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CLASSIFICATION OF 2-STEP NILPOTENT LIE ALGEBRAS OF DIMENSION 9 WITH 2-DIMENSIONAL CENTER

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Abstract. A Lie algebra L is called 2-step nilpotent if L is not abelian and $[L, L]$ lies in the center of L. 2-step nilpotent Lie algebras are useful in the study of some geometric problems, and their classification has been an important problem in Lie theory. In this paper, we give a classification of 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center.

Keywords: related set; basis; derivation

MSC 2010: 17B05, 17B30

1. Introduction

It is well known that the classification problem of nilpotent Lie algebras is rather difficult. The first important research about classification of the nilpotent Lie algebras is due to Umlauf (see [16]) in later 19th century. Since then, several attempts have been made to develop some machinery whereby the classification problem can be reformulated. However, the progress towards a complete classification of nilpotent Lie algebras has been quite slow. Seeley in [15] and Gong in [5] gave the classification of nilpotent Lie algebras of dimension 7 over $\mathbb{C}$ and $\mathbb{R}$. In dimension 8, there are only partial results, see [1], [3], [4], [11].

2-step nilpotent Lie algebras are useful in the study of some geometric problems, such as commutative Riemannian manifolds, weakly symmetric Riemannian manifolds, etc. Moreover, the classification of 2-step nilpotent Lie algebras has been an important problem in Lie theory. Among 2-step nilpotent Lie algebras, a special one is the Heisenberg algebra which has 1-dimensional center. The classification problem of Heisenberg algebras is rather simple. The classification of 2-step nilpotent Lie

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algebras of dimension 8 was given in [12], [17]. In this paper, we give a classification of 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center.

Troughout the paper, all nilpotent Lie algebras discussed are finite dimensional and over the complex field $\mathbb{C}$.

2. Preliminaries

In this section, we recall some elementary facts about nilpotent Lie algebras.

Lemma 2.1 ([14]). *If N is a nilpotent Lie algebra, the following two assertions are equivalent:*
(1) $\{x_1, x_2, \dots, x_n\}$ *is a minimal system of generators;*
(2) $\{x_1 + N^2, x_2 + N^2, \dots, x_n + N^2\}$ *is a basis for the vector space* N/N^2.

If H is a maximal torus on N (a maximal abelian subalgebra of $\operatorname{Der} N$ consisting of semisimple linear transformations), then N can be decomposed into a direct sum of root spaces with respect to H: $N = \sum\limits_{\alpha \in H^*} N_\alpha$. The scalar $\operatorname{mult}(\alpha) := \dim N_\alpha$ is called the multiplicity of the root α. We also denote $\dim[x] = \dim N_\alpha$ if $x \in N_\alpha$.

Definition 2.1 ([14]). Let H be a maximal torus on N. One calls H-msg a minimal system of generators which consists of root vectors for H.

Definition 2.2. Let $\{x_1, x_2, \dots, x_n\}$ be a minimal system of generators of a nilpotent Lie algebra N. The related set of x_i is defined to be the set $G(x_i) = \{x_j : [x_i, x_j] \neq 0\}$, and the number $p_i = |G(x_i)|$ is called the related number of x_i. The n-tuple of integers $(p_1, p_2, \dots, p_n)$ is called the related sequence of $\{x_1, x_2, \dots, x_n\}$.

Definition 2.3 ([11]). A minimal system of generators is called a $(p_1, p_2, \dots, p_n)$-msg if its related sequence is $(p_1, p_2, \dots, p_n)$. It is called a $(p_1, p_2, \dots, p_n)$-H-msg if it is also an H-msg.

Lemma 2.2 ([11]). *If N is a 2-step nilpotent Lie algebra, then $N^2 = C(N)$ (the center of N) if and only if $p_i > 0$ for any $(p_1, p_2, \dots, p_n)$-msg.*

Definition 2.4 ([9]). A nilpotent Lie algebra N is called quasi-cyclic if N has a subspace U such that $N = U \dotplus U^2 \dotplus \dots \dotplus U^k$, where $U^i = [U, U^{i-1}]$.

If N is quasi-cyclic, it is clear that there exists a derivation I such that $I|_{U^s} = s \cdot \mathrm{id}$ (where id denotes the identity map). Hence there exists a maximal torus on N.

Lemma 2.3 ([11]). *Let N be a quasi-cyclic nilpotent Lie algebra, $\{x_1, x_2, \dots, x_n\}$ an H_1-msg of N, $\{y_1, y_2, \dots, y_n\}$ an H_2-msg of N. Then there exists $\theta \in \operatorname{Aut} N$ such* *that*

$$(y_1 \;\; y_2 \;\; \dots \;\; y_n) = (\theta(x_1) \;\; \theta(x_2) \;\; \dots \;\; \theta(x_n))A,$$

where A is an $n \times n$ invertible matrix. In particular, if $\dim[x_i] = 1$ for all i, then A is a monomial matrix (i.e. each row and each column has exactly one nonzero entry).

In what follows, all Lie algebras discussed are 2-step nilpotent Lie algebras.

Lemma 2.4 ([11]). *Let $\{x_1, x_2, \dots, x_n\}$ be a $(p_1, p_2, \dots, p_n)$-msg of N. If a linear transformation h of N satisfies $h[x_i, x_j] = [h(x_i), x_j] + [x_i, h(x_j)]$, $1 \leqslant i, j \leqslant n$, then $h \in \operatorname{Der} N$.*

Lemma 2.5 ([11]). *Let $\{x_1, x_2, \dots, x_n\}$ be a minimal system of generators of N. If there exists $h \in \operatorname{Der} N$ such that $h(x_i) = a_i x_i$, and $a_i \neq a_j$, $i \neq j$, then there exists a maximal torus H on N such that $\{x_1, x_2, \dots, x_n\}$ is an H-msg, and $\dim[x_i] = 1$ for all i.*

Definition 2.5 ([12]). Let $\{x_1, x_2, \dots, x_n\}$ be a $(p_1, p_2, \dots, p_n)$-msg of N. We define $[[x_i]] = \left\langle [x_i, x_{i_1}], \dots, [x_i, x_{i_{p_i}}] \right\rangle$ to be the vector space spanned by $\{[x_i, x_{i_1}], \dots, [x_i, x_{i_{p_i}}]\}$, where $G(x_i) = \{x_{i_1}, x_{i_2}, \dots, x_{i_{p_i}}\}$.

