CLASSIFICATION OF EXTREMAL ELLIPTIC $K3$ SURFACES AND FUNDAMENTAL GROUPS OF OPEN $K3$ SURFACES

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Abstract. We present a complete list of extremal elliptic $K3$ surfaces (Theorem 1.1). As an application, we give a sufficient condition for the topological fundamental group of complement to an ADE-configuration of smooth rational curves on a $K3$ surface to be trivial (Proposition 4.1 and Theorems 4.3).

1. Introduction

A complex elliptic $K3$ surface $f : X \to \mathbb{P}^1$ with a section O is said to be *extremal* if the Picard number $\rho(X)$ of X is 20 and the Mordell-Weil group MW_f of f is finite. The purpose of this paper is to present the complete list of all extremal elliptic $K3$ surfaces. As an application, we show that, if an ADE-configuration of smooth rational curves on a $K3$ surface satisfies a certain condition, then the topological fundamental group of the complement is trivial. (See Theorem 4.3 for the precise statement.)

Let $f : X \to \mathbb{P}^1$ be an elliptic $K3$ surface with a section O. We denote by R_f the set of all points $v \in \mathbb{P}^1$ such that $f^{-1}(v)$ is reducible. For a point $v \in R_f$, let $f^{-1}(v)^{\#}$ be the union of irreducible components of $f^{-1}(v)$ that are disjoint from the zero section O. It is known that the cohomology classes of irreducible components of $f^{-1}(v)^{\#}$ form a negative definite root lattice $S_{f,v}$ of type A_l, D_m or E_n in $H^2(X;\mathbb{Z})$. Let $\tau(S_{f,v})$ be the type of this lattice. We define Σ_f to be the formal sum of these types;

$$\Sigma_f := \sum_{v \in R_f} \tau(S_{f,v}).$$

The Néron-Severi lattice NS_X of X is defined to be $H^{1,1}(X) \cap H^2(X;\mathbb{Z})$, and the transcendental lattice T_X of X is defined to be the orthogonal complement of NS_X in $H^2(X;\mathbb{Z})$. We call the triple (Σ_f, MW_f, T_X) the *data* of the elliptic $K3$ surface $f : X \to \mathbb{P}^1$. When $f : X \to \mathbb{P}^1$ is extremal, the transcendental lattice T_X is a positive definite even lattice of rank 2.

Theorem 1.1. *There exists an extremal elliptic $K3$ surface $f : X \to \mathbb{P}^1$ with data (Σ_f, MW_f, T_X) if and only if (Σ_f, MW_f, T_X) appears in Table 2 given at the end of this paper.*

In Table 2, the transcendental lattice T_X is expressed by the coefficients of its Gram matrix

$$\begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

See Subsection 2.1 on how to recover the $K3$ surface X from T_X.

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The classification of *semi-stable* extremal elliptic $K3$ surfaces has been done by Miranda and Persson[7] and complemented by Artal-Bartolo, Tokunaga and Zhang[1]. We can check that the semi-stable part of our list (No. 1- No. 112) coincides with theirs. Nishiyama[12] classified all elliptic fibrations (not necessarily extremal) on certain $K3$ surfaces. On the other hand, Ye[19] has independently classified all extremal elliptic $K3$ surfaces with no semi-stable singular fibers by different methods from ours.

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2. Preliminaries

2.1. **Transcendental lattice of singular $K3$ surfaces.** Let $\mathcal{Q}$ be the set of symmetric matrices

$$Q = \begin{pmatrix} a & b \\ b & c \end{pmatrix}$$

of integer coefficients such that a and c are even and that the corresponding quadratic forms are positive definite. The group $GL_2(\mathbb{Z})$ acts on $\mathcal{Q}$ from right by

$$Q \mapsto {}^t g \cdot Q \cdot g,$$

where $g \in GL_2(\mathbb{Z})$. Let Q_1 and Q_2 be two matrices in $\mathcal{Q}$, and let L_1 and L_2 be the positive definite even lattices of rank 2 whose Gram matrices are Q_1 and Q_2, respectively. Then L_1 and L_2 are isomorphic as lattices if and only if Q_1 and Q_2 are in the same orbit under the action of $GL_2(\mathbb{Z})$. On the other hand, each orbit in $\mathcal{Q}$ under the action of $SL_2(\mathbb{Z})$ contains a unique matrix with coefficients satisfying

$$-a < 2\,b \le a \le c, \qquad \text{with} \quad b \ge 0 \quad \text{if} \quad a = c.$$

(See, for example, Conway and Sloane[3, p. 358].) Hence each orbit in $\mathcal{Q}$ under the action of $GL_2(\mathbb{Z})$ contains a unique matrix with coefficients satisfying

$$0 \le 2\,b \le a \le c. \tag{2.1}$$

In Table 2, the transcendental lattice is represented by the Gram matrix satisfying the condition(2.1).

Let X be a $K3$ surface with $\rho(X) = 20$; that is, X is a singular $K3$ surface in the terminology of Shioda and Inose[16]. The transcendental lattice T_X can be naturally oriented by means of a holomorphic two form on X (cf. [16, p.128]). Let $\mathcal{S}$ denote the set of isomorphism classes of singular $K3$ surfaces. Using the natural orientation on the transcendental lattice, we can lift the map $\mathcal{S} \to \mathcal{Q}/GL_2(\mathbb{Z})$ given by $X \mapsto T_X$ to the map $\mathcal{S} \to \mathcal{Q}/SL_2(\mathbb{Z})$.

Proposition 2.1 (Shioda and Inose[16]). *This map $\mathcal{S} \to \mathcal{Q}/SL_2(\mathbb{Z})$ is bijective.* □

Moreover, Shioda and Inose[16] gave us a method to construct explicitly the singular $K3$ surface corresponding to a given element of $\mathcal{Q}/SL_2(\mathbb{Z})$ by means of Kummer surfaces. The injectivity of the map $\mathcal{S} \to \mathcal{Q}/SL_2(\mathbb{Z})$ had been proved by Piateskii-Shapiro and Shafarevich[14].

Suppose that an orbit $[Q] \in \mathcal{Q}/GL_2(\mathbb{Z})$ is represented by a matrix Q satisfying (2.1). Let $\rho : \mathcal{Q}/SL_2(\mathbb{Z}) \to \mathcal{Q}/GL_2(\mathbb{Z})$ be the natural projection. Then we have

$$|\rho^{-1}([Q])| = \begin{cases} 2 & \text{if } 0 < 2\,b < a < c \\ 1 & \text{otherwise.} \end{cases}$$

Therefore, if a data in Table 2 satisfies $a = c$ or $b = 0$ or $2\,b = a$ (resp. $0 < 2\,b < a < c$), then the number of the isomorphism classes of $K3$ surfaces that possess a structure of the extremal elliptic $K3$ surfaces with the given data is one (resp. two).

2.2. **Roots of a negative definite even lattice.** Let M be a negative definite even lattice. A vector of M is said to be a *root* of M if its norm is -2. We denote by $\mathrm{root}(M)$ the number of roots of M, and by M_{root} the sublattice of M generated by the roots of M. Suppose that a Gram matrix (a_{ij}) of M is given. Then $\mathrm{root}(M)$ can be calculated by the following method. Let

$$g_r(x) = -\sum_{i,j=1}^{r} a_{ij} x_i x_j$$

be the positive definite quadratic form associated with the opposite lattice M^- of M, where r is the rank of M. We consider the bounded closed subset

$$E(g_r, 2) := \{x \in \mathbb{R}^r \ ; \ g_r(x) \le 2\}$$

of $\mathbb{R}^r$. Then we have

$$\mathrm{root}(M) + 1 = |E(g_r, 2) \cap \mathbb{Z}^r|,$$

where $+1$ comes from the origin. For a positive integer k less than r, we write by $p_k : \mathbb{R}^r \to \mathbb{R}^k$ the projection $(x_1, \dots, x_r) \mapsto (x_1, \dots, x_k)$. Then there exists a positive definite quadratic form g_k of variables $(x_1, \dots, x_k)$ and a positive real number σ_k such that

$$p_k(E(g_r, 2)) = E(g_k, \sigma_k) := \{y \in \mathbb{R}^k \ ; \ g_k(y) \le \sigma_k\}.$$

The projection $(x_1, \dots, x_{k+1}) \mapsto (x_1, \dots, x_k)$ maps $E(g_{k+1}, \sigma_{k+1})$ to $E(g_k, \sigma_k)$. Hence, if we have the list of the points of $E(g_k, \sigma_k) \cap \mathbb{Z}^k$, then it is easy to make the list of the points of $E(g_{k+1}, \sigma_{k+1}) \cap \mathbb{Z}^{k+1}$. Thus, starting from $E(g_1, \sigma_1) \cap \mathbb{Z}$, we can make the list of the points of $E(g_r, 2) \cap \mathbb{Z}^r$ by induction on k.

2.3. **Root lattices of type ADE.** A *root type* is, by definition, a finite formal sum Σ of A_l, D_m and E_n with non-negative integer coefficients;

$$\Sigma = \sum_{l \ge 1} a_l A_l + \sum_{m \ge 4} d_m D_m + \sum_{n=6}^{8} e_n E_n.$$

We denote by $L(\Sigma)$ the negative definite root lattice corresponding to Σ. The rank of $L(\Sigma)$ is given by

$$\mathrm{rank}(L(\Sigma)) = \sum_{l \ge 1} a_l l + \sum_{m \ge 4} d_m m + \sum_{n=6}^{8} e_n n,$$

and the number of roots of $L(\Sigma)$ is given by

$$\text{(2.2)} \quad \mathrm{root}(L(\Sigma)) = \sum_{l \ge 1} a_l (l^2 + l) + \sum_{m \ge 4} d_m (2\,m^2 - 2\,m) + 72\,e_6 + 126\,e_7 + 240\,e_8.$$

(See, for example, Bourbaki[2].) Because of $L(\Sigma)_{root} = L(\Sigma)$, we have

$$L(\Sigma_1) \cong L(\Sigma_2) \Longleftrightarrow \Sigma_1 = \Sigma_2. \tag{2.3}$$

We also define $eu(\Sigma)$ by

$$eu(\Sigma) := \sum_{l \geq 1} a_l(l+1) + \sum_{m \geq 4} d_m(m+2) + \sum_{n=6}^{8} e_n(n+2).$$

Lemma 2.2. *Let $f : X \to \mathbb{P}^1$ be an elliptic $K3$ surface. Then $eu(\Sigma_f)$ is at most 24. Moreover, if $eu(\Sigma_f) < 24$, then there exists at least one singular fiber of type* I_1, II, III *or* IV*.*

Proof. Let $e(Y)$ denote the topological euler number of a CW-complex Y. Then $e(X) = 24$ is equal to the sum of topological euler numbers of singular fibers of f. Every singular fiber has a positive topological euler number. We have defined $eu(\Sigma)$ in such a way that, if $v \in R_f$, then $eu(\tau(S_{f,v})) \leq e(f^{-1}(v))$ holds, and if $eu(\tau(S_{f,v})) < e(f^{-1}(v))$, then the type of the fiber $f^{-1}(v)$ is either III or IV. Hence $eu(\Sigma_f)$ does not exceed the sum of the topological euler numbers of reducible singular fibers, and if $eu(\Sigma_f) < 24$, then there is an irreducible singular fiber or a singular fiber of type III or IV. □

2.4. **Discriminant form and overlattices.** Let L be an even lattice, $L^\vee$ the dual of L, D_L the discriminant group $L^\vee/L$ of L, and q_L the discriminant form on D_L. (See Nikulin[11, n. 4] for the definitions.) An overlattice of L is, by definition, an integral sublattice of the $\mathbb{Q}$-lattice $L^\vee$ containing L.

