Classification of pivotal tensor categories with fusion rules related to SO(4)

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Abstract

In this paper we classify all semisimple tensor categories with the same fusion rules as Rep(SO(4)), or one of the associated truncations. We show that such categories are explicitly classified by two non-zero complex numbers. Furthermore we show these tensor categories are always braided, and there exist exactly 8 braidings.

1 Introduction

In this note we continue the program to classify tensor categories with fusion rules the same as $\operatorname{Rep}(G)$ for G a semisimple Lie group (or of the associated fusion categories). Classification is currently known for the majority of the classical Lie groups. The known results are for: SU(2) [FK93], SU(N) [KW93], O(N) and Sp(N) [TW05], and SO(N) ($N \neq 4$) [Cop20]. The latter two results apply to ribbon categories, while the first two do not require any assumption of braiding and provide a classification for pivotal tensor categories. Our technique for SO(4)-type categories also does not require a braiding assumption.

The standard technique for attacking these classification problems is to identify the endomorphism algebras of tensor powers of the "vector representation" in an arbitrary tensor category with the same fusion rules of Rep(G), and to show that this algebra must agree with the known examples. In the case of SU(N)this gives well-known quotients of the Hecke algebras [Wen88], and in the O(N) and SO(N) cases we find quotients of BMW algebras [BW89]. For SO(N) with $N \neq 4$ the endomorphism algebras also afford representations of the BMW algebra, but the image of the BMW algebra does not generate the endomorphism algebra for SO(2n) for n > 2.

The gap at SO(4) is due to the fact that the tensor square of the vector representation splits into four simples, rather than three (as is the case for every other SO(N) with $N \ge 3$). This means that a braid element on $X^{\otimes 2}$ need not satisfy the cubic BMW skein relation, which was required for the method of [Cop20].

There is another important distinction between SO(4) and SO(2n) with n > 2, which is that the root system for SO(4) is not irreducible (its root system is the product $A_1 \times A_1$). As we shall see, this manifests in categorifications of SO(4) fusion rules being described by two independent parameters q_1 , q_2 , rather than a single parameter q.

In this paper we close this gap by studying a known SO(4)-type category and identifying the monoidal subcategory whose objects are tensor powers of the vector representation. This subcategory is essentially a planar algebra, and we describe it by generators and relations in a planar algebraic way, although we do not use that language. The planar algebras we describe can be seen as natural extensions of the Fuss-Catalan planar algebras [BJ97]. We then show that the corresponding subcategory of any category with SO(4)-type fusion rules must have the same presentation. We then obtain the classification of tensor categories with SO(4) fusion rules from standard reconstruction arguments.

We say a tensor category has SO(4) fusion rules if its Grothiendieck ring is isomorphic to $K(\operatorname{Rep}(SO(4)))$, or isomorphic to the Grothendieck ring of one of the associated fusion categories. We label these fusion rings by K_{n_1,n_2} where $n_i \in \mathbb{N} \cup \{\infty\}$ (see Definition 2.2 for a precise definition). The fusion graph of K_{n_1,n_2} for the vector representation is given by (shown here with $n_1 = 5$ and $n_2 = 8$):



The classification of such categories is given in our main theorem.

Theorem 1.1. Let C be a pivotal tensor category with $K(C) = K_{n_1,n_2}$ where $n_1, n_2 \in \mathbb{N}_{\geq 2} \cup \infty$. We have the following:

1. The category C is monoidally equivalent to C_{q_1,q_2} where $q_1, q_2 \in \mathbb{C}^{\times}$ with the order of q_i^2 equal to $n_i + 1$ (or possibly $q_i^2 = 1$ if $n_i = \infty$). Further we have the monoidal equivalences

$$\mathcal{C}_{q_1,q_2} \simeq \mathcal{C}_{q_2,q_1} \simeq \mathcal{C}_{q_1,q_2^{-1}} \simeq \mathcal{C}_{q_1^{-1},q_2} \simeq \mathcal{C}_{-q_1,-q_2}.$$

2. The category C_{q_1,q_2} is braided, and the possible braidings on these categories are parameterised by the set

$$\{(s_1, s_2) : s_1^2 = -q_1^{\pm 1} \text{ and } s_2^2 = -q_2^{\pm 1}\} / \{(s_1, s_2) = (-s_1, -s_2)\}.$$

When both $n_1, n_2 > 2$, these eight braidings are all distinct. If either n_1 or n_2 are equal to 2, then four of these braidings are distinct. If both n_1 and n_2 are equal to 2, then two of these braidings are distinct.

Constructions of these categories are given in Definition 2.1.

Remark 1.2. The above classification is up to equivalences which preserve the distinguished objects $f^{(1)} \boxtimes f^{(1)}$ in the categories C_{q_1,q_2} . The equivalences given in Theorem 1.1 are all the possible equivalences which preserve $f^{(1)} \boxtimes f^{(1)}$. There can exist additional equivalences between the categories C_{q_1,q_2} which don't preserve $f^{(1)} \boxtimes f^{(1)}$.

An illustrating example is seen in the case when q_2^2 is a root of unity of even order $n_2 + 1$ such that $[n_2]_{q_2} = -1$. For these parameters, we have that \mathcal{C}_{q_1,q_2} is monoidally equivalent to $\mathcal{C}_{q_1,-q_2}$ with the map sending $f^{(1)} \boxtimes f^{(1)} \boxtimes f^{(1)} \boxtimes f^{(n_2-1)}$.

This paper is outlined as follows.

In Section 2 we define the categories C_{q_1,q_2} which are examples of categories with SO(4) fusion rules. We define what it means to give a semisimple presentation of a pivotal tensor category, and give such a presentation for the categories C_{q_1,q_2} .

