phi}{|g|}\partial_\mu|g|\partial_\nu\phi + \frac{(n-2)^2}{16n^2}\frac{\phi^2}{|g|^2}\partial_\mu|g|\partial_\nu|g|\right] - V\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}|g|^{\frac{n-2}{4n}}\phi\right)\right\}. \tag{127}$$

From this action, the equations of motion for the scalar field and the metric tensor field are, respectively, calculated to be

$$-\frac{1}{8}\frac{n-2}{n-1}|g|^{\frac{1}{2}}\left[\frac{(n-2)(5n-2)}{8n^2}\frac{\phi}{|g|^2}(\partial_\rho|g|)^2 - \frac{n-2}{2n}\frac{\phi}{|g|}\nabla_\rho\nabla^\rho|g| - 2\nabla_\rho\nabla^\rho\phi\right] - \frac{1}{2}\sqrt{\frac{n-2}{n-1}}|g|^{\frac{n-2}{4n}}V'\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}|g|^{\frac{n-2}{4n}}\phi\right) = 0, \tag{128}$$

with $V'(\phi) \equiv \frac{dV(\phi)}{d\phi}$, and

$$R_{\mu\nu} - \frac{1}{n}g_{\mu\nu}R = T_{(g,\phi)\mu\nu} - \frac{1}{n}g_{\mu\nu}T_{(g,\phi)}, \tag{129}$$

where the energy-momentum tensor $T_{(g,\phi)\mu\nu}$ is defined as

$$T_{(g,\phi)\mu\nu} = \frac{(n-2)(2n-1)}{4n^2}\frac{1}{|g|^2}\partial_\mu|g|\partial_\nu|g| - \frac{n-2}{2n}\frac{1}{|g|}\nabla_\mu\nabla_\nu|g| + \frac{1}{4}\frac{n-2}{n-1}\left(\partial_\mu\phi + \frac{n-2}{4n}\frac{\phi}{|g|}\partial_\mu|g|\right) \times \left(\partial_\nu\phi + \frac{n-2}{4n}\frac{\phi}{|g|}\partial_\nu|g|\right). \tag{130}$$

In deriving the energy-momentum tensor (130), we have used the equation of motion for ϕ, (128).

The energy-momentum tensor $T_{(g,\phi)\mu\nu}$ is not covariantly conserved, either, but it is again possible to construct a covariantly conserved energy-momentum tensor as before. The covariantly conserved energy-momentum tensor is now given by

$$\begin{aligned} T_{\mu\nu} &= T_{(g,\phi)\mu\nu} + \frac{n-2}{2n} g_{\mu\nu} \\ &\quad \times \left[-\frac{5n-3}{4n} \frac{1}{|g|^2} (\partial_\rho |g|)^2 + \frac{1}{|g|} \nabla_\rho \nabla^\rho |g| \right] \\ &\quad + g_{\mu\nu} \left[-\frac{1}{8} \frac{n-2}{n-1} \left(\partial_\rho \phi + \frac{n-2}{4n} \frac{\phi}{|g|} \partial_\rho |g| \right)^2 \right. \\ &\quad \left. - |g|^{-\frac{1}{2}} V \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} |g|^{\frac{n-2}{4n}} \phi \right) \right]. \end{aligned} \tag{131}$$

It turns out that this covariantly conserved energy-momentum tensor $T_{\mu\nu}$ satisfies the desired relation,

$$T_{\mu\nu} - \frac{1}{n} g_{\mu\nu} T = T_{(g,\phi)\mu\nu} - \frac{1}{n} g_{\mu\nu} T_{(g,\phi)}. \tag{132}$$

Hence, although we add the scalar field to the WTDiff gravity, we can derive the standard Einstein equations (29) where the cosmological constant appears as an integration constant.

To simplify the energy-momentum tensor, we will again select the unimodular condition (88) as a gauge condition for the Weyl symmetry. This choice of the gauge condition provides us with an enormous simplification, since the energy-momentum tensor (131) is reduced to the tractable form

$$\begin{aligned} T_{\mu\nu} &= \frac{1}{4} \frac{n-2}{n-1} \partial_\mu \phi \partial_\nu \phi + g_{\mu\nu} \\ &\quad \times \left[-\frac{1}{8} \frac{n-2}{n-1} (\partial_\rho \phi)^2 - V \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \phi \right) \right]. \end{aligned} \tag{133}$$

We now proceed to the study of cosmological solutions. It is usually assumed that our universe is described in terms of an expanding, homogeneous and isotropic Friedmann–Lemaitre–Robertson–Walker (FLRW) universe, given by the line element

$$\begin{aligned} ds^2 &= g_{\mu\nu} dx^\mu dx^\nu \\ &= -dt^2 + a^2(t) \gamma_{ij}(x) dx^i dx^j, \end{aligned} \tag{134}$$

where $a(t)$ is a scale factor and $\gamma_{ij}(x)$ is the spatial metric of the unit $(n-1)$-sphere, unit $(n-1)$-hyperboloid or $(n-1)$-plane, and i, j run over spatial coordinates ($i = 1, 2, \ldots, n-1$). However, this metric ansatz does not satisfy the gauge condition (88), so the line element should be somewhat modified. A suitable modification, which satisfies the gauge condition (88), is to consider the following line element:

