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Classification of N-(Super)-Extended Poincaré Algebras and Bilinear Invariants of the Spinor Representation of $Spin\ (p, q)$

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Abstract: We classify extended Poincaré Lie super algebras and Lie algebras of any signature (p, q), that is Lie super algebras (resp. Z_2-graded Lie algebras) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, where $\mathfrak{g}_0 = \mathfrak{so}(V)+V$ is the (generalized) Poincaré Lie algebra of the pseudo-Euclidean vector space $V = \mathbb{R}^{p,q}$ of signature (p, q) and $\mathfrak{g}_1 = S$ is the spinor $\mathfrak{so}(V)$-module extended to a $\mathfrak{g}_0$-module with kernel V. The remaining super commutators $\{\mathfrak{g}_1, \mathfrak{g}_1\}$ (respectively, commutators $[\mathfrak{g}_1, \mathfrak{g}_1]$) are defined by an $\mathfrak{so}(V)$-equivariant linear mapping

$$\vee^2\mathfrak{g}_1 \to V \quad \text{(respectively,} \quad \wedge^2\mathfrak{g}_1 \to V)\,.$$

Denote by $\mathcal{P}^+(n, s)$ (respectively, $\mathcal{P}^-(n, s)$) the vector space of all such Lie super algebras (respectively, Lie algebras), where $n = p + q = \dim V$ and $s = p - q$ is the classical signature. The description of $\mathcal{P}^\pm(n, s)$ reduces to the construction of all $\mathfrak{so}(V)$-invariant bilinear forms on S and to the calculation of three $\mathbb{Z}_2$-valued invariants for some of them.

This calculation is based on a simple explicit model of an irreducible Clifford module S for the Clifford algebra $Cl_{p,q}$ of arbitrary signature (p, q). As a result of the classification, we obtain the numbers $L^\pm(n, s) = \dim \mathcal{P}^\pm(n, s)$ of independent Lie super algebras and algebras, which take values 0,1,2,3,4 or 6. Due to Bott periodicity, $L^\pm(n, s)$ may be considered as periodic functions with period 8 in each argument. They are invariant under the group Γ generated by the four reflections with respect to the axes $n = -2$, $n = 2$, $s - 1 = -2$ and $s - 1 = 2$. Moreover, the reflection $(n, s) \to (-n, s)$ with respect to the axis $n = 0$ interchanges L^+ and L^- :

$$L^+(-n, s) = L^-(n, s)\,.$$

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Contents

Introduction

General relativity is a gauge theory with the Poincaré group $P(1,3) = \mathbb{R}^{1,3} \rtimes Lor(1,3)$ of Minkowski space $\mathbb{R}^{1,3}$ as gauge group. In N-extended supergravity the N-extended Poincaré supergroup plays the role of (super) gauge group.

The Lie super algebra of this super group for $N = 1$ is defined as follows: $\mathfrak{p}^{(1)}(1,3) = \mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1 = \mathfrak{p}(1,3) + S$, where $\mathfrak{p}(1,3) = \mathbb{R}^{1,3} + \mathfrak{so}(1,3)$ is the Poincaré Lie algebra and $S = \mathbb{C}^2$ is the spinor module of the Lorentz algebra $\mathfrak{so}(1,3) \cong \mathfrak{sl}(2,\mathbb{C})$ trivially extended to a $\mathfrak{p}(1,3)$-module. The supercommutator $\{\cdot,\cdot\} : S \otimes S \to \mathbb{R}^{1,3}$ is defined as projection onto the unique vector submodule $V \cong \mathbb{R}^{1,3}$ in the symmetric square $\vee^2 S$.

We remark that in this case there exists also a unique vector submodule in $\wedge^2 S$, which defines on $\mathfrak{p}(1,3) + S$ the structure of a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{p}^{(-1)}(1,3)$.

Our goal is to classify for any pseudo-Euclidean space $V = \mathbb{R}^{p,q}$ all similar extensions of the (generalized) Poincaré algebra $\mathfrak{p}(V) = \mathfrak{p}(p,q) = \mathbb{R}^{p,q} + \mathfrak{so}(p,q)$ to a super Lie algebra or to a $\mathbb{Z}_2$-graded Lie algebra. The super Lie algebra extensions of the Poincaré algebra $\mathfrak{p}(p,q)$ are the natural gauge algebras for supergravity theories over space times of signature (p,q). Since the time when the classical (i.e. $(p,q) = (1,3)$) super Poincaré algebra was discovered [G-L] these (generalized) super Poincaré algebras play a mayor role in many super symmetric field theories, see e.g [O-S and F] for further reference. However, despite the various realizations of particular super Poincaré algebras as infinitesimal symmetries of supergravity theories (for special dimensions and signatures of the space time), a systematic classification, as given in our paper, was missing.

Another motivation to study such extensions is that extended Poincaré Lie algebras are closely related to the full isometry algebra $\mathfrak{isom}(M)$ of homogeneous quaternionic Kähler manifolds M (see [dW-V-VP, A-C1]). In fact, $\mathfrak{isom}(M) = \mathfrak{p} + \mathbb{R}A$, where $\mathfrak{p}$ is an extension of the Poincaré algebra $\mathfrak{p}(3, 3+k)$ of the pseudo-Euclidean space $\mathbb{R}^{3,3+k}$ of signature $(3, 3+k)$, $k = -1, 0, 1, \ldots$, and A is a derivation of $\mathfrak{p}$ defining a natural gradation.

Definition 1. *A super Lie algebra (respectively a $\mathbb{Z}_2$-graded Lie algebra) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is called an N***-extended** *(respectively $-N$***-extended***)* **Poincaré algebra** *of $V = \mathbb{R}^{p,q}$ if the following conditions hold*

1) $\mathfrak{g}_0 \cong \mathfrak{p}(V)$.
2) $\mathfrak{g}_1$ is a sum of N irreducible spinor or semi spinor modules of $\mathfrak{p}(V) = V + \mathfrak{so}(V)$ with trivial action of the vector group V.
3) The super bracket $\{S, S\} \subset V$ (respectively Lie bracket $[S, S] \subset V$).

Let S be a $\mathfrak{p}(V)$-module with trivial action of the vector group V. Then defining on $\mathfrak{g} = \mathfrak{p}(V) + S$ the structure of a super Lie algebra (respectively of a $\mathbb{Z}_2$-graded Lie algebra) such that $\mathfrak{g}_0 \cong \mathfrak{p}(V)$, $\mathfrak{g}_1 = S$ and $\{S, S\} \subset V$ (respectively $[S, S] \subset V$) is equivalent to defining an $\mathfrak{so}(V)$-equivariant mapping $j : V^* \to \vee^2 S^*$ (respectively $j : V^* \to \wedge^2 S^*$). The super bracket (respectively the Lie bracket) is given by $j^* : \vee^2 S \to V$ (respectively $j^* : \wedge^2 S \to V$). Remark that under these assumptions the Jacobi identities are automatically satisfied since $[[x, y], z] = 0$ for $x, y, z \in \mathfrak{g}_1$.

We show that the classification of N-extended ($N \in \mathbb{Z}$) Poincaré algebras easily reduces to the classification of equivariant embeddings $V^* \hookrightarrow \vee^2 S^*$ if $N > 0$ and $V^* \hookrightarrow \wedge^2 S^*$ if $N < 0$, where V is the vector module and S the spinor module of $\mathfrak{so}(V)$. In other words, we reduce the classification to the cases $N = \pm 1, \pm 2$.

We prove that the following three vector spaces are isomorphic:

1) the space $\mathcal{J}$ of $\mathfrak{so}(V)$-equivariant mappings $j : V^* \to S^* \otimes S^*$,
2) the space $\mathcal{M}$ of $\mathfrak{so}(V)$-equivariant multiplications $\mu : V^* \otimes S \to S$, and
3) the space $\mathcal{B}$ of $\mathfrak{so}(V)$-invariant bilinear forms β on S.

Let $\rho : V^* \otimes S \to S$ be the (standard) Clifford multiplication, where we have identified $V \cong V^*$ using the scalar product on $V = \mathbb{R}^{p,q}$. Then an isomorphism $j_\rho : \mathcal{B} \to \mathcal{J}$ is given by

$$j_\rho(\beta) : v^* \in V^* \mapsto \beta \circ \rho(v^*) = \beta(\rho(v^*)\cdot, \cdot) \in S^* \otimes S^* .$$

In particular, the classification of $\mathfrak{so}(V)$-equivariant mappings $V^* \to S^* \otimes S^*$ is equivalent to the classification of $\mathfrak{so}(V)$-invariant bilinear forms on the spinor module S. The latter amounts to the description of the Schur algebra $\mathcal{C}$ of $\mathfrak{so}(V)$-invariant endomorphisms of S. The structure of $\mathcal{C}$ as abstract algebra depends only on the signature $s = p - q$ of $\mathbb{R}^{p,q}$ modulo 8; it is a simple real, complex or quaternionic matrix algebra of rank 1 or 2 or a sum of two isomorphic such algebras.

To construct equivariant embeddings of the vector module V^* into the symmetric square $\vee^2 S^*$ (or into the exterior square $\wedge^2 S^*$) we introduce the notion of an admissible bilinear form β on S and also the corresponding notion of an admissible endomorphism of S, which depends on the choice of an admissible bilinear form β.

Definition 2. *An $\mathfrak{so}(V)$-invariant bilinear form β on the spinor module S is called* **admissible** *if it has the following properties:*

1) *Clifford multiplication $\rho(v)$ is either β-symmetric or β-skew symmetric. We define the type τ of β to be $\tau(\beta) = +1$ in the first case and $\tau(\beta) = -1$ in the second.*
2) *β is symmetric or skew symmetric. Accordingly, we define the symmetry σ of β to be $\sigma(\beta) = \pm 1$.*
3) *If the spinor module is reducible, $S = S^+ + S^-$, then $S^\pm$ are either mutually orthogonal or isotropic. We put $\iota(\beta) = +1$ in the first case, $\iota(\beta) = -1$ in the second and call $\iota(\beta)$ the isotropy of β.*

Every admissible form β defines an $\mathfrak{so}(V)$-equivariant embedding $j_\rho(\beta) : V^* \to \vee^2 S^*$ if $\tau(\beta)\sigma(\beta) = +1$ or $j_\rho(\beta) : V^* \to \wedge^2 S^*$ if $\tau(\beta)\sigma(\beta) = -1$. Moreover, if $S = S^+ + S^-$, then either $S^\pm$ are orthogonal or isotropic for every bilinear form in the image of $j_\rho(\beta)$.

The main part of the paper is the construction of an admissible basis for the space $\mathcal{J}$ of equivariant mappings $V^* \to S^* \otimes S^*$, i.e. a basis consisting of embeddings $j_\rho(\beta)$, where β are admissible bilinear forms on S.

To describe all admissible forms β we make use of very simple explicit models of the irreducible Clifford modules inspired by Raševskiĭ [R]. We prove that the problem reduces to the three fundamental cases $V = \mathbb{R}^{m,m}, \mathbb{R}^{k,0}$ and $\mathbb{R}^{0,k}$ using the isomorphisms $C\ell_{m+k,m} \cong C\ell_{m,m} \hat{\otimes} C\ell_k$ and $C\ell_{m,m+k} \cong C\ell_{m,m} \hat{\otimes} C\ell_{0,k}$ and the algebraic properties of the fundamental invariants τ, σ and ι with respect to $\mathbb{Z}_2$-graded tensor products.

Moreover, we establish that for every pseudo-Euclidean vector space $V = \mathbb{R}^{p,q}$ there is a preferred non-degenerate $\mathfrak{so}(V)$-invariant bilinear form h on the spinor module S. This allows us to define *canonically* the notion of an admissible endomorphism of S and the invariants τ, σ and ι for such endomorphisms. They are multiplicative with respect to the composition $h \circ A = h(A\cdot, \cdot)$, $A \in \mathcal{C}$ admissible.

Finally, we explicitly construct in all the cases an admissible basis for the Schur algebra $\mathcal{C}$. This canonically yields admissible bases for the space $\mathcal{B}$ of invariant bilinear forms and the space $\mathcal{J}$ of equivariant mappings.

This gives an explicit description of all extended Poincaré algebras $\mathfrak{g} = \mathfrak{p}(V) + S$, where S is the spinor module. The super (respectively Lie) brackets $\vee^2 S \to V$ (respectively $\wedge^2 S \to V$) are given as linear combinations of mappings j_i^*, where the $j_i : V^* \to \vee^2 S^*$ (respectively $V^* \to \wedge^2 S^*$) form an admissible basis for the space of $\mathfrak{so}(V)$-equivariant mappings $V^* \to \vee^2 S^*$ (respectively $V^* \to \wedge^2 S^*$).

If the spinor module S is an irreducible $\mathfrak{so}(V)$-module, we obtain all $N = \pm 1$ extended Poincaré algebras. If S is reducible, then we obtain all $N = \pm 2$ extended Poincaré algebras and using the invariant ι we can determine all $N = \pm 1$ extended Poincaré algebras. Sometimes there exist only trivial $N = 1$ (or $N = -1$) extended Poincaré algebras, i.e. $\{S, S\} = 0$ (or $[S, S] = 0$).

Given a pseudo-Euclidean vector space $V = \mathbb{R}^{p,q}$, let $|N| = 1$ or 2 denote the number of irreducible summands of the spinor module S of $\mathfrak{so}(V)$. For fixed $N = +|N|$ or $N = -|N|$ we give now the dimension d_N of the vector space of N-extended Poincaré algebra structures on $\mathfrak{g} = \mathfrak{p}(V) + S$.

The function d_N, which depends only on the signature (p, q), admits a symmetry group Γ generated by reflections. Moreover, there is an additional supersymmetry which relates the dimension $L^+ := d_{+|N|}$ of the space of super algebras to the dimension $L^- := d_{-|N|}$ of the space of Lie algebras.

More precisely: Denote by $n = p + q$ the dimension and by $s = p - q$ the signature of $V = \mathbb{R}^{p,q}$ and let $L^+ = L^+(n, s)$ (respectively $L^-(n, s)$) be the maximal number of linearly independent super algebra structures $\vee^2 S \to V$ (respectively Lie algebra structures $\wedge^2 S \to V$) on $\mathfrak{g} = \mathfrak{p}(V) + S$. The functions L^+ and L^- are periodic with period 8 in each argument, hence we may consider them as functions on $\mathbb{Z}^2 = \mathbb{Z} \times \mathbb{Z}$. The value of the pair (L^+, L^-) is given in Table 1.

Table 1. The numbers L^+ of super algebras and L^- of Lie algebras $\mathfrak{g} = \mathfrak{p}(V) + S$ are given as functions of the dimension n and signature s of V. A fundamental domain for the reflection group Γ is emphasized in boldface. The supersymmetry axis is given by the equation $n = 0$.

s:	$(L^+(n,s), L^-(n,s))$								
5		1,3		1,3		3,1		3,1	
4	4,4		2,6		4,4		6,2		4,4
3		1,3		**1,3**		**3,1**		3,1	
2	4,4		**2,6**		**4,4**		**6,2**		4,4
1		1,3		**1,3**		**3,1**		3,1	
0	1,1		**0,2**		**1,1**		**2,0**		1,1
-1		0,1		**0,1**		**1,0**		1,0	
-2	1,1		0,2		1,1		2,0		1,1
-3		1,3		1,3		3,1		3,1	
n:	-4	-3	-2	-1	0	1	2	3	4

It follows from the inspection of this table, that the function (L^+, L^-) is invariant under the group Γ generated by the reflections with respect to the 4 axes defined by the equations $n = -2$, $n = 2$, $s' := s - 1 = -2$ and $s' = 2$. A fundamental domain F for Γ is

$$F = \{(n,s) \in \mathbb{Z}^2 | -2 \leq n \leq 2, \quad -2 \leq s' = s - 1 \leq 2\} \cap G,$$

$$G = \{(n,s) | \exists (p,q) \in \mathbb{Z}^2 : n = p + q, \quad s = p - q\} = \{(n,s) \in \mathbb{Z}^2 | n + s \quad \text{even}\}$$

and consists of 12 points. The values of the pair (L^+, L^-) at these points are typed in boldface in Table 1.

Moreover, the reflection θ with respect to the axis $\{n = 0\}$, $\theta : (n,s) \mapsto (-n, s)$, is a supersymmetry of the pair (L^+, L^-), that is it interchanges the number of Lie algebras and Lie super algebras:

$$(L^+(+n,s), L^-(+n,s)) = (L^-(-n,s), L^+(-n,s)).$$

In short:

$$\boxed{L^\pm = L^\mp \circ \theta}$$

A fundamental domain $\tilde{F}$ for the group $\tilde{\Gamma} =< \Gamma, \theta >$ is given by

$$\tilde{F} = \{(n,s) = (0,0),\ (0,2),\ (1,-1),\ (1,1),\ (1,3),\ (2,0),\ (2,2)\}.$$

In terms of the coordinates (p,q) a fundamental domain with $p \geq 0$ and $q \geq 0$ is given by

$$\tilde{D} = \{(p,q) = (2,0),\ (1,1),\ (3,0),\ (2,1),\ (1,2),\ (3,1),\ (2,2)\}.$$

1. (Super) Extensions of the Poincaré Algebra $\mathfrak{p}(p, q)$ and $Spin(p, q)$-Equivariant Embeddings $\mathbb{R}^{p,q} \hookrightarrow S^* \otimes S^*$

1.1. Extending the Poincaré algebra. Let $V = \mathbb{R}^{p,q}$ be the pseudo-Euclidean space with the metric $< x, y > = \sum_{i=1}^{p} x^i y^i - \sum_{j=p+1}^{p+q} x^j y^j$. We denote by $\mathfrak{so}(V) = \mathfrak{so}(p, q)$ the pseudo-orthogonal Lie algebra and by $\mathfrak{p}(V) = \mathfrak{p}(p, q) = \mathfrak{so}(V) + V$ the semidirect sum of $\mathfrak{so}(V)$ and the Abelian ideal V, it is the Lie algebra of the isometry group of $(V, < \cdot, \cdot >)$. We call $\mathfrak{p}(V)$ the **Poincaré algebra** of the space V.

Definition 1.1. *A $\mathbb{Z}_2$-graded Lie algebra (respectively a super algebra) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is called an* **extension** *(respectively a* **super extension***) of $\mathfrak{p}(V)$ if $\mathfrak{g}_0 = \mathfrak{p}(V)$, V is in the kernel of the representation of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ and $[\mathfrak{g}_1, \mathfrak{g}_1] \subset V$ (respectively $\{\mathfrak{g}_1, \mathfrak{g}_1\} \subset V$).*

Remark 1. Sometimes, for unification, we will refer to $\mathbb{Z}_2$-graded Lie algebras and to super algebras as ϵ-algebras, where $\epsilon = -1$ or $+1$ respectively. Correspondingly, we will speak of ϵ-extensions.

