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# CLASSIFICATION OF HYPERSURFACES WITH CONSTANT MÖBIUS RICCI CURVATURE IN $\mathbb{R}^{n+1}$

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**Abstract.** Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersed umbilic-free hypersurface in an $(n+1)$-dimensional Euclidean space $\mathbb{R}^{n+1}$ with standard metric $I = df \cdot df$. Let $II$ be the second fundamental form of the hypersurface $f$. One can define the Möbius metric $g = \frac{n}{n-1}(\|II\|^2 - n|\mathrm{tr} II|^2)I$ on $f$ which is invariant under the conformal transformations (or the Möbius transformations) of $\mathbb{R}^{n+1}$. The sectional curvature, Ricci curvature with respect to the Möbius metric $g$ is called Möbius sectional curvature, Möbius Ricci curvature, respectively. The purpose of this paper is to classify hypersurfaces with constant Möbius Ricci curvature.

**1. Introduction.** Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface without umbilics. Given the induced metric $I = df \cdot df$ as well as a local orthonormal basis $\{e_i\}$ and the dual basis $\{\theta_i\}$, we denote $II = \sum_{ij} h_{ij}\theta_i \otimes \theta_j$ the second fundamental form and $H = \frac{1}{n}\sum_i h_{ii}$ the mean curvature. The so-called Möbius metric

$$g = \rho^2 df \cdot df = \frac{n}{n-1}(\|II\|^2 - nH^2)df \cdot df$$

is an invariant under the conformal (or Möbius) transformations of $\mathbb{R}^{n+1}$ [11]. Together with the Möbius second fundamental form (for definition, see Section 2) they form a complete system of invariants for hypersurfaces (dim $\geq 3$) in Möbius geometry [11]. Note that the conformal compactification space $\mathbb{S}^{n+1}$ unifies the space forms $\mathbb{S}^{n+1}, \mathbb{R}^{n+1}, \mathbb{H}^{n+1}$ by the Möbius diffeomorphism $\sigma : \mathbb{R}^{n+1} \to \mathbb{S}^{n+1}\backslash\{(-1, 0, \ldots, 0)\}$, $\tau : \mathbb{H}^{n+1} \to \mathbb{S}^{n+1}_+ \subset \mathbb{S}^{n+1}$ defined by

$$\sigma(u) = \left(\frac{1-|u|^2}{1+|u|^2}, \frac{2u}{1+|u|^2}\right),$$
$$\tau(y) = \left(\frac{1}{y_0}, \frac{\bar{y}}{y_0}\right), \quad y = (y_0, y_1, \ldots, y_{n+1}) := (y_0, \bar{y}) \in H^{n+1},$$

where $\mathbb{S}^{n+1}_+ = \{(x_1, \ldots, x_{n+2}) \in \mathbb{S}^{n+1} | x_1 > 0\} \subset \mathbb{S}^{n+1}$ is the upper hemisphere. And the formula above defining the Möbius metric $g$ is the same for any of them.

Recent years the study of hypersurfaces (and various submanifolds) based on these Möbius invariants becomes quite active (see [1, 2, 4, 5]). A notable class of hypersurfaces are those with constant Möbius curvature, i.e., constant sectional curvature with respect to the

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Möbius metric $g$. In a point of view of Möbius geometry, one of the basic questions in the differential geometry of hypersurfaces is to classify these hypersurfaces in $\mathbb{R}^{n+1}$. In [3], the authors have classified them up to Möbius transformations when dimension of hypersurfaces $n \geq 4$.

THEOREM 1.1 ([3]). *Let $f : M^n \to \mathbb{R}^{n+1}$ $(n \geq 4)$ be a hypersurface with constant Möbius curvature. Then locally $f$ is Möbius equivalent to one of the following examples:*

$(i)$ *$S^1 \times \mathbb{R}^{n-1}$;*

$(ii)$ *a cylinder over a logarithmic spiral in a Euclidean 2-plane $\mathbb{R}^2$;*

$(iii)$ *a cone over a curvature-spiral in a 2-sphere $\mathbb{S}^2$;*

$(iv)$ *a rotation hypersurface over a curvature-spiral in a hyperbolic 2-plane $\mathbb{R}^2_+$ (upper half-space model).*

For the definition of the so-called *curvature-spiral* in Theorem 1.1, see [3]. The hypothesis of constant Möbius curvature implies that the hypersurface is conformally flat. A classical result says that when the dimension $n \geq 4$ this happens if and only if a principle curvature has multiplicity at least $n-1$ everywhere. On the other hand, a 3-dimensional hypersurface $f : M^3 \to \mathbb{R}^4$ with constant Möbius sectional curvature may have three distinct principal curvatures. In [6], the authors have classified three dimensional hypersurfaces with constant Möbius curvature and three distinct principal curvatures.

THEOREM 1.2 ([6]). *Let $f : M^3 \to \mathbb{R}^4$ be an immersed hypersurface with three distinct principal curvatures. If $f$ is of constant Möbius curvature $c$, Then $f$ is Möbius equivalent to a cone over a flat torus $x : \mathbb{S}^1(r) \times \mathbb{S}^1(\sqrt{1-r^2}) \to \mathbb{S}^3$, which Möbius curvature $c = 0$.*

Here we need to point out that Theorem 1.1 is valid for three dimensional hypersurfaces so long as the hypersurfaces has only two distinct principal curvatures. Combining Theorem 1.1, the hypersurfaces (dim $\geq 3$) with constant Möbius curvature were classified completely. For surfaces of constant Möbius curvature there are already many results, see [8, 9, 12].

Another notable class of hypersurfaces are those with constant Möbius Ricci curvature, i.e., constant Ricci curvature with respect to $g$. Clearly the hypersurfaces with constant Möbius sectional curvature are of constant Möbius Ricci curvature, but the converse may not true when the dimension of the hypersurfaces $n \geq 4$. In this paper, our purpose is to classify these hypersurfaces of dimension $n \geq 4$. We note that some of such examples come from cones, cylinders, or rotational hypersurfaces over $(\lambda, n, \varepsilon)$-surfaces in 3-sphere $\mathbb{S}^3$, Euclidean space $\mathbb{R}^3$ and hyperbolic space $\mathbb{R}^3_+$ (upper half-space model), respectively.

DEFINITION 1.3. Let $u : M^2 \to N^3(-\varepsilon)$ be an umbilic free surface in 3-dimensional space form $N^3(-\varepsilon)$, and $H_u$, $K_u$ the mean curvature, Gauss curvature, respectively.
For positive integer $n \geq 4$, let

$$\mu = \frac{1}{\sqrt{4H_u^2 - \frac{2n}{n-1}(K_u + \varepsilon)}}, \quad \nu = \mu\varepsilon - \frac{\mu(\varepsilon + K_u)}{n-2}.$$

A surface $u$ is called a $(\lambda, n, \varepsilon)$-surface for some $\lambda$=constant, if

$$Hess(\mu)(e_i, e_j) = \nu I_u(e_i, e_j), \quad e_i, e_j \in TM^2,$$
$$|\nabla\mu|^2 = \mu^2\varepsilon - \frac{\mu^2(K_u + \varepsilon)}{(n-1)(n-2)} - \frac{\lambda}{n-1}.$$

Here $Hess$ and $\nabla$ are the Hessian operator and the gradient with respect to the induced metric $I_u$.

Our main results is given as follows:

THEOREM 1.4. *Let $f : M^n \to \mathbb{R}^{n+1}(n \geq 3)$ be an immersed hypersurface in $\mathbb{R}^{n+1}$ without umbilics. If $f$ is of constant Möbius Ricci curvature $\lambda$, Then locally $f$ is Möbius equivalent to one of the following examples:*

(1) *$f$ is of constant Möbius curvature;*
(2) *the image of $\sigma^{-1}$ of the torus $\mathbb{S}^k(\sqrt{\frac{k-1}{n-2}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k-1}{n-2}}) \subset \mathbb{S}^{n+1}$, $1 < k < n-1$;*
(3) *a cylinder over a $(\lambda, n, 0)$-surface in a Euclidean 3-plane $\mathbb{R}^3$, $(n \geq 4)$;*
(4) *a cone over a $(\lambda, n, 1)$-surface in a 3-sphere $\mathbb{S}^3$, $(n \geq 4)$;*
(5) *a rotation hypersurface over a $(\lambda, n, -1)$-surface in a hyperbolic 3-plane $\mathbb{R}^3_+$, $(n \geq 4)$.*

For the purpose of making the procedure of the proof of our main Theorem clear, We organize the paper as follows. In Section 2, we review the basic invariants and equations in Möbius geometry for hypersurfaces in $\mathbb{R}^{n+1}$. In Section 3 we give some special examples of hypersurfaces and compute the Ricci curvature. In Section 4, we prove our main Theorem 1.4. In Section 5, we prove a special case of Theorem 1.4, since the proof of the case is very long.

**2. Möbius invariants for hypersurfaces in $\mathbb{R}^{n+1}$.** In this section we briefly review the theory of hypersurfaces in Möbius geometry. For details we refer to [11], [7].

Let $\mathbb{R}^{n+3}_1$ be the Lorentz space, i.e., $\mathbb{R}^{n+3}$ with inner product $\langle\cdot,\cdot\rangle$ defined by

$$\langle x, y\rangle = -x_0y_0 + x_1y_1 + \cdots + x_{n+2}y_{n+2},$$

for $x = (x_0, x_1, \ldots, x_{n+2})$, $y = (y_0, y_1, \ldots, y_{n+2}) \in \mathbb{R}^{n+3}$.