$$ds^2 = -N^2(t) dt^2 + a^2(t) (dx^i)^2, \tag{135}$$

where $N(t)$ is a lapse function and the spatial geometry is chosen to be the $(n-1)$-plane, i.e. the $(n-1)$-dimensional Euclidean space. Note that the existence of the lapse function $N(t)$ means that the time coordinate t does not coincide with the proper time of particles at rest. With this line element, the gauge condition (88) provides a relation between the lapse function $N(t)$ and the scale factor $a(t)$,

$$N(t) = a^{-(n-1)}(t). \tag{136}$$

Given the line element (135) and Eq. (136), it turns out that the non-vanishing components of the *traceless* Einstein tensor defined as $G^T_{\mu\nu} = R_{\mu\nu} - \frac{1}{n} g_{\mu\nu} R$ are given by

$$\begin{aligned} G^T_{tt} &= -\frac{(n-1)(n-2)}{n} \left[\dot{H} + (n-1) H^2 \right], \\ G^T_{ij} &= -\frac{n-2}{n} a^{2n} \left[\dot{H} + (n-1) H^2 \right] \delta_{ij}, \end{aligned} \tag{137}$$

where $H = \frac{\dot{a}}{a}$ is the Hubble parameter and we have defined $\dot{a} = \frac{da(t)}{dt}$. In a similar way, the non-vanishing components of the *traceless* energy-momentum tensor, which is defined as $T^T_{\mu\nu} = T_{\mu\nu} - \frac{1}{n} g_{\mu\nu} T$, read

$$\begin{aligned} T^T_{tt} &= \frac{n-2}{4n} (\dot{\phi})^2, \\ T^T_{ij} &= \frac{1}{n-1} \frac{n-2}{4n} a^{2n} (\dot{\phi})^2 \delta_{ij}, \end{aligned} \tag{138}$$

where we have specified the scalar field ϕ to be spatially homogeneous, that is, $\phi = \phi(t)$. As a result, the *traceless* Einstein equations are cast into the form of a single equation,

$$\dot{H} + (n-1) H^2 = -\frac{1}{4(n-1)} (\dot{\phi})^2. \tag{139}$$

Moreover, using the line element (135) and Eq. (136), the equation of motion for the scalar matter field ϕ, Eq. (128), is simplified to

$$\ddot{\phi} + 2(n-1) H \dot{\phi} + 2 \sqrt{\frac{n-1}{n-2}} a^{-2(n-1)} V' \left(\frac{1}{2} \sqrt{\frac{n-2}{n-1}} \phi \right) = 0. \tag{140}$$

It is of interest to see that the *traceless* Einstein equations have yielded only the single Eq. (139), which is similar to the Raychaudhuri equation or the first Friedmann equation [52,53], which comes from all ij-components of the Einstein equations in general relativity though there is a slight difference in Eq. (139), which will be commented on shortly. However, in the present formalism, the (second) Friedmann equation stemming from the 00-components of the Einstein equations is missing. In order to solve Eq. (139), we need the conservation law of the energy-momentum tensor. In this respect, recall that in general relativity the first Friedmann equation can be viewed as a consequence of the (second) Friedmann equation and covariant conservation of energy, so that the combination of the (second) Friedmann equation and the conservation law, supplemented by the equation of state $p = p(\rho)$ (which will appear later), forms a complete system of equations that determines the two unknown functions, the scale factor $a(t)$ and the energy density ρ. In our formalism, instead of the (second) Friedmann equation, we have to use the first Friedmann equation like Eq. (139).

At this stage, we meet a new situation: As mentioned above, to solve Eq. (139), we must set up the conservation law of the energy-momentum tensor as an additional equation. It is the covariantly conserved energy-momentum tensor $T_{\mu\nu}$ in Eq. (133) that we have to deal with in this context. So far the covariantly conserved energy-momentum tensors are needed to make contact with the standard Einstein equations, but in the present situation, we must make use of the concrete expressions to find classical solutions. Then the non-vanishing components of $T_{\mu\nu}$ are easily evaluated to be

$$T^t{}_t = -\frac{1}{8}\frac{n-2}{n-1}a^{2(n-1)}(\dot{\phi})^2 - V\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\phi\right) \equiv -\rho(t),$$
$$T^i{}_j = \left[\frac{1}{8}\frac{n-2}{n-1}a^{2(n-1)}(\dot{\phi})^2 - V\left(\frac{1}{2}\sqrt{\frac{n-2}{n-1}}\phi\right)\right]\delta^i{}_j \equiv p(t)\delta^i{}_j, \tag{141}$$

where we have introduced energy density $\rho(t)$ and pressure $p(t)$ in a conventional way. Using these expressions, the covariant conservation law $\nabla_\mu T^{\mu\nu} = 0$ leads to an equation

$$\dot{\rho} + (n-1)H(\rho + p) = 0. \tag{142}$$

To close the system of equations, which determines the dynamics of homogeneous and isotropic universe, we have to specify the equation of state of matter as usual,