Proposition 1.1. *There exists a natural one-to-one correspondence between extensions (respectively super extensions) of $\mathfrak{p}(V)$ up to isomorphisms and equivalence classes of pairs (ρ, π), where*

$$\rho : \mathfrak{so}(V) \to \mathfrak{gl}(W)$$

is a representation and

$$\pi : \wedge^2 W \to V \quad (resp. \quad \vee^2 W \to V)$$

is a $\mathfrak{so}(V)$-equivariant linear map from the space of skew symmetric (respectively symmetric) bilinear forms on W^* to the vector module V. Two pairs (ρ, π) and (ρ', π') $(\rho' : \mathfrak{so}(V) \to \mathfrak{gl}(W'))$ are **equivalent** *if there exists an automorphism $\phi : \mathfrak{p}(V) \to \mathfrak{p}(V)$ and a linear map $\psi : W \to W'$ such that the following diagrams are commutative (for pairs of skew symmetric type):*

$$\begin{array}{ccc} \mathfrak{so}(V) & \xrightarrow{\rho} & \mathfrak{gl}(V) \\ \downarrow \bar{\phi} & & \downarrow \psi \\ \mathfrak{so}(V) & \xrightarrow{\rho'} & \mathfrak{gl}(W') \end{array} \qquad \begin{array}{ccc} \wedge^2 W & \xrightarrow{\pi} & V \\ \downarrow \psi & & \downarrow \phi|V \\ \wedge^2 W' & \xrightarrow{\pi'} & V \end{array},$$

where $\bar{\phi}$ is the induced automorphism of $\mathfrak{so}(V) = \mathfrak{p}(V)/V$. For pairs of symmetric type $\wedge^2$ must be replaced by $\vee^2$.

Proof. Given a pair (ρ, π) of skew symmetric type, we define a $\mathbb{Z}_2$-graded Lie algebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$, $\mathfrak{g}_0 = \mathfrak{p}(V) = \mathfrak{so}(V) + V$, $\mathfrak{g}_1 = W$ by

$$\begin{aligned} [A, w] &= \rho(A)w\,, \\ [w_1, w_2] &= \pi(w_1 \wedge w_2)\,, \\ [v, w] &= 0\,, \end{aligned}$$

where $A \in \mathfrak{so}(V)$, $v \in V$ and $w, w_1, w_2 \in W$. For a pair of symmetric type we define a super algebra $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ by the same formulas replacing only the middle equation by

$$\{w_1, w_2\} = \pi(w_1 \vee w_2)\,.$$

The Jacobi identity is satisfied because ρ is a representation, π is equivariant and the (anti)commutator of W with W is contained in V and hence commutes with W. The other statements can be checked easily. □

Recall that the spinor representation is the representation of $\mathfrak{so}(V)$ on an irreducible module S of the Clifford algebra $C\ell(V)$. It is either irreducible or a sum of two irreducible semi spinor modules $S^{\pm}$.

Definition 1.2. *(cf. Def. 1) Let $\mathfrak{g} = \mathfrak{g}(\rho, \pi)$ be an ϵ-extension of $\mathfrak{p}(V)$ associated with a pair (ρ, π). We say that $\mathfrak{g}$ is an ϵN-***extended Poincaré algebra** *if ρ is a sum of $N = 0, 1, 2, \ldots$ irreducible spin 1/2 representations, i.e. irreducible spinor or semi-spinor representations.*

The purpose of this paper is to classify all N-extended ($N \in \mathbb{Z}$) Poincaré algebras. Before starting this classification we explain how, given a (super) extension of the Poincaré algebra, we can construct more complicated ϵ-algebras.

1.2. Internal symmetries and charges.

Definition 1.3. *Let $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ be an ϵ-algebra. An* **internal symmetry** *of $\mathfrak{g}$ is an automorphism of $\mathfrak{g}$ which acts trivially on $\mathfrak{g}_0$.*

Now we give a simple construction which associates with an ϵ-extension $\mathfrak{g} = \mathfrak{g}(\rho, \pi)$ of the Poincaré algebra $\mathfrak{p}(V)$ and $l \in \mathbb{N}$ an ϵ-extension $\mathfrak{g}^{(+l)}$ and also a $-\epsilon$-extension $\mathfrak{g}^{(-2l)}$ which admit $O(l)$, respectively, $Sp(2l, \mathbb{R})$ as internal symmetry groups. We define $\mathfrak{g}^{(+l)} = \mathfrak{g}(\rho^{(+l)}, \pi^{(+l)})$, where

$$\rho^{(+l)} = l\rho : \mathfrak{so}(V) \to lW = W \otimes \mathbb{R}^l ,$$

$$\pi^{(+l)}(w_1 \otimes v_1, w_2 \otimes v_2) = \pi(w_1, w_2) < v_1, v_2 > ,$$

$< \cdot, \cdot >$ is the standard Euclidean scalar product on $\mathbb{R}^l$. Similarly, we define

$$\mathfrak{g}^{(-2l)} = 2l\rho : \mathfrak{so}(V) \to 2lW = W \otimes \mathbb{R}^{2l} ,$$

$$\pi^{(-2l)}(w_1 \otimes v_1, w_2 \otimes v_2) = \pi(w_1, w_2)\omega(v_1, v_2) ,$$

where ω is the standard symplectic form on $\mathbb{R}^{2l}$. Here we have used the convention that $\pi(w_1, w_2) = \pi(w_1 \vee w_2)$ if $\epsilon = +1$ and $\pi(w_1, w_2) = \pi(w_1 \wedge w_2)$ if $\epsilon = -1$.

Proposition 1.2. *If $\mathfrak{g}$ is an ϵ-extension of the Poincaré algebra $\mathfrak{p}(V)$, then $\mathfrak{g}^{(+l)}$ is an ϵ-extension and $\mathfrak{g}^{(-2l)}$ is a $-\epsilon$-extension. The standard actions of $O(l)$ (respectively $Sp(2l, \mathbb{R})$) on $\mathbb{R}^l$ (respectively $\mathbb{R}^{2l}$) are naturally extended to actions on $\mathfrak{g}^{(+l)}$ (respectively $\mathfrak{g}^{(-2l)}$) by internal symmetries.*

Proof. The first statement follows immediately from Prop. 1.1 and the remark that the bilinear map $\pi^{(+l)}$ (respectively $\pi^{(-2l)}$) has the same (respectively the opposite) symmetry as π. The last statement is immediate. □

Example 1: Applying this construction to an ϵ-extended (see Def. 1.2) Poincaré algebra, we obtain an ϵl-extended Poincaré algebra and also an $-\epsilon 2l$-extended Poincaré algebra with internal symmetry groups $O(l)$ and $Sp(2l, \mathbb{R})$ respectively.

Definition 1.4. *A $\mathbb{Z}_2$-graded Lie algebra (respectively a super algebra) $\mathfrak{g} = \mathfrak{g}_0 + \mathfrak{g}_1$ is called a* **charged extension** *(respectively a* **charged super extension***) of the Poincaré algebra $\mathfrak{p}(V)$ if*

1) $\mathfrak{g}_0 = \mathfrak{p}(V) + C$ is a trivial extension of $\mathfrak{p}(V)$, i.e. $[C, C] = 0$.

2) *The action of $V + C$ on the $\mathfrak{g}_0$-module $W = \mathfrak{g}_1$ is trivial.*
3) *The Lie (respectively super) bracket $\pi : \wedge^2 W \to \mathfrak{g}_0$ (respectively $\vee^2 W \to \mathfrak{g}_0$) is a sum $\pi = \pi_V + \pi_C$, where $\pi_V : \wedge^2 W \to V$ and $\pi_C : \wedge^2 W \to C$ (respectively $\pi_V : \vee^2 W \to V$ and $\pi_C : \vee^2 W \to C$). In particular, $(\mathfrak{p}(V)+W, \pi_V)$ is an extension (respectively super extension) of $\mathfrak{p}(V)$.*

If moreover, $[\mathfrak{so}(V), C] = 0$, and hence $[C, \mathfrak{g}] = 0$, then $\mathfrak{g}$ is called a **central charge extension** *(respectively a* **central charge super extension***) of $\mathfrak{p}(V)$.*

Let an extension (respectively super extension) $\mathfrak{p}(V) + W$ admitting a connected Lie group H of internal symmetries be given. Without restriction of generality we can assume that H is simply connected and we denote the Lie algebra of H by $\mathfrak{h}$. To construct a charged extension (respectively super extension) $(\mathfrak{p}(V)+C)+W$ preserving the internal symmetry group H it is necessary and sufficient to define an $(\mathfrak{so}(V) + \mathfrak{h})$-equivariant map π_C from the exterior (respectively symmetric) square of W to an $(\mathfrak{so}(V) + \mathfrak{h})$-module C.

Example 2. Let $\mathfrak{p}(V)+W$ be an extension of $\mathfrak{p}(V)$. Consider the extension $\mathfrak{g}^{(+l)} = \mathfrak{p}(V)+ W \otimes \mathbb{R}^l$ with internal symmetry group $H = O(l)$ defined above. Let $h \in \vee^2 W^* \otimes \mathbb{R}^r$ be a symmetric $\mathfrak{so}(V)$-invariant (possibly trivial) vector valued bilinear form on W and $\eta \in \wedge^2 W^* \otimes \mathbb{R}^s$ a skew symmetric such form. Define

$$\pi_C : \wedge^2(W \otimes \mathbb{R}^l) \to C = \mathbb{R}^r \otimes \wedge^2\mathbb{R}^l + \mathbb{R}^s \otimes \vee^2\mathbb{R}^l\,,$$

$$\pi_C(w_1 \otimes x_1, w_2 \otimes x_2) = h(w_1, w_2) x_1 \wedge x_2 + \eta(w_1, w_2) x_1 \vee x_2\,,$$

where $w_1, w_2 \in W$ and $x_1, x_2 \in \mathbb{R}^l$. Then π_C defines on $(\mathfrak{p}(V) + C) + W \otimes \mathbb{R}^l$ the structure of central charge extension of $\mathfrak{p}(V)$ with symmetry group $O(l)$.

Analogously, we can define on $(\mathfrak{p}(V)+C)+W\otimes\mathbb{R}^{2l}$, $C = \mathbb{R}^r\otimes\vee^2\mathbb{R}^{2l}+\mathbb{R}^s\otimes\wedge^2\mathbb{R}^{2l}$, the structure of central charge super extension of $\mathfrak{p}(V)$ with symmetry group $Sp(2l, \mathbb{R})$ by

$$\pi_C : \vee^2(W \otimes \mathbb{R}^{2l}) \to C\,,$$

$$\pi_C(w_1 \otimes x_1, w_2 \otimes x_2) = h(w_1, w_2) x_1 \vee x_2 + \eta(w_1, w_2) x_1 \wedge x_2\,.$$

Example 3. Let $\mathfrak{p}(V) + W$ be a super extension of $\mathfrak{p}(V)$. Consider the super extension $\mathfrak{g}^{(+l)} = \mathfrak{p}(V)+W\otimes\mathbb{R}^l$ with internal symmetry group $H = O(l)$ and let h be a symmetric and η a skew symmetric vector valued $\mathfrak{so}(V)$-invariant bilinear form on W, as above. Define

$$\pi_C : \vee^2(W \otimes \mathbb{R}^l) \to C = \mathbb{R}^r \otimes \vee^2\mathbb{R}^l + \mathbb{R}^s \otimes \wedge^2\mathbb{R}^l\,,$$

$$\pi_C(w_1 \otimes x_1, w_2 \otimes x_2) = h(w_1, w_2) x_1 \vee x_2 + \eta(w_1, w_2) x_1 \wedge x_2\,.$$

Then π_C defines on $(\mathfrak{p}(V)+C)+W\otimes\mathbb{R}^l$ the structure of central charge super extension of $\mathfrak{p}(V)$ with symmetry group $O(l)$.

Analogously, we can define on $(\mathfrak{p}(V)+C)+W\otimes\mathbb{R}^{2l}$, $C = \mathbb{R}^r\otimes\wedge^2\mathbb{R}^{2l}+\mathbb{R}^s\otimes\vee^2\mathbb{R}^{2l}$ the structure of central charge extension of $\mathfrak{p}(V)$ with symmetry group $Sp(2l, \mathbb{R})$ by

$$\pi_C : \wedge^2(W \otimes \mathbb{R}^{2l}) \to C\,,$$

$$\pi_C(w_1 \otimes x_1, w_2 \otimes x_2) = h(w_1, w_2) x_1 \wedge x_2 + \eta(w_1, w_2) x_1 \vee x_2\,.$$

In the physical literature (see [F]) the expression "central charges" is used for a special case of Example 3.

1.3. Reduction of the classification of N-extended Poincaré algebras to the cases $N = \pm 1, \pm 2$. Let $\mathfrak{g} = \mathfrak{g}(\rho, \pi) = \mathfrak{p}(V) + W$ be a $\pm N$-extended Poincaré algebra, $N = 1, 2, \ldots$. Then either the spinor representation $\rho_0 : \mathfrak{so}(V) \to \mathfrak{gl}(S)$ is irreducible and $\rho = N\rho_0$, $W = NS = S \otimes \mathbb{R}^N$, or it decomposes into two irreducible subrepresentations $\rho_0 = \rho_+ + \rho_-$, $S = S^+ + S^-$ and $\rho = N_+\rho_+ + N_-\rho_-$, $W = N_+S^+ + N_-S^- = S^+ \otimes \mathbb{R}^{N_+} + S^- \otimes \mathbb{R}^{N_-}$, $N = N_+ + N_-$. The description of all ϵN-extended Poincaré algebras $\mathfrak{g}(\rho, \pi)$ reduces to the description of all $\mathfrak{so}(V)$-equivariant mappings $\pi : \wedge^2 W \to V$ if $\epsilon = -1$ and $\pi : \vee^2 W \to V$ if $\epsilon = +1$. If $\pi \neq 0$, the dual mapping defines an $\mathfrak{so}(V)$-equivariant embedding $\pi^* : V^* \hookrightarrow \wedge^2 W^*$ if $\epsilon = -1$ or $\pi^* : V^* \hookrightarrow \vee^2 W^*$ if $\epsilon = +1$. To find all such embeddings it is sufficient to determine all submodules isomorphic to V^* in $\wedge^2 W^*$ and $\vee^2 W^*$ or, equivalently, all vector submodules V in $\wedge^2 W$ and $\vee^2 W$. Tables 2 and 3 reduce this problem to the cases $N = 1$ or 2.

Table 2. Decomposition of the symmetric square of W

ρ:	$N\rho_0$	$N_+\rho_+ + N_-\rho_-$
W:	$NS = S \otimes \mathbb{R}^N$	$N_+S^+ + N_-S^- =$ $S^+ \otimes \mathbb{R}^{N_+} + S^- \otimes \mathbb{R}^{N_-}$
$\vee^2 W$	$\vee^2 S \otimes \vee^2 \mathbb{R}^N + \wedge^2 S \otimes \wedge^2 \mathbb{R}^N$	$\vee^2 S^+ \otimes \vee^2 \mathbb{R}^{N_+} + \vee^2 S^- \otimes \vee^2 \mathbb{R}^{N_-} +$ $\wedge^2 S^+ \otimes \wedge^2 \mathbb{R}^{N_+} + \wedge^2 S^- \otimes \wedge^2 \mathbb{R}^{N_-} +$ $S^+ \otimes S^- \otimes \mathbb{R}^{N_+ N_-}$

Table 3. Decomposition of the exterior square of W

ρ:	$N\rho_0$	$N_+\rho_+ + N_-\rho_-$
W:	$NS = S \otimes \mathbb{R}^N$	$N_+S^+ + N_-S^- =$ $S^+ \otimes \mathbb{R}^{N_+} + S^- \otimes \mathbb{R}^{N_-}$
$\wedge^2 W$	$\wedge^2 S \otimes \vee^2 \mathbb{R}^N + \vee^2 S \otimes \wedge^2 \mathbb{R}^N$	$\wedge^2 S^+ \otimes \vee^2 \mathbb{R}^{N_+} + \wedge^2 S^- \otimes \vee^2 \mathbb{R}^{N_-} +$ $\vee^2 S^+ \otimes \wedge^2 \mathbb{R}^{N_+} + \vee^2 S^- \otimes \wedge^2 \mathbb{R}^{N_-} +$ $S^+ \otimes S^- \otimes \mathbb{R}^{N_+ N_-}$

If ρ_+ and ρ_- are equivalent then $\rho = N_+\rho_+ + N_-\rho_- \cong N\rho_0$, $\rho_0 \cong \rho_\pm$,

$$\begin{aligned} \vee^2 W &\cong \vee^2 S_0 \otimes \vee^2 \mathbb{R}^N + \wedge^2 S_0 \otimes \wedge^2 \mathbb{R}^N , \\ \wedge^2 W &\cong \vee^2 S_0 \otimes \wedge^2 \mathbb{R}^N + \wedge^2 S_0 \otimes \vee^2 \mathbb{R}^N , \end{aligned}$$

where $S_0 \cong S^\pm$ and $N = N_+ + N_-$. Table 2 shows that the classification of all equivariant embeddings $V \hookrightarrow \vee^2 W$ (case $\epsilon = +1$) reduces to finding all equivariant embeddings $V \hookrightarrow \vee^2 S$ and $V \hookrightarrow \wedge^2 S$ if S is irreducible and equivariant embeddings $V \hookrightarrow \vee^2 S^\pm$, $V \hookrightarrow \wedge^2 S^\pm$ and $V \hookrightarrow S^+ \otimes S^-$ if $S = S^+ + S^-$. Table 3 shows that the same reduction applies to the case $\epsilon = -1$, i.e. to the problem of finding all equivariant embeddings $V \hookrightarrow \wedge^2 S$. We see that e.g. the classification of N-extended Poincaré algebras for $N > 0$ (i.e. super algebra extensions) reduces to the classification of $N = \pm 1$-extended Poincaré algebras in case there is only one irreducible spin 1/2 representation of $\mathfrak{so}(V)$. The same is true for $N < 0$, i.e. for Lie algebra extensions.

To illustrate this reduction we consider the case $\epsilon = +1$ and $\rho = N\rho_0$ in more detail.