Definition 2.6 ([12]). Let $\{x_1, x_2, \dots, x_n\}$ be a $(p_1, p_2, \dots, p_n)$-msg of N. If there exists k such that $p_k = 1$, then $\{x_1, x_2, \dots, x_n\}$ is called a [1]-msg. If there exist $p_k = p_t = 1$, $k \neq t$, then $\{x_1, x_2, \dots, x_n\}$ is called a [1,1]-msg.

Lemma 2.6 ([12]). *Let $\{x_1, x_2, \dots, x_n\}$ be a $(p_1, p_2, \dots, p_n)$-msg of N. If there exists an x_j such that $\dim[[x_j]] = 1$, then N has a* [1]*-msg.*

Definition 2.7 ([12]). Let $\{x_1, \dots, x_n\}$ be a $(p_1, \dots, p_n)$-msg of N. If $G(x_1) = \{x_2, x_3\}$, $\dim[[x_1]] = 2$, and $[x_2, x_3] = 0$, then $\{x_1, x_2, \dots, x_n\}$ is called a [2]-msg.

Lemma 2.7 ([12]). *Let $\{x_1, \dots, x_n\}$ be a $(p_1, \dots, p_n)$-msg of N. If $\dim N^2 = 2$, and $[x_1, x_2]$,$[x_1, x_3]$ are linearly independent, then there exist $a, b \in C$ such that $[x_2 + a x_1, x_3 + b x_1] = 0$.*

Lemma 2.8 ([12]). *If $\dim N^2 = 2$, then N has a* [1]*-msg, or a* [2]*-msg.*

Definition 2.8 ([12]). If N has a [2]-msg $\{x_1, x_2, \dots, x_n\}$ such that $G(x_2) = \{x_1, x_k\}$, $\dim[[x_2]] = 2$, and $[x_3, x_k] = 0$, then $\{x_1, x_2, \dots, x_n\}$ is called a [2,2]-msg.

Lemma 2.9 ([12]). *If $\dim N^2 = 2$, then N has a* [1]*-msg, or a* $[2, 2]$*-msg.*

Recently, several authors investigated some new invariants, such as characteristic sequence, and breadth. Using the characteristic sequence, many results about classification of nilpotent Lie algebras were obtained in [1], [2], [6], [7], [10]. The notion of breadth of a nilpotent Lie algebra was introduced in [8]. In [8], authors gave a characterization of finite dimensional nilpotent Lie algebras of breadth less than or equal to two, and determined the isomorphism classes of these algebras. This shows that the two new invariants are helpful when discussing the classification of nilpotent Lie algebras. It is worth mentioning that some other invariants of 2-step Lie algebras were used by Revoy in [13] to study 2-step nilpotent Lie algebras. Revoy determined the classes of 2-step nilpotent Lie algebras with small generating sets. Unfortunately, the characteristic sequences of all 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center are of the type $(2,2,1,1,1,1,1)$, the breadth of these algebras is 2. In this paper, we will apply Lemma 2.3 to determine whether two 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center are isomorphic. The key here is to find an H-msg of N using our method.

3. Main results

In this section, all Lie algebras discussed are 2-step nilpotent, indecomposable (i.e. cannot be decomposed into the direct sum of their ideals), and $\dim N^2 = 2$. Let $\{x_1, x_2, \ldots, x_9\}$ be a basis of N.

3.1. Isomorphism theorem.

Theorem 3.1. *The following 2-step nilpotent Lie algebras of dimension 9 with 2-dimensional center are mutually nonisomorphic:*

$N_1^{9,2}$: *There exists a $(2,1,1,1,1,1,1)$-msg: $\{x_1, x_2, \ldots, x_7\}$ such that*

$$[x_1, x_2] = [x_4, x_5] = [x_6, x_7] = x_8, \quad [x_1, x_3] = x_9.$$

$N_2^{9,2}$: *There exists a $(2,1,1,1,1,1,1)$-msg: $\{x_1, x_2, \ldots, x_7\}$ such that*

$$[x_1, x_2] = [x_4, x_5] = x_8, \quad [x_1, x_3] = [x_6, x_7] = x_9.$$

$N_3^{9,2}$: *There exists a $(1,1,1,2,2,2,1)$-msg: $\{x_1, x_2, \ldots, x_7\}$ such that*

$$[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = x_8, \quad [x_4, x_5] = [x_6, x_7] = x_9.$$

$N_4^{9,2}$: *There exists a $(2,1,1,1,2,2,1)$-msg: $\{x_1, x_2, \ldots, x_7\}$ such that*

$$[x_1, x_2] = [x_4, x_5] = [x_6, x_7] = x_8, \quad [x_1, x_3] = [x_5, x_6] = x_9.$$

$N_5^{9,2}$: *There exists a* $(1,2,2,2,2,2,1)$*-msg:* $\{x_1, x_2, \ldots, x_7\}$ *such that*

$$[x_1, x_2] = [x_3, x_4] = [x_5, x_6] = x_8, \quad [x_2, x_3] = [x_4, x_5] = [x_6, x_7] = x_9.$$

In particular, $\{x_1, x_2, \ldots, x_7\}$ *in each* $N_i^{9,2}$ *is an* H_i*-msg,* $\dim[x_i] = 1$ *for any* x_i*, and* $\dim H_1 = \dim H_2 = 5$, $\dim H_3 = \dim H_4 = 4$, $\dim H_5 = 3$.