$$p = w\rho, \tag{143}$$

where w is a certain constant. Of course, the equation of state is not a consequence of equations of our formalism, but should be determined by the matter content in our universe. With the help of Eq. (143), Eq. (142) is exactly solved to be

$$\rho(t) = \rho_0 a^{-(n-1)(w+1)}(t), \tag{144}$$

where ρ_0 is an integration constant. Equations (142)–(144) are the same expressions as in general relativity. Now, using Eqs. (141), (143) and (144), our Friedmann equation (139) is rewritten as

$$\dot{H} + (n-1)H^2 = -\frac{w+1}{n-2}\rho_0 a^{-(n-1)(w+3)}. \tag{145}$$

Since it is difficult to find a general solution to this Eq. (145), we will refer only to special solutions which are physically interesting. Looking at the RHS in Eq. (145), one soon notices that at $w = -1$ and $w = -3$, specific situations occur. Actually, at $w = -1$, Eq. (145) can be exactly integrated to be

$$a(t) = a_0 t^{\frac{1}{n-1}}, \tag{146}$$

where a_0 is an integration constant and this solution describes the decelerating universe in four dimensions owing to $\ddot{a} < 0$.

In the case $w = -3$, Eq. (145) is reduced to the form

$$\dot{H} + (n-1)H^2 = \frac{2}{n-2}\rho_0. \tag{147}$$

This equation includes a special solution describing an exponentially expanding universe,

$$a(t) = a_0 e^{H_0 t}, \tag{148}$$

where H_0 is a constant defined as

$$H_0 = \sqrt{\frac{2\rho_0}{(n-1)(n-2)}}. \tag{149}$$

Finally, one can find a special solution such that the scale factor $a(t)$ has the form of a polynomial in t,

$$a(t) = a_0 t^\alpha, \tag{150}$$

where α is a constant to be determined by the Friedmann equation (145). It is easy to verify that the constant α is given by

$$\alpha = \frac{2}{(n-1)(w+3)}, \tag{151}$$

so that in this case the scale factor takes the form

$$a(t) = a_0 t^{\frac{2}{(n-1)(w+3)}}. \tag{152}$$

Then the accelerating universe $\ddot{a}(t) > 0$ requires

$$w < \frac{-3n+5}{n-1}, \tag{153}$$

while the decelerating universe requires

$$w > \frac{-3n+5}{n-1}. \tag{154}$$

One might wonder how the obtained solutions are related to the solutions in general relativity. In particular, in general relativity we are familiar with the fact that the case $w = -1$ corresponds to the cosmological constant and the solution is then an exponentially expanding universe, whereas in our case the corresponding solution belongs to the case $w = -3$, which appears to be strange. But this is just an illusion, since we do not use the conventional form (134) of the line element but the line element (135) involving the nontrivial lapse function $N(t)$.

In order to show that our result coincides with that in general relativity, let us focus our attention on the Friedmann equation (139). By means of Eq. (141), this equation is rewritten as

$$\dot{H} + (n-1)H^2 = -\frac{1}{n-2}N^2(\rho + p), \tag{155}$$

where we recovered the lapse function $N(t)$ by using Eq. (136).

On the other hand, with the conventional notation of the energy-momentum tensor

$$T^{\mu}{}_{\nu} = \mathrm{diag}(-\rho, p, \ldots, p), \tag{156}$$

and the line element (135), the Einstein equations in general relativity,

$$G^{\mu}{}_{\nu} \equiv R^{\mu}{}_{\nu} - \frac{1}{2}\delta^{\mu}{}_{\nu} R = T^{\mu}{}_{\nu}, \tag{157}$$

become a set of Friedmann equations,

$$H^2 = \frac{2}{(n-1)(n-2)} N^2 \rho, \tag{158}$$

$$\dot{H} + \frac{n-1}{2} H^2 - \frac{\dot{N}}{N} H = -\frac{1}{n-2} N^2 p. \tag{159}$$

By using Eq. (136), Eq. (159) is written as

$$\dot{H} + \frac{3(n-1)}{2} H^2 = -\frac{1}{n-2} N^2 p. \tag{160}$$

Equation (158) allows us to rewrite this equation in the form

$$\dot{H} + (n-1) H^2 = -\frac{1}{n-2} N^2 (\rho + p), \tag{161}$$

which precisely coincides with our Friedmann equation (155). This demonstration clearly indicates that our cosmological solution is just equivalent to that of general relativity specified in such a way that the line element is (135) and the lapse function is given by Eq. (136).

7 Discussions

In this article, we have clarified various classical aspects of the Weyl transverse (WTDiff) gravity in a general space-time dimension. We have found that the Schwarzschild black hole is a classical solution to the equations of motion of the WTDiff gravity when expressed in the Cartesian coordinate system. We have also shown that the Reissner–Nordstrom black hole is a solution in the same coordinate system in four space-time dimensions. The generalization to higher space-time dimensions has required us to extend the conventional Maxwell action to the Weyl-invariant action since the Maxwell action is invariant under the Weyl (local conformal) transformation only in four dimensions. It is of interest that even in such an extended electro-magnetic action plus the WTDiff gravity action in higher dimensions there is a charged black hole solution which shares all features with the conventional Reissner–Nordstrom charge black hole solution in four dimensions. Furthermore, we have investigated the Friedmann–Lemaitre–Robertson–Walker (FLRW) cosmology and seen that the FLRW cosmology is a classical solution when the shift factor has a nontrivial scale factor and the spatial geometry is flat.