Let $f : M^n \to \mathbb{R}^{n+1}$ be an immersed hypersurface without umbilics and assume that $\{e_i\}$ is an orthonormal basis with respect to the induced metric $I = df \cdot df$ with $\{\theta_i\}$ the dual basis. Let $II = \sum_{ij} h_{ij}\theta_i\theta_j$ and $H = \sum_i \frac{h_{ii}}{n}$ be the second fundamental form and the mean curvature of $f$, respectively. We define the Möbius position vector $Y : M^n \to \mathbb{R}^{n+3}_1$ of $f$ by

$$Y = \rho\left(\frac{1+|f|^2}{2}, \frac{1-|f|^2}{2}, f\right), \quad \rho^2 = \frac{n}{n-1}(|II|^2 - nH^2).$$

THEOREM 2.1 ([11]). *Two hypersurfaces $f, \bar{f} : M^n \to \mathbb{R}^{n+1}$ are Möbius equivalent if and only if there exists $T$ in the Lorentz group $O(n+2, 1)$ in $\mathbb{R}^{n+3}_1$ such that $\bar{Y} = YT$.*

It follows immediately from Theorem 2.1 that

$$g = \langle dY, dY\rangle = \rho^2 df \cdot df$$

is a Möbius invariant, called the Möbius metric of $f$.

Let $\Delta$ be the Laplacian with respect to $g$. Define

$$N = -\frac{1}{n}\Delta Y - \frac{1}{2n^2}\langle \Delta Y, \Delta Y\rangle Y ,$$

which satisfies $\langle Y, Y\rangle = 0 = \langle N, N\rangle,\ \langle N, Y\rangle = 1$ .

Let $\xi$ be the mean curvature sphere of $f$ written as

$$\xi = \left(\frac{1+|f|^2}{2}H + f\cdot e_{n+1}, \frac{1-|f|^2}{2}H - f\cdot e_{n+1}, Hf + e_{n+1}\right) ,$$

where $e_{n+1}$ is the unit normal vector field of $f$ in $\mathbb{R}^{n+1}$.

Let $\{E_1, \dots, E_n\}$ be a local orthonormal basis for $(M^n, g)$ with dual basis $\{\omega_1, \dots, \omega_n\}$. Write $Y_i = E_i(Y)$. Then $\{Y, N, Y_1, \dots, Y_n, \xi\}$ forms a moving frame in $\mathbb{R}^{n+3}_1$ along $M^n$. We will use the following range of indices in this section: $1 \le i, j, k \le n$. We can write the structure equations as following:

$$\begin{aligned}
dY &= \sum_i Y_i\omega_i ,\\
dN &= \sum_{ij} A_{ij}\omega_i Y_j + \sum_i C_i\omega_i\xi ,\\
dY_i &= -\sum_j A_{ij}\omega_j Y - \omega_i N + \sum_j \omega_{ij}Y_j + \sum_j B_{ij}\omega_j\xi ,\\
d\xi &= -\sum_i C_i\omega_i Y - \sum_{ij}\omega_i B_{ij} Y_j ,
\end{aligned}$$

where $\omega_{ij}$ is the connection form of the Möbius metric $g$ and $\omega_{ij} + \omega_{ji} = 0$. The tensors

$$\mathbf{A} = \sum_{ij} A_{ij}\omega_i\otimes\omega_j , \quad \mathbf{B} = \sum_{ij} B_{ij}\omega_i\otimes\omega_j , \quad \mathbf{C} = \sum_i C_i\omega_i$$

are called the Blaschke tensor, the Möbius second fundamental form and the Möbius form of $f$, respectively. The covariant derivative of $C_i$, $A_{ij}$, $B_{ij}$ are defined by

$$\begin{aligned}
\sum_j C_{i,j}\omega_j &= dC_i + \sum_j C_j\omega_{ji} ,\\
\sum_k A_{ij,k}\omega_k &= dA_{ij} + \sum_k A_{ik}\omega_{kj} + \sum_k A_{kj}\omega_{ki} ,\\
\sum_k B_{ij,k}\omega_k &= dB_{ij} + \sum_k B_{ik}\omega_{kj} + \sum_k B_{kj}\omega_{ki} .
\end{aligned}$$

The integrability conditions for the structure equations are given by

$$A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k , \tag{2.1}$$

$$C_{i,j} - C_{j,i} = \sum_k (B_{ik}A_{kj} - B_{jk}A_{ki}), \tag{2.2}$$

$$B_{ij,k} - B_{ik,j} = \delta_{ij}C_k - \delta_{ik}C_j, \tag{2.3}$$

$$R_{ijkl} = B_{ik}B_{jl} - B_{il}B_{jk} + \delta_{ik}A_{jl} + \delta_{jl}A_{ik} - \delta_{il}A_{jk} - \delta_{jk}A_{il}, \tag{2.4}$$

$$R_{ij} := \sum_k R_{ikjk} = -\sum_k B_{ik}B_{kj} + (tr\mathbf{A})\delta_{ij} + (n-2)A_{ij}, \tag{2.5}$$

$$\sum_i B_{ii} = 0, \quad \sum_{ij} (B_{ij})^2 = \frac{n-1}{n}, \quad tr\mathbf{A} = \sum_i A_{ii} = \frac{1}{2n}(1 + n^2\kappa), \tag{2.6}$$

where $R_{ijkl}$ denote the curvature tensor of $g$, $\kappa = \frac{1}{n(n-1)}\sum_{ij} R_{ijij}$ is its normalized Möbius scalar curvature. We know that all coefficients in the structure equations are determined by $\{g, \mathbf{B}\}$ and we have

THEOREM 2.2 ([11]). *Two hypersurfaces $f : M^n \to \mathbb{R}^{n+1}$ and $\bar{f} : M^n \to \mathbb{R}^{n+1}$ $(n \geq 3)$ are Möbius equivalent if and only if there exists a diffeomorphism $\varphi : M^n \to M^n$ which preserves the Möbius metric and the Möbius second fundamental form.*

The second covariant derivative of $B_{ij}$ are defined by

$$dB_{ij,k} + \sum_m B_{mj,k}\omega_{mi} + \sum_m B_{im,k}\omega_{mj} + \sum_m B_{ij,m}\omega_{mk} = \sum_m B_{ij,km}\omega_m .$$

We have the following Ricci identities

$$B_{ij,kl} - B_{ij,lk} = \sum_m B_{mj}R_{mikl} + \sum_m B_{im}R_{mjkl} .$$

Coefficients of Möbius invariants and Euclidean invariants are related by [7]

$$\begin{aligned} B_{ij} =& \rho^{-1}(h_{ij} - H\delta_{ij}), \\ C_i =& -\rho^{-2}[e_i(H) + \sum_j (h_{ij} - H\delta_{ij})e_j(\log\rho)], \\ A_{ij} =& -\rho^{-2}[Hess_{ij}(\log\rho) - e_i(\log\rho)e_j(\log\rho) - Hh_{ij}] \\ & -\frac{1}{2}\rho^{-2}[|\nabla(\log\rho)|^2 + H^2]\delta_{ij}, \end{aligned} \tag{2.7}$$

where $Hess_{ij}$ and $\nabla$ are the Hessian matrix and the gradient with respect to $I = df \cdot df$. Then

$$\mathbf{A} = \rho^2 \sum_{ij} A_{ij}\theta_i \otimes \theta_j, \quad \mathbf{B} = \rho^2 \sum_{ij} B_{ij}\theta_i \otimes \theta_j, \quad \mathbf{C} = \rho \sum_i C_i\theta_i .$$

We call eigenvalues of $(B_{ij})$ as *Möbius principal curvatures* of $f$. Clearly the number of distinct Möbius principal curvatures is the same as that of its distinct Euclidean principal curvatures.

Let $k_1, \dots, k_n$ be the principal curvatures of $f$, and $\{b_1, \dots, b_n\}$ the corresponding Möbius principal curvatures, then the curvature sphere of principal curvature $k_i$ is

$$\xi_i = b_i Y + \xi = \left( \frac{1+|f|^2}{2} k_i + f \cdot e_{n+1}, \frac{1-|f|^2}{2} k_i - f \cdot e_{n+1}, k_i f + e_{n+1} \right) .$$

Note that $k_i = 0$ if, and only if,

$$\langle \xi_i, (1, -1, 0, \dots, 0) \rangle = 0 .$$

This means that the curvature sphere of principal curvature $k_i = 0$ is a hyperplane in $\mathbb{R}^{n+1}$.