Lemma 2.3 (Nikulin[11] Proposition 1.4.2)**.** (1) *Let A be an isotropic subgroup of (D_L, q_L). Then the pre-image $M := \phi_L^{-1}(A)$ of A by the natural projection $\phi_L : L^\vee \to D_L$ is an overlattice of L, and the discriminant form (D_M, q_M) of M is isomorphic to $(A^\perp/A, q_L|_{A^\perp/A})$, where $A^\perp$ is the orthogonal complement of A in D_L, and $q_L|_{A^\perp/A}$ is the restriction of q_L to $A^\perp/A$.* (2) *The correspondence $A \mapsto M$ gives a bijection from the set of isotropic subgroups of (D_L, q_L) to the set of even overlattices of L.* □

Lemma 2.4 (Nikulin[11] Corollary 1.6.2)**.** *Let S and K be two even lattices. Then the following two conditions are equivalent.* (i) *There is an isomorphism $\gamma : D_S \xrightarrow{\sim} D_K$ of abelian groups such that $\gamma^* q_K = -q_S$.* (ii) *There is an even unimodular overlattice of $S \oplus K$ into which S and K are primitively embedded.* □

2.5. **Néron-Severi groups of elliptic $K3$ surfaces.** Let $f : X \to \mathbb{P}^1$ be an elliptic $K3$ surface with the zero section O. In the Néron-Severi lattice NS_X of X, the cohomology classes of the zero section O and a general fiber of f generate a sublattice U_f of rank 2, which is isomorphic to the hyperbolic lattice

$$H := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Let W_f be the orthogonal complement of U_f in NS_X. Because U_f is unimodular, we have $NS_X = U_f \oplus W_f$. Because U_f is of signature $(1,1)$ and NS_X is of signature $(1, \rho(X) - 1)$, W_f is negative definite of rank $\rho(X) - 2$. Note that W_f contains the sublattice

$$S_f := \bigoplus_{v \in R_f} S_{f,v}$$

generated by the cohomology classes of irreducible components of reducible fibers of f that are disjoint from the zero section. By definition, S_f is isomorphic to $L(\Sigma_f)$.

Lemma 2.5 (Nishiyama[12] Lemma 6.1)**.** *The sublattice S_f of W_f coincides with $(W_f)_{root}$, and the Mordell-Weil group MW_f of f is isomorphic to W_f/S_f. In particular,* $\mathrm{root}(L(\Sigma_f))$ *is equal to* $\mathrm{root}(W_f)$. □

Because $W_f \oplus U_f \oplus T_X$ has an even unimodular overlattice $H^2(X;\mathbb{Z})$ into which $NS_X = W_f \oplus U_f$ and T_X are primitively embedded, and because the discriminant form of NS_X is equal to the discriminant form of W_f by $D_{U_f} = (0)$, Lemma 2.4 implies the following:

Corollary 2.6. *There is an isomorphism $\gamma : D_{W_f} \xrightarrow{\sim} D_{T_X}$ of abelian groups such that $\gamma^* q_{T_X}$ coincides with $-q_{W_f}$.* □

2.6. **Existence of elliptic $K3$ surfaces.** Let Λ be the $K3$ lattice $L(2E_8) \oplus H^{\oplus 3}$.

Lemma 2.7 (Kondō[5] Lemma 2.1)**.** *Let T be a positive definite primitive sublattice of Λ with* $\mathrm{rank}(T) = 2$*, and $T^{\perp}$ the orthogonal complement of T in Λ. Suppose that $T^{\perp}$ contains a sublattice H_T isomorphic to the hyperbolic lattice. Let M_T be the orthogonal complement of H_T in $T^{\perp}$. Then there exists an elliptic $K3$ surface $f : X \to \mathbb{P}^1$ such that $T_X \cong T$ and $W_f \cong M_T$.*

Proof. By the surjectivity of the period map of the moduli of $K3$ surfaces (cf. Todorov[17]), there exist a $K3$ surface X and an isomorphism $\alpha : H^2(X;\mathbb{Z}) \cong \Lambda$ of lattices such that $\alpha^{-1}(T) = T_X$. By Kondō[5, Lemma 2.1], the $K3$ surface X has an elliptic fibration $f : X \to \mathbb{P}^1$ with a section such that $\mathbb{Z}[F]^{\perp}/\mathbb{Z}[F] \cong M_T$, where $[F] \in U_f$ is the cohomology class of a fiber of f, and $\mathbb{Z}[F]^{\perp}$ is the orthogonal complement of $[F]$ in the Néron-Severi lattice NS_X. Because NS_X coincides with $U_f \oplus W_f$, and because $\mathbb{Z}[F]^{\perp} \cap U_f$ coincides with $\mathbb{Z}[F]$, we see that $\mathbb{Z}[F]^{\perp}/\mathbb{Z}[F]$ is isomorphic to W_f. □

2.7. **Datum of extremal elliptic $K3$ surfaces.**

Proposition 2.8. *A triple (Σ, MW, T) consisting of a root type Σ, a finite abelian group MW and a positive definite even lattice T of rank 2 is a data of an extremal elliptic $K3$ surface if and only if the following hold:*

$(D1)$ $\mathrm{length}(MW) \le 2$, $\mathrm{rank}(L(\Sigma)) = 18$ *and* $eu(\Sigma) \le 24$.
$(D2)$ *There exists an overlattice M of $L(\Sigma)$ satisfying the following:*
- $(D2\text{-}a)$ $M/L(\Sigma) \cong MW$,
- $(D2\text{-}b)$ *there exists an isomorphism $\gamma : D_M \xrightarrow{\sim} D_T$ of abelian groups such that $\gamma^* q_T = -q_M$, and*
- $(D2\text{-}c)$ $\mathrm{root}(L(\Sigma)) = \mathrm{root}(M)$.

Proof. Suppose that there exists an extremal elliptic $K3$ surface $f : X \to \mathbb{P}^1$ with data equal to (Σ, MW, T). It is obvious that Σ and MW satisfies the condition $(D1)$. Via the isomorphism $S_f \cong L(\Sigma)$, the overlattice W_f of S_f corresponds to an overlattice M of $L(\Sigma)$, which satisfies the conditions $(D2\text{-}a)$-$(D2\text{-}c)$ by Lemma 2.5 and Corollary 2.6. Conversely, suppose that (Σ, MW, T) satisfies the conditions $(D1)$ and $(D2)$. By Lemma 2.4, the condition $(D2\text{-}b)$ and $D_H = 0$ imply that there exists an even unimodular overlattice of $M \oplus H \oplus T$ into which $M \oplus H$ and T are primitively embedded. By the theorem of Milnor (see, for example, Serre[15]) on the classification of even unimodular lattices, any even unimodular lattice of signature $(3, 19)$ is isomorphic to the $K3$ lattice Λ. Then Lemma 2.7 implies that there exists an elliptic $K3$ surface $f : X \to \mathbb{P}^1$ satisfying $W_f \cong M$ and $T_X \cong T$. The condition $(D2\text{ - }c)$ implies $M_{root} = L(\Sigma)$. Combining this with Lemma 2.5, we see that $S_f \cong L(\Sigma)$. Then (2.2) implies that $\Sigma_f = \Sigma$. Using Lemma 2.5 and the condition $(D2\text{ - }a)$, we see that $MW_f \cong MW$. Thus the data of $f : X \to \mathbb{P}^1$ coincides with (Σ, MW, T). □

Remark 2.9. In the light of Lemma 2.3, the condition $(D2)$ is equivalent to the following:

$(D3)$ There exists an isotropic subgroup A of $(D_{L(\Sigma)}, q_{L(\Sigma)})$ satisfying the following:
- $(D3\text{ - }a)$ A is isomorphic to MW,
- $(D3\text{ - }b)$ there exists an isomorphism $\gamma : A^\perp/A \xrightarrow{\sim} D_T$ of abelian groups such that $\gamma^* q_T = -q_{L(\Sigma)}|_{A^\perp/A}$, and
- $(D3\text{ - }c)$ $\text{root}(\phi^{-1}_{L(\Sigma)}(A))$ is equal to $\text{root}(L(\Sigma))$, where $\phi_{L(\Sigma)} : L(\Sigma)^\vee \to D_{L(\Sigma)}$ is the natural projection.

Remark 2.10. We did not use the conditions $\text{length}(MW) \le 2$ and $eu(\Sigma) \le 24$ in the proof of the "if" part of Proposition 2.8. It follows that, if (Σ, MW, T) satisfies $\text{rank}(L(\Sigma)) = 18$ and the condition $(D2)$, then $\text{length}(MW) \le 2$ and $eu(\Sigma) \le 24$ follow automatically. This fact can be used when we check the computer program described in the next section.

3. Making the list

First we list up all root types Σ satisfying $\text{rank}(L(\Sigma)) = 18$ and $eu(\Sigma) \le 24$. This list $\mathcal{L}$ consists of 712 elements.

Next we run a program that takes an element Σ of the list $\mathcal{L}$ as an input and proceeds as follows.