In Section 3 we use planar algebraic inspired techniques to completely presentation for an arbitrary pivotal tensor category with SO(4) fusion rules. These techniques were inspired by similar results in [BJ00, Liu16]. The presentation we describe is exactly the same as the category C_{q_1,q_2} , hence reconstruction techniques allow us to deduce that the arbitrary pivotal tensor category must be C_{q_1,q_2} . Our methods to describe the arbitrary presentation rely heavily on the SO(4) fusion rules for objects appearing in the tensor square, and the tensor cube, of the "vector representation". By working in the idempotent basis, we are able to use these fusion rules to pin down a large number of relations in our arbitrary category. The hard part of the argument is determining the Fourier transformation of our generators. By playing off the standard algebra multiplication in $End(X \otimes X)$ against the special convolution algebra structure, we are able to fully pin down the Fourier transform, and finish our presentation.

We finish the paper with Section 4, where we classify all the braidings on the monoidal categories C_{q_1,q_2} . The key idea to classify these braidings is to consider the adjoint subcategory C_{q_1,q_2}^{ad} , which we know is equivalent to a product of SO(3) type categories. The braidings on the SO(3) type categories are fully classified [TW05], and we can leverage this information up via some technical computations to classify all braidings on the full category C_{q_1,q_2} .

2 Preliminaries

We refer the reader to [EGNO15] for the basics on tensor categories.

2.1 Tensor categories with SO(4) fusion rules

In this subsection we present a family of pivotal tensor categories with SO(4) fusion rules. We build these categories using Deligne products of SU(2) categories.

Categories with SU(2) fusion rules (and their truncations) are known as type A categories. In the generic case there are infinitely many simple isotypes labeled $\mathbf{1} = X_0, X_1, X_2, \ldots$ and the fusion graph for multiplication by X_1 is

$$1 \qquad X_1 \qquad \cdots$$

In the fusion case there are finitely many simples $1, X_1, X_2, \ldots, X_{n-1}$ and the fusion graph for multiplication by X_1 is the truncated graph

$$X_{1}$$
 X_{1} X_{n-1}

Fusion categories with these fusion rules are known as A_n categories.

Type A and A_n categories are classified up to monoidal equivalence [FK93] by the dimension of the object X_1 , which can be expressed as

$$\dim(X_1) = [2]_q = q + q^{-1} \tag{1}$$

where q is a non-zero complex number which is not a root of unity in the generic case, and is a primitive root of unity in the fusion case. These categories are spherical and there is a unique choice of spherical structure such that X_1 is symmetrically self-dual. We denote a type A or A_n category with parameter q by \mathcal{A}_q . Note that $\mathcal{A}_q = \mathcal{A}_{q^{-1}}$.

The categories \mathcal{A}_q are all braided. The A and A_n are classified up to braided equivalence (which fixes distinguished object X_1), by the two eigenvalues of the braid σ_{X_1,X_1} . These eigenvalues are s and $-s^{-3}$ where s is a solution to either $s^2 = -q$ or $s^2 = -q^{-1}$. Hence there are four distinct braidings on each of the monoidal categories \mathcal{A}_q .

With the categories \mathcal{A}_q in hand, we can define the categories \mathcal{C}_{q_1,q_2} which appear in our main theorem.

Definition 2.1. Let C_{q_1,q_2} denote the sub-tensor category of $\mathcal{A}_{q_1} \boxtimes \mathcal{A}_{q_2}$ generated by $X := X_1 \boxtimes Y_1$ (we use X_1 , resp. Y_1 , to denote the generating object of \mathcal{A}_{q_1} , resp. \mathcal{A}_{q_2}).

The categories C_{q_1,q_2} inherit 16 braidings from the four braidings on each of \mathcal{A}_{q_1} and \mathcal{A}_{q_2} . These are parameterised by solutions to $s_1^2 = -q_1^{\pm 1}$ and $s_2^2 = -q_2^{\pm 1}$. The braided categories corresponding to the solutions (s_1, s_2) and $(-s_1, -s_2)$ are braided equivalent. Hence we get 8 distinct braidings on the categories C_{q_1,q_2} .

We say a category has SO(4) type fusion rules if its Grotheindeick ring is isomorphic to the Grotheindeick ring of a category C_{q_1,q_2} .

Definition 2.2. For $n_1, n_2 \in \mathbb{N} \cup \{\infty\}$ we define the fusion ring K_{n_1, n_2} by

$$K_{n_1,n_2} := K(\mathcal{C}_{q_1,q_2})$$

where each q_i is a non-zero complex number with order $2(n_i + 1)$.

Let us expand on the fusion rules K_{n_1,n_2} further, as an explicit description is useful later on in this paper. The simple elements of K_{n_1,n_2} are those $X_i \boxtimes Y_j$ with $0 \le i \le n_1 - 1, 0 \le j \le n_2 - 1$ and $i + j \in 2\mathbb{Z}$.

From the fusion graphs we see that all the fusion rings are \mathbb{Z}_2 -graded (since 1 only appears in even powers of X). The adjoint subcategories have fusion rules of $SO(3) \times SO(3)$ type, an important fact we will use later.

2.2 Presentations for semisimple tensor categories.

We recall some basic facts regarding presentations of semisimple spherical tensor categories, before providing a presentation of the categories C_{q_1,q_2} .

In this note a *based tensor category* will be a pair (\mathcal{C}, X) where X is a chosen tensor generator of a spherical tensor category \mathcal{C} . The A_n categories are conventionally based by picking a simple object corresponding to the vector representation of SU(2). Likewise, we consider any SO(4)-category based by a simple object X corresponding to the vector rep of SO(4).

Given a spherical tensor category C, let $\mathcal{N}(C)$ denote the tensor ideal of negligible morphisms in C. It is well-known that the quotient $C / \mathcal{N}(C)$ is a semisimple spherical tensor category, called the *semisimplification* of C [EO18].

A presentation of a (small) spherical based tensor category (\mathcal{C}, X) is a set of morphisms F between tensor powers of X, and a set of relations R satisfied in \mathcal{C} such that

$$\mathcal{C} \cong \overline{\mathcal{C}(F)/\mathcal{R}}$$

where $\mathcal{C}(F)$ is the free (based, strictly pivotal and strict monoidal) spherical \mathbb{C} -linear monoidal category generated by one object and the morphisms F, \mathcal{R} is the smallest tensor ideal of $\mathcal{C}(F)$ containing R, and the notation $\overline{\mathcal{C}}$ denotes the *Cauchy completion* (additive and idempotent completion) of a category \mathcal{C} .