In the classical analysis of the WTDiff gravity, a novel feature is the classical relation among three gravitational theories, those are, the conformally invariant scalar–tensor gravity, Einstein's general relativity and the Weyl transverse (WTDiff) gravity, in a general space-time dimension. To put it concretely, starting with the conformally invariant scalar–tensor gravity, which is invariant under both the local Weyl transformation and the diffeomorphisms (Diff), we have gauge-fixed the longitudinal diffeomorphism, by which the full diffeomorphisms (Diff) are broken to the transverse diffeomorphisms (TDiff), and we have obtained the WTDiff gravity. It is explicitly verified that not only the resultant action of the WTDiff gravity but also its equations of motion are invariant under both the local Weyl transformation and the TDiff. On the other hand, beginning with the conformally invariant scalar–tensor gravity and gauge-fixing the Weyl transformation have yielded general relativity, which is invariant under Diff. In this sense, the three gravitational theories are classically equivalent and we then conjecture that this equivalence holds even in the quantum regime. In other words, the conformally invariant scalar–tensor gravity is the underlying theory with the maximum symmetry behind Einstein's general relativity and the WTDiff gravity.

As a bonus, the equivalence of the three theories has made it possible to construct covariantly conserved energy-momentum tensors, by which we can prove that the traceless Einstein equations in the WTDiff gravity become equivalent with the standard Einstein equations in general relativity. Here one of the most remarkable things is that the cosmological constant emerges as an integration constant. This interesting phenomenon has already been observed in unimodular gravity and is expected to lead to a resolution to the cosmological constant problem. However, afterwards, it was revealed that this is not indeed the case for the following reason: In unimodular gravity, the unimodular condition plays an important role and this condition must be properly implemented by the method of Lagrange multipliers. Then it turns out that the Lagrange multiplier field is nothing but the cosmological constant and it receives huge radiative corrections.

On the other hand, in the WTDiff gravity under consideration, we have a chance of utilizing the phenomenon of the emergence of the cosmological constant as an integration constant for solving the cosmological constant problem. In the WTDiff gravity, we have neither additional conditions like the unimodular condition nor Lagrange multiplier fields, so we have no counterpart of the cosmological constant in the action. Moreover, the Weyl symmetry forbids the emergence of the cosmological constant of dimension zero in a quantum effective action, and if it were not violated at the quantum level, the cosmological constant appearing as an integration constant in the Einstein equations would keep its classical value in all energy scales. In this sense, the cosmological constant in the WTDiff gravity is radiatively stable. Thus, important remaining works amount to giving a proof that the *fake* Weyl symmetry is not broken by quantum effects and determining the initial value of the cosmological constant from some still unknown principle.

In a pioneering paper by Englert et al. [9], it is stated that the Weyl symmetry in the conformally invariant scalar–tensor gravity is free of Weyl anomaly when the Weyl symmetry is spontaneously broken, and this situation is unchanged when the Weyl-invariant matter fields are incorporated into the theory. Here "spontaneously broken" needs an explanation. Usually, it is necessary to have a Higgs potential to trigger the spontaneous symmetry breakdown, but it is in general difficult to set up such a potential for breaking the Weyl symmetry. Thus, as commented on around the end of Sect. 3, the meaning of "spontaneously broken" should be understood in the sense that the spurion field φ is assumed to be divided into two terms, $\varphi = \langle\varphi\rangle + \sigma$ where $\langle\varphi\rangle$ is the vacuum expectation value and σ is the Goldstone boson restoring conformal symmetry, respectively. Then the key technical idea in [9–18] is that the vacuum expectation value $\langle\varphi\rangle$ plays a role as the renormalization scale instead of the conventional fixed renormalization scale, by which the Weyl-invariant effective action can be obtained. Our conjecture that the fake Weyl symmetry has no anomaly is interpreted as a supplementary statement from the symmetry side, which supports this technical idea.

Anyway, as an important feature problem, we must understand quantum aspects of the WTDiff gravity. This is a very important step for the cosmological constant problem. We wish to consider this problem in the near future.

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Appendix A: Notation and conventions

A.1 Gravity

We follow the notation and conventions by Misner et al.'s textbook [54], for instance, the flat Minkowski metric $\eta_{ab} = \mathrm{diag}(-1, 1, 1, 1)$, the Riemann curvature tensor $R^\mu{}_{\nu\alpha\beta} = \partial_\alpha\Gamma^\mu_{\nu\beta} - \partial_\beta\Gamma^\mu_{\nu\alpha} + \Gamma^\mu_{\sigma\alpha}\Gamma^\sigma_{\nu\beta} - \Gamma^\mu_{\sigma\beta}\Gamma^\sigma_{\nu\alpha}$, and the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. The Latin indices label the flat space-time coordinates, while the Greek ones run over the curved space-time coordinates. The reduced Planck mass is defined as $M_p = \sqrt{\frac{c\hbar}{8\pi G}} = 2.4 \times 10^{18}$ GeV. Throughout this article, we adopt reduced Planck units where we set $c = \hbar = M_p = 1$. In these units, all quantities become dimensionless. Finally, note that in reduced Planck units, the Einstein–Hilbert Lagrangian density takes the form $\mathcal{L}_{EH} = \frac{1}{2}\sqrt{-g}R$.