**3. Some examples of Hypersurfaces in $\mathbb{R}^{n+1}$ with constant Möbius Ricci curvature.** In this section we give some examples of special hypersurfaces in $\mathbb{R}^{n+1}$, and compute the Möbius Ricci curvature. Some other properties of these hypersurfaces were studied in [4, 5]

EXAMPLE 3.1. Let $x : \mathbb{S}^k(a) \times \mathbb{S}^{n-k}(\sqrt{1-a^2}) \to \mathbb{S}^{n+1}$ be the isoparametric torus defined by

$$x = (a x_1, \sqrt{1-a^2} x_2) , \quad 0 < a < 1 ,$$

where $x_1 : \mathbb{S}^k \to \mathbb{R}^{k+1}$, $x_2 : \mathbb{S}^{n-k} \to \mathbb{R}^{n-k+1}$ are unit spheres. We define the hypersurface in $\mathbb{R}^{n+1}$

$$f = \sigma^{-1} \circ x : \mathbb{S}^k(a) \times \mathbb{S}^{n-k}(\sqrt{1-a^2}) \to \mathbb{R}^{n+1} .$$

PROPOSITION 3.2. *If $f = \sigma^{-1} \circ x : \mathbb{S}^k(a) \times \mathbb{S}^{n-k}(\sqrt{1-a^2}) \to \mathbb{R}^{n+1}$ is of constant Möbius Ricci curvature, then*

$$a^2 = \frac{k-1}{n-2} , \quad R_{ij} = \frac{(n-k-1)(n-1)(k-1)}{k(n-k)(n-2)} \delta_{ij} , \quad 1 < k < n-1 .$$

PROOF. Using the relations (2.7), by direct computation, $f$ has two distinct Möbius principle curvatures

$$b_1 = \frac{-1}{n} \sqrt{\frac{(n-1)(n-k)}{k}} , \quad b_2 = \frac{1}{n} \sqrt{\frac{(n-1)k}{n-k}}$$

with multiplicity $k$ and $n-k$. We take a local orthonormal basis $\{E_1, \dots, E_n\}$ such that

$$(B_{ij}) = diag\{b_1, \dots, b_1, b_2, \dots, b_2\} .$$

Under such basis using relations (2.7) we have

$$A_{ij} = \frac{n-1}{2k(n-k)n^2} \{k(2n-k) - n^2 a^2\} \delta_{ij} , \quad 1 \le i, j \le k ,$$
$$A_{ij} = \frac{n-1}{2k(n-k)n^2} \{n^2 a^2 - k^2\} \delta_{ij} , \quad k+1 \le i, j \le n ,$$
$$A_{ij} = 0 , \quad 1 \le i \le k , \quad k+1 \le j \le n .$$

Thus the sectional curvature with respect to $g$ are given by

$$
\begin{aligned}
&R_{ijij} = \frac{n-1}{k(n-k)}(1-a^2)\,, \quad 1 \le i\,, j \le k\,,\\
(3.8)\qquad &R_{ijij} = \frac{n-1}{k(n-k)}a^2\,, \quad k+1 \le i\,, j \le n\,,\\
&R_{ijij} = 0\,, \quad 1 \le i \le k\,, \quad k+1 \le j \le n\,.
\end{aligned}
$$

From (3.8) we finish the proof. □

PROPOSITION 3.3. *Let $f : \mathbb{S}^k(1) \times \mathbb{R}^{n-k} \to R^{n+1}$ be the standard cylinder in $\mathbb{R}^{n+1}$. If $f : \mathbb{S}^k(1) \times \mathbb{R}^{n-k} \to \mathbb{R}^{n+1}$ is of constant Möbius Ricci curvature*, *then $k = 1$*, *and $f$ is of constant Möbius curvature $\lambda = 0$.*

The proof is similar to the proof of Proposition 3.2. For detailed computation of Möbius invariants of the hypersurface $f$ we refer to [5].

EXAMPLE 3.4. Let $x : \mathbb{S}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1+r^2}) \to \mathbb{H}^{n+1}$ be the standard embedding given by

$$x = (\sqrt{1+r^2}v, ru, \sqrt{1+r^2}w) \in R^+ \times \mathbb{R}^{k+1} \times \mathbb{R}^{n-k}\,,$$

where $-v^2 + w \cdot w = -1, u \cdot u = 1$. We define the hypersurface in $\mathbb{R}^{n+1}$:

$$f = \sigma^{-1} \circ \tau \circ x : \mathbb{S}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1+r^2}) \to \mathbb{R}^{n+1}\,.$$

PROPOSITION 3.5. *The hypersurface $f = \sigma^{-1} \circ \tau \circ x : \mathbb{S}^k(r) \times \mathbb{H}^{n-k}(\sqrt{1+r^2}) \to \mathbb{R}^{n+1}$ can not be of constant Möbius Ricci curvature.*

The proof is also similar to the proof of Proposition 3.2. For detailed computation of Möbius invariants of the hypersurface $f$ we refer to [5].

EXAMPLE 3.6 ([4]). Let $f : \mathbb{S}^p(a) \times \mathbb{S}^q(b) \times R^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1}$ be the warped product embedding given by

$$f = (tu_1, tu_2, u_3)\,, \quad u_1 \in \mathbb{S}^p(a)\,, \ u_2 \in \mathbb{S}^q(b)\,, \ t \in R^+\,, \ u_3 \in \mathbb{R}^{n-p-q-1}\,, \ a^2 + b^2 = 1\,.$$

By direct computation (or see [4]), we have the following results:

PROPOSITION 3.7. *If $f : \mathbb{S}^p(a) \times \mathbb{S}^q(b) \times R^+ \times \mathbb{R}^{n-p-q-1} \to \mathbb{R}^{n+1}$ is of constant Möbius Ricci curvature*, *then $p = q = 1, n = 3$*, *and $f$ is of constant Möbius curvature $\lambda = 0$.*

The Möbius second fundamental form of the hypersurfaces given by the standard cylinder in $R^{n+1}$, Example 3.1, Example 3.4, and Example 3.6 are parallel. In fact these hypersurfaces exhaust the hypersurfaces with parallel Möbius second fundamental form **B** (see [4]).

EXAMPLE 3.8. Let $u : M^2 \to \mathbb{R}^3$ be an immersed surface. We define *the cylinder over $u$* in $\mathbb{R}^{n+1}$ as

$$f = (u, id) : M^2 \times \mathbb{R}^{n-2} \to \mathbb{R}^3 \times \mathbb{R}^{n-2} = \mathbb{R}^{n+1}\,, \quad f(x, y) = (u(x), y)\,,$$

where $id : \mathbb{R}^{n-2} \to \mathbb{R}^{n-2}$ is the identity map.

Let $\eta$ be the unit normal vector of the surface $u$. Then $e_{n+1} = (\eta, \vec{0}) \in \mathbb{R}^{n+1}$ is the unit normal vector of the hypersurface $f$. The first fundamental form $I$ and the second fundamental form $II$ of the hypersurface $f$ are given by

$$I = I_u + I_{R^{n-2}}\,, \quad II = II_u\,, \tag{3.9}$$

where $I_u$, $II_u$ are the first and second fundamental forms of $u$, respectively, and $I_{\mathbb{R}^{n-2}}$ denotes the standard metric of $\mathbb{R}^{n-2}$. Let $k_1, k_2$ be principal curvatures of the surface $u$. The principal curvatures of the hypersurface $f$ are obviously

$$k_1, k_2, 0, \dots, 0\,.$$

The Möbius metric $g$ of the hypersurface $f$ is

$$g = \rho^2 I = \frac{n}{n-1}(|II|^2 - nH^2)I = \left(4H_u^2 - \frac{2n}{n-1}K_u\right)(I_u + I_{\mathbb{R}^{n-2}})\,, \tag{3.10}$$

where $H_u$, $K_u$ are the mean curvature and Gauss curvature of $u$, respectively.

EXAMPLE 3.9. Let $u : M^2 \to \mathbb{S}^3 \subset \mathbb{R}^4$ be an immersed surface. We define *the cone over* $u$ in $\mathbb{R}^{n+1}$ as

$$f : M^2 \times R^+ \times \mathbb{R}^{n-3} \to \mathbb{R}^{n+1}\,, \quad f(x, t, y) = (tu(x), y)\,.$$

The first and second fundamental forms of the hypersurface $f$ are, respectively,

$$I = t^2 I_u + I_{\mathbb{R}^{n-2}}\,, \quad II = t\, II_u\,,$$

where $I_u$, $II_u$, $I_{\mathbb{R}^{n-2}}$ are understood as before. Let $k_1, k_2$ be principal curvatures of the surface $u$. The principal curvatures of the hypersurface $f$ are

$$\frac{1}{t}k_1, \frac{1}{t}k_2, 0, \dots, 0\,.$$

Thus the Möbius metric $g$ of the hypersurface $f$ is

$$\begin{aligned} g = \rho^2 I &= \frac{1}{t^2}\left[4H_u^2 - \frac{2n}{n-1}(K_u - 1)\right](t^2 I_u + I_{\mathbb{R}^{n-2}}) \\ &= \left[4H_u^2 - \frac{2n}{n-1}(K_u - 1)\right](I_u + I_{\mathbb{H}^{n-2}})\,, \end{aligned} \tag{3.11}$$

where $H_u$, $K_u$ are the mean curvature and Gauss curvature of $u$, respectively, $I_{\mathbb{H}^{n-2}}$ is the standard hyperbolic of $\mathbb{R}^{n-2}_+ = R^+ \times \mathbb{R}^{n-3}$.

EXAMPLE 3.10. Let $\mathbb{R}^3_+ = \{(x_1, x_2, x_3) \in \mathbb{R}^3 | x_3 > 0\}$ be the upper half-space endowed with the standard hyperbolic metric

$$ds^2 = \frac{1}{x_3^2}[dx_1^2 + dx_2^2 + dx_3^2]\,.$$

Let $u = (x_1, x_2, x_3) : M^2 \to \mathbb{R}^3_+$ be an immersed surface. We define *rotational hypersurface over* $u$ in $\mathbb{R}^{n+1}$ as

$$f : M^2 \times \mathbb{S}^{n-2} \to \mathbb{R}^{n+1}, \quad f(x_1, x_2, x_3, \phi) = (x_1, x_2, x_3\phi),$$

where $\phi : \mathbb{S}^{n-2} \to \mathbb{R}^{n-1}$ is the standard sphere.