For instance, an A_n category has a presentation with no generators and the relations

$$\bigcirc = [2]_q \qquad \text{and} \qquad f^{(n)} = 0$$

where q is a primitive 2(n + 1)-st root of 1 and $f^{(n)}$ denotes the n-th Jones-Wenzl projection. Note that here we have chosen a spherical structure which makes the generating object symmetrically self-dual. This allows us to draw unorientated strands.

Definition 2.3. A semisimple presentation of a based semisimple spherical tensor category C is a set of morphisms F and a set of relations R satisfied in C such that

$$\mathcal{C}' = \overline{\mathcal{C}(F)/\mathcal{R}}.$$

is a tensor category (in particular its tensor unit is simple), and

$$\mathcal{C} \cong \mathcal{C}' / \mathcal{N}(\mathcal{C}').$$

A semisimple presentation generally contains less information than a presentation (since we do not need to provide relations for the negligible ideal). For example, an A_n category has a semisimple presentation with no generators and the single relation

$$\bigcirc = [2]_q$$

where q is a primitive 2(n + 1)-st root of 1. The relation $f^{(n)} = 0$ is not necessary since the element $f^{(n)}$ gets sent to 0 when we quotient by negligibles.

The condition that $\overline{\mathcal{C}(F)/\mathcal{R}}$ (or equivalently $\mathcal{C}(F)/\mathcal{R}$) has a simple tensor unit is often summarized as "having enough relations to evaluate closed diagrams". The following well-known fact states that having enough relations to evaluate closed diagrams is a sufficient condition to produce a semisimple presentation.

Lemma 2.4. [BPMS12, Proposition 3.5] Suppose a semisimple spherical tensor category C is generated by morphisms F and satisfies relations R such that $C(F)/\mathcal{R}$ has a simple tensor unit. Then (F, R) is a semisimple presentation for C.

We can outline our argument for classifying SO(4)-type categories:

Step 1. Provide a semisimple presentation for the categories C_{q_1,q_2} (the presentation depends on q_1, q_2).

Step 2. Given an arbitrary category \mathcal{D} with SO(4)-type fusion rules, find parameters q_1, q_2 and morphisms in \mathcal{D} which satisfy the relations for \mathcal{C}_{q_1,q_2} from Step 1.

Step 3. Conclude that $\mathcal{D} \cong \mathcal{C}_{q_1,q_2}$, as follows. Let $\mathcal{C}' = \overline{\mathcal{C}(F)/\mathcal{R}}$ where (F, R) is the semisimple presentation of \mathcal{C}_{q_1,q_2} from Step 1. Observe that Step 2 provides a tensor functor

$$\Phi: \mathcal{C}' \to \mathcal{D}.$$

The kernel of Φ is a tensor ideal of \mathcal{C}' , which must be contained in $\mathcal{N}(\mathcal{C}')$ since $\mathcal{N}(\mathcal{C}')$ is the unique maximal tensor ideal of \mathcal{C}' . Let $\operatorname{Im}(\Phi)$ denote the image of Φ , a tensor subcategory of \mathcal{D} . If $X^{\otimes i}$ and $X^{\otimes j}$ are any objects of \mathcal{C}' , then we may also consider them objects of \mathcal{C}_{q_1,q_2} and \mathcal{D} (through mild abuse of notation), and the previous two sentences give inequalities

 $\dim \operatorname{Hom}_{\mathcal{C}_{q_i,q_0}}(X^{\otimes i}, X^{\otimes j}) \leq \dim \operatorname{Hom}_{\operatorname{Im}(\Phi)}(X^{\otimes i}, X^{\otimes j}) \leq \dim \operatorname{Hom}_{\mathcal{D}}(X^{\otimes i}, X^{\otimes j}).$

On the other hand,

$$\dim \operatorname{Hom}_{\mathcal{D}}(X^{\otimes i}, X^{\otimes j}) = \dim \operatorname{Hom}_{\mathcal{C}_{q_1, q_2}}(X^{\otimes i}, X^{\otimes j})$$

since \mathcal{D} and \mathcal{C}_{q_1,q_2} have the same fusion rules. Hence both inequalities above are equalities and in particular $\operatorname{Im}(\Phi) \simeq \mathcal{D}$. Since \mathcal{D} is semisimple, all negligible morphisms are zero so the kernel of Φ must be equal to $\mathcal{N}(\mathcal{C}')$. In conclusion, this shows $\mathcal{D} \simeq \mathcal{C}' / \mathcal{N}(\mathcal{C}') \simeq \mathcal{C}_{q_1,q_2}$.

With the above ansatz in mind, let's give a semisimple presentation for the categories C_{q_1,q_2} . To reduce clutter, we abbreviate the quantum numbers

$$[n]_{q_1}$$
 by $[n]_1$, and $[n]_{q_2}$ by $[n]_2$

Given a morphism $f \in \text{End}(X^{\otimes 2})$, we let $\rho(f)$ denote the *Fourier transform*, or one-click rotation of f:

$\rho(f)$	=	$\rho(f)$	=	$\rho(f)$
		Ŀ		$\left \begin{array}{c} \left \begin{array}{c} \left \begin{array}{c} \left \end{array}\right\rangle \right\rangle \right\rangle$

The second equality expresses that we assume our categories are strictly pivotal and that every object is self-dual (this is also equivalent to $\rho^2(f) = f$). Our presentation for \mathcal{C}_{q_1,q_2} will use two generators P and Qin $\operatorname{End}(X^{\otimes 2})$. They are defined by

$$P = \frac{1}{[2]_2} f^{(2)} \boxtimes \stackrel{\smile}{\frown} \text{and } Q = \frac{1}{[2]_1} \stackrel{\smile}{\frown} \boxtimes f^{(2)}, \tag{2}$$

where $f^{(2)}$ denotes the second Jones-Wenzl projection in the respective factors. With these definitions, P is the projection with image $X_2 \boxtimes \mathbf{1} \subset X^{\otimes 2}$ and Q is the projection with image $\mathbf{1} \boxtimes Y_2 \subset X^{\otimes 2}$.