A.2 Spinor

In this subsection, we gather some notation and definitions relevant to spinor fields. The Dirac spinor ψ is a $2^{[\frac{n}{2}]}$ dimensional spinor where $[\frac{n}{2}]$ is the Gauss symbol. The Clifford algebra is defined as $\{\gamma^a, \gamma^b\} = 2\eta^{ab}$. The gamma matrices in a curved space-time are related to those in a flat space-time with the help of the vielbein by $\gamma^\mu = e^\mu_a\gamma^a$. The metric tensor $g_{\mu\nu}$ is composed of the vielbein e^μ_a by the conventional relation $g_{\mu\nu} = \eta^{ab}e_{a\mu}e_{b\nu}$. Therefore, we have $\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu}$.

In writing down the Dirac action, we need to define the Dirac adjoint and the covariant derivative. The Dirac adjoint is defined as $\bar\psi = -i\psi^\dagger\gamma_{a=0} = i\psi^\dagger\gamma^{a=0}$ where $\gamma_{a=0}$ or $\gamma^{a=0}$ denotes the zero component of the flat space-time gamma matrices. Using the spin connection ω^{ab}_μ, the covariant derivative is of the form

$$D_\mu\psi = \left(\partial_\mu + \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\right)\psi, \tag{A.1}$$

where $\gamma_{ab} = \frac{1}{2}[\gamma_a, \gamma_b]$. Similarly, the covariant derivative for the Dirac adjoint can be derived from (A.1) to be

$$\bar\psi\overleftarrow{D}_\mu = \bar\psi\left(\overleftarrow{\partial}_\mu - \frac{1}{4}\omega^{ab}_\mu\gamma_{ab}\right). \tag{A.2}$$

We will use the torsion-free spin connection. Then it is defined through the Ricci rotation coefficient as

$$\omega_{a,bc} = e^\mu_a\omega_{\mu,bc} = \frac{1}{2}(\Delta_{a,bc} - \Delta_{b,ca} + \Delta_{c,ba}) = -\omega_{a,cb}, \tag{A.3}$$

where the Ricci rotation coefficient is defined as

$$\begin{aligned}\Delta_{a,bc} &= -\Delta_{a,cb} = (e^\mu_b e^\nu_c - e^\mu_c e^\nu_b)\partial_\nu e_{a\mu}\\ &= -e_{a\mu}(e^\nu_c\partial_\nu e^\mu_b - e^\nu_b\partial_\nu e^\mu_c).\end{aligned} \tag{A.4}$$

Appendix B: Proof of invariance

In this appendix, let us explicitly show that the action (14) and the equations of motion (15) are invariant under the Weyl transformation (3) and the transverse group of diffeomorphisms.

For this purpose, let us explain the transverse diffeomorphisms (TDiff) in more detail. Under the general coordinate transformation or Diff, the metric tensor transforms as

$$g_{\mu\nu}(x) \to g'_{\mu\nu}(x') = \frac{\partial x^\alpha}{\partial x^{\mu'}} \frac{\partial x^\beta}{\partial x^{\nu'}} g_{\alpha\beta}(x) \equiv J^\alpha_{\mu'} J^\beta_{\nu'} g_{\alpha\beta}(x), \tag{B.1}$$

where the Jacobian matrix $J^\alpha_{\mu'}$, which is defined as $J^\alpha_{\mu'} = \frac{\partial x^\alpha}{\partial x^{\mu'}}$, was introduced. Denoting the determinant of the Jacobian matrix as $J = \det J^\alpha_{\mu'} = \det \frac{\partial x^\alpha}{\partial x^{\mu'}}$, taking the determinant of Eq. (B.1) produces

$$g'(x') = J^2(x) g(x). \tag{B.2}$$

Then the transverse diffeomorphisms (TDiff), or equivalently the volume preserving diffeomorphisms, are defined as a subgroup of the full diffeomorphisms such that the determinant of the Jacobian matrix is the unity

$$J(x) = 1. \tag{B.3}$$

With this condition (B.3), the volume element is preserved under Diff, and Eq. (B.2) shows that $g(x)$ is a dimensionless scalar field. In the infinitesimal form of diffeomorphisms $x^\mu \to x^{\mu'} = x^\mu - \xi^\mu(x)$, using Eq. (B.3), TDiff can be expressed in terms of Eq. (12) since we can derive the following equation:

$$1 = J(x) = \det \frac{\partial x^\alpha}{\partial x^{\mu'}} = \det\left(\delta^\alpha_\mu + \partial_\mu \xi^\alpha\right) = e^{\mathrm{Tr}\log\left(\delta^\alpha_\mu + \partial_\mu \xi^\alpha\right)} = e^{\partial_\mu \xi^\mu}. \tag{B.4}$$

