CLASSIFICATION OF FANO 4-FOLDS WITH LEFSCHETZ DEFECT 3 AND PICARD NUMBER 5

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1. INTRODUCTION

The classification of (smooth, complex) Fano manifolds has been achieved up to dimension 3 and attracts a lot of attention also in higher dimensions, especially due to the Minimal Model Program. Indeed we recall that Fano manifolds appear in the birational classification of varieties of negative Kodaira dimension: in this case the MMP is expected to end up with a fiber type morphism whose fibers are (mildly singular) Fano varieties.

In the early 80's the classification of Fano 3-folds in [MM81] due to Mori and Mukai was the starting point to study Fano manifolds via their contractions. In fact, the Fano condition makes the situation special, because the Cone and the Contraction Theorems hold for the whole cone of effective curves. Nevertheless, there is still no complete classification of Fano varieties in dimension 4 and higher.

In this paper we focus on classification results of some Fano 4-folds. Let us fix some notation. Given a smooth complex projective variety X, we denote by $\mathcal{N}_1(X)$ the \mathbb{R} -vector space of one-cycles with real coefficients, modulo numerical equivalence, whose dimension is the *Picard number* ρ_X .

Let D be a prime divisor in X. The inclusion $i: D \hookrightarrow X$ induces a pushforward of one-cycles $i_*: \mathcal{N}_1(D) \to \mathcal{N}_1(X)$. We set $\mathcal{N}_1(D, X) := i_*(\mathcal{N}_1(D)) \subseteq \mathcal{N}_1(X)$, which is the linear subspace of $\mathcal{N}_1(X)$ spanned by numerical classes of curves contained in D. In [Cas12] the following invariant, called *Lefschetz defect*, was introduced:

 $\delta_X := \max\{\operatorname{codim} \mathcal{N}_1(D, X) | D \subset X \text{ prime divisor}\}.$

By [loc. cit., Th. 1.1], if X is a Fano manifold of arbitrary dimension with $\delta_X \ge 4$, then $X \cong S \times T$, with S a del Pezzo surface. As a consequence, all Fano 4-folds with $\delta_X \ge 4$ are well known, being product of two del Pezzo surfaces.

In this paper we deal with the case in which X is a Fano 4-fold with $\delta_X = 3$. Under this assumption, by [Cas13, Th. 1.1] we know that if X is not a product of two del Pezzo surfaces, then $\rho_X \in \{5, 6\}$. Therefore in order to complete the classification of Fano 4-folds with $\delta_X = 3$ we are left to study the cases in which $\rho_X = 5$ and $\rho_X = 6$. This setting has already been studied in [MR19], where several properties of these 4-folds are shown (see Th. 2.1).

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Our main result is the complete classification of Fano 4-folds with $\rho_X = 5$ and $\delta_X = 3$. To give the statement, we first need to introduce two examples; the former is due to Tsukioka, while the latter is new.

Example 1.1 ([Tsu10]). Let $p, q \in \mathbb{P}^4$ be distinct points, and $Q \subset \mathbb{P}^4$ a smooth quadric surface disjoint from the line \overline{pq} . Let Z be the blow-up of \mathbb{P}^4 along \overline{pq} , $F_p, F_q \subset Z$ the fibers over p and q respectively, and $S \subset Z$ the transform of Q.

The surfaces S, F_p, F_q are pairwise disjoint in Z; let X be the blow-up of Z along S, F_p , and F_q . Then X is a Fano 4-fold with $\rho_X = 5$ and $\delta_X = 3$.

We can also describe X as follows: let X' be the blow-up of \mathbb{P}^4 along p, q, and Q. Then X is the blow-up of X' along the transform of the line \overline{pq} .

Example 1.2. The 4-fold $Z := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2))$ has a divisorial contraction $Z \to W := \mathbb{P}(1, 1, 1, 2, 2)$ which sends the exceptional divisor D to a curve (the singular locus of W); let $F_1, F_2 \subset D$ be two distinct fibers.

Let moreover $\mathcal{O}_W(1)$ be the ample generator of $\operatorname{Pic}(W)$, and $H \in \operatorname{Pic}(Z)$ the pullback of $\mathcal{O}_W(1)$. Consider $S \subset Z$ a general complete intersection of elements in the linear systems |H| and |2H|; S is a del Pezzo surface of degree 2 (see §3.1).

The surfaces S, F_1, F_2 are pairwise disjoint in Z; let X be the blow-up of Z along S, F_1 , and F_2 . Then X is a Fano 4-fold with $\rho_X = 5$ and $\delta_X = 3$ (see Lemma 3.3).

We recall that toric Fano 4-folds have been classified by Batyrev [Bat99] and Sato [Sat00]; we refer to [Bat99] for the terminology concerning toric varieties and their combinatorial type.

Theorem 1.3. Let X be a Fano 4-fold with $\rho_X = 5$ and $\delta_X = 3$. Then X is one of the following varieties: the toric Fano 4-folds K_1 , K_2 , K_3 , $K_4 \cong \mathbb{P}^2 \times S_4$ (S_4 the blow-up of \mathbb{P}^2 in three non collinear points), or one of the 4-folds of Examples 1.1 and 1.2.

We describe these varieties and their invariants in §3, see Table 3.4. They are all rational, as already shown in [MR19, Cor. 1.3]. The first five are rigid, while Example 1.2 yields a positive dimensional family (see Lemma 3.3).

We note that the assumptions $\rho_X = 5$ and $\delta_X = 3$ imply that X contains a prime divisor D with dim $\mathcal{N}_1(D, X) = 2$; in fact it is easy to see that all the varieties listed in Theorem 1.3 also contain a prime divisor D' with $\rho_{D'} = 2$. We obtain the following application to Fano 4-folds containing a prime divisor with $\rho = 2$; an analogous result for the case of a prime divisor with $\rho = 1$ is given in [CD15, Th. 3.8].

Corollary 1.4. Let X be a Fano 4-fold containing a prime divisor D with $\rho_D = 2$, or more generally with dim $\mathcal{N}_1(D, X) = 2$. Then either $X \cong \mathbb{P}^2 \times S$, or $\rho_X \leq 5$. Moreover, $\rho_X = 5$ if and only if X is one of the 4-folds listed in Theorem 1.3.

Corollary 1.4 is also related to the study of Fano 4-folds having an elementary divisorial contraction $\sigma: X \to X'$ such that $\sigma(\text{Exc}(\sigma))$ is a curve, because then

automatically dim $\mathcal{N}_1(\text{Exc}(\sigma), X) = 2$. It follows from [Cas12] that $\rho_X \leq 5$, and Tsukioka has classified the case $\rho_X = 5$ when σ is a smooth blow-up, as follows.

Theorem 1.5 ([Tsu10]). Let X be a Fano 4-fold obtained as a blow-up $\sigma: X \to X'$ of a smooth, irreducible curve in a smooth 4-fold X', and assume that $\rho_X = 5$. Then X is either the toric 4-fold K_3 , or Example 1.1. In both cases $\text{Exc}(\sigma) \cong \mathbb{P}^1 \times \mathbb{P}^2$, $\mathcal{N}_{\text{Exc}(\sigma)/X} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, -1)$, and X' is not Fano.