Lemma 2.5. The morphisms P and Q generate C_{q_1,q_2} as a spherical tensor category.

$$P = \frac{1}{[2]_2} \left(g - \frac{1}{[2]_1} \overleftrightarrow{} \right) \text{ and } Q = \frac{1}{[2]_1} \left(h - \frac{1}{[2]_2} \overleftrightarrow{} \right),$$

the result will follow.

To show that g and h generate, it suffices to check they generate all the morphisms in the full tensor subcategory of C_{q_1,q_2} with objects $\mathbf{1}, X, X^{\otimes 2}, X^{\otimes 3}, \ldots$ (since X tensor generates C_{q_1,q_2}). Furthermore, C_{q_1,q_2} is \mathbb{Z}_2 -graded, so by Frobenius reciprocity it's enough to show that g and h generate the endomorphism algebras $\operatorname{End}(X^{\otimes k})$. We have

$$\operatorname{End}(X^{\otimes k}) \cong \operatorname{End}_{\mathcal{A}_{q_1}}(X_1^{\otimes k}) \otimes_{\mathbb{C}} \operatorname{End}_{\mathcal{A}_{q_2}}(Y_1^{\otimes k}).$$

The subalgebra $\operatorname{End}_{\mathcal{A}_{q_1}}(X_1^{\otimes k}) \boxtimes \operatorname{id}_k$ is generated (as an algebra) by the cup/cap elements $g_1, g_2, \ldots, g_{k-1}$ where

$$g_i = \mathrm{id}_{i-1} \otimes g \otimes \mathrm{id}_{k-i-1}$$
.

Similarly, $\operatorname{id}_k \boxtimes \operatorname{End}_{\mathcal{A}_{q_2}}(Y_1^{\otimes k})$ is generated (as an algebra) by the corresponding h_i 's. Hence g and h generate $\operatorname{End}(X^{\otimes k})$ (as a Hom space in a spherical tensor category).

Now that we know P and Q generate C_{q_1,q_2} , we can give a semisimple presentation generated by these two elements. This presentation is closely related to the Fuss-Catalan algebras of [BJ97]. By choosing spherical structures on the categories \mathcal{A}_{q_1} and \mathcal{A}_{q_2} , we can ensure that \mathcal{C}_{q_1,q_2} is generated by a symmetrially self-dual object.

Proposition 2.6. For q_1, q_2 non-zero complex numbers, the pivotal category C_{q_1,q_2} is tensor generated by the symmetrically self-dual object $X = X_1 \boxtimes Y_1$, and has a semisimple presentation with two generators $P, Q \in End(X^{\otimes 2})$ and the following relations:

(a)
$$\bigcirc = [2]_1 [2]_2$$

(b)
$$P^2 = P, Q^2 = Q$$
 and $PQ = QP = 0$

(c) Fourier equation:

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \Big| \Big| + \frac{1}{[2]_2^2} \underbrace{\smile}_{} + \frac{[2]_1}{[2]_2} Q.$$

(d) Bubble popping:

(e) Triangle popping:





Furthermore, for (s_1, s_2) solutions to $s_1^2 = -q_1^{\pm 1}$ and $s_2^2 = -q_2^{\pm 1}$, we have a braiding on \mathcal{C}_{q_1, q_2} defined by

$$= s_1 s_2 \left| \right| + \frac{\frac{q_1 s_1^2}{q_1^2 + 1} + \frac{q_2 s_2^2}{q_2^2 + 1} + 1}{s_1 s_2} + \frac{(q_2^2 + 1) s_1}{q_2 s_2} P + \frac{(q_1^2 + 1) s_2}{q_1 s_1} Q.$$

Remark 2.7. Note that the Fourier equation (c) implies

$$\rho(Q) = \frac{-1}{[2]_1[2]_2} | + \frac{1}{[2]_1^2} \stackrel{\smile}{\frown} + \frac{[2]_2}{[2]_1} P.$$

Proof. By Lemma 2.5, the morphisms P and Q generate the category. Checking that they satisfy the given relations we leave as an exercise in type A skein theory.

We must check that we have enough relations to describe the category C_{q_1,q_2} . By Lemma 2.4, it suffices to show that we can use the provided relations to evaluate any closed planar diagram made from P's and Q's to a scalar. We can represent such a diagram as a planar 4-valent graph with vertices labeled by P, Q, $\rho(P)$ or $\rho(Q)$. A simple modification of [MPS17, Lemma 6.15] shows that a planar 4-valent graph must contain either a loop, bigon, or triangle. We prove by induction on the number of vertices that the diagram can be reduced to a scalar using the relations. If there are no vertices, then relation (a) reduces any diagram (made of cups/caps) to a scalar. For the inductive step, note that if the graph contains any self-loops then the bubble popping relations allow one to reduce the number of vertices. If there are no self-loops, then the graph must contain a bigon or a triangle. The relations (b) and (c) imply that any diagram with a bigon can be reduced to a sum of diagrams with fewer vertices. Finally, any triangle can be reduced in a similar way using the triangle popping relations and relation (c) (possibly after applying a 2 or 4-click rotation to the triangle).

The braidings described in the final statement come from the known braidings on $\mathcal{A}_{q_1} \boxtimes \mathcal{A}_{q_2}$.

3 Monoidal Classification

In this section we classify pivotal categories \mathcal{C} with $K(\mathcal{C}) \cong K_{n_1,n_2}$. We may identify the Grothendieck ring of \mathcal{C} with that of \mathcal{C}_{q_1,q_2} thus use the symbols $X_a \boxtimes Y_b$ to denote simple objects in \mathcal{C} .