Armed with the knowledge of TDiff, we are ready to show explicitly that the action (14) and the equations of motion (15) of the WTDiff gravity are indeed invariant under both TDiff and Weyl transformation. In fact, under Diff, the Lagrangian density of (14) is transformed as

$$\mathcal{L}'(x') = \frac{1}{2}|J^2 g|^{\frac{1}{n}} \left[R + \frac{(n-1)(n-2)}{4n^2} \frac{1}{|g|^2} g^{\mu\nu} (\partial_\mu |g| + \frac{2|g|}{J}\partial_\mu J)(\partial_\nu |g| + \frac{2|g|}{J}\partial_\nu J) \right]. \tag{B.5}$$

It is obvious that the Lagrangian density $\mathcal{L}$ is not invariant under Diff owing to the presence of the terms with J while it is invariant under TDiff because of Eq. (B.3), which means that TDiff is in fact a symmetry of the action (14) of the WTDiff gravity. Now let us show that the traceless Einstein equations (15) are also invariant under TDiff. To do so, let us perform the general coordinate transformation to Eq. (15), whose result is described as

$$\begin{aligned} G^{T\prime}_{\mu\nu} - T^{T\prime}_{(g)\mu\nu} = J^\alpha_{\mu'} J^\beta_{\nu'} \Big\{ & G^T_{\alpha\beta} - T^T_{(g)\alpha\beta} + \frac{n-2}{2n} \\ & \times \left[\frac{1}{n}\frac{1}{J|g|}(\partial_\alpha J \partial_\beta |g| + \partial_\beta J \partial_\alpha |g|) + \frac{2(1-n)}{n}\frac{1}{J^2}\partial_\alpha J \partial_\beta J + \frac{2}{J} D_\alpha D_\beta J \right] \\ & - \frac{n-2}{n^2}\left[\frac{1}{n}\frac{1}{J|g|}\partial_\rho J \partial^\rho |g| + \frac{1-n}{n}\frac{1}{J^2}(\partial_\rho J)^2 + \frac{1}{J} D_\rho D^\rho J \right] g_{\alpha\beta} \Big\}. \end{aligned} \tag{B.6}$$

From this expression, we see that (15) is not invariant under Diff, but with $J = 1$, that is, under TDiff, it becomes invariant.

Next, we will prove Weyl invariance of the action (14) and the equations of motion (15). Under the Weyl transformation (3), the Lagrangian density of (14) is changed as

$$\mathcal{L}' = \mathcal{L} - (n-1)\partial_\mu \left(|g|^{\frac{1}{n}} g^{\mu\nu} \frac{1}{\Omega} \partial_\nu \Omega \right), \tag{B.7}$$

which implies that the WTDiff gravity is invariant under the Weyl transformation up to a surface term. Now, under the Weyl transformation, the traceless Einstein tensor $G^T_{\mu\nu}$ and $T^T_{(g)\mu\nu}$ are transformed by the same quantity

$$\begin{aligned} G^{T\prime}_{\mu\nu} &= G^T_{\mu\nu} + A^T_{\mu\nu}, \\ T^{T\prime}_{(g)\mu\nu} &= T^T_{(g)\mu\nu} + A^T_{\mu\nu}, \end{aligned} \tag{B.8}$$

where $A^T_{\mu\nu}$ is defined as

$$\begin{aligned} A^T_{\mu\nu} = 2(n-2)\frac{1}{\Omega^2}\left[\partial_\mu \Omega \partial_\nu \Omega - \frac{1}{n} g_{\mu\nu} (\partial_\rho \Omega)^2 \right] \\ - (n-2)\frac{1}{\Omega}\left[\nabla_\mu \nabla_\nu \Omega - \frac{1}{n} g_{\mu\nu} \nabla_\rho \nabla^\rho \Omega \right]. \end{aligned} \tag{B.9}$$

It is therefore obvious that Eq. (15) is invariant under the Weyl transformation.

Appendix C: Derivations of Eq. (15)

In this appendix, we will present two different derivations of the equations of motion (15) for the metric tensor in the WTDiff gravity.

C.1 Derivation from Eq. (20)

This derivation method utilizes the equivalence relation between the conformally invariant scalar–tensor gravity and the WTDiff gravity via the gauge-fixing procedure, and the fact that the equations of motion for the metric tensor in the WTDiff gravity are traceless equations.

As mentioned in the article, the equations of motion in the WTDiff gravity is entirely described in Eq. (20), or equivalently Eq. (18). The equivalence between the conformally invariant scalar–tensor gravity and the WTDiff gravity via the gauge-fixing procedure demands that the equations of motion in the WTDiff gravity should be obtained from Eq. (20) by substituting the gauge condition (6). After a straightforward calculation, we find that

$$G_{\mu\nu} = \frac{(n-2)(2n-1)}{4n^2}\frac{1}{|g|^2}\partial_\mu|g|\partial_\nu|g| - \frac{n-2}{2n}\frac{1}{|g|}\nabla_\mu\nabla_\nu|g| - \frac{(n-2)(5n-3)}{8n^2}g_{\mu\nu}\frac{1}{|g|^2}\left(\partial_\rho|g|\right)^2 + \frac{n-2}{2n}g_{\mu\nu}\frac{1}{|g|}\nabla_\rho\nabla^\rho|g|. \tag{C.1}$$

It is easy to see that taking its traceless part, i.e. calculating $G^T_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{n}g_{\mu\nu}R$, yields the equations of motion in the WTDiff gravity, Eq. (15) with the definition of the energy-momentum tensor (16).