Lemma 1.1. *Assume $\epsilon = +1$ and $\rho = N\rho_0$, where ρ_0 is an irreducible spin 1/2 representation on S_0. Then any $\mathfrak{so}(V)$-equivariant embedding*

$$j : V \hookrightarrow \vee^2 W = \vee^2 S_0 \otimes \vee^2 \mathbb{R}^N + \wedge^2 S_0 \otimes \wedge^2 \mathbb{R}^N$$

is given by

$$j(v) = \sum_a \phi_a(v) \otimes A_a + \sum_b \psi_b(v) \otimes B_b ,$$

where $\phi_a : V \to \vee^2 S_0$ and $\psi_b : V \to \wedge^2 S_0$ are equivariant embeddings, $A_a \in \vee^2 \mathbb{R}^N$ and $B_b \in \wedge^2 \mathbb{R}^N$.

Proof. Choose bases (A_a) and (B_b) of $\vee^2\mathbb{R}^N$ and $\wedge^2\mathbb{R}^N$ respectively. Then $j(v)$ can be decomposed as above and the coefficients ϕ_a and ψ_b are equivariant embeddings or zero. □

1.4. Equivariant embeddings $V^* \hookrightarrow S^* \otimes S^*$, modified Clifford multiplications and Dirac operators. We reduced the problem of the classification of N-extended Poincaré algebras to the description of $\mathfrak{so}(V)$-equivariant mappings $V^* \to S^* \otimes S^*$, where S is the spinor module of $\mathfrak{so}(V)$. We will denote by $\mathcal{J}$ the vector space of all such mappings.

Now we will show that this space is closely related to two other vector spaces:

- the space $\mathcal{B}$ of all $\mathfrak{so}(V)$-invariant bilinear forms on S, and
- the space $\mathcal{M}$ of $\mathfrak{so}(V)$-equivariant multiplications $\mu : V^* \otimes S \to S$.

Denote by $\mathcal{C}$ the **Schur algebra** of $\mathfrak{so}(V)$-invariant endomorphisms of S. We define two natural anti-representations of $\mathcal{C}$ on $\mathcal{B}$ and $\mathcal{J}$ and also a representation and an anti-representation of $\mathcal{C}$ on $\mathcal{M}$ by:

$$\begin{aligned}
\xi_A^{\mathcal{B}}\beta &= \beta(A\cdot,\cdot),\\
\eta_A^{\mathcal{B}}\beta &= \beta(\cdot,A\cdot),\\
(\xi_A^{\mathcal{J}} j)(v^*) &= \xi_A^{\mathcal{B}}(j(v^*)),\\
(\eta_A^{\mathcal{J}} j)(v^*) &= \eta_A^{\mathcal{B}}(j(v^*)),\\
(\xi_A^{\mathcal{M}}\mu)(v^*) &= A\circ\mu(v^*),\\
(\eta_A^{\mathcal{M}}\mu)(v^*) &= \mu(v^*)\circ A,
\end{aligned}$$

where $A \in \mathcal{C}$, $v^* \in V^*$, $\beta \in \mathcal{B}$, $j \in \mathcal{J}$ and $\mu \in \mathcal{M} \subset Hom(V^*, End\,S)$. Remark that a non zero equivariant mapping $j : V^* \to S^* \otimes S^*$ is automatically an embedding.

Definition 1.5. *An equivariant embedding $j : V^* \to S^* \otimes S^*$ is called* **non-degenerate**, *if $j(V^*)S = S^*$ and $j(S) \cong S$, where we consider j as mapping $j : S \to V \otimes S^*$. An equivariant multiplication $\mu : V^* \otimes S \to S$ is called* **non-degenerate**, *if $\mu(V^*)S = S$.*

Using the following identifications, we define mappings from two of the spaces $\mathcal{B}$, $\mathcal{J}$ and $\mathcal{M}$ into the third:

$$\begin{aligned}
\mathcal{B} &= (S^*\otimes S^*)^{\mathfrak{so}(V)},\\
\mathcal{J} &= Hom(V^*, S^*\otimes S^*)^{\mathfrak{so}(V)} \overset{(*)}{\cong} Hom(S, V^*\otimes S^*)^{\mathfrak{so}(V)},\\
\mathcal{M} &= Hom(V^*\otimes S, S)^{\mathfrak{so}(V)} \cong Hom(V^*, End\,S)^{\mathfrak{so}(V)}\\
&\cong Hom(V^*\otimes S^*, S^*)^{\mathfrak{so}(V)}.
\end{aligned}$$

At $(*)$ we used the metric identification $V^* \cong V$. The mappings are defined as follows:

$$\begin{array}{rcl} \mathcal{B}\times\mathcal{M} & \to & \mathcal{J} \\ (\beta,\mu) & \mapsto & j(\beta,\mu)=\beta\circ\mu \\ j(\beta,\mu)(v^*) & = & \beta(\mu(v^*)\cdot,\cdot)\,, \quad v^*\in V^*\,; \\ \mathcal{M}\times\mathcal{J} & \to & \mathcal{B} \\ (\mu,j) & \mapsto & \beta(\mu,j)=\mu\circ j\,, \\ \beta(\mu,j)(s,t) & = & <\mu(j(s)),t>\,, \quad s,t\in S\,; \\ \mathcal{B}\times\mathcal{J} & \to & \mathcal{M} \\ (\beta,j) & \mapsto & \mu(\beta,j)=\beta\circ j \\ \mu(\beta,j)(v^*) & = & \beta(j(v^*)\cdot,\cdot)\in S\otimes S^*\cong End\,S\,, \end{array}$$

where $<\cdot,\cdot>$ denotes the natural duality pairing $S^*\times S\to\mathbb{R}$ and for the last mapping we have used that $j(v^*)\in S^*\otimes S^*\cong Hom(S^*,S)$.

Theorem 1.1. *The choice of a non-degenerate element β_0, j_0 or μ_0 in any of the spaces $\mathcal{B}$, $\mathcal{J}$ and $\mathcal{M}$ defines vector space isomorphisms between the two others:*

$$\begin{array}{rcl} j_{\beta_0}:\mathcal{M} & \to & \mathcal{J} \\ \mu & \mapsto & j(\beta_0,\mu)=\beta_0\circ\mu\,, \\ \mu_{\beta_0}:\mathcal{J} & \to & \mathcal{M} \\ j & \mapsto & \mu(\beta_0,j)=\beta_0\circ j\,; \\ \beta_{j_0}:\mathcal{M} & \to & \mathcal{B} \\ \mu & \mapsto & \beta(\mu,j_0)=\mu\circ j_0\,, \\ \mu_{j_0}:\mathcal{B} & \to & \mathcal{M} \\ \beta & \mapsto & \mu(\beta,j_0)=\beta\circ j_0\,; \\ j_{\mu_0}:\mathcal{B} & \to & \mathcal{J} \\ \beta & \mapsto & j(\beta,\mu_0)=\beta\circ\mu_0\,, \\ \beta_{\mu_0}:\mathcal{J} & \to & \mathcal{B} \\ j & \mapsto & \beta(\mu_0,j)=\mu_0\circ j\,. \end{array}$$

Proof. The statement is trivial for j_{β_0} and μ_{β_0}, because these mappings amount to "raising and lowering" indices of tensors via the non-degenerate form β_0.

It is clear that μ_{j_0} and j_{μ_0} are injective, since j_0 and μ_0 are non-degenerate. Hence, it is sufficient to prove that β_{j_0} and β_{μ_0} are injective.

Consider first $\beta_{\mu_0}(j)=\mu_0\circ j$, where $j:S\to V^*\otimes S^*$ and $\mu_0:V^*\otimes S^*\to S^*$. The kernel of β_{μ_0} equals

$$ker\,\beta_{\mu_0}=\{j\in\mathcal{J}|j(S)\subset ker\,\mu_0\}.$$

If $0\neq j\in ker\,\beta_{\mu_0}$, then $ker\,\mu_0$ contains the non-trivial submodule $j(S)$. This is impossible, because $ker\,\mu_0$ does not contain spin 1/2 submodules. Indeed, after complexification the $\mathfrak{so}(V^{\mathbb{C}})$-module $(V^*)^{\mathbb{C}}\otimes(S^*)^{\mathbb{C}}$ has the decomposition

$$(V^*)^{\mathbb{C}}\otimes(S^*)^{\mathbb{C}}=\Sigma\oplus(S^*)^{\mathbb{C}}=(ker\,\mu_0^{\mathbb{C}})\oplus(S^*)^{\mathbb{C}}\,,$$

where $\Sigma=ker\,\mu_0^{\mathbb{C}}$ contains only spin 3/2 modules, i.e. Kronecker product of the vector module $V^{\mathbb{C}}\cong(V^*)^{\mathbb{C}}$ (spin 1) and an irreducible spin 1/2 module.

Consider now $\beta_{j_0}(\mu) = \mu \circ j_0$, where $j_0 : S \to V^* \otimes S^*$ and $\mu : V^* \otimes S^* \to S^*$. As before we have the decomposition $(V^*)^{\mathbb{C}} \otimes (S^*)^{\mathbb{C}} = \Sigma \oplus (S^*)^{\mathbb{C}}$, where Σ has no submodules isomorphic to submodules of $(S^*)^{\mathbb{C}}$. If $\mu \neq 0$, $ker\, \mu^{\mathbb{C}} = \Sigma \oplus S_1^{\mathbb{C}}$, where $S_1^{\mathbb{C}} \neq (S^*)^{\mathbb{C}}$ is a proper submodule of $(S^*)^{\mathbb{C}}$. Since j_0 is non-degenerate $j_0(S) \cong S$ cannot be contained in $ker\, \mu$. □

Lemma 1.2. *Let S be the spinor module of $\mathfrak{so}(V)$. There always exists a non-degenerate $\mathfrak{so}(V)$-invariant bilinear form β on S.*

Proof. The existence of β is equivalent to the self duality of S, i.e. to the condition $S^* \cong S$ as $\mathfrak{so}(V)$-modules.

The self duality of the complex $\mathfrak{so}(V^{\mathbb{C}})$ spinor module $\mathbb{S}$ follows from the criterion of self duality given in [O-V], p. 195.

Now we discuss the real case. Assume first $S^{\mathbb{C}}$ has the same number of irreducible summands as S. Then the self duality of S follows from that of $S^{\mathbb{C}}$, see [O-V], p. 291. In the opposite case S admits an invariant complex structure J and $(S, J) \cong \mathbb{S}$ (complex spinor module of $\mathfrak{so}(V^{\mathbb{C}})$). Then the real part of a non-degenerate complex $\mathfrak{so}(V^{\mathbb{C}})$-invariant bilinear form on $S = \mathbb{S}$ gives a real $\mathfrak{so}(V)$-invariant bilinear form on S and it is easy to check that this form is non-degenerate. □

From Theorem 1.1 and this lemma we now derive an important consequence. Recall that by definition the spinor module S is an irreducible module over the Clifford algebra $C\ell(V)$. The restriction of the multiplication mapping $C\ell(V) \times S \to S$ to $V \times S$ defines a non-degenerate $\mathfrak{so}(V)$-equivariant multiplication $\rho : V \otimes S \cong V^* \otimes S \to S$, which is called Clifford multiplication (as above V and V^* are identified using the pseudo-Euclidean scalar product of V). The composition $j(\beta, \rho) = \beta \circ \rho$ with a non-degenerate $\mathfrak{so}(V)$-invariant form β gives a non-degenerate $\mathfrak{so}(V)$-equivariant embedding $V^* \hookrightarrow S^* \otimes S^*$. Using the lemma and this remark, we obtain the following corollary from Theorem 1.1.

Corollary 1.1. *The spaces $\mathcal{B}$ of $\mathfrak{so}(V)$-invariant bilinear forms on S, $\mathcal{J}$ of $\mathfrak{so}(V)$-equivariant mappings $V^* \to S^* \otimes S^*$ and $\mathcal{M}$ of $\mathfrak{so}(V)$-equivariant multiplications $V^* \otimes S \to S$ are isomorphic. In particular, Clifford multiplication ρ defines the isomorphism $j_\rho : \mathcal{B} \to \mathcal{J}$ and hence any $\mathfrak{so}(V)$-equivariant embedding $V^* \hookrightarrow S^* \otimes S^*$ is of the form*

$$j = j_\rho(\beta) : v^* \mapsto \beta(\rho(v^*)\cdot, \cdot)\,, \quad \beta \in \mathcal{B}\,, \quad v^* \in V^*\,.$$

Remark 2. Using an $\mathfrak{so}(V)$-equivariant multiplication $\mu : V^* \otimes S \to S$ one can define a Dirac type operator D^μ on a pseudo-Riemannian spin manifold M as follows. Let $\mu_x : T_x^* M \otimes S_x \to S_x$ be a field of equivariant multiplications, where $S(M) = \cup_{x \in M} S_x \to M$ is the spinor bundle. Then

$$(D^\mu s)_x = \mu_x(\nabla s) = \mu_x(\sum_i e^i \otimes \nabla_{e_i} s)\,,$$

where (e_i) is a basis of $T_x M$, (e^i) the dual basis of $T_x^* M$ and ∇ is the spinor connection induced by the Levi Civita connection.

1.5. $\mathbb{Z}_2$-graded type and Schur algebra $\mathcal{C}$. It is well known (see [L-M]), that every Clifford algebra $C\ell(V)$, $V = \mathbb{R}^{p,q}$, is isomorphic to $\mathbb{K}(l)$ or to $2\mathbb{K}(l) = \mathbb{K}(l) \oplus \mathbb{K}(l)$, where $\mathbb{K}(l)$ is the full matrix algebra over $\mathbb{K}$ of rank l depending on (p, q) and where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$.

Definition 1.6. *We say that a Clifford algebra $C\ell(V)$ has* **type** *$r\mathbb{K}$, $r = 1$ or 2, if $C\ell(V) \cong r\mathbb{K}(l)$ for some $l \in \mathbb{N}$.*

Recall that the Clifford algebra $C\ell(V)$ has a natural $\mathbb{Z}_2$-grading $C\ell(V) = C\ell^0(V) + C\ell^1(V)$. If $V = \mathbb{R}^{p,q}$ ($\neq 0$), then the even part $C\ell^0(V)$ is isomorphic to the Clifford algebra $C\ell(V')$ of $V' = \mathbb{R}^{p-1,q}$ if $p \geq 1$ and $V' = \mathbb{R}^{q-1}$ if $p = 0$. Remark that $\dim C\ell^0(V) = \dim C\ell(V)/2$. By the preceding remarks, the following definition makes sense.

Definition 1.7. *The pair $t(C\ell(V)) = (r_0\mathbb{K}_0, r\mathbb{K}) = (type\ C\ell^0(V), type\ C\ell(V))$ is called the $\mathbb{Z}_2$-***graded type** *of the Clifford algebra $C\ell(V)$.*

The following proposition describes the periodicity of the type t of the $\mathbb{Z}_2$-graded Clifford algebras $C\ell_{p,q} = C\ell(\mathbb{R}^{p,q})$.

Proposition 1.3. *The $\mathbb{Z}_2$-graded type $t_{p,q} = t(C\ell_{p,q})$ depends only on the signature $s = p - q$ modulo 8 and $t(s) = t(p - q) = t_{p,q}$ is given in the table.*

s	1	2	3	4	5	6	7	8
$t(s)$	$\mathbb{R}, \mathbb{C}$	$\mathbb{C}, \mathbb{H}$	$\mathbb{H}, 2\mathbb{H}$	$2\mathbb{H}, \mathbb{H}$	$\mathbb{H}, \mathbb{C}$	$\mathbb{C}, \mathbb{R}$	$\mathbb{R}, 2\mathbb{R}$	$2\mathbb{R}, \mathbb{R}$

Proof. The proof reduces to the investigation of [L-M], Table II. □

Corollary 1.2. *The $\mathbb{Z}_2$-graded type $t_{p,q} = t(s = p - q)$ is mirror symmetric with respect to the diagonal $\{p + q = 0\}$: $t_{p,q} = t_{-q,-p}$; in other words, $t(C\ell_{p,q}) = t(C\ell_{8k-q,8k-p})$, $8k \geq p, q$.*

Moreover, the $\mathbb{Z}_2$-graded type $t_{p,q} = t(s) = (t^0(s), t^1(s))$ is mirror super symmetric with respect to the axis $\{s = p - q = 3.5\}$, i.e.

$$(t^0(7 - s), t^1(7 - s)) = (t^1(s), t^0(s))\,.$$

The type $r\mathbb{C}$ and $\mathbb{Z}_2$-graded type $t_m = (r_0\mathbb{C}, r\mathbb{C})$ of a complex Clifford algebra $\mathbb{C}\ell_m = \mathbb{C}\ell(\mathbb{C}^m)$ are defined by putting $V = \mathbb{C}^m$ in Definition 1.6 and 1.7, where $\mathbb{C}^m$ is equipped with a non-degenerate (complex) bilinear form, e.g. the standard one: $< z, w >= \sum_{j=1}^m z_j w_j$, $z, w \in \mathbb{C}^m$.

Proposition 1.4. *The $\mathbb{Z}_2$-graded type $t_m = t(\mathbb{C}\ell_m)$ depends only on the parity of m:*

$$t_m = \begin{cases} (2\mathbb{C}, \mathbb{C}) & \textit{if} \quad m \textit{ is even} \\ (\mathbb{C}, 2\mathbb{C}) & \textit{if} \quad m \textit{ is odd} \end{cases}$$

Let $S = S_{p,q}$ be an irreducible $C\ell_{p,q}$-module. Recall that by definition the Schur algebra $\mathcal{C} = \mathcal{C}_{p,q}$ of S is the algebra of all its $\mathfrak{so}(V)$-invariant endomorphisms; it is the algebra of endomorphisms which commute with $C\ell^0_{p,q}$. Analogously, we define the Schur algebra $\mathcal{C}^c_m$ of the complex spinor module $\mathbb{S}$; it is the algebra of endomorphism of $\mathbb{S}$ commuting with $\mathbb{C}\ell^0_m$.

