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CLASSIFICATION OF ANCIENT COMPACT SOLUTIONS TO THE RICCI FLOW ON SURFACES

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Abstract

We consider an ancient solution $g(\cdot, t)$ of the Ricci flow on a compact surface that exists for $t \in (-\infty, T)$ and becomes spherical at time $t = T$. We prove that the metric $g(\cdot, t)$ is either a family of contracting spheres, which is a type I ancient solution, or a King–Rosenau solution, which is a type II ancient solution.

1. Introduction

We consider an ancient solution of the Ricci flow

$$\frac{\partial g_{ij}}{\partial t} = -2\, R_{ij} \tag{1.1}$$

on a compact two-dimensional surface that exists for time $t \in (-\infty, T)$ and becomes singular at $t = T$, for some $T < \infty$. In two dimensions we have $R_{ij} = \frac{1}{2} R\, g_{ij}$, where R is the scalar curvature of the surface. Moreover, on an ancient non-flat solution we have $R > 0$. It is well known [**2**, **7**] that the surface also becomes extinct at T and becomes spherical, which means that after a normalization, the normalized flow converges to a spherical metric, to which we will refer as to the limiting sphere.

Since $R > 0$, by the Uniformization theorem and the fact that the Ricci flow in dimension 2 preserves the conformal class, we can parametrize the Ricci flow by the limiting sphere at time T, that is, we can write

$$g(\cdot, t) = u(\cdot, t)\, g_{S^2}.$$

The spherical metric can be written as

$$g_{S^2} = d\psi^2 + \cos^2 \psi\, d\theta^2 \tag{1.2}$$

where ψ, θ denote the global coordinates on the sphere. An easy computation shows that (1.1) is equivalent to the following evolution equation

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for the conformal factor $u(\cdot, t)$, namely,

$$u_t = \Delta_{S^2} \log u - 2 \qquad \text{on } S^2 \times (-\infty, T) \tag{1.3}$$

where Δ_{S^2} denotes the Laplacian on S^2. Let us recall, for future references, that the only nonzero Christoffel symbols for the spherical metric (1.2) are

$$\Gamma^2_{12} = \Gamma^2_{21} = -\tan\psi, \quad \Gamma^1_{22} = \frac{\sin 2\psi}{2},$$

where we use the indices $1, 2$ for the ψ, θ variables, respectively. It follows that for any function f on the sphere we have

$$\Delta_{S^2} f = f_{\psi\psi} - \tan\psi\, f_\psi + \sec^2\psi\, f_{\theta\theta}$$

which, in the case of a radially symmetric function $f = f(\psi)$, becomes

$$\Delta_{S^2} f = f_{\psi\psi} - \tan\psi\, f_\psi.$$

We will assume, throughout this paper, that $g = u\, ds_p^2$ is an ancient solution to the Ricci flow (1.3) on the sphere that becomes extinct at time $T = 0$.

It is natural to consider the pressure function $v = u^{-1}$ that evolves by

$$v_t = v^2 \left(\Delta_{S^2} \log v + 2\right) \qquad \text{on } S^2 \times (-\infty, 0) \tag{1.4}$$

or, after expanding the Laplacian of $\log v$,

$$v_t = v\, \Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2v^2 \qquad \text{on } S^2 \times (-\infty, 0). \tag{1.5}$$

Definition 1.1. We will say that an ancient solution to the Ricci flow (1.1) on a compact surface M is of type I if it satisfies

$$\limsup_{t\to-\infty} \left(|t| \max_M R(\cdot, t)\right) < \infty.$$

A solution that is not of type I will be called of type II.

Explicit examples of ancient solutions to the Ricci flow in two dimensions are:

i. **The contracting spheres**
They are described on S^2 by a pressure v_S that is given by

$$v_S(\psi, t) = \frac{1}{2(-t)} \tag{1.6}$$

and they are examples of ancient type I shrinking Ricci solitons.

ii. **The King–Rosenau solutions**
These solutions were discovered by J.R. King [**13**, **14**] and later, independently, by P. Rosenau ([**16**]). They are described on S^2 by a pressure v_K that has the form

$$v_K(\psi, t) = a(t) - b(t) \sin^2\psi \tag{1.7}$$

with $a(t) = -\mu \coth(2\mu t)$, $b(t) = -\mu \tanh(2\mu t)$, for some $\mu > 0$. These solutions are *not solitons*. We can visualize them as two cigars "glued" together to form a compact solution to the Ricci flow. They are type II ancient solutions.

Our goal in this paper is to prove the following classification result:

Theorem 1.2. *Let $g = u\, g_{S^2}$ be an ancient compact solution to the Ricci flow* (1.1). *Then u is either one of the contracting spheres or one of the King–Rosenau solutions.*

Remark 1.3. The classification of two-dimensional, complete, noncompact ancient solutions of the Ricci flow was recently given in [**6**] (see also [**9**, **3**]). The result in Theorem 1.2, together with the results in [**6**] and [**3**], provides a complete classification of all ancient two-dimensional complete solutions to the Ricci flow, with the scalar curvature uniformly bounded at each time slice.

The outline of the paper is as follows:

i. In section 2 we will show a priori derivative estimates on any ancient solution v of (1.5), which hold uniformly in time, up to $t = -\infty$. These estimates will play a crucial role throughout the rest of the paper.
ii. In section 3 we will introduce a suitable Lyapunov functional and will use it to show that the solution $v(\cdot, t)$ of (1.5) converges, as $t \to -\infty$, in the $C^{1,\alpha}$ norm, to a steady state v_∞.
iii. Section 4 will be devoted to the classification of all backward limits v_∞. We will show that there is a parametrization of the flow by a sphere, in which $v_\infty(\psi, \theta) = \mu \cos^2 \psi$, for some $\mu \geq 0$ (ψ, θ are the global coordinates on S^2). When $\mu > 0$, then v_∞ represents the cylindrical metric.
iv. In section 5 we will show that if $v_\infty(\psi, \theta) = \mu \cos^2 \psi$, with $\mu > 0$, then v must be one of the King–Rosenau solutions.
v. Finally in section 6 we will show that if $v_\infty \equiv 0$, then the solution v must be one of the contracting spheres.

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1.1. Change of variables. Throughout the paper we will be performing different changes of variables that we summarize below. We may write the evolving metric

$$g(\cdot, t) = u(\cdot, t) g_{S^2} = \bar{u}(\cdot, t) g_e = \hat{u}(\cdot, t) g_c$$

where g_{S^2}, g_e, g_c denote the metrics on the standard sphere S^2, the euclidean plane, and the cylinder, respectively. Denote by ψ, θ the global coordinates on S^2, by r, θ the polar coordinates on the plane, and by

s, θ the global coordinates on the cylinder. Mercator's projection gives us a relation between the sphere and the cylinder, that is,

$$g(\cdot, t) = u(\psi, \theta, t)\,(d\psi^2 + \cos^2\psi\, d\theta^2) = \hat{u}(s, \theta, t)\,(ds^2 + d\theta^2)$$

with

$$\hat{u}(s, \theta, t) = u(\psi, \theta, t)\cos^2\psi \qquad \cos\psi = (\cosh s)^{-1}. \tag{1.8}$$

It follows that

$$\cosh s = \sec\psi \qquad \text{and} \qquad \sinh s = \tan\psi. \tag{1.9}$$

A simple computation shows that if u is a solution of (1.3), then $\hat{u}$ satisfies the equation

$$\hat{u}_t = \Delta_c \log \hat{u} \qquad \text{on } \mathbb{R} \times [0, 2\pi] \times (-\infty, 0), \tag{1.10}$$

where Δ_c is the cylindrical Laplacian; that is,

$$\Delta_c f = f_{ss} + f_{\theta\theta}$$

for a function f defined on $\mathbb{R} \times [0, 2\pi]$. Furthermore, if

$$g(\cdot, t) = \bar{u}(r, \theta, t)(dr^2 + r^2\, d\theta^2)$$

where (r, θ) denote the polar coordinates of the plane obtained by projecting $S^2 \setminus \{\psi = \frac{\psi}{2}\}$ or $S^2 \setminus \{\psi = -\frac{\psi}{2}\}$ via stereographic projection, it is easy to compute that

$$\hat{u}(s, \theta, t) = r^2\, \bar{u}(r, \theta, t), \qquad r = e^s. \tag{1.11}$$

The function $\bar{u}$ satisfies the equation

$$\bar{u}_t = \Delta \log \bar{u} \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0) \tag{1.12}$$

where Δ is the euclidean Laplacian in polar coordinates.
Equivalently, the pressure functions in cylindrical and polar coordinates given by

$$\hat{v}(s, \theta, t) := \frac{1}{\hat{u}(s, \theta, t)} \quad \text{and} \quad \bar{v}(r, \theta, t) := \frac{1}{\bar{u}(r, \theta, t)}$$

satisfy the evolution equations

$$\hat{v}_t = \hat{v}\,\Delta_c \hat{v} - |\nabla \hat{v}|^2 \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0)$$

and

$$\bar{v}_t = \bar{v}\,\Delta \bar{v} - |\nabla \bar{v}|^2 \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0),$$

respectively.

The change of variables between the sphere $S^2 \setminus \{\psi = \frac{\pi}{2}\}$ and the plane that maps the south pole $\psi = -\pi/2$ to the origin is given via stereographic projection, that is,

$$r = \frac{1 + \sin\psi}{\cos\psi},$$

and if $g(\cdot,t) = u(\psi,\theta,t)g_{S^2} = \bar{u}(r,\theta,t)g_e$, then

$$\bar{u}(r,\theta,t) = \frac{\cos^4\psi}{(1+\sin\psi)^2}\,u(\psi,\theta,t).$$

2. A priori estimates

We will assume, throughout this section, that v is an ancient solution of the Ricci flow (1.5) on $S^2 \times (-\infty, 0)$ that becomes extinct at $T = 0$. We fix $t_0 < 0$. We will establish a priori derivative estimates on v that hold uniformly on $S^2 \times (-\infty, t_0]$. We will denote by C various constants that depend on t_0 and may vary from line to line but are always independent of time $t \in (-\infty, t_0]$.

Since our solution is ancient, the scalar curvature $R = v_t/v$ is strictly positive. This in particular implies that $v_t > 0$. Hence, we have the bound

$$v(\cdot,t) \le C(t_0) \qquad \text{on } (-\infty, t_0) \tag{2.1}$$

for any $t_0 < 0$. Define the backward limit

$$v_\infty := \lim_{t\to-\infty} v(\cdot,t)$$

that exists because of the inequality $v_t > 0$ but may vanish at some points on S^2. This actually happens in our model, the King–Rosenau solution. As a result, equation (1.5) fails to be uniformly parabolic near those points, as $t \to -\infty$, and the standard parabolic and elliptic derivative estimates fail as well. Nevertheless, it is essential for our classification result, to establish a priori derivative estimates that hold uniformly in time, as $t \to -\infty$.

We recall that on our ancient solution, the Harnack estimate for the scalar curvature shown in [**7**], takes the form

$$R_t \ge \frac{|\nabla R|_g^2}{R},$$

which in particular implies that $R_t \ge 0$. Hence, we also have

$$R(\cdot,t) \le C \qquad \text{on } (-\infty, t_0] \tag{2.2}$$

for any $t_0 < 0$, because $R(\cdot,t) \le R(\cdot,t_0)$. Once we have the uniform curvature bound along the flow, Shi's derivative estimates in [**17**] (in the compact case, the curvature derivative estimates have been obtained by Hamilton in [**12**] as well) imply that

$$|\nabla^k R| \le C(k) \;\; t \le t_0 < 0.$$

In addition, the limit

$$R_\infty := \lim_{t\to-\infty} R(\cdot,t)$$

exists and is bounded. We will actually show in the next section that $R_\infty = 0$ a.e. on S^2.

It is well known that R evolves by

$$R_t = \Delta_g R + R^2.$$

Expressing $g = v^{-1}\, g_{S^2}$, in which case $\Delta_g = v\,\Delta_{S^2}$ and $|\nabla\cdot|_g^2 = v|\nabla\cdot|_{S^2}^2$, we may rewrite the Harnack estimate for R as

$$v\,\Delta_{S^2} R + R^2 \geq \frac{v\,|\nabla_{S^2} R|^2}{R}$$

or equivalently,

$$\Delta_{S^2} R + \frac{R^2}{v} \geq \frac{|\nabla_{S^2} R|^2}{R}. \tag{2.3}$$

The pressure v satisfies the elliptic equation

$$v\,\Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2v^2 = R\,v. \tag{2.4}$$

We will next use this equation and the bounds (2.1), (2.2), and (2.3) to establish uniform first- and second-order derivative estimates on v.

Lemma 2.1. *There exists a uniform constant C, independent of time, so that*

$$\sup_{S^2}\left(|\Delta_{S^2} v| + \frac{|\nabla_{S^2} v|^2}{v}\right)(\cdot, t) \leq C \qquad \textit{for all } t \leq t_0 < 0.$$

Proof. To simplify the notation we will set, throughout the proof, $\Delta := \Delta_{S^2}$ and $\nabla := \nabla_{S^2}$. We first differentiate (2.4) twice to compute the equation for Δv. After a direct computation, where we also use the Bochner formula on S^2, we find that

$$\Delta(\Delta v) + \frac{(\Delta v)^2 - 2\,|\nabla^2 v|^2}{v} + 4\Delta v + 2\frac{|\nabla v|^2}{v} = \frac{\Delta(Rv)}{v},$$

which implies the inequality

$$\Delta(\Delta v + 4v) \geq -2\,\frac{|\nabla v|^2}{v} + \Delta R + 2\frac{\nabla R\cdot\nabla v}{v} + \frac{R\,\Delta v}{v} \tag{2.5}$$

since by the trace formula we have

$$(\Delta v)^2 \leq 2|\nabla^2 v|^2.$$

By (2.4) we also have

$$\Delta v = \frac{|\nabla v|^2}{v} - 2v + R, \tag{2.6}$$

and if we use this to replace Δv from the last term on the right-hand side of (2.5), we obtain

$$\begin{aligned}\Delta(\Delta v + 4v) &\geq -2\frac{|\nabla v|^2}{v} + \Delta R + 2\frac{\nabla R\cdot\nabla v}{v} + \frac{R}{v}\left(\frac{|\nabla v|^2}{v} - 2v + R\right)\\ &= -2\frac{|\nabla v|^2}{v} + (\Delta R + \frac{R^2}{v}) + 2\frac{\nabla R\cdot\nabla v}{v} + \frac{R\,|\nabla v|^2}{v^2} - 2R.\end{aligned}$$

Combining the last inequality and the Harnack estimate, (2.3) we obtain

$$\begin{aligned}\Delta(\Delta v + 4v) \geq & -2\frac{|\nabla v|^2}{v} + \frac{|\nabla R|^2}{R} + 2\frac{\nabla R\cdot\nabla v}{v} + \frac{R|\nabla v|^2}{v^2} - 2R \\ = & -2\frac{|\nabla v|^2}{v} + \frac{1}{R}\left(|\nabla R|^2 + 2\,\nabla R\cdot R\,\frac{\nabla v}{v} + R^2\,\frac{|\nabla v|^2}{v^2}\right) - 2R \\ = & -2\frac{|\nabla v|^2}{v} + \frac{1}{R}\left|\nabla R + R\frac{\nabla v}{v}\right|^2 - 2R.\end{aligned}$$

Since $R > 0$, we conclude the estimate

$$\Delta(\Delta v + 4v) \geq -2\,\frac{|\nabla v|^2}{v} - 2R. \tag{2.7}$$

If we multiply (2.6) by $m = 2$ and add it to (2.7), we get

$$\Delta(\Delta v + 6\,v) \geq -4v \geq -C$$

for a uniform constant C (independent of time). By (2.6) we also have

$$\Delta v \geq -2v + R > -\bar{C}, \tag{2.8}$$

and therefore

$$X := \Delta v + \bar{C} + 6v > 0$$

and

$$\Delta X \geq -C. \tag{2.9}$$

Standard Moser iteration applied to (2.9) yields to the bound

$$\sup X \leq C_1 \int_{S^2} X\,da + C_2. \tag{2.10}$$

Observe that

$$\int_{S^2} X\,da = \int_{S^2} (\Delta v + \bar{C} + 6v)\,da = \int_{S^2} (\bar{C} + 6v)\,da \leq C.$$

The last estimate combined with (2.10) yields to the bound

$$\Delta v \leq C.$$

This, together with (2.8), implys

$$\sup_{S^2} |\Delta v|(\cdot, t) \leq C \qquad \text{for all } \ t \in (-\infty, t_0], \tag{2.11}$$

for C is a uniform constant. Since

$$\frac{|\nabla v|^2}{v} = \Delta v + 2v - R$$

the estimate (2.11) and $R > 0$ readily imply the bound

$$\sup_{S^2} \frac{|\nabla v|^2}{v} \leq C \qquad \text{for all } \ t \in (-\infty, t_0]. \tag{2.12}$$

q.e.d.