In fact this result is shown in [Tsu10] in arbitrary dimension ≥ 4 . Here we extend the classification as follows.

Corollary 1.6. Let X be a Fano 4-fold having an elementary divisorial contraction $\sigma: X \to X'$ such that $\sigma(\text{Exc}(\sigma))$ is a curve, and assume that $\rho_X = 5$. Then X is one of the toric Fano 4-folds K_1 , K_3 , or one of the 4-folds of Examples 1.1 and 1.2. In all cases, $\text{Exc}(\sigma) \cong \mathbb{P}^1 \times \mathbb{P}^2$, and X' is not Fano.

In cases K_1 and Example 1.2, we have $\mathcal{N}_{\operatorname{Exc}(\sigma)/X} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, -2)$, and $\sigma(\operatorname{Exc}(\sigma))$ is a curve of singular points.

Let us briefly discuss the strategy used to show Theorem 1.3. We build on results from [Cas12, Rom19, MR19], which give a structure theorem for Fano 4-folds X with $\rho_X = 5$ and $\delta_X = 3$. More precisely, X has always a flat fibration $X \to \mathbb{P}^2$, that factors as $X \to Y \to \mathbb{P}^2$, where the first map is a conic bundle, and the second one a \mathbb{P}^1 -bundle. We collect these results in Theorem 2.1.

We show that the fibration $X \to \mathbb{P}^2$ has also a different factorization as $X \to Z \xrightarrow{\varphi} \mathbb{P}^2$, where φ is a \mathbb{P}^2 -bundle, and $X \to Z$ is the blow-up of three pairwise disjoint smooth surfaces $S_i \subset Z$, horizontal for φ . When all the surfaces S_i are sections of φ , it turns out that X is toric. Otherwise, we prove that two surfaces S_i are always sections, and the third one has degree 2 over \mathbb{P}^2 . In this case, we show that X is one of Examples 1.1 or 1.2.

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Notations. We work over the field of complex numbers. Let X be a smooth projective variety.

 \sim denotes linear equivalence for divisors.

 $\mathcal{N}_1(X)$ is the \mathbb{R} -vector space of one-cycles with real coefficients, modulo numerical equivalence, and dim $\mathcal{N}_1(X) = \rho_X$ is the Picard number of X.

We denote by [C] the numerical equivalence class in $\mathcal{N}_1(X)$ of a one-cycle C of X.

 $NE(X) \subset \mathcal{N}_1(X)$ is the convex cone generated by classes of effective curves.

A contraction of X is a surjective morphism $\varphi \colon X \to Y$ with connected fibers, where Y is normal and projective.

The relative cone $NE(\varphi)$ of φ is the convex subcone of NE(X) generated by classes of curves contracted by φ .

A conic bundle $h: X \to Y$ is a fiber type contraction such that every fiber is one-dimensional and $-K_X$ is *h*-ample; then every fiber is isomorphic to a plane conic.

2. Proof of the main result

2.1. **Preliminaries.** We begin by collecting in a unique statement the known results on the structure of Fano 4-folds X with $\rho_X = 5$ and $\delta_X = 3$. This is our starting point to prove Theorem 1.3.

Theorem 2.1 ([Cas12, Rom19, MR19]). Let X be a Fano 4-fold with $\rho_X = 5$ and $\delta_X = 3$. Then there exists a diagram:

$$X \xrightarrow{f} X_2 \xrightarrow{\psi} Y \xrightarrow{\xi} \mathbb{P}^2$$

with the following properties:

- (a) $Y \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a))$ with $a \in \{0, 1, 2\}$, and ξ is the natural \mathbb{P}^1 -bundle;
- (b) ψ is a \mathbb{P}^1 -bundle;
- (c) f is the blow-up of two disjoint smooth, irreducible surfaces $B_1, B_2 \subset X_2$;
- (d) for i = 1, 2 set $A_i := \psi(B_i) \subset Y$; A_1 and A_2 are disjoint smooth surfaces, and A_1 is a nef divisor;
- (e) if a = 0, then $Y \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $A_i \cong \{pt\} \times \mathbb{P}^2$ for i = 1, 2; if $a \in \{1, 2\}$, then A_2 is the negative section of $\xi \colon Y \to \mathbb{P}^2$ (namely $\mathcal{N}_{A_2/Y} \cong \mathcal{O}_{\mathbb{P}^2}(-a)$); in any case A_2 is a section of ξ and $\xi_{|A_1} \colon A_1 \to \mathbb{P}^2$ is finite;
- (f) for i = 1, 2 set $T_i := \psi^{-1}(A_i) \subset X_2$; B_i is a section of $\psi_{|T_i|}: T_i \to A_i$, for i = 1, 2.

Proof. The existence of the diagram, together with properties (b) and (c), and the additional fact that $\psi \circ f \colon X \to Y$ is a conic bundle, are shown in [Cas12, Th. 1.1 and its proof, in particular 3.3.15-3.3.17]. Then applying [Rom19, Prop. 3.5(1)] to $\psi \circ f$ we get (f). Finally (a) is [MR19, Prop. 1.2(a)], and (d) and (e) are proved in [MR19, proof of Prop. 1.2].

We note that after (d) and (e), the role of the two surfaces A_1 and A_2 is not symmetric if a > 0.

In the toric case, the classification is already known, and relies on Batyrev's classification of toric Fano 4-folds [Bat99], see [MR19, Prop. 5.1]. We follow the notation of [Bat99].

Proposition 2.2. There are four toric Fano 4-folds X with $\rho_X = 5$ and $\delta_X = 3$; they are the 4-folds K_1 , K_2 , K_3 , and K_4 .

2.2. Proof of Theorem 1.3. We keep the same notation as in Th. 2.1.

Step 2.3. We can assume that there exists a commutative diagram:



where $\varphi: Z \to \mathbb{P}^2$ is a \mathbb{P}^2 -bundle, g is the blow-up along a section $S_3 \subset Z$ of φ , $E := \operatorname{Exc}(g)$ is a section of ψ , and $E \cap (B_1 \cup B_2) = \emptyset$.

Proof. Consider the natural factorization of f as a sequence of two blow-ups:

$$X \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2$$

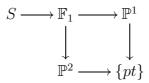
where f_2 is the blow-up of B_2 and f_1 is the blow-up of the transform of B_1 .