Corollary 1.3. *The Schur algebra $C_{p,q} = C(p-q)$ depends only on $s = p-q$ modulo 8 and is given in the table. In particular, it admits the mirror symmetry $(p,q) \mapsto (-q,-p)$.*

s	1	2	3	4	5	6	7	8
$C(s)$	$\mathbb{R}(2)$	$\mathbb{C}(2)$	$\mathbb{H}$	$\mathbb{H}\oplus\mathbb{H}$	$\mathbb{H}$	$\mathbb{C}$	$\mathbb{R}$	$\mathbb{R}\oplus\mathbb{R}$

Proof. Remark that if $t(C\ell_{p,q}) = (r_0\mathbb{K}_0, r\mathbb{K})$, and hence $C\ell^0_{p,q} \cong r_0\mathbb{K}_0(l_0)$, $C\ell_{p,q} \cong r\mathbb{K}(l)$, then l is completely determined by l_0 and vice versa; $l = l_0$ or $2l_0$. This follows from $\dim C\ell_{p,q} = 2\dim C\ell^0_{p,q}$.

Using this remark, Proposition 1.3 shows that the pair $(Cl^0_{p,q}, Cl_{p,q})$ is isomorphic to one of the following:

$$\begin{array}{lcl} (\mathbb{K}(l), \mathbb{K}'(l)) & , & S = \mathbb{K}'^{l}\,, \\ (\mathbb{K}(l), 2\mathbb{K}(l)) & , & S = \mathbb{K}^{l}\,, \\ (\mathbb{K}'(l), \mathbb{K}(2l)) & , & S = \mathbb{K}^{2l}\,, \\ (2\mathbb{K}(l), \mathbb{K}(2l)) & , & S = \mathbb{K}^{2l}\,, \end{array}$$

where $\mathbb{K} = \mathbb{R}, \mathbb{C}$ or $\mathbb{H}$ and $\mathbb{R}' = \mathbb{C}$, $\mathbb{C}' = \mathbb{H}$.

In the first case the $\mathbb{K}(l)$-module $S = \mathbb{K}'^{l}$ is a sum of two irreducible equivalent modules $S^{\pm} \cong \mathbb{K}^l$ and hence the Schur algebra $C \cong \mathbb{K}(2)$.

In the second (respectively third) case $S = \mathbb{K}^l$ (respectively $\mathbb{K}^{2l}$) is irreducible as $\mathbb{K}(l)$- (respectively $\mathbb{K}'(l)$-) module and hence $C \cong \mathbb{K}$ (respectively $\mathbb{K}'$).

In the last case $C \cong \mathbb{K} \oplus \mathbb{K}$, which follows from the next lemma. □

Lemma 1.3. *Let $S = \mathbb{K}^{2l}$ be the irreducible module of the algebra $\mathbb{K}(2l)$ and $\mathcal{A} \cong 2\mathbb{K}(l)$ a subalgebra of $\mathbb{K}(2l)$, then the $\mathcal{A}$-module S is decomposed into a sum of two nonequivalent submodules $S^{\pm}$.*

Proof. It is clear that the $\mathcal{A}$-module S is the sum of two irreducible submodules S^+ and S^-. They are not equivalent because $\mathcal{A}|S^+$ and $\mathcal{A}|S^-$ have different kernels, namely the two ideals $\mathbb{K}(l) \subset \mathcal{A}$. □

Remark that the algebras $\mathbb{C}\oplus\mathbb{C}$ and $\mathbb{H}(2)$ do not occur as Schur algebras of the real spinor module S.

Corollary 1.4. *The Schur algebra C^c_m of the complex spinor module $\mathbb{S}$ depends only on the parity of m:*

$$C^c_m = \begin{cases} \mathbb{C}\oplus\mathbb{C} & \text{if } m \text{ is even} \\ \mathbb{C} & \text{if } m \text{ is odd} \end{cases}.$$

The proof of Corollary 1.3 shows that the structure of the matrix algebra C contains the following information about the $C\ell^0(V)$-module S.

Proposition 1.5. *C is a simple $\mathbb{K}$-matrix algebra (respectively a sum of two isomorphic $\mathbb{K}$-matrix algebras) if and only if $C\ell^0(V)$ is a simple $\mathbb{K}$-matrix algebra (respectively a sum of two isomorphic such algebras). S is an irreducible $C\ell^0(V)$-module if and only if $C \cong \mathbb{K}(= \mathbb{R}, \mathbb{C}$ or $\mathbb{H})$. S is decomposed into a sum of two equivalent (respectively inequivalent) $C\ell^0(V)$-modules if and only if $C \cong \mathbb{K}(2)$ (respectively $C \cong \mathbb{K}\oplus\mathbb{K}$).*

The corresponding statement in the complex case is given for the sake of completeness:

Proposition 1.6. *If m is even, then the spinor module $\mathbb{S} = \mathbb{S}_m$ is the sum $\mathbb{S} = \mathbb{S}^+ + \mathbb{S}^-$ of two inequivalent irreducible $\mathbb{C}l_m^0$-modules. In this case, $\mathbb{C}l_m^0$ and the Schur algebra $\mathcal{C}_m^c$ are the direct sum of two isomorphic simple (complex) matrix algebras.*

If m is odd, then the spinor module is an irreducible module of the simple matrix algebra $\mathbb{C}l_m^0$ and its Schur algebra is also simple.

Since, due to Lemma 1.2, S admits a non-degenerate $\mathfrak{so}(p,q)$-invariant bilinear form, by Schur's Lemma the dimension $b_{p,q}$ of the space $\mathcal{B} = \mathcal{B}_{p,q}$ of $\mathfrak{so}(p,q)$-invariant bilinear forms on S equals

$$b_{p,q} = \dim \mathcal{B}_{p,q} = \dim \mathcal{C}_{p,q} .$$

Hence we have:

Corollary 1.5. *$b_{p,q} = b(p-q)$ is a periodic function of $s = p - q$ with period 8. In particular, it admits the mirror symmetry $(p,q) \mapsto (-q,-p)$. Its values are given in the following table:*

s	1	2	3	4	5	6	7	8
$b(s)$	4	8	4	8	4	2	1	2

Denote by b_m the (complex) dimension of the space of $\mathfrak{so}(m,\mathbb{C})$-invariant bilinear forms on the complex spinor module $\mathbb{S}$, then $b_m = \dim_{\mathbb{C}} \mathcal{C}_m^c$ and we have:

$$b_m = \begin{cases} 2 & \text{if} \quad m \text{ is even} \\ 1 & \text{if} \quad m \text{ is odd.} \end{cases}$$

2. Fundamental Invariants τ, σ and ι and Reduction to the Basic Signatures (m,m), $(k,0)$ and $(0,k)$

2.1. Fundamental invariants. As before let V denote a pseudo-Euclidean vector space and S its spinor module. In Corollary 1.1 we have established that every $\mathfrak{so}(V)$-equivariant embedding $j : V^* \hookrightarrow S^* \otimes S^*$ is of the form

$$j = j_\rho(\beta) : v^* \mapsto \beta(\rho(v^*)\cdot, \cdot), \quad v^* \in V^* ,$$

where ρ is Clifford multiplication and $\beta \in \mathcal{B}$. The dimension of the space $\mathcal{B}$ of $\mathfrak{so}(V)$-invariant bilinear forms on S was given in Corollary 1.5.

Now we will concentrate on a class of bilinear forms $\beta \in \mathcal{B}$ for which $j_\rho(\beta)V^* \subset \vee^2 S^*$ or $j_\rho(\beta)V^* \subset \wedge^2 S^*$ and define fundamental invariants τ, σ and ι for this class.

Definition 2.1. *A bilinear form β on the spinor module S is called* **admissible** *if it has the following properties:*

1) *Clifford multiplication $\rho(v)$, $v \in V$, is either β-symmetric or β-skew symmetric. We define the* **type** *τ of β to be $\tau(\beta) = +1$ in the first case and $\tau(\beta) = -1$ in the second.*
2) *The bilinear form β is symmetric or skew symmetric. Accordingly, we define the* **symmetry** *σ of β to be $\sigma(\beta) = \pm 1$.*

3) *If the spinor module is reducible,* $S = S^+ + S^-$*, then* $S^\pm$ *are either mutually orthogonal or isotropic. We put* $\iota(\beta) = +1$ *in the first case,* $\iota(\beta) = -1$ *in the second and call* $\iota(\beta)$ *the* **isotropy** *of* β.

Due to 1) every admissible form β is $\mathfrak{so}(V)$-invariant and hence defines an $\mathfrak{so}(V)$-equivariant embedding $j_\rho(\beta) : V \cong V^* \hookrightarrow S^* \otimes S^*$. In addition, $j_\rho(\beta)V \subset \vee^2 S^*$ if $\tau(\beta)\sigma(\beta) = +1$ and $j_\rho(\beta)V \subset \wedge^2 S^*$ if $\tau(\beta)\sigma(\beta) = -1$. If $S = S^+ + S^-$, then for every bilinear form $\gamma \in j_\rho(\beta)V$ the semi spinor modules $S^\pm$ are either γ-isotropic (if $\iota(\gamma) = -\iota(\beta) = -1$) or mutually γ-orthogonal (if $\iota(\gamma) = -\iota(\beta) = +1$).

Given an admissible form $\beta \in \mathcal{B}$ and $A \in \mathcal{C}$, the composition $\beta \circ A = \beta(A\cdot, \cdot) \in \mathcal{B}$ is in general not admissible. However, if A is β-admissible (see Definition 2.2 below) then $\beta \circ A$ is admissible.

Definition 2.2. *Let* $\beta \in \mathcal{B}$ *be admissible. An endomorphism* A *of* S *is called* β-**admissible** *if it has the following properties:*

1) *Clifford multiplication* $\rho(v)$, $v \in V$, *either commutes or anticommutes with* A. *We define the* **type** τ *of* A *to be* $\tau(A) = +1$ *in the first case and* $\tau(A) = -1$ *in the second.*
2) A *is* β*-symmetric or* β*-skew symmetric. Accordingly, we define the* β-**symmetry** σ *of* A *to be* $\sigma_\beta(A) = \pm 1$.
3) *If the spinor module is reducible,* $S = S^+ + S^-$*, then either* $AS^\pm \subset S^\pm$ *or* $AS^\pm \subset S^\mp$. *We put* $\iota(A) = +1$ *in the first case,* $\iota(A) = -1$ *in the second and call* $\iota(A)$ *the* **isotropy** *of* A.

Due to 1) every β-admissible endomorphism A is $\mathfrak{so}(V)$-invariant and hence $\beta \circ A \in \mathcal{B}$. Moreover, $\beta \circ A$ is admissible and the fundamental invariants are multiplicative:

$$\begin{aligned} \tau(\beta \circ A) &= \tau(\beta)\tau(A), \\ \sigma(\beta \circ A) &= \sigma(\beta)\sigma(A), \\ \iota(\beta \circ A) &= \iota(\beta)\iota(A). \end{aligned}$$

In Sect. 3.1 (see Definition 3.1), for every pseudo-Euclidean space V, we will construct a canonical non-degenerate $\mathfrak{so}(V)$-invariant bilinear form h on the spinor module S. We will define that an endomorphism A of S is admissible of symmetry $\sigma(A) = \pm 1$, if A is h-admissible and $\sigma_h(A) = \pm 1$.

Remark 3. The complete classification of admissible forms $\beta \in \mathcal{B}$, which we will give in this paper, implies the following. Let $\gamma \in \mathcal{B}$ be non-degenerate and admissible. Then a γ-admissible endomorphism $A \in \mathcal{C}$ is β-admissible for every admissible $\beta \in \mathcal{B}$. In particular, admissibility (i.e. h-admissibility) implies β-admissibility.

2.2. Reduction to the basic signatures. Let V_1 and V_2 be pseudo-Euclidean spaces and $V = V_1 + V_2$ their orthogonal sum. We recall (see [L-M] I. Prop. 1.5) that there is a canonical isomorphism of $\mathbb{Z}_2$-graded algebras

$$\mathcal{C}\ell(V) \cong \mathcal{C}\ell(V_1)\hat{\otimes}\mathcal{C}\ell(V_2),$$

where $\hat{\otimes}$ denotes the $\mathbb{Z}_2$-graded tensor product of $\mathbb{Z}_2$-graded algebras.

Proposition 2.1. *Let $M_1 = M_1^0 + M_1^1$ be a $\mathbb{Z}_2$-graded $C\ell(V_1)$-module and M_2 a (not necessarily $\mathbb{Z}_2$-graded) $C\ell(V_2)$-module. Then $M = M_1 \otimes M_2$ carries a natural structure of $C\ell(V)$-module, $V = V_1 + V_2$, given by:*

$$(a_1 \otimes a_2)(m_1 \otimes m_2) = (-1)^{\deg(a_2)\deg(m_1)} a_1 m_1 \otimes a_2 m_2 ,$$

where $a_i \in C\ell(V_i)$, $m_i \in M_i$, $i = 1, 2$. If $M_2 = M_2^0 + M_2^1$ is a $\mathbb{Z}_2$-graded $C\ell(V_2)$-module, then this formula defines on M the structure of $\mathbb{Z}_2$-graded $C\ell(V)$-module: $M^0 = M_1^0 \otimes M_2^0 + M_1^1 \otimes M_2^1$, $M^1 = M_1^0 \otimes M_2^1 + M_1^1 \otimes M_2^0$.

Corollary 2.1. *Let S_i be an irreducible $C\ell(V_i)$-module, $i = 1, 2$, and assume that $S_1 = S_1^+ + S_1^-$ is reducible as $C\ell^0(V_1)$-module. Then $S = S_1 \otimes S_2$ is an irreducible $(C\ell(V) = C\ell(V_1)\hat{\otimes}C\ell(V_2))$-module. The $C\ell^0(V)$-module S is reducible, $S = S^+ + S^-$, if and only if S_2 is reducible as $C\ell^0(V_2)$-module, $S_2 = S_2^+ + S_2^-$.*

Proof. Let S_1 be an irreducible $C\ell(V_1)$-module which is reducible as $C\ell^0(V_1)$-module and let S_1^+ be an irreducible $C\ell^0(V_1)$-submodule. Then

$$S_1' := C\ell(V_1) \otimes_{C\ell^0(V_1)} S_1^+$$

is an irreducible $C\ell(V_1)$-module, hence without restriction of generality $S_1 \cong S_1'$ as $C\ell(V_1)$-modules. Moreover, S_1' is a $\mathbb{Z}_2$-graded $C\ell(V_1)$-module (see [L-M] I. Prop. 5.20): $S_1' = S_1'^0 + S_1'^1$, $S_1'^0 = C\ell^0(V_1) \otimes_{C\ell^0(V_1)} S_1^+ \cong S_1^+$ and $S_1'^1 = C\ell^1(V_1) S_1'^0 = C\ell^1(V_1) \otimes_{C\ell^0(V_1)} S_1^+$.

Therefore, we may assume (as usual) that $S_1 = S_1^+ + S_1^-$ is a $\mathbb{Z}_2$-graded $C\ell(V_1)$-module: $S_1^0 = S_1^+$, $S_1^1 = S_1^- = C\ell^1(V_1)S_1^+$, reducing the first statement to Proposition 2.1. The remaining statements also follow from the structure of $\mathbb{Z}_2$-graded Clifford module on S_1 and on S_2 (in the reducible case). □

Now we investigate the algebraic properties of the fundamental invariants with respect to $\mathbb{Z}_2$-graded tensor products.

Proposition 2.2. *Under the assumptions of Corollary 2.1 let β_i be admissible bilinear forms on S_i, $i = 1, 2$.*

If $\tau(\beta_1) = \iota(\beta_1)\tau(\beta_2)$, then $\beta = \beta_1 \otimes \beta_2$ is admissible and

$$\begin{aligned} \tau(\beta) &= \tau(\beta_1) = \iota(\beta_1)\tau(\beta_2) , \\ \sigma(\beta) &= \sigma(\beta_1)\sigma(\beta_2) , \\ \iota(\beta) &= \iota(\beta_1)\iota(\beta_2) , \end{aligned}$$

where $\iota(\beta)$ and $\iota(\beta_2)$ are defined if and only if S_2 (and hence S) is reducible as a module of the even part of the corresponding Clifford algebra.

Let A_i be β_i-admissible endomorphisms of S_i, $i = 1, 2$. If $\tau(A_1) = \iota(A_1)\tau(A_2)$, then $A = A_1 \otimes A_2$ is admissible and

$$\begin{aligned} \tau(A) &= \tau(A_1) = \iota(A_1)\tau(A_2) , \\ \sigma_\beta(A) &= \sigma_{\beta_1}(A_1)\sigma_{\beta_2}(A_2) , \\ \iota(A) &= \iota(A_1)\iota(A_2) , \end{aligned}$$

where $\iota(A)$ and $\iota(A_2)$ are defined if and only if S_2 is reducible as $C\ell^0(V_2)$-module.

Proof. The only non-trivial statements are the ones concerning the type τ. For $s_i, t_i \in S_i$ and $v_i \in V_i$ we compute:

$$\begin{aligned} \beta((v_1 \otimes 1)(s_1 \otimes s_2), t_1 \otimes t_2) &= \beta(v_1 s_1 \otimes s_2, t_1 \otimes t_2) = \\ \beta_1(v_1 s_1, t_1)\beta_2(s_2, t_2) &= \tau(\beta_1)\beta_1(s_1, v_1 t_1)\beta_2(s_2, t_2) = \\ \tau(\beta_1)\beta(s_1 \otimes s_2, v_1 t_1 \otimes t_2) &= \tau(\beta_1)\beta(s_1 \otimes s_2, (v_1 \otimes 1)(t_1 \otimes t_2)) \end{aligned}$$

and

$$\begin{aligned} \beta((1 \otimes v_2)(s_1 \otimes s_2), t_1 \otimes t_2) &= (-1)^{\deg s_1}\beta(s_1 \otimes v_2 s_2, t_1 \otimes t_2) = \\ (-1)^{\deg s_1}\beta_1(s_1, t_1)\beta_2(v_2 s_2, t_2) &= (-1)^{\deg s_1}\tau(\beta_2)\beta_1(s_1, t_1)\beta_2(s_2, v_2 t_2) = \end{aligned}$$

$$(-1)^{\deg s_1}\tau(\beta_2)\beta(s_1 \otimes s_2, t_1 \otimes v_2 t_2) =$$

$$(-1)^{\deg s_1 + \deg t_1}\tau(\beta_2)\beta(s_1 \otimes s_2, (1 \otimes v_2)(t_1 \otimes t_2)).$$

If $\iota(\beta_1) = (-1)^{\deg s_1 + \deg t_1}$ we obtain

$$\beta((1 \otimes v_2)(s_1 \otimes s_2), t_1 \otimes t_2) = \iota(\beta_1)\tau(\beta_2)\beta(s_1 \otimes s_2, (1 \otimes v_2)(t_1 \otimes t_2)). \quad (1)$$

Otherwise, both sides of (1) vanish. Hence, Eq. (1) is always true.