C.2 Derivation from variation of WTDiff gravity action (14)

In this subsection, we will derive the equations of motion (15) of the WTDiff gravity by taking the variation for the metric tensor step by step.

Let us first divide the action of the WTDiff gravity, Eq. (14), into two parts

$$S = S_R + S_g = \int d^n x\mathcal{L}_R + \int d^n x\mathcal{L}_g, \tag{C.2}$$

where we have defined

$$\mathcal{L}_R = \frac{1}{2}|g|^{\frac{1}{n}}R, \quad \mathcal{L}_g = \frac{(n-1)(n-2)}{8n^2}|g|^{\frac{1}{n}-2}g^{\mu\nu}\partial_\mu|g|\partial_\nu|g|. \tag{C.3}$$

Using the formulas

$$\delta|g| = -|g|g_{\mu\nu}\delta g^{\mu\nu}, \quad \delta R = R_{\mu\nu}\delta g^{\mu\nu} + (g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}, \tag{C.4}$$

the metric variation of $\mathcal{L}_R$ reads

$$\delta\mathcal{L}_R = \frac{1}{2}|g|^{\frac{1}{n}}G^T_{\mu\nu}\delta g^{\mu\nu} + \frac{1}{2}|g|^{\frac{1}{n}}(g_{\mu\nu}\Box - \nabla_\mu\nabla_\nu)\delta g^{\mu\nu}. \tag{C.5}$$

Next, let us divide the second term in (C.5) into two parts and evaluate each term separately

$$I_1 = \frac{1}{2}|g|^{\frac{1}{n}}g_{\mu\nu}\Box\delta g^{\mu\nu}, \quad I_2 = -\frac{1}{2}|g|^{\frac{1}{n}}\nabla_\mu\nabla_\nu\delta g^{\mu\nu}. \tag{C.6}$$

In what follows, to convert the covariant derivative ∇_μ to the partial derivative ∂_μ we repeatedly use the well-known formula

$$|g|^{\frac{1}{2}}\nabla_\mu A^\mu = \partial_\mu\left(|g|^{\frac{1}{2}}A^\mu\right), \tag{C.7}$$

where A^μ is a generic vector field which includes $\partial^\mu|g|$ and $\nabla_\nu\delta g^{\mu\nu}$ etc. We will give a detailed derivation of I_1 below and only give the result of I_2, since the calculation of I_2 is similar to that of I_1. Neglecting total derivative terms and using Eq. (C.7) twice, we can proceed to calculate I_1 as follows:

$$\begin{aligned} I_1 &= \frac{1}{2}|g|^{\frac{1}{n}-\frac{1}{2}}|g|^{\frac{1}{2}}\nabla_\rho\left(g_{\mu\nu}g^{\rho\sigma}\nabla_\sigma\delta g^{\mu\nu}\right) \\ &= \frac{1}{2}|g|^{\frac{1}{n}-\frac{1}{2}}\partial_\rho\left(|g|^{\frac{1}{2}}g_{\mu\nu}g^{\rho\sigma}\nabla_\sigma\delta g^{\mu\nu}\right) \\ &= -\frac{1}{2}\left(\frac{1}{n}-\frac{1}{2}\right)|g|^{\frac{1}{n}-\frac{3}{2}}\partial_\rho|g||g|^{\frac{1}{2}}g_{\mu\nu}g^{\rho\sigma}\nabla_\sigma\delta g^{\mu\nu} \\ &= \frac{n-2}{4n}|g|^{\frac{1}{n}-\frac{3}{2}}\Big[|g|^{\frac{1}{2}}\nabla_\sigma\left(\partial_\rho|g|g_{\mu\nu}g^{\rho\sigma}\delta g^{\mu\nu}\right) \\ &\quad -|g|^{\frac{1}{2}}\nabla_\rho\nabla_\sigma|g|g_{\mu\nu}g^{\rho\sigma}\delta g^{\mu\nu}\Big] \\ &= \frac{n-2}{4n}|g|^{\frac{1}{n}-\frac{3}{2}}\Big[\partial_\sigma\left(|g|^{\frac{1}{2}}\partial_\rho|g|g_{\mu\nu}g^{\rho\sigma}\delta g^{\mu\nu}\right) \\ &\quad -|g|^{\frac{1}{2}}\nabla_\rho\nabla_\sigma|g|g_{\mu\nu}g^{\rho\sigma}\delta g^{\mu\nu}\Big] \\ &= \frac{n-2}{4n}\Big[-\left(\frac{1}{n}-\frac{3}{2}\right)|g|^{\frac{1}{n}-\frac{5}{2}}\partial_\sigma|g||g|^{\frac{1}{2}}\partial_\rho|g|g_{\mu\nu}g^{\rho\sigma}\delta g^{\mu\nu} \\ &\quad -|g|^{\frac{1}{n}-1}g_{\mu\nu}\nabla_\rho\nabla^\rho|g|\delta g^{\mu\nu}\Big] \\ &= \frac{n-2}{4n}|g|^{\frac{1}{n}}g_{\mu\nu}\delta g^{\mu\nu}\left[\frac{3n-2}{2n}\frac{1}{|g|^2}(\partial_\rho|g|)^2 - \frac{1}{|g|}\nabla_\rho\nabla^\rho|g|\right]. \end{aligned} \tag{C.8}$$