As a consequence of the previous lemma we have:

Corollary 2.2. *For any $p \geq 1$, we have*

$$\|v(\cdot,t)\|_{W^{2,p}(S^2)} \leq C(p) \qquad \text{for all } t \in (-\infty, t_0]. \tag{2.13}$$

It follows that for any $\alpha < 1$, we have

$$\|v(\cdot,t)\|_{C^{1,\alpha}(S^2)} \leq C(\alpha) \qquad \text{for all } t \in (-\infty, t_0]. \tag{2.14}$$

Proof. Since $\Delta_{S^2} v = f$ in S^2, with $f \in L^\infty$, standard $W^{2,p}$ estimates for Laplace's equation imply that $v \in W^{2,p}(S^2)$ for all $p \geq 1$. Hence, (2.14) follows by the Sobolev embedding theorem. q.e.d.

We will now use the estimates proven above to improve the regularity of the function v.

Lemma 2.3. *For every $0 < \alpha < 1$, there is a uniform constant $C(\alpha)$ so that*

$$\||\nabla_{S^2} v(\cdot,t)|^2\|_{C^{1,\alpha}(S^2)} \leq C(\alpha) \qquad \text{for all } t \leq t_0 < 0. \tag{2.15}$$

Proof. To simplify the notation, we will set $\Delta := \Delta_{S^2}$ and $\nabla := \nabla_{S^2}$. A direct computation shows that $|\nabla v|^2$ satisfies the evolution equation

$$\frac{\partial}{\partial t} |\nabla v|^2 = v\, \Delta(|\nabla v|^2) - 6v\, |\nabla^2 v|^2 + 2v|\nabla v|^2 + 2|\nabla v|^2\, \Delta v - 2\, \nabla(|\nabla v|^2) \cdot \nabla v.$$

On the other hand, differentiating the equation $v_t = R\, v$ gives

$$\frac{\partial}{\partial t} |\nabla v|^2 = 2\, \nabla(Rv) \cdot \nabla v.$$

Combining the above yields

$$\Delta(|\nabla v|^2) = f \tag{2.16}$$

with f given by

$$f = 2|\nabla^2 v|^2 - 6\, |\nabla v|^2 - 2\frac{|\nabla v|^2}{v}\, \Delta v + \frac{2\nabla(|\nabla v|^2) \cdot \nabla v}{v} + \frac{2\nabla(Rv) \cdot \nabla v}{v}. \tag{2.17}$$

We will show that for every $p \geq 1$, we have

$$\|f(\cdot,t)\|_{L^p(S^2)} \leq C(p) \qquad \text{for all } t \in (-\infty, t_0] \tag{2.18}$$

with $C(p)$ independent of t. We will denote in the sequel by $C(p)$ various constants that are independent of t. We begin by recalling that by (2.13), we have

$$\|\nabla^2 v(\cdot,t)\|_{L^p} \leq C(p) \qquad \text{for all } t \in (-\infty, t_0].$$

Also, by Lemma 2.1, we have

$$\|2\frac{|\nabla v|^2}{v}\, \Delta v\|_{L^p(S^2)} + \||\nabla v|^2\|_{L^p(S^2)} \leq C(p) \qquad \text{for all } t \in (-\infty, t_0].$$

Since

$$|\frac{\nabla(|\nabla v|^2) \cdot \nabla v}{v}| \leq 2|\nabla^2 v|\, \frac{|\nabla v|^2}{v}$$

by the previous estimates, we have

$$\|\frac{\nabla|(\nabla v|^2)\cdot\nabla v}{v}\|_{L^p(S^2)}\leq C(p)\qquad\text{for all } t\leq t_0<0.$$

We also have

$$\begin{aligned}|\frac{\nabla(Rv)\cdot\nabla v}{v}| &\leq R\,\frac{|\nabla v|^2}{v}+|\nabla R|\,|\nabla v|\\ &\leq C+(\sqrt{v}|\nabla R|)\left(\frac{|\nabla v|}{\sqrt{v}}\right)\leq C\end{aligned}$$

for all $t\leq t_0<0$, since $\sqrt{v}\,|\nabla R|=|\nabla R|_{g(t)}\leq C$. We can now conclude that (2.18) holds for $p\geq 1$. Standard elliptic regularity estimates applied to (2.16) imply the bound

$$\||\nabla v|^2\|_{W^{2,p}}\leq C(p)\qquad\text{for all } t\in(-\infty,t_0].$$

Since the previous estimate holds for any $p\geq 1$, by the Sobolev embedding theorem, we conclude (2.15). q.e.d.

Lemma 2.4. *For any $0<\alpha<1$, there is a uniform in time constant $C(\alpha)$, so that*

$$\|\sqrt{v}\,\nabla^2_{S^2}v\|_{C^{0,\alpha}(S^2)}\leq C(\alpha)\qquad\textit{for all } t\leq t_0<0.$$

Moreover,

$$\|v\,\nabla^3_{S^2}v\|_{L^\infty(S^2)}\leq C\qquad\textit{for all } t\leq t_0<0$$

for a uniform in time constant C.

Proof. To simplify the notation, we will set $\Delta:=\Delta_{S^2}$ and $\nabla:=\nabla_{S^2}$. To prove the estimate on $\|\sqrt{v}\,\nabla^2 v\|_{C^{0,\alpha}(S^2)}$, we observe that we can rewrite (2.4) in the form

$$\Delta v^{3/2}=\frac{9|\nabla v|^2}{4\sqrt{v}}-3v^{3/2}+\frac{3}{2}\,R\sqrt{v}. \tag{2.19}$$

We claim that the right-hand side of the previous identity has uniformly in time bounded $C^{0,\alpha}$ norm, for any $0<\alpha<1$. To see this, observe that for every $p\geq 1$ and for any i,j, we have

(2.20)

$$\begin{aligned}\|\nabla(\frac{\nabla_i v\nabla_j v}{\sqrt{v}})\|_{L^p(S^2)} &\leq C\left(\|\nabla^2 v\|_{L^p(S^2)}\,\|\frac{|\nabla v|}{\sqrt{v}}\|_{L^\infty(S^2)}+\|\frac{|\nabla v|^2}{v}\|^{3/2}_{L^\infty(S^2)}\right)\\ &\leq C(p)\end{aligned}$$

and also

$$\|\nabla(R\sqrt{v})\|_{L^\infty(S^2)}\leq C$$

since

$$|\nabla(R\sqrt{v})|\leq|\nabla R|\,\sqrt{v}+R\,\frac{|\nabla v|}{2\sqrt{v}}\leq|\nabla R|_{g(t)}+C\leq\tilde{C}.$$

All of the above inequalities hold uniformly on $t \le t_0 < 0$. By the Sobolev embedding theorem, we conclude that the right-hand side of (2.19) has uniformly bounded $C^{0,\alpha}$ norm, for any $\alpha < 1$. Standard elliptic regularity theory applied to (2.19) implies that

$$\|v^{3/2}\|_{C^{2,\alpha}(S^2)} \le C(\alpha),$$

which in particular yields the estimate

$$\|\sqrt{v}\,\nabla^2 v\|_{C^\alpha} \le C(\alpha) \qquad \text{for all } t \in (-\infty, t_0]$$

since

$$\nabla^2_{ij} v^{3/2} = \frac{3}{4}\,\frac{\nabla_i v \nabla_j v}{\sqrt{v}} + \frac{3}{2}\sqrt{v}\,\nabla^2_{ij} v$$

and the first term on the right-hand side is in $C^{0,\alpha}$ by (2.20).

To prove the second estimate, we now rewrite (2.4) as

$$\Delta v^2 = 4|\nabla v|^2 - 4v^2 + 2R\,v. \tag{2.21}$$

Lemma 2.3 implies that $4\,|\nabla v|^2 - 4v^2$ has uniformly in time bounded $C^{1,\alpha}$ norm. We claim the same is true for the term Rv. To see this, we differentiate it twice and use the inequality

$$|\nabla^2(R\,v)| \le |\nabla^2 v|\,R + |\nabla^2 R|\,v + 2|\nabla R|\,|\nabla v|.$$

By Lemmas 2.1 and 2.3 and the bounds

$$v\,|\nabla^2 R| = |\nabla^2 R|_g \le C, \quad \sqrt{v}\,|\nabla R| = |\nabla R|_g \le C, \quad R \le C,$$

we conclude that for all $p \ge 1$, we have

$$\|\nabla^2(R\,v)\|_{L^p(S^2)} \le C(p), \qquad \text{for all } t \in (-\infty, t_0].$$

The Sobolev embedding theorem now implies that $\|R\,v\|_{C^{1,\alpha}(S^2)}$ is uniformly bounded in time, for every $\alpha < 1$. Standard elliptic theory applied to (2.21) yields the bound $\|v^2\|_{C^{3,\alpha}} \le C(\alpha)$, for all $t \le t_0 < 0$. In particular,

$$\|\nabla^3 v^2\|_{L^\infty(S_2)} \le C$$

for a uniform constant C. Since

$$\|v\,\nabla^3 v\|_{L^\infty(S_2)} \le C\left(\|\nabla^3 v^2\|_{L^\infty(S_2)} + \|\sqrt{v}\,\nabla^2 v\|_{L^\infty(S_2)}\,\|\frac{\nabla v}{\sqrt{v}}\|_{L^\infty(S_2)}\right),$$

this readily implies the bound $\|v\,\nabla^3 v\|_{L^\infty(S^2)} \le C$. q.e.d.

3. Lyapunov functional and convergence

In this section, we introduce the Lyapunov functional

$$J(t) = \int_{S^2} \left(\frac{|\nabla_{S^2} v|^2}{v} - 4\,v \right) da. \tag{3.1}$$

We will show next that $J(t)$ is nondecreasing and bounded. In the sequel, we will combine these properties with the a priori estimates shown in the previous section to show that $v(\cdot,t)$ converges, as $t \to -\infty$, in the $C^{1,\alpha}$ norm to a steady-state solution v_∞.

Lemma 3.1. *The Lyapunov functional $J(t)$ is monotone under (1.5), and in particular, we have*

$$\frac{d}{dt}J(t) = -2\int_{S^2} \frac{v_t^2}{v^2}\,da - \int_{S^2} \frac{|\nabla_{S^2} v|^2}{v^2}\,v_t\,da \tag{3.2}$$

Proof. To simplify the notation, we will set $\Delta := \Delta_{S^2}$ and $\nabla := \nabla_{S^2}$. Equation (1.5) and a direct calculation show

$$\begin{aligned} \frac{d}{dt}\int_{S^2} \frac{|\nabla v|^2}{v}\,da &= 2\int_{S^2} \frac{\nabla v\,\nabla v_t}{v}\,da - \int_{S^2} \frac{|\nabla v|^2}{v^2}\,v_t\,da \\ &= -2\int_{S^2} \frac{\Delta v}{v}\,v_t\,da + \int_{S^2} \frac{|\nabla v|^2}{v^2}\,v_t\,da \\ &= -2\int_{S^2} (\frac{v_t}{v^2} + \frac{|\nabla v|^2}{v^2} - 2)\,v_t\,da + \int_{S^2} \frac{|\nabla v|^2}{v^2}\,v_t\,da \\ &= -2\int_{S^2} \frac{v_t^2}{v^2}\,da - \int_{S^2} \frac{|\nabla v|^2}{v^2}\,v_t\,da + 4\int_{S^2} v_t\,da. \end{aligned}$$

We then conclude that

$$\frac{d}{dt}\int_{S^2} \left(\frac{|\nabla v|^2}{v} - 4v\right)\,da = -2\int_{S^2} \frac{v_t^2}{v^2}\,da - \int_{S^2} \frac{|\nabla v|^2}{v^2}\,v_t\,da \tag{3.3}$$

that is,

$$\frac{d}{dt}J(t) = -2\int_{S^2} \frac{v_t^2}{v^2}\,da - \int_{S^2} \frac{|\nabla v|^2}{v^2}\,v_t\,da \tag{3.4}$$

where both terms on the right-hand side of (3.4) are nonpositive, since on an ancient solution of (1.5) we have $v_t \geq 0$. q.e.d.

As an immediate consequence of the estimate in Lemma 2.1 and the inequality

$$v\,\Delta v - |\nabla v|^2 + 2\,v^2 \geq 0,$$

we have:

Lemma 3.2. *There exists a uniform constant C so that*

$$-C \leq J(t) \leq 0 \qquad \textit{for all } -\infty < t \leq t_0 < 0.$$

We will next use Lemma 3.1 to show that on our ancient solution the backward time limit

$$R_\infty = \lim_{t\to-\infty} R(\cdot,t)$$

of the scalar curvature R is equal to zero almost everywhere on S^2.

Lemma 3.3. *On an ancient solution v of equation (1.5), we have $R_\infty = 0$ a.e. on S^2.*

Proof. It is enough to show that

$$\int_{S^2} R_\infty^2 \, da = 0.$$

Indeed, assume the opposite, namely, that $\int_{S^2} R_\infty^2 \, da := c > 0$. Then, since $R_t \geq 0$, we have $\int_{S^2} R^2(\cdot, t)\, da \geq c$, that is,

$$\int_{S^2} \frac{v_t^2}{v^2} \, da \geq c.$$

Integrating (3.2) in time while using the above inequality and the fact that $v_t \geq 0$ gives

$$J(t_2) - J(t_1) \leq - \int_{t_1}^{t_2} \int_{S^2} \frac{v_t^2}{v^2} \, da \, dt \leq -c\,(t_2 - t_1)$$

for every $-\infty < t_1 < t_2 < t_0 < 0$. This obviously contradicts the uniform bound $-C \leq J(t) \leq 0$ shown in Lemma 3.2. q.e.d.

We will now combine some of the a priori estimates of the previous section with Lemma 3.3 to prove the following convergence result.

Proposition 3.4. *The solution $v(\cdot, t)$ of (1.5) converges, as $t \to -\infty$, to a limit $v_\infty \in C^{1,\alpha}(S^2)$, for any $\alpha < 1$. Moreover, $\| v_\infty \nabla^2_{S^2} v_\infty \|_{C^\alpha(S^2)} < \infty$, for all $\alpha < 1$, and v_∞ satisfies the steady-state equation*

$$v_\infty \, \Delta_{S^2} v_\infty - |\nabla_{S^2} v_\infty|^2 + 2\, v_\infty^2 = 0. \tag{3.5}$$

Proof. Since $v_t \geq 0$ and $v > 0$, the pointwise limit

$$v_\infty := \lim_{t \to -\infty} v(\cdot, t)$$

exists. Lemmas 2.3 and 2.4 ensure that for every $\alpha < 1$ and every sequence $t_i \to -\infty$, along a subsequence still denoted by t_i, we have $v(\cdot, t_i) \overset{C^{1,\alpha}(S^2)}{\longrightarrow} \tilde{v}$ and $v\, \nabla^2_{S^2} v(\cdot, t_i) \overset{C^\alpha(S^2)}{\longrightarrow} \tilde{v} \nabla^2_{S^2} \tilde{v}$. By the uniqueness of the limit, $\tilde{v} = v_\infty$, which means that for every $\alpha < 1$, we have

$$v(\cdot, t) \overset{C^{1,\alpha}(S^2)}{\longrightarrow} v_\infty \quad \text{and} \quad (v\, \nabla^2_{S^2} v)(\cdot, t) \overset{C^\alpha(S^2)}{\longrightarrow} v_\infty \nabla^2_{S^2} v_\infty, \quad \text{as } t \to -\infty.$$

We can now let $t \to -\infty$ in equation

$$v\, \Delta_{S^2} v - |\nabla_{S^2} v|^2 + 2\, v^2 = R\, v$$

and use Lemma 3.3 to conclude that v_∞ satisfies equation (3.5). q.e.d.

4. The backward limit

In this section we will classify all the backward limits $v_\infty = \lim_{t \to -\infty} v(\cdot, t)$ proving:

Theorem 4.1. *There exists a conformal change of S^2 in which the limit*

$$v_\infty(\psi,\theta) := \lim_{t\to-\infty} v(\psi,\theta,t) = \mu \cos^2\psi$$

for some constant $\mu \geq 0$, where ψ, θ denote global coordinates on a conformally changed sphere. Moreover, the convergence is in $C^{1,\alpha}$ on S^2, for any $0 < \alpha < 1$, and in C^∞ on every compact subset of $S^2\backslash\{S,N\}$, where S, N denote the south, north poles of S^2, respectively (the points that correspond to $\psi = \pm\frac{\pi}{2}$).