Similarly we have:

$$(v_1 \otimes 1)((A_1 \otimes A_2)(s_1 \otimes s_2)) = \tau(A_1)(A_1 \otimes A_2)((v_1 \otimes 1)(s_1 \otimes s_2))$$

and

$$\begin{aligned} (1 \otimes v_2)((A_1 \otimes A_2)(s_1 \otimes s_2)) &= (1 \otimes v_2)(A_1 s_1 \otimes A_2 s_2) = \\ (-1)^{\deg(A_1 s_1)} A_1 s_1 \otimes v_2 A_2 s_2 &= (-1)^{\deg(A_1 s_1)}\tau(A_2) A_1 s_1 \otimes A_2 v_2 s_2 = \end{aligned}$$

$$(-1)^{\deg(A_1 s_1)}\tau(A_2)(A_1 \otimes A_2)(s_1 \otimes v_2 s_2) =$$

$$(-1)^{\deg(A_1 s_1) + \deg s_1}\tau(A_2)(A_1 \otimes A_2)((1 \otimes v_2)(s_1 \otimes s_2)) =$$

$$\iota(A_1)\tau(A_2)(A_1 \otimes A_2)((1 \otimes v_2)(s_1 \otimes s_2)). \quad \square$$

Now we point out that every pseudo-Euclidean space V can be decomposed as the orthogonal sum $V = V_1 + V_2$ such that the assumptions of Corollary 2.1 are satisfied, i.e. such that the spinor $C\ell^0(V_1)$-module S_1 is reducible. In fact, we can decompose V into $V_1 = \mathbb{R}^{m,m}$ and $V_2 = \mathbb{R}^{k,0}$ or $\mathbb{R}^{0,k}$.

Proposition 2.3. *Let $V = V_1 + V_2$ be the orthogonal sum of the pseudo Euclidean spaces $V_1 = \mathbb{R}^{m,m}$ and V_2. Let S_1 be an irreducible $C\ell(V_1)$-module. Then $S_1 = S_1^+ + S_1^-$ is a sum of two inequivalent irreducible $C\ell^0(V_1)$-submodules $S_1^\pm$ and an irreducible $(C\ell(V) = C\ell(V_1)\hat{\otimes}C\ell(V_2))$-module S is given by $S = S_1 \otimes S_2$, where S_2 is an irreducible $C\ell(V_2)$-module. S is reducible as $C\ell^0(V)$-module if and only if S_2 is reducible as $C\ell^0(V_2)$-module.*

Proof. The first statement follows from the fact that the Schur algebra of S_1 is $C_{m,m} = C(s = m - m = 0) = \mathbb{R} \oplus \mathbb{R}$. Now all other statements follow immediately from Corollary 2.1. $\square$

3. Case of Signature (m, m) and Complex Case

3.1. Signature (m, m). Let U and U^* denote two complementary isotropic subspaces of $V = \mathbb{R}^{m,m}$, so $V = U + U^*$. We denote by $< \cdot, \cdot >$ the scalar product of V and identify U^* with the dual space to U by

$$u^*(u) = 2< u, u^* >, \quad u^* \in U^*, u \in U.$$

Proposition 3.1. *The following formulas define an irreducible $\mathcal{C}\ell_{m,m}$-module on $S = \wedge U$:*

$$\rho(u)s = u \wedge s,$$
$$\rho(u^*)s = -u^* \angle s, \quad s \in \wedge U, \ u \in U, \ u^* \in U^*,$$

where $\angle$ is the interior multiplication.

Proof. This follows from the obvious identities $\rho(u)^2 = \rho(u^*)^2 = 0$ and $\rho(u)\rho(u^*) + \rho(u^*)\rho(u) = -2< u, u^* > Id$. □

For any $a \in \wedge U$ and $\alpha \in \wedge U^*$ we define nilpotent endomorphisms ϵ_a and ι_α of $S = \wedge U$ by:

$$\epsilon_a = a \wedge s,$$
$$\iota_\alpha = \alpha \angle s.$$

Proposition 3.2. *The Lie algebra $\mathfrak{so}(m, m) \hookrightarrow End\, S$ of the spinor group admits the following graded decomposition:*

$$\mathfrak{so}(m, m) = \mathfrak{g}^{-2} + \mathfrak{g}^0 + \mathfrak{g}^2 = \iota_{\wedge^2 U^*} + \mathfrak{sl}(U) + \epsilon_{\wedge^2 U},$$

$\mathfrak{sl}(U) = [\iota_{U^*}, \epsilon_U]$, $[\mathfrak{g}^i, \mathfrak{g}^j] \subset \mathfrak{g}^{i+j}$ *($\mathfrak{g}^{i+j} = 0$ for $|i + j| > 2$). In particular, $\iota_{\wedge^2 U^*}$ and $\epsilon_{\wedge^2 U}$ are Abelian subalgebras.*

It is very easy to describe the semi spinor modules $S^\pm$ in our model of the spinor module S.

Lemma 3.1. *$S = \wedge U$ is the sum of the two inequivalent irreducible $\mathfrak{so}(m, m)$-submodules $S^+ = \wedge^{ev} U$ and $S^- = \wedge^{odd} U$.*

Proof. It is clear that $\wedge^{ev} U$ and $\wedge^{odd} U$ are irreducible $\mathfrak{so}(m, m)$-submodules and we already know that they are inequivalent, see e.g. Proposition 2.3. □

Remark 4. The statement that $\wedge^{ev} U$ and $\wedge^{odd} U$ are inequivalent $\mathfrak{so}(m, m)$-modules follows also from the fact that these are eigenspaces of the volume element $\omega_{m,m} = e_1 \cdots e_{2m} \in \mathcal{C}\ell^0_{m,m}$, (e_i) an orthonormal basis of $\mathbb{R}^{m,m}$.

We can define an $\mathfrak{so}(m, m)$-invariant endomorphism E of S by

$$E|S^\pm = \pm Id.$$

To construct an admissible bilinear form f on $S = \wedge U$ we fix a volume form $vol \in \wedge^m U$ on U^* and define

$$f(\wedge^i U, \wedge^j U) = 0, \quad \text{if} \quad i + j \neq m,$$
$$f(s, t) vol = \epsilon_i s \wedge t, \quad s \in \wedge^i U, \ t \in \wedge^{m-i} U,$$

where $\epsilon_i = (-1)^{i(i+1)/2}$. Remark that $\epsilon_{i+1} = (-1)^{i+1}\epsilon_i$.

Proposition 3.3. *The space $\mathcal{B}$ of $\mathfrak{so}(m,m)$-invariant bilinear forms on $S = S_{m,m}$ is spanned by the admissible elements f and $f_E = f(E\cdot,\cdot)$. Their fundamental invariants (τ,σ,ι) depend only on $m \pmod 4$ and are given in the next table:*

f	$---$	$--+$	$-+-$	$-++$
f_E	$++-$	$+-+$	$+--$	$+++$
$m:$	1	2	3	4

An f- and f_E-admissible basis for the Schur algebra $\mathcal{C} \cong \mathbb{R}\oplus\mathbb{R}$ is given by the endomorphisms Id and E of S:

$$\tau(E) = -1\,, \quad \sigma_f(E) = \sigma_{f_E}(E) = (-1)^m\,, \quad \iota(E) = +1\,.$$

Proof. We first check that $\rho(v)$, $v \in U + U^*$, is f-skew symmetric. For $v = u \in U$, $s \in \wedge^i U, t \in \wedge^{m-i-1}U$:

$$(f(\rho(u)s,t) + f(s,\rho(u)t))vol = \epsilon_{i+1}(u\wedge s)\wedge t + \epsilon_i s\wedge(u\wedge t) = 0\,.$$

For $v = u^* \in U^*$, $s \in \wedge^i U, t \in \wedge^{m-i+1}U$:

$$\begin{aligned}-(f(\rho(u^*)s,t) + f(s,\rho(u^*)t))vol &= \epsilon_{i-1}(u^*\angle s)\wedge t + \epsilon_i s\wedge(u^*\angle t) = \\ &\epsilon_{i-1}(u^*\angle s)\wedge t + \epsilon_i(-1)^i(u^*\angle(s\wedge t) - (u^*\angle s)\wedge t) = \\ &(\epsilon_{i-1} - (-1)^i\epsilon_i)(u^*\angle s)\wedge t = 0\,.\end{aligned}$$

The symmetry properties of f follow from the computation

$$f(t,s)vol = \epsilon_j t\wedge s = \epsilon_j\epsilon_i(-1)^{ij}f(s,t)vol = (-1)^{m(m+1)/2}f(s,t)vol\,,$$

where $s \in \wedge^i U, t \in \wedge^j U$ and $i + j = m$.
Finally, $f(\wedge^{ev}U, \wedge^{odd}U) = 0$ if m is even and $f(\wedge^{ev}U, \wedge^{ev}) = f(\wedge^{odd}U, \wedge^{odd}U) = 0$ if m is odd. This proves all the statements about f. It is immediate to see that E is f-admissible with fundamental invariants given above. Since f is admissible and E is f-admissible, f_E is admissible and its fundamental invariants are computed by multiplicativity:

$$\tau(f_E) = \tau(f)\tau(E)\,, \quad \sigma(f_E) = \sigma(f)\sigma_f(E)\,, \quad \iota(f_E) = \iota(f)\iota(E)\,.$$

This proves the proposition. □

Proposition 3.3 implies the following theorem:

Theorem 3.1. *Every $\mathfrak{so}(m,m)$-equivariant embedding $V^* \hookrightarrow S^*\otimes S^*$, where $S = S_{m,m}$ is the spinor $\mathfrak{so}(m,m)$-module, is a linear combination of the embeddings $j_\rho(f)$ and $j_\rho(f_E)$. Their image is contained in the dual of the subspaces indicated in the table depending on $m \pmod 4$.*

$j_\rho(f)$	$\vee^2S^+ + \vee^2S^-$	$S^+\vee S^-$	$\wedge^2S^+ + \wedge^2S^-$	$S^+\wedge S^-$
$j_\rho(f_E)$	$\vee^2S^+ + \vee^2S^-$	$S^+\wedge S^-$	$\wedge^2S^+ + \wedge^2S^-$	$S^+\vee S^-$
m	1	2	3	4

Now put $V_1 = \mathbb{R}^{m,m} \neq 0$ and let V_2 be an arbitrary pseudo-Euclidean space. Denote the spinor module of $\mathfrak{so}(V_i)$ by S_i, $i = 1, 2$.

Proposition 3.4. *Let β_2 be an admissible bilinear form on S_2. Then there is a unique (up to scaling) admissible form β_1 on S_1 such that $\tau(\beta_2) = \iota(\beta_1)\tau(\beta_1)$. In particular, $\beta_1 \otimes \beta_2$ is an admissible bilinear form on the spinor $\mathfrak{so}(V_1 + V_2)$-module $S_1 \otimes S_2$.*

If moreover, A_2 is a β_2-admissible endomorphism of S_2, then there is a unique β_1-admissible endomorphism A_1 of S_1 such that $\tau(A_2) = \iota(A_1)\tau(A_1)$, in particular, $A_1 \otimes A_2$ is a $\beta_1 \otimes \beta_2$-admissible endomorphism of $S_1 \otimes S_2$.

The fundamental invariants of $\beta_1 \otimes \beta_2$ and $A_1 \otimes A_2$ are easily computed using the rules given in Proposition 2.2.

Proof. This follows from $\iota(f_E)\tau(f_E) = -\iota(f)\tau(f)$, $\iota(E)\tau(E) = -\iota(Id)\tau(Id)$ and Sect. 2.2. □

If we assume that V_2 is of definite signature, i.e. $V_2 = \mathbb{R}^{k,0}$ or $\mathbb{R}^{0,k}$, then there is a unique (up to scaling) $Pin(V_2)$-invariant symmetric bilinear form h_2 on the irreducible module S_2 of the compact group $Pin(V_2)$.

Lemma 3.2. *The $Pin(V_2)$-invariant scalar product h_2 is admissible: $\tau(h_2) = -1$ if $V_2 = \mathbb{R}^{k,0}$ and $\tau(h_2) = +1$ if $V_2 = \mathbb{R}^{0,k}$; $\sigma(h_2) = +1$ and if S_2 is reducible, $S_2 = S_2^+ + S_2^-$, $S_2^- = C\ell^1(V_2)S_2^+$, then $\iota(h_2) = +1$.*

Proof. Let $\rho(v)$ denote Clifford multiplication by a unit vector $v \in V_2$. Then h_2 is $\rho(v)$-invariant and $\rho(v)^2 = -Id$ if $V_2 = \mathbb{R}^{k,0}$ and $\rho(v)^2 = +Id$ if $V_2 = \mathbb{R}^{0,k}$. This implies $\tau(h_2) = \mp 1$.

To see that $\iota(h_2) = +1$ in the reducible case, consider the scalar product h_2' on S_2 defined by

$$h_2'(S_2^+, S_2^-) = 0\,, \quad h_2'|S_2^\pm = h_2|S_2^\pm \; (\neq 0)\,.$$

It is easy to check that h_2' is invariant under Clifford multiplication by unit vectors $v \in V_2$ using that $S^- = vS^+$. This implies $h_2' = h_2$. □

By Proposition 3.4 for every $V_1 = \mathbb{R}^{m,m} \neq 0$ there is a unique admissible bilinear form h_1 on the spinor module S_1 of $\mathfrak{so}(V_1)$ such that $\tau(h_2) = \iota(h_1)\tau(h_1)$.

Definition 3.1. *The* **canonical bilinear form** *on the spinor module $S = S_1 \otimes S_2$ of $\mathfrak{so}(V_1 + V_2)$ is $h = h_1 \otimes h_2$, where h_2 is the canonical bilinear form on the spinor module S_2 of $\mathfrak{so}(V_2) \cong \mathfrak{so}(k)$, i.e. the $Pin(V_2)$-invariant scalar product. In line with this definition we say that an endomorphism A of S (respectively A_2 of S_2) is* **admissible** *of* **symmetry** *$\sigma(A) = \pm 1$ (respectively $\sigma(A_2) = \pm 1$) if A is h-admissible (respectively h_2-admissible) and $\sigma_h(A) = \pm 1$ (respectively $\sigma_{h_2}(A_2) = \pm 1$).*

Remark 5. For $V_1 = \mathbb{R}^{m,m}$ we have two (non-degenerate) admissible bilinear forms f and f_E on $S_1 = S_{m,m}$. If we want to choose a *canonical* one, which is not necessary for our purpose, we can consider on S_1 the structure of irreducible $C\ell_{m,m+1}$-module defined in Sect. 3.2. Then only one of the forms remains admissible for the $C\ell_{m,m+1}$-module $S_1 = S_{m,m+1}$, it is in fact the canonical bilinear form on this module. Moreover, its complex bilinear extension is the unique (up to scaling) $\mathfrak{so}(2m+1, \mathbb{C})$-invariant complex bilinear form on the irreducible $\mathbb{C}\ell_{2m+1}$-module $\mathbb{S}_{2m+1} = S_{m,m+1} \otimes \mathbb{C}$, s. Corollary 3.1.

3.2. Complex case. *Case of even dimension.* The following theorem follows immediately from the fact that an irreducible module $\mathbb{S}_{2m}$ of $\mathbb{C}\ell_{2m}$ can be obtained as $\mathbb{S}_{2m} = S_{m,m} \otimes \mathbb{C}$ and that $\mathbb{S}_{2m}$ splits as $\mathbb{C}\ell^0_{2m}$-module: $\mathbb{S}_{2m} = \mathbb{S}^+_{2m} + \mathbb{S}^-_{2m}$, where $\mathbb{S}^{\pm}_{2m} = S^{\pm}_{m,m} \otimes \mathbb{C}$.

Theorem 3.2. *Every $\mathfrak{so}(2m, \mathbb{C})$-equivariant embedding $\mathbb{C}^{2m} \hookrightarrow \mathbb{S}_{2m} \otimes \mathbb{S}_{2m}$ is a linear combination of the embeddings $j_\rho(f)^{\mathbb{C}}$ and $j_\rho(f_E)^{\mathbb{C}}$. Their image is contained in the dual of the subspaces indicated in the table depending on $m \pmod 4$, where we have put $\mathbb{S} = \mathbb{S}_{2m}$.*

$j_\rho(f)^{\mathbb{C}}$	$\vee^2\mathbb{S}^+ + \vee^2\mathbb{S}^-$	$\mathbb{S}^+ \vee \mathbb{S}^-$	$\wedge^2\mathbb{S}^+ + \wedge^2\mathbb{S}^-$	$\mathbb{S}^+ \wedge \mathbb{S}^-$
$j_\rho(f_E)^{\mathbb{C}}$	$\vee^2\mathbb{S}^+ + \vee^2\mathbb{S}^-$	$\mathbb{S}^+ \wedge \mathbb{S}^-$	$\wedge^2\mathbb{S}^+ + \wedge^2\mathbb{S}^-$	$\mathbb{S}^+ \vee \mathbb{S}^-$
m	1	2	3	4

Case of odd dimension. The odd dimensional complex case can be obtained from the real case of signature $(m, m+1)$ by complexification.

We fix the orthogonal decomposition $(\mathbb{R}^{m,m+1}, < \cdot, \cdot >) = \mathbb{R}e_0 + \mathbb{R}^{m,m}$, where $< e_0, e_0 > = -1$, and denote by ρ the irreducible representation of $C\ell_{m,m}$ on $S_{m,m}$ constructed in Proposition 3.1.

Proposition 3.5. *An irreducible representation $\tilde{\rho}$ of $C\ell_{m,m+1}$ on $S_{m,m+1} = S_{m,m}$ is defined by*

$$\tilde{\rho}|\mathbb{R}^{m,m} = \rho|\mathbb{R}^{m,m}, \quad \tilde{\rho}(e_0) = \rho(\omega_{m,m}),$$

where $\omega_{m,m}$ is the volume element of $C\ell_{m,m}$. The $C\ell^0_{m,m+1}$-module $S_{m,m+1}$ is irreducible and has Schur algebra $C_{m,m+1} = \mathbb{R}\, Id$.