Proof. For $N_1^{9,2}$, by Lemma 2.4, there exists $h_1 \in \operatorname{Der} N$ such that the matrix of h_1 relative to $\{x_1, x_2, \ldots, x_7, [x_1, x_2], [x_1, x_3]\}$ is $\operatorname{diag}(1, -1, 4, 2, -2, 3, -3, 0, 5)$. By Lemma 2.5, there exists a maximal torus H_1 on N such that $\{x_1, x_2, \ldots, x_7\}$ is an H_1-msg, and $\dim[x_i] = 1$ for each x_i.

For any $h \in H_1$, the matrix of h relative to $\{x_1, x_2, \ldots, x_7, [x_1, x_2], [x_1, x_3]\}$ is $\operatorname{diag}(a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_1 + a_2, a_1 + a_3)$. By $[x_1, x_2] = [x_4, x_5] = [x_6, x_7]$, we have $a_1 + a_2 = a_4 + a_5 = a_6 + a_7$. It is easy to know that $\dim H_1 = 5$.

Similarly we have the following results:

For $N_2^{9,2}$: There exists $h_2 \in \operatorname{Der} N$ such that the matrix of h_2 relative to $\{x_1, x_2, \ldots, x_7, [x_1, x_2], [x_1, x_3]\}$ is $\operatorname{diag}(1, -1, 4, 2, -2, -3, 8, 0, 5)$. There exists a maximal torus H_2 on N such that $\{x_1, x_2, \ldots, x_7\}$ is an H_2-msg, and $\dim[x_i] = 1$ for all i, $\dim H_2 = 5$.

For $N_3^{9,2}$: There exists $h_3 \in \operatorname{Der} N$ such that the matrix of h_3 relative to $\{x_1, x_2, \ldots, x_7, [x_1, x_2], [x_4, x_5]\}$ is $\operatorname{diag}(1, -1, 2, -2, 3, -3, 4, 0, 1)$. There exists a maximal torus H_3 on N such that $\{x_1, x_2, \ldots, x_7\}$ is an H_3-msg, and $\dim[x_i] = 1$ for all i, $\dim H_3 = 4$.

For $N_4^{9,2}$: There exists $h_4 \in \operatorname{Der} N$ such that the matrix of h_4 relative to $\{x_1, x_2, \ldots, x_7, [x_1, x_2], [x_1, x_3]\}$ is $\operatorname{diag}(1, -1, 3, 2, -2, 6, -6, 0, 4)$. There exists a maximal torus H_4 on N such that $\{x_1, x_2, \ldots, x_7\}$ is an H_4-msg, and $\dim[x_i] = 1$ for all i, $\dim H_4 = 4$.

For $N_5^{9,2}$: There exists $h_5 \in \operatorname{Der} N$ such that the matrix of h_5 relative to $\{x_1, x_2, \ldots, x_7, [x_1, x_2], [x_2, x_3]\}$ is $\operatorname{diag}(1, -1, 2, -2, 3, -3, 4, 0, 1)$. There exists a maximal torus H_5 on N such that $\{x_1, x_2, \ldots, x_7\}$ is an H_5-msg, and $\dim[x_i] = 1$ for all i, $\dim H_5 = 3$.

By Lemma 2.3, these five algebras are mutually nonisomorphic. □

Theorem 3.2. *If* $\dim N = 9$*, and* $\dim N^2 = 2$*, then* N *is isomorphic to one of the Lie algebras in Theorem* 3.1.

Theorem 3.2 will be proved in the next section.

3.2. Proof of Theorem 3.2. In order to prove Theorem 3.2, we need the following lemmas.

For ease of exposition, we introduce the following notation: $u \parallel v$ denotes that u and v are linearly dependent, and $u \nparallel v$ that they are linearly independent.

Lemma 3.1. *Let* $\{x_1, \ldots, x_n\}$ *be a* $(p_1, \ldots, p_n)$*-msg of* N*. If* $[x_1, x_2] \nparallel [x_2, x_3]$ *and* $[x_2, x_3] \nparallel [x_3, x_4]$*, then there exist* $k, s \in C$ *such that* $[x_4 + kx_2, x_1 + sx_3] = 0$.

P r o o f. Since $\dim N^2 = 2$, we have

$$[x_1, x_2] = a[x_2, x_3] + b[x_3, x_4], \quad [x_4, x_1] = c[x_2, x_3] + d[x_3, x_4].$$

Let $s = d - kb$, and let k satisfy $bk^2 - (d - a)k - c = 0$, then

$$\begin{aligned}[x_4 + kx_2, x_1 + sx_3] &= [x_4, x_1] + s[x_4, x_3] + k[x_2, x_1] + ks[x_2, x_3] \\ &= (c - ka + ks)[x_2, x_3] + (d - s - kb)[x_3, x_4] = 0.\end{aligned}$$

□

Lemma 3.2. *If* N *has a* $(2, 2, \ldots, 2)$*-msg:* $\{x_1, x_2, \ldots, x_7\}$*, where* $[x_i, x_{i+1}] \neq 0$*, and* $[x_1, x_7] \neq 0$*, then* N *has a* $[1]$*-msg.*

P r o o f. By Lemma 2.6, we only consider the case of $\dim[[x_i]] = 2$ for each x_i. Since $\dim N^2 = 2$, we have

$$a_i[x_i, x_{i+1}] + b_i[x_{i+1}, x_{i+2}] + c_i[x_{i+2}, x_{i+3}] = 0.$$

Note that $a_i c_i \neq 0$ and $\dim N/N^2 = 7$, hence it is easy to show that there exits a b_i such that $b_i \neq 0$. By a permutation of subscripts, we may assume that $b_1 \neq 0$, and

$$[x_1, x_2] + [x_2, x_3] + [x_3, x_4] = 0.$$

Since $\dim[[x_i]] = 2$ for each x_i, we set

$$[x_1, x_2] = s[x_4, x_5] + t[x_5, x_6] = m[x_6, x_7] + n[x_7, x_1].$$

If $s \neq 0$, note that $[x_2 - x_4, x_3] = -[x_1, x_2] = [x_2 - x_4 + s^{-1}tx_6, sx_5]$. Let $x_2' = x_2 - x_4 + s^{-1}tx_6 - m^{-1}ns^{-1}tx_1$, thus $\{x_1, x_2', x_3, \ldots, x_7\}$ is a $(2, 4, 2, 2, 3, 2, 3)$-msg, and $[x_2', x_3] = -[x_1, x_2'] = s[x_2', x_5]$, $[x_2', x_7] = m^{-1}s^{-1}t[x_1, x_2']$, i.e. $\dim[[x_2']] = 1$. By Lemma 2.6, N has a $[1]$-msg.

