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CLASSIFICATION OF A CONFORMALLY FLAT K-SPACE

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1. Introduction. Let (M, F, g) be an n-dimensional $(n \geqq 4)$ almost-Hermitian manifold with almost-Hermitian structure (F, g), i.e. with an almost-complex structure F and a positive definite Riemannian metric g satisfying $F^2 = -I$, $g(FX, FY) = g(X, Y)$ for all tangent vectors X and Y, where I denotes the identity transformation. If an almost-Hermitian manifold (M, F, g) satisfies $(\nabla_X F)Y + (\nabla_Y F)X = 0$ for all tangent vectors X and Y, where ∇_X denotes the operation of the covariant differentiation with respect to the Riemannian connection, then the manifold is called a K-space (or Tachibana space, nearly Kähler manifold). Of course, a Kählerian space is a K-space, but in the sequel, by a proper K-space we shall mean a K-space which is not Kählerian. It is well known that there does not exist a 4-dimensional proper K-space [1], [6]. For any tensor T on M, by the components of T we shall mean the ones with respect to a local coordinate system $\{x^i\}$, $1 \leqq k, j, i, \cdots \leqq n$.

In a conformally flat space, the curvature tensor has the following form:

$$(1.1) \qquad (n-2)R_{kjih} = g_{kh}R_{ji} - g_{jh}R_{ki} + R_{kh}g_{ji} - R_{jh}g_{ki} - \frac{R}{n-1}(g_{kh}g_{ji} - g_{jh}g_{ki}) ,$$

where $R_{kji}{}^h$, $R_{ji} = R_{sji}{}^s$ and $R = g^{ji}R_{ji}$ are the Riemannian curvature tensor, Ricci tensor and scalar curvature respectively.

For a conformally flat K-space, we have already known that there exists no conformally flat proper K-space of dimension $\neq 6, 8, 10$, a 6-dimensional conformally flat proper K-space is a space of constant curvature and a conformally flat K-space is locally symmetric $(n \geqq 6)$ [5], [7]. A conformally flat K-space of dimension $n \geqq 12$ is Kählerian and it is necessarily locally flat.

The main purpose of the present paper is to prove that there exists no conformally flat proper K-space of dimension 10 and the following

THEOREM. *Let (M, F, g) be an n-dimensional connected conformally flat proper K-space. Then it is* (I) *a 6-dimensional K-space of positive* *constant curvature or* (II) *locally of the form* $(M_1, F_1, g_1) \times (M_2, F_2, g_2)$, *where the former is a* 6-*dimensional K-space of positive constant curvature C and the latter is a* 2-*dimensional Kählerian space of negative constant curvature* $-C$ *and,* (F_1, g_1) *and* (F_2, g_2) *are the restrictions of* (F, g) *to* M_1 *and* M_2 *respectively.*

Recently, we received an information from S. Tanno, in which he proved following result [8], stating that *a 4-dimensional conformally flat Kählerian manifold is either locally flat, or locally, a product space of 2-dimensional Kählerian manifolds of constant curvature K and* $-K$, *respectively.*

Taking account of the above result, we would conclude that the classification of conformally flat K-space is completed.

Now, to prove Theorem, we need the following

LEMMA 1.1. (Sumitomo [3]) *Let M be an n-dimensional* $(n > 3)$ *conformally flat Riemannian space satisfying*

$$\nabla_k \nabla_h R_{ji} - \nabla_h \nabla_k R_{ji} = 0 \,, \tag{1.2}$$

then each characteristic root of Ricci tensor must be one of the roots of the following equation:

$$\lambda^2 - \frac{R}{n-1}\lambda - \frac{1}{n}\left(R_{rs}R^{rs} - \frac{R^2}{n-1}\right) = 0 \,. \tag{1.3}$$

From (1.3), we have

$$\begin{aligned} \lambda &= \frac{1}{2}\left\{\frac{R}{n-1} \pm \left(\frac{R^2}{(n-1)^2} + \frac{4}{n}\left(R_{rs}R^{rs} - \frac{R^2}{n-1}\right)\right)^{1/2}\right\} \\ &= \frac{1}{2}\left\{\frac{R}{n-1} \pm \left(\frac{4}{n}R_{rs}R^{rs} - \frac{3n-4}{n(n-1)^2}R^2\right)^{1/2}\right\} . \end{aligned} \tag{1.4}$$