We have shown in the previous section that $v(\cdot,t) \overset{C^{1,\alpha}(S^2)}{\longrightarrow} v_\infty$, for any $\alpha \in (0,1)$, where v_∞ is a weak solution of the steady-state equation

$$v_\infty\,\Delta v_\infty - |\nabla v_\infty|^2 + 2\,v_\infty^2 = 0.$$

To classify all the backward limits v_∞, we will need the following Proposition, which constitutes the main step in the proof of Theorem 4.1.

Proposition 4.2. *The limit v_∞ is either identically equal to zero or has at most two zeros.*

For a fixed $t_0 < 0$, the conformal factor u of our evolving metric on S^2 is uniformly bounded from above and below away from zero. Set

$$m(t_0) := \inf_{z\in S^2} u(z,t_0).$$

Assume that v_∞ is not identically equal to zero. Then, since v_∞ is a continuous function, there exist two points $P_1, P_2 \in S^2$ such that $\lim_{t\to\infty} v(P_i,t) > 0$, $i = 1,2$, or equivalently, $\lim_{t\to\infty} u(P_1,t) < \infty$, $i = 1,2$. By performing a conformal change of coordinates, we may assume that P_1 is the south pole S and P_2 is the north pole N of the background sphere S^2. Let ψ,θ be global coordinates on S^2, where $\psi = \frac{\pi}{2}$ and $\psi = -\frac{\pi}{2}$ correspond to the poles (denote them by S and N). Denote by $\bar{u}$ the conformal factor of our metric in plane coordinates, after the stereographic projection that maps S to the origin. It follows that

$$\lim_{t\to-\infty} \bar{u}(0,t) = \bar{u}_\infty(0) < \infty.$$

We have seen that $\bar{u}$ satisfies equation (1.12). We will show:

Lemma 4.3. *Given any $r_0 > 0$ and $t_0 < 0$, there exists a uniform in time constant $C(r_0)$ that also depends on $\bar{u}_\infty(0)$ and $m(t_0)$ such that*

$$\int_{|x|\leq r_0} (\log\bar{u})^+(x,t)\,dx \leq C(r_0) \qquad \textit{for all } t \leq t_0. \tag{4.1}$$

Proof. Set

$$\bar{U}(r,t) = \int_0^{2\pi} \log\bar{u}(r,\theta,t)\,d\theta.$$

Since $\Delta \log \bar{u} = -R\,\bar{u} \le 0$, integrating the inequality $\Delta \log \bar{u} \le 0$ in θ yields the inequality

$$r^{-1}\big(r\,\bar{U}_r\big)_r \le 0.$$

Integration in r (using the fact that $\bar{u}(\cdot,t)$ is smooth at the origin) shows that $U_r \le 0$, implying the bound

$$\bar{U}(r,t) \le \log \bar{u}(0,t) \le \log \bar{u}_\infty(0) < \infty.$$

Hence, for any $t \le t_0$, we have

$$\int_{|x|\le r_0} \log \bar{u}(x,t)\,dx \le C\,\log \bar{u}_\infty(0)\,r_0^2 \le C_1(r_0)$$

for a uniform in t constant $C_1(r_0)$. In addition, the inequality $u_t \le 0$ implies the bound

$$\inf_{x\in B_{r_0}(0)} \bar{u}(x,t) \ge \inf_{x\in B_{r_0}(0)} \bar{u}(x,t_0) \ge c(m(t_0),r_0) > 0,$$

which gives

$$\int_{|x|\le r_0} (\log \bar{u})^-(x,t)\,dx \le C_2(r_0).$$

Combining the previous two integral bounds yields (4.1). q.e.d.

The following L^∞ bound is inspired by the beautiful paper of Brezis and Merle [**1**]. It will play a crucial role in the proof of Proposition 4.2.

Lemma 4.4. *Let $\delta > 0$ be a given small number. If for some $t \le t_0$, $\rho < 1$ and $x_0 \in R^2$, with $|x_0| \le r_0$, we have*

$$\int_{B_\rho(x_0)} R\,\bar{u}(x,t)\,dx \le 4\pi - 2\delta, \tag{4.2}$$

then

$$\sup_{B_{\rho/4}(x_0)} u(\cdot,t) \le C(r_0,\rho,\delta) \tag{4.3}$$

for a constant $C(r_0,\rho,\delta)$ that depends on r_0, ρ, δ, $\bar{u}_\infty(0)$, and $m(t_0)$ but is independent of time t.

The proof of the bound (4.3) will use the ideas of Brezis and Merle [**1**], including the following result, which we state for the reader's convenience.

Theorem 4.5 (Brezis–Merle)**.** *Assume that $\Omega \subset \mathrm{R}^2$ is a bounded domain and let w be a solution of*

$$\begin{cases} -\Delta w = f(x) & \text{in } \Omega, \\ \quad u = 0 & \text{on } \partial\Omega, \end{cases} \tag{4.4}$$

with $f \in L^1(\Omega)$. Then, for every $\delta \in (0,4\pi)$, we have

$$\int_\Omega e^{\frac{(4\pi-\delta)|w(x)|}{\|f\|_{L^1(\Omega)}}}\,dx \le \frac{4\pi^2}{\delta}(\mathrm{diam}\Omega)^2.$$

Proof of Lemma 4.4. Fix $t \leq t_0$ so that (4.2) holds, according to the statement of the lemma. Throughout the proof of the lemma, we will denote by $C(\rho, \delta)$ various constants that depend on ρ and δ but are independent of time t.

Set $w := \log \bar{u}(\cdot, t)$ and observe that w solves the elliptic equation

$$-\Delta w = R\, e^{w} \qquad \text{in } B_\rho(x_0),$$

with R denoting the scalar curvature. Let w_1 solve problem (4.4) in $\Omega := B_\rho(x_0)$ with $f := R\, e^w$. Since

$$\|f\|_{L^1(\Omega)} \leq 4\pi - 2\delta$$

by our assumption (4.2), Theorem 4.5 implies the bound

$$\int_{B_\rho(x_0)} e^{p\,|w_1(x)|}\, dx \leq C(\rho, \delta) \tag{4.5}$$

with

$$p := \frac{4\pi - \delta}{4\pi - 2\delta} > 1.$$

Combining (4.5) and Jensen's inequality gives the estimate

$$\|w_1\|_{L^1(B_\rho(x_0))} \leq C(\rho, \delta). \tag{4.6}$$

The difference $w_2 := w - w_1$ satisfies $\Delta w_2 = 0$ on $B_\rho(x_0)$. Hence, by the mean value inequality,

$$\|w_2^+\|_{L^\infty(B_{\rho/2}(x_0))} \leq C(\rho)\, \|w_2^+\|_{L^1(B_\rho(x_0))}. \tag{4.7}$$

Since $w_2^+ \leq w^+ + |w_1|$ combining, (4.6) and (4.1) yields the bound

$$\|w_2^+\|_{L^1(B_\rho(x_0))} \leq C(r_0, \rho, \delta)$$

if $|x_0| \leq r_0$, with $C(r_0, \rho, \delta)$ also depending on $m(t_0)$ and $\bar{u}_\infty(0)$, as in the statement of Lemma 4.3. Express $R\, e^w = R\, e^{w_2}\, e^{w_1}$ and recall that $-\Delta w = R\, e^w$ with R is uniformly bounded, so that $R\, e^w \leq C\, e^{w_1}$ on $B_{\rho/2}(x_0)$ by (4.7). We conclude by standard elliptic estimates and (4.5) that

$$\|w^+\|_{L^\infty(B_{\rho/4}(x_0))} \leq C\left(\|w^+\|_{L^1(B_{\rho/2}(x_0))} + \|e^{w_1}\|_{L^p(B_{\rho/2}(x_0))}\right) \leq C(\rho, \delta, \rho_0),$$

finishing the proof of the Lemma. q.e.d.

We will now prove Proposition 4.2.

Proof. We argue by contradiction. Assume that there exist at least three distinct points p_i, $i = 1, 2, 3$, such that $\lim_{t\to\infty} v(p_i, t) = 0$, $i = 1, 2, 3$, or equivalently, $\lim_{t\to\infty} u(p_i, t) = +\infty$. By our choice of the north and south poles in our given coordinates (as in the beginning of this section), these points belong to $S^2 \setminus \{S, N\}$; hence, they are mapped to three distinct points x_i, $i = 1, 2, 3$ on R^2 via the stereographic projection that maps S to the origin. We choose $0 < \rho < 1$ so that all balls $B_\rho(x_i)$ are disjoint.

Let $\delta < 1/2$ be a given positive number. Given any sequence $t_k \to \infty$ and any of the three points x_i, we may choose a subsequence, still denoted by $\{t_k\}$, such that

$$\int_{B_\rho(x_i)} R\,\bar{u}(x,t_k)\,dx > 4\pi - 2\delta$$

for the particular point x_i. Otherwise, $\lim_{t\to\infty} u(x_i,t) < \infty$ by (4.3), which would contradict the choice of x_i. This readily implies the existence of a sequence $t_k \to -\infty$ for which

$$\int_{B_\rho(x_i)} R\,\bar{u}(x,t_k)\,dx > 4\pi - 2\delta, \qquad \forall k \tag{4.8}$$

and for all three points x_i, $i = 1,2,3$.

Recall that the balls $B_\rho(x_i)$ are chosen to be disjoint. It follows that the total curvature for our metric $g(t_k) := \bar{u}(\cdot,t_k)$ satisfies

$$\int_{\mathbb{R}^2} R\,\bar{u}\,dx \geq \sum_{i=1}^{3} \int_{B_\rho(x_i)} R\,\bar{u}(x,t_k)\,dx > 12\pi - 6\delta > 8\pi$$

if $\delta < 1/2$, a contradiction to the total curvature of $g(t_k)$ being equal to 8π. This completes the proof of the proposition. q.e.d.

We are now in a position to classify all backward limits u_∞.

Proof of Theorem 4.1. Assume from now on that the backward limit v_∞ is not identically zero. We have just shown in Proposition 4.2 that v_∞ has at most two zeros. Choose a conformal change of S^2 that brings those two zeros to two antipodal poles on S^2 (if there is only one zero, we bring it to the south pole and choose for a north pole its antipodal point). Let ψ, θ be global coordinates on S^2, where $\psi = \frac{\pi}{2}$ and $\psi = -\frac{\pi}{2}$ correspond to the poles (denote them by S and N). Observe that equation (1.5) is strictly parabolic away from the poles, uniformly as $t \to -\infty$. It follows by standard parabolic PDE arguments that the convergence $v(\cdot,t) \to v_\infty$, as $t \to -\infty$, is in C^∞ on compact subsets of $S^2 \backslash \{S,N\}$. Perform Mercator's transformation (1.8) and denote by $\hat{v}(s,\theta,t) = v(\psi,\theta,t)\,\cosh^2 x$ the pressure in cylindrical coordinates. We conclude that

$$\lim_{t\to-\infty} \hat{v}(s,\theta,t) = \hat{v}_\infty(s,\theta) := v_\infty(\psi,\theta)\,\cosh^2 x > 0$$

and the convergence is smooth on compact subsets of $\mathbb{R} \times [0,2\pi]$. The limit $\hat{v}_\infty$ satisfies

$$\hat{v}_\infty \Delta \hat{v}_\infty - |\nabla \hat{v}_\infty|^2 = 0,$$

or (since $\hat{v}_\infty(s,\theta) > 0$ on $\mathbb{R} \times [0,2\pi]$) equivalently,

$$\Delta_c \log \hat{v}_\infty = 0 \tag{4.9}$$

where Δ_c denotes the cylindrical laplacian on $\mathbb{R}^2$. To finish the proof of Theorem 4.1, we need to classify the solutions of equation (4.9) that come as limits of ancient solutions $\hat{v}(\cdot, t)$.

To this end, set $w := \log \hat{v}_\infty$ so that $\Delta_c w = 0$, by (4.9). We can view w as a harmonic function on $\mathbb{R}^2$, after extending it in the θ direction so that it remains 2π-periodic. In addition, since $\hat{v}_\infty(s,\theta) = v_\infty(\psi,\theta)\cosh^2 s$ and $v_\infty = \lim_{t\to-\infty} v(\cdot,t) \le C$, it follows that there are uniform constants C_1, C_2 such that

$$w(s,\theta) \le C_1 + C_2\,|s|. \tag{4.10}$$

Since w is harmonic on $\mathbb{R}^2$, it follows by the mean value formula (via a standard argument) that the same bound (4.10) holds for $|w|$. It is now a well-known fact that the only harmonic functions on $\mathbb{R}^2$ with linear growth at infinity are the linear functions. Since our function w is periodic in θ, it follows that $w(s,\theta) = a_1 + a_2\,s$, for some constants a_1, a_2, and after exponentiating we obtain

$$\hat{v}_\infty(s,\theta) = \mu\, e^{\lambda s} \tag{4.11}$$

for some constants $\mu \ge 0$ and $\lambda \in \mathbb{R}$. Since we have assumed that the function $\hat{v}_\infty$ is not identically zero, we have $\mu > 0$.

To finish our argument, we need to show that $\lambda = 0$. We recall the estimate $|\nabla_{S^2} v| \le C\sqrt{v}$, shown in Lemma 2.1, or equivalently, $|v_\psi| \le C\sqrt{v}$, which in cylindrical coordinates gives the bound

$$|\hat{v}_s(s,\theta,t) - 2\hat{v}(s,\theta,t)\tanh s| \le C\sqrt{\hat{v}(s,\theta,t)},$$

which holds for $t \le t_0 < 0$. Taking $t \to -\infty$, we obtain

$$|(\hat{v}_\infty)_s(s,\theta) - 2\hat{v}_\infty(s,\theta)\,\tanh s| \le C\sqrt{\hat{v}_\infty(s,\theta)},$$

or equivalently,

$$\sqrt{\mu}\,|\lambda - 2\,\tanh s| \le Ce^{-\lambda s/2} \qquad \forall s \in (-\infty,+\infty),$$

which is impossible unless either $\lambda = 2$ or $\lambda = 0$.

In the case that $\lambda = 2$, then if $\bar{u}$ is the conformal factor of our metric g parametrized by the standard plane and $\bar{v} := \bar{u}^{-1}$ the corresponding pressure function, then $\lim_{t\to-\infty}\bar{v}(\cdot,t) = \mu$, on $\mathbb{R}^2\setminus\{0\}$. Since $\mu > 0$, we have

$$\lim_{t\to-\infty}\bar{u}(x,t) = \gamma := \mu^{-1} < \infty \qquad \text{on } \mathbb{R}^2\setminus\{0\},$$

which in particular implies that $\bar{u}(\cdot,t)$ is bounded from above and below away from zero on any compact subset $K \subset \mathbb{R}^2$. Standard parabolic PDE arguments imply that $\bar{u}(\cdot,t) \to \gamma$, as $t \to -\infty$, in C^∞ on compact subsets of $\mathbb{R}^2$. By Lemma 6.9 (which will be proven at the end of Section 6), this is impossible. We conclude that $\lambda = 0$.

The above discussion yields that $\hat{v}_\infty(s,\theta) = \mu$, with $\mu \geq 0$. Going back to the sphere S^2, via Mercator's transformation, we conclude that

$$v_\infty(\psi,\theta) := \lim_{t\to-\infty} v(\psi,\theta,t) = \mu \cos^2\psi.$$

Moreover, the convergence is smooth on compact subsets of $S^2\backslash\{S,N\}$. This finishes the proof of the theorem. q.e.d.

5. The King–Rosenau Solutions

We have shown in the previous section that there exists a parametrization of our evolving metric $g(t)$ on S^2, namely, $g(t) = u(\psi,\theta,t)\,ds_p^2$ for which the backward limit of the pressure function $v := u^{-1}$ satisfies

$$v_\infty(\psi,\theta) := \lim_{t\to-\infty} v(\psi,\theta,t) = \mu \cos^2\psi$$

with $\mu \geq 0$. Assuming, throughout this section that $\mu > 0$, we will show:

Theorem 5.1. *If the backward limit $v_\infty(\psi,\theta) = \mu \cos^2\psi$, with $\mu > 0$, then v is one of the King–Rosenau solutions* (**??**).

The case where $\mu = 0$ will be treated separately in the last section of the paper. Observe that when $\mu > 0$, the metric $g_\infty := v_\infty^{-1}\,ds_p$ is just the cylindrical metric written on S^2. By performing a simple rescaling in t and v, we may assume, without loss of generality, that $\mu = 1$.