Let $\mathbb{R}^4_1$ be the Lorentz space with inner product

$$\langle y, y\rangle = -y_1^2 + y_2^2 + y_3^2 + y_4^2, \quad y = (y_1, y_2, y_3, y_4).$$

Let $\mathbb{H}^3 = \{y \in \mathbb{R}^4_1 | \langle y, y\rangle = -1, y_1 > 0\}$ be the hyperbolic space. Introduce isometry $\tau : \mathbb{R}^3_+ \to \mathbb{H}^3$ as below:

$$\tau(x_1, x_2, x_3) = \left( \frac{1 + x_1^2 + x_2^2 + x_3^2}{2x_3}, \frac{1 - x_1^2 - x_2^2 - x_3^2}{2x_3}, \frac{x_1}{x_3}, \frac{x_2}{x_3} \right).$$

The inverse $\tau^{-1} : \mathbb{H}^3 \to \mathbb{R}^3_+$ is $\tau^{-1}(y_1, y_2, y_3, y_4) = (\frac{y_3}{y_1+y_2}, \frac{y_4}{y_1+y_2}, \frac{1}{y_1+y_2})$.

Let $\eta$ be the unit normal vector of the surface $u$ in $\mathbb{R}^3_+$. Write $\eta = (\eta_1, \eta_2, \eta_3)$. Then the unit normal vector of the hypersurface $f$ in $\mathbb{R}^{n+1}$ is

$$\xi = \frac{1}{x_3}(\eta_1, \eta_2, \eta_3\phi).$$

The first fundamental form and the second fundamental form of $u$ is, respectively,

$$I_u = \frac{1}{x_3^2}(dx_1 \cdot dx_1 + dx_2 \cdot dx_2 + dx_3 \cdot dx_3),$$
$$II_u = -\langle \tau_*(du), \tau_*(d\eta)\rangle = \frac{1}{x_3^2}(dx_1 \cdot d\eta_1 + dx_2 \cdot d\eta_2 + dx_3 \cdot d\eta_3) - \frac{\eta_3}{x_3} I_u.$$

Now we can write out the first and the second fundamental forms of $f$:

$$I = dx \cdot dx = x_3^2(I_u + I_{\mathbb{S}^{n-2}}), \quad II = x_3 II_u - \eta_3 I_u - \eta_3 I_{\mathbb{S}^{n-2}},$$

where $I_{\mathbb{S}^{n-2}}$ is the standard metric of $\mathbb{S}^{n-2}$. Let $k_1, k_2$ be principal curvatures of $u$. Then principal curvatures of the hypersurface $f$ are

$$\frac{k_1}{x_3} - \frac{\eta_3}{x_3^2}, \ \frac{k_2}{x_3} - \frac{\eta_3}{x_3^2}, \ \frac{-\eta_3}{x_3^2}, \dots, \frac{-\eta_3}{x_3^2}.$$

Thus

$$\rho^2 = \frac{n}{n-1}(|II|^2 - nH^2) = \frac{1}{x_3^2}\left[4H_u^2 - \frac{2n}{n-1}(K_u + 1)\right],$$

where $H_u, K_u$ are the mean curvature and Gauss curvature of $u$, respectively. So the Möbius metric of the hypersurface $f$ is

$$g = \rho^2 I = \left[4H_u^2 - \frac{2n}{n-1}(K_u + 1)\right](I_u + I_{\mathbb{S}^{n-2}}). \tag{3.12}$$

From Examples 3.8, 3.9, or 3.10, we have

$$f : M^2 \times N^{n-2}(\varepsilon) \to \mathbb{R}^{n+1},$$

when $f$ is a cylinder over a surface $u(M^2) \subset \mathbb{R}^3$, $\varepsilon = 0$ and $N^{n-2}(\varepsilon) = \mathbb{R}^{n-2}$; a cone over a surface $u(M^2) \subset \mathbb{S}^3$, $\varepsilon = -1$ and $N^{n-2}(\varepsilon) = R^+ \times \mathbb{R}^{n-3} = \mathbb{H}^{n-2}$; and a rotation hypersurface over a surface $u(M^2) \subset \mathbb{R}^3_+$, $\varepsilon = 1$ and $N^{n-2}(\varepsilon) = \mathbb{S}^{n-2}$.

Let the induced metric, Gauss curvature, and mean curvature of the surface $u$, be denoted by $I_u$, $K_u$, and $H_u$, respectively. From (3.10), (3.11) and (3.12), the Möbius metric of the hypersurface $f$ is

$$g = \left[4H_u^2 - \frac{2n}{n-1}(K_u + \varepsilon)\right](I_u + I_{N^{n-2}(\varepsilon)}) := \phi^2(I_u + I_{N^{n-2}(\varepsilon)})\,, \tag{3.13}$$

where $I_{N^{n-2}(\varepsilon)}$ is the Riemannian metric of an $(n-2)$-dimensional space form of constant curvature $\varepsilon$.

PROPOSITION 3.11. *Let $f : M^n \to \mathbb{R}^{n+1} (n \geq 4)$ be a cylinder, or a cone, or a rotation hypersurface over an umbilic-free surface $u : M^2 \to N^3(-\varepsilon)$, which was constructed as Examples* 3.8, 3.9, *or* 3.10. *If the hypersurface $f$ is of constant Möbius Ricci curvature $\lambda$, then $u$ is a $(\lambda, n, \varepsilon)$-surface in $N^3(-\varepsilon)$.*

PROOF. Now we take the local orthonormal basis $\{e_1, e_2\}$ on $TM^2$ with respect to $I_u$, consisting of principal vectors. Let $\{e_3, \dots, e_n\}$ be a local orthonormal basis on $TN^{n-2}(\varepsilon)$, then $\{e_1, e_2, e_3, \dots, e_n\}$ is a orthonormal basis on $T(M^2 \times N^{n-2}(\varepsilon))$ with respect to the product metric $I_u + I_{N^{n-2}(\varepsilon)}$.

Let $\tilde{R}_{ijkl}$ denote the curvature tensor for $I_u + I_{N^{n-2}(\varepsilon)}$, and $R_{ijkl}$ the curvature tensor for the Möbius metric $g$. Setting $\mu = \frac{1}{\phi} = \frac{1}{\sqrt{4H_u^2 - \frac{2n}{n-1}(K_u+\varepsilon)}}$, then by direct computation (also see [13]), we have

$$\begin{aligned} &R_{ijij} = \mu^2 \tilde{R}_{ijij} + \mu\mu_{ii} + \mu\mu_{jj} - |\nabla\mu|^2\,, \quad i \neq j\,, \\ &R_{ijik} = \mu^2 \tilde{R}_{ijik} + \mu\mu_{jk}\,, \quad \textit{when } \{i, j, k\} \textit{ are distinct}\,, \end{aligned} \tag{3.14}$$

where $\mu_{ij}$ and $\nabla\mu$ are the Hessian matrix and the gradient of $\mu$ with respect to the metric $I_u + I_{N^{n-2}(\varepsilon)}$.

Next we assume that $f$ is of constant Möbius Ricci curvature $\lambda$. Note the metric $I_u + I_{N^{n-2}(\varepsilon)}$ is a Riemannian product metric, from (3.14), we have

$$\begin{aligned} &\lambda = \mu^2(n-3)\varepsilon + \mu\Delta\mu - (n-1)|\nabla\mu|^2\,, \\ &\lambda = \mu^2 K_u + \mu\Delta\mu + (n-2)\mu\mu_{11} - (n-1)|\nabla\mu|^2\,, \\ &\lambda = \mu^2 K_u + \mu\Delta\mu + (n-2)\mu\mu_{22} - (n-1)|\nabla\mu|^2\,. \end{aligned} \tag{3.15}$$

Note that $\Delta\mu = \mu_{11} + \mu_{22}$, from (3.15) we get

$$\begin{aligned} &\Delta\mu = \frac{2\mu}{n-2}[(n-3)\varepsilon - K_u] = 2\mu_{11} = 2\mu_{22}\,, \\ &|\nabla\mu|^2 = \mu_1^2 + \mu_2^2 = \frac{\mu^2[n(n-3)\varepsilon - 2K_u]}{(n-1)(n-2)} - \frac{\lambda}{n-1}\,. \end{aligned} \tag{3.16}$$

From (3.14), we also have $\mu_{12} = 0$. □

**4. Proof of Theorem 1.4.** Let $f : M^n \to \mathbb{R}^{n+1} (n \geq 4)$ be an immersed hypersurface without umbilical points, which is of constant Möbius Ricci curvature. Since three dimensional Einstein manifolds are of constant sectional curvature, in this section we assume $n \geq 4$. Because of the local nature of our results, we can assume that the multiplicities of all principal curvatures are locally constant. In fact there always exists an open dense subset $U$ of $M^n$ on which the multiplicities of the principal curvatures are locally constant (see [10]).