Now, let λ_1, λ_2 be the roots of the equation (1.3) and m be the multiplicity of λ_1, then the multiplicity of λ_2 is $n - m$ where $0 \leqq m \leqq n$. As the scalar curvature R is a trace of Ricci tensor, we have

$$R = m\lambda_1 + (n - m)\lambda_2 \,. \tag{1.5}$$

LEMMA 1.2. (Sekigawa-Takagi [2]) *Let M be an n-dimensional* $(n \geqq 3)$ *connected conformally flat Riemannian space satisfying the condition* (1.2). *If the Ricci form is non-degenerate and indefinite of signature* $2m - n$ *at least at one point of M, then M is a locally product space of an m-dimensional space of constant curvature C and an* $(n - m)$*-dimensional space of constant curvature* $-C$, *where* $1 < m < n - 1$.

In §2, we shall prepare some lemma and identities for a K-space and a conformally flat K-space. §3 will be devoted to the proof of the theorem.

2. Some lemma and identities. In a K-space, we have the following identities [4]:

$$R^*_{ji} = R^*_{ij},\ F_j{}^iR_k{}^j = F_k{}^jR_j{}^i,\ F_j{}^iR^{*j}_k = F_k{}^jR^{*i}_j\ , \tag{2.1}$$

where $R^*_{ji} = (1/2)F^{kh}R_{khsi}F_j{}^s$,

$$R - R^* = \nabla_j F_{ih}(\nabla^j F^{ih}) = \text{constant} \tag{2.2}$$

where $R^* = g^{ji}R^*_{ji}$.

Recently, Takamatsu [6] proved the following

LEMMA. *In a K-space, we have*

$$(R_{ji} - R^*_{ji})(R^{ji} - 5R^{*ji}) = 0\ . \tag{2.3}$$

Next, let M be a conformally flat K-space, then from (1.1), by making use of (2.1), we have

$$2R_{ji} - (n-2)R^*_{ji} = \frac{R}{n-1}g_{ji}\ , \tag{2.4}$$

or

$$R^*_{ji} = \frac{2}{n-2}R_{ji} - \frac{R}{(n-1)(n-2)}g_{ji}\ , \tag{2.5}$$

transvecting (2.5) with g^{ji}, we have

$$R = (n-1)R^*\ . \tag{2.6}$$

Taking account of (2.2) and (2.6), we get

$$\frac{n-2}{n-1}R = \nabla_j F_{ih}(\nabla^j F^{ih})\ . \tag{2.7}$$

From (2.3), we have the following

$$R_{ji}R^{*ji} = \frac{1}{6}(R_{ji}R^{ji} + 5R^*_{ji}R^{*ji}) \tag{2.8}$$

and from the square of the both sides of (2.4), we get

$$4R_{ji}R^{ji} - 4(n-2)R_{ji}R^{*ji} + (n-2)^2R^*_{ji}R^{*ji} = \frac{nR^2}{(n-1)^2}\ . \tag{2.9}$$

Similarly from (2.5), we get

$$R^*_{ji}R^{*ji} = \frac{4}{(n-2)^2}R_{ji}R^{ji} - \frac{3n-4}{(n-1)^2(n-2)^2}R^2\ . \tag{2.10}$$

Substituting (2.8) into (2.9), we have

$$(2.11)\qquad 2(8-n)R_{ji}R^{ji}+(3n^2-22n+32)R^*_{ji}R^{*ji}=\frac{3nR^2}{(n-1)^2}$$

and substituting (2.10) into (2.11), we have

$$(2.12)\qquad (n-4)(n-12)R_{ji}R^{ji}+(6n^2-33n+32)\frac{R^2}{(n-1)^2}=0\,,$$

or

$$(2.13)\qquad (n-4)(n-12)R_{ji}R^{ji}+6\left(n-\frac{33+\sqrt{321}}{12}\right)\times\left(n-\frac{33-\sqrt{321}}{12}\right)\frac{R^2}{(n-1)^2}=0\,.$$

3. Proof of Theorem. By virture of (2.13), we easily see that there exists no conformally flat proper K-space of dimension $\neq 6, 8, 10$ [5].

If we put $n=6$ in (2.12), we have

$$R_{ji}R^{ji}=\frac{1}{6}R^2$$

or

$$\left(R_{ji}-\frac{R}{6}g_{ji}\right)\left(R^{ji}-\frac{R}{6}g^{ji}\right)=0\,,$$

from which we have

$$R_{ji}=\frac{R}{6}g_{ji}\,,$$

that is, M is an Einstein space. Hence (1.1) reduces to

$$(3.1)\qquad R_{kjih}=\frac{R}{30}(g_{ji}g_{kh}-g_{ki}g_{jh})$$

and by (2.7), $R>0$, that is, a 6-dimensional conformally flat proper K-space is a space of positive constant curvature.