Step 1. The program calculates the intersection matrix of $L(\Sigma)^\vee$. Using this matrix, it calculates the discriminant form of $L(\Sigma)$, and decomposes it into p-parts;

$$(D_{L(\Sigma)}, q_{L(\Sigma)}) = \bigoplus_p (D_{L(\Sigma)}, q_{L(\Sigma)})_p,$$

where p runs through the set $\{p_1, \dots, p_k\}$ of prime divisors of the discriminant $|D_{L(\Sigma)}|$ of $L(\Sigma)$. We write the p_i-part of $(D_{L(\Sigma)}, q_{L(\Sigma)})$ by $(D_{L(\Sigma),i}, q_{L(\Sigma),i})$.

Step 2. For each p_i, it calculates the set $I(p_i)$ of all pairs $(A, A^\perp)$ of an isotropic subgroup A of $(D_{L(\Sigma),i}, q_{L(\Sigma),i})$ and its orthogonal complement $A^\perp$ such that $\text{length}(A) \le 2$.

Step 3. For each element

$$\mathcal{A} := ((A_1, A_1^\perp), \dots, (A_k, A_k^\perp)) \in I(p_1) \times \cdots \times I(p_k),$$

it calculates the $\mathbb{Q}/2\mathbb{Z}$-valued quadratic form

$$q_{\mathcal{A}} := q_{L(\Sigma),1}|_{A_1^\perp/A_1} \times \cdots \times q_{L(\Sigma),k}|_{A_k^\perp/A_k}$$

on the finite abelian group

$$D_{\mathcal{A}} := A_1^\perp/A_1 \times \cdots \times A_k^\perp/A_k.$$

Let $d(\mathcal{A})$ be the order of $D_{\mathcal{A}}$.

Step 4. It generates the list $\mathcal{T}(d(\mathcal{A}))$ of positive definite even lattices of rank 2 with discriminant equal to $d(\mathcal{A})$. For each $T \in \mathcal{T}(d(\mathcal{A}))$, it calculates the discriminant form of T and decomposes it into p-parts. If D_T is isomorphic to $D_{\mathcal{A}}$ and q_T is isomorphic to $-q_{\mathcal{A}}$, then it proceeds to the next step. Note that the automorphism group of a finite abelian p-group of length ≤ 2 is easily calculated, and hence it is an easy task to check whether two given quadratic forms on the finite abelian p-group of length ≤ 2 are isomorphic or not.

Step 5. It calculates the Gram matrix of the sublattice $\widetilde{L}(\mathcal{A})$ of $L(\Sigma)^\vee$ generated by $L(\Sigma) \subset L(\Sigma)^\vee$ and the pull-backs of generators of the subgroups $A_i \subset D_{L(\Sigma),i}$ by the projection $L(\Sigma)^\vee \to D_{L(\Sigma)} \to D_{L(\Sigma),i}$. Then it calculates $\mathrm{root}(\widetilde{L}(\mathcal{A}))$ by the method described in the subsection 2.2. If $\mathrm{root}(\widetilde{L}(\mathcal{A}))$ is equal to $\mathrm{root}(L(\Sigma))$ calculated by (2.2), then it puts out the pair of the finite abelian group

$$MW := A_1 \times \cdots \times A_k$$

and the lattice T.

Then (Σ, MW, T) satisfies the conditions $(D1)$ and $(D3)$, and all triples (Σ, MW, T) satisfying $(D1)$ and $(D3)$ are obtained by this program.

4. Fundamental groups of open $K3$ surfaces

A simple normal crossing divisor Δ on a $K3$ surface X is said to be an *ADE-configuration of smooth rational curves* if each irreducible component of Δ is a smooth rational curve and the intersection matrix of the irreducible components of Δ is a direct sum of the Cartan matrices of type A_l, D_m or E_n multiplied by -1. It is known that Δ is an ADE-configuration of smooth rational curves if and only if each connected component of Δ can be contracted to a rational double point. We consider the following quite plausible hypothesis. Let Δ be an ADE-configuration of smooth rational curves on a $K3$ surface X.

Hypothesis. If $\pi_1^{alg}(X \setminus \Delta)$ is trivial, then so is $\pi_1(X \setminus \Delta)$.

Here $\pi_1^{alg}(X \setminus \Delta)$ is the algebraic fundamental group of $X \setminus \Delta$, which is the pro-finite completion of the topological fundamental group $\pi_1(X \setminus \Delta)$.

Proposition 4.1. *Suppose that Hypothesis is true for any ADE-configuration of smooth rational curves on an arbitrary $K3$ surface. Let Δ be an ADE-configuration of smooth rational curves on a $K3$ surface X. Then $\pi_1(X \setminus \Delta)$ satisfies one of the following*:

(i) $\pi_1(X \setminus \Delta)$ *is trivial.*

(ii) *There exist a complex torus T of dimension 2 and a finite automorphism group G of T such that T/G is birational to X and that $\pi_1(X \setminus \Delta)$ fits in the exact sequence*

$$1 \longrightarrow \pi_1(T) \longrightarrow \pi_1(X \setminus \Delta) \longrightarrow G \longrightarrow 1.$$

(iii) $\pi_1(X \setminus \Delta)$ *is isomorphic to a symplectic automorphism group of a $K3$ surface.*

Remark 4.2. Fujiki[4] classified the automorphism groups of complex tori of dimension 2. In particular, the G in (ii) is either one of $\mathbb{Z}/(n)$ $(n = 2, 3, 4, 6)$, Q_8 (Quaternion of order 8), D_{12} (Dihedral of order 12) and T_{24} (Tetrahedral of order 24), whence the $\pi_1(X \setminus \Delta)$ in (ii) is a soluble group. Mukai[9] presented the complete list of symplectic automorphism groups of $K3$ surfaces. (See also Kondō[6] and Xiao[18].) Under Hypothesis, therefore, we know what groups can appear as $\pi_1(X \setminus \Delta)$.

Proof of Proposition 4.1. Suppose that $\pi_1(X \setminus \Delta)$ is non-trivial. By Hypothesis, $\pi_1^{alg}(X \setminus \Delta)$ is also non-trivial. For a surjective homomorphism $\phi : \pi_1(X \setminus \Delta) \to G$ from $\pi_1(X \setminus \Delta)$ to a finite group G, we denote by

$$\psi_\phi : \widetilde{Y}_\phi \longrightarrow X$$

the finite Galois cover of X corresponding to ϕ, which is étale over $X \setminus \Delta$ and whose Galois group is canonically isomorphic to G. Let $\rho : \widetilde{Y}'_\phi \to \widetilde{Y}_\phi$ be the resolution of singularities, and $\gamma : \widetilde{Y}'_\phi \to Y_\phi$ the contraction of (-1)-curves. We denote by Δ_ϕ the union of one-dimensional irreducible components of $\gamma(\rho^{-1}(\psi_\phi^{-1}(\Delta)))$. Then it is easy to see that Y_ϕ is either a $K3$ surface or a complex torus of dimension 2, and that the Galois group G of ψ_ϕ acts on Y_ϕ symplectically. Moreover, Δ_ϕ is an empty set or an ADE-configuration of smooth rational curves. We have an exact sequence

$$1 \longrightarrow \pi_1(Y_\phi \setminus \Delta_\phi) \longrightarrow \pi_1(X \setminus \Delta) \longrightarrow G \longrightarrow 1,$$

because $\pi_1(\widetilde{Y}_\phi \setminus \psi_\phi^{-1}(\Delta))$ is isomorphic to $\pi_1(Y_\phi \setminus \Delta_\phi)$. Suppose that there exists a homomorphism $\phi : \pi_1(X \setminus \Delta) \to G$ such that Y_ϕ is a complex torus of dimension 2. Then Δ_ϕ is empty, and hence (ii) occurs. Suppose that no complex tori of dimension 2 appear as a finite Galois cover of X branched in Δ. Then any finite quotient group of $\pi_1(X \setminus \Delta)$ must appear in Mukai's list of symplectic automorphism groups of $K3$ surfaces. Because this list consists of finite number of isomorphism classes of finite groups, there exists a maximal finite quotient $\phi_{max} : \pi_1(X \setminus \Delta) \to G_{max}$ of $\pi_1(X \setminus \Delta)$. Then $\pi_1(Y_{\phi_{max}} \setminus \Delta_{\phi_{max}})$ has no non-trivial finite quotient group, and hence it is trivial by Hypothesis. Thus (iii) occurs. □

For an ADE-configuration Δ of smooth rational curves on a $K3$ surface X, we denote by $\mathbb{Z}[\Delta]$ the sublattice of $H^2(X;\mathbb{Z})$ generated by the cohomology classes of the irreducible components of Δ, which is isomorphic to a negative definite root lattice of type ADE. We denote by Σ_Δ the root type such that $\mathbb{Z}[\Delta]$ is isomorphic to $L(\Sigma_\Delta)$. Using the list of extremal elliptic $K3$ surfaces, we prove the following theorem. We consider the following conditions on a root type Σ.

$(N1)$ $\mathrm{rank}(L(\Sigma)) \le 18$, and
$(N2)$ $\mathrm{length}(D_{L(\Sigma)}) \le \min\{\mathrm{rank}(L(\Sigma)), 20 - \mathrm{rank}(L(\Sigma))\}$.

Theorem 4.3. *Suppose that a root type Σ_Δ satisfies the conditions $(N1)$ and $(N2)$. If $\mathbb{Z}[\Delta]$ is primitive in $H^2(X;\mathbb{Z})$ then $\pi_1(X \setminus \Delta)$ is trivial.*

By virtue of Lemma 4.6 below, we can easily derive the following:

Corollary 4.4. *Suppose that Σ satisfies the conditions $(N1)$ and $(N2)$. Then Hypothesis is true for any (X, Δ) with $\Sigma_\Delta = \Sigma$.* □

Remark 4.5. The conditions $(N1)$ and $(N2)$ come from Nikulin[11, Theorem 1.14.1] (see also Morrison[8, Theorem 2.8]), which gives a sufficient condition for the uniqueness of the primitive embedding of $L(\Sigma)$ into the $K3$ lattice Λ.

First we prepare some lemmas. Let $\overline{\mathbb{Z}[\Delta]}$ be the primitive closure of $\mathbb{Z}[\Delta]$ in $H^2(X;\mathbb{Z})$.