The subcategories tensor generated by $X_2 \boxtimes 1$ and $1 \boxtimes Y_2$ have SO(3)-type fusion rules. A result of Etingof and Ostrik ([EO18], Thms. A.1, A.3 and Remark A.4) states that any pivotal category with SO(3) type fusion rules is monoidally equivalent to $\operatorname{Rep}(SO(3)_q) \cong \mathcal{A}_q^{\operatorname{ad}}$ where q is not a root of unity or $q^2 = \pm 1$

(if there are infinitely many simples) or q is an appropriate root of unity in the fusion case.¹

$$\langle X_2 \boxtimes \mathbf{1} \rangle \cong \mathcal{A}_{q_1}^{\mathrm{ad}} \text{ and } \langle \mathbf{1} \boxtimes Y_2 \rangle \cong \mathcal{A}_{q_2}^{\mathrm{ad}}$$

where

- In the $K_{\infty,\infty}$ case, q_1 and q_2 are not roots of unity and/or $q_1^2 = 1$ and/or $q_2^2 = 1$.
- In the K_{n_1,n_2} case, q_1^2 is a primitive $(n_1 + 1)$ -st root of 1 and q_2^2 is a primitive $(n_2 + 1)$ -st root of 1.
- In the $K_{n_1,\infty}$ case, q_1^2 a primitive $(n_1 + 1)$ -st root of unity, and q_2 is not a root of 1 (or $q_2^2 = 1$).

In particular we have

$$\dim(X_1 \boxtimes \mathbf{1}) = [3]_1 \text{ and } \dim(\mathbf{1} \boxtimes Y_1) = [3]_2 \tag{3}$$

again using the convention $[n]_i = [n]_{q_i}$. Since $\mathcal{A}_{q_i}^{\mathrm{ad}} \simeq \mathcal{A}_{-q_i}^{\mathrm{ad}}$ we are free to replace q_i by $-q_i$.

Lemma 3.1. By possible replacing q_1 with $-q_1$ and/or modifying the spherical structure, we may assume $X = X_1 \boxtimes Y_1$ is symmetrically self-dual and

$$\dim(X) = [2]_1 [2]_2.$$

Proof. By changing the pivotal structure by an element of $\text{Hom}(\mathbb{Z}_2 \to \mathbb{C}^{\times})$ we can assume that X is symmetrically self-dual.

The fusion rules for ${\mathcal C}$ dictate

$$X^{\otimes 2} \cong \mathbf{1} \oplus X_2 \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes Y_2 \oplus X_2 \boxtimes Y_2. \tag{4}$$

Taking dimensions we find

$$\dim(X)^2 = 1 + [3]_1 + [3]_2 + [3]_1[3]_2.$$

Hence

$$\dim(X) = \pm [2]_1 [2]_2.$$

By possibly replacing q_1 with $-q_1$ we can ensure that $\dim(X) = [2]_1 [2]_2$

Remark 3.2. We note some small degenerate cases, which will allows us to restrict q_1 and q_2 . If either of n_1 or n_2 is equal to 2, then K_{n_1,n_2} has either type A or type A_n fusion rules. Classification is already known in these cases [FK93], so we can assume both q_1^2 and q_2^2 have orders larger than three. If both n_1 and n_2 are equal to 3, then K_{n_1,n_2} is a Tambara-Yamagami fusion ring with group $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, which is another case where classification is known [TY98]. Hence we can assume that the order of q_2^2 is greater than four.

3.1 Planar calculations

We now wish to obtain a semisimple presentation for the category C. To do this we first need to find generators. Using the fusion rule Eq. (4) we can define morphisms P and Q:

Definition 3.3. Let P and Q in $\operatorname{End}_{\mathcal{C}}(X^{\otimes 2})$ denote the minimal idempotents with images isomorphic to $X_2 \boxtimes \mathbf{1}$ and $\mathbf{1} \boxtimes Y_2$, respectively.

Note that $\{P, Q, | |, \bigcirc \}$ forms a basis for $\operatorname{End}_{\mathcal{C}}(X^{\otimes 2})$.

Our goal will be to show that P and Q generate a category with the same semisimple presentation as C_{q_1,q_2} . As mentioned previously, this will show that C must be monoidally equivalent to C_{q_1,q_2} .

Note that relation (a) is true from our choice of normalization and (b) follows from the fact P and Q are orthogonal idempotents. We show the rest of the relations hold in a series of lemmas.

¹This is only true when q is not a primitive eight root of unity. When q is a primitive eight root of unity we have $K(\operatorname{Rep}(SO(3)_q) \cong K(\operatorname{Vec}(\mathbb{Z}_2)))$, and there is an additional category coming from the twisting of the associator. However, in our situation we have that $(X_1 \boxtimes Y_1) \otimes (X_1 \boxtimes Y_1)$ is an algebra object which contains the simple object $X_2 \boxtimes \mathbf{1}$. Hence we can not have that $\langle X_2 \boxtimes \mathbf{1} \rangle \simeq \operatorname{Vec}^{\omega}(\mathbb{Z}_2)$. Hence we can still assume that $\langle X_2 \boxtimes \mathbf{1} \rangle \simeq \mathcal{A}_q^{ad}$.

Lemma 3.4. The bubble popping relations are satisfied in C.

Proof. If we cap off P or Q on the top or bottom, we must get 0 since P and Q are projections onto nontrivial objects of C. Capping the sides of P or Q must result in a scalar times the identity of X, and taking traces yields the result.

Lemma 3.5. The triangle popping relations are satisfied in C.

Proof. The relations that include both P and Q follow from the fusion rules. Let us prove the triangle relation involving three P's. By the fusion rules, $\text{Hom}_{\mathcal{C}}(P \otimes X^{\otimes 2}, P)$ is 2-dimensional if $n_1 > 3$, and 1-dimensional if $n_1 = 3$, and spanned by the following diagrams:



By turning the lower right strand upwards, it is seen that these diagrams are linearly independent if $n_1 > 3$ (as $P \otimes X \ncong X$). Therefore the triangle with 3 P's is a linear combination of these two diagrams. By precomposing with $\operatorname{id}_{X^{\otimes 2}} \otimes P$ and $\operatorname{id}_{X^{\otimes 2}} \otimes \bigcap$, the coefficients are determined and give the triangle popping relation.