If $s = 0$, then $n \neq 0$, note that $[nx_7 + x_2, x_1 - mn^{-1}x_6] = 0$. Let $x_7' = nx_7 + x_2$, $x_1' = x_1 - mn^{-1}x_6$, $x_5' = x_5 - mn^{-1}t^{-1}x_2$, then we have $[x_1', x_7'] = [x_1', x_5'] = 0$, so $\{x_1', x_2, x_3, x_4, x_5', x_6, x_7'\}$ is a $[1]$-msg $(G(x_1') = \{x_2\})$. □

Lemma 3.3. *If* $\dim N = 9$*, then* N *has a* $[1]$*-msg.*

P r o o f. Let $\{x_1, x_2, \ldots, x_7\}$ be a $(p_1, p_2, \ldots, p_7)$-msg. We may assume

$$\dim[[x_i]] = 2 \quad \text{for all } i.$$

By Lemma 2.9, assume that $G(x_1) = \{x_2, x_3\}$, $G(x_2) = \{x_1, x_4\}$, and $[x_3, x_4] = 0$. Thus $G(x_3) \subseteq \{x_1, x_5, x_6, x_7\}$. As $\dim[[x_3]] = 2$, we may assume $p_3 \leqslant 3$.

Case 1: $p_3 = 2$. We may assume $G(x_3) = \{x_1, x_5\}$, then

$$G(x_6) \subseteq \{x_4, x_5, x_7\}, \quad G(x_7) \subseteq \{x_4, x_5, x_6\}.$$

Case 1.1: $[x_6, x_7] = 0$. If $[x_5, x_6] = a[x_5, x_7]$, we have $[x_5, x_6'] = [x_5, x_6 - ax_7] = 0$, then $\{x_1, x_2, x_3, x_4, x_5, x_6', x_7\}$ is a [1]-msg $(G(x_6') = \{x_4\})$.

If $[x_5, x_6] \nparallel [x_5, x_7]$, by Lemma 3.1 we have $[x_4', x_6'] = [x_4 + mx_5, x_6 + nx_7] = 0$, then $\{x_1, x_2, x_3, x_4', x_5, x_6', x_7\}$ is a [1]-msg $(G(x_6') = \{x_5\})$.

Case 1.2: $[x_6, x_7] \neq 0$.

Case 1.2.1: $p_6 = 2$, or $p_7 = 2$. We may assume $G(x_7) = \{x_5, x_6\}$. By Lemma 2.7, we may assume $[x_5, x_6] = 0$, then $G(x_6) = \{x_7, x_4\}$. As $\dim[[x_6]] = \dim[[x_7]] = 2$, by Lemma 3.1, assume $[x_4, x_5] = 0$, thus $\{x_1, x_2, \ldots, x_7\}$ is a $(2, 2, \ldots, 2)$-msg as in Lemma 3.2, so N has a [1]-msg.

Case 1.2.2: $p_6 = p_7 = 3$. If $[x_5, x_6] = a[x_6, x_7]$, then $[x_5', x_6] = [x_5 + ax_7, x_6] = 0$, so this case can be reduced to Case 1.2.1. Otherwise, by Lemma 2.7, we have $[x_5', x_7'] = [x_5 + sx_6, x_7 + tx_6] = 0$, thus this case can be reduced to Case 1.2.1.

Case 2: $p_3 = 3$. We may assume $G(x_3) = \{x_1, x_5, x_6\}$. If $[x_3, x_5] \parallel [x_3, x_6]$, or $p_4 = 2$, this case can be reduced to Case 1. We now assume that $[x_3, x_5] \nparallel [x_3, x_6]$, and $p_4 > 2$.

Case 2.1: $[x_4, x_7] = 0$. If $[x_3, x_5] \parallel [x_5, x_7]$, this case can be reduced to Case 1. Otherwise, by Lemma 3.1 we have $[x_3', x_6'] = [x_3 + mx_7, x_6 + nx_5] = 0$, so this case can be reduced to Case 1.

Case 2.2: $[x_4, x_7] \neq 0$. *Case 2.2.1*: $p_7 = 2$.

We may assume $G(x_7) = \{x_4, x_5\}$. By Lemma 2.7, we may assume $[x_4, x_5] = 0$.

If $[x_5, x_6] = 0$, since $[x_1, x_3] = a[x_3, x_5] + b[x_3, x_6]$, let $x_1' = x_1 + ax_5 + bx_6$, then $\{x_3, x_5, x_6, x_7, x_4, x_2, x_1'\}$ is a $(2, 2, 2, q_7, q_4, q_2, q_1)$-msg, and by a permutation of subscripts, this case can be reduced to Case 1.

If $[x_5, x_6] \neq 0$, we may assume $[x_5, x_6] \nparallel [x_5, x_7]$ (otherwise this case can be reduced to case of $[x_5, x_6] = 0$), thus we have $[x_3, x_5] = c[x_5, x_6] + d[x_5, x_7]$. Let $x_3' = x_3 + cx_6 + dx_7$, then $[x_3', x_5] = 0$. If $[x_3', x_4] = 0$, this case can be reduced to Case 1. If $[x_3', x_4] \neq 0$, by Lemma 3.1, this case can be reduced to the case of $[x_3', x_4] = 0$.