Proof. It is sufficient to check that $\{\tilde{\rho}(e_0), \rho(x)\} = 0$ for $x \in \mathbb{R}^{m,m}$ and that $\tilde{\rho}(e_0)^2 = Id$. This follows from the next lemma. □

Lemma 3.3. *The volume element $\omega = \omega_{m,m} = e_1 e_2 \cdots e_{2m}$ ((e_i) an orthonormal basis of $\mathbb{R}^{m,m}$) of $C\ell_{m,m}$ satisfies $\{\omega, x\} = 0$ for all $x \in \mathbb{R}^{m,m}$ and $\omega^2 = +1$.*

Proposition 3.6. *If m is even, then every $\mathfrak{so}(m, m+1)$-invariant bilinear form on $S = S_{m,m+1}$ is a multiple of the admissible (canonical) form f_E (see Proposition 3.3) and hence every $\mathfrak{so}(m, m+1)$-equivariant embedding $\mathbb{R}^{m,m+1} \hookrightarrow (S \otimes S)^*$ is proportional to the embedding $j_{\tilde{\rho}}(f_E)$, which maps $\mathbb{R}^{m,m+1}$ into $\vee^2 S^*$ if $m \equiv 0 \pmod 4$ and into $\wedge^2 S^*$ if $m \equiv 2 \pmod 4$. If m is odd, then every $\mathfrak{so}(m, m+1)$-invariant bilinear form on $S = S_{m,m+1}$ is a multiple of the admissible (canonical) form f (see Proposition 3.3) and hence every $\mathfrak{so}(m, m+1)$-equivariant embedding $\mathbb{R}^{m,m+1} \hookrightarrow (S \otimes S)^*$ is proportional to the embedding $j_{\tilde{\rho}}(f)$, which maps $\mathbb{R}^{m,m+1}$ into $\vee^2 S^*$ if $m \equiv 1 \pmod 4$ and into $\wedge^2 S^*$ if $m \equiv 3 \pmod 4$.*

Proof. If m is even, then $\tilde{\rho}(e_0) = \rho(\omega_{m,m})$ is f_E-symmetric and $\tau(f_E) = +1$. If m is odd, then $\tilde{\rho}(e_0)$ is f-skew symmetric and $\tau(f) = -1$. □

Corollary 3.1. *If m is even, then every $\mathfrak{so}(2m+1, \mathbb{C})$-invariant bilinear form on $\mathbb{S} = \mathbb{S}_{2m+1} = S_{m,m+1} \otimes \mathbb{C}$ is a multiple of the form $f_E^{\mathbb{C}}$ and every $\mathfrak{so}(2m+1, \mathbb{C})$-equivariant embedding $\mathbb{C}^{2m+1} \hookrightarrow (\mathbb{S} \otimes \mathbb{S})^*$ is proportional to the embedding $j_{\tilde{\rho}}(f_E)^{\mathbb{C}}$. If m is odd, then every $\mathfrak{so}(2m+1, \mathbb{C})$-invariant bilinear form on $\mathbb{S} = \mathbb{S}_{2m+1} = S_{m,m+1} \otimes \mathbb{C}$ is a multiple of the form $f^{\mathbb{C}}$ and every $\mathfrak{so}(2m+1, \mathbb{C})$-equivariant embedding $\mathbb{C}^{2m+1} \hookrightarrow (\mathbb{S} \otimes \mathbb{S})^*$ is proportional to the embedding $j_{\tilde{\rho}}(f)^{\mathbb{C}}$.*

4. Case of Signature $(k, 0)$

4.1. Case of even dimension. We fix the orthogonal decomposition $\mathbb{R}^{2m} = \mathbb{R}^m + \widetilde{\mathbb{R}^m}$, where $\tilde{}: \mathbb{R}^m \to \widetilde{\mathbb{R}^m}$ is an isometry. Denote by α the involution of $C\ell_m$ (respectively $\mathbb{C}\ell_m$) extending $x \mapsto -x$ on $\mathbb{R}^m$ (respectively $\mathbb{C}^m$).

Proposition 4.1. *If $m \equiv 0$ or $3 \pmod 4$ the following formulas define on $S = S_{2m,0} = C\ell_m$ the structure of irreducible $C\ell_{2m}$-module:*

$$\begin{aligned} \rho(x)s &= xs, \\ \rho(\tilde{x})s &= \omega s x \quad \text{if} \quad m \equiv 0 \pmod 4, \\ \rho(\tilde{x})s &= \omega\alpha(s)x \quad \text{if} \quad m \equiv 3 \pmod 4, \end{aligned}$$

where $x \in \mathbb{R}^m$, $s \in S$ and ω is the volume element of $C\ell_m$, i.e. $\omega = e_1 \cdots e_m$ for an orthonormal basis (e_i) of $\mathbb{R}^m$. The $\mathfrak{so}(2m)$-module S is the sum $S = S^+ + S^-$ of the two inequivalent irreducible modules $S^+ = C\ell_m^0$ and $S^- = C\ell_m^1$ if $m \equiv 0 \pmod 4$ and is irreducible if $m \equiv 3 \pmod 4$.

If $m \equiv 1$ or $2 \pmod 4$ the structure of irreducible $C\ell_{2m}$-module on $S = S_{2m,0} = \mathbb{S}_{2m} = \mathbb{C}\ell_m$ is given by:

$$\begin{aligned} \rho(x)s &= xs, \\ \rho(\tilde{x})s &= i\alpha(s)x, \quad x \in \mathbb{R}^m, \quad s \in S. \end{aligned}$$

As $\mathfrak{so}(2m)$-module $S = S^+ + S^-$ is the sum of the two irreducible modules $S^+ = \mathbb{C}\ell_m^0$ and $S^- = \mathbb{C}\ell_m^1$, which are equivalent for $m \equiv 1 \pmod 4$ and inequivalent for $m \equiv 2 \pmod 4$.

Proof. It is sufficient to check the identities $\rho(x)^2 = -< x, x > Id, \rho(\tilde{x})^2 = -< x, x > Id$ and $\{\rho(x), \rho(\tilde{y})\} = 0$ for $x, y \in \mathbb{R}^m$. This is straightforward using the following lemma. □

Lemma 4.1. *The volume element $\omega = \omega_m = e_1 \cdots e_m$ of $C\ell_m$ satisfies $\{\omega, x\} = 0$ if m is even and $[\omega, x] = 0$ if m is odd, $x \in \mathbb{R}^m \subset C\ell_m$. Moreover,*

$$\omega^2 = \begin{cases} +1 & \text{if} \quad m \equiv 0 \quad \text{or} \quad 3 \pmod 4 \\ -1 & \text{if} \quad m \equiv 1 \quad \text{or} \quad 2 \pmod 4. \end{cases}$$

Now we describe the $Pin(2m)$-invariant symmetric bilinear form h on S using the canonical identification $\wedge\mathbb{R}^m \to C\ell_m$ of $\mathbb{Z}_2$-graded vector spaces given by

$$e_{i_1} \wedge \ldots \wedge e_{i_k} \mapsto e_{i_1} \cdots e_{i_k}$$

with respect to an orthonormal basis (e_i), $i = 1, \ldots, m$, of $\mathbb{R}^m$.

The standard scalar product $< \cdot, \cdot >$ on $\wedge\mathbb{R}^m$ induced by the scalar product on $\mathbb{R}^m$ is invariant under exterior $x \wedge \cdot$ and interior $x \angle \cdot$ multiplication with unit vectors $x \in \mathbb{R}^m$.

Lemma 4.2. *Using the identification $C\ell_m = \wedge\mathbb{R}^m$, Clifford multiplication of $x \in \mathbb{R}^m$ and $\phi \in C\ell_m$ is given by:*

$$\begin{aligned} x\phi &= x \wedge \phi - x \angle \phi, \\ \phi x &= x \wedge \alpha(\phi) + x \angle \alpha(\phi). \end{aligned}$$

Proof. The proof is similar to [L-M] I. Prop. 3.9. □

Corollary 4.1. *The standard scalar product* $< \cdot, \cdot >$ *on* $\wedge\mathbb{R}^m = C\ell_m$ *is invariant under left and right multiplications by unit vectors* $x \in \mathbb{R}^m$. *In particular, if* $m \equiv 0$ *or* $3 \pmod 4$, $h = < \cdot, \cdot >$ *is the (admissible)* $Pin(2m)$*-invariant scalar product on the irreducible* $C\ell_{2m}$*-module* $S = C\ell_m$.

If $m \equiv 1$ or $2 \pmod 4$, we extend the standard scalar product on $\wedge\mathbb{R}^m$ to a symmetric complex bilinear form $< \cdot, \cdot >_{\mathbb{C}}$ on $S = \wedge\mathbb{C}^m$. Using the operator c of complex conjugation, we define a symmetric real bilinear form $h = Re < c\,\cdot, \cdot >_{\mathbb{C}}$ on S.

Lemma 4.3. *Let* $m \equiv 1$ *or* $2 \pmod 4$. *Then* $h = Re < c\,\cdot, \cdot >_{\mathbb{C}}$ *is the (admissible)* $Pin(2m)$*-invariant scalar product on the irreducible* $C\ell_{2m}$*-module* $S = \mathbb{C}\ell_m$.

Proof. We check that $\rho(x)$ and $\rho(\tilde{x})$, $x \in \mathbb{R}^m$, are $< c\,\cdot, \cdot >_{\mathbb{C}}$-skew symmetric and hence h-skew symmetric. By Corollary 4.1 left and right multiplication, L_x and R_x, by $x \in \mathbb{R}^m$ are $< \cdot, \cdot >_{\mathbb{C}}$-skew symmetric endomorphisms of $S = \mathbb{C}\ell_m$, in particular, $\rho(x)$ is $< \cdot, \cdot >_{\mathbb{C}}$-skew symmetric. It is easy to see that α and the operator I of multiplication by i are $< \cdot, \cdot >_{\mathbb{C}}$-symmetric endomorphisms. Moreover,

$$[I, R_x] = [I, \alpha] = \{\alpha, R_x\} = 0$$

and hence $\rho(\tilde{x}) = I \circ R_x \circ \alpha$ is $< \cdot, \cdot >_{\mathbb{C}}$-symmetric. From the relations

$$[c, L_x] = [c, R_x] = [c, \alpha] = \{c, I\} = 0$$

we obtain that $[\rho(x), c] = \{\rho(\tilde{x}), c\} = 0$, which implies that $\rho(x)$ and $\rho(\tilde{x})$ are $< c\,\cdot, \cdot >_{\mathbb{C}}$-skew symmetric. □

Now we construct admissible, i.e. h-admissible, bases of the Schur algebra $\mathcal{C} = \mathcal{C}_{2m,0}$ for all the values of $m \pmod 4$.

Proposition 4.2. *If* $m \equiv 0 \pmod 4$, *an admissible basis of the Schur algebra* $\mathcal{C}_{2m,0} \cong \mathbb{R} \oplus \mathbb{R}$ *is given by the endomorphisms* Id *and* $E = \alpha$ *of* $S = C\ell_m$: $\tau(E) = -1$, $\sigma(E) = \sigma_h(E) = +1$, $\iota(E) = +1$.

If $m \equiv 3 \pmod 4$, *an admissible basis of* $\mathcal{C}_{2m,0} \cong \mathbb{C}$ *is given by the endomorphisms* Id *and* $J = L_\omega \circ \alpha$ *of* $S = C\ell_m$: $\tau(J) = -1$, $\sigma(J) = -1$.

The space $\mathcal{B}$ *of* $\mathfrak{so}(2m)$*-invariant bilinear forms on* S *is spanned by admissible elements:*

$$\mathcal{B} = span\,\{h, h_E\} \quad if \quad m \equiv 0 \pmod 4\,,$$

$$\mathcal{B} = span\,\{h, h_J\} \quad if \quad m \equiv 3 \pmod 4\,.$$

The fundamental invariants (τ, σ, ι) *are given by* $(\tau, \sigma, \iota)(h) = (-1, +1, +1)$, $(\tau, \sigma, \iota)(h_E) = (+1, +1, +1)$ *if* $m \equiv 0 \pmod 4$ *and* $(\tau, \sigma)(h) = (-1, +1)$, $(\tau, \sigma)(h_J) = (+1, -1)$ *if* $m \equiv 3 \pmod 4$.

Proof. We show that J is admissible and $\tau(J) = \sigma(J) = -1$. All other statements are immediate.

Let $m \equiv 3 \pmod 4$. From $[L_x, L_\omega] = [R_x, L_\omega] = \{L_x, \alpha\} = \{R_x, \alpha\} = 0$ (see Lemma 4.1) it follows that $\{L_x, J\} = \{R_x, J\} = 0$. Since $\rho(x) = L_x$ and $\rho(\tilde{x}) = R_x \circ J$, we conclude $\{\rho(x), J\} = \{\rho(\tilde{x}), J\} = 0$.

The operator J is skew symmetric as the product of two anticommuting symmetric operators, namely L_ω and α (the scalar product is L_ω-invariant and $L_\omega^2 = +Id$). □

If $m \equiv 1$ or $2 \pmod 4$, we consider the following operators on $S = \mathbb{C}l_m$:

$$I : s \mapsto is\,,\; J = L_\omega \circ c\,,\; K = IJ \quad \text{and} \quad E = \alpha\,,$$

where $\omega = e_1 \cdots e_m \in Cl_m \subset \mathbb{C}l_m$ is the volume element.

Proposition 4.3. *Let $m \equiv 1$ or $2 \pmod 4$. The Schur algebra $C_{2m,0}$ ($\cong \mathbb{C}(2)$ if $m \equiv 1 \pmod 4$ and $\cong \mathbb{H} \oplus \mathbb{H}$ if $m \equiv 2 \pmod 4$) is generated by the admissible operators I, J and E satisfying the following (anti) commutator relations:*

$$I^2 = J^2 = L_\omega^2 = -1\,, \quad E^2 = c^2 = +1\,,$$

$$\{I, J\} = [I, E] = [I, L_\omega] = \{I, c\} = 0\,,$$

$$[J, L_\omega] = [J, c] = [E, c] = [L_\omega, c] = 0\,,$$

$$\{J, E\} = \{L_\omega, E\} = 0 \quad \text{if} \quad m \equiv 1 \pmod 4\,,$$

$$[J, E] = [L_\omega, E] = 0 \quad \text{if} \quad m \equiv 2 \pmod 4\,.$$

An admissible basis of the Schur algebra is given by the endomorphisms Id, I, J, K, E, EI, EJ, EK. Their fundamental invariants (τ, σ, ι) are given in the next table, where the value of m is modulo 4.

m:	Id	I	J	K	E	EI	EJ	EK
1	$+++$	$+-+$	$+--$	$+--$	$-++$	$--+$	$-+-$	$-+-$
2	$+++$	$+-+$	$--+$	$--+$	$-++$	$--+$	$+-+$	$+-+$

The fundamental invariants of the corresponding admissible basis of $\mathcal{B}$ are also listed for convenience:

m:	h	h_I	h_J	h_K	h_E	h_{EI}	h_{EJ}	h_{EK}
1	$-++$	$--+$	$---$	$---$	$+++$	$+-+$	$++-$	$++-$
2	$-++$	$--+$	$+-+$	$+-+$	$+++$	$+-+$	$--+$	$--+$

Proof. The proof is similar to the proof of Proposition 3.3 and 4.2. One uses the multiplication rules for the invariants and also that L_ω is skew symmetric, c is symmetric and they commute. □

Theorem 4.1. *Every $\mathfrak{so}(2m)$-equivariant embedding $\mathbb{R}^{2m} \hookrightarrow (S \otimes S)^*$, $S = S_{2m,0}$, is a linear combination of the embeddings*

$$j_\rho(h) : \mathbb{R}^{2m} \hookrightarrow (S^+ \wedge S^-)^* \quad \textit{and} \quad j_\rho(h_E) : \mathbb{R}^{2m} \hookrightarrow (S^+ \vee S^-)^*$$

if $m \equiv 0 \pmod 4$ and a linear combination of

$$j_\rho(h) : \mathbb{R}^{2m} \hookrightarrow \wedge^2 S^* \quad \textit{and} \quad j_\rho(h_J) : \mathbb{R}^{2m} \hookrightarrow \wedge^2 S^*$$

if $m \equiv 3 \pmod 4$.

If $m \equiv 1$ or $2 \pmod 4$ every $\mathfrak{so}(2m)$-equivariant embedding $\mathbb{R}^{2m} \hookrightarrow (S \otimes S)^*$ is a linear combination of the embeddings $j_A = j_\rho(h_A)$, $A \in C$ admissible, whose image is contained in the dual of the subspaces indicated in Table 4 depending on $m \pmod 4$.

Table 4. $SO(2m)$-equivariant embeddings $j_A = j_\rho(h_A) : \mathbb{R}^{2m} \hookrightarrow (S \otimes S)^*$

j_{Id}	$S^+ \wedge S^-$	$S^+ \wedge S^-$
j_I	$S^+ \vee S^-$	$S^+ \vee S^-$
j_J	$\vee^2 S^+ + \vee^2 S^-$	$S^+ \wedge S^-$
j_K	$\vee^2 S^+ + \vee^2 S^-$	$S^+ \wedge S^-$
j_E	$S^+ \vee S^-$	$S^+ \vee S^-$
j_{EI}	$S^+ \wedge S^-$	$S^+ \wedge S^-$
j_{EJ}	$\vee^2 S^+ + \vee^2 S^-$	$S^+ \vee S^-$
j_{EK}	$\vee^2 S^+ + \vee^2 S^-$	$S^+ \vee S^-$
m:	1	2

4.2. Case of odd dimension. To reduce the odd dimensional case to the even dimensional, we consider the orthogonal decomposition $\mathbb{R}^{2m+1} = \mathbb{R}e_0 + \mathbb{R}^{2m}$, where e_0 is a unit vector. Let ρ denote the irreducible representation of $C\ell_{2m}$ on $S_{2m,0}$ defined in Sect. 4.1. We will extend ρ to an irreducible representation $\tilde{\rho}$ of $C\ell_{2m+1}$ on $S = S_{2m+1,0}$, where $S_{2m+1,0} = S_{2m,0}$ if $m \equiv 1, 2$ or $3 \pmod 4$ and $S_{2m+1,0} = S_{2m,0} \otimes \mathbb{C} = \mathbb{S}_{2m}$ if $m \equiv 0 \pmod 4$. If $m \equiv 1$ or $2 \pmod 4$, $S_{2m,0} = \mathbb{S}_{2m}$ admits the $C\ell_{2m}$-invariant complex structure I. For $m \equiv 0 \pmod 4$ multiplication by i is a $C\ell_{2m}$-invariant complex structure on $S_{2m,0} \otimes \mathbb{C}$ and will also be denoted by I.