The case of a triangle with three Q's is very similar.

The trickiest relation to prove is the Fourier transform equation (c). In order to do this we need to study the convolution algebra of $\operatorname{End}_{\mathcal{C}}(X^{\otimes 2})$. This is the algebra obtained by taking horizontal multiplication, which is denoted by *. Note that we have $x * y = \rho(\rho(x)\rho(y))$.

First observe we can compute the structure coefficients of the convolution algebra of $\operatorname{End}_{\mathcal{C}}(X^{\otimes 2})$ in the $\{P, Q, \bigcup_{i=1}^{i} | i\}$ basis. The convolution of anything with $\bigcup_{i=1}^{i}$ or | i is easy to figure out, so it suffices to compute $P \star P$, $P \star Q$ and $Q \star Q$. At this point it's useful to name another element of $\operatorname{End}(X^{\otimes 2})$:

Definition 3.6. Denote by R the minimal projection of $\operatorname{End}(X^{\otimes 2})$ of type $X_2 \boxtimes Y_2$.

The element R has the expression

$$R = \left| \left| -\frac{1}{[2]_1[2]_2} \widecheck{\frown} - P - Q \right| \right|$$

since the set $\{\frac{1}{[2]_1[2]_2} \overset{\bigcirc}{\frown}, P, Q, R\}$ form a complete set of minimal idempotents of $\operatorname{End}_{\mathcal{C}}(X \otimes X)$.

Lemma 3.7. We have the following relations in C:

$$P \star P = \frac{[3]_1}{[2]_1^2 [2]_2^2} \overleftrightarrow{} + \frac{[3]_1 - 1}{[2]_1 [2]_2} P$$

$$Q \star Q = \frac{[3]_2}{[2]_1^2 [2]_2^2} \overleftrightarrow{} + \frac{[3]_2 - 1}{[2]_1 [2]_2} Q$$

$$P \star Q = \frac{1}{[2]_1 [2]_2} R = \frac{1}{[2]_1 [2]_2} \left(\left| -\frac{1}{[2]_1 [2]_2} \overleftrightarrow{} - P - Q \right) \right|.$$

Proof. First consider the diagram for $P \star P$. It must be contained in the span of \subset and P (this follows from the fusion rules). The coefficients are determined by applying caps to the bottom and side (and using $\dim(P) = [3]_1$ and $\dim(X) = [2]_1[2]_2$).

The equation for $Q \star Q$ is verified similarly. To derive the equation for $P \star Q$, note that $P \otimes Q$ is a minimal idempotent of $\operatorname{End}_{\mathcal{C}}(X^{\otimes 4})$ whose image is a simple object of type $X_2 \boxtimes Y_2$. Since $X_2 \boxtimes Y_2$ appears with multiplicity 1 in $X \otimes X$ and $P \star Q$ factors through $P \otimes Q$, we see that $P \star Q$ is a scalar multiple of R. The scalar is computed by taking traces.

Remark 3.8. The above lemma shows that the structure constants of the convolution algebra in the $P, Q, [], \bigcirc$ basis depend only on q_1 and q_2 .

Now that we know the multiplication structure on the convolution algebra, it is routine to compute the minimal idempotents.

Lemma 3.9. A complete set of minimal idempotents for the convolution algebra $(End_{\mathcal{C}}(X \otimes X), \star)$ is given by

$$\begin{split} & \left\{ \frac{1}{[2]_1[2]_2} \middle| \quad \middle|, \\ & \frac{-1}{[2]_1[2]_2} \middle| \quad \middle| + \frac{1}{[2]_2^2} \bigvee + \frac{[2]_1}{[2]_2} Q, \\ & \frac{-1}{[2]_1[2]_2} \middle| \quad \middle| + \frac{1}{[2]_1^2} \bigvee + \frac{[2]_2}{[2]_1} P, \\ & \frac{1}{[2]_1[2]_2} \middle| \quad \middle| + (1 - \frac{1}{[2]_1^2} - \frac{1}{[2]_2^2}) \bigvee - \frac{[2]_2}{[2]_1} P - \frac{[2]_1}{[2]_2} Q \right\}. \end{split}$$

Proof. This can be checked directly using the structure constants given in the previous lemma.

As the Fourier transform preserves minimal idempotents, we can now pin down the Fourier transform of P (and hence Q) to one of two possibilities.

Lemma 3.10. We have have two possibilies for the Fourier transform of P. Either

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \left| + \frac{1}{[2]_2^2} \overleftrightarrow{} + \frac{[2]_1}{[2]_2} Q \right|$$
$$\rho(P) = \frac{-1}{[2]_1[2]_2} \left| + \frac{1}{[2]_2^2} \overleftrightarrow{} + \frac{[2]_2}{[2]_2} P \right|$$

or

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \left| + \frac{1}{[2]_1^2} + \frac{[2]_2}{[2]_1} F \right|$$

with the latter case only occuring when $q_1 = \pm q_2$.

Proof. The Fourier transform ρ intertwines the standard product and convolution product in End_c(X \otimes X), so $\rho(P)$ must be a minimal idempotent with respect to the convolution product. Hence it must belong to the set listed in the previous lemma. A simple computation shows that

$$\rho\left(\frac{1}{[2]_1[2]_2} \overleftrightarrow\right) = \frac{1}{[2]_1[2]_2} |$$

in the space $\operatorname{End}_{\mathcal{C}}(X \otimes X)$, and thus

$$\begin{split} \rho(P) &\in \{\frac{-1}{[2]_1[2]_2} | \ | + \frac{1}{[2]_2^2} \overleftrightarrow{} + \frac{[2]_1}{[2]_2} Q, \\ &\frac{-1}{[2]_1[2]_2} | \ | + \frac{1}{[2]_1^2} \overleftrightarrow{} + \frac{[2]_2}{[2]_1} P, \\ &\frac{1}{[2]_1[2]_2} | \ | + (1 - \frac{1}{[2]_1^2} - \frac{1}{[2]_2^2}) \overleftrightarrow{} - \frac{[2]_2}{[2]_1} P - \frac{[2]_1}{[2]_2} Q \} \end{split}$$

We want to rule out the third listed solution. Indeed, if $\rho(P)$ was equal to that solution then taking traces gives

$$[3]_1 = [3]_1 [3]_2,$$

which implies $[3]_2 = 1$, a contradiction to Remark 3.2.