Let us consider the morphism $\zeta := \xi \circ \psi \circ f_2 \colon X_1 \to \mathbb{P}^2$. Since both ψ and ξ are smooth by Th. 2.1(*a*)-(*b*), the composition $\xi \circ \psi \colon X_2 \to \mathbb{P}^2$ is smooth. Moreover, since $A_2 \subset Y$ is a section of ξ , and the center B_2 of the blow-up $f_2 \colon X_1 \to X_2$ is a section over A_2 (see Th. 2.1(*e*) and (*f*)), we conclude that B_2 is a section of $\xi \circ \psi \colon X_2 \to \mathbb{P}^2$. This implies that $\zeta \colon X_1 \to \mathbb{P}^2$ is smooth. We show that $-K_{X_1}$ is ζ -ample. Let $C \subset X_1$ be an irreducible curve such that

We show that $-K_{X_1}$ is ζ -ample. Let $C \subset X_1$ be an irreducible curve such that $-K_{X_1} \cdot C \leq 0$. If $\widetilde{C} \subset X$ is its transform, we have $-K_X \cdot \widetilde{C} > 0$ which implies that $\operatorname{Exc}(f_1) \cdot \widetilde{C} < 0$, hence $C \subset f_1(\operatorname{Exc}(f_1)), f_2(C) \subset B_1$, and $\psi(f_2(C)) \subset A_1$. Since ψ is finite on B_1 and ξ is finite on A_1 by Th. 2.1(f) and (e), we conclude that $\zeta(C)$ is a curve. This shows that $-K_{X_1}$ is positive on every curve contracted by ζ . Being X Fano, the cone NE(X) is closed and polyhedral, and this easily implies that NE(X_1) is closed. By the relative Kleiman's criterion, $-K_{X_1}$ is ζ -ample.

Hence $\zeta \colon X_1 \to \mathbb{P}^2$ is a smooth contraction of relative Picard number 3 with $-K_{X_1}$ relatively ample, and every fiber of ζ is isomorphic to the del Pezzo surface S with $\rho_S = 3$, the blow-up of \mathbb{P}^2 in two points.

If $i: S \hookrightarrow X_1$ is the inclusion of a fiber, the pushforward of 1-cycles $i_*: \mathcal{N}_1(S) \to \mathcal{N}_1(X_1)$ is injective, and yields and isomorphism $\mathcal{N}_1(S) \cong \ker \zeta_*$. It is clear that $i_*(\operatorname{NE}(S)) \subseteq \operatorname{NE}(\zeta)$; conversely it follows from [Wiś91, Prop. 1.3] that equality holds, so that every contraction of the fiber S extends to a global contraction of X_1 over \mathbb{P}^2 . Therefore the sequence of elementary contractions:



yields a corresponding factorization of ζ :

$$\begin{array}{ccc} X_1 \xrightarrow{f_2'} X_2' \xrightarrow{\psi'} Y' \\ g & & & \downarrow^{\xi'} \\ Z \xrightarrow{\varphi} \mathbb{P}^2 \end{array}$$

We have:

- $\xi': Y' \to \mathbb{P}^2$ and $\psi': X'_2 \to Y'$ are \mathbb{P}^1 -bundles, and $\varphi: Z \to \mathbb{P}^2$ is a \mathbb{P}^2 -bundle;
- g is the blow-up of a smooth surface $S_3 \subset Z$ which is a section of φ ;
- f'_2 is the blow-up of a smooth surface $B'_2 \subset X'_2$ which is a section of $\varphi \circ g \colon X'_2 \to \mathbb{P}^2$, and is disjoint from $E := \operatorname{Exc}(g)$;
- E is a section of $\psi' \colon X'_2 \to Y'$.

Notice also that the center of the blow-up $f_1: X \to X_1$ cannot meet any (-1)-curve in the fiber S, otherwise X would not be Fano. Hence $B'_1 := f'_2(f_1(\operatorname{Exc}(f_1)))$ is disjoint from E.

Finally, using [MR19, Prop. 1.2(a)], we still get $Y' \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a))$ with $a \in \{0, 1, 2\}$, so that we can replace the original factorization of ζ with the new one keeping all the previous properties.

Set $S_i := g(B_i) \subset Z$ for i = 1, 2. Then S_1, S_2 , and S_3 are pairwise disjoint smooth surfaces, and X is the blow-up of Z along $S_1 \cup S_2 \cup S_3$. We set $Z_p := \varphi^{-1}(p)$ for every $p \in \mathbb{P}^2$. Moreover we denote by d the degree of the finite morphism $\xi_{|A_1} : A_1 \to \mathbb{P}^2$ (see Th. 2.1(e)).

Step 2.4. S_2 is a section of φ , and $\varphi_{|S_1} \colon S_1 \to \mathbb{P}^2$ is finite of degree d.

Proof. For i = 1, 2, since B_i is a section over A_i by Th. 2.1(f), the degree of S_i over \mathbb{P}^2 is equal to the degree of A_i over \mathbb{P}^2 ; in particular S_2 is a section of φ by Th. 2.1(e).

Step 2.5. The points $(S_1 \cup S_2 \cup S_3) \cap Z_p$ (with the reduced structure) are in general linear position in $Z_p \cong \mathbb{P}^2$, for every $p \in \mathbb{P}^2$.

Indeed, if there were three of them on a line ℓ , the transform of ℓ in X would have non-positive anticanonical degree.

Step 2.6. If d = 1, then X is toric, and it is one of the 4-folds K_1 , K_2 , K_3 , or K_4 , in the notation of [Bat99].

Proof. If d = 1, then $\varphi: Z \to \mathbb{P}^2$ has three pairwise disjoint sections S_i , which are fiberwise in general linear position, by Steps 2.3, 2.4, and 2.5. This implies that $Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3)$ in such a way that the three sections S_i correspond to the projections $\mathcal{L}_1 \oplus \mathcal{L}_2 \oplus \mathcal{L}_3 \twoheadrightarrow \mathcal{L}_i$. This means that Z is a toric 4-fold, and that S_1, S_2 , and S_3 are invariant for the torus action, so that X is toric. Then the statement follows from Prop. 2.2.

From now on we assume that $d \ge 2$, in particular this implies that $a \in \{1, 2\}$ by Th. 2.1(e).

For $q_1, q_2 \in Z_p$ distinct points, we denote by $\overline{q_1q_2} \subset Z_p$ the line spanned by q_1 and q_2 .

Step 2.7. Let $H \subset Z$ be the relative secant variety of S_1 in Z, namely the closure in Z of the set:

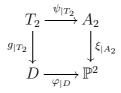
$$\left\{\overline{q_1q_2} \subset Z_p \,|\, q_1, q_2 \in S_1 \cap Z_p, q_1 \neq q_2, p \in \mathbb{P}^2\right\}.$$

For p general, we have $|S_1 \cap Z_p| = d \ge 2$, so that H is non-empty. It is not difficult to see that dim H = 3, and Step 2.5 implies that $H \cap (S_2 \cup S_3) = \emptyset$.

Recall that $T_2 = \psi^{-1}(A_2) \subset X_2$ and that $E = \text{Exc}(g) \subset X_2$ (see Th. 2.1(f) and Step 2.3).

Step 2.8. Set $D := g(T_2) \subset Z$. Then $T_2 \cong D \cong \mathbb{P}^1 \times \mathbb{P}^2$, and $D \cap Z_p$ is a line in Z_p for every $p \in \mathbb{P}^2$. Moreover D contains S_2 and S_3 , while $D \cap S_1 = \emptyset$.

Proof. We have a commutative diagram:



where the vertical maps are isomorphisms by Step 2.3 and Th. 2.1(e), and the horizontal maps are \mathbb{P}^1 -bundles.