Case 2.2.2: $p_7 = 3$. We may assume $[x_5, x_7] \nparallel [x_6, x_7]$, we set $[x_4, x_7] = e[x_5, x_7] + f[x_6, x_7]$, then we have $[x_4', x_7] = [x_4 - ex_5 - fx_6, x_7] = 0$, and $[x_3, x_4'] \neq 0$. By Lemma 3.1, this case can be reduced to Case 2.1. □

Lemma 3.4. *If* $\dim N = 9$, *then* N *has a* $[1,1]$*-msg.*

Proof. By Lemma 3.3, let $\{x_1, x_2, \ldots, x_7\}$ be a $(1, p_2, \ldots, p_7)$-msg.

If $\dim[[x_i]] = 1$ for some $i > 1$, obviously N has a [1,1]-msg. We now assume $\dim[[x_i]] = 2$, $i > 1$. Hence we may assume that $G(x_2) = \{x_1, x_3\}$, and $p_3 \leqslant 3$.

Case 1: $p_3 = 2$. We may assume that $G(x_3) = \{x_2, x_4\}$, and $p_4 \leqslant 3$.

If $p_4 = 2$, assume $G(x_4) = \{x_3, x_5\}$, then $p_6 = 2$, and by Lemma 2.7, N has a [1,1]-msg.

If $p_4 = 3$, assume $G(x_7) = \{x_5, x_6\}$. As $\dim[[x_7]] = 2$, we may assume $[x_5, x_6] = 0$. By Lemma 3.1, N has a [1,1]-msg.

Case 2: $p_3 = 3$. We may assume that $G(x_3) = \{x_2, x_4, x_5\}$, and $[x_3, x_4] \nparallel [x_3, x_5]$.

When $[x_6, x_7] = 0$, by Lemma 2.7, we may assume $[x_4, x_5] = 0$. By Lemma 3.1, N has a [1,1]-msg. We now consider the case of $[x_6, x_7] \neq 0$.

As $\dim[[x_6]] = 2$, assume $G(x_6) = \{x_4, x_7\}$. By Lemma 2.7, assume $[x_4, x_7] = 0$. By Lemma 3.1, assume $[x_4, x_5] = 0$. So $\{x_1, x_2, \ldots, x_7\}$ is a $(1, 2, 3, 2, 2, 2, 2)$-msg.

If $[x_4, x_6] \nparallel [x_5, x_7]$, we set $[x_6, x_7] = m[x_4, x_6] + n[x_5, x_7]$, then we have $[x_6', x_7'] = [x_6 - nx_5, x_7 + nx_4] = 0$, thus $\{x_1, x_2, x_3, x_4, x_5, x_6', x_7'\}$ is a $(1, 2, 5, 2, 2, 2, 2)$-msg. By Lemma 2.7, N has a [1,1]-msg.

If $[x_4, x_6] \parallel [x_5, x_7]$, then $[x_4, x_6] \nparallel [x_3, x_5]$. We set $[x_3, x_4] = s[x_4, x_6] + t[x_3, x_5]$, then we have $[x_3', x_4'] = [x_3 + sx_6, x_4 - tx_5] = 0$. Thus $\{x_1, x_2, x_3', x_4', x_5, x_6, x_7\}$ is a $(1, 2, 3, 2, 2, 2, 4)$-msg. By Lemma 2.7, N has a [1,1]-msg. $\square$

Lemma 3.5. *If* N *has a* $(1, 1, p_3, \ldots, p_7)$*-msg:* $\{x_1, x_2, \ldots, x_7\}$, *and* $[x_1, x_2] \neq 0$, *then* $N \cong N_i^{9,2}$, $1 \leqslant i \leqslant 3$.

Proof. Let L be the subalgebra generated by $\{x_3, x_4, \ldots, x_7\}$, then $\dim L^2 = 2$ as N is indecomposable. By the classification of nilpotent Lie algebras of dimension 7, L is isomorphic to one of the following two Lie algebras:

N_1: There exists a $(1, 1, 1, 2, 1)$-msg: $\{x_3', x_4', \ldots, x_7'\}$ such that $[x_5', x_6'] \nparallel [x_6', x_7']$, and $[x_3', x_4'] = [x_5', x_6']$.

N_2: There exists a $(1, 2, 2, 2, 1)$-msg: $\{x_3', x_4', \ldots, x_7'\}$ such that $[x_3', x_4'] \nparallel [x_4', x_5']$, and $[x_3', x_4'] = [x_5', x_6']$, $[x_4', x_5'] = [x_6', x_7']$.

When $L \cong N_1$, we have $[x_1, x_2] = a[x_5', x_6'] + b[x_6', x_7']$.

If $b = 0$, then $N \cong N_1^{9,2}$. Otherwise, let $x_7'' = bx_7' - ax_5'$, then $N \cong N_2^{9,2}$.

When $L \cong N_2$, we have $[x_1, x_2] = s[x_3', x_4'] + t[x_4', x_5']$.

If $s = 0$, or $t = 0$, then $N \cong N_3^{9,2}$.

If $st \neq 0$, assume $t = s = 1$. Note that $[x_1, x_2] = [x_3' - x_5', x_4']$. Let $x_3'' = x_3' - x_5' + x_7'$, $x_5'' = x_5' - 2x_7'$, $x_6'' = x_6' + x_4'$, then we have $[x_3'', x_6''] = 0$, $[x_1, x_2] = [x_3'', x_4'] = [x_5'', x_6'']$ and $[x_4', x_5''] = [x_6'', x_7']$, thus $N \cong N_3^{9,2}$. $\square$

Lemma 3.6. *If N has a $(1,2,1,p_4,p_5,p_6,p_7)$-msg: $\{x_1,x_2,\ldots,x_7\}$, and $G(x_2)=\{x_1,x_3\}$, then $N\cong N_i^{9,2}$, $1\leqslant i\leqslant 4$.*

Proof. Let L be the subalgebra generated by $\{x_4,x_5,x_6,x_7\}$, then $\dim L^2\leqslant 2$.