Lemma 4.6 (Xiao[18] Lemma 2)**.** *The dual of the abelianisation of* $\pi_1(X \setminus \Delta)$ *is canonically isomorphic to* $\overline{\mathbb{Z}[\Delta]}/\mathbb{Z}[\Delta]$. *In particular, if* $\pi_1^{alg}(X \setminus \Delta)$ *is trivial, then* $\mathbb{Z}[\Delta]$ *is primitive in* $H^2(X;\mathbb{Z})$. □

Let Γ_1 and Γ_2 be graphs with the set of vertices denoted by $\mathrm{Vert}(\Gamma_1)$ and $\mathrm{Vert}(\Gamma_2)$, respectively. An embedding of Γ_1 into Γ_2 is, by definition, an injection $f : \mathrm{Vert}(\Gamma_1) \to \mathrm{Vert}(\Gamma_2)$ such that, for any $u, v \in \mathrm{Vert}(\Gamma_1)$, $f(u)$ and $f(v)$ are connected by an edge of Γ_2 if and only if u and v are connected by an edge of Γ_1.

Let $\Gamma(\Sigma)$ denote the Dynkin graph of Σ.

Lemma 4.7. *Suppose that* Σ *satisfies the conditions* $(N1)$ *and* $(N2)$. *Then there exists* Σ' *satisfying* $\mathrm{rank}(L(\Sigma')) = 18$ *and the condition* $(N2)$ *such that* $\Gamma(\Sigma)$ *can be embedded in* $\Gamma(\Sigma')$.

Proof. This is checked by listing up all Σ satisfying the conditions $(N1)$ and $(N2)$ using computer. □

Lemma 4.8. *Let* $f : X \to \mathbb{P}^1$ *be an elliptic surface with the zero section* O. *Suppose that a fiber* $f^{-1}(v)$ *over* $v \in \mathbb{P}^1$ *is a singular fiber of type* III *or* IV. *Let* Ξ *be a union of some irreducible components of* $f^{-1}(v)$ *that does not coincide with the whole fiber* $f^{-1}(v)$. *If* U *is a small open disk on* $\mathbb{P}^1$ *with the center* v, *then* $f^{-1}(U) \setminus (\Xi \cup (f^{-1}(U) \cap O))$ *has an abelian fundamental group.*

Proof. This can be proved easily by the van-Kampen theorem. □

Lemma 4.9. *Let* Σ *be satisfying the conditions* $(N1)$ *and* $(N2)$. *Suppose that* (X, Δ) *and* (X', Δ') *satisfy the following*:

(a) $\Sigma_\Delta = \Sigma_{\Delta'} = \Sigma$,
(b) $\overline{\mathbb{Z}[\Delta]} = \mathbb{Z}[\Delta]$ *and* $\overline{\mathbb{Z}[\Delta']} = \mathbb{Z}[\Delta']$.

Then there exists a connected continuous family (X_t, Δ_t) *parameterized by* $t \in [0,1]$ *such that* $(X_0, \Delta_0) = (X, \Delta)$, $(X_1, \Delta_1) = (X', \Delta')$ *and that* (X_t, Δ_t) *are diffeomorphic to one another. In particular,* $\pi_1(X \setminus \Delta)$ *is isomorphic to* $\pi_1(X' \setminus \Delta')$.

Proof. By Nikulin[11, Theorem 1.14.1], the primitive embedding of $L(\Sigma)$ into the $K3$ lattice Λ is unique up to $\mathrm{Aut}(\Lambda)$. Hence the assertion follows from Nikulin's connectedness theorem[10, Theorem 2.10]. □

Proof of Theorem 4.3. Let us consider the following:

Claim 1. Suppose that Σ satisfies $\mathrm{rank}(L(\Sigma)) = 18$ and the condition $(N2)$. Then there exists an ADE-configuration of smooth rational curves Δ_Σ on a $K3$ surface X_Σ such that $\Sigma_{\Delta_\Sigma} = \Sigma$ and $\pi_1(X_\Sigma \setminus \Delta_\Sigma) = \{1\}$.

We deduce Theorem 4.3 from Claim 1. Suppose that Δ is an ADE-configuration of smooth rational curves on a $K3$ surface X such that Σ_Δ satisfies the conditions $(N1)$ and $(N2)$, and that $\mathbb{Z}[\Delta]$ is primitive in $H^2(X;\mathbb{Z})$. By Lemma 4.7, there exists Σ_1 satisfying $\mathrm{rank}(L(\Sigma_1)) = 18$ and the condition $(N2)$ such that $\Gamma(\Sigma_\Delta)$ is embedded into $\Gamma(\Sigma_1)$. By Claim 1, we have (X_1, Δ_1) such that $\Sigma_{\Delta_1} = \Sigma_1$ and $\pi_1(X_1 \setminus \Delta_1) = \{1\}$. Let $\Delta' \subset \Delta_1$ be the sub-configuration of smooth rational curves on X_1 which corresponds to the subgraph $\Gamma(\Sigma_\Delta) \hookrightarrow \Gamma(\Sigma_1) = \Gamma(\Sigma_{\Delta_1})$. There is a surjection from $\pi_1(X_1 \setminus \Delta_1)$ to $\pi_1(X_1 \setminus \Delta')$, and hence $\pi_1(X_1 \setminus \Delta')$ is trivial. In particular, $\mathbb{Z}[\Delta']$ is primitive in $H^2(X_1;\mathbb{Z})$. Since $\Sigma_{\Delta'} = \Sigma_\Delta$, Lemma 4.9 implies that $\pi_1(X \setminus \Delta)$ is isomorphic to $\pi_1(X_1 \setminus \Delta')$. Thus $\pi_1(X \setminus \Delta)$ is trivial.

Let $f : X \to \mathbb{P}^1$ be an extremal elliptic $K3$ surface. For a point $v \in R_f$, we denote the total fiber of f over v by

$$\sum_{i=1}^{r_v} m_{v,i}\, C_{v,i},$$

where $m_{v,i}$ is the multiplicity of the irreducible component $C_{v,i}$ of $f^{-1}(v)$. We denote by Γ_f the union of the zero section and all irreducible fibers $f^{-1}(v)$ $(v \in R_f)$.

Claim 2. Suppose that $MW_f = (0)$. Suppose that a sub-configuration Δ of Γ_f satisfies the following two conditions.

$(Z1)$ The number of $v \in R_f$ such that $m_{v,i} = 1 \Longrightarrow$ The number of $C_{v,i} \subset \Delta$ is at most one.

$(Z2)$ Either one of the following holds:

$(Z2\text{-}a)$ The configuration Δ does not contain the zero section,

$(Z2\text{-}b)$ there is a point $v_1 \in R_f$ such that the type $\tau(S_{f,v_1})$ is A_1 and that $F_1 := f^{-1}(v_1)$ and Δ have no common irreducible components, or

$(Z2\text{-}c)$ $eu(\Sigma_f) \leq 23$.

Then $\pi_1(X \setminus \Delta)$ is trivial.

Proof of Claim 2. By Lemma 2.5, the assumption $MW_f = (0)$ implies that the cohomology classes $[O]$ and $[C_{v,i}]$ $(v \in R_f, i = 1, \dots, r_v)$ of the irreducible components of Γ_f span NS_X. The relations among these generators are generated by

$$\sum_{i=1}^{r_v} m_{v,i}\, C_{v,i} = \sum_{i=1}^{r_{v'}} m_{v',i}\, C_{v',i} \quad (v, v' \in R_f).$$

Therefore the condition $(Z1)$ implies that the cohomology classes of the irreducible components of Δ constitute a subset of a $\mathbb{Z}$-basis of NS_X. Hence $\mathbb{Z}[\Delta]$ is primitive in $H^2(X;\mathbb{Z})$. In particular, $\pi_1(X \setminus \Delta)$ is a perfect group by Lemma 4.6. On the other hand, the condition $(Z1)$ implies that there exists a point $v_0 \in \mathbb{P}^1$ such that every fiber of the restriction

$$f|_{X \setminus (\Delta \cup f^{-1}(v_0))} : X \setminus (\Delta \cup f^{-1}(v_0)) \longrightarrow \mathbb{P}^1 \setminus \{v_0\}$$

of f has a reduced irreducible component. Then, by Nori's lemma[13, Lemma 1.5 (C)], if U is a non-empty connected classically open subset of $\mathbb{P}^1 \setminus \{v_0\}$, then the inclusion of $f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta))$ into $X \setminus (\Delta \cup f^{-1}(v_0))$ induces a surjection on the fundamental groups. The inclusion of $X \setminus (\Delta \cup f^{-1}(v_0))$ into $X \setminus \Delta$ also induces a surjection on the fundamental groups. We shall show that there exists a small open disk U on $\mathbb{P}^1 \setminus \{v_0\}$ such that

$$G_U := \pi_1(f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta))$$

is abelian. When $(Z2\text{-}a)$ occurs, we take a small open disk disjoint from R_f as U. Then G_U is abelian, because of $f^{-1}(U) \cap \Delta = \emptyset$. Suppose that $(Z2\text{-}b)$ occurs. We can take v_0 from $\mathbb{P}^1 \setminus \{v_1\}$, because F_1 has no irreducible components of multiplicity ≥ 2. We choose a small open disk U with the center v_1. There is a contraction from $f^{-1}(U) \setminus (f^{-1}(U) \cap \Delta)$ to $F_1 \setminus (F_1 \cap \Delta)$. Because $\pi_1(F_1 \setminus (F_1 \cap \Delta))$ is abelian, so is G_U. Suppose that $(Z2\text{-}c)$ occurs. By Lemma 2.2, there exists a singular fiber $F_2 := f^{-1}(v_2)$ of type I_1, II, III or IV. Because F_2 has no irreducible components of multiplicity ≥ 2, we can choose v_0 from $\mathbb{P}^1 \setminus \{v_2\}$. If F_2 is of type I_1 or II, then $F_2 \cap \Delta$ consists of a nonsingular point of F_2, and $\pi_1(F_2 \setminus (F_2 \cap \Delta))$ is abelian. Hence

G_U is also abelian. If F_2 is of type III or IV, then $F_2 \cap \Delta$ cannot coincide with the whole fiber F_2. Hence Lemma 4.8 implies that G_U is abelian. Therefore we see that $\pi_1(X \setminus \Delta)$ is abelian. Being both perfect and abelian, $\pi_1(X \setminus \Delta)$ is trivial. □

Now we proceed to the proof of Claim 1. We list up all Σ satisfying the condition $(N2)$ and $\mathrm{rank}(L(\Sigma)) = 18$. It consists of 297 elements. Among them, 199 elements can be the type Σ_f of singular fibers of some extremal elliptic $K3$ surface $f : X \to \mathbb{P}^1$ with $MW_f = 0$. For these configurations, $\pi_1(X \setminus \Delta)$ is trivial by Claim 2. The remaining 98 configurations are listed in the second column of Table 1 below. Each of them is a sub-configuration of Γ_f satisfying the conditions $(Z1)$ and $(Z2)$, where $f : X \to \mathbb{P}^1$ is the extremal elliptic $K3$ surface with $MW_f = 0$ whose number in Table 2 is given in the third column of Table 1. The fourth and fifth columns of Table 1 indicate Σ_f and $eu(\Sigma_f)$, respectively. In the case nos. 20, 28, 39, 41 and 85 in Table 1, we can choose the embedding of Δ into Γ_f in such a way that $(Z2\text{ - }b)$ holds. In the case nos. 30, 37, 57 and 63 in Table 1, we can choose the embedding of Δ into Γ_f in such a way that $(Z2\text{ - }a)$ holds. By Claim 2 again, $\pi_1(X \setminus \Delta)$ is trivial for these 98 configurations Δ. □

Remark 4.10. The graph $\Gamma(A_{19})$ (resp. $\Gamma(D_{19})$) can be embedded into Γ_f in such a way that $(Z1)$ and $(Z2)$ are satisfied, where $f : X \to \mathbb{P}^1$ is the extremal elliptic $K3$ surfaces whose number in Table 2 is 312 (resp. 320). Therefore, if $\Gamma(\Delta)$ is embedded in $\Gamma(A_{19})$ or $\Gamma(D_{19})$, then $\Gamma(\Delta)$ can be embedded in Γ_f in such a way that $(Z1)$ and $(Z2)$ are satisfied.