Let S, N denote the north and south pole of the sphere S^2 corresponding to $\psi = -\frac{\pi}{2}$ and $\psi = \frac{\pi}{2}$, respectively. Consider the stereographic projections $\Phi_S : S^2 \setminus \{N\} \to \mathbb{R}^2$ and $\Phi_N : S^2 \setminus \{S\} \to \mathbb{R}^2$ such that $\Phi_S(S) = 0$ and $\Phi_N(N) = 0$ and set

$$\bar{v}_S(x,y,t) = v(\Phi_S^{-1}(x,y),t) \qquad \text{and} \qquad \bar{v}_N(\zeta,\xi,t) = v(\Phi_N^{-1}(\zeta,\xi),t)$$

where $v(\psi,\theta,t)$ denotes the pressure function on S^2. Then if

$$\zeta = \frac{x}{x^2+y^2} \quad \text{and} \quad \xi = \frac{y}{x^2+y^2}, \tag{5.1}$$

we have

$$\bar{v}_S(x,y,t) = (x^2+y^2)^2\,\bar{v}_N(\zeta,\xi,t). \tag{5.2}$$

Observe that, after stereographic projection, the pressure function in the King–Rosenau solutions takes the form

$$\bar{v}_S(x,y,t) = b(t) + c(t)(x^2+y^2) + b(t)(x^2+y^2)^2$$

and, similarly,

$$\bar{v}_N(\zeta,\xi,t) = b(t) + c(t)(\zeta^2+\xi^2) + b(t)(\zeta^2+\xi^2)^2$$

where $\lim_{t\to-\infty} c(t) = \mu$ and $\lim_{t\to-\infty} b(t) = 0$.

We consider the quantities

$$Q_S(x,y,t) := \bar{v}_S\left[\left((\bar{v}_S)_{xxx} - 3(\bar{v}_S)_{xyy}\right)^2 + \left((\bar{v}_S)_{yyy} - 3(\bar{v}_S)_{xxy}\right)^2\right]$$

and, similarly,

$$Q_N(\zeta,\xi,t) := \bar{v}_S\,\left[\left((\bar{v}_N)_{\zeta\zeta\zeta} - 3(\bar{v}_N)_{\zeta\xi\xi}\right)^2 + \left((\bar{v}_N)_{\xi\xi\xi} - 3(\bar{v}_N)_{\zeta\zeta\xi}\right)^2\right].$$

Both Q_N and Q_S are identically equal to zero on the King–Rosenau solutions. A direct calculation (where we make use of (5.1) and (5.2)) shows that

$$Q_S(x,y,t) = Q_N(\zeta,\xi,t). \tag{5.3}$$

Hence, the quantity

$$Q(\psi,\theta,t) := \begin{cases} Q_S(\Phi_S(\psi,\theta),t) & (\psi,\theta)\in S^2\setminus\{N\} \\ Q_N(\Phi_N(\psi,\theta),t) & (\psi,\theta)\in S^2\setminus\{S\} \end{cases}$$

is a well-defined and smooth function on $S^2\times(-\infty,0)$.

Sketch of proof: We will show next that $Q(\cdot,t)\equiv 0$, for all $t<0$, by showing that its maximum is decreasing in time and is equal to zero at $t=-\infty$. Following this, we will prove that any solution of equation (1.5) that satisfies $Q(\cdot,t)\equiv 0$ must be one of the King–Rosenau solutions, yielding the statement of Proposition 5.1.

Let

$$Q_{\max}(t) := \max_{(\psi,\theta)\in S^2} Q(\psi,\theta,t), \qquad t\in(-\infty,0).$$

Lemma 5.2. *The function $Q_{\max}(t)$ is decreasing in t.*

Proof. To show that $Q_{\max}(t)$ is decreasing, we will compute the evolution equation of Q. We may assume, without loss of generality, that $Q_{\max}(t)$ at an instant t is achieved on the southern hemisphere corresponding to $-\pi/2\le\psi\le 0$ so that

$$Q_{\max}(t) = \sup_{(x,y)\in\mathbb{R}^2} Q_S(x,y,t) = Q_S(x_0,y_0,t)$$

for some point $(x_0,y_0)\in\mathbb{R}^2$.

To simplify the notation, set $\bar{v} := \bar{v}_S$ and

$$A := \bar{v}_{xxx} - 3\,\bar{v}_{xyy} \qquad B := \bar{v}_{yyy} - 3\,\bar{v}_{xxy}$$

so that

$$A_x := \bar{v}_{xxxx} - 3\,\bar{v}_{xxyy} \qquad B_y := \bar{v}_{yyyy} - 3\,\bar{v}_{xxyy},$$

and also set

$$\bar{Q}(x,y,t) := \frac{1}{2}\,Q_S(x,y,t) = \frac{\bar{v}}{2}\,(A^2+B^2)$$

and

$$D_1 := 2\bar{v}\,A_x + \bar{v}_x\,A - 3\bar{v}_y\,B \qquad D_2 := 2\bar{v}\,B_y + \bar{v}_y\,B - 3\bar{v}_x\,A.$$

A direct computation shows that

$$L\bar{Q} := \bar{Q}_t - \bar{v}\,\Delta\bar{Q} = -a_1\,\bar{Q}_x^2 - b_1\,\bar{Q}_y^2 - a_2\,\bar{Q}_x - b_2\,\bar{Q}_y - C\,\bar{Q} - 4R\,\bar{Q}$$

where $R \geq 0$ denotes the scalar curvature of our metric, and

$$a_1 = \frac{1}{B^2}, \quad b_1 = \frac{1}{A^2}, \quad a_2 = \frac{A}{B^2}\, D_1, \quad b_2 = \frac{B}{A^2}\, D_2,$$

and

$$C = \frac{2}{\bar{v}}\,\Big(\frac{D_1^2}{4B^2} + \frac{D_2^2}{4A^2}\Big).$$

Observe next that

$$C\bar{Q} = \frac{2}{\bar{v}}\,\Big(\frac{D_1^2}{4B^2} + \frac{D_2^2}{4A^2}\Big)\,\bar{Q} = \frac{A^2 D_1^2}{4B^2} + \frac{B^2 D_2^2}{4A^2} + \frac{1}{4}(D_1^2 + D_2^2)$$

and

$$a_1 \bar{Q}_x^2 + a_2 \bar{Q}_x + \frac{A^2 D_1^2}{4B^2} = \frac{1}{B^2}\,\Big(\bar{Q}_x + \frac{AD_1}{2}\Big)^2$$

and, similarly,

$$b_1 \bar{Q}_y^2 + b_2 \bar{Q}_y + \frac{B^2 D_2^2}{4A^2} = \frac{1}{A^2}\,\Big(\bar{Q}_y + \frac{BD_2}{2}\Big)^2.$$

Hence,

$$L\bar{Q} = -\frac{1}{B^2}\,\Big(\bar{Q}_x + \frac{AD_1}{2}\Big)^2 - \frac{1}{A^2}\,\Big(\bar{Q}_y + \frac{BD_2}{2}\Big)^2 - \frac{1}{4}(D_1^2 + D_2^2) - R\,\bar{Q},$$

where we recall that $R \geq 0$ everywhere. Since $\bar{Q}$ is smooth (because $\bar{v}$ is), it follows that all quantities on the right-hand side of the above equation are bounded at any given point $(x, y, t) \in \mathbb{R}^2 \times (-\infty, 0)$ and

$$L\bar{Q} = \bar{Q}_t - v\,\Delta\bar{Q} \leq 0 \qquad \text{for all } (x, y, t) \in \mathbb{R}^2 \times (-\infty, 0).$$

This readily implies that $Q_{\max}(t)$ is decreasing in t, finishing the proof of the lemma. q.e.d.

We will next show that the backward limit as $t \to -\infty$ of $Q_{\max}(t)$ is zero.

Lemma 5.3. *We have*

$$\lim_{t \to -\infty} Q_{\max}(t) = 0.$$

As above, we set $\bar{v}(x, y, t) = \bar{v}_S(x, y, t)$ and consider the conformal factor $\bar{u} = \bar{v}^{-1}$. Our evolving metric $g(t)$ is then given by $g(t) = \bar{u}(\cdot, t)\,(dx^2 + dy^2)$, where $dx^2 + dy^2$ denotes the standard metric on the plane. Recall that the function $\bar{u}$ satisfies the evolution equation (1.12).

To simplify the notation, we will also denote by $\bar{u}$ the conformal factor of our metric over the plane $\mathbb{R}^2$ expressed in polar coordinates. Then

$$g(t) = \bar{u}(\cdot, t)\,(dr^2 + r^2\, d\theta^2) = \hat{u}(\cdot, t)\,(ds^2 + d\theta^2)$$

where $\hat{u}$ is the conformal factor in cylindrical coordinates, defined in terms of u by (1.8).

In the proof of Lemma 5.3, we will use the following estimate.

Lemma 5.4. *For any $t_0 < 0$, there exists a uniform in time constant C, depending only on t_0, such that*

$$|(\log \bar{u})_\theta(\cdot, t)| = |(\log \hat{u})_\theta(\cdot, t)| \le C \qquad on \ -\infty < t \le t_0. \tag{5.4}$$

In addition, $\hat{u}(\cdot, t) \le 1$ and $r^2\, \bar{u}(\cdot, t) \le 1$, for all $-\infty < t \le t_0 < 0$.

Proof. We have seen in Lemma 2.1 that the pressure function v written on S^2 satisfies the bound

$$|\nabla_{S^2} v|^2 \le C\, v \qquad \text{on } -\infty < t \le t_0$$

for a uniform constant C. This readily gives us the bound

$$\sec^2 \psi\, |v_\theta(\psi, \theta, t)|^2 \le C\, v(\psi, \theta, t),$$

or equivalently,

$$|v_\theta(\psi, \theta, t)|^2 \le C\, v(\psi, \theta, t)\, \cos^2 \psi.$$

However, since $v_t \ge 0$, we have $v(\psi, \theta, t) \ge \lim_{t \to -\infty} v(\psi, \theta, t) = \cos^2 \psi$. It follows that

$$|v_\theta(\psi, \theta, t)| \le C\, v(\psi, \theta, t),$$

or equivalently, $|(\log v)_\theta(\cdot, t)| \le C$. Hence, the conformal factor $u := v^{-1}$ also satisfies

$$|(\log u)_\theta(\cdot, t)| \le C$$

The bounds (5.4) now readily follow from (1.8) and (1.11).

For the L^∞ bounds on $\hat{u}$ and $\bar{u}$, we use that $\hat{u}_t \le 0$, which implies $\hat{u}(\cdot, t) \le \lim_{t \to -\infty} \hat{u}(\cdot, t) = 1$ giving the bound $\hat{u}(\cdot, t) \le 1$ and also yielding that $r^2\, \bar{u}(\cdot, t) \le 1$. q.e.d.

For a given sequence $t_k \to -\infty$, we define the re-scaled solutions of (1.12) given by

$$\bar{u}_k(x, y, t) := \rho_k^2\, \bar{u}(\rho_k x, \rho_k y, t + t_k) \tag{5.5}$$

where $\rho_k^2 = (\bar{u}(0, t_k))^{-1}$ is chosen so that

$$\bar{u}_k(0, 0) = 1.$$

Before we give the proof of Lemma 5.3, we will show:

Lemma 5.5. *Passing to a subsequence, $\{\bar{u}_k\}$ converges uniformly on compact subsets of $\mathbb{R} \times (-\infty, \infty)$ to a cigar solution $\bar{\bar{u}}$ given by*

$$\bar{\bar{u}}(x, y, t) = \frac{\alpha}{\beta e^{2\lambda t} + (x - x_0)^2 + (y - y_0)^2} \tag{5.6}$$

for some constants $\alpha, \beta > 0$, λ and some point $(x_0, y_0) \in \mathbb{R}^2$.

Proof. It is more convenient to switch for the moment to polar coordinates, defining $\bar{u}_k(r, \theta, t) = \bar{u}(\rho_k r, \theta, t + t_k)\, \rho_k^2$. We will first show the bounds

$$-C\,(1 + r^2) \le \log \bar{u}_k(r, \theta, 0) \le C \tag{5.7}$$

for a uniform constant C (independent of k). To this end, we begin by observing that $\log \bar{u}_k$ satisfies the elliptic equation

$$\Delta \log \bar{u}_k = -R_k\, \bar{u}_k,$$

where $R_k(r,\theta,t) = R(\rho_k\, r, \theta, t+t_k)$ satisfies the uniform bound

$$0 < R_k \le M.$$

Set

$$\bar{U}_k(r) = \int_0^{2\pi} \log \bar{u}_k(r,\theta,0)\, d\theta, \qquad r \ge 0,$$

and observe that by integrating the inequality

$$\Delta \log \bar{u}_k(\cdot, 0) \le 0$$

in θ we obtain the differential inequality

$$\Delta \bar{U}_k = r^{-1}(r\, \bar{U}_k'(r))' \le 0.$$

Since $\lim_{r\to 0} r\, \bar{U}_k'(r) = 0$, we readily conclude that $\bar{U}_k(r)$ is decreasing in r, and hence

$$\int_0^{2\pi} \log \bar{u}_k(r,\theta,0)\, d\theta \le \log \bar{u}_k(0,0) = 0.$$

In addition, by (5.4) we have $|(\log \bar{u}_k)_\theta(\cdot, 0)| \le C$, for a uniform constant C. The last two inequalities clearly imply the bound from above in (5.7). For the bound from below, observe that

$$-\Delta \log \bar{u}_k(\cdot,0) = R_k \bar{u}_k(\cdot,0) \le C$$

for a uniform constant C, which gives (after integration in θ) the differential inequality

$$-r^{-1}(r\, \bar{U}_k'(r))' \le C.$$

The desired bound now readily follows by integrating in r and using (5.4). This proves (5.7).

Now for a given $\tau > 0$, we choose k sufficiently large so that $t_k + \tau < -1$, and hence

$$\max_{\mathbb{R}^2 \times (-\infty, \tau]} R_k \le \max_{\mathbb{R}^2 \times (-\infty, -1]} R \le M$$

for a uniform constant M. Since $(\log \bar{u}_k)_t = -R_k$, from (5.7) we readily conclude the bounds

$$-C(\tau)\,(1+r^2) \le \log \bar{u}_k(r,\theta,t) \le C(\tau) \qquad \text{on } \mathbb{R}^2 \times [-\tau, \tau]$$

for a constant $C(\tau)$ that depends on τ but is uniform in k. Exponentiating gives us the bounds

$$0 < c(\tau, r) \le \bar{u}_k(r,\theta,t) \le C(\tau) < \infty \qquad \text{on } \mathbb{R}^2 \times [-\tau,\tau].$$

Standard parabolic PDE arguments imply that the sequence $\{\bar{u}_k\}$ is equicontinuous on compact subsets of $\mathbb{R}^2 \times (-\infty, \infty)$; hence, passing to a subsequence, $\{\bar{u}_k\}$ converges, uniformly on compact subsets of $\mathbb{R}^2 \times (-\infty, \infty)$, to an eternal solution $\bar{\bar{u}}$ of equation

$$\bar{\bar{u}}_t = \Delta \log \bar{\bar{u}} \qquad \text{on } \mathbb{R}^2 \times (-\infty, \infty)$$

that in addition satisfies the bound $0 < \bar{\bar{u}}(\cdot, t) \le C(t)$, for all t.

The result in [**6**] now shows that $\bar{\bar{u}}$ is either a cigar solution or the constant solution $\bar{\bar{u}} \equiv \alpha$. In the latter case, given $r > 0$, we may find k sufficiently large (depending on r) so that

$$\bar{u}_k(r, \theta, 0) := \rho_k^2\, u(\rho_k r, \theta, t_k) \ge \frac{\alpha}{2}$$

for all $\theta \in [0, 2\pi]$. It follows then that

$$(\rho_k r)^2\, \bar{u}(\rho_k r, \theta, t_k) \ge \frac{r^2 \alpha}{2}.$$

This will contradict our uniform bound $r^2\, \bar{u}(r, t) \le 1$ shown in Lemma 5.4 if we choose $r^2 = 4/a$.

We conclude that our limit $\bar{\bar{u}}$ is a cigar solution that in standard plane coordinates (x, y) takes the form (5.6). The proof of the lemma is now complete. q.e.d.

We will now proceed to the proof of Lemma 5.3.

Proof of Lemma 5.3. We begin by noticing that our quantity $Q(\psi, \theta, t)$ becomes identically equal to zero if v is either the cigar solution or the cylinder. Hence, the convergence of $v(\cdot, t)$ to the cylindrical metric in $C^\infty(S^2 \setminus \{S, N\})$ readily shows that $Q(\cdot, t)$ converges uniformly to zero as $t \to -\infty$ on compact subsets of $S^2 \setminus \{S, N\}$.