Proposition 4.4. *The following formulas define an irreducible representation $\tilde{\rho}$ of $C\ell_{2m+1}$ on $S_{2m+1,0}$.*

$$\tilde{\rho}|\mathbb{R}^{2m} = \rho|\mathbb{R}^{2m},$$

$$\tilde{\rho}(e_0) = \begin{cases} \rho(\omega_{2m}) & \text{if} \quad m \equiv 1 \quad \text{or} \quad 3 \pmod 4 \\ I \circ \rho(\omega_{2m}) & \text{if} \quad m \equiv 0 \quad \text{or} \quad 2 \pmod 4, \end{cases}$$

where, in the case $m \equiv 0 \pmod 4$, ρ has been extended complex linearly to a representation on $S_{2m,0} \otimes \mathbb{C}$, denoted by the same symbol. $S = S_{2m+1,0}$ is irreducible as $C\ell^0_{2m+1}$-module if $m \not\equiv 0 \pmod 4$ and the sum $S = S^+ + S^-$ of the two equivalent irreducible $C\ell^0_{2m+1}$-modules $S^+ = S^+_{2m,0} + iS^-_{2m,0} = C\ell^0_m + iC\ell^1_m$ and $S^- = iS^+$ if $m \equiv 0 \pmod 4$.

Proof. It is sufficient to check that $\tilde{\rho}(e_0)^2 = -Id$ and $\{\tilde{\rho}(e_0), \rho(x)\} = 0$ for $x \in \mathbb{R}^{2m}$, since all other information can be extracted from the Schur algebra, see Corollary 1.3. These identities follow immediately from Lemma 4.1 and the fact that I is a $C\ell_{2m}$-invariant complex structure. □

Now we describe the $Pin(2m+1)$-invariant scalar product h on $S = S_{2m+1,0}$. Let $h_{2m,0}$ denote the $Pin(2m)$-invariant scalar product on $S_{2m+1,0} = S_{2m,0}$ if $m \equiv 1, 2$ or $3 \pmod 4$ and by $h^{\mathbb{C}}_{2m,0}$ the complex bilinear extension of the $Pin(2m)$-invariant scalar product on $S_{2m,0}$ to a $Pin(2m)$-invariant complex bilinear form on $S_{2m+1,0} = \mathbb{S}_{2m} = S_{2m,0} \otimes \mathbb{C}$ if $m \equiv 4 \pmod 4$.

Lemma 4.4. *The $Pin(2m+1)$-invariant scalar product $h = h_{2m+1,0}$ on $S = S_{2m+1,0}$ is given by $h = h_{2m,0}$ if $m \equiv 1, 2$ or $3 \pmod 4$ and by $h = Re\, h^{\mathbb{C}}_{2m,0}(c\,\cdot, \cdot)$ if $m \equiv 4 \pmod 4$, where c is complex conjugation with respect to $S_{2m,0} \subset S_{2m,0} \otimes \mathbb{C}$.*

Proof. If $m \not\equiv 4 \pmod 4$, the statement follows from Schur's Lemma, since $S_{2m+1,0} = S_{2m,0}$. If $m \equiv 4 \pmod 4$, the Hermitian form $h^{\mathbb{C}}_{2m,0}(c\,\cdot,\cdot)$ is I-invariant and hence invariant under $\tilde{\rho}(e_0) = I \circ \rho(\omega_{2m})$ and the same is true for $h = Re\, h^{\mathbb{C}}_{2m,0}(c\,\cdot,\cdot)$. □

If $m \not\equiv 3 \pmod 4$, we have on $S_{2m+1,0} = \mathbb{C}\ell_m = C\ell_m + iC\ell_m$ the operator c of complex conjugation. Hence, we can define an endomorphism J of $S_{2m+1,0} = \mathbb{C}\ell_m$ by the formulas

$$J := \begin{cases} L_\omega \circ c & \text{if} \quad m \equiv 1 \quad \text{or} \quad 2 \pmod 4 \\ \alpha \circ c & \text{if} \quad m \equiv 0 \pmod 4, \end{cases}$$

where L_ω is left multiplication by the volume element $\omega = \omega_m$ of $C\ell_m$ and $\alpha|\mathbb{C}\ell^0_m = +Id$, $\alpha|\mathbb{C}\ell^1_m = -Id$.

Proposition 4.5. *Let $m \not\equiv 3 \pmod 4$. An admissible basis of the Schur algebra $\mathcal{C} = \mathcal{C}_{2m+1,0}$ is given by the endomorphisms Id, I, J and $K = IJ$ of $S_{2m+1,0} = \mathbb{C}\ell_m$. If $m \equiv 1$ or $2 \pmod 4$, then $I^2 = J^2 = -Id$, $\{I, J\} = 0$ and $\mathcal{C}_{2m+1,0} \cong \mathbb{H}$. If $m \equiv 0 \pmod 4$, then $I^2 = -J^2 = -Id$, $\{I, J\} = 0$ and $\mathcal{C}_{2m+1,0} \cong \mathbb{R}(2)$. The space $\mathcal{B}$ of $\mathfrak{so}(2m+1)$-invariant bilinear forms on $S_{2m+1,0}$ has the admissible basis (h, h_I, h_J, h_K). If $m \equiv 3 \pmod 4$, then the Schur algebra $\mathcal{C}_{2m+1,0} = \mathbb{R}\, Id$ and $\mathcal{B} = \mathbb{R}h$.*

Proof. Straightforward, cf. Proposition 4.2. □

Theorem 4.2. *If $m \equiv 3 \pmod 4$, every $\mathfrak{so}(2m+1)$-equivariant embedding $\mathbb{R}^{2m+1} \hookrightarrow S^* \otimes S^*$, $S = S_{2m+1,0}$, is a multiple of $j_\rho(h) : \mathbb{R}^{2m+1} \hookrightarrow \wedge^2 S^*$. If $m \not\equiv 3 \pmod 4$, every $\mathfrak{so}(2m+1)$-equivariant embedding $\mathbb{R}^{2m+1} \hookrightarrow (S \otimes S)^*$ is a linear combination of the embeddings $j_A = j_\rho(h_A)$, $A = Id$, I, J or K, whose image is contained in the dual of the subspaces indicated in Table 5 depending on $m \pmod 4$.*

Table 5. $\mathfrak{so}(2m+1)$-equivariant embeddings $j_A : \mathbb{R}^{2m+1} \hookrightarrow (S \otimes S)^*$

m:	j_{Id}	j_I	j_J	j_K
1	$\wedge^2 S$	$\vee^2 S$	$\vee^2 S$	$\vee^2 S$
2	$\wedge^2 S$	$\vee^2 S$	$\wedge^2 S$	$\wedge^2 S$
4	$S^+ \wedge S^-$	$\vee^2 S^+ + \vee^2 S^-$	$S^+ \vee S^-$	$\vee^2 S^+ + \vee^2 S^-$

5. Case of Signature (0, k)

Now we discuss the case of signature $(0, k)$. The proofs are similar to the proofs in the case of signature $(k, 0)$ and will mostly be omitted.

5.1. Case of even dimension. As in the positively defined case, we fix the orthogonal decomposition $\mathbb{R}^{0,2m} = \mathbb{R}^{0,m} + \widetilde{\mathbb{R}^{0,m}}$, where $\tilde{}: \mathbb{R}^{0,m} \to \widetilde{\mathbb{R}^{0,m}}$ is an isometry.

Lemma 5.1. *The volume element $\omega = \omega_{0,m} = e_1 \cdots e_m$ ((e_i) an orthonormal basis of $\mathbb{R}^{0,m}$) of $C\ell_{0,m}$ satisfies $\{\omega, x\} = 0$ if m is even and $[\omega, x] = 0$ if m is odd, $x \in \mathbb{R}^{0,m} \subset C\ell_{0,m}$. Moreover,*

$$\omega^2 = \begin{cases} +1 & \textit{if} \quad m \equiv 0 \quad \textit{or} \quad 1 \pmod 4 \\ -1 & \textit{if} \quad m \equiv 2 \quad \textit{or} \quad 3 \pmod 4. \end{cases}$$

The next proposition is checked using Lemma 5.1.

Proposition 5.1. *If $m \equiv 0$ or $1 \pmod 4$ the following formulas define on $S = S_{0,2m} = C\ell_{0,m}$ the structure of irreducible $C\ell_{0,2m}$-module:*

$$\begin{aligned} \rho(x)s &= xs, \\ \rho(\tilde{x})s &= \omega s x \quad \text{if} \quad m \equiv 0 \pmod 4, \\ \rho(\tilde{x})s &= \omega\alpha(s)x \quad \text{if} \quad m \equiv 1 \pmod 4), \end{aligned}$$

where $x \in \mathbb{R}^{0,m}$, $s \in S$ and ω is the volume element of $C\ell_{0,m}$. The $\mathfrak{so}(0,2m)$-module S is the sum $S = S^+ + S^-$ of the two inequivalent irreducible modules $S^+ = C\ell^0_{0,m}$ and $S^- = C\ell^1_{0,m}$ if $m \equiv 0 \pmod 4$ and is irreducible if $m \equiv 1 \pmod 4$.

If $m \equiv 2$ or $3 \pmod 4$ the structure of irreducible $C\ell_{0,2m}$-module on $S = S_{0,2m} = \mathbb{S}_{2m} = \mathbb{C}\ell_m$ is given by:

$$\begin{aligned} \rho(x)s &= xs, \\ \rho(\tilde{x})s &= i\alpha(s)x, \quad x \in \mathbb{R}^{0,m} \subset \mathbb{C}\ell_m = C\ell_{0,m} \otimes \mathbb{C}, \ s \in S = \mathbb{C}\ell_m . \end{aligned}$$

As $\mathfrak{so}(0,2m)$-module $S = S^+ + S^-$ is the sum of the two irreducible submodules $S^+ = \mathbb{C}\ell^0_m$ and $S^- = \mathbb{C}\ell^1_m$, which are inequivalent for $m \equiv 2 \pmod 4$ and equivalent for $m \equiv 3 \pmod 4$.

Recall (see Corollary 4.1) that the standard scalar product on $\wedge\mathbb{R}^m = C\ell_m = C\ell_{m,0}$ is invariant under left and right multiplications by unit vectors $x \in \mathbb{R}^m = \mathbb{R}^{m,0}$. We can consider $\mathbb{R}^{0,m}$ as subspace

$$\mathbb{R}^{0,m} = i\mathbb{R}^m \subset \mathbb{C}\ell_m = C\ell_m \otimes \mathbb{C} = C\ell_m + iC\ell_m .$$

Then $C\ell_{0,m} = C\ell^0_{0,m} + C\ell^1_{0,m} = C\ell^0_m + iC\ell^1_m$. We define an isomorphism of $\mathbb{Z}_2$-graded vector spaces $\varphi : C\ell_m \to C\ell_{0,m}$ on elements $a \in C\ell_m$ of pure degree $\deg(a) = 0$ or 1 by:

$$a \mapsto i^{\deg(a)}a .$$

A scalar product $< \cdot, \cdot >$ on $C\ell_{0,m}$ is defined by the condition that $\varphi : C\ell_m \to C\ell_{0,m}$ is an isometry for the standard scalar product on $\wedge\mathbb{R}^m = C\ell_m$. The following lemma is true by construction.

Lemma 5.2. *The scalar product $< \cdot, \cdot >$ on $C\ell_{0,m}$ is invariant under left and right multiplications by unit vectors $x \in \mathbb{R}^{0,m}$. In particular, if $m \equiv 0$ or $1 \pmod 4$, $h = < \cdot, \cdot >$ is the (admissible) $Pin(0,2m)$-invariant scalar product on the irreducible $C\ell_{0,2m}$-module $S = S_{0,2m} = C\ell_{0,m}$.*

If $m \equiv 2$ or $3 \pmod 4$, we extend the scalar product $< \cdot, \cdot >$ on $C\ell_{0,m}$ to a symmetric complex bilinear form $< \cdot, \cdot >_{\mathbb{C}}$ on $S = \wedge\mathbb{C}^m$. Using the operator $c = c_{0,m}$ of complex conjugation with respect to the real form $C\ell_{0,m} = C\ell^0_m + iC\ell^1_m$ of $\mathbb{C}\ell_m$, we define a (real) scalar product $h = Re < c\,\cdot, \cdot >_{\mathbb{C}}$ on S.

Lemma 5.3. *Let $m \equiv 2$ or $3 \pmod 4$. Then $h = Re < c\,\cdot, \cdot >_{\mathbb{C}}$ is the (admissible) $Pin(0,2m)$-invariant scalar product on the irreducible $C\ell_{0,2m}$-module $S = \mathbb{C}\ell_m$.*

Now we construct (h-)admissible bases of the Schur algebra $\mathcal{C} = \mathcal{C}_{0,2m}$ for all the values of $m \pmod 4$.

Proposition 5.2. *If* $m \equiv 0 \pmod 4$, *an admissible basis of the Schur algebra* $\mathcal{C}_{0,2m} \cong \mathbb{R} \oplus \mathbb{R}$ *is given by the endomorphisms* Id *and* $E = \alpha$ *of* $S = \mathcal{C}\ell_{0,m}$: $\tau(E) = -1$, $\sigma(E) = \sigma_h(E) = +1$, $\iota(E) = +1$.

If $m \equiv 1 \pmod 4$, *an admissible basis of* $\mathcal{C}_{0,2m} \cong \mathbb{C}$ *is given by the endomorphisms* Id *and* $J = L_\omega \circ \alpha$ *of* $S = \mathcal{C}\ell_{0,m}$ *(where* ω *is a volume element of* $\mathcal{C}\ell_{0,m}$*):* $\tau(J) = -1$, $\sigma(J) = -1$.

The space $\mathcal{B}$ *of* $\mathfrak{so}(0,2m)$*-invariant bilinear forms on* S *is spanned by the admissible elements* h *and* h_E *if* $m \equiv 0 \pmod 4$ *and by* h *and* h_J *if* $m \equiv 1 \pmod 4$. *Their fundamental invariants* (τ, σ, ι) *are* $(\tau,\sigma,\iota)(h) = (+1,+1,+1)$, $(\tau,\sigma,\iota)(h_E) = (-1,+1,+1)$ *if* $m \equiv 0 \pmod 4$ *and* $(\tau,\sigma)(h) = (+1,+1)$, $(\tau,\sigma)(h_J) = (-1,-1)$ *if* $m \equiv 1 \pmod 4$.

If $m \equiv 2$ or $3 \pmod 4$, we consider the following operators on $S = \mathcal{C}\ell_m$:

$$I : s \mapsto is\,,\ J = L_\omega \circ c\,,\ K = IJ \text{ and } E = \alpha \quad (\omega = \omega_{0,m})\,.$$

Proposition 5.3. *Let* $m \equiv 2$ *or* $3 \pmod 4$. *The Schur algebra* $\mathcal{C}_{0,2m}$ *(*$\cong \mathbb{H} \oplus \mathbb{H}$ *if* $m \equiv 2 \pmod 4$ *and* $\cong \mathbb{C}(2)$ *if* $m \equiv 3 \pmod 4$*) is generated by the admissible operators* I, J *and* E, *which satisfy the following identities:*

$$\begin{aligned} &I^2 = J^2 = L_\omega^2 = -1\,, \quad E^2 = c^2 = +1\,,\\ &\{I,J\} = [I,E] = [I,L_\omega] = \{I,c\} = 0\,,\\ &[J,L_\omega] = [J,c] = [E,c] = [L_\omega,c] = 0\,,\\ &[J,E] = [L_\omega,E] = 0 \quad \text{if} \quad m \equiv 2 \pmod 4\,,\\ &\{J,E\} = \{L_\omega,E\} = 0 \quad \text{if} \quad m \equiv 3 \pmod 4\,. \end{aligned}$$

An admissible basis of the Schur algebra is given by the endomorphisms Id, I, J, K, E, EI, EJ, EK. *Their fundamental invariants* (τ,σ,ι) *are given in the next table, where the value of* m *is modulo 4.*

m:	Id	I	J	K	E	EI	EJ	EK
2	$+++$	$+-+$	$--+$	$--+$	$-++$	$--+$	$+-+$	$+-+$
3	$+++$	$+-+$	$+--$	$+--$	$-++$	$--+$	$-+-$	$-+-$

The fundamental invariants of the corresponding admissible basis for the space $\mathcal{B} = \mathcal{B}_{0,2m}$ *(of* $\mathfrak{so}(0,2m)$*-invariant bilinear forms on* $S_{0,2m}$*) are as follows:*

m:	h	h_I	h_J	h_K	h_E	h_{EI}	h_{EJ}	h_{EK}
2	$+++$	$+-+$	$--+$	$--+$	$-++$	$--+$	$+-+$	$+-+$
3	$+++$	$+-+$	$+--$	$+--$	$-++$	$--+$	$-+-$	$-+-$

Theorem 5.1. *Every* $\mathfrak{so}(0,2m)$*-equivariant embedding* $\mathbb{R}^{0,2m} \hookrightarrow (S \otimes S)^*$, $S = S_{0,2m}$, *is a linear combination of the embeddings*

$$j_\rho(h) : \mathbb{R}^{0,2m} \hookrightarrow (S^+ \vee S^-)^* \quad \textit{and} \quad j_\rho(h_E) : \mathbb{R}^{0,2m} \hookrightarrow (S^+ \wedge S^-)^*$$

if $m \equiv 0 \pmod 4$ *and a linear combination of*

$$j_\rho(h) \text{ and } j_\rho(h_J) : \mathbb{R}^{0,2m} \hookrightarrow \vee^2 S^* \quad \text{if} \quad m \equiv 1 \pmod 4\,.$$

If $m \equiv 2$ *or* $3 \pmod 4$ *every* $\mathfrak{so}(0,2m)$*-equivariant embedding* $\mathbb{R}^{0,2m} \hookrightarrow (S \otimes S)^*$ *is a linear combination of the embeddings* $j_A = j_\rho(h_A)$, $A \in \mathcal{C} = \mathcal{C}_{0,2m}$ *admissible, whose image is contained in the dual of the subspaces indicated in Table 6 depending on* $m \pmod 4$.