Table 1. List of embedding of Δ in Γ_f

no	Δ	No	Σ_f	$eu(\Sigma_f)$
1	$A_2 + A_3 + 2\ A_4 + A_5$	19	$A_2 + 2\ A_3 + A_4 + A_6$	23
2	$A_1 + A_2 + A_3 + 2\ A_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
3	$2\ A_1 + A_4 + 2\ A_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
4	$2\ A_2 + 2\ A_4 + A_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
5	$A_1 + A_5 + 2\ A_6$	40	$A_1 + A_4 + A_6 + A_7$	22
6	$A_4 + 2\ A_7$	52	$A_4 + A_6 + A_8$	21
7	$A_1 + A_2 + 2\ A_4 + A_7$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
8	$A_3 + 2\ A_4 + A_7$	24	$A_3 + A_4 + A_5 + A_6$	22
9	$A_2 + 2\ A_4 + A_8$	36	$A_2 + A_4 + A_5 + A_7$	22
10	$2\ A_3 + A_4 + A_8$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
11	$A_3 + A_7 + A_8$	53	$A_1 + A_2 + A_7 + A_8$	22
12	$A_1 + 2\ A_2 + A_4 + A_9$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
13	$A_2 + A_3 + A_4 + A_9$	71	$2\ A_2 + A_4 + A_{10}$	22
14	$A_3 + A_4 + A_{11}$	93	$A_2 + A_4 + A_{12}$	21
15	$A_7 + A_{11}$	312	$A_{10} + E_8$	21
16	$2\ A_3 + A_{12}$	93	$A_2 + A_4 + A_{12}$	21
17	$A_3 + A_{15}$	312	$A_{10} + E_8$	21
18	$A_2 + 2\ A_6 + D_4$	99	$A_2 + A_3 + A_{13}$	21
19	$2\ A_4 + A_6 + D_4$	18	$A_1 + A_3 + 2\ A_4 + A_6$	23
20	$2\ A_2 + A_4 + A_6 + D_4$	20	$A_1 + 2\ A_2 + A_3 + A_4 + A_6$	24
21	$A_2 + A_4 + A_8 + D_4$	44	$2\ A_1 + 2\ A_4 + A_8$	23
22	$A_6 + A_8 + D_4$	50	$2\ A_1 + A_2 + A_6 + A_8$	23
23	$2\ A_2 + A_{10} + D_4$	72	$2\ A_1 + A_2 + A_4 + A_{10}$	23
24	$A_4 + A_{10} + D_4$	72	$2\ A_1 + A_2 + A_4 + A_{10}$	23
25	$A_2 + A_{12} + D_4$	90	$2\ A_1 + 2\ A_2 + A_{12}$	23
26	$A_{14} + D_4$	320	$D_{10} + E_8$	22
27	$2\ A_2 + A_4 + 2\ D_5$	210	$2\ A_2 + D_{14}$	22
28	$A_1 + 2\ A_2 + 2\ A_4 + D_5$	157	$A_1 + A_2 + 2\ A_4 + D_7$	24
29	$A_2 + A_3 + 2\ A_4 + D_5$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
30	$A_2 + A_6 + 2\ D_5$	193	$A_2 + A_6 + D_{10}$	22
31	$A_3 + A_4 + A_6 + D_5$	18	$A_1 + A_3 + 2\ A_4 + A_6$	23
32	$A_2 + A_4 + A_7 + D_5$	72	$2\ A_1 + A_2 + A_4 + A_{10}$	23
33	$A_6 + A_7 + D_5$	50	$2\ A_1 + A_2 + A_6 + A_8$	23
34	$A_2 + A_3 + A_8 + D_5$	50	$2\ A_1 + A_2 + A_6 + A_8$	23
35	$A_3 + A_{10} + D_5$	69	$A_1 + 2\ A_2 + A_3 + A_{10}$	23
36	$A_2 + A_{11} + D_5$	90	$2\ A_1 + 2\ A_2 + A_{12}$	23
37	$A_4 + 2\ D_7$	213	$A_4 + D_{14}$	21
38	$A_3 + 2\ A_4 + D_7$	44	$2\ A_1 + 2\ A_4 + A_8$	23
39	$2\ A_2 + A_3 + A_4 + D_7$	20	$A_1 + 2\ A_2 + A_3 + A_4 + A_6$	24
40	$A_2 + A_4 + A_5 + D_7$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
41	$A_1 + 2\ A_2 + A_6 + D_7$	14	$2\ A_1 + 2\ A_2 + 2\ A_6$	24

Table 1. List of embedding of Δ in Γ_f

no	Δ	No	Σ_f	$eu(\Sigma_f)$
42	$2\ A_2 + A_7 + D_7$	90	$2\ A_1 + 2\ A_2 + A_{12}$	23
43	$A_4 + A_7 + D_7$	44	$2\ A_1 + 2\ A_4 + A_8$	23
44	$A_1 + A_2 + A_8 + D_7$	50	$2\ A_1 + A_2 + A_6 + A_8$	23
45	$A_3 + A_8 + D_7$	44	$2\ A_1 + 2\ A_4 + A_8$	23
46	$A_{11} + D_7$	320	$D_{10} + E_8$	22
47	$A_2 + A_4 + D_5 + D_7$	200	$A_2 + A_5 + D_{11}$	22
48	$A_6 + D_5 + D_7$	186	$A_9 + D_9$	21
49	$A_2 + 2\ A_4 + D_8$	66	$A_2 + A_7 + A_9$	21
50	$A_4 + A_6 + D_8$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
51	$A_2 + A_8 + D_8$	50	$2\ A_1 + A_2 + A_6 + A_8$	23
52	$A_{10} + D_8$	320	$D_{10} + E_8$	22
53	$A_1 + 2\ A_4 + D_9$	44	$2\ A_1 + 2\ A_4 + A_8$	23
54	$A_2 + A_3 + A_4 + D_9$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
55	$A_3 + A_6 + D_9$	76	$2\ A_1 + A_6 + A_{10}$	22
56	$A_2 + A_7 + D_9$	50	$2\ A_1 + A_2 + A_6 + A_8$	23
57	$2\ A_2 + D_5 + D_9$	210	$2\ A_2 + D_{14}$	22
58	$A_2 + D_7 + D_9$	186	$A_9 + D_9$	21
59	$2\ A_2 + A_4 + D_{10}$	72	$2\ A_1 + A_2 + A_4 + A_{10}$	23
60	$A_3 + A_4 + D_{11}$	44	$2\ A_1 + 2\ A_4 + A_8$	23
61	$A_7 + D_{11}$	320	$D_{10} + E_8$	22
62	$A_2 + D_5 + D_{11}$	186	$A_9 + D_9$	21
63	$D_7 + D_{11}$	218	D_{18}	20
64	$A_2 + A_4 + D_{12}$	72	$2\ A_1 + A_2 + A_4 + A_{10}$	23
65	$A_6 + D_{12}$	320	$D_{10} + E_8$	22
66	$A_1 + 2\ A_2 + D_{13}$	90	$2\ A_1 + 2\ A_2 + A_{12}$	23
67	$A_2 + A_3 + D_{13}$	72	$2\ A_1 + A_2 + A_4 + A_{10}$	23
68	$A_3 + D_{15}$	320	$D_{10} + E_8$	22
69	$A_2 + D_{16}$	320	$D_{10} + E_8$	22
70	$2\ A_1 + A_4 + 2\ E_6$	303	$A_1 + A_4 + A_5 + E_8$	23
71	$2\ A_1 + A_2 + 2\ A_4 + E_6$	23	$A_1 + A_2 + A_4 + A_5 + A_6$	23
72	$A_2 + 2\ A_3 + A_4 + E_6$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
73	$2\ A_6 + E_6$	37	$A_1 + 2\ A_2 + A_6 + A_7$	23
74	$2\ A_3 + A_6 + E_6$	41	$A_5 + A_6 + A_7$	21
75	$A_2 + A_3 + A_7 + E_6$	37	$A_1 + 2\ A_2 + A_6 + A_7$	23
76	$2\ A_4 + D_4 + E_6$	182	$A_4 + A_5 + D_9$	22
77	$A_2 + A_6 + D_4 + E_6$	183	$A_1 + A_2 + A_6 + D_9$	23
78	$A_8 + D_4 + E_6$	186	$A_9 + D_9$	21
79	$A_1 + D_5 + 2\ E_6$	320	$D_{10} + E_8$	22
80	$A_2 + 2\ D_5 + E_6$	320	$D_{10} + E_8$	22
81	$A_1 + A_2 + A_4 + D_5 + E_6$	193	$A_2 + A_6 + D_{10}$	22
82	$A_2 + A_3 + D_7 + E_6$	200	$A_2 + A_5 + D_{11}$	22