We assume that the hypersurface has $(s+t)$ distinct principal curvatures. Since the multiplicities of all principal curvatures are locally constant, we can choose a local orthonormal basis $\{E_1, \ldots, E_n\}$, such that

$$(B_{ij}) = diag\{b_1, b_2, \ldots, b_s, b_{s+1}, \ldots, b_{s+1}, \ldots, b_{s+t}, \ldots, b_{s+t}\}\,. \tag{4.17}$$

Here the Möbius principal curvatures $b_1, \ldots, b_s$ are simple, the multiplicities of the Möbius principal curvatures $b_{s+1}, \ldots, b_{s+t}$ are great than one. From (2.5), we have

$$\begin{aligned} &R_{ij} = \lambda\delta_{ij} = -\sum_k B_{ik}B_{kj} + \mathrm{tr(A)}\delta_{ij} + (n-2)\mathrm{A}_{ij}\,, \\ &(A_{ij}) = diag\{a_1, \ldots, a_n\}\,, \quad a_i = \frac{1}{n-2}(\lambda + b_i^2 - \mathrm{tr(A)})\,, \quad 1 \leq i \leq n\,. \end{aligned} \tag{4.18}$$

Since $f$ is of constant Möbius Ricci curvature, $\lambda$ and $\mathrm{tr}(\mathbf{A})$ are constant.

By covariant derivative for the first equation of (4.18), we get that

$$A_{ij,k} = \frac{1}{n-2}\left(\sum_m B_{im,k}B_{mj} + \sum_m B_{im}B_{mj,k}\right). \tag{4.19}$$

Using (4.17) and (4.19), we have

$$(b_i + b_j)B_{ij,k} = (n-2)A_{ij,k}\,. \tag{4.20}$$

LEMMA 4.1. *Under the basis* $\{E_1, \ldots, E_n\}$, *set* $[i] = \{m | b_m = b_i\}$. *The Möbius invariants of* $f$ *have the following relations:*

$$\begin{aligned} &C_i = 0\,; \quad B_{ij,k} = 0\,, \quad i, j > s\,, \quad i \neq j; \quad 1 \leq k \leq n\,, \\ &B_{ij,k} = 0\,, \quad i \neq j\,, \quad j \neq k\,, \quad k \neq i\,, \\ &B_{jj,i} = \frac{b_i + (n-1)b_j}{b_i - b_j}C_i\,, \quad B_{ij,j} = \frac{nb_j}{b_i - b_j}C_i\,, \quad [i] \neq [j]\,, \\ &\omega_{ij} = \frac{B_{ij,i}}{b_i - b_j}\omega_i + \frac{B_{ij,j}}{b_i - b_j}\omega_j = \frac{nb_jC_i}{(b_i - b_j)^2}\omega_j - \frac{nb_iC_j}{(b_i - b_j)^2}\omega_i\,, \quad [i] \neq [j]\,. \end{aligned} \tag{4.21}$$

PROOF. Using $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$, setting $[i] = [j], i \neq j$, so $b_i = b_j$, we get

$$B_{ij,k} = 0\,, \quad [i] = [j]\,, \quad i \neq j\,, \quad 1 \leq k \leq n\,. \tag{4.22}$$

Particularly, $B_{ij,j} = 0$. Using (2.1) and (2.3)

$$A_{ij,j} - A_{jj,i} = b_jC_i\,, \quad B_{ij,j} - B_{jj,i} = -C_i\,,$$

and (4.20), we obtain

$$\frac{n}{n-2} b_j C_i = 0 .$$

If $b_j \neq 0$, then $C_i = 0$. If for all $b_j = 0,\ j > s$ then $E_i(b_j) = B_{jj,i} = 0$, and $C_i = 0$. Thus

$$C_i = 0 , \quad i > s .$$

When $i \neq j, j \neq k, i \neq k$, then $B_{ij,k} = B_{ik,j}$, $A_{ij,k} = A_{ik,j}$. Moreover if $b_i \neq b_j$ *or* $b_i \neq b_k$, using (4.20) we get

$$B_{ij,k} = A_{ij,k} = 0 , \quad i \neq j , \quad j \neq k , \quad i \neq k . \tag{4.23}$$

Combining (4.22) and (4.23), we obtain

$$B_{ij,k} = 0 , \quad i, j > s , \quad i \neq j; \quad 1 \leq k \leq n .$$

If $[i] \neq [j]$, using (2.1), (2.3) and (4.20), we obtain that

$$A_{ij,j} - A_{jj,i} = b_j C_i = \frac{b_i + b_j}{n-2} B_{ij,j} - \frac{2b_j}{n-2} B_{jj,i} = \frac{b_i + b_j}{n-2} B_{ij,j} - \frac{2b_j}{n-2} (B_{ij,j} + C_i) ,$$

and

$$B_{jj,i} = \frac{b_i + (n-1) b_j}{b_i - b_j} C_i , \quad B_{ij,j} = \frac{n b_j}{b_i - b_j} C_i .$$

Using $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$, we have

$$(b_i - b_j)\omega_{ij} = \sum_k B_{ij,k}\omega_k .$$

Since $b_i \neq b_j$, we have

$$\omega_{ij} = \frac{B_{ij,i}}{b_i - b_j}\omega_i + \frac{B_{ij,j}}{b_i - b_j}\omega_j = \frac{n b_j C_i}{(b_i - b_j)^2}\omega_j - \frac{n b_i C_j}{(b_i - b_j)^2}\omega_i .$$

We complete the proof of Lemma 4.1. □

PROPOSITION 4.2. *Let $f : M^n \to \mathbb{R}^{n+1}$ be an umbilic free hypersurface with constant Möbius Ricci curvature. If the Möbius form $\mathbf{C} = 0$, then locally $f$ is Möbius equivalent to one of the following examples:*

(1) *cylinder $f : \mathbb{S}^1 \times \mathbb{R}^{n-1} \to \mathbb{R}^{n+1}$;*

(2) *the image of $\sigma^{-1}$ of the torus $\mathbb{S}^k(\sqrt{\frac{k-1}{n-2}}) \times \mathbb{S}^{n-k}(\sqrt{\frac{n-k-1}{n-2}}) \subset \mathbb{S}^{n+1}$, $1 < k < n-1$.*

*Particularly, $f$ has only two distinct principal curvatures.*

PROOF. Since $\mathbf{C} = 0$, i.e., $C_i = 0$, from Lemma 4.1, we know that $B_{jj,i} = 0,\ i \neq j$. Since $\mathrm{tr}(\mathbf{B}) = 0$, we have $\sum_m B_{mm,i} = 0$ and $B_{ii,i} = 0$. Thus $B_{ij,k} = 0 (1 \leq i, j, k \leq n)$ is constant. So the Möbius second fundamental form of $f$ is parallel. From reference [4] and Proposition 3.2, Proposition 3.3, Proposition 3.5 and Proposition 3.7, we finish the proof. □

THEOREM 4.3. *Let $f : M^n \to \mathbb{R}^{n+1}$ $(n \geq 4)$ be an immersed hypersurface without umbilical points. If $f$ is of constant Möbius Ricci curvature, then $f$ has three distinct principal curvatures at most.*

PROOF. We assume that $s+t \geq 4$. Next we prove that there exists a contradiction.

Now we fix the indices $i, j, k$ such that $[i] \neq [j]$, $[j] \neq [k]$, $[k] \neq [i]$, then

$$B_{ij,k} = 0\,, \quad i \in [i]\,, \quad j \in [j]\,, \quad k \in [k]\,.$$

Noting $E_k(b_i) = B_{ii,k}$, and Using definition of $C_{i,j}$ and Lemma 4.1, we can obtain

$$\begin{aligned} B_{ij,jk} &= E_k(B_{ij,j}) + B_{kj,j}\omega_{ki}(E_k) \\ &= n\frac{b_k + (n-1)b_j}{(b_i - b_j)(b_k - b_j)}C_i C_k + \frac{nb_j}{b_i - b_j}C_{i,k}\,. \end{aligned}$$

Similarly we have

$$B_{ij,kj} = \frac{n^2 b_j}{(b_i - b_j)(b_k - b_j)}C_i C_k\,.$$

From Ricci identity $B_{ij,jk} - B_{ij,kj} = (b_i - b_j)R_{jijk} = 0$ we get

$$C_i C_k + b_j C_{i,k} = 0\,. \tag{4.24}$$

Since $s+t \geq 4$, there is $[l]$ such that $[l] \neq [i], [j], [k]$. Similarly we have $C_i C_k + b_l C_{i,k} = 0$, from (4.24) we get

$$(b_j - b_l)C_{i,k} = 0\,, \quad C_i C_k = 0\,.$$

This implies that there are at least $n-1$ zero elements in $\{C_1, \ldots, C_n\}$, and we assume that

$$C_2 = \cdots = C_n = 0\,.$$

If the multiplicity of $b_1$ is greater than one, then from Lemma 4.1 we have

$$C_1 = 0\,, \quad B_{ij,k} = 0\,, \quad 1 \leq i, j, k \leq n\,, \tag{4.25}$$

thus $B$ is parallel. From ([4]) we know that $M^n$ has at most three distinct Möbius principal curvature. This is a contradiction.

Now we assume that the multiplicity of $b_1$ is one. Since $s+t \geq 4$, we take $i, j, k > 1$. Noting $[i] \neq [j]$, $[j] \neq [k]$, $[k] \neq [i]$, so we have

$$\begin{gathered} C_i = C_j = C_k = 0\,, \quad \omega_{ij} = 0\,, \quad \omega_{ik} = 0\,, \quad \omega_{jk} = 0\,, \\ \omega_{1i} = \frac{nb_i C_1}{(b_1 - b_i)^2}\omega_i\,, \quad \omega_{1j} = \frac{nb_j C_1}{(b_1 - b_j)^2}\omega_j\,, \quad \omega_{1k} = \frac{nb_k C_1}{(b_1 - b_k)^2}\omega_k\,. \end{gathered}$$

Using $d\omega_{ij} - \sum_l \omega_{il} \wedge \omega_{lj} = -\frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l$, we get

$$\begin{aligned} R_{ijij} &= b_i b_j + a_i + a_j = \frac{-n^2 b_i b_j}{(b_1 - b_i)^2(b_1 - b_j)^2}C_1^2\,, \\ R_{ikik} &= b_i b_k + a_i + a_k = \frac{-n^2 b_i b_k}{(b_1 - b_i)^2(b_1 - b_k)^2}C_1^2\,, \end{aligned} \tag{4.26}$$

where $i \in [i]$, $j \in [j]$, $k \in [k]$.