We also have $S_3 = g(E \cap T_2) \subset D$; moreover $B_2 \subset T_2$ by Th. 2.1(d), hence $S_2 = g(B_2) \subset D$. By Steps 2.3 and 2.4, S_2 and S_3 are disjoint sections of the \mathbb{P}^1 -bundle $\varphi_{|D} \colon D \to \mathbb{P}^2$; this implies that $D \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(c))$ for some $c \in \mathbb{Z}$.

Now we have $H \cap D \neq \emptyset$, because both contain a line in Z_p , so that $H \cap D$ yields a non-zero effective divisor in D. On the other hand, this divisor is disjoint from both sections S_2 and S_3 by Step 2.7. This easily implies that c = 0 and $D \cong \mathbb{P}^1 \times \mathbb{P}^2$.

Finally, we note that for every $p \in \mathbb{P}^2$, $D \cap Z_p$ is the line spanned by the points $S_2 \cap Z_p$ and $S_3 \cap Z_p$, so that $D \cap S_1 = \emptyset$ by Step 2.5.

We denote by $L \in \operatorname{Pic}(Y)$ the tautological line bundle for $Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a))$; note that L is nef and big, and $L_{|A_2} \cong \mathcal{O}_{\mathbb{P}^2}$ by Th. 2.1(e).

Step 2.9. There exists $b \in \mathbb{Z}$ such that $\mathcal{N}_{S_3/Z}^{\vee} \cong \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}(a+b)$ and $\mathcal{N}_{E/X_2}^{\vee} \cong L \otimes \xi^*(\mathcal{O}_{\mathbb{P}^2}(b)) \in \operatorname{Pic}(Y).$

Proof. By Step 2.3 we have a commutative diagram:



where the horizontal maps are isomorphisms, and the vertical maps are \mathbb{P}^1 bundles. Since $q: X_2 \to Z$ is the blow-up of S_3 , using Th. 2.1(a) we get:

$$\mathbb{P}_{S_3}(\mathcal{N}_{S_3/Z}^{\vee}) \cong E \cong Y \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(a)),$$

therefore $\mathcal{N}_{S_3/Z}^{\vee} \cong \mathcal{O}_{\mathbb{P}^2}(b) \oplus \mathcal{O}_{\mathbb{P}^2}(a+b)$, with $b \in \mathbb{Z}$. Moreover $\mathcal{N}_{E/X_2}^{\vee}$ is the tautological line bundle of $\mathbb{P}_{\mathbb{P}^2}(\mathcal{O}(b) \oplus \mathcal{O}(a+b))$, which gives the statement.

Step 2.10. We have b = 0, $X_2 \cong \mathbb{P}_Y(\mathcal{O} \oplus L)$, and E corresponds (as a section of ψ) to the projection $\mathcal{O} \oplus L \twoheadrightarrow \mathcal{O}$.

Proof. Let \mathcal{E} be a rank 2 vector bundle on Y such that $X_2 = \mathbb{P}_Y(\mathcal{E})$. We know by Step 2.3 that E is a section of ψ ; this section corresponds to a surjection $\sigma: \mathcal{E} \twoheadrightarrow \mathcal{F}$ with $\mathcal{F} \in \operatorname{Pic}(Y)$, and up to replacing \mathcal{E} with $\mathcal{E} \otimes \mathcal{F}^{\vee}$ we may assume that $\mathcal{F} = \mathcal{O}_Y$, so that ker $\sigma \cong \mathcal{N}_{E/X_2}^{\vee} \cong L \otimes \xi^*(\mathcal{O}_{\mathbb{P}^2}(b))$ by Step 2.9. We obtain the following exact sequence over Y:

(2.1)
$$0 \longrightarrow \ker \sigma \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Y \longrightarrow 0.$$

Now let us consider $A_2 \subset Y$; we have ker $\sigma_{|A_2} \cong L_{|A_2} \otimes \xi^*(\mathcal{O}_{\mathbb{P}^2}(b))_{|A_2} \cong \mathcal{O}_{\mathbb{P}^2}(b)$, so by restricting to A_2 the above exact sequence we get:

$$0 \longrightarrow \mathcal{O}(b) \longrightarrow \mathcal{E}_{|A_2} \longrightarrow \mathcal{O} \longrightarrow 0.$$

On the other hand $\mathbb{P}_{A_2}(\mathcal{E}_{|A_2}) = T_2 \cong \mathbb{P}^2 \times \mathbb{P}^1$ by Step 2.8, and we deduce that b = 0 and ker $\sigma \cong L$.

We note that Y is Fano, and L is nef and big, so that $-K_Y + L$ is ample. Therefore $\operatorname{Ext}^1(\mathcal{O}_Y, \ker \sigma) \cong H^1(Y, L) = H^1(Y, K_Y - K_Y + L) = 0$ by Kodaira vanishing, hence the sequence (2.1) splits, so that $\mathcal{E} \cong \mathcal{O}_Y \oplus L$.

Step 2.11. There exists a section K of $\psi \colon X_2 \to Y$ containing B_1 and disjoint from E.

Proof. By Th. 2.1(f) and Step 2.10 we have $B_1 \subset T_1 = \mathbb{P}_{A_1}(\mathcal{O}_{A_1} \oplus L_{|A_1})$ and B_1 is a section of $\psi_{|T_1}: T_1 \to A_1$. Moreover by Steps 2.3 and 2.10 we deduce that $T_1 \cap E$ is another section of $\psi_{|T_1}$, disjoint from B_1 , and corresponding to the projection $\mathcal{O}_{A_1} \oplus L_{|A_1} \to \mathcal{O}_{A_1}$. Then it is not difficult to see that B_1 corresponds, as a section, to a surjection $\tau: \mathcal{O}_{A_1} \oplus L_{|A_1} \to L_{|A_1}$.

Let us consider the restriction $r: \operatorname{Hom}(\mathcal{O}_Y \oplus L, L) \to \operatorname{Hom}(\mathcal{O}_{A_1} \oplus L_{|A_1}, L_{|A_1}).$ We have

$$\operatorname{Hom}(\mathcal{O}_{A_1} \oplus L_{|A_1}, L_{|A_1}) \cong \operatorname{Hom}(L_{|A_1}^{\vee} \oplus \mathcal{O}_{A_1}, \mathcal{O}_{A_1}) \cong H^0(A_1, L_{|A_1}) \oplus H^0(A_1, \mathcal{O}_{A_1}),$$

and similarly $\operatorname{Hom}(\mathcal{O}_Y \oplus L, L) \cong H^0(Y, L) \oplus H^0(Y, \mathcal{O}_Y)$. Since the restriction $H^0(Y, \mathcal{O}_Y) \to H^0(A_1, \mathcal{O}_{A_1})$ is an isomorphism, r is surjective if the restriction $H^0(Y, L) \to H^0(A_1, L_{|A_1})$ is.