To prove the lemma, we argue by contradiction. If the conclusion of the lemma doesn't hold, then there exists a sequence of times t_k and points $P_k \in S^2$ such that

$$Q(P_k, t_k) \ge \epsilon > 0. \tag{5.8}$$

It follows from the above discussion that we may assume, without loss of generality, that $P_k \to S$ as $k \to \infty$, where S denotes the south pole of the sphere corresponding to $\psi = -\pi/2$ in the chosen coordinates. Denote by $\bar{P}_k = (r_k, \theta_k)$ the polar coordinates of the points P_k on the plane obtained by projecting $S^2 \setminus \{N\}$ onto $\mathbb{R}^2$ and mapping S to the origin.

Set $\rho_k^2 := (\bar{u}(0, t_k))^{-1}$ and let $\bar{u}_k$ be the sequence of rescaled solutions defined by (5.5) and used in Lemma 5.5. We will separate between the following two cases:

Case 1: We have $\liminf_{k \to \infty} r_k/\rho_k < \infty$.

In this case, we may assume, without loss of generality, that $(\bar{r}_k, \theta_k) := (r_k/\rho_k, \theta_k) \to (r_0, 0)$, with $r_0 < \infty$ (otherwise we pass to a subsequence and rotate in θ). Since $\bar{u}_k(\bar{r}_k, \theta_k, 0) = \rho_k^2\, \bar{u}(\bar{r}_k \rho_k, \theta_k, t_k)$, the convergence of $\bar{u}_k$ to the cigar solution readily implies that

$$\lim_{k\to\infty} \bar{Q}_k(\bar{r}_k, \theta_k, 0) = 0$$

(where $\bar{Q}_k$ is our given quantity corresponding to $\bar{u}_k$ when expressed in polar coordinates on the plane). However, since this quantity is dilation invariant, we have

$$\bar{Q}_k(\bar{r}_k, \theta_k, 0) = Q(r_k, \theta_k, t_k) = Q(P_k, t_k) \geq \epsilon > 0,$$

which contradicts that the limit is zero.
Case 2: We have $\liminf_{k\to\infty} r_k/\rho_k = +\infty$.

It is more convenient to work in cylindrical coordinates and set

$$\hat{u}(s, \theta, t) = r^2\, \bar{u}(r, \theta, t), \qquad r = e^s,$$

recalling that $\hat{u}$ satisfies the equation (1.10). We set

$$\hat{U}(s,t) := \int_0^{2\pi} \log \hat{u}(s, \theta, t)\, d\theta$$

and observe that that since $\Delta_c \log \hat{u} = \hat{u}_t \leq 0$, we have $\hat{U}_{ss}(s,t) \leq 0$; hence, $\hat{U}_s$ is nonincreasing in s. In addition, by (1.8), we have

$$\log \hat{u}(s, \theta, t) = \log u(\psi, \theta, t) - 2 \log \cosh s,$$

and hence

$$\lim_{s\to-\infty} \hat{U}_s(\cdot, t) = 2 \qquad \text{and} \qquad \lim_{s\to\infty} \hat{U}_s(\cdot, t) = -2, \qquad \forall t \in (-\infty, 0).$$

It follows that

$$|\hat{U}_s(\cdot, t)| \leq 2, \qquad \forall t \in (-\infty, 0).$$

We claim that if $s_k := \log r_k$, we have

$$\hat{U}(s_k, t_k) \geq -C \tag{5.9}$$

for some constant $C > 0$. To this end, choose $\hat{r}$ sufficiently large so that if $\bar{\bar{u}}(r, \theta, t)$ is the cigar solution given in (5.6) expressed in polar coordinates, then

$$\hat{r}^2\, \bar{\bar{u}}(\hat{r}, \theta, 0) \geq \frac{2\alpha}{3}.$$

This is possible because $\lim_{r\to+\infty} r^2\, \bar{\bar{u}}(r, \theta, 0) = \alpha$. Since $\hat{r}^2\, \bar{u}_k(\hat{r}, \theta, 0) \to \hat{r}^2\, \bar{\bar{u}}(\hat{r}, \theta, 0)$, as $k \to \infty$, we must have

$$\hat{r}^2 \rho_k^2\, \bar{u}(\rho_k \hat{r}, \theta, t_k) \geq \frac{\alpha}{2} \tag{5.10}$$

if k is sufficiently large and $\theta \in [0, 2\pi]$. It follows that if $\hat{s}_k := \log(\hat{r}\rho_k)$, then

$$\hat{u}(\hat{s}_k, \theta_k, t_k) \geq \frac{\alpha}{2}.$$

Since $r_k/\rho_k \to +\infty$, we may assume that $\hat{r}\rho_k << r_k$, which in particular implies that $\hat{s}_k < s_k$. By (5.4) and the bound from below on $\hat{u}$, we have $\hat{U}(\hat{s}_k, t_k) \geq -C$, for a uniform constant C.

We will now conclude that the bound (5.9) holds. If $\hat{U}(s_k, t_k) \to 0$, as $k \to \infty$, then it obviously holds. Otherwise, since $\lim_{s\to-\infty} \hat{U}(s, t_k) = -\infty$ and $\lim_{t_k\to-\infty} \hat{U}(s, t_k) = 0$ on compact subsets of $\mathbb{R}$ (remember $\lim_{t\to-\infty} \hat{u}(s, \theta, t) = \mu$ on compact subsets of $\mathbb{R} \times [0, 2\pi]$ and we have assumed that $\mu = 1$), we easily conclude that $\hat{U}_s(s, t_k) \geq 0$ for $s \leq s_k$ (recall that $s_k := \log r_k \to -\infty$). It follows that $\hat{U}(s_k, t_k) \geq \hat{U}(\hat{s}_k, t_k) \geq -C$, which proves (5.9).

For the given sequences $t_k \to -\infty$ and $s_k \to -\infty$, we define the translating solutions

$$\hat{u}_k(s, \theta, t) := \hat{u}(s + s_k, \theta, t + t_k),$$

which also satisfy equation (1.10) on $-\infty < t < |t_k|$. Set

$$\hat{U}_k(s, t) := \int_0^{2\pi} \log \hat{u}_k(s, \theta, t)\, d\theta.$$

Then $|(\hat{U}_k)_s| \leq 2$ and $\hat{U}_k \leq 0$ on $\mathbb{R} \times (-\infty, |t_k| - 1)$, since $|\hat{U}_s| \leq 2$ and $\hat{u} \leq 1$ on $\mathbb{R} \times (-\infty, -1)$. In addition, by (5.9), $\hat{U}_k(0, 0) \geq -C$ and also $|(\hat{U}_k)_t| \leq C$, for all $s \in R$ and $t < |t_k| - 1$ for a uniform constant C (since the scalar curvature $R(\cdot, t)$ is uniformly bounded on $t < -1$).

It follows that the sequence $\{\hat{U}_k\}$ is uniformly bounded on compact sets in space and time, and by (5.4) the same holds for the sequence $\log \hat{u}_k$. Hence, for a given compact set $K \subset \mathbb{R} \times [0, 2\pi] \times (-\infty, \infty)$, we have

$$0 < c < \hat{u}_k(s, \theta, t) \leq 1, \qquad (s, \theta, t) \in K$$

if k is chosen sufficiently large so that $K \subset \mathbb{R} \times [0, 2\pi] \times (-\infty, |t_k| - 1)$. Standard parabolic PDE arguments imply that, passing to a subsequence, $\hat{u}_k \to \tilde{\hat{u}}$ in C^∞ on compact sets of $\mathbb{R} \times [0, 2\pi] \times (-\infty, \infty)$. The function $\tilde{\hat{u}}$ is a smooth eternal solution of equation (1.10) on $\mathbb{R} \times [0, 2\pi] \times (-\infty, \infty)$.

We will next show that $\tilde{\hat{u}} \equiv \gamma$, for some constant γ, which implies that $\lim_{k\to\infty} Q(r_k, \theta_k, t_k) = 0$, contradicting our assumption (5.8).

Claim 5.6. *If $R(s, \theta, t)$ is the scalar curvature in cylindrical coordinates, then we have*

$$\lim_{t_k\to-\infty} R(s_k, \theta_k, t_k) = 0. \tag{5.11}$$

Proof. Assume the claim is not true, that is; there exists a $\delta > 0$ and a subsequence (s_k, θ_k, t_k) so that $R(s_k, \theta_k, t_k) \geq \delta > 0$, for all k. Passing to a subsequence, $\theta_k \to \theta_0$. Since $R_k := -\Delta \log \hat{u}_k/\hat{u}_k$ satisfies $R_k(s, \theta, t) = R(s + s_k, \theta, t + t_k)$ and $R_k \to \tilde{R} := -\Delta \log \tilde{\hat{u}}/\tilde{\hat{u}}$ uniformly on compact sets, we conclude that $\tilde{R}(0, \theta_0, 0) := \lim_{k\to\infty} R_k(0, \theta_k, 0) \geq \delta$.

It follows that there exists an $\epsilon > 0$ and k_0 so that

(5.12)
$$R(s+s_k,\theta,t_k) \geq \frac{\delta}{2}, \qquad \text{for all } (s,\theta) \in I_\epsilon := [-\epsilon,\epsilon]\times[\theta_0-\epsilon,\theta_0+\epsilon].$$

On the other hand, as we have proved earlier, we have $0 < c < \hat{u}_k(s,\theta,0) \leq 1$, for all $(s,\theta) \in I_\epsilon$. Combining this with (5.12) yields

$$\iint_{I_\epsilon} R_k(\cdot,t_k)\,\hat{u}_k(\cdot,0)\,ds\,d\theta \geq \hat{\delta} > 0 \qquad k \geq k_0,$$

or equivalently,

$$\iint_{I_\epsilon(s_k)} R(\cdot,t_k)\,\hat{u}(\cdot,t_k)\,ds\,d\theta \geq \hat{\delta} > 0 \qquad k \geq k_0, \tag{5.13}$$

where $I_\epsilon(s_k) := [s_k-\epsilon, s_k+\epsilon]\times[\theta_0-\epsilon,\theta_0+\epsilon]$.

Recall that by Lemma 5.5, $\bar{u}_k(x,y,t) := \rho_k^2\,\bar{u}(\rho_k x, \rho_k y, t+t_k)$ converges uniformly on compact subsets of $\mathbb{R}\times(-\infty,\infty)$ to a cigar solution. This implies that there exists a compact ball $B(0,\bar{r})$ (with $\bar{r}$ sufficiently large depending on η and η chosen arbitrarily small) so that

$$\Big|\iint_{B(0,\bar{r})} R_k\bar{u}_k(\cdot,0)\,dx\,dy - 4\pi\Big| < \eta,$$

or equivalently,

$$\Big|\iint_{B(0,\rho_k\bar{r})} R(\cdot,t_k)\,\bar{u}(\cdot,t_k)\,dx\,dy - 4\pi\Big| < \eta.$$

We may also choose $\bar{r}$ so that $\bar{r} \geq \hat{r}$, where $\hat{r}$ is chosen as before so that (5.10) holds.

Set $r_k = e^{s_k}$ and $\bar{r}\rho_k = e^{\hat{s}_k}$. Recall that since we are in the case where $r_k/\rho_k \to +\infty$, we may assume that $\hat{s}_k < s_k - 1$. The last integral inequality in cylindrical coordinates gives

$$\Big|\int_{-\infty}^{\hat{s}_k}\int_0^{2\pi} R(\cdot,t_k)\,\hat{u}(\cdot,t_k)\,d\theta\,ds - 4\pi\Big| < \eta. \tag{5.14}$$

Combining (5.13) with (5.14) and choosing $\eta << \hat{\delta}$, we obtain that

$$\int_{-\infty}^{s_k+\epsilon}\int_0^{2\pi} R(\cdot,t_k)\hat{u}(\cdot,t_k)\,d\theta\,ds > 4\pi + \frac{\eta}{2}. \tag{5.15}$$

Recall that $s_k \to -\infty$. Lemma 5.5 may be applied near the north pole N of S^2 corresponding to $\psi = \pi/2$, so that we may also conclude that, after passing to a subsequence, the rescaled solutions converge to a cigar. In our chosen cylindrical coordinates, this would imply that for the given sequence of times $t_k \to -\infty$, after passing to a subsequence, there exists a sequence $\tilde{s}_k \to +\infty$ for which

$$\Big|\int_{\tilde{s}_k}^{+\infty}\int_0^{2\pi} R(\cdot,t_k)\,\hat{u}(\cdot,t_k)\,d\theta\,ds - 4\pi\Big| < \frac{\eta}{4}. \tag{5.16}$$

Combining (5.15) with (5.16), we conclude that the total curvature

$$\int_{-\infty}^{+\infty}\int_0^{2\pi} R(\cdot,t_k)\hat{u}(\cdot,t_k)\,d\theta\,ds > 8\pi,$$

which is a contradiction to the total curvature of our evolving compact surface being equal always to 8π. This concludes the proof of the claim. q.e.d.

To finish the proof of the lemma, we will first show that $\tilde{\hat{u}} \equiv \gamma$, for a constant $\gamma > 0$. To this end, we will first prove that the scalar curvature $\tilde{R} := -\Delta \log \tilde{\hat{u}}/\tilde{\hat{u}}$ of the metric $\tilde{g} := \hat{u}\,(ds^2 + d\theta^2)$ is identically equal to zero. Clearly $\tilde{R} \geq 0$. If we prove that $\tilde{R}(0,\theta_0,0) = 0$, for some point $\theta_0 \in [0,2\pi]$, then $\tilde{R} \equiv 0$ by the strong maximum principle. But this readily follows from (5.11) by choosing a subsequence so that $\theta_k \to \theta_0$ and passing to the limit, similarly as in the proof of the previous claim.

To conclude that $\tilde{\hat{u}}$ is a constant, for a fixed t, set $w := \log \tilde{\hat{u}}$ and observe that w satisfies $\Delta_c w = 0$ and $w \leq 0$ on $\mathbb{R}\times[0,2\pi]$ (since $\hat{u}_k \leq 1$). We may view w as a harmonic function on $\mathbb{R}^2$ by extending it in the θ direction so that it remains 2π periodic. The bound $w \leq 0$ then implies that w must be a constant function, which shows that $\log \tilde{\hat{u}}(\cdot,t) = c(t)$, for all t. Since $R \equiv 0$, we conclude that $c(t)$ is constant in t, and hence $\log \tilde{\hat{u}} \equiv c$.

We will now conclude the proof of Lemma 5.3. We have just shown that $\hat{u}_k := \hat{u}(s+s_k,\theta,t_k) \to \gamma$, for some constant $\gamma > 0$, and the convergence is in C^∞ on compact subsets of $\mathbb{R}\times[0,2\pi]$. Going back to the plane coordinates, we conclude that $u_k := r_k^2\,u(r_k r,\theta,t_k) \to \gamma/r^2$ in C^∞ on compact subsets of the punctured plane $0 < r < \infty$. Notice that γ/r^2 is the cylindrical metric in plane coordinates. Since our quantity Q is dilation invariant and vanishes identically on the cylinder, this implies that $Q(r_k,\theta,t_k) \to 0$, which contradicts (5.8). q.e.d.

As an immediate consequence of Lemmas 5.2 and 5.3, we obtain:

Corollary 5.7. *We have $Q(\cdot,t) \equiv 0$, for all $-\infty < t < 0$. Consequently, the pressure function $\bar{v} := \bar{v}_N$ satisfies the identities*

$$(5.17)\qquad (a)\ \bar{v}_{xxx} = 3\,\bar{v}_{xyy} \quad \textit{and} \quad (b)\ \bar{v}_{yyy} = 3\,\bar{v}_{xxy}.$$

The above identities also imply the identities

$$(5.18)\qquad (a)\ \bar{v}_{xxxx} = \bar{v}_{yyyy} \quad \textit{and} \quad (b)\ \bar{v}_{xxxy} = \bar{v}_{yyyx} = 0.$$

We will now show that $\bar{v}(\cdot,t)$ must be a fourth-order polynomial of a certain form.