The first formula of (4.26) minus the second formula of (4.26) we get

$$b_i(b_j - b_k) + (a_j - a_k) = \frac{n^2 b_i C_1^2 (b_j - b_k)(b_j b_k - b_1^2)}{(b_1 - b_i)^2(b_1 - b_k)^2(b_1 - b_j)^2}\,. \tag{4.27}$$

From (4.18), we have $a_j - a_k = \frac{b_j^2 - b_k^2}{n-2}$. Combining (4.27) we get

$$\frac{(n-2)b_i + b_j + b_k}{n-2} = \frac{n^2 b_i C_1^2 (b_j b_k - b_1^2)}{(b_1 - b_i)^2 (b_1 - b_k)^2 (b_1 - b_j)^2}\,. \tag{4.28}$$

Similarly

$$\frac{(n-2)b_j + b_i + b_k}{n-2} = \frac{n^2 b_j C_1^2 (b_i b_k - b_1^2)}{(b_1 - b_j)^2 (b_1 - b_k)^2 (b_1 - b_i)^2}\,. \tag{4.29}$$

Using (4.28)–(4.30), we have

$$\frac{n^2 C_1^2 b_1^2}{(b_1 - b_j)^2 (b_1 - b_k)^2 (b_1 - b_i)^2} = -\frac{n-3}{n-2}\,,$$

which is equivalent to

$$\frac{n^2 C_1^2 b_1^2}{(b_1 - b_j)^2 (b_1 - b_k)^2 (b_1 - b_i)^2} + \frac{n-3}{n-2} = 0\,. \tag{4.30}$$

Clearly the equation (4.30) does not hold, which is a contradiction. Thus we complete the proof of Theorem 4.3. □

THEOREM 4.4. *Let $f : M^n \to \mathbb{R}^{n+1} (n \geq 4)$ be an immersed hypersurface with three distinct principal curvatures. If $f$ has constant Möbius Ricci curvature $\lambda$, then locally $f$ is Möbius equivalent to one of the following examples:*

(1) *a cylinder over a $(\lambda, n, 0)$-surface in a Euclidean 3-plane $\mathbb{R}^3$;*
(2) *a cone over a $(\lambda, n, 1)$-surface in a 3-sphere $\mathbb{S}^3$;*
(3) *a rotation hypersurface over a $(\lambda, n, -1)$-surface in a hyperbolic 3-plane $\mathbb{R}^3_+$.*

Since the proof is long, we will give the proof at the end of this paper.

Next we consider that $f$ has only two distinct principle curvatures. Let $b_1, b_2$ denote the two Möbius principal curvatures with multiplicity $k$ and $n-k$, respectively. Using (2.6) we get

$$b_1 = -\frac{1}{n}\sqrt{\frac{(n-1)(n-k)}{k}}\,, \quad b_2 = \frac{1}{n}\sqrt{\frac{(n-1)k}{(n-k)k}}\,. \tag{4.31}$$

When the multiplicity $k$ and $n-k$ are greater than 1. We take a local orthonormal $\{E_1, \dots, E_n\}$ such that

$$(B_{ij}) = diag\{\underbrace{b_1, \dots, b_1}_{k}, \underbrace{b_2, \dots, b_2}_{n-k}\}\,.$$

Using $\sum_m B_{ij,m}\omega_m = dB_{ij} + \sum_m B_{im}\omega_{mj} + \sum_m B_{mj}\omega_{mi}$ and $b_1, b_2$ are constant, we get

$$\begin{aligned} &B_{ij,m} = 0\,, \quad 1 \leq i\,, j \leq k\,, \quad 1 \leq m \leq n;\\ &B_{ij,m} = 0\,, \quad k+1 \leq i\,, j \leq n\,, \quad 1 \leq m \leq n\,. \end{aligned} \tag{4.32}$$

When $1 \leq i, j \leq k, i \neq j$, using (2.3) we have

$$C_j = B_{ii,j} - B_{ij,i} = 0\,.$$

Similarly when $k+1 \le i \le n$ we can get $C_i = 0$. Thus $\mathbf{C} = 0$. From Proposition 4.2, we know that $f$ is Möbius equivalent to the image of $\sigma^{-1}$ of the torus

$$\mathbb{S}^k\left(\sqrt{\frac{k-1}{n-2}}\right) \times \mathbb{S}^{n-k}\left(\sqrt{\frac{n-k-1}{n-2}}\right) \subset \mathbb{S}^{n+1}\,, \quad 1 < k < n-1\,.$$

When $k = 1$. It is a well-known fact that an $n$-dimensional hypersurface in space form is conformally flat if and only if it has a principle curvature of multiplicity at least $n-1$ everywhere. Thus $f$ is conformally flat, Since $f$ is of constant Möbius Ricci curvature, so $f$ is of constant Möbius sectional curvature.

Sum together we complete the proof of our main Theorem 1.4.

**5. Appendix: Details of proof to Theorem 4.4.** In the section, we prove Theorem 4.4. We assume that $f$ has three distinct principal curvatures. From Lemma 4.1, Proposition 4.2 and $n \ge 4$, we need to consider the following two cases:

Case1 $\{b_1, \dots, b_n\} = \{b_1, \mu, \dots, \mu, \nu, \dots, \nu\}$, Case2 $\{b_1, \dots, b_n\} = \{b_1, b_2, \nu, \dots, \nu\}$.

In the following two propositions, we show that Case 1 can not appear, and Case 2 is Möbius equivalent to a cone, a cylinder, or rotational hypersurface over a $(\lambda, n, \varepsilon)$-surface.

PROPOSITION 5.1. *Let $f : M^n \to \mathbb{R}^{n+1} (n \ge 4)$ be an umbilic free hypersurface. If $f$ has three distinct principal curvatures and one of the principal curvatures is simple, i.e.,*

$$\{b_1, \dots, b_n\} = \{b_1, \underbrace{\mu, \dots, \mu}_{s}, \underbrace{\nu, \dots, \nu}_{t}\}\,, \quad 1+s+t = n\,, \quad s, t \ge 2\,.$$

*Then the Möbius Ricci curvature of $f$ can not be constant.*

PROOF. We assume that $f$ is of constant Möbius Ricci curvature. From Lemma 4.1, setting $i \in \{m | b_m = \mu\}$, $j \in \{m | b_m = \nu\}$, we have

$$\begin{aligned} &C_2 = \cdots = C_n = 0\,,\\ &B_{1i,i} = \frac{n\mu}{b_1-\mu}C_1\,, \quad B_{1j,j} = \frac{n\nu}{b_1-\nu}C_1\,,\\ &\omega_{1i} = \frac{B_{1i,i}}{b_1-\mu}\omega_i\,, \quad \omega_{1j} = \frac{B_{1j,j}}{b_1-\nu}\omega_j\,, \quad \omega_{ij} = 0\,. \end{aligned} \tag{5.33}$$

Since $B_{jj,1} = B_{1j,j} + C_1$, from (5.33), we obtain

$$B_{ii,1} = \frac{b_1 + (n-1)\mu}{b_1-\mu}C_1\,, \quad B_{jj,1} = \frac{b_1+(n-1)\nu}{b_1-\nu}C_1\,. \tag{5.34}$$

Since $\mathrm{tr}(\mathbf{B}) = 0$, $\nabla_{E_1}\mathrm{tr}(\mathbf{B}) = \mathrm{tr}(\nabla_{E_1}\mathbf{B}) = 0$, i.e., $\sum_m B_{mm,1} = 0$. Combining $b_1 + s\mu + t\nu = 0$ and $b_1^2 + s\mu^2 + t\nu^2 = \frac{n-1}{n}$, we obtain

$$B_{11,1} = -sB_{ii,1} - tB_{jj,1} = \frac{nb_1^2 - \frac{n-1}{n}}{(b_1-\mu)(b_1-\nu)}C_1\,. \tag{5.35}$$

Using $d\omega_{ij} - \sum_l \omega_{il} \wedge \omega_{lj} = -\frac{1}{2}\sum_{kl} R_{ijkl}\omega_k \wedge \omega_l$, we obtain

$$R_{ijij} = \frac{-n^2\mu\nu}{(b_1-\mu)^2(b_1-\nu)^2}C_1^2\,. \tag{5.36}$$

Using the definition of $B_{ij,kl}$ and Lemma 4.1, we obtain

$$B_{1i,i1} = \frac{b_1 B_{ii,1} - \mu B_{11,1}}{(b_1-\mu)^2}nC_1 + \frac{n\mu}{b_1-\mu}C_{1,1}\,,\quad B_{1i,1i} = (B_{11,1} - B_{ii,1} - B_{1i,i})\frac{n\mu C_1}{(b_1-\mu)^2}\,,$$
$$B_{1j,j1} = \frac{b_1 B_{jj,1} - \nu B_{11,1}}{(b_1-\nu)^2}nC_1 + \frac{n\nu}{b_1-\nu}C_{1,1}\,,\ B_{1j,1j} = (B_{11,1} - B_{jj,1} - B_{1j,j})\frac{n\nu C_1}{(b_1-\nu)^2}\,.$$