In a similar fashion, if $\rho(P)$ was equal to the second solution, then taking traces shows $[3]_1 = [3]_2$. This can only happen if $q_1 = \pm q_2^{\pm 1}$. Finally, by considering fusion of depth three objects, we can deduce the Fourier transform equation (c): Lemma 3.11. In C we have the equation

$$\rho(P) = \frac{-1}{[2]_1[2]_2} | + \frac{1}{[2]_2^2} + \frac{[2]_1}{[2]_2} Q$$

Proof. It suffices to prove that the second solution for $\rho(P)$ and $\rho(Q)$ in the previous lemma is not possible. So assume for contradiction that

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \left| + \frac{1}{[2]_1^2} \stackrel{\checkmark}{\smile} + \frac{[2]_2}{[2]_1} P \right|$$

To find a contradiction, consider $(Q \otimes 1)(1 \otimes P)(Q \otimes 1)$. Note that $Q \otimes 1$ is a sum of two minimal idempotents, one a projection onto a simple isomorphic to X and the other a projection onto a simple of type $X_1 \boxtimes Y_3$. Since $X_1 \boxtimes Y_3$ does not occur in the image of $1 \boxtimes P$, we have $(Q \otimes 1)(1 \otimes P)(Q \otimes 1)$ must be a scalar times the projection onto X. Taking traces, this proves that



On the other hand, we have:



In the second equality we used our assumption about $\rho(P)$ and also the triangle popping relation to remove a triangle with two Q's and a P. These two expressions for $(Q \otimes 1)(1 \otimes P)(Q \otimes 1)$ can only be equal if $n_2 = 3$. However by Remark 3.2 we can assume $n_2 > 3$.

Remark 3.12. We remark that there do exist categories satisfying the relations of Proposition 2.6, except with the different Fourier transformation

$$\rho(P) = \frac{-1}{[2]_1[2]_2} \left| + \frac{1}{[2]_1^2} + \frac{[2]_2}{[2]_1} \right| + \frac{1}{[2]_1^2} + \frac{[2]_2}{[2]_1} P$$

This category is constructed as follows.

If $q_1 = q_2^{\pm 1}$, then the category \mathcal{C}_{q_1,q_2} has an order two monoidal auto-equivalence, which is the restriction of the swap auto-equivalence on $\mathcal{A}_{q_1} \boxtimes \mathcal{A}_{q_2}$. This auto-equivalence exchanges the minimal idempotents Pand Q. We claim that the subcategory of $\mathcal{C}_{q_1,q_2} \rtimes \mathbb{Z}_2$ generated by the object X in the non-trivial grading gives the desired category. We leave the proof of this fact to an interested reader.

Note that this subcategory of $C_{q_1,q_2} \rtimes \mathbb{Z}_2$ does not have SO(4)-type fusion rules. This differing of fusion rules can first be seen in the third tensor power of X, which explains why we have to consider 3 box relations in order to prove Lemma 3.11.

Putting everything together, we have given a semisimple presentation for a subcategory of C which is equivalent to the semisimple presentation of C_{q_1,q_2} for some $q_1, q_2 \in \mathbb{C}^{\times}$. As explained in the preliminaries, this implies that C is equivalent to C_{q_1,q_2} as a pivotal tensor category.

4 Braided Classification

In this section we classify all braidings on the fixed monoidal category C_{q_1,q_2} . We will show that the eight braidings given in Definition 2.1 and described in Proposition 2.6 are the only braidings on C_{q_1,q_2} .

We begin by considering the two distinguished subcategories $\mathcal{A}_{q_1}^{\mathrm{ad}}$ and $\mathcal{A}_{q_2}^{\mathrm{ad}}$. As these subcategories are equivalent to SO(3) type categories, we know that if the order of q_1 is greater than 8, then their braidings are classified by a choice of $q_1^{\pm 1}$ and $q_2^{\pm 1/2}$

The next lemma shows that the braidings on these subcategories determine the braiding on their product (which is the adjoint subcategory of C_{q_1,q_2}).

Lemma 4.1. Let C and D be semisimple tensor categories. Then braidings on $C \boxtimes D$ are determined by braidings on C and D, together with a bicharacter

$$a: U(\mathcal{C}) \times U(\mathcal{D}) \to \mathbb{C}$$
.

Proof. First we show how a braiding on $\mathcal{C} \boxtimes \mathcal{D}$ gives rise to braidings on \mathcal{C} and \mathcal{D} and a bicharacter. Clearly the braiding on the product gives braidings on the factors. Now suppose X is an object of \mathcal{C} and Y an object of \mathcal{D} . Then the braiding

$$c_{\mathbf{1}\boxtimes Y,X\boxtimes \mathbf{1}_{\circ}}:\mathbf{1}\boxtimes Y\otimes X\boxtimes \mathbf{1}\to X\boxtimes \mathbf{1}\otimes \mathbf{1}\boxtimes Y$$

describes a morphism $a_{X,Y} \in \operatorname{End}_{\mathcal{C} \boxtimes \mathcal{D}}(X \boxtimes Y)$. The naturality of the braiding on $\mathcal{C} \boxtimes \mathcal{D}$ implies $a_{X,Y}$ is an automorphism of the identity functor of $\mathcal{C} \boxtimes \mathcal{D}$. If we fix one of the factors (say fix an object X in \mathcal{C}) then the hexagon identity for the braiding implies $a_{X,-}$ is identified with a monoidal isomorphism of the identity functor of \mathcal{D} . In other words, the morphisms $a_{X,Y}$ for X fixed are described by a character of $U(\mathcal{D})$. The same considerations hold when fixing an object Y of \mathcal{D} , and the conclusion is that $a_{X,Y}$ may be identified with a bicharacter of $U(\mathcal{C}) \times U(\mathcal{D})$.