Lemma 5.8. *Let $\bar{v}(x,y)$ be a smooth function on $\mathbb{R}^2$ satisfying* (5.17). *Then $\bar{v}$ has the form*

$$\bar{v}(x,y) = a\,((x-x_1)^2 + (y-y_1)^2)^2 + q(x,y)$$

for some constants a, x_1, x_2*, and a quadratic polynomial* $q(x,y)$*.*

Proof. We will omit the details of calculations that can be checked in a straightforward manner by the reader. We will also denote by C, C_i various fixed constants. Identity (5.18)(b) implies that $\bar v_{xxx} = f_1(x)$ and $\bar v_{yyy} = g_1(y)$, and by (5.18)(a) we have $f_1(x) = C\,x + C_1$, $g_1(y) = C\,y + C_2$; hence,

$$\bar v_{xx} = \frac{C}{2}\,x^2 + C_1\,x + g_2(y), \qquad \bar v_{yy} = \frac{C}{2}\,y^2 + C_2\,y + f_2(x).$$

Combining the above identities with (5.17) gives

$$\bar v_{xx} = \frac{C}{2}x^2 + \frac{C}{6}y^2 + C_1 x + \frac{C_2}{3}y + C_3, \quad \bar v_{yy} = \frac{C}{2}y^2 + \frac{C}{6}x^2 + C_2 y + \frac{C_1}{3}x + C_4.$$

Differentiating these last identities in y, x, respectively gives

$$\bar v_{xxy} = \frac{C}{3}y + \frac{C_2}{3}, \qquad \bar v_{xyy} = \frac{C}{3}x + \frac{C_1}{3},$$

which after integration in x, y, respectively yield

$$\bar v_{xy} = \frac{C}{3}xy + \frac{C_2}{3}x + g_3(y) = \frac{C}{3}xy + \frac{C_1}{3}y + f_3(x).$$

It follows that

$$\bar v_{xy} = \frac{C}{3}xy + \frac{C_2}{3}x + \frac{C_1}{3}y + C_5.$$

If we set $q := \bar v - V$, where

$$V(x,y) = a\,((x-x_1)^2 + (y-y_1)^2)^2$$

with

$$a = \frac{C}{24},\ x_1 = -\frac{C_1}{24a},\ y_1 = -\frac{C_2}{24a},$$

then a direct computation shows that q satisfies

$$q_{xx} = C_3, \quad q_{yy} = C_4, \quad q_{xy} = C_5,$$

from which the lemma readily follows. q.e.d.

We will next show that our solution v has the particular form of the King–Rosenau solutions.

Lemma 5.9. *Let* $\bar v(x,y,t)$ *be an ancient solution of the equation*

$$\bar v_t = \bar v\,\Delta \bar v - |\nabla \bar v|^2 \qquad \text{on } \mathbb{R}^2 \times (-\infty, 0) \tag{5.19}$$

of the form

$$\bar v(x,y,t) = a\,((x-x_1)^2 + (y-y_1)^2)^2 + b\,(x-x_2)^2 + d\,(y-y_2)^2 + \rho\,xy + c,$$

where all a, b, c, d, ρ*, and* x_i, y_i *are functions of* t*. Assume in addition that*

$$\lim_{t\to-\infty} \bar v(x,y,t) = x^2 + y^2 \tag{5.20}$$

uniformly on compact subsets of $\mathbb{R}^2$*. Then*

$$\bar v(x,y,t) = a(t)\,(x^2+y^2)^2 + b(t)\,(x^2+y^2) + c(t) \tag{5.21}$$

for some functions of time $a(t), b(t), c(t)$ *that are defined on* $-\infty < t < 0$*.*

Proof. The lemma follows from a direct calculation where you plug a solution $\bar v(x,y,t)$ of the given form into equation (5.19) and compute the relation between all coefficients a, b, c, d, ρ, and x_i, y_i.

Indeed, by doing so, we first find the following equations relating the coefficients a, b, c, d, ρ:

$$a' = 2a\,(b+d), \quad (b-d)' = -2\,(b-d)\,(b+d), \quad \rho' = -2\rho\,(b+d). \tag{5.22}$$

From (5.20) we have

$$\lim_{t\to-\infty} a(t) = \lim_{t\to-\infty} c(t) = \lim_{t\to-\infty} \rho(t) = 0, \quad \lim_{t\to-\infty} b(t) = \lim_{t\to-\infty} d(t) = 1,$$

which, in particular, imply that $1 \le b+d \le 3$, if $t < t_0$ with t_0 sufficiently close to $-\infty$. Hence, the last two equations in (5.22) readily imply that $b \equiv d$ and $\rho \equiv 0$. Hence, $\bar v$ is now of the simpler form

$$\bar v(x,y,t) = a\,((x-x_1)^2 + (y-y_1)^2)^2 + b\,((x-x_2)^2 + (y-y_2)^2) + c,$$

where all $a, b, c,$ and x_i, y_i are all functions of t. Observe that since $\bar v(x,y,t) > 0$ on $\mathbb{R}^2 \times (-\infty, 0)$ and $\lim_{t\to-\infty} b(t) = 1$, all coefficients a, b, c are positive and $3/4 \le b(t) \le 5/4$, for $t \le t_0 < 0$. By (5.22), we now have

$$a' = 4\,ab \le\ 5\,a, \qquad t \le t_0 < 0,$$

which readily gives the bound

$$a(t) \ge C_1\, e^{5t} \tag{5.23}$$

for a constant $C_1 > 0$. Now, plugging $\bar v$ back into the equation, we find by direct calculation that

$$x_1' = -4b\,(x_1 - x_2), \qquad y_1' = -4b\,(y_1 - y_2) \tag{5.24}$$

and that $X(t) := x_1(t) - x_2(t)$ and $Y(t) := y_1(t) - y_2(t)$ both satisfy the same equation

$$X' = -\frac{4X}{b}\,(b^2 + 4ac + 4ab\,(X^2+Y^2)),$$

and the same for Y. It follows that $\phi(t) := X^2 + Y^2 > 0$ satisfies the equation

$$\phi' = -\frac{8\phi}{b}\,(b^2 + 4ac + 4ab\,\phi), \tag{5.25}$$

where $b^2 + 4ac + 4ab\,\phi \ge b^2 > 0$ for $t < t_0$. Since $\lim_{t\to-\infty} b(t) = 1$, we have $3/4 \le b(t) \le 5/4$, for $t < t_0 < 0$. It follows from (5.25) that

$$\phi' = -8\,\phi b \le -6\,\phi b, \qquad t \le t_0 < 0,$$

which implies the bound

$$\phi(t) \geq C_2 \, e^{6|t|}, \qquad t \leq t_0 < 0, \tag{5.26}$$

for a constant $C_2 > 0$, unless $\phi \equiv 0$.

We will next show that $\phi \equiv 0$. Observe first that from (5.20) and the fact that $\lim_{t \to -\infty} b(t) = 1$, we have

$$\lim_{t \to -\infty} a(t)\,(x_1^2(t) + y_1^2(t))^2 = 0, \qquad \lim_{t \to -\infty} (x_2^2(t) + y_2^2(t)) = 0,$$

which yields

$$\lim_{t \to -\infty} a(t)\, \phi^2(t) = 0. \tag{5.27}$$

On the other hand, it follows from (5.26) and (5.23) that

$$a(t)\, \phi^2(t) \geq C\, e^{5t + 12|t|} = C\, e^{7|t|},$$

which contradicts (5.27). Hence, $\phi \equiv 0$. Once we know that $\phi \equiv 0$, (5.24) and (5.20) yield $x_1(t) = x_2(t) = 0$ and $y_1(t) = y_2(t) = 0$ for all t.

We conclude from the above discussion that the solution v_∞ is of the form (5.21). q.e.d.

We will now conclude that our solution is one of the King–Rosenau solution in plane coordinates. Such solutions were first discovered by King [**13**].

Lemma 5.10. *Let $\bar{v}(x, y, t)$ be an ancient solution of the equation* (5.19) *of the form* (5.21)*. Then, up to a dilation constant, which makes $a(t) = c(t)$ for all t, we have*

$$a(t) = -\frac{\mu}{2} \operatorname{csch}(4\mu t) \quad \text{and} \quad b(t) = -\mu \coth(4\mu t). \tag{5.28}$$

Proof. If we plug a solution of the form (5.21) into the equation (5.19), we find that the coefficients a, b, c must satisfy the equations

$$a' = 4\, b\, a, \qquad c' = 4\, b\, c, \qquad b' = 16\, a\, c. \tag{5.29}$$

Since $a(t) > 0$ and $c(t) > 0$, the first two equations imply that

$$(\log a(t))' = (\log c(t))',$$

which shows that

$$c(t) = \lambda^2\, a(t)$$

for a constant $\lambda > 0$. By performing a dilation $\bar{v}_\lambda(x, y, t) = \lambda^{-2}\, \bar{v}(\lambda x, \lambda y, t)$ (which leaves $b(t)$ unchanged), we may assume that $\lambda = 1$; i.e., $a \equiv c$. The functions a, b satisfy the system

$$a' = 4\, b\, a \qquad \text{and} \qquad b' = 16\, a^2. \tag{5.30}$$

Solving this system gives us (5.28) for a given constant $\mu > 0$ (if we assume that $\lim_{t \to -\infty} b(t) = 1$, then $\mu = 1$). q.e.d.

We will now conclude the proof of Theorem 5.1.

Proof of Theorem 5.1. We observe that if $\bar{v}(r,t) = a(t)\,r^4 + b(t)\,r^2 + a(t)$ is the King–Rosenau solution in polar coordinates, then in cylindrical coordinates it takes the form

$$\hat{v}(s,t) = 2a(t)\cosh^2 s + b(t).$$

Recalling that $a(t)$ and $b(t)$ are given by (5.28) and using (1.8), we conclude, by direct calculation, that

$$v(\psi,t) = -\mu\coth(2\mu t) + \mu\,\tanh(2\mu t)\,\sin^2\psi,$$

finishing the proof of the theorem. q.e.d.

6. The Contracting Spheres

Throughout this section we will assume that the backward lmit

$$v_\infty := \lim_{t\to-\infty} v(\cdot,t) \equiv 0. \tag{6.1}$$

Our goal is to show that in this case the ancient solution v must be a family of contracting spheres, as stated in the following theorem.

Theorem 6.1. *If the backward limit $v_\infty \equiv 0$, then*

$$v(\cdot,t) = \frac{1}{(-2t)},$$

that is, our ancient solution is a family of contracting spheres.

To prove the theorem, we will use an isoperimetric estimate for the Ricci flow that was proven by R. Hamilton in [**10**]. Let M be any compact surface. Any simple closed curve γ on M of length $L(\gamma)$ divides the compact surface M into two regions with areas $A_1(\gamma)$ and $A_2(\gamma)$. We define the isoperimetric ratio as in [**4**], namely,

$$I = \frac{1}{4\pi}\,\inf_\gamma L^2(\gamma)\,\left(\frac{1}{A_1(\gamma)} + \frac{1}{A_2(\gamma)}\right). \tag{6.2}$$

It is well known that $I \le 1$ always, and that $I \equiv 1$ if and only if the surface M is a sphere.

We will briefly outline the proof of Theorem 6.1 whose steps will be proven in detail afterwards. We consider our evolving surfaces at each time $t < 0$ and define the isoperimetric ratio $I(t)$ as above. Our goal is to show that our assumption (6.1) implies that $I(t) \equiv 1$, which forces $(M, g(t))$ to be a family of contracting spheres. We will argue by contradiction and assume that $I(t_0) < 1$, for some $t_0 < 0$. In that case we will show that there exists a sequence $t_k \to -\infty$ and closed curves β_k on S^2 so that simultaneously we have

$$L_{S^2}(\beta_k) \ge \delta > 0 \quad \text{and} \quad L_{g(t_k)}(\beta_k) \le C \qquad \forall k, \tag{6.3}$$

where L_{S^2} and $L_{g(t_k)}$ denote the length of a curve in the round metric on S^2 and in the metric $g(t_k)$, respectively. This clearly contradicts the fact that $u(\cdot, t_k) \to \infty$, uniformly in S^2 (implied by (6.1)) and finishes the proof.

We will now outline how we will find the curves β_k. For each $t < t_0$, let γ_t be a curve for which the isoperimetric ratio $I(t)$ is achieved.

i. If $I(t_0) < 1$, for some $t_0 < 0$, we will show that $I(t) \le \frac{C}{|t|}$, for $t < t_0$. We will use that to show $L_{g(t)}(\gamma_t) \le C$, for all $t < t_0$.
ii. For any sequence $t_k \to -\infty$ and $p_k \in \gamma_{t_k}$, we will show that there exists a subsequence such that $(M, g(t_k), p_k)$ converges to $(M_\infty, g_\infty, p_\infty)$, where $M_\infty = S^1 \times \mathbb{R}$ and $\gamma_\infty := \lim_{k\to\infty} \gamma_{t_k}$ is a closed geodesic on M_∞, one of the cross circles of $S^1 \times \mathbb{R}$.
iii. Let t_k be as above. If $A_1(t_k), A_2(t_k)$ are the areas of the two regions into which γ_{t_k} divides S^2, we show that both of them are comparable to $|t_k| = -t_k$.
iv. We show that the maximal distances from γ_{t_k} to the points of the two regions of areas $A_1(t_k), A_2(t_k)$, respectively, are both of length comparable to $|t_k|$.
v. The curves γ_{t_k} do not necessarily satisfy (6.3). However, we use them and (ii) to define a foliation $\{\beta^k_w\}$ of our surfaces $(M, g(t_k))$, and we choose the curve β_k from this foliation that splits S^2 into two parts of equal areas with respect to the round metric. We prove that this is the curve that satisfies (6.3) by using that $I_{S^2} = 1$, the Bishop–Gromov volume comparison principle, (iii) and (iv).

Lemma 6.2. *If $I(t_0) < 1$, for some $t_0 < 0$, then there exist positive constants C_1, C_2 so that*

$$I(t) \le \frac{C_1}{|t| + C_2} \qquad \text{for all } t < t_0.$$

Moreover, if γ_t is the curve at which the infimum in (6.2) is attained, then

$$L(t) := L(\gamma_t) \le C \qquad \text{for all } t < t_0.$$

Proof. Let $t < t_0$, with t_0 as in the statement of the lemma. It has been shown in [**10**] that

$$I'(t) \ge \frac{4\pi\,(A_1^2 + A_2^2)}{A_1 A_2 (A_1 + A_2)}\, I\,(1 - I^2).$$

Since $A_1 + A_2 = 8\pi|t|$ and $A_1^2 + A_2^2 \ge 2A_1A_2$, we conclude the differential inequality

$$I'(t) \ge \frac{1}{|t|}\, I\,(1 - I^2).$$

Since $I(t_0) < 1$, the above inequality implies the bound

$$I(t) \le \frac{C_1}{|t| + C_2} \qquad \text{for all } t < t_0$$

for uniform in time constants C_1 and C_2. Using that $\frac{1}{A_1} + \frac{1}{A_2} \geq \frac{1}{4\pi|t|}$, we will conclude that that the length $L(t)$ of a curve γ_t at which the infimum in (6.2) is attained satisfies $L(t) \leq C$, for all $t < t_0$. q.e.d.

We also have the following estimate from below on the length $L(t)$ of the curve at which the infimum in (6.2) is attained.

Lemma 6.3. *There is a uniform constant $c > 0$, independent of time so that*

$$L(t) \geq c \qquad \textit{for all } \; t \leq t_0 < 0.$$

Proof. Recall that for $t_0 < 0$, the scalar curvature R satisfies $0 < R(\cdot, t) \leq C$, for all $t \leq t_0$. The Klingenberg injectivity radius estimate for even-dimensional manifolds implies the bound

$$\text{injrad}(g(t)) \geq \frac{c}{\sqrt{R_{\max}}} \geq \delta > 0 \qquad \text{for all } \; t \leq t_0 < 0 \tag{6.4}$$

for a uniform in time constant $c > 0$. We will prove the lemma by contradiction. Assume that there is a sequence $t_i \to -\infty$, so that $L_i := L(t_i) \to 0$, as $i \to \infty$, and denote by γ_{t_i} a curve at which the isoperimetric ratio is attained, i.e., $L(t_i) = L(\gamma_{t_i})$.

Define a new sequence of re-scaled Ricci flows, $g_i(t) := L_i^{-2}\, g(t_i + L_i^2\, t)$, and take a sequence of points $p_i \in \gamma_{t_i}$. The bound (6.4) implies a lower bound on the injectivity radius at p_i with respect to metric g_i, namely,

$$\text{injrad}_{g_i}(p_i) = \frac{\text{injrad}_{g(t_i)}(p_i)}{L_i^2} \geq \frac{\delta}{L_i^2} \to \infty \qquad \text{as } \; i \to \infty. \tag{6.5}$$

Also, since $R_i(\cdot, t) = L_i^2\, R(\cdot, t_i + L_i^2\, t) \leq C\, L_i^2$ and $L_i \to 0$, we get

$$\max R_i(\cdot, t) \to 0 \qquad \text{as } \; i \to \infty. \tag{6.6}$$

Hamilton's compactness theorem (c.f. in [**11**]) implies, passing to a subsequence, the pointed smooth convergence of $(M, g_i(0), p_i))$ to a complete manifold $(M_\infty, g_\infty, p_\infty)$, which is, due to (6.5) and (6.6), a standard plane. Moreover,

$$I(t_i) = \frac{1}{4\pi} L_i^2 \left(\frac{1}{A_1(t_i)} + \frac{1}{A_2(t_i)} \right) = \frac{1}{4\pi} \left(\frac{1}{A_1(g_i(0))} + \frac{1}{A_2(g_i(0))} \right),$$

where $A_1(g_i(0))$ and $A_2(g_i(0))$ are the areas inside and outside the curve γ_{t_i}, respectively, both computed with respect to metric $g_i(0)$. Since $g_i(0)$ converges to the euclidean metric and γ_{t_i} converges to a curve of length 1, it follows that $\lim_{i\to\infty} A_1(g_i(0)) = \alpha > 0$ and $\lim_{i\to\infty} A_2(g_i(0)) = \infty$, which implies that

$$\lim_{i\to\infty} I(t_i) \geq \delta > 0$$

and obviously contradicts Lemma 6.2. q.e.d.