We have an exact sequence of sheaves on Y:

$$0 \longrightarrow L \otimes \mathcal{O}_Y(-A_1) \longrightarrow L \longrightarrow L_{|A_1|} \longrightarrow 0.$$

Since $A_1 \cap A_2 = \emptyset$ by Th. 2.1(*d*), and A_1 has degree *d* over \mathbb{P}^2 , it is not difficult to see that $\mathcal{O}_Y(A_1) = L^{\otimes d}$ in $\operatorname{Pic}(Y)$. Then, using Serre duality and Kawamata-Viehweg vanishing:

$$H^{1}(Y, L \otimes \mathcal{O}_{Y}(-A_{1})) = H^{1}(Y, L^{\otimes (1-d)}) = H^{2}(Y, K_{Y} \otimes L^{\otimes (d-1)}) = 0$$

because $d \geq 2$ and L is nef and big.

We conclude that r is surjective, so that τ extends to a morphism $\overline{\tau} \colon \mathcal{O}_Y \oplus L \to L$.

We show that $\overline{\tau}$ is surjective. By contradiction, suppose that $\operatorname{Im} \overline{\tau} \subsetneq L$; then $\operatorname{Im} \overline{\tau} \cong L \otimes \mathcal{O}_Y(-D_0)$ with D_0 a non-zero effective divisor, and $\ker \overline{\tau} \cong \mathcal{O}_Y(D_0)$.

We have $\operatorname{Hom}(L, L \otimes \mathcal{O}_Y(-D_0)) = 0$, hence $\ker \overline{\tau} \supseteq \{0\} \oplus L$. On the other hand $\operatorname{Hom}(\mathcal{O}_Y(D_0), \mathcal{O}_Y) = 0$, hence $\ker \overline{\tau} \subseteq \{0\} \oplus L$. We conclude that $\ker \overline{\tau} = \{0\} \oplus L$ and $\overline{\tau}$ factors through the projection $\mathcal{O}_Y \oplus L \twoheadrightarrow \mathcal{O}_Y$. Then the same happens by restricting to A_1 , which is impossible, because τ is surjective.

Thus we have a surjection $\overline{\tau} \colon \mathcal{O}_Y \oplus L \twoheadrightarrow L$ which yields a section $K \subset X_2$ extending B_1 .

We show that $K \cap E = \emptyset$. Let us consider the projection $\mathcal{O}_Y \oplus L \twoheadrightarrow L$ and the corresponding section $K' \subset X_2$. Since E corresponds to the projection onto the other summand, we have $K' \cap E = \emptyset$. On the other hand, it is easy to check that $K \sim K'$ in X_2 , hence for every curve $C \subset E$ we have $K \cdot C = 0$. Since $K \neq E$, this implies that $K \cap E = \emptyset$.

Step 2.12. We have d = 2 and H = g(K).

Proof. Consider $g(K) \subset Z$, so that $K \cong g(K)$ and $g(K) \supset S_1$ by Step 2.11. If $p \in \mathbb{P}^2$ is general, then $g^{-1}(Z_p) \cong \mathbb{F}_1$, and $K \cap g^{-1}(Z_p)$ is a section of $\mathbb{F}_1 \to \mathbb{P}^1$, disjoint from the (-1)-curve $E \cap g^{-1}(Z_p)$. Thus $g(K) \cap Z_p$ is a line in $Z_p \cong \mathbb{P}^2$, and this line contains the d points $S_1 \cap Z_p$. Since these points are in general linear position (see Step 2.5), we conclude that d = 2. We also deduce that g(K) = H (see Step 2.7).

Step 2.13. We have $Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(a))$, and under the isomorphism $D \cong \mathbb{P}^1 \times \mathbb{P}^2$ one has $\mathcal{O}_Z(D)_{|\{pt\} \times \mathbb{P}^2} \cong \mathcal{O}_{\mathbb{P}^2}(-a)$.

Proof. We know by Steps 2.3, 2.9 and 2.10 that S_3 is a section of $\varphi \colon Z \to \mathbb{P}^2$ with conormal bundle $\mathcal{O} \oplus \mathcal{O}(a)$. As in the proof of Step 2.10, using that $H^1(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(c)) = 0$ for every $c \in \mathbb{Z}$, one shows that $Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O} \oplus \mathcal{O}(a))$.

Recall from Step 2.8 that $D \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $S_3 \cong \{\text{pt}\} \times \mathbb{P}^2$. Thus $\mathcal{N}_{S_3/Z} \cong \mathcal{O}_{\mathbb{P}^2} \oplus \mathcal{O}_{\mathbb{P}^2}(-a)$ and $\mathcal{N}_{S_3/D} \cong \mathcal{O}_{\mathbb{P}^2}$; using the exact sequence on S_3 :

$$0 \longrightarrow \mathcal{N}_{S_3/D} \longrightarrow \mathcal{N}_{S_3/Z} \longrightarrow \mathcal{N}_{D/Z|S_3} \longrightarrow 0$$

we conclude that $\mathcal{N}_{D/Z|S_3} \cong \mathcal{O}_{\mathbb{P}^2}(-a).$

Step 2.14. If a = 1, then X is the 4-fold of Example 1.1.

Proof. Set a = 1. Applying Step 2.13 we have $Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(1))$, that is the blow-up of \mathbb{P}^4 along a line, with exceptional divisor D; moreover S_2 and S_3 are

non-trivial fibers of the blow-up $Z \to \mathbb{P}^4$. The image of H (see Steps 2.7 and 2.12) in \mathbb{P}^4 is a hyperplane, and the image of S_1 is a smooth quadric surface. This is Example 1.1.

Step 2.15. If a = 2, then X is the 4-fold of Example 1.2.

Proof. Set a = 2. Using Step 2.13 one has $Z \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2))$, then there is a divisorial contraction $Z \to W$ sending D to a curve; moreover $S_2, S_3 \subset D$ are fibers of this contraction.

Consider the divisor $H \subset Z$ (see Step 2.7), and let $\mathcal{O}_W(1)$ be the ample generator of $\operatorname{Pic}(W)$. Using Steps 2.11 and 2.12, and Th. 2.1(*a*) we have $H \cong K \cong Y = \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$; and it is not difficult to see that $\mathcal{O}_Z(H)$ is the pullback of $\mathcal{O}_W(1)$. Moreover, $S_1 \subset H$ is disjoint from the negative section $D \cap H$ (see Step 2.8) and has degree 2 over \mathbb{P}^2 ; this implies that $S_1 \in |\mathcal{O}_Z(2H)|_{|H|}$ in H. Since Z is Fano and $\mathcal{O}_Z(H)$ is nef, we have $h^1(Z, H) = h^1(Z, K_Z - K_Z + H) = 0$ by Kodaira vanishing, therefore the restriction $H^0(Z, \mathcal{O}_Z(2H)) \to H^0(H, \mathcal{O}_Z(2H)|_{|H|})$ is surjective. We conclude that S_1 is a complete intersection of elements in the linear systems |H| and |2H|, and this is Example 1.2.

This concludes the proof of Th. 1.3.

Remark 2.16. A posteriori, we see that the varieties X_1 , X_2 , Z and Y are always toric; moreover it is not difficult to see that there is always a choice of the ordering of the three blow-ups $X \to Z$ such that X_1 is toric and Fano.