Table 1. List of embedding of Δ in Γ_f

no	Δ	No	Σ_f	$eu(\Sigma_f)$
83	$A_5 + D_7 + E_6$	320	$D_{10} + E_8$	22
84	$A_2 + D_{10} + E_6$	193	$A_2 + A_6 + D_{10}$	22
85	$A_1 + A_2 + 2\ A_4 + E_7$	17	$2\ A_1 + A_2 + 2\ A_4 + A_6$	24
86	$A_3 + 2\ A_4 + E_7$	18	$A_1 + A_3 + 2\ A_4 + A_6$	23
87	$2\ A_2 + D_7 + E_7$	210	$2\ A_2 + D_{14}$	22
88	$A_2 + 2\ A_4 + E_8$	36	$A_2 + A_4 + A_5 + A_7$	22
89	$2\ A_1 + 2\ A_2 + A_4 + E_8$	30	$2\ A_2 + A_3 + A_4 + A_7$	23
90	$2\ A_3 + A_4 + E_8$	24	$A_3 + A_4 + A_5 + A_6$	22
91	$A_3 + A_7 + E_8$	46	$A_1 + A_2 + A_3 + A_4 + A_8$	23
92	$A_2 + A_4 + D_4 + E_8$	182	$A_4 + A_5 + D_9$	22
93	$A_6 + D_4 + E_8$	186	$A_9 + D_9$	21
94	$A_1 + 2\ A_2 + D_5 + E_8$	210	$2\ A_2 + D_{14}$	22
95	$A_2 + A_3 + D_5 + E_8$	198	$2\ A_2 + A_3 + D_{11}$	23
96	$A_3 + D_7 + E_8$	213	$A_4 + D_{14}$	21
97	$A_2 + D_8 + E_8$	210	$2\ A_2 + D_{14}$	22
98	$2\ A_1 + A_2 + E_6 + E_8$	320	$D_{10} + E_8$	22

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
1	$6\,A_3$	$\mathbb{Z}/(4)\times\mathbb{Z}/(4)$	4	0	4
2	$2\,A_1+4\,A_4$	$\mathbb{Z}/(5)$	10	0	10
3	$2\,A_2+2\,A_3+2\,A_4$	(0)	60	0	60
4	$3\,A_1+3\,A_5$	$\mathbb{Z}/(2)\times\mathbb{Z}/(6)$	2	0	6
5	$4\,A_2+2\,A_5$	$\mathbb{Z}/(3)\times\mathbb{Z}/(3)$	6	0	6
6	$A_3+3\,A_5$	$\mathbb{Z}/(6)$	4	0	6
7	$2\,A_1+2\,A_3+2\,A_5$	$\mathbb{Z}/(2)\times\mathbb{Z}/(2)$	12	0	12
8	$A_1+2\,A_2+A_3+2\,A_5$	$\mathbb{Z}/(6)$	6	0	12
9	$2\,A_4+2\,A_5$	(0)	30	0	30
10	$2\,A_2+A_4+2\,A_5$	$\mathbb{Z}/(3)$	6	0	30
11	$A_1+A_3+A_4+2\,A_5$	$\mathbb{Z}/(2)$	12	0	30
12	$A_1+A_2+2\,A_3+A_4+A_5$	$\mathbb{Z}/(2)$	24	12	36
13	$3\,A_6$	$\mathbb{Z}/(7)$	2	1	4
14	$2\,A_1+2\,A_2+2\,A_6$	(0)	42	0	42
15	$2\,A_3+2\,A_6$	(0)	28	0	28
16	$A_2+A_4+2\,A_6$	(0)	28	7	28
17	$2\,A_1+A_2+2\,A_4+A_6$	(0)	50	20	50
18	$A_1+A_3+2\,A_4+A_6$	(0)	10	0	140
			20	0	70
19	$A_2+2\,A_3+A_4+A_6$	(0)	24	12	76
20	$A_1+2\,A_2+A_3+A_4+A_6$	(0)	30	0	84
21	$2\,A_1+2\,A_5+A_6$	$\mathbb{Z}/(2)$	12	6	24
22	$A_1+2\,A_3+A_5+A_6$	$\mathbb{Z}/(2)$	4	0	84
23	$A_1+A_2+A_4+A_5+A_6$	(0)	30	0	42
			18	6	72
24	$A_3+A_4+A_5+A_6$	(0)	12	0	70
25	$4\,A_1+2\,A_7$	$\mathbb{Z}/(2)\times\mathbb{Z}/(4)$	4	0	4
26	$2\,A_2+2\,A_7$	(0)	24	0	24
		$\mathbb{Z}/(2)$	12	0	12
27	$A_1+A_3+2\,A_7$	$\mathbb{Z}/(8)$	2	0	4
28	$2\,A_1+3\,A_3+A_7$	$\mathbb{Z}/(2)\times\mathbb{Z}/(4)$	4	0	8
29	$A_2+3\,A_3+A_7$	$\mathbb{Z}/(4)$	4	0	24
30	$2\,A_2+A_3+A_4+A_7$	(0)	12	0	120
31	$2\,A_1+A_2+A_3+A_4+A_7$	$\mathbb{Z}/(2)$	20	0	24
32	$A_1+2\,A_5+A_7$	$\mathbb{Z}/(2)$	6	0	24
33	$3\,A_1+A_3+A_5+A_7$	$\mathbb{Z}/(2)\times\mathbb{Z}/(2)$	8	0	12

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
34	$A_1 + A_2 + A_3 + A_5 + A_7$	$\mathbb{Z}/(2)$	12	0	24
35	$2\,A_1 + A_4 + A_5 + A_7$	$\mathbb{Z}/(2)$	2	0	120
36	$A_2 + A_4 + A_5 + A_7$	(0)	6	0	120
			24	0	30
37	$A_1 + 2\,A_2 + A_6 + A_7$	(0)	24	0	42
38	$2\,A_1 + A_3 + A_6 + A_7$	$\mathbb{Z}/(2)$	12	4	20
39	$A_2 + A_3 + A_6 + A_7$	(0)	4	0	168
40	$A_1 + A_4 + A_6 + A_7$	(0)	2	0	280
			18	4	32
41	$A_5 + A_6 + A_7$	(0)	16	4	22
42	$2\,A_1 + 2\,A_8$	(0)	18	0	18
		$\mathbb{Z}/(3)$	4	2	10
43	$A_1 + 3\,A_2 + A_3 + A_8$	$\mathbb{Z}/(3)$	12	0	18
44	$2\,A_1 + 2\,A_4 + A_8$	(0)	20	10	50
45	$3\,A_2 + A_4 + A_8$	$\mathbb{Z}/(3)$	12	3	12
46	$A_1 + A_2 + A_3 + A_4 + A_8$	(0)	6	0	180
47	$A_1 + 2\,A_2 + A_5 + A_8$	$\mathbb{Z}/(3)$	6	0	18
48	$A_2 + A_3 + A_5 + A_8$	$\mathbb{Z}/(3)$	4	0	18
49	$A_1 + A_4 + A_5 + A_8$	(0)	18	0	30
50	$2\,A_1 + A_2 + A_6 + A_8$	(0)	18	0	42
51	$A_1 + A_3 + A_6 + A_8$	(0)	10	4	52
52	$A_4 + A_6 + A_8$	(0)	18	9	22
53	$A_1 + A_2 + A_7 + A_8$	(0)	18	0	24
54	$2\,A_9$	(0)	10	0	10
		$\mathbb{Z}/(5)$	2	0	2
55	$A_1 + A_2 + 2\,A_3 + A_9$	$\mathbb{Z}/(2)$	4	0	60
56	$2\,A_1 + 2\,A_2 + A_3 + A_9$	$\mathbb{Z}/(2)$	6	0	60
57	$A_1 + 2\,A_4 + A_9$	$\mathbb{Z}/(5)$	2	0	10
58	$3\,A_1 + A_2 + A_4 + A_9$	$\mathbb{Z}/(2)$	20	10	20
59	$2\,A_1 + A_3 + A_4 + A_9$	$\mathbb{Z}/(2)$	10	0	20
60	$2\,A_1 + A_2 + A_5 + A_9$	$\mathbb{Z}/(2)$	12	6	18
61	$A_1 + A_3 + A_5 + A_9$	$\mathbb{Z}/(2)$	10	0	12
62	$A_4 + A_5 + A_9$	(0)	10	0	30
		$\mathbb{Z}/(2)$	10	5	10
63	$3\,A_1 + A_6 + A_9$	$\mathbb{Z}/(2)$	4	2	36
64	$A_1 + A_2 + A_6 + A_9$	(0)	10	0	42

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
65	$A_3 + A_6 + A_9$	(0)	2	0	140
66	$A_2 + A_7 + A_9$	(0)	10	0	24
67	$A_1 + A_8 + A_9$	(0)	10	0	18
68	$A_2 + 2\,A_3 + A_{10}$	(0)	24	12	28
69	$A_1 + 2\,A_2 + A_3 + A_{10}$	(0)	12	0	66
70	$2\,A_4 + A_{10}$	(0)	10	5	30
71	$2\,A_2 + A_4 + A_{10}$	(0)	6	3	84
			24	9	24
72	$2\,A_1 + A_2 + A_4 + A_{10}$	(0)	2	0	330
73	$A_1 + A_3 + A_4 + A_{10}$	(0)	20	0	22
			12	4	38
74	$A_1 + A_2 + A_5 + A_{10}$	(0)	6	0	66
			18	6	24
75	$A_3 + A_5 + A_{10}$	(0)	4	0	66
			12	0	22
76	$2\,A_1 + A_6 + A_{10}$	(0)	12	2	26
77	$A_2 + A_6 + A_{10}$	(0)	4	1	58
			16	5	16
78	$A_1 + A_7 + A_{10}$	(0)	2	0	88
			10	2	18
79	$A_8 + A_{10}$	(0)	10	1	10
80	$A_1 + 3\,A_2 + A_{11}$	$\mathbb{Z}/(3)$	6	0	12
81	$3\,A_1 + 2\,A_2 + A_{11}$	$\mathbb{Z}/(6)$	2	0	12
82	$A_1 + 2\,A_3 + A_{11}$	$\mathbb{Z}/(4)$	4	0	6
83	$2\,A_2 + A_3 + A_{11}$	$\mathbb{Z}/(3)$	4	0	12
		$\mathbb{Z}/(6)$	4	2	4
84	$2\,A_1 + A_2 + A_3 + A_{11}$	$\mathbb{Z}/(4)$	6	0	6
		$\mathbb{Z}/(2)$	12	0	12
85	$3\,A_1 + A_4 + A_{11}$	$\mathbb{Z}/(2)$	6	0	20
86	$A_1 + A_2 + A_4 + A_{11}$	(0)	12	0	30
87	$2\,A_1 + A_5 + A_{11}$	$\mathbb{Z}/(2)$	6	0	12
		$\mathbb{Z}/(6)$	2	0	4
88	$A_2 + A_5 + A_{11}$	$\mathbb{Z}/(3)$	4	0	6
89	$A_1 + A_6 + A_{11}$	(0)	4	0	42
90	$2\,A_1 + 2\,A_2 + A_{12}$	(0)	12	6	42
91	$A_1 + A_2 + A_3 + A_{12}$	(0)	6	0	52