In a perfectly similar way, we have

$$I_2 = -\frac{n-2}{4n}|g|^{\frac{1}{n}}\delta g^{\mu\nu}\left[\frac{3n-2}{2n}\frac{1}{|g|^2}\partial_\mu|g|\partial_\nu|g| - \frac{1}{|g|}\nabla_\mu\nabla_\nu|g|\right]. \tag{C.9}$$

Then from Eqs. (C.5), (C.8) and (C.9), the variation of S_R with respect to the metric tensor becomes

$$\begin{aligned} \frac{\delta S_R}{\delta g^{\mu\nu}} &= \frac{1}{2}|g|^{\frac{1}{n}}G^T_{\mu\nu} + \frac{n-2}{4n}|g|^{\frac{1}{n}}g_{\mu\nu} \\ &\quad \times\left[\frac{3n-2}{2n}\frac{1}{|g|^2}(\partial_\rho|g|)^2 - \frac{1}{|g|}\nabla_\rho\nabla^\rho|g|\right] \\ &\quad -\frac{n-2}{4n}|g|^{\frac{1}{n}}\left[\frac{3n-2}{2n}\frac{1}{|g|^2}\partial_\mu|g|\partial_\nu|g| - \frac{1}{|g|}\nabla_\mu\nabla_\nu|g|\right]. \end{aligned} \tag{C.10}$$

The variation of S_g with respect to the metric tensor can be calculated in a similar manner to be

$$\frac{\delta S_g}{\delta g^{\mu\nu}} = \frac{(n-1)(n-2)}{8n^2}|g|^{\frac{1}{n}}$$

$$\times \left\{ \frac{1}{|g|^2} \partial_\mu |g| \partial_\nu |g| + g_{\mu\nu} \left[-\frac{3n-1}{n} \frac{1}{|g|^2} (\partial_\rho |g|)^2 + \frac{2}{|g|} \nabla_\rho \nabla_\rho |g| \right] \right\}. \tag{C.11}$$

It is easy to check that adding the two results (C.10) and (C.11) leads to the equations of motion of the WTDiff gravity

$$\frac{\delta S}{\delta g^{\mu\nu}} = \frac{\delta S_R}{\delta g^{\mu\nu}} + \frac{\delta S_g}{\delta g^{\mu\nu}} = \frac{1}{2} |g|^{\frac{1}{n}} \left(G^T_{\mu\nu} - T^T_{\mu\nu} \right). \tag{C.12}$$

Appendix D: Covariantly conserved energy-momentum tensors

In this article, we mainly work with the Weyl transverse (WTDiff) gravity, which is not invariant under the general coordinate transformation (Diff) but only invariant under the Weyl transformation and TDiff. We find that the energy-momentum tensor derived from the WTDiff gravity is not covariantly conserved, thereby making it unclear to make a connection with the standard Einstein equations. However, as shown in this paper, the WTDiff gravity can be obtained by gauge-fixing the longitudinal diffeomorphism existing in the conformally invariant scalar–tensor gravity, which is generally covariant, so there should be a covariantly conserved energy-momentum tensor. In this appendix, for completeness, we will give a (well-known) proof for the existence of the covariantly conserved energy-momentum tensor if the underlying gravitational theory is invariant under the general coordinate transformation (Diff).

Suppose that a generic action S is invariant under Diff,

$$S = \int d^n x \sqrt{-g} \mathcal{L}. \tag{D.1}$$

Under Diff, the metric tensor transforms as

$$\delta g^{\mu\nu} = \nabla^\mu \xi^\nu + \nabla^\nu \xi^\mu, \tag{D.2}$$

where ξ^μ is a local parameter of Diff. Under Diff, the action S is transformed into

$$\delta S = -\int d^n x \sqrt{-g} T_{\mu\nu} \nabla^\mu \xi^\nu, \tag{D.3}$$

where the energy-momentum tensor $T_{\mu\nu}$ is defined as

$$T_{\mu\nu} = -\frac{2}{\sqrt{-g}} \frac{\delta(\sqrt{-g}\mathcal{L})}{\delta g^{\mu\nu}} = -2 \frac{\delta \mathcal{L}}{\delta g^{\mu\nu}} + g_{\mu\nu} \mathcal{L}. \tag{D.4}$$

By using Eq. (C.7) and integrating by parts, Eq. (D.3) can be recast into the form

$$\delta S = \int d^n x \sqrt{-g} \nabla_\mu T^{\mu\nu} \xi_\nu, \tag{D.5}$$

from which we can arrive at the covariant conservation law of the energy-momentum tensor

$$\nabla_\mu T^{\mu\nu} = 0. \tag{D.6}$$

Let us note that only general coordinate invariance of the action plays a critical role in this proof.

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