If $\dim L^2=1$, then L is a Heisenberg algebra, and by Lemma 3.5, $N\cong N_i^{9,2}$, $1\leqslant i\leqslant 3$.

If $\dim L^2=2$, by the classification of nilpotent Lie algebras of dimension 6, L is isomorphic to one of the following two Lie algebras:

M_1: There exists a $(1,1,1,1)$-msg: $\{x_4',x_5',x_6',x_7'\}$ such that $[x_4',x_5']\nparallel[x_6',x_7']$.

M_2: There exists a $(1,2,2,1)$-msg: $\{x_4',x_5',x_6',x_7'\}$ such that $[x_4',x_5']\nparallel[x_5',x_6']$, and $[x_4',x_5']=[x_6',x_7']$.

When $L\cong M_1$, by Lemma 3.5, $N\cong N_i^{9,2}$, $1\leqslant i\leqslant 3$.

When $L\cong M_2$, note that $[x_2,x_1]\nparallel[x_2,x_3]$ as N is indecomposable, we set

$$[x_4',x_5']=c[x_2,x_1]+d[x_2,x_3],\quad [x_5',x_6']=e[x_2,x_1]+f[x_2,x_3].$$

Let $x_1'=cx_1+dx_3$, $x_3'=ex_1+fx_3$, we have $[x_2,x_1']=[x_4',x_5']=[x_6',x_7']$, and $[x_2,x_3']=[x_5',x_6']$. As $cf-ed\neq 0$, $\{x_1',x_2,x_3',\ldots,x_7'\}$ is a $(1,2,1,1,2,2,1)$-msg. Thus $N\cong N_4^{9,2}$. $\square$

Lemma 3.7. *If N has a $(1,2,\ldots,2,1)$-msg: $\{x_1,x_2,\ldots,x_7\}$, and $[x_i,x_{i+1}]\neq 0$, then $N\cong N_i^{9,2}$, $1\leqslant i\leqslant 5$.*

Proof. By Lemma 3.5, we may assume that $\dim[[x_2]]=\dim[[x_6]]=2$.

Case 1: $\dim[[x_j]]=1$, $j=3$ or 5.

We may assume that $\dim[[x_3]]=1$, and $[x_2,x_3]=[x_3,x_4]$.

Case 1.1: $\dim[[x_4]]=1$. We may assume $[x_3,x_4]=[x_4,x_5]$. Note that $[x_2+x_4,x_3]=0=[x_4,x_5+x_3]$. Let $x_2'=x_2+x_4$, $x_5'=x_5+x_3$, then $[x_2',x_5']\neq 0$. Therefore $\{x_3,x_4,x_1,x_2',x_5',x_6,x_7\}$ is a $(1,1,1,2,2,2,1)$-msg as in Lemma 3.5, thus $N\cong N_i^{9,2}$, $1\leqslant i\leqslant 3$.

Case 1.2: $\dim[[x_4]]=2$.

Case 1.2.1: $\dim[[x_5]]=1$. We may assume $[x_4,x_5]=[x_5,x_6]$. Since $\dim[[x_4]]=2$, we set

$$[x_1,x_2]=a[x_3,x_4]+b[x_4,x_5],\quad [x_6,x_7]=c[x_3,x_4]+d[x_4,x_5].$$

Obviously $bc\neq 0$. When $a=0$ or $d=0$, we may assume $a=0$. Since $[x_2',x_3]=[x_2+x_4,x_3]=0$, $[x_2',x_5']=[x_2+x_4,bx_5+x_1]=0$, so $\{x_1,x_2',x_3,x_4,x_5',x_6,x_7\}$ is a $(1,1,1,2,2,2,1)$-msg as in Lemma 3.5. We now consider the case of $ad\neq 0$.

If $ad=bc$, then $d[x_1,x_2]=b[x_6,x_7]$. Let $x_2'=x_2+x_4+x_6$, $x_7'=x_7+db^{-1}x_1$, then we have $[x_2',x_3]=[x_2',x_5]=[x_2',x_7']=0$, thus $\{x_1,x_2',x_3,x_4,x_5,x_6,x_7'\}$ is a $(1,1,1,2,2,2,1)$-msg as in Lemma 3.5.

If $ad \neq bc$, let $x_3' = ax_3$, $x_5' = bx_5$, $x_7' = bd^{-1}x_7$, $s = bca^{-1}d^{-1}$, then we have $[x_2, x_3'] = [x_3', x_4]$, $[x_4, x_5'] = [x_5', x_6]$, and

$$[x_1, x_2] = [x_3', x_4] + [x_4, x_5'], \quad [x_6, x_7'] = s[x_3', x_4] + [x_4, x_5'].$$

Note that $[x_1, x_2] = [x_3' - x_5', x_4]$, and $[x_6, x_7'] = [x_4, x_5' - sx_3']$. Let $x_2' = x_2 + stx_4$, $x_3'' = x_3' - x_5' + tx_7'$, $x_5'' = x_5' - sx_3' + stx_1$, $x_6' = x_6 + stx_4$, where $t = (s-1)^{-1}$, i.e.

$$\begin{bmatrix} x_1 \\ x_2' \\ x_3'' \\ x_4 \\ x_5'' \\ x_6' \\ x_7' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & st & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & t \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ st & 0 & -s & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & st & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3' \\ x_4 \\ x_5' \\ x_6 \\ x_7' \end{bmatrix}.$$

Then we have

$$[x_3'', x_6'] = st[x_3', x_4] - [x_5', x_6] + st[x_4, x_5'] - t[x_6, x_7'] = 0,$$

$$[x_2', x_5''] = -s[x_2, x_3'] - st[x_1, x_2] + st[x_4, x_5'] + s^2t[x_3', x_4] = 0.$$

Thus $\{x_1, x_2', x_3'', x_4, x_5'', x_6', x_7'\}$ is a $(1,2,2,2,2,2,1)$-msg, and $[x_1, x_2'] = [x_3'', x_4]$, $[x_4, x_5''] = [x_6', x_7']$. If $[x_1, x_2'] \parallel [x_2', x_3'']$ or $[x_5'', x_6'] \parallel [x_6', x_7']$, by Lemma 3.5, our assertion holds. Otherwise we have

$$[x_4, x_5''] = m[x_1, x_2'] + n[x_2', x_3''], \quad [x_3'', x_4] = p[x_5'', x_6'] + q[x_6', x_7'].$$

Obviously $[x_3'', x_4] \nparallel [x_4, x_5'']$, hence $np \neq 0$. Note that $[x_4, x_5''] = [x_2', nx_3'' - mx_1]$, and $[x_3'', x_4] = [px_5'' - qx_7', x_6']$, hence it is easy to know that $N \cong N_5^{9,2}$.