Next, if we put $n=10$ in (2.12), then we have

$$(3.2)\qquad R_{rs}R^{rs}=\frac{151}{9^2\times 6}R^2\,.$$

Therefore, from (1.4) we have

$$(3.3)\qquad \lambda=\frac{1}{2}\left\{\frac{R}{9}\pm\left(\frac{2\times 151}{5\times 9^2\times 6}-\frac{13}{5\times 9^2}\right)^{1/2}|R|\right\}=\frac{R}{18}\left(1\pm 4\sqrt{\frac{7}{15}}\right)\,.$$

Now, we put

$$(3.4)\qquad \lambda_1 = \frac{R}{18}\left(1 + 4\sqrt{\frac{7}{15}}\right), \quad \lambda_2 = \frac{R}{18}\left(1 - 4\sqrt{\frac{7}{15}}\right).$$

Substituting (3.4) into (1.5), we obtain

$$\sqrt{\frac{7}{15}}m = 1 + 5\sqrt{\frac{7}{15}},$$

which contradicts that m is integer, therefore R must be zero. Consequently, we can see that any 10-dimensional conformally flat proper K-space does not exist.

Lastly, we shall consider an 8-dimensional conformally flat proper K-space. If we put $n = 8$ in (1.4), then we have

$$(3.5)\qquad \lambda = \frac{1}{2}\left\{\frac{R}{7} \pm \left(\frac{1}{2}R_{rs}R^{rs} - \frac{5}{2 \times 7^2}R^2\right)^{1/2}\right\}.$$

On the other hand, if we put $n = 8$ in (2.12), then we have

$$(3.6)\qquad R_{rs}R^{rs} = \frac{19}{7^2 \times 2}R^2 .$$

Substituting (3.6) into (3.5), we get

$$(3.7)\qquad \lambda = \frac{R}{28}(2 \pm 3),$$

because of $R > 0$.

In this place, if we put

$$(3.8)\qquad \lambda_1 = \frac{5}{28}R, \quad \lambda_2 = -\frac{1}{28}R,$$

where $R > 0$ and substitute (3.8) into (1.5), then we have

$$(3.9)\qquad m = 6 .$$

Hence we can find that the Ricci form of an 8-dimensional conformally flat proper K-space is indefinite of signature $2m - n = 4$. Moreover, as our space is locally symmetric, i.e. $\nabla_l R_{kji}{}^h = 0$, it satisfies the condition (1.2).

Thus, by virtue of Lemma 1.2, we can conclude that an 8-dimensional connected conformally flat proper K-space is locally of the form $(M_1, g_1) \times (M_2, g_2)$ as a Riemannian manifold, where (M_1, g_1) and (M_2, g_2) are a 6-dimensional space of positive constant curvature $C = (\lambda_1/(m-1)) = (1/28)R$ and a 2-dimensional space of negative constant curvature $-C$ respectively.

Let $\{x^a\}$ and $\{x^p\}$ be local coordinate systems on M_1 and M_2 respectively, where $1 \leqq a, b, \cdots \leqq 6, 7 \leqq p, q, \cdots \leqq 8$. Then, with respect to this coordinate system $\{x^i\} = \{x^a, x^p\}$, we can put

$$(R_j{}^i) = \begin{pmatrix} R_b{}^a & 0 \\ 0 & R_q{}^p \end{pmatrix} \tag{3.10}$$

$$(F_j{}^i) = \begin{pmatrix} F_b{}^a & F_q{}^a \\ F_b{}^p & F_q{}^p \end{pmatrix} . \tag{3.11}$$

On the other hand, by (2.1) we know that

$$F_j{}^i R_k{}^j = F_k{}^j R_j{}^i$$

and therefore, from (3.10) and (3.11), after some calculation, we have

$$F_q{}^a = 0 , \qquad F_b{}^p = 0 . \tag{3.12}$$

Furthermore, we get, taking account of (3.11)

$$\nabla_q F_b{}^a = -\nabla_b F_q{}^a = 0 .$$

Similarly we have $\nabla_b F_q{}^p = 0$. These facts show that the space is compatible to the decomposition of the space.

Consequently, summarizing the above arguments, the proof of our theorem is completed.

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