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
92	$2\,A_1 + A_4 + A_{12}$	(0)	2	0	130
			18	8	18
93	$A_2 + A_4 + A_{12}$	(0)	6	3	34
94	$A_1 + A_5 + A_{12}$	(0)	10	2	16
95	$A_6 + A_{12}$	(0)	2	1	46
96	$A_1 + 2\,A_2 + A_{13}$	(0)	6	0	42
		$\mathbb{Z}/(2)$	6	3	12
97	$3\,A_1 + A_2 + A_{13}$	$\mathbb{Z}/(2)$	2	0	42
98	$2\,A_1 + A_3 + A_{13}$	$\mathbb{Z}/(2)$	6	2	10
99	$A_2 + A_3 + A_{13}$	(0)	4	0	42
100	$A_1 + A_4 + A_{13}$	(0)	2	0	70
			8	2	18
		$\mathbb{Z}/(2)$	2	1	18
101	$A_5 + A_{13}$	(0)	4	2	22
102	$2\,A_2 + A_{14}$	$\mathbb{Z}/(3)$	4	1	4
103	$2\,A_1 + A_2 + A_{14}$	(0)	12	6	18
		$\mathbb{Z}/(3)$	2	0	10
104	$A_1 + A_3 + A_{14}$	(0)	10	0	12
105	$A_4 + A_{14}$	(0)	10	5	10
106	$3\,A_1 + A_{15}$	$\mathbb{Z}/(4)$	2	0	4
107	$A_1 + A_2 + A_{15}$	(0)	10	2	10
		$\mathbb{Z}/(2)$	4	0	6
108	$A_3 + A_{15}$	$\mathbb{Z}/(4)$	2	0	2
109	$2\,A_1 + A_{16}$	(0)	2	0	34
			4	2	18
110	$A_2 + A_{16}$	(0)	6	3	10
111	$A_1 + A_{17}$	(0)	4	2	10
		$\mathbb{Z}/(3)$	2	0	2
112	A_{18}	(0)	2	1	10
113	$2\,A_4 + 2\,D_5$	(0)	20	0	20
114	$A_3 + 2\,A_5 + D_5$	$\mathbb{Z}/(2)$	12	0	12
115	$2\,A_4 + A_5 + D_5$	(0)	20	0	30
116	$A_1 + A_3 + A_4 + A_5 + D_5$	$\mathbb{Z}/(2)$	12	0	20
117	$A_1 + 2\,A_6 + D_5$	(0)	14	0	28
118	$2\,A_2 + A_3 + A_6 + D_5$	(0)	12	0	84
119	$A_1 + A_2 + A_4 + A_6 + D_5$	(0)	20	0	42

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
120	$A_2 + A_5 + A_6 + D_5$	(0)	6	0	84
			12	0	42
121	$A_1 + A_7 + 2\,D_5$	$\mathbb{Z}/(4)$	2	0	8
122	$A_1 + A_2 + A_3 + A_7 + D_5$	$\mathbb{Z}/(4)$	6	0	8
123	$2\,A_1 + A_4 + A_7 + D_5$	$\mathbb{Z}/(2)$	8	0	20
124	$A_8 + 2\,D_5$	(0)	8	4	20
125	$A_1 + A_4 + A_8 + D_5$	(0)	2	0	180
			18	0	20
126	$A_5 + A_8 + D_5$	(0)	12	0	18
127	$2\,A_2 + A_9 + D_5$	(0)	6	0	60
128	$2\,A_1 + A_2 + A_9 + D_5$	$\mathbb{Z}/(2)$	2	0	60
129	$A_1 + A_3 + A_9 + D_5$	$\mathbb{Z}/(2)$	8	4	12
130	$A_4 + A_9 + D_5$	(0)	10	0	20
131	$A_1 + A_2 + A_{10} + D_5$	(0)	14	4	20
132	$2\,A_1 + A_{11} + D_5$	$\mathbb{Z}/(4)$	2	0	6
133	$A_2 + A_{11} + D_5$	$\mathbb{Z}/(2)$	6	0	6
134	$A_1 + A_{12} + D_5$	(0)	2	0	52
			6	2	18
135	$A_{13} + D_5$	(0)	6	2	10
136	$3\,D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	2
137	$2\,A_3 + 2\,D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	0	4
138	$2\,A_2 + 2\,A_4 + D_6$	(0)	30	0	30
139	$2\,A_1 + 2\,A_5 + D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	6	0	6
140	$A_1 + 2\,A_3 + A_5 + D_6$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	0	12
141	$A_3 + A_4 + A_5 + D_6$	$\mathbb{Z}/(2)$	4	0	30
142	$2\,A_6 + D_6$	(0)	14	0	14
143	$A_2 + A_4 + A_6 + D_6$	(0)	6	0	70
144	$A_1 + 2\,A_2 + A_7 + D_6$	$\mathbb{Z}/(2)$	6	0	24
145	$A_2 + A_3 + A_7 + D_6$	$\mathbb{Z}/(2)$	4	0	24
146	$A_1 + A_4 + A_7 + D_6$	$\mathbb{Z}/(2)$	6	2	14
147	$A_4 + A_8 + D_6$	(0)	4	2	46
148	$A_1 + A_2 + A_9 + D_6$	$\mathbb{Z}/(2)$	6	0	10
		$\mathbb{Z}/(2)$	4	2	16
149	$A_3 + A_9 + D_6$	$\mathbb{Z}/(2)$	4	0	10
150	$A_2 + A_{10} + D_6$	(0)	6	0	22
151	$A_1 + A_{11} + D_6$	$\mathbb{Z}/(2)$	4	0	6

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
152	$A_{12} + D_6$	(0)	4	2	14
153	$A_2 + A_5 + D_5 + D_6$	$\mathbb{Z}/(2)$	6	0	12
154	$A_7 + D_5 + D_6$	$\mathbb{Z}/(2)$	4	0	8
155	$2\,A_2 + 2\,D_7$	(0)	12	0	12
156	$A_2 + 3\,A_3 + D_7$	$\mathbb{Z}/(4)$	8	4	8
157	$A_1 + A_2 + 2\,A_4 + D_7$	(0)	10	0	60
158	$A_2 + A_3 + A_6 + D_7$	(0)	8	4	44
159	$A_1 + A_4 + A_6 + D_7$	(0)	4	0	70
160	$A_5 + A_6 + D_7$	(0)	2	0	84
161	$2\,A_1 + A_2 + A_7 + D_7$	$\mathbb{Z}/(2)$	4	0	24
162	$A_1 + A_3 + A_7 + D_7$	$\mathbb{Z}/(4)$	2	0	8
163	$2\,A_1 + A_9 + D_7$	$\mathbb{Z}/(2)$	4	0	10
164	$A_2 + A_9 + D_7$	(0)	2	0	60
165	$A_1 + A_{10} + D_7$	(0)	4	0	22
166	$A_{11} + D_7$	$\mathbb{Z}/(4)$	2	1	2
167	$A_1 + A_5 + D_5 + D_7$	$\mathbb{Z}/(2)$	4	0	12
168	$A_5 + D_6 + D_7$	$\mathbb{Z}/(2)$	2	0	12
169	$2\,A_1 + 2\,D_8$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	2
170	$2\,A_2 + 2\,A_3 + D_8$	$\mathbb{Z}/(2)$	12	0	12
171	$2\,A_5 + D_8$	$\mathbb{Z}/(2)$	6	0	6
172	$2\,A_1 + A_3 + A_5 + D_8$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	12
173	$A_1 + A_4 + A_5 + D_8$	$\mathbb{Z}/(2)$	2	0	30
174	$2\,A_2 + A_6 + D_8$	(0)	12	6	24
175	$A_1 + A_2 + A_7 + D_8$	$\mathbb{Z}/(2)$	2	0	24
176	$A_1 + A_9 + D_8$	$\mathbb{Z}/(2)$	2	0	10
177	$2\,D_5 + D_8$	$\mathbb{Z}/(2)$	4	0	4
178	$A_1 + A_3 + D_6 + D_8$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	2	0	4
179	$2\,D_9$	(0)	4	0	4
180	$A_1 + 2\,A_2 + A_4 + D_9$	(0)	12	0	30
181	$A_1 + A_3 + A_5 + D_9$	$\mathbb{Z}/(2)$	4	0	12
182	$A_4 + A_5 + D_9$	(0)	4	0	30
183	$A_1 + A_2 + A_6 + D_9$	(0)	4	0	42
184	$2\,A_1 + A_7 + D_9$	$\mathbb{Z}/(2)$	4	0	8
185	$A_1 + A_8 + D_9$	(0)	4	0	18
186	$A_9 + D_9$	(0)	4	0	10
187	$A_4 + D_5 + D_9$	(0)	4	0	20