Using Ricci identity $B_{1i,i1} - B_{1i,1i} = (\mu - b_1)R_{1i1i}$ and Lemma 4.1, we get

$$\begin{aligned}(b_1-\mu)^2 R_{1i1i} &= \frac{nC_1}{b_1-\mu}[2\mu B_{11,1} - (b_1+\mu)B_{ii,1} - \mu B_{1i,i}] - n\mu C_{1,1}\,,\\ (b_1-\nu)^2 R_{1j1j} &= \frac{nC_1}{b_1-\nu}[2\nu B_{11,1} - (b_1+\nu)B_{jj,1} - \nu B_{1j,j}] - n\nu C_{1,1}\,.\end{aligned} \tag{5.37}$$

From (5.37), (5.33) and (5.35), we can get

$$\begin{aligned}&(b_1-\mu)^2\nu R_{1i1i} - (b_1-\nu)^2\mu R_{1j1j} = \frac{n(\mu-\nu)C_1^2}{(b_1-\mu)^2(b_1-\nu)^2}\chi\,,\\ &\chi \triangleq b_1^2[\mu^2+\nu^2+b_1^2 - 2b_1(\mu+\nu) - 4(n-1)\mu\nu] + (3n-2)b_1\mu\nu(\mu+\nu)\\ &\quad + (2n^2-2n+1)\mu^2\nu^2\,.\end{aligned}$$

Combining (5.36), we have

$$(b_1-\mu)^2\nu R_{1i1i} - (b_1-\nu)^2\mu R_{1j1j} + \frac{\mu-\nu}{n\mu\nu}\chi R_{ijij} = 0\,. \tag{5.38}$$

Using (2.4), (5.38) and $b_1 + s\mu + t\nu = 0$, $b_1^2 + s\mu^2 + t\nu^2 = \frac{n-1}{n}$, we know that $b_1$, $\mu$, and $\nu$ are constant.

Therefore $b_1, \mu, \nu$ are constant, then from Lemma 4.1 we get $C_1 = 0$. Therefore $\mathbf{C} = 0$. Using Proposition 4.2, we know that $f$ has only two distinct principal curvatures, which is a contradiction. we finish the proof. □

PROPOSITION 5.2. *Let $f : M^n \to \mathbb{R}^{n+1} (n \geq 4)$ be an immersed hypersurface. Assume we can diagonize the Möbius second fundamental form under an orthonormal frame $\{E_1, \dots, E_n\}$ with respect to the Möbius metric $g$ such that*

$$(B_{ij}) = diag\{b_1, b_2, \mu, \dots, \mu\}\,,\quad b_1 \neq b_2,\ \ b_1 \neq \mu\,,\ \ b_2 \neq \mu\,.$$

*If $B_{pq,\alpha} = 0,\ C_\alpha = 0,\ 1 \leq p, q \leq 2,\ 3 \leq \alpha \leq n$.*
*Then $f$ is Möbius equivalent to a cone, a cylinder, or rotational hypersurface over a surface in sphere $\mathbb{S}^3$, Euclidean space $\mathbb{R}^3$, and hyperbolic space $\mathbb{R}^3_+$ constructed by the examples* (3.8), (3.9) *and* (3.10).

PROOF. Let $\{Y, N, Y_1, \ldots, Y_n, \xi\}$ be a moving frame in $\mathbb{R}^{n+3}_1$ (see Section 2). In the proof below we adopt the convention on the range of indices as below:

$$1 \le p, q \le 2\,, \quad 3 \le \alpha, \beta, \gamma \le n\,, \quad 1 \le i, j, k, l \le n\,.$$

Without loss of generality we make a new choice of frame vectors such that

$$A_{\alpha\beta} = a_\alpha \delta_{\alpha\beta}\,. \tag{5.39}$$

Applying $dB_{ij} + \sum_k B_{kj}\omega_{ki} + \sum_k B_{ik}\omega_{kj} = \sum_k B_{ij,k}\omega_k$ for off-diagonal element $B_{\alpha\beta}$ ($\alpha \neq \beta$) and using the fact $B_{\alpha\alpha} = B_{\beta\beta} = \mu$, $B_{\alpha\beta} = 0$ we get

$$B_{\alpha\beta,k} = 0 = B_{k\alpha,\beta}\,, \quad \forall\, \alpha \neq \beta\,, \quad 1 \le k \le n\,. \tag{5.40}$$

The second equality is by the integrability equation. Since $n - 2 \ge 2$, we can always choose indices $\alpha \neq \beta$. Then by integrability equation and the assumption $C_\beta = 0$ one has

$$E_\beta(\mu) = B_{\alpha\alpha,\beta} = B_{\alpha\beta,\alpha} + \delta_{\alpha\alpha}C_\beta - \delta_{\alpha\beta}C_\alpha = C_\beta = 0\,, \quad \forall \beta\,. \tag{5.41}$$

Here $B_{\alpha\beta,\alpha} = 0$ due to (5.40). Similarly we have $B_{p\alpha,q} = B_{pq,\alpha} + \delta_{p\alpha}C_q - \delta_{pq}C_\alpha = B_{pq,\alpha}$ and $B_{p\alpha,\alpha} = B_{\alpha\alpha,p} - C_p = E_p(\mu) - C_p$. Together with the assumption $B_{pq,\alpha} = 0$ we summarize that

$$B_{pq,\alpha} = B_{p\alpha,q} = 0\,, \quad B_{p\alpha,\alpha} = E_p(\mu) - C_p\,, \quad \forall\, p, q, \alpha\,. \tag{5.42}$$

Now with the help of (5.40) and (5.42) we compute the covariant derivatives of off-diagonal components $B_{p\alpha}$ and find

$$\omega_{p\alpha} = \frac{B_{p\alpha,\alpha}}{b_p - \mu}\omega_\alpha\,, \quad \forall\, p, \alpha\,. \tag{5.43}$$

Differentiating once more we obtain the curvature tensor. Compare the coefficient of the component $\omega_p \wedge \omega_q$ for any given $p \neq q$. We find that

$$R_{p\alpha pq} = 0\,.$$

From the integrability equation (2.4) we get

$$A_{q\alpha} = 0\,, \quad 1 \le q \le 2\,, \quad 3 \le \alpha \le n\,. \tag{5.44}$$

Similarly by comparing the component $\omega_p \wedge \omega_\alpha$ we observe that $R_{p\alpha p\alpha}$ is independent of $\alpha$ (here we use (5.42)). Equation (2.4) yields $R_{p\alpha p\alpha} = b_p\mu + A_{pp} + A_{\alpha\alpha}$ and

$$A_{\alpha\alpha} = a\,, \quad \forall\, \alpha\,. \tag{5.45}$$

Next we compute the covariant derivatives of tensor $A$ and $C$. By the condition $C_\alpha = 0$ and the integrability equation (2.1) $A_{ij,k} - A_{ik,j} = B_{ik}C_j - B_{ij}C_k$,

$$E_\alpha(a) = E_\alpha(A_{\beta\beta}) = A_{\beta\beta,\alpha} = A_{\alpha\beta,\beta} = 0\,, \quad \forall\, \alpha \neq \beta\,. \tag{5.46}$$

As a consequence of (5.43) and $dC_i + \sum_k C_k\omega_{ki} = \sum_k C_{i,k}\omega_k$ we get that

$$E_\alpha(C_p) = C_{p,\alpha} = C_{\alpha,p} = 0\,, \quad \forall\, p, \alpha\,. \tag{5.47}$$

Let's look at the geometric meaning of these results. From the formula in (5.43) we know that distributions

$$D_1 \triangleq \mathrm{Span}\{E_1, E_2\}, \quad D_2 \triangleq \mathrm{Span}\{E_\alpha | 3 \le \alpha \le n\}$$

are integrable. Any integral submanifold of distribution $D_1$ is an $m$-dimensional submanifold. On the other hand, along any integral submanifold of $D_2$ the hypersurface $Y$ is tangent to

$$F \triangleq \mu Y + \xi, \tag{5.48}$$

the principal curvature sphere of multiplicity $n-2$. Using (5.41), $E_p(\mu) = B_{\alpha\alpha,p} = B_{p\alpha,\alpha} + C_p$ and the structure equation it is easy to get that

$$E_\alpha(F) = 0, \quad E_p(F) = B_{p\alpha,\alpha} Y + (\mu - b_p) Y_p. \tag{5.49}$$

Then principal curvature sphere $F$ induces a 2-dimensional submanifold in the de-Sitter space $S_1^{n+2}$

$$F : \widetilde{M}^2 = M^n / L \to \mathbb{S}_1^{n+2},$$

where fibers $L$ are integral submanifolds of distribution $D_2$. In other words, $F$ form a 2-parameter family of $n$-spheres enveloped by the hypersurface $Y$.