Case 1.2.2: $\dim[[x_5]] = 2$. When $[x_4, x_5] \nparallel [x_6, x_7]$, by $[x_5, x_6] = e[x_4, x_5] + f[x_6, x_7]$, it is easy to know that this case can be reduced to Case 1.2.1. We now assume $[x_4, x_5] = [x_6, x_7]$.

In this case, by $[x_5, x_6] = p[x_3, x_4] + q[x_6, x_7]$, we may assume $[x_3, x_4] = [x_5, x_6]$. We now set

$$[x_1, x_2] = a[x_3, x_4] + b[x_4, x_5].$$

If $a = 0$, then we have $[x_2', x_5'] = [x_2 + x_4, x_5 + b^{-1}x_1] = 0$ and $[x_2', x_3] = 0$, thus $\{x_1, x_2', x_3, x_4, x_5', x_6, x_7\}$ is a $(1,1,1,2,2,2,1)$-msg as in Lemma 3.5.

If $a \neq 0$, we have $[x_2', x_5'] = [x_2 + 2x_4, x_5 + 2b^{-1}x_1 - 2ab^{-1}x_3 - (2a)^{-1}bx_7] = 0$, so $\{x_1, x_2', x_3, x_4, x_5', x_6, x_7\}$ is a $(1,2,2,2,2,2,1)$-msg. As $[x_4, x_5'] = (2a)^{-1}b[x_5', x_6]$, this case can be reduced to Case 1.2.1.

Case 2: $\dim[[x_i]] = 2$, $i = 3, 5$. If $[x_1, x_2] \nparallel [x_3, x_4]$ or $[x_4, x_5] \nparallel [x_6, x_7]$, it is easy to know that this case can be reduced to Case 1. We now assume that $[x_1, x_2] = [x_3, x_4]$, and $[x_4, x_5] = [x_6, x_7]$.

Case 2.1: $\dim[[x_4]] = 1$. We may assume $[x_3, x_4] = [x_4, x_5]$. By $[x_2, x_3] = p[x_1, x_2] + q[x_5, x_6]$, we may assume $[x_2, x_3] = [x_5, x_6]$. Let $x_5' = x_5 + x_3$, $x_2' = x_2 + x_6$, $x_7' = x_7 + x_1$, hence we have $[x_4, x_5'] = [x_2', x_5'] = [x_2', x_7'] = 0$, then $\{x_1, x_2', x_3, x_4, x_5', x_6, x_7'\}$ is a $(1, 2, 2, 1, 1, 2, 1)$-msg as in Lemma 3.6. Thus $N \cong N_i^{9,2}$, $1 \leqslant i \leqslant 4$.

Case 2.2: $\dim[[x_4]] = 2$. Since $\dim[[x_2]] = \dim[[x_6]] = 2$, we set

$$[x_3, x_4] = m[x_5, x_6] + n[x_6, x_7], \quad [x_4, x_5] = u[x_1, x_2] + v[x_2, x_3].$$

Obviously $mv \neq 0$. Note that $[x_3, x_4] = [mx_5 - nx_7, x_6]$, $[x_4, x_5] = [x_2, vx_3 - ux_1]$, and $[vx_3 - ux_1, v^{-1}x_4] = [x_3, x_4]$, $[v^{-1}x_4, mx_5 - nx_7] = mv^{-1}[x_4, x_5]$. Let $x_4' = v^{-1}x_4$, $x_3' = vx_3 - ux_1$, $x_5' = mx_5 - nx_7$, $x_2' = mv^{-1}x_2$, $x_1' = m^{-1}vx_1$, $x_7' = mv^{-1}x_7$, hence $\{x_1, x_2', x_3', x_4', x_5', x_6, x_7'\}$ is a $(1, 2, 2, 2, 2, 2, 1)$-msg, and $[x_1', x_2'] = [x_3', x_4'] = [x_5', x_6]$, $[x_2', x_3'] = [x_4', x_5'] = [x_6, x_7']$, thus $N \cong N_5^{9,2}$. □

We now give the proof of Theorem 2.

Proof of Theorem 2. By Lemma 3.4, assume that $\{x_1, x_2, \ldots, x_7\}$ is a $(1, p_2, p_3, p_4, p_5, p_6, 1)$-msg.

If $[x_1, x_7] \neq 0$ or $\{x_1, x_7\} \subseteq G(x_j)$, then $N \cong N_i^{9,2}$, $i \leqslant 4$, because of Lemma 3.5 and Lemma 3.6. We now assume that $G(x_1) = \{x_2\}$ and $G(x_7) = \{x_6\}$.

If $\dim[[x_2]] = 1$ or $\dim[[x_6]] = 1$, obviously this case can be reduced to the case of $[x_1, x_7] \neq 0$. Now we also assume that $\dim[[x_2]] = \dim[[x_6]] = 2$.

Case 1: $[x_2, x_6] = 0$. Since $\dim[[x_2]] = 2$, we may assume $G(x_2) = \{x_1, x_3\}$.