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
188	$2\,A_1 + 2\,A_3 + D_{10}$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	0	4
189	$2\,A_4 + D_{10}$	(0)	10	0	10
190	$A_1 + A_3 + A_4 + D_{10}$	$\mathbb{Z}/(2)$	2	0	20
191	$3\,A_1 + A_5 + D_{10}$	$\mathbb{Z}/(2) \times \mathbb{Z}/(2)$	4	2	4
192	$A_3 + A_5 + D_{10}$	$\mathbb{Z}/(2)$	2	0	12
193	$A_2 + A_6 + D_{10}$	(0)	2	0	42
194	$A_8 + D_{10}$	(0)	2	0	18
195	$A_1 + A_2 + D_5 + D_{10}$	$\mathbb{Z}/(2)$	4	0	6
196	$A_2 + D_6 + D_{10}$	$\mathbb{Z}/(2)$	2	0	6
197	$A_1 + D_7 + D_{10}$	$\mathbb{Z}/(2)$	2	0	4
198	$2\,A_2 + A_3 + D_{11}$	(0)	12	0	12
199	$A_1 + A_2 + A_4 + D_{11}$	(0)	6	0	20
200	$A_2 + A_5 + D_{11}$	(0)	6	0	12
201	$A_1 + A_6 + D_{11}$	(0)	6	2	10
202	$2\,A_1 + 2\,A_2 + D_{12}$	$\mathbb{Z}/(2)$	6	0	6
203	$A_1 + A_2 + A_3 + D_{12}$	$\mathbb{Z}/(2)$	4	0	6
204	$2\,A_1 + A_4 + D_{12}$	$\mathbb{Z}/(2)$	4	2	6
205	$A_1 + D_5 + D_{12}$	$\mathbb{Z}/(2)$	2	0	4
206	$D_6 + D_{12}$	$\mathbb{Z}/(2)$	2	0	2
207	$A_1 + A_4 + D_{13}$	(0)	2	0	20
208	$A_5 + D_{13}$	(0)	2	0	12
209	$D_5 + D_{13}$	(0)	4	0	4
210	$2\,A_2 + D_{14}$	(0)	6	0	6
211	$2\,A_1 + A_2 + D_{14}$	$\mathbb{Z}/(2)$	2	0	6
212	$A_1 + A_3 + D_{14}$	$\mathbb{Z}/(2)$	2	0	4
213	$A_4 + D_{14}$	(0)	4	2	6
214	$A_1 + A_2 + D_{15}$	(0)	4	0	6
215	$2\,A_1 + D_{16}$	$\mathbb{Z}/(2)$	2	0	2
216	$A_2 + D_{16}$	$\mathbb{Z}/(2)$	2	1	2
217	$A_1 + D_{17}$	(0)	2	0	4
218	D_{18}	(0)	2	0	2
219	$3\,E_6$	$\mathbb{Z}/(3)$	2	1	2
220	$2\,A_3 + 2\,E_6$	(0)	12	0	12
221	$A_1 + A_3 + 2\,A_4 + E_6$	(0)	20	0	30
222	$A_1 + A_5 + 2\,E_6$	$\mathbb{Z}/(3)$	2	0	6
223	$A_2 + 2\,A_5 + E_6$	$\mathbb{Z}/(3)$	6	0	6

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
224	$2\,A_2 + A_3 + A_5 + E_6$	$\mathbb{Z}/(3)$	6	0	12
225	$A_3 + A_4 + A_5 + E_6$	(0)	12	0	30
226	$A_6 + 2\,E_6$	(0)	6	3	12
227	$A_1 + A_2 + A_3 + A_6 + E_6$	(0)	6	0	84
			12	0	42
228	$2\,A_1 + A_4 + A_6 + E_6$	(0)	20	10	26
229	$A_2 + A_4 + A_6 + E_6$	(0)	18	3	18
230	$A_1 + A_5 + A_6 + E_6$	(0)	6	0	42
231	$A_1 + A_4 + A_7 + E_6$	(0)	2	0	120
232	$A_5 + A_7 + E_6$	(0)	6	0	24
233	$2\,A_2 + A_8 + E_6$	$\mathbb{Z}/(3)$	6	3	6
234	$2\,A_1 + A_2 + A_8 + E_6$	$\mathbb{Z}/(3)$	2	0	18
235	$A_1 + A_3 + A_8 + E_6$	(0)	12	0	18
236	$A_4 + A_8 + E_6$	(0)	12	3	12
237	$A_1 + A_2 + A_9 + E_6$	(0)	12	6	18
238	$A_3 + A_9 + E_6$	(0)	10	0	12
239	$2\,A_1 + A_{10} + E_6$	(0)	2	0	66
240	$A_2 + A_{10} + E_6$	(0)	6	3	18
241	$A_1 + A_{11} + E_6$	(0)	6	0	12
		$\mathbb{Z}/(3)$	2	0	4
242	$A_{12} + E_6$	(0)	4	1	10
243	$A_3 + A_4 + D_5 + E_6$	(0)	12	0	20
244	$A_1 + A_6 + D_5 + E_6$	(0)	2	0	84
245	$A_7 + D_5 + E_6$	(0)	8	0	12
246	$D_6 + 2\,E_6$	(0)	6	0	6
247	$A_2 + A_4 + D_6 + E_6$	(0)	6	0	30
248	$A_6 + D_6 + E_6$	(0)	4	2	22
249	$A_1 + A_4 + D_7 + E_6$	(0)	4	0	30
250	$D_5 + D_7 + E_6$	(0)	4	0	12
251	$A_4 + D_8 + E_6$	(0)	8	2	8
252	$A_1 + A_2 + D_9 + E_6$	(0)	6	0	12
253	$A_3 + D_9 + E_6$	(0)	4	0	12
254	$A_1 + D_{11} + E_6$	(0)	2	0	12
255	$D_{12} + E_6$	(0)	4	2	4
256	$2\,A_2 + 2\,E_7$	(0)	6	0	6
257	$A_1 + A_3 + 2\,E_7$	$\mathbb{Z}/(2)$	2	0	4

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
258	$A_4 + 2\,E_7$	(0)	4	2	6
259	$A_1 + 2\,A_3 + A_4 + E_7$	$\mathbb{Z}/(2)$	4	0	20
260	$2\,A_2 + A_3 + A_4 + E_7$	(0)	12	0	30
261	$2\,A_3 + A_5 + E_7$	$\mathbb{Z}/(2)$	4	0	12
262	$A_1 + A_2 + A_3 + A_5 + E_7$	$\mathbb{Z}/(2)$	6	0	12
263	$2\,A_1 + A_4 + A_5 + E_7$	$\mathbb{Z}/(2)$	8	2	8
264	$A_2 + A_4 + A_5 + E_7$	(0)	6	0	30
265	$A_1 + 2\,A_2 + A_6 + E_7$	(0)	6	0	42
266	$A_2 + A_3 + A_6 + E_7$	(0)	4	0	42
267	$A_1 + A_4 + A_6 + E_7$	(0)	2	0	70
			8	2	18
268	$A_5 + A_6 + E_7$	(0)	4	2	22
269	$2\,A_2 + A_7 + E_7$	(0)	6	0	24
270	$2\,A_1 + A_2 + A_7 + E_7$	$\mathbb{Z}/(2)$	2	0	24
271	$A_1 + A_3 + A_7 + E_7$	$\mathbb{Z}/(2)$	4	0	8
272	$A_4 + A_7 + E_7$	(0)	6	2	14
273	$A_1 + A_2 + A_8 + E_7$	(0)	6	0	18
274	$A_3 + A_8 + E_7$	(0)	4	0	18
275	$2\,A_1 + A_9 + E_7$	$\mathbb{Z}/(2)$	2	0	10
276	$A_2 + A_9 + E_7$	(0)	6	0	10
		$\mathbb{Z}/(2)$	4	1	4
277	$A_1 + A_{10} + E_7$	(0)	2	0	22
			6	2	8
278	$A_{11} + E_7$	(0)	4	0	6
279	$D_4 + 2\,E_7$	$\mathbb{Z}/(2)$	2	0	2
280	$A_2 + A_4 + D_5 + E_7$	(0)	6	0	20
281	$A_1 + A_5 + D_5 + E_7$	$\mathbb{Z}/(2)$	2	0	12
282	$A_6 + D_5 + E_7$	(0)	6	2	10
283	$A_2 + A_3 + D_6 + E_7$	$\mathbb{Z}/(2)$	4	0	6
284	$A_5 + D_6 + E_7$	$\mathbb{Z}/(2)$	4	2	4
285	$D_5 + D_6 + E_7$	$\mathbb{Z}/(2)$	2	0	4
286	$A_1 + A_3 + D_7 + E_7$	$\mathbb{Z}/(2)$	4	0	4
287	$A_4 + D_7 + E_7$	(0)	2	0	20
288	$A_1 + A_2 + D_8 + E_7$	$\mathbb{Z}/(2)$	2	0	6
289	$A_2 + D_9 + E_7$	(0)	4	0	6
290	$A_1 + D_{10} + E_7$	$\mathbb{Z}/(2)$	2	0	2

Table 2. List of extremal elliptic $K3$ surfaces

No	Σ	MW	a	b	c
291	$D_{11} + E_7$	(0)	2	0	4
292	$A_2 + A_3 + E_6 + E_7$	(0)	6	0	12
293	$A_1 + A_4 + E_6 + E_7$	(0)	2	0	30
294	$A_5 + E_6 + E_7$	(0)	6	0	6
295	$D_5 + E_6 + E_7$	(0)	2	0	12
296	$2\,A_1 + 2\,E_8$	(0)	2	0	2
297	$A_2 + 2\,E_8$	(0)	2	1	2
298	$2\,A_2 + 2\,A_3 + E_8$	(0)	12	0	12
299	$2\,A_1 + 2\,A_4 + E_8$	(0)	10	0	10
300	$A_1 + A_2 + A_3 + A_4 + E_8$	(0)	6	0	20
301	$2\,A_5 + E_8$	(0)	6	0	6
302	$A_2 + A_3 + A_5 + E_8$	(0)	6	0	12
303	$A_1 + A_4 + A_5 + E_8$	(0)	2	0	30
304	$2\,A_2 + A_6 + E_8$	(0)	6	3	12
305	$2\,A_1 + A_2 + A_6 + E_8$	(0)	2	0	42
306	$A_1 + A_3 + A_6 + E_8$	(0)	6	2	10
307	$A_4 + A_6 + E_8$	(0)	2	1	18
308	$A_1 + A_2 + A_7 + E_8$	(0)	2	0	24
309	$2\,A_1 + A_8 + E_8$	(0)	2	0	18
310	$A_2 + A_8 + E_8$	(0)	6	3	6
311	$A_1 + A_9 + E_8$	(0)	2	0	10
312	$A_{10} + E_8$	(0)	2	1	6
313	$2\,D_5 + E_8$	(0)	4	0	4
314	$A_1 + A_4 + D_5 + E_8$	(0)	2	0	20
315	$A_5 + D_5 + E_8$	(0)	2	0	12
316	$2\,A_2 + D_6 + E_8$	(0)	6	0	6
317	$A_4 + D_6 + E_8$	(0)	4	2	6
318	$A_1 + A_2 + D_7 + E_8$	(0)	4	0	6
319	$A_1 + D_9 + E_8$	(0)	2	0	4
320	$D_{10} + E_8$	(0)	2	0	2
321	$A_1 + A_3 + E_6 + E_8$	(0)	2	0	12
322	$A_4 + E_6 + E_8$	(0)	2	1	8
323	$D_4 + E_6 + E_8$	(0)	4	2	4
324	$A_1 + A_2 + E_7 + E_8$	(0)	2	0	6
325	$A_3 + E_7 + E_8$	(0)	2	0	4

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