More precisely, in the two non-toric cases one can choose an ordering of the blow-ups in such a way that, following the notation of [Bat99] for toric Fano 4-folds:

- in Example 1.1, X_1 is H_4 and X_2 is D_8 ;

- in Example 1.2, X_1 is H_1 and X_2 is D_2 .

Remark 2.17 (conic bundles of Fano manifolds). The varieties X of Examples 1.1 and 1.2 give new examples of Fano varieties with a conic bundle $X \to Y$ such that $\rho_X - \rho_Y = 3$. It is shown in [Rom19, Th. 1.1] that if X is a Fano manifold (of arbitrary dimension) which is not a product of varieties of smaller dimension, and has a conic bundle $X \to Y$, then $\rho_X - \rho_Y \leq 3$.

Given a conic bundle $h: X \to Y$, let $\triangle := \{y \in Y | h^{-1}(y) \text{ is singular}\}$ be its *discriminant divisor*. As a consequence of our main result, we find an explicit description of the discriminant divisors of conic bundles encoded by the varieties of Th. 1.3. See also [MR19, Cor. 3.4] for some partial results in this direction.

Corollary 2.18. Let X be a Fano 4-fold with $\rho_X = 5$, admitting a conic bundle $h: X \to Y$ such that $\rho_X - \rho_Y = 3$. Denote by \triangle the discriminant divisor of h. Then one of the following holds:

(i) $\triangle \cong \mathbb{P}^2 \sqcup \mathbb{P}^2$, and X is toric of combinatorial type K;

(ii) $\triangle \cong (\mathbb{P}^1 \times \mathbb{P}^1) \sqcup \mathbb{P}^2$, and X is the variety of Example 1.1;

(iii) $\triangle \cong S \sqcup \mathbb{P}^2$ where S is a del Pezzo surface of degree 2, and X is the variety of Example 1.2.

Proof. We first show that $\delta_X = 3$. In view of [MR19, Th. 1.1] we are left to analyze the case in which $X \cong S_1 \times S_2$ with S_i del Pezzo surfaces. In this situation, h is induced by a conic bundle on one of the two del Pezzo surfaces S_i , say S_1 , and $Y \cong \mathbb{P}^1 \times S_2$ (see e.g. the proof of [Rom19, Th. 4.2(1)]). Being $\rho_Y = \rho_X - 3 = 2$, we conclude that $S_2 \cong \mathbb{P}^2$, $\rho_{S_1} = 4$, and $X \cong S_1 \times \mathbb{P}^2 = K_4$, thus $\delta_X = 3$.

We observe that $h: X \to Y$ satisfies all the properties (a)-(f) listed in Th. 2.1. Indeed, by [Rom19, Prop. 3.5(1) and Prop. 4.2(2)] we may take a factorization for h such that (b), (c), and (f) hold. All the remaining properties follow by arguing as in the proof of Th. 2.1. This implies that we can run the arguments of the proof of Th. 1.3 replacing $\psi \circ f$ by h. Let us keep the notation as in that theorem.

Then $\triangle = A_1 \sqcup A_2$, and using Step 2.3 we have $A_i \cong B_i \cong S_i$ for i = 1, 2. At this point, Steps 2.6, 2.14, 2.15 and their proofs give respectively (i), (ii), and (iii), hence the statement.

Proof of Cor. 1.4. If $\delta_X \leq 3$, then $\rho_X \leq \delta_X + \dim \mathcal{N}_1(D, X) \leq 5$, and the statement follows from Th. 1.3.

If instead $\delta_X \geq 4$, by [Cas12, Th. 1.1] we have $X \cong S \times T$ where S and T are del Pezzo surfaces. We can assume that $\pi(D) = T$, where $\pi: X \to T$ is the projection. Let us consider the pushforward $\pi_*: \mathcal{N}_1(X) \to \mathcal{N}_1(T)$. Then $\pi_*(\mathcal{N}_1(D,X)) = \mathcal{N}_1(T)$; on the other hand π cannot be finite on D, so that $\mathcal{N}_1(D,X) \cap \ker \pi_* \neq 0$, and dim $\mathcal{N}_1(T) < \dim \mathcal{N}_1(D,X) = 2$. We conclude that $\rho_T = 1$, hence $T \cong \mathbb{P}^2$, and being $\delta_X = \rho_S - 1$ (cf. [Cas12, Ex. 3.1]) we get $\rho_S \geq 5$ and $\rho_X \geq 6$.

Proof of Cor. 1.6. Let us take the push-forward $\sigma_* \colon \mathcal{N}_1(X) \to \mathcal{N}_1(X')$. We have $\sigma_*(\mathcal{N}_1(\operatorname{Exc}(\sigma), X)) = \mathbb{R}[\sigma(\operatorname{Exc}(\sigma))]$, and dim ker $\sigma_* = 1$, thus dim $\mathcal{N}_1(\operatorname{Exc}(\sigma), X) = 2$. By Cor. 1.4, X is one of the 4-folds appearing in Th. 1.3, so we only have to check which of these varieties admit a contraction as in the statement. We already know by Th. 1.5 that the toric 4-fold K_3 and Example 1.1 do.

It is easy to check from [Bat99], using the primitive relations of the varieties K_i , that K_2 and $K_4 \cong \mathbb{P}^2 \times S_4$ do not have any elementary divisorial contraction such that the image of the exceptional divisor is a curve, while K_1 does, and satisfies the statement.

Concerning Example 1.2, keeping the same notation as in the example, we have $D \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathcal{N}_{D/Z} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, -2)$. If $\tilde{D} \subset X$ is the transform of D, it is not difficult to check that $\tilde{D} \cong \mathbb{P}^1 \times \mathbb{P}^2$ and $\mathcal{N}_{\tilde{D}/X} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, -2)$. Then $-2K_X + \tilde{D}$ is nef, and has intersection zero only with the curves in $\{pt\} \times \mathbb{P}^2 \subset \tilde{D}$, so the classes of these curves belong to an extremal ray of NE(X) which gives the desired contraction.

3. The examples

3.1. Example 1.2. The 4-fold $Z := \mathbb{P}_{\mathbb{P}^2}(\mathcal{O}^{\oplus 2} \oplus \mathcal{O}(2))$ has two contractions, the \mathbb{P}^2 -bundle $\varphi \colon Z \to \mathbb{P}^2$, and a divisorial contraction $Z \to W := \mathbb{P}(1, 1, 1, 2, 2)$. We note that $\operatorname{Pic}(W)$ is generated by a very ample line bundle $\mathcal{O}_W(1)$ which embeds W in \mathbb{P}^7 as the cone over the Veronese surface, with vertex a line; the birational morphism $Z \to W$ sends the exceptional divisor D to this line.

Let $H \subset Z$ be the pullback of a general element in $|\mathcal{O}_W(1)|$; then H is a resolution of a cone over the Veronese surface (with vertex a point), and $H \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$. We have:

$$H \sim D + \varphi^* \mathcal{O}_{\mathbb{P}^2}(2)$$
 and $-K_Z = 3H + \varphi^* \mathcal{O}_{\mathbb{P}^2}(1).$

Let $F_1, F_2 \subset D$ be two distinct fibers of the contraction $Z \to W$, so that $F_i \cong \mathbb{P}^2$ are sections of $\varphi \colon Z \to \mathbb{P}^2$. In D we have $F_i \in |H_{|D}|$.