Case 1.1: $[x_3, x_6] = 0$. We may assume $G(x_6) = \{x_5, x_7\}$, then $G(x_4) \subseteq \{x_3, x_5\}$. If $p_3 = 1$ or $p_5 = 1$, this case can be reduced to the case of $\{x_1, x_7\} \subseteq G(x_j)$. We now assume $p_i > 1$, $i = 3, 5$.

In this case, if $[x_3, x_5] = 0$, by Lemma 3.7 our assertion holds. We now consider the case of $[x_3, x_5] \neq 0$.

When $p_4 = 1$, we may assume $[x_3, x_4] \neq 0$. If $[x_2, x_3] \parallel [x_3, x_4]$ or $[x_3, x_4] \parallel [x_3, x_5]$, this case can be reduced to the case of $[x_1, x_7] \neq 0$ or $\{x_1, x_7\} \subseteq G(x_j)$. Otherwise, we have $[x_3, x_5'] = [x_3, x_5 + ax_4 + bx_2] = 0$, and $[x_1, x_5'] \neq 0$, so $\{x_7, x_6, x_5', x_1, x_2, x_3, x_4\}$ is a $(1, 2, 2, 2, 2, 2, 1)$-msg as in Lemma 3.7.

When $p_4 = 2$, if $[x_3, x_4] \parallel [x_3, x_5]$, or $[x_4, x_5] \parallel [x_3, x_5]$, or $\dim[[x_4]] = 2$, this case can be reduced to the case of $[x_3, x_5] = 0$. So we now assume that $[x_3, x_4] \nparallel [x_3, x_5]$, and $[x_4, x_5] \nparallel [x_3, x_5]$, and $\dim[[x_4]] = 1$.

If $[x_1, x_2] \parallel [x_3, x_4]$, or $[x_4, x_5] \parallel [x_6, x_7]$, we may assume $[x_4, x_5] = [x_6, x_7]$. We set $[x_5, x_6] = c[x_3, x_5] + d[x_6, x_7]$, thus $[cx_3 + x_6, x_5 + dx_7] = 0$. Let $x_3' = cx_3 + x_6$,

$x_5' = x_5 + dx_7 - c^{-1}x_6$, $x_7' = x_7 - c^{-1}x_4$, then we have $[x_3', x_5'] = [x_3', x_7'] = [x_5', x_7'] = 0$, hence $\{x_1, x_2, x_3', x_4, x_5', x_6, x_7'\}$ is a $(1,2,2,2,2,2,1)$-msg as in Lemma 3.7.

If $[x_1, x_2] \nparallel [x_3, x_4]$ and $[x_4, x_5] \nparallel [x_6, x_7]$, we have

$$[x_2, x_3] = m[x_1, x_2] + n[x_3, x_4], \quad [x_5, x_6] = s[x_4, x_5] + t[x_6, x_7].$$

We may assume that $[x_2, x_3] = [x_3, x_4]$ and $[x_4, x_5] = [x_5, x_6]$. As $\dim[[x_4]] = 1$, we may assume $[x_3, x_4] = [x_4, x_5]$. Thus we have $[x_2', x_3] = [x_2 + x_4 + x_6, x_3] = 0$, $[x_4, x_5'] = [x_4, x_5 + x_3] = 0$, $[x_2', x_5'] = 0$, $[x_2', x_7] \neq 0$. Hence $\{x_4, x_3, x_5', x_6, x_7, x_2', x_1\}$ is a $(1,2,2,2,2,2,1)$-msg as in Lemma 3.7.

Case 1.2: $[x_3, x_6] \neq 0$. If $[x_4, x_6] \nparallel [x_6, x_7]$, or $[x_5, x_6] \nparallel [x_6, x_7]$, this case can be reduced to Case 1.1. Otherwise, we may assume $[x_4, x_6] = [x_5, x_6] = 0$, i.e. $G(x_6) = \{x_3, x_7\}$.

If $p_3 = 2$, obviously this case can be reduced to the case of $[x_1, x_7] \neq 0$.

If $p_3 = 3$, we may assume $[x_3, x_4] \neq 0$.

In this case, when $[x_2, x_3] \parallel [x_3, x_6]$, we have $[x_3, x_6'] = [x_3, kx_6 + x_2] = 0$ and $[x_1, x_6'] \neq 0$, then $\{x_7, x_6', x_1, x_2, x_3, x_4, x_5\}$ is a $(1,2,2,2,2,2,1)$-msg as in Lemma 3.7. Otherwise, it is easy to see that $[x_4, x_5] \nparallel [x_2, x_3]$, or $[x_4, x_5] \nparallel [x_3, x_6]$, and we may assume $[x_4, x_5] \nparallel [x_3, x_6]$. We set $[x_3, x_4] = m[x_3, x_6] + n[x_4, x_5]$, then we have $[x_3', x_4'] = [x_3 + nx_5, x_4 - mx_6] = 0$ and $[x_4', x_7] \neq 0$, so $\{x_1, x_2, x_3', x_6, x_7, x_4', x_5\}$ is a $(1,2,2,2,2,2,1)$-msg as in Lemma 3.7.

If $p_3 = 4$, we have $a[x_3, x_4] + b[x_3, x_5] + c[x_3, x_6] = 0$. It is easy to know that this case can be reduced to the case of $p_3 = 3$ when $c = 0$, or Case 1.1 when $c \neq 0$.

Case 2: $[x_2, x_6] \neq 0$.

Case 2.1: $p_2 = p_6 = 2$.

Let N_s be the subalgebra generated by $\{x_3, x_4, x_5\}$, it is easy to show that N_s has a $(1,2,1)$-msg, so N has a $(1,2,1,1,2,2,1)$-msg as in Lemma 3.6.

Case 2.2: $p_2 > 2$, or $p_6 > 2$.

If $[x_2, x_i] \nparallel [x_1, x_2]$ for some x_i, or $[x_6, x_j] \nparallel [x_6, x_7]$ for some x_j, then this case can be reduced to Case 1. Otherwise, this case can be reduced to Case 2.1. □

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