The next crucial observation is that $F$ is located in a fixed 4-dimensional linear subspace of $\mathbb{R}_1^{n+3}$. To show that we compute the repeated derivatives of $F$, which contains all information of the envelope $Y$. Straightforward yet tedious computation shows that the frames of

$$V_1 \triangleq \mathrm{Span}\{F, E_1(F), E_2(F), P\} \tag{5.50}$$

$$\text{where} \quad P \triangleq A_{\alpha\alpha} Y - N + \sum_{p=1}^{2} \frac{B_{p\alpha,\alpha}}{(\mu - b_p)^2} E_p(F) + \mu F$$

satisfy a linear first order PDE system. Hence these vectors, including $F$ itself, are contained in a fixed 4-dimensional subspace $V_1$ endowed with degenerate, Lorentzian, or positive definite inner product. This agrees with the geometry of cylinders, cones, and rotational hypersurfaces (see Examples 3.4, 3.5, 3.6), where the principal curvature sphere $F$ is orthogonal to an $(n-1)$-parameter family of hyperplanes/hyperspheres. Moreover, the orthogonal complement $V_1^\perp$ of $dim = n-1$ contains all $Y_\alpha$, $3 \le \alpha \le n$.

The final fact above inspires us to proceed in an alternative and easier way. Differentiate any given $Y_\alpha$ and modulo components in the subspace $\mathrm{Span}\{Y_\gamma,\ 3 \le \gamma \le n\}$. By (5.39), (5.44), (5.43) one finds

$$\begin{aligned} E_i(Y_\alpha) &= -A_{\alpha i} Y - \delta_{\alpha i} N + \sum\nolimits_j \omega_{\alpha j}(E_i) Y_j + B_{\alpha i} \xi \\ &= \begin{cases} -T \ (\mathrm{mod}\ Y_\gamma, & \text{when } i = \alpha; \\ 0 \ (\mathrm{mod}\ Y_\gamma), & \text{otherwise}, \end{cases} \end{aligned} \tag{5.51}$$

where

$$T \triangleq A_{\alpha\alpha} Y + N + \sum_{p=1}^{2} \frac{B_{p\alpha,\alpha}}{b_p - \mu} Y_p - \mu \xi \tag{5.52}$$

is independent of $\alpha$ by (5.42), (5.45). Then we assert that the subspace

$$V_2 \triangleq \mathrm{Span}\{T, Y_\gamma | 3 \le \gamma \le n\} \tag{5.53}$$

is parallel along $M$. According to our previous computation, $E_i(Y_\alpha) = 0 \pmod{V_2}$, $\forall \alpha$ . So we need only to consider $E_i(T)$. Fix $i$ and choose $\alpha \ne i$. (Such $\alpha$ exists by the assumption $n - 2 \ge 2$, which is the third and final time that we use it. Recall that this condition has been used to derive (5.41), (5.46), i.e., $E_\alpha(\mu) = 0 = E_\alpha(a)$.) Rewrite the first equality of (5.51) as

$$T = -E_\alpha(Y_\alpha) + \sum\nolimits_\gamma (\cdots) Y_\gamma \,. \tag{5.54}$$

By this clever choice of index $\alpha$ we may prove in a unified way that

$$\begin{aligned} E_i(T) &= -E_i(E_\alpha(Y_\alpha)) + \sum\nolimits_\gamma (\cdots) E_i(Y_\gamma) \pmod{Y_\gamma} \\ &= -E_\alpha(E_i(Y_\alpha)) + [E_\alpha, E_i](Y_\alpha) + \sum\nolimits_\gamma (\cdots) E_i(Y_\gamma) \pmod{Y_\gamma} \\ &= -E_\alpha\Big(\sum\nolimits_\beta (\cdots) Y_\beta\Big) + [E_\alpha, E_i](Y_\alpha) + \sum\nolimits_\gamma (\cdots) E_i(Y_\gamma) \pmod{Y_\gamma} \\ &= 0 \pmod{V_2} \,. \end{aligned}$$

This verifies our previous assertion. More precisely, we have

$$E_p(T) = \frac{B_{p\alpha,\alpha}}{b_p - \mu} T \,, \quad E_\alpha(T) = Q Y_\alpha \,, \quad \forall\, p, \alpha \tag{5.55}$$

where

$$Q \triangleq \langle T, T \rangle = 2A_{\alpha\alpha} + \mu^2 + \sum_{p=1}^{2} \frac{B^2_{p\alpha,\alpha}}{(b_p - \mu)^2}$$

satisfies

$$E_p(Q) = \frac{2B_{p\alpha,\alpha}}{b_p - \mu} Q \,, \quad E_\alpha(Q) = 0 \,. \tag{5.56}$$

One could verify (5.55) directly. But the easy way is using $\langle T, Y_\alpha \rangle = 0$ and (5.51) to get

$$\langle E_i(T), Y_\alpha \rangle = -\langle T, E_i(Y_\alpha) \rangle = \begin{cases} Q \,, & \text{when } i = \alpha \,; \\ 0 \,, & \text{otherwise} \,. \end{cases} \tag{5.57}$$

This implies $E_p(T) \parallel T$ for any $1 \le p \le 2$. Then $E_p(T)$ as in (5.55) is derived by differentiating (5.52) and comparing the $\xi$ component with $T$. The formula for $E_p(Q)$ in (5.56) follows directly. On the other hand, we know

$$\langle E_\alpha(T), T \rangle = \frac{1}{2} E_\alpha(Q) = 0 \,,$$

where we used (5.42) and its consequence $[E_p, E_\alpha] \in D_2$ together with (5.41), (5.46), (5.47). Combined with (5.57) we have $E_\alpha(T) = Q Y_\alpha$.

Regarding (5.56) as a linear first-order ODE for $Q$ we see that $Q \equiv 0$ or $Q \ne 0$ on the connected manifold $M^n$. Thus there are three possibilities for the induced metric on the fixed subspace $V_2 \subset \mathbb{R}^{n+3}_1$.

CASE 1. $Q = 0$ on $M^n$; $V_2$ is endowed with a degenerate inner product.

In this case, $\langle T, T\rangle = 0$. By (5.55), $E_p(T) \parallel T$, so $T$ determines a fixed light-like direction in $\mathbb{R}^{n+3}_1$, which we may take to be

$$[T] = [1, -1, 0, \dots, 0] \in \mathbb{R}^{n+3}_1 .$$

This corresponds to $\infty$, the point at infinity of $\mathbb{R}^{n+1}$. Choose space-like vectors $X_3, \dots, X_n$ so that $V_2 = \mathrm{Span}\{T, X_3, \dots, X_n\}$. We interpret the geometry of hypersurface $f : M^n \to \mathbb{R}^{n+1}$ as below:

1) Any $X_\alpha$ determines a hyperplane in $\mathbb{R}^{n+1}$ because $\langle T, X_\alpha\rangle = 0$;
2) $\mathrm{Span}\{X_\alpha, (3 \le \alpha \le n)\}$ corresponds to an $(n-2)$-dimensional plane $\Sigma$ in $\mathbb{R}^{n+1}$.
3) $F$ is a 2-parameter family of hyperplanes orthogonal to the fixed plane $\Sigma$.

$f(M)$, as the envelope of this family of hyperplanes $F$, is clearly a cylinder over a hypersurface $\widetilde{M} \subset \mathbb{R}^3$.

CASE 2. $Q < 0$ on $M^n$; $V_2$ is a Lorentz subspace in $\mathbb{R}^{n+3}_1$.

Fix a basis $\{P_0, P_\infty, X_4, \dots, X_n\}$ of the $(n-1)$-dimensional $V_2$ so that $P_0, P_\infty$ are light-like. Without loss of generality we may assume

$$P_0 = (1, 1, 0, \dots, 0), \quad P_\infty = (1, -1, 0, \dots, 0).$$

Using the stereographic projection $\sigma$ they correspond to the origin $O$ and the point at infinity $\infty$ of the flat $\mathbb{R}^{n+1}$, respectively. We interpret $F$ and $V_2$ in terms of the geometry of $\mathbb{R}^{n+1}$:

1) $\mathrm{Span}\{X_\alpha | 4 \le \alpha \le n\}$ corresponds to a coordinate plane $\mathbb{R}^{n-3} \subset \mathbb{R}^{n+1}$, because $X_\alpha$ must be space-like and orthogonal to $P_0, P_\infty$.
2) $F$ is an $m$-parameter family of hyperplanes (passing $O$ and $\infty$) and orthogonal to this fixed $\mathbb{R}^{n-3}$.

Based on the fact 1), $f(M)$, the envelope of $F$, is a cylinder over a 3-dimensional hypersurface in $\mathbb{R}^4$ (the orthogonal complement of the previous $\mathbb{R}^{n-3}$); moreover, the fact 2) means that $f(M)$ is a cone (with vertex $O$) over a 2-dimensional hypersurface in $S^3$.

CASE 3. $Q > 0$ on $M^n$; $V_2$ is a space-like subspace.

Without loss of generality we assume that $P_\infty = (1, -1, 0, \dots, 0)$ is contained in the orthogonal complement of $V_2$. As before we make the following interpretation:

1) $V_2$ corresponds to a 2-dimensional plane $\mathbb{R}^2 \subset \mathbb{R}^{n+1}$.
2) $F$ is an $(n-2)$-parameter family of hyper-spheres orthogonal to this fixed plane $\mathbb{R}^2$ with centers locating on it. Thus $F$ envelops a rotational hypersurface $f(M)$ (over a hypersurface in half-space $\mathbb{R}^3_+$).

Sum together we complete the proof to the proposition. □

THE PROOF OF THEOREM 4.4. Since the Case 1 can not appear. In Case 2, from Lemma 4.1, we have $B_{pq,\alpha} = 0,\ C_\alpha = 0(1),\ 1 \le p, q \le 2,\ 3 \le \alpha \le n$. From Proposition 3.11 and Proposition 5.2, we finish the proof of Theorem 4.4.

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