Let $S \subset Z$ be a general complete intersection of elements in the linear systems |H| and |2H|; notice that $\varphi_{|S} \colon S \to \mathbb{P}^2$ is finite of degree 2. By adjunction $-K_S = (-K_Z - 3H)_{|S} = \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)_{|S}$, and we deduce that S is a del Pezzo surface of degree 2.

The surfaces S, F_1, F_2 are pairwise disjoint in Z. Let $\sigma: X \to Z$ be the blow-up of Z along S, F_1 , and F_2 .

Lemma 3.1. $-K_X$ has positive intersection with every curve in X.

Proof. Let $D \subset X$ be the transform of D, E_i the exceptional divisor over F_i , and E_0 the exceptional divisor over S. Using that the surface S has degree 2 over \mathbb{P}^2 , it is not difficult to see that there exists a unique $H_0 \in |H|$ containing it; let $\widetilde{H}_0 \subset X$ be its transform. By the generality of S, H_0 is smooth and disjoint from F_1 and F_2 .

Let $\Gamma \subset X$ be an irreducible curve not contained in any of the divisors D, H_0 , E_1 , or E_2 . If $\sigma(\Gamma)$ is a point, then $-K_X \cdot \Gamma = 1$. Otherwise, we set $\Gamma' := \sigma(\Gamma) \subset Z$, so that Γ' is not contained in D, nor in H_0 , nor in F_i .

For i = 1, 2 let $H_i \in |H|$ be a general element containing F_i , so that these divisors do not contain Γ' . Then $(\sigma^* H_i - E_i) \cdot \Gamma \geq 0$ for i = 0, 1, 2, and we get

$$-K_X \cdot \Gamma = \left(\sigma^*(-K_Z) - \sum_{i=0}^2 E_i\right) \cdot \Gamma = \left(\sigma^*\left(3H + \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)\right) - \sum_{i=0}^2 E_i\right) \cdot \Gamma$$
$$= \sum_{i=0}^2 (\sigma^*H - E_i) \cdot \Gamma + \sigma^*\varphi^*\mathcal{O}_{\mathbb{P}^2}(1) \cdot \Gamma \ge 0.$$

We show that the intersection is in fact positive. If $-K_X \cdot \Gamma = 0$, then $\sigma^* \varphi^* \mathcal{O}_{\mathbb{P}^2}(1) \cdot \Gamma = 0$, so that Γ' is contained in a fiber Z_p of the \mathbb{P}^2 -bundle $\varphi: Z \to \mathbb{P}^2$. Let $i \in \{1, 2\}$. We also have $(\sigma^* H_i - E_i) \cdot \Gamma = 0$, so that the transform of H_i in X is disjoint from Γ . Note that $(H_i)_{|Z_p}$ is a line in $Z_p \cong \mathbb{P}^2$; this means that H_i meets Γ' only at the point $z_i := F_i \cap Z_p$, transversally. Thus Γ'

must be the line $\overline{z_1 z_2}$ in Z_p . However this line is contained in the exceptional divisor D, indeed $D \cap Z_p$ is a line and contains both points z_i , against our assumptions on Γ .

Now we show that the restriction of $-K_X$ to the divisors \widetilde{D} , \widetilde{H}_0 , E_1 , E_2 is ample.

We have $\widetilde{D} \cong \mathbb{P}^1 \times \mathbb{P}^2$ with $\mathcal{N}_{\widetilde{D}/X} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(-1, -2)$, so that $(-K_X)_{|\widetilde{D}|} = -K_{\widetilde{D}} + \mathcal{N}_{\widetilde{D}/X} \cong \mathcal{O}_{\mathbb{P}^1 \times \mathbb{P}^2}(1, 1)$ is ample.

Similarly, for $i = 1, 2, E_i \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$ and $\mathcal{N}_{E_i/X} \cong -A - 2B$, where A is the negative section of $E_i \to \mathbb{P}^2$ and B is the pullback of $\mathcal{O}_{\mathbb{P}^2}(1)$. Moreover $-K_{E_i} = 2A + 5B$, so that $(-K_X)_{|E_i} = -K_{E_i} + \mathcal{N}_{E_i/X} \cong A + 3B$ is ample.

Finally we have $\widetilde{H}_0 \cong H_0 \cong \mathbb{P}_{\mathbb{P}^2}(\mathcal{O} \oplus \mathcal{O}(2))$, and the negative section is $\widetilde{H}_0 \cap \widetilde{D}$; we know that $(-K_X)_{|\widetilde{H}_0 \cap \widetilde{D}}$ is ample.

If $\ell \subset \widetilde{H}_0$ is a fiber of the \mathbb{P}^1 -bundle, then $\ell \cdot E_i = 0$ for $i = 1, 2, \ell \cdot E_0 = 2$, and $-K_Z \cdot \sigma(\ell) = 3$, thus $-K_X \cdot \ell = 1$.

Note that $NE(\tilde{H}_0)$ is generated by $[\ell]$ and by the class of a curve in $\tilde{H}_0 \cap \tilde{D}$; we conclude that $(-K_X)_{|\tilde{H}_0}$ is ample.

The following is a standard computation.

Lemma 3.2. Let $f: X \to Y$ be the blow-up of a smooth projective 4-fold along a smooth irreducible surface S. Then we have the following:

$$K_X^4 = K_Y^4 - 3(K_{Y|S})^2 - 2K_S \cdot K_{Y|S} + c_2(\mathcal{N}_{S/Y}) - K_S^2,$$

$$K_X^2 \cdot c_2(X) = K_Y^2 \cdot c_2(Y) - 12\chi(\mathcal{O}_S) + 2K_S^2 - 2K_S \cdot K_{Y|S} - 2c_2(\mathcal{N}_{S/Y}),$$

$$\chi(X, -K_X) = \chi(Y, -K_Y) - \chi(\mathcal{O}_S) - \frac{1}{2} ((K_{Y|S})^2 + K_S \cdot K_{Y|S}).$$

Proof. Let $E \subset X$ be the exceptional divisor, set $\pi := f_{|E} \colon E \to S$, and let $F \subset E$ be a fiber of π .

We have $(f^*K_Y)^4 = K_Y^4$, and for $i \in \{0, 1, 2, 3\}$ one has $(f^*K_Y)^i \cdot E^{4-i} = (\pi^*K_{Y|S})^i \cdot (E_{|E})^{3-i}$. This gives

$$(f^*K_Y)^3 \cdot E = 0$$
 and $(f^*K_Y)^2 \cdot E^2 = -(K_{Y|S})^2$.