Now we show that braidings c_{X_1,X_2} on \mathcal{C} and d_{Y_1,Y_2} on \mathcal{D} together with a bicharacter a uniquely determine a braiding on $\mathcal{C} \boxtimes \mathcal{D}$. Suppose X_1, X_2 are in \mathcal{C} and Y_1, Y_2 are in \mathcal{D} . Then the braiding in $\mathcal{C} \boxtimes \mathcal{D}$ on $(X_1 \boxtimes Y_1) \otimes (X_2 \boxtimes Y_2)$ factors as

$$(c_{X_1,X_2} \boxtimes d_{Y_1,Y_2}) \circ (1 \otimes a_{X_2,Y_1} \otimes 1)$$

which shows how the braiding on the product is completely determined by c, d and a.

Corollary 4.2. There exist four distinct braidings on the subcategory

$$\mathcal{C}_{q_1,q_2}^{ad} = \mathcal{A}_{q_1}^{ad} \boxtimes \mathcal{A}_{q_2}^{ad}$$

These are parameterised by the four choices of $q_1^{\pm 1}$ and $q_2^{\pm 1}$.

Proof. The universal grading group of $\mathcal{A}_q^{\mathrm{ad}}$ is trivial, so by the previous lemma the braiding on $\mathcal{C}_{q_1,q_2}^{\mathrm{ad}}$ is determined by the braidings on the factors. By the classification of braidings on SO(3) type categories by Tuba and Wenzl [TW05] there are exactly two braidings on $\mathcal{A}_q^{\mathrm{ad}}$, parametrized by the choice of q or q^{-1} . \Box

Let us fix one of these four possible braidings. As the monoidal category C_{q_1,q_2} is determined up to $q_1 \rightarrow q_1^{-1}$ and $q_2 \rightarrow q_2^{-1}$, we can freely choose q_1 and q_2 so that this braiding corresponds to the choice q_1^{+1} and q_2^{+1} in the above lemma. In particular this gives us the following twists in C_{q_1,q_2} :

$$\theta_1 = 1, \qquad \theta_P = q_1^4, \qquad \theta_Q = q_2^4, \quad \text{and} \quad \theta_R = (q_1 q_2)^4.$$

With these twists in hand, it is straightforward to determine all possible braidings on C_{q_1,q_2} compatible with the fixed braiding on C_{q_1,q_2}^{ad} .

²In the case of $n_i \in \{3, 5\}$, there exist additional Tannakian braidings on the categories $\mathcal{A}_{q_i}^{ad}$. We can repeat the analysis of this section for these special cases. We find that these Tannakian braidings can not lift to braidings of the categories \mathcal{C}_{q_1,q_2} . Furthermore, in the case of $n_1 = 3$, we have that only two of the braidings on the subcategory $\mathcal{A}_{q_1}^{ad} \boxtimes \mathcal{A}_{q_2}^{ad}$ lift to the category \mathcal{C}_{q_1,q_2} . However in this case each of these two braidings on $\mathcal{A}_{q_1}^{ad} \boxtimes \mathcal{A}_{q_2}^{ad}$ has four extensions to \mathcal{C}_{q_1,q_2} . Hence these special cases are still covered by Theorem 1.1. We leave the details to a motivated reader.

Lemma 4.3. There exist two braidings on C_{q_1,q_2} which restrict to a fixed braiding on C_{q_1,q_2}^{ad} .

Proof. For this proof it is more convenient to work in the idempotent basis of $\operatorname{End}_{\mathcal{C}_{q_1,q_2}}(X \otimes X)$. The braiding on \mathcal{C}_{q_1,q_2} is determined by

$$\sum = \alpha_1 \frac{1}{[2]_1[2]_2} + \alpha_P P + \alpha_Q Q + \alpha_R R,$$

where $\alpha_1, \alpha_P, \alpha_Q, \alpha_R \in \mathbb{C}$. As we know the twists on 1, P, Q, and R we can use the balancing equation to find

$$1 = \theta_X^2 \alpha_1^2, \qquad q_1^4 = \theta_X^2 \alpha_P^2, \qquad q_2^4 = \theta_X^2 \alpha_Q^2, \text{ and } (q_1 q_2)^4 = \theta_X^2 \alpha_R^2.$$

This allows us to determine α_P, α_Q and α_R in terms of α_1 , up to sign. For some $\epsilon_P, \epsilon_Q, \epsilon_R \in \{-1, 1\}$ we have

$$\alpha_P = \epsilon_P q_1^2 \alpha_1, \qquad \alpha_Q = \epsilon_Q q_2^2 \alpha_1, \quad \text{and} \quad \alpha_R = \epsilon_R (q_1 q_2)^2 \alpha_1.$$

To determine α_1 and the three signs, we solve for the inverse of the braiding being equal to its Fourier transform. This gives us the following equations:

$$\alpha_R^{-1} = \frac{1}{[2]_1[2]_2} (\alpha_1 - \alpha_P - \alpha_Q + \alpha_R)$$
$$\frac{\alpha_1^{-1} - \alpha_R^{-1}}{[2]_1[2]_2} = \frac{\alpha_P}{[2]_2^2} + \frac{\alpha_Q}{[2]_1^2} + \alpha_R \left(1 - \frac{1}{[2]_1^2} - \frac{1}{[2]_2^2}\right)$$
$$\alpha_P^{-1} - \alpha_R^{-1} = \frac{[2]_2}{[2]_1} (\alpha_Q - \alpha_R)$$