Table 6. $\mathfrak{so}(0,2m)$-equivariant embeddings $j_A : \mathbb{R}^{0,2m} \hookrightarrow (S \otimes S)^*$

j_{Id}	$S^+ \vee S^-$	$S^+ \vee S^-$
j_I	$S^+ \wedge S^-$	$S^+ \wedge S^-$
j_J	$S^+ \vee S^-$	$\wedge^2 S^+ + \wedge^2 S^-$
j_K	$S^+ \vee S^-$	$\wedge^2 S^+ + \wedge^2 S^-$
j_E	$S^+ \wedge S^-$	$S^+ \wedge S^-$
j_{EI}	$S^+ \vee S^-$	$S^+ \vee S^-$
j_{EJ}	$S^+ \wedge S^-$	$\wedge^2 S^+ + \wedge^2 S^-$
j_{EK}	$S^+ \wedge S^-$	$\wedge^2 S^+ + \wedge^2 S^-$
m:	2	3

5.2. Case of odd dimension. Consider the orthogonal decomposition $(\mathbb{R}^{0,2m+1}, < \cdot, \cdot >) = \mathbb{R}e_0 + \mathbb{R}^{0,2m}$, where $< e_0, e_0 > = -1$. Let ρ denote the irreducible representation of $C\ell_{0,2m}$ on $S_{0,2m}$ defined in Sect. 5.1. We will extend ρ to an irreducible representation $\tilde{\rho}$ of $C\ell_{0,2m+1}$ on $S = S_{0,2m+1}$, where $S_{0,2m+1} = S_{0,2m}$ if $m \equiv 0, 2$ or 3 (mod 4) and $S_{0,2m+1} = S_{0,2m} \otimes \mathbb{C} = \mathbb{S}_{2m}$ if $m \equiv 1 \pmod 4$. If $m \equiv 2$ or $3 \pmod 4$, $S_{0,2m} = \mathbb{S}_{2m}$ admits the $C\ell_{0,2m}$-invariant complex structure I. For $m \equiv 1 \pmod 4$ multiplication by i is a $C\ell_{0,2m}$-invariant complex structure on $S_{0,2m} \otimes \mathbb{C}$ and will also be denoted by I.

Proposition 5.4. *The following formulas define an irreducible representation $\tilde{\rho}$ of $C\ell_{0,2m+1}$ on $S_{0,2m+1}$.*

$$\tilde{\rho}|\mathbb{R}^{0,2m} = \rho|\mathbb{R}^{0,2m},$$

$$\tilde{\rho}(e_0) = \begin{cases} \rho(\omega_{0,2m}) & \text{if} \quad m \equiv 0 \quad \text{or} \quad 2 \pmod 4 \\ I \circ \rho(\omega_{0,2m}) & \text{if} \quad m \equiv 1 \quad \text{or} \quad 3 \pmod 4, \end{cases}$$

where, in the case $m \equiv 1 \pmod 4$, ρ has been extended complex linearly to a representation on $S_{0,2m+1} = S_{0,2m} \otimes \mathbb{C}$. $S = S_{0,2m+1}$ is irreducible as a $C\ell^0_{0,2m+1}$-module if $m \not\equiv 3 \pmod 4$ and the sum $S = S^+ + S^-$ of the two equivalent irreducible $C\ell^0_{0,2m+1}$-modules $S^+ = S^{\hat{J}}$ and $S^- = iS^{\hat{J}}$ if $m \equiv 3 \pmod 4$, where $S^{\hat{J}}$ is the fixed point set of a $\mathfrak{so}(0, 2m+1)$-invariant real structure $\hat{J}$ on S (the explicit expression for $\hat{J}$ will be given below).

Next we describe the $Pin(0, 2m + 1)$-invariant scalar product $h = h_{0,2m+1}$ on $S = S_{0,2m+1}$. Let $h_{0,2m}$ denote the $Pin(0, 2m)$-invariant scalar product on $S_{0,2m+1} = S_{0,2m}$ if $m \equiv 0, 2$ or $3 \pmod 4$ and by $h^{\mathbb{C}}_{0,2m}$ the complex bilinear extension of the $Pin(0, 2m)$-invariant scalar product on $S_{0,2m}$ to a $Pin(0, 2m)$-invariant complex bilinear form on $S_{0,2m+1} = \mathbb{S}_{2m} = S_{0,2m} \otimes \mathbb{C}$ if $m \equiv 1 \pmod 4$.

Lemma 5.4. *The $Pin(0, 2m+1)$-invariant scalar product $h = h_{0,2m+1}$ on $S = S_{0,2m+1}$ is given by $h = h_{0,2m}$ if $m \equiv 0, 2$ or $3 \pmod 4$ and by $h = Re\, h^{\mathbb{C}}_{0,2m}(c\,\cdot, \cdot)$ if $m \equiv 1 \pmod 4$, where c is complex conjugation with respect to $S_{0,2m} \subset S_{0,2m} \otimes \mathbb{C}$.*

If $m \not\equiv 0 \pmod 4$, we have on $S_{0,2m+1} = \mathbb{C}\ell_m = C\ell_{0,m} + iC\ell_{0,m}$ the operator $c = c_{0,m}$ of complex conjugation. Using it we define an endomorphism $\hat{J}$ of $S_{0,2m+1} = \mathbb{C}\ell_m$ by

$$\hat{J} := L_\omega \circ \alpha \circ c,$$

where $\omega = \omega_{0,m}$ is a volume element of $C\ell_{0,m}$ and $\alpha|\mathbb{C}\ell^0_m = +Id$, $\alpha|\mathbb{C}\ell^1_m = -Id$.

Proposition 5.5. *Let $m \not\equiv 0 \pmod 4$. The Schur algebra $\mathcal{C} = \mathcal{C}_{0,2m+1}$ is generated by the endomorphisms I and $\hat{J}$ of $S = S_{0,2m+1} = \mathbb{C}l_m$, which satisfy the following relations: $I^2 = -1$, $\{I, \hat{J}\} = 0$. Moreover, $\hat{J}^2 = +Id$ and $\mathcal{C}_{0,2m+1} \cong \mathbb{R}(2)$ if $m \equiv 3 \pmod 4$ and $\hat{J}^2 = -Id$ and $\mathcal{C}_{0,2m+1} \cong \mathbb{H}$ if $m \equiv 1$ or $2 \pmod 4$. An admissible basis of $\mathcal{C}_{0,2m+1}$ is given by the endomorphisms Id, I, $\hat{J}$ and $\hat{K} = I\hat{J}$. Their fundamental invariants (τ, σ, ι) together with the invariants of the associated admissible basis for the space $\mathcal{B}$ of $\mathfrak{so}(0, 2m+1)$-invariant bilinear forms are given in Table 7 (ι is only defined if $m \equiv 3 \pmod 4$). If $m \equiv 0 \pmod 4$, $\mathcal{C}_{0,2m+1} = \mathbb{R}Id$.*

Table 7. Fundamental invariants of admissible endomorphisms and bilinear forms of $S_{0,2m+1}$

m:	Id	I	$\hat{J}$	$\hat{K}$	h	h_I	$h_{\hat{J}}$	$h_{\hat{K}}$
1	++	+−	−−	−−	++	+−	−−	−−
2	++	+−	+−	+−	++	+−	+−	+−
3	+++	+−−	−++	−+−	+++	+−−	−++	−+−

Theorem 5.2. *Every $\mathfrak{so}(0, 2m+1)$-equivariant embedding $\mathbb{R}^{0,2m+1} \hookrightarrow (S \otimes S)^*$ is proportional to $j_\rho(h) : \mathbb{R}^{0,2m+} \hookrightarrow \vee^2 S^*$ if $m \equiv 0 \pmod 4$ and a linear combination of the embeddings $j_A = j_\rho(h_A)$, $A = Id$, I, $\hat{J}$ and $\hat{K}$ if $m \not\equiv 0 \pmod 4$. The image of the j_A is contained in the dual of the subspaces indicated in Table 8.*

Table 8. $\mathfrak{so}(0, 2m+1)$-equivariant embeddings $j_A : \mathbb{R}^{0,2m+1} \hookrightarrow (S \otimes S)^*$

j_{Id}	$\vee^2 S$	$\vee^2 S$	$S^+ \vee S^-$
j_I	$\wedge^2 S$	$\wedge^2 S$	$S^+ \wedge S^-$
$j_{\hat{J}}$	$\vee^2 S$	$\wedge^2 S$	$\wedge^2 S^+ + \wedge^2 S^-$
$j_{\hat{K}}$	$\vee^2 S$	$\wedge^2 S$	$\wedge^2 S^+ + \wedge^2 S^-$
m:	1	2	3

6. Complete Classification

Every pseudo-Euclidean space V admits a (unique up to an isometry) orthogonal decomposition $V = V_1 + V_2$, where $V_1 = \mathbb{R}^{m,m}$ and the scalar product of V_2 is positively or negatively defined. Now we consider the case when $V_1 \neq 0$ and $V_2 \neq 0$, the other cases were treated in Sects. 3.1, 4 and 5. We denote by S_i, $i = 1, 2$, the irreducible $\mathcal{C}l(V_i)$-module constructed in Sects. 3.1 and 4, 5 respectively. Then $S = S_1 \otimes S_2$ carries the structure of irreducible module for the Clifford algebra $\mathcal{C}l(V) = \mathcal{C}l(V_1)\hat{\otimes}\mathcal{C}l(V_2)$, see Proposition 2.3. By Proposition 3.4, to every admissible bilinear form β_2 (respectively endomorphism A_2) on S_2 we associate an admissible bilinear form $\beta = \beta_1 \otimes \beta_2$ (respectively endomorphism $A_1 \otimes A_2$) on S. In Sects. 4 and 5 we have contructed admissible bases for the space $\mathcal{B}_2$ of $\mathfrak{so}(V_2)$-invariant bilinear forms on S_2 and for the Schur algebra $\mathcal{C}_2$ of S_2. Therefore, this explicit correspondence defines an injective linear mapping $\phi : \beta_2 \mapsto \beta = \phi(\beta_2)$ (respectively $\psi : A_2 \mapsto A = \psi(A_2)$) from $\mathcal{B}_2$ into the space $\mathcal{B}$ of $\mathfrak{so}(V)$-invariant bilinear forms on S (respectively from $\mathcal{C}_2$ into the Schur algebra $\mathcal{C}$ of S). Moreover, ϕ and ψ are actually isomorphisms, because the Schur algebras of S and S_2 are isomorphic, due to the fact that V and V_2 have the same signature s, see Corollary 1.3. So we have essentially proved:

Theorem 6.1. *There exist natural isomorphisms $\phi : \mathcal{B}_2 \to \mathcal{B}$ of vector spaces and $\psi : \mathcal{C}_2 \to \mathcal{C}$ of algebras mapping admissible elements onto admissible elements. Under these maps, the fundamental invariants of admissible elements transform according to the rules given in Proposition 2.2. In particular, if $m \equiv 0 \pmod 4$, then ϕ and ψ preserve the fundamental invariants (***(4,4)-periodicity***).*

Proof. We recall that by Proposition 3.3 the Schur algebra $\mathcal{C}_{m,m}$ of $S_1 = S_{m,m}$ has the admissible basis (Id, E) and $E^2 = +Id$. This implies that the vector space isomorphism ψ is actually an isomorphism of *algebras*. The (4,4)-periodicity follows from

$$\sigma(f_E) = \iota(f_E) = \sigma_f(E) = \sigma_{f_E}(E) = \iota(E) = +1 . \quad \square$$

Recall that $\mathcal{B}_{p,q}$ denotes the space of $\mathfrak{so}(p,q)$-invariant bilinear forms on the $\mathfrak{so}(p,q)$ spinor module $S_{p,q}$ and $\mathcal{C}_{p,q}$ is the Schur algebra of $S_{p,q}$.

Corollary 6.1. **((8,0)- and (0,8)-periodicity)** *There exist natural isomorphisms*

$$\phi_{8,0} : \mathcal{B}_{p,q} \to \mathcal{B}_{p+8,q} \quad \textit{and} \quad \phi_{0,8} : \mathcal{B}_{p,q} \to \mathcal{B}_{p,q+8}$$

of vector spaces and

$$\psi_{8,0} : \mathcal{C}_{p,q} \to \mathcal{C}_{p+8,q} \quad \textit{and} \quad \psi_{0,8} : \mathcal{C}_{p,q} \to \mathcal{C}_{p,q+8}$$

of algebras mapping the admissible elements onto admissible elements preserving their fundamental invariants.

Proof. By Theorem 6.1 $\mathcal{B}_{p,q}$ and $\mathcal{C}_{p,q}$ have admissible bases. Now we recall from Sect. 4 and 5 that if $k \equiv 0 \pmod 8$, then $\mathcal{C}_{k,0} \cong \mathcal{C}_{0,k}$ has an admissible basis, which was denoted by (Id, E), such that $(\tau, \sigma, \iota)(E) = (-1, +1, +1)$ and, of course, $(\tau, \sigma, \iota)(Id) = (+1, +1, +1)$. The existence of the maps $\psi_{8,0}$ and $\psi_{0,8}$ follows from $\tau(Id)\iota(Id) = -\tau(E)\iota(E)$. They preserve the fundamental invariants, because $\sigma(Id) = \iota(Id) = \sigma(E) = \iota(E) = +1$. The existence and properties of $\phi_{8,0}$ and $\phi_{0,8}$ are proved similarly. $\square$

Corollary 6.2. *Every $\mathfrak{so}(V)$-equivariant mapping $j : V \to (S \otimes S)^*$ is a linear combination of the embeddings $j_A = j_\rho(h_A)$, where h is the canonical bilinear form on the spinor module S of $\mathfrak{so}(V)$ and A are admissible elements of the Schur algebra $\mathcal{C}$ of S.*

To obtain an overview over all possible N-extended Poincaré algebras $\mathfrak{p}(V) + S$, $N = \pm 1, \pm 2$, it is useful to define the invariants σ and ι for embeddings $j : V \hookrightarrow (S \otimes S)^*$ having special properties. More precisely, we put $\sigma(j) = +1$ if $jV \subset \vee^2 S^*$ and $\sigma(j) = -1$ if $jV \subset \wedge^2 S^*$. If $S = S^+ + S^-$, we define $\iota(j) = +1$ if $jV \subset (S^+ \otimes S^+ + S^- \otimes S^-)^*$ and $\iota(j) = -1$ if $jV \subset (S^+ \otimes S^-)^*$.

Note that the fundamental invariants of $j_A = j_\rho(h_A)$, $A \in \mathcal{C}$ admissible, are easily computable:

$$\sigma(j_A) = \tau(h_A)\sigma(h_A) = \tau(h)\tau(A)\sigma(h)\sigma(A) \quad \text{and} \quad \iota(j_A) = -\iota(h_A) = -\iota(h)\iota(A) .$$

Recall that $\mathcal{J}$ denotes the space of $\mathfrak{so}(V)$-equivariant mappings $j : V \to (S \otimes S)^*$. We define the subspaces

$$\mathcal{J}^{\sigma_0} := \{ j \in \mathcal{J} | \sigma(j) = \sigma_0 \} \cup \{0\} \quad \text{and}$$

$$\mathcal{J}^{\sigma_0 \iota_0} := \{ j \in \mathcal{J}^{\sigma_0} | \iota(j) = \iota_0 \} \cup \{0\}$$

and put

$$L^{\sigma_0} := \dim \mathcal{J}^{\sigma_0}, \quad L^{\sigma_0 \iota_0} := \dim \mathcal{J}^{\sigma_0 \iota_0}.$$

We shall write L^+, L^{+-}, ... instead of the more cumbersome L^{+1}, $L^{+1\,-1}$, ...

Remark that L^+ ($= L^{++} + L^{+-}$ if $S = S^+ + S^-$) is the maximal number of linearly independent super algebra structures on $\mathfrak{p}(V) + S$ and that L^- ($= L^{-+} + L^{--}$) is the number of $\mathbb{Z}_2$-graded Lie algebra structures on $\mathfrak{p}(V) + S$.

Theorem 6.2. *The numbers* (L^+, L^-) *and* $(L^{++}, L^{+-}, L^{-+}, L^{--})$ *depend only on the dimension* $n = \dim V = p + q$ *and the signature* $s = p - q$ *of* $V = \mathbb{R}^{p,q}$ *modulo 8. Moreover, they admit the* **mirror super symmetry** $n \mapsto -n$. *More precisely,*

$$\begin{aligned} L^+(-n, s) &= L^-(n, s) \quad \textit{and} \\ L^{+\iota_0}(-n, s) &= L^{-\iota_0}(n, s), \quad \iota_0 = \pm . \end{aligned}$$

Their values are given in Table 9.

Table 9. Numbers of extended Poincaré algebras $\mathfrak{p}(p, q) + S_{p,q}$ of different types depending on $n = p + q$ and $s = p - q$ modulo 8

s:	$(L^{++}, L^{+-}, L^{-+}, L^{--})(n, s)$ or $(L^+, L^-)(n, s)$							
4		2,0,6,0		0,4,0,4		6,0,2,0		0,4,0,4
3	1,3		1,3		3,1		3,1	
2		0,2,4,2		2,2,2,2		4,2,0,2		2,2,2,2
1	0,1,2,1		0,1,2,1		2,1,0,1		2,1,0,1	
0		0,0,2,0		0,1,0,1		2,0,0,0		0,1,0,1
-1	0,1		0,1		1,0		1,0	
-2		0,2		1,1		2,0		1,1
-3	1,3		1,3		3,1		3,1	
n:	-3	-2	-1	0	1	2	3	4

Proof. This follows from Theorem 6.1 and the tables of Sects. 3.1, 4 and 5 by straightforward computation. □

In the complex case we consider the space $\mathcal{J}_c$ of $\mathfrak{so}(m, \mathbb{C})$-equivariant mappings $\mathbb{C}^m \to (\mathbb{S}_m \otimes \mathbb{S}_m)^*$ and define the invariants σ, ι and the spaces $\mathcal{J}_c^+$, $\mathcal{J}_c^{+-}$, etc. as in the real case (ι is only defined if the complex $\mathfrak{so}(m, \mathbb{C})$ spinor module $\mathbb{S}_m$ is reducible $\mathbb{S}_m = \mathbb{S}_m^+ + \mathbb{S}_m^-$). Their dimensions are denoted by L_c^+, L_c^{+-}, etc.

Theorem 6.3. *The numbers* (L_c^+, L_c^-) *and* $(L_c^{++}, L_c^{+-}, L_c^{-+}, L_c^{--})$ *depend only on* m (mod 8). *Moreover, they admit the mirror super symmetry* $m \mapsto -m$. *More precisely,*

$$\begin{aligned} L_c^+(-m) &= L_c^-(m) \quad \textit{and} \\ L_c^{+\iota_0}(-m) &= L_c^{-\iota_0}(m), \quad \iota_0 = \pm . \end{aligned}$$

Their values are given in the next table.

	0, 1	0, 0, 2, 0	0, 1	0, 1, 0, 1	1, 0	2, 0, 0, 0	1, 0	0, 1, 0, 1
m:	−3	−2	−1	0	1	2	3	4

Proof. Follows from Sect. 3.2. □

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