We recall that at each time t, a curve γ_t at which the isoperimetric ratio is achieved splits the surface into two regions of areas $A_1(t)$ and $A_2(t)$. Lemma 6.3 yields to the following conclusion.

Corollary 6.4. *There are uniform constants $c > 0$ and $C > 0$ so that*

$$c\,|t| \le A_1(t) \le C\,|t| \quad \textit{and} \quad c\,|t| \le A_2(t) \le C\,|t|$$

for all $t < t_0 < 0$.

Proof. It is well known that the total area of our evolving surface is $A(t) = 8\pi\,|t|$. Hence, $A_1(t) \le 8\pi\,|t|$ and $A_2(t) \le 8\pi\,|t|$. On the other hand, by Lemmas 6.2 and 6.3, we have

$$\frac{c}{A_j(t)} \le \frac{L^2(t)}{A_j(t)} \le I(t) \le \frac{C}{|t|} \qquad j = 1, 2$$

for all $t < t_0$, which shows that $A_j(t) \ge c\,|t|$, $j = 1, 2$, for a uniform constant $c > 0$, therefore proving the corollary. q.e.d.

We will fix in the sequel a sequence $t_k \to -\infty$. Let γ_{t_k} be, as before, a curve at which the isoperimetric ratio is achieved. From now on we will refer to γ_{t_k} as an isoperimetric curve at time t_k. To simplify the notation, we will set $A_{1k} := A_1(t_k)$, $A_{2k} := A_2(t_k)$ and $L_k = L(t_k)$. It follows from Corollary 6.4 that

$$\lim_{k\to\infty} A_{1k} = +\infty \qquad \text{and} \qquad \lim_{k\to\infty} A_{2k} = +\infty. \tag{6.7}$$

Pick a sequence of points $p_k \in \gamma_{t_k}$ and look at the pointed sequence of solutions $(M, g(t_k + t), p_k)$. Since the curvature is uniformly bounded and since the injectivity radius at p_k is uniformly bounded from below, by Hamilton's compactness theorem we can find a subsequence of pointed solutions that converge, in the Cheeger–Gromov sense, to a complete smooth solution $(M_\infty, g_\infty, p_\infty)$. This means that for every compact set $K \subset M_\infty$ there are compact sets $K_k \subset M$ and diffeomorphisms $\phi_k : K \to K_k$ so that $\phi_k^* g(t_k)$ converges to g_∞. From Lemma 6.2, $L(t_k) \le C$, for all k, and therefore our curves γ_{t_k} converge to a curve γ_∞ (this convergence is induced by the manifold convergence) which by (6.7) has the property that it splits M_∞ into two parts (call them $M_{1\infty}$ and $M_{2\infty}$), each of which has infinite area. It follows that we can choose points $x_j \in M_{1\infty}$ and $y_j \in M_{2\infty}$ so that $\mathrm{dist}_{g_\infty}(x_j, p_\infty) = \mathrm{dist}_{g_\infty}(p_\infty, y_j) = \rho_j$, where ρ_j is an arbitrary sequence so that $\rho_j \to \infty$. Since (M_∞, g_∞) is complete, there exists a minimal geodesic β_j from x_j to y_j. This geodesic β_j intersects γ_∞ at some point q_j. Since $q_j \in \gamma_\infty$ and γ_∞ is a closed curve of finite length, the set $\{q_j\}$ is compact and therefore there is a subsequence so that $q_j \to q_\infty \in \gamma_\infty$. This implies that there is a subsequence of geodesics $\{\beta_j\}$ so that, as $j \to \infty$, it converges to a minimal geodesic $\beta_\infty : (-\infty, \infty) \to M_\infty$ (minimal geodesic means a globally distance minimizing geodesic). It follows that our limiting manifold M_∞ contains a straight line. Since the curvature of M_∞ is zero, by the splitting theorem our manifold splits off a line and therefore is diffeomorphic to the cylinder $S^1 \times \mathbb{R}$.

We next observe that the limiting curve γ_∞ is a geodesic, as shown in the following lemma.

Lemma 6.5. *The geodesic curvature κ of the curve γ_∞ is zero.*

Proof. As in [**10**], at each time $t < t_0 < 0$, we start with the isoperimetric curve γ_t and we construct the one-parameter family of parallel curves γ_t^r at distance r from γ_t on either side. We take $r > 0$ when the curve moves from the region of area $A_1(t)$ to the region of area $A_2(t)$, and $r < 0$ when it moves the other way. We then regard L, A_1, A_2, and

$$I = I(\gamma_t^r) = \frac{1}{4\pi} L^2(\gamma_t^r) \left(\frac{1}{A_1(\gamma_t^r)} + \frac{1}{A_2(\gamma_t^r)} \right)$$

as functions of r and t. By the computation in [**10**], we have

$$\frac{\partial A_1}{\partial r} = L \qquad \frac{\partial A_2}{\partial r} = -L \qquad \frac{dL}{dr} = \int \kappa \, ds = \kappa \, L,$$

where κ is the geodesic curvature of the curve γ_t^r. By a standard variational argument, κ is constant on γ_t. If $A := A_1 + A_2$ is the total surface area, we have

$$\log I = 2 \log L + \log A - \log A_1 - \log A_2 - \log(4\pi).$$

Since $\frac{\partial I}{\partial r}|_{r=0} = 0$, we conclude that

$$0 = \frac{2}{L} \frac{\partial L}{\partial r} + \frac{1}{A} \frac{\partial A}{\partial r} - \frac{1}{A_1} \frac{\partial A_1}{\partial r} - \frac{1}{A_2} \frac{\partial A_2}{\partial r} = \frac{2}{L} \kappa L - \frac{1}{A_1} L + \frac{1}{A_2} L,$$

which leads to

$$\kappa = \frac{L}{2} \left(\frac{1}{A_1} - \frac{1}{A_2} \right).$$

By Lemmas 6.2 and 6.3 and (6.7), we conclude that

$$\kappa_\infty := \lim_{t \to -\infty} \kappa = \lim_{t \to -\infty} \frac{L}{2} \left(\frac{1}{A_1} - \frac{1}{A_2} \right) = 0,$$

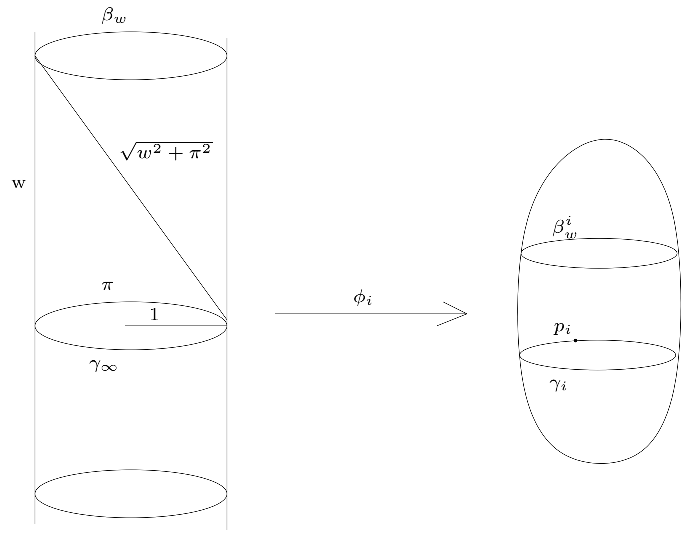
which means the geodesic curvature κ_∞ of the limiting curve γ_∞ is zero.

q.e.d.

We have just shown that our limiting manifold is a cylinder $M_\infty = S^1 \times \mathbb{R}$ and γ_∞ is a closed geodesic on M_∞. Hence, γ_∞ is one of the cross circles of M_∞.

We have the following picture, assuming that the radius of γ_∞ is 1.

Assume that we have a foliation of our limiting cylinder M_∞ by circles β_w, where $|w|$ is the distance from β_w to γ_∞, taking $w > 0$ if β_w lies on the upper side of the cylinder and $w < 0$ if β_w lies on its lower side. Denote by β_w^k the curve on M such that $\phi_k^* \beta_w^k = \beta_w$.



One of the properties of the cylinder is that for every $\delta > 0$ there is a $w_0 > 0$, so that for every $|w| \geq w_0$ we have

$$|\sup_{x\in\beta_w, y\in\gamma_\infty} \mathrm{dist}(x,y) - \inf_{x\in\beta_w, y\in\gamma_\infty} \mathrm{dist}(x,y)| \leq \sqrt{w^2+\pi^2} - |w| \leq \frac{C}{|w|} \leq \frac{C}{w_0} < \frac{\delta}{2}$$

where the distance is computed in the cylindrical metric on M_∞.

Since for every sequence $t_k \to -\infty$, there exists a subsequence for which we have uniform convergence of our metrics $\{g(t_k)\}$ on bounded sets around the points $p_k \in \gamma_{t_k}$, the previous observation implies the following claim, which will be used frequently from now on.

Claim 6.6. *For every sequence $t_k \to -\infty$ and every $\delta > 0$ there exists k_0 and w so that for $k \geq k_0$,*

$$|\sup_{x\in\beta_k^w, y\in\gamma_{t_k}} \mathrm{dist}_{g(t_k)}(x,y) - \inf_{x\in\beta_k^w, y\in\gamma_{t_k}} \mathrm{dist}_{g(t_k)}(x,y)| < \delta.$$

The variant of the Bishop–Gromov volume comparison principle (since $R \geq 0$) implies the following area comparison of the annuli, for each $t < 0$,

$$\frac{\mathrm{area}(b_1 \leq s \leq b_2)}{\mathrm{area}(a_1 \leq s \leq a_2)} \leq \frac{b_2^2 - b_1^2}{a_2^2 - a_1^2} \tag{6.8}$$

where $a_1 \leq a_2 \leq b_1 \leq b_2$ and s is the distance from a fixed point on $(M, g(t))$, computed with respect to the metric $g(t)$. We are going to use this fact in the lemma that follows.

For each k, γ_{t_k} splits our manifold in two parts; call them M_{1k} and M_{2k} with areas A_{1k} and A_{2k}, respectively. Choose points $x_k \in M_{1k}$ and

$y_k \in M_{2k}$ so that

$$\mathrm{dist}_{g(t_k)}(x_k, \gamma_{t_k}) = \max_{z \in M_{1k}} \mathrm{dist}_{g(t_k)}(\gamma_{t_k}, z) =: \rho_k$$

and

$$\mathrm{dist}_{g(t_k)}(y_k, \gamma_{t_k}) = \max_{z \in M_{2k}} \mathrm{dist}_{g(t_k)}(\gamma_{t_k}, z) =: \sigma_k.$$

By the definition of σ_k and ρ_k and from the convergence of $(M, g(t_k), p_k)$ to an infinite cylinder, we have

$$\lim_{k\to\infty} \sigma_k = +\infty \qquad \text{and} \qquad \lim_{k\to\infty} \rho_k = +\infty.$$

Lemma 6.7. *There are uniform constants $k_0 > 0$ and $c > 0$ so that the*

$$\mathrm{area}\,(B_{\rho_k}(x_k)) \geq c\,\rho_k \quad \textit{and} \quad \mathrm{area}\,(B_{\sigma_k}(y_k)) \geq c\,\sigma_k, \qquad \textit{for all } k \geq k_0$$

where both the distance and the area are computed with respect to the metric $g(t_k)$.

Proof. We take $a_1 = 0$, $a_2 = b_1 = \sigma_k \geq 1$, and $b_2 = \sigma_k + 1$ in (6.8). Then, if s is the distance from y_k computed with respect to $g(t_k)$, we have

$$\frac{\mathrm{area}\,(\sigma_k \leq s \leq \sigma_k + 1)}{\mathrm{area}\,(0 \leq s \leq \sigma_k)} \leq \frac{(\sigma_k+1)^2 - \sigma_k^2}{\sigma_k^2} \leq \frac{3}{\sigma_k}.$$

Hence,

$$\mathrm{area}\,(B_{\sigma_k}(y_k)) \geq \frac{\sigma_k}{3}\,\mathrm{area}\,(\sigma_k \leq s \leq \sigma_k + 1). \tag{6.9}$$

Having (6.9), the proof of Lemma 6.7 is finished once we show the following estimate: there are uniform constants $c > 0$ and k_1 so that for $k \geq k_1$,

$$\mathrm{area}\,(\sigma_k \leq s \leq \sigma_k + 1) \geq c. \tag{6.10}$$

To prove the estimate, denote by $U_k := \{z \mid \sigma_k \leq s \leq \sigma_k + 1\}$. We consider the set

$$V_k := \{z \mid \mathrm{dist}_{t_k}(z, \gamma_{t_k}) \leq \frac{1}{2},\ \overline{y_k z} \cap \gamma_{t_k} \neq \emptyset\}$$

where $\overline{y_k z}$ denotes a geodesic connecting the points y_k and z. It is enough to show that $V_k \subset U_k$, for k sufficiently large, and that area $(V_k) \geq c > 0$. To prove that $V_k \subset U_k$, take $z \in V_k$ and let $w_k \in \gamma_{t_k}$ be such that $\mathrm{dist}_{t_k}(z, w_k) = \mathrm{dist}_{t_k}(z, \gamma_{t_k}) \leq \frac{1}{2}$. If $q_k := \gamma_{t_k} \cap \overline{z y_k}$, then

$$\sigma_k = \mathrm{dist}_{t_k}(y_k, \gamma_{t_k}) \leq \mathrm{dist}_{t_k}(y_k, q_k) \leq \mathrm{dist}_{t_k}(z, y_k),$$

which implies that $\sigma_k \leq \mathrm{dist}_{t_k}(z, y_k)$. On the other hand, by Claim 6.6 we have $\mathrm{dist}_{t_k}(w_k, y_k) \leq \sigma_k + \frac{1}{2}$, for k sufficiently large. Hence,

$$\mathrm{dist}_{t_k}(z, y_k) \leq \mathrm{dist}_{t_k}(y_k, w_k) + \mathrm{dist}_{t_k}(w_k, z) \leq \sigma_k + \frac{1}{2} + \frac{1}{2} < \sigma_k + 1$$

for k sufficiently large. This proves that $V_k \subset U_k$ and hence

$$\text{area}\,(U_k) \geq \text{area}\,(V_k).$$

To estimate area$\,(V_k)$ from below, we recall that for $p_k \in \gamma_{t_k}$, we have pointed convergence of $(M, g(t_k), p_k)$ to a cylinder which is uniform on compact sets around p_k. To use this we need to show there is a constant $C > 0$, for which

$$V_k \subset B_{t_k}(p_k, C) \qquad \text{for all } k \geq k_0.$$

Let $z \in V_k$ and let $q_k \in \overline{y_k z} \cap \gamma_{t_k}$. Then by Claim 6.6 for k sufficiently large, we have

$$\sigma_k - 1 \leq \text{dist}_{g(t_k)}(y_k, q_k) \leq \sigma_k + 1. \tag{6.11}$$

We also have

$$\text{dist}_{g(t_k)}(p_k, z) \leq \text{dist}_{g(t_k)}(p_k, q_k) + \text{dist}_{g(t_k)}(q_k, z).$$

Since, $z \in V_k \subset U_k$ and (6.11) holds, we get

$$\text{dist}_{g(t_k)}(q_k, z) \leq \text{dist}_{g(t_k)}(y_k, z) - \text{dist}_{g(t_k)}(y_k, q_k) \leq \sigma_k + 1 - \sigma_k + 1 = 2,$$

which combined with $\text{dist}_{g(t_k)}(p_k, q_k) \leq L(\gamma_{t_k}) \leq C$ gives us the bound

$$\text{dist}_{g(t_k)}(p_k, z) \leq C.$$

This guarantees that, as $k \to \infty$, V_k converges to a part of the cylinder $S^1 \times \mathbb{R}$, while $(M, g(t_k), p_k) \to (S^1 \times \mathbb{R}, g_\infty, p_\infty)$ and g_∞ is the cylindrical metric. Recall that $\gamma_{t_k} \to \gamma_\infty$ and γ_∞ is one of the cross circles on $S^1 \times \mathbb{R}$. It follows that V_k converges as $k \to \infty$ to the upper or lower part of the set $\{z \in S^1 \times \mathbb{R} \mid \text{dist}_{g_\infty}(z, \gamma_\infty) \leq \frac{1}{2}\}$ with respect to γ_∞. This implies that

$$c \leq \text{area}\,(\{\sigma_k \leq s \leq \sigma_k + 1\}) \leq C \qquad \text{for } k \geq k_0, \tag{6.12}$$

for some uniform constants $c, C > 0$, finishing the proof of (6.10) and therefore Lemma 6.7. q.e.d.