In $H^4(E,\mathbb{Z})$ we have $\sum_{i=0}^2 (-1)^i \pi^* c_i(\mathcal{N}_{S/Y}^{\vee})(-E_{|E})^{2-i} = 0$, which yields

$$E_{|E}^{2} = -(\pi^{*}c_{1}(\mathcal{N}_{S/Y}^{\vee})) \cdot E_{|E} - c_{2}(\mathcal{N}_{S/Y}^{\vee})F$$

Recall also that $c_1(\mathcal{N}_{S/Y}^{\vee}) = K_{Y|S} - K_S$. Using these formulas, we get

$$f^* K_Y \cdot E^3 = (K_{Y|S})^2 - K_S \cdot K_{Y|S},$$

$$E^4 = c_2(\mathcal{N}_{S/Y}) + 2K_S \cdot K_{Y|S} - (K_{Y|S})^2 - K_S^2$$

Finally we have $K_X^4 = (f^*K_Y + E)^4 = \sum_{i=0}^4 {4 \choose i} (f^*K_Y)^i \cdot E^{4-i}$, which yields the formula for K_X^4 .

By [Ful98, Ex. 15.4.3] we have $c_2(X) = f^*c_2(Y) + j_*(\pi^*(K_S) - E_{|E})$ in $H^4(X, \mathbb{Z})$, where $j: E \hookrightarrow X$ is the inclusion and $j_*: H^2(E, \mathbb{Z}) \to H^4(X, \mathbb{Z})$ is the Gysin homomorphism. Using that $j_*\alpha \cdot \beta = \alpha \cdot \beta_{|E}$ for every $\alpha \in H^2(E, \mathbb{Z})$ and $\beta \in$ $H^4(X, \mathbb{Z})$, a computation similar to the previous one gives the formula for $K_X^2 \cdot c_2(X)$. Finally, $\chi(X, -K_X)$ is given by the Riemann-Roch formula, which in this setting is as follows:

$$\chi(X, -K_X) = \frac{1}{12} \left(2K_X^2 + K_X^2 \cdot c_2(X) \right) + \chi(X, \mathcal{O}_X).$$

Lemma 3.3. The 4-fold X is Fano with $\rho_X = 5$ and $\delta_X = 3$. We also have:

$$K_X^4 = 250, \quad K_X^2 \cdot c_2(X) = 172, \quad h^0(X, -K_X) = 57,$$

 $b_3(X) = 0, \quad b_4(X) = h^{2,2}(X) = 13.$

Moreover $h^1(X, T_X) = h^0(X, T_X) + 6 \ge 6$, where T_X is the tangent bundle.

Proof. We have $K_Z^4 = 594$, $K_Z^2 \cdot c_2(Z) = 240$, and $\chi(Z, -K_Z) = 120$; this can be computed using toric geometry, or see [Bat99, Table 4, n. 7].

We compute K_X^4 using Lemma 3.2.

We calculate the contribution of $F_i \cong \mathbb{P}^2$. We have $K_{Z|F_i} \cong \mathcal{O}_{\mathbb{P}^2}(-1)$ and $\mathcal{N}_{F_i/Z} \cong \mathcal{O}_{\mathbb{P}^2}(-2) \oplus \mathcal{O}_{\mathbb{P}^2}$; this gives that the blow-up along each surface F_i makes K^4 decrease by 18.

Now we compute the contribution of S. In Z we have $H^4 = 4$, as the degree of $W \subset \mathbb{P}^7$ is equal to the degree of the Veronese surface. Moreover $H^3 \cdot \varphi^* \mathcal{O}_{\mathbb{P}^2}(1) = 2$, because if $\ell \subset \mathbb{P}^2$ is a line, the image of $\varphi^{-1}(\ell)$ under the birational map $Z \to W$ is a quadric cone. Finally $H^2 \cdot \varphi^* \mathcal{O}_{\mathbb{P}^2}(1)^2 = 1$, because $H_{|Z_p} \cong \mathcal{O}_{\mathbb{P}^2}(1)$ for every fiber Z_p of φ . We get:

$$(K_{Z|S})^{2} = K_{Z}^{2} \cdot 2H^{2} = (3H + \varphi^{*}\mathcal{O}_{\mathbb{P}^{2}}(1))^{2} \cdot 2H^{2} = 98,$$

$$K_{Z|S} \cdot K_{S} = (3H + \varphi^{*}\mathcal{O}_{\mathbb{P}^{2}}(1)) \cdot \varphi^{*}\mathcal{O}_{\mathbb{P}^{2}}(1) \cdot 2H^{2} = 14.$$

Moreover $\mathcal{N}_{S/Z} = \mathcal{O}_S(H) \oplus \mathcal{O}_S(2H)$, so that $c_2(\mathcal{N}_{S/Z}) = H_{|S} \cdot 2H_{|S} = 2H^2 \cdot 2H^2 = 16$. In the end the blow-up along S makes K^4 decrease by 308, and we get:

$$K_X^4 = K_Z^4 - 36 - 308 = 250.$$

Together with Lemma 3.1, this shows that $-K_X$ is nef and big, hence semiample by the base-point-free theorem, and hence ample. Therefore X is a Fano 4-fold, and $\rho_X = 5$.

For i = 1, 2 the divisor E_i has $\rho_{E_i} = 2$, so that $\delta_X \ge 5 - 2 = 3$. On the other hand X is not a product, so [Cas12, Th. 1.1] implies that $\delta_X \le 3$, and we conclude that $\delta_X = 3$.

The values of $K_X^2 \cdot c_2(X)$ and $h^0(X, -K_X) = \chi(X, -K_X)$ can be computed from the ones of Z using Lemma 3.2, as done for K_X^4 . Also the Hodge numbers of X can be easily computed by the explicit description of X as a blow-up. Finally, since X is Fano, by Nakano vanishing we have $h^i(X, T_X) = 0$ for every $i \ge 2$, so that $\chi(X, T_X) = h^0(X, T_X) - h^1(X, T_X)$. We can compute $\chi(X, T_X)$ from the other invariants using Riemann Roch, see for instance [CCF19, Lemma 6.25] for an explicit formula.

3.2. Numerical invariants. Table 3.4 gives some relevant invariants of the varieties listed in Th. 1.3; T is the tangent bundle, and S_4 is the del Pezzo surface with $\rho_{S_4} = 4$, namely the blow-up of \mathbb{P}^2 in three non collinear points. The invariants for the toric cases are given in [Bat99], and the ones for Example 1.2 are given in Lemma 3.3. For Example 1.1, the invariants are computed as for Example 1.2, using the same technique as in the proof of Lemma 3.3.

4-fold	b_3	$h^{2,2}$	$h^{1,3}$	K^4	$K^2 \cdot c_2$	$h^0(-K)$	$\chi(T)$	
K_1	0	6	0	364	196	78	10	toric
K_2	0	6	0	354	192	76	10	toric
K_3	0	6	0	334	184	72	10	toric
$K_4 \cong \mathbb{P}^2 \times S_4$	0	6	0	324	180	70	10	toric
Ex. 1.1	0	7	0	253	166	57	3	non toric
Ex. 1.2	0	13	0	250	172	57	-6	non toric

TABLE 3.4. Fano 4-folds with $\rho = 5$ and $\delta = 3$

References

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