Let us denote briefly by $A_{\sigma_k} := \text{area}\,(B_{\sigma_k}(y_k))$ and $A_{\rho_k} := \text{area}\,(B_{\rho_k}(x_k))$.

Lemma 6.8. *There exist a number k_0 and constants $c_1 > 0, c_2 > 0$, so that*

$$c_1\,|t_k| \leq A_{\rho_k} \leq c_2\,|t_k| \quad \textit{and} \quad c_1\,|t_k| \leq A_{\sigma_k} \leq c_2\,|t_k| \qquad \textit{for all } k \geq k_0.$$

Proof. Notice that

$$A_{\rho_k} + A_{\sigma_k} \leq 2\,A(t_k) = 16\pi\,|t_k| \tag{6.13}$$

since $A(t_k) = 8\pi|t_k|$ is the total surface area. Hence,

$$A_{\rho_k} \leq C\,|t_k| \qquad \text{and} \qquad A_{\sigma_k} \leq C\,|t_k|. \tag{6.14}$$

To establish the bounds from below, we will use Lemma 6.7 and show that there is a uniform constant c so that

$$\sigma_k \geq c\,|t_k| \qquad \text{and} \qquad \rho_k \geq c\,|t_k| \qquad \text{for all } k \geq k_0. \tag{6.15}$$

We will first show there are uniform constants $c > 0$ and $C < \infty$, so that

$$c\,\rho_k \leq \sigma_k \leq C\,\rho_k. \tag{6.16}$$

Recall that $\sigma_k = \mathrm{dist}_{g(t_k)}(y_k, \gamma_{t_k})$. By our choice of points x_k, y_k and the figure we have that the $\mathrm{diam}(M, g(t_k)) \leq \sigma_k + \rho_k + 1$ for $k \geq k_0$, sufficiently large. We also have that the subset of M that corresponds to area $A_2(t_k)$ contains a ball $B_{\sigma_k}(y_k)$. By Corollary 6.4 and the comparison inequality (6.8), we have

$$\begin{aligned} c &\leq \frac{A_1(t_k)}{A_2(t_k)} \leq \frac{\mathrm{area}(B_{\rho_k+\sigma_k+1}(y_k)\backslash B_{\sigma_k}(y_k))}{\mathrm{area}(B_{\sigma_k}(y_k))} \\ &= \frac{\mathrm{area}(\sigma_k \leq s \leq \rho_k + \sigma_k + 1)}{\mathrm{area}(0 \leq s \leq \sigma_k)} \leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{\sigma_k^2}. \end{aligned}$$

Using the previous inequality, we obtain the bound

$$c\,\sigma_k^2 - 2\,\rho_k - 2\sigma_k - 2\rho_k\,\sigma_k - 1 \leq \rho_k^2.$$

We claim there is a uniform constant c so that $\sigma_k \leq c\rho_k$. If not, then $\rho_k << \sigma_k$ for $k >> 1$, and from the inequality above we get

$$\frac{c}{2}\,\sigma_k^2 \leq \rho_k^2 \qquad \text{for } k >> 1.$$

In any case, there are k_1 and $C_1 > 0$ so that

$$\rho_k \leq C_1 \sigma_k \qquad \text{for } k \geq k_1. \tag{6.17}$$

By a similar analysis, as above, there are $k_2 \geq k_1$ and $C_2 > 0$ such that

$$\sigma_k \leq C_2 \rho_k \qquad \text{for } k \geq k_1. \tag{6.18}$$

We will now conclude the proof of Lemma 6.8. By Lemma 6.7 and (6.13) it follows that

$$\rho_k + \sigma_k \leq C|t_k| \qquad \text{for } k >> 1.$$

By (6.17) and (6.18) it follows that

$$\rho_k \leq C\,|t_k| \quad \text{and} \quad \sigma_k \leq C\,|t_k| \qquad \text{for } k >> 1.$$

Moreover, by (6.8), we have

$$\begin{aligned} \frac{A_1(t_k)}{\mathrm{area}\,(\sigma_k - 1 \leq s \leq \sigma_k)} &\leq \frac{\mathrm{area}\,(B_{\rho_k+\sigma_k+1}(y_k)\backslash B_{\sigma_k}(y_k))}{\mathrm{area}\,(\sigma_k - 1 \leq s \leq \sigma_k)} \\ &\leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{\sigma_k^2 - (\sigma_k - 1)^2} \leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{2\sigma_k - 1} \\ &\leq \frac{(\rho_k + \sigma_k + 1)^2 - \sigma_k^2}{2\sigma_k - 1} \\ &\leq C\,\rho_k, \end{aligned} \tag{6.19}$$

where we have used (6.17) and (6.18). The same analysis that yielded to (6.12) can be applied again to conclude that

$$\text{area}(\sigma_k - 1 \le s \le \sigma_k) \le C.$$

This, together with Corollary 6.4 and (6.19), implys

$$\rho_k \ge c\,|t_k| \qquad \text{for } \; k \ge k_0.$$

Claim 6.16 implies the same conclusion about σ_k. This is sufficient to conclude the proof of Lemma 6.8, as we have explained at the beginning of it. q.e.d.

We will now finish the proof of Theorem 6.1.

Proof of Theorem 6.1. If the isoperimetric constant $I(t) \equiv 1$, it follows by a well-known result that our solution is a family of contracting spheres. Hence, we will assume that $I(t_0) < 1$, for some $t_0 < 0$, which implies all the results in this section are applicable. We will show that this contradicts the fact that $\lim_{t\to-\infty} v(\cdot,t) = 0$, uniformly on S^2.

As explained at the beginning of this section, it suffices to find positive constants δ, C and curves β_k, so that

$$L_{S^2}(\beta_k) \ge \delta > 0 \qquad \text{and} \qquad L_k(\beta_k) \le C < \infty \tag{6.20}$$

where L_{S^2} denotes the length of a curve computed in the round spherical metric and L_k denotes the length of a curve computed in the metric $g(t_k)$. If we manage to find those curves β_k, that would imply

$$C \ge L_k(\beta_k) = \int_{\beta_k} \sqrt{u(t_k)}\, d_{S^2} \ge M\, L_{S^2}(\beta_k) \ge M\,\delta, \quad \text{for } \; k \ge k_0,$$

where d_{S^2} is the length element with respect to the standard round spherical metric, $M > 0$ is an arbitrary big constant, and k_0 is sufficiently large so that $\sqrt{u(t_k)} \ge M$, for $k \ge k_0$, uniformly on S^2 (which is justified by the fact $v(\cdot,t)$ converges uniformly to zero on S^2, in $C^{1,\alpha}$ norm). The last estimate is impossible, when M is taken larger than C/δ, hence finishing the proof of our theorem.

We will now prove (6.20). Our isoperimetric curves γ_{t_k} have the property that $L_k(\gamma_{t_k}) \le C$ for all k, but we do not know whether $L_{S^2}(\gamma_{t_k}) \ge \delta > 0$, uniformly in k. For each k, we will choose the curve β_k that will satisfy (6.20) from a constructed family of curves $\{\beta_\alpha^k\}$ that foliate our solution $(M, g(t_k))$. Define the foliation of $(M, g(t_k))$ by the curves $\{\beta_\alpha^k\}$ so that for every α and every $x \in \beta_\alpha^k$, $\text{dist}_{t_k}(x, y_k) = \alpha$. Choose a curve β_k from that foliation so that the corresponding curve $\tilde\beta_k$ on S^2 splits S^2 in two parts of equal areas, where the area is computed with respect to the round metric.

Since the isperimetric constant for the sphere $I_{S^2} = 1$, that is,

$$1 \le L_{S^2}(\tilde\beta_k)\,(\frac{1}{A_1} + \frac{1}{A_2}) = L_{S^2}(\tilde\beta_k)\,\frac{4}{A_{S^2}},$$

we have

$$L_{S^2}(\tilde{\beta}_k) \geq \delta > 0 \qquad \text{for all } k.$$

To finish the proof of the theorem, we will now show that there exists a uniform constant C so that

$$L_k(\beta_k) \leq C \qquad \text{for all } k.$$

To this end, we observe first that the area element of $g(t_k)$, when computed in polar coordinates, is

$$da_k = J_k(r,\theta)\, r\, dr\, d\theta,$$

where $J_k(r,\theta)$ is the Jacobian and r is the radial distance from y_k. The length of β_r^k is given by

$$L_k^r = \int_0^{2\pi} J_k(r,\theta)\, r\, d\theta,$$

which implies that

$$\frac{L_k^r}{r} = \int_0^{2\pi} J_k(r,\theta)\, d\theta.$$

By the Jacobian comparison theorem, for each fixed θ, we have

$$\frac{J_k'(r,\theta)}{J_k(r,\theta)} \leq \frac{J_a'(r,\theta)}{J_a(r,\theta)}, \tag{6.21}$$

where the derivative is in the r direction, $J_a(r,\theta)$ denotes the Jacobian for the model space, and a refers to a lower bound on Ricci curvature (the model space is a simply connected space of constant sectional curvature equal to a). In our case $a = 0$ (since $R \geq 0$) and the model space is the euclidean plane, which implies that the right-hand side of (6.21) is zero and therefore $J_k(r,\theta)$ decreases in r. Hence, L_k^r/r decreases in r.

In the proof of Lemma 6.8 we showed that there are uniform constants C_1, C_2 so that

$$C_1|t_k| \leq \rho_k \leq C_2|t_k| \qquad \text{and} \qquad c_1|t_k| \leq \sigma_k \leq C_2|t_k|.$$

We have shown that $\gamma_{t_k} \to \gamma_\infty$ and γ_∞ is a circle in M_∞. Let y_k, p_k be the points that we have chosen previously. We may assume that $\text{dist}_{t_k}(y_k, p_k) = \sigma_k$. Choose a curve $\bar{\gamma}_k \in M$ so that $p_k \in \bar{\gamma}_k$ and that for every $x \in \bar{\gamma}_k$ we have the $\text{dist}_{t_k}(y_k, x) = \sigma_k$. Observe that, for every $x \in \gamma_{t_k}$, by the figure we have

$$\sigma_k \leq \text{dist}_{t_k}(x, y_k) \leq \sigma_k + \frac{C}{\sigma_k},$$

for sufficiently big k. For $x \in \gamma_{t_k}$ let $z = \bar{\gamma}_{t_k} \cap \overline{y_k x}$. Then

$$\text{dist}_{t_k}(x, \bar{\gamma}_k) \leq \text{dist}_{t_k}(x, z) \leq \text{dist}_{t_k}(y_k, x) - \text{dist}(y_k, z) \leq \sigma_k + \frac{C}{\sigma_k} - \sigma_k = \frac{C}{\sigma_k}.$$

This implies that the curves $\bar{\gamma}_k$ converge to γ_∞ as $k \to \infty$. Moreover, this also implies the curve $\bar{\gamma}_k$ is at distance $\sigma_k = O(|t_k|)$ from y_k and if

$s_k = \mathrm{dist}_{t_k}(\beta_k, y_k)$, then $s_k = O(|t_k|)$ and we also know $L_k(\bar\gamma_k) \le C$, for all k. We may assume $s_k \le \sigma_k$ for infinitely many k; otherwise, we can consider point x_k instead of y_k and do the same analysis as above but with respect to x_k. Since $J_k(r,\theta)$ decreases in r, we have

$$\frac{L_k^{s_k}}{s_k} \le \frac{L_k^{\sigma_k}}{\sigma_k},$$

that is,

$$L_k(\beta_k) = L_k^{s_k} \le \frac{s_k}{\sigma_k}\, L_k^{\sigma_k} = \frac{s_k}{\sigma_k}\, L_k(\bar\gamma_k) \le C \qquad \text{for all } k,$$

finishing the proof of (6.20) and the theorem. q.e.d.

Based on the arguments of the proof of Theorem 6.1, we will show the following lemma, which was used in the proof of Theorem 4.1.

Lemma 6.9. *Assuming that our evolving metric $g(t) = \bar u\, g_e$, where g_e denotes the standard euclidean metric, it is impossible to have that the backward limit*

$$\bar u_\infty := \lim_{t \to -\infty} \bar u(\cdot, t) = \gamma$$

for a constant $\gamma > 0$.

Proof. We will use the arguments from the proof of Theorem 6.1 presented above. For a given time t, which will be chosen sufficiently close to $-\infty$, we denote by (M, g) our evolving surface at time t (for simplicity we omit t in all considered quantities below in the proof of the Lemma) and by γ the isoperimetric curve that divides M into two regions M_1 and M_2. We have seen in the proof of Theorem 6.1 that

$$L_g(\gamma) \le C$$

for a uniform constant C and that the areas of M_1 and M_2 are comparable to $|t|$.

Let $\mathcal{R} > 0$ be a large but uniform in time constant, which will be chosen in the sequel. By our assumption, there exist a $t_0 < 0$ and a point $O \in M$, so that the metric g is very close to the flat metric on the ball $B_{\mathcal{R}}(O)$ that is taken with respect to the metric g, for $t \le t_0 < 0$. Notice that since g is very close to the flat metric, $B_{\mathcal{R}}(O)$ is also close to the euclidean ball.

We may assume, without loss of generality, that $O \in M_2$. Since M_1 and M_2 have unbounded areas, as $|t| \to \infty$, the curve γ cannot be entirely contained in $B_{\mathcal{R}}(O)$. Hence, $\gamma \cap B_{\mathcal{R}}(O)^c \ne \emptyset$. By choosing $\mathcal{R}$ larger than $2C$, we then have that

$$\gamma \cap B_{\mathcal{R}/2}(O) = \emptyset.$$

As in the proof of Theorem 6.1, consider the point $x \in M_1$ that is the furthest from γ and the family of curves β_r of radial distance r from x which foliate our surface M. Let $\sigma = \mathrm{dist}_g(x,\gamma) = \mathrm{dist}_g(x,p)$, for some point $p \in \gamma$. Let $\bar{\gamma}$ be the curve such that $p \in \bar{\gamma}$ and such that for all $y \in \bar{\gamma}$ we have $\text{dist}_g(p, y) = \sigma$. As in the proof of Theorem 6.1, we have $L_g(\bar{\gamma}) \le C$,

$$\text{dist}_g(y, \bar{\gamma}) \le \frac{C}{\sigma}$$

for all $y \in \gamma$, and σ is comparable to $|t|$. These all together, combined with the fact that $\gamma \cap B_{\mathcal{R}/2}(O) = \emptyset$, imply that $\bar{\gamma} \cap B_{\mathcal{R}/2}(O) = \emptyset$, for $|t| \ge |t_0|$ and $|t_0|$ chosen sufficiently large. Let β_{r_1} be the curve that contains the point O. Based on the previous analysis, since all the curves in the foliation $\{\beta_r\}_{r \ge 0}$ of our surface M are mutually disjoint, we conclude $r_1 > \sigma$. In the proof of Theorem 6.1 we have argued that L_g^r/r decreases in r. This implies

$$\frac{L_g^r}{r} \le \frac{L_g^\sigma}{\sigma} = \frac{L(\bar{\gamma})}{\sigma},$$

finally yielding the bound

$$L_g(\beta_{r_1}) \le C\,\frac{r_1}{\sigma} \le \tilde{C}$$

for a uniform constant $\tilde{C}$, since σ is comparable to $|t|$ and $r_1 \le \text{diam}(M, g) \le C|t|$.

If $\beta_{r_1} \cap \partial B_{\mathcal{R}/4}(O) \ne \emptyset$, then $L_g(\beta_{r_1}) \ge \mathcal{R}/4$, which will lead to a contradiction if we choose $\mathcal{R} > 4\bar{C}$. Otherwise, β_{r_1} is entirely contained in $B_{\mathcal{R}/4}(O)$, which means that there exists another curve β_{r_2}, which encloses β_{r_1} and is contained in the closure of $B_{\mathcal{R}/4}(O)$ and touches the boundary of $B_{\mathcal{R}/4}(O)$. Since our metric on $B_{\mathcal{R}}(O)$ is very close to the euclidean metric, this would imply that $L_g(\beta_{r_2}) > \mathcal{R}/8$, which would also lead to a contradiction if we choose $\mathcal{R} > 8\bar{C}$. This finishes the proof of the lemma. q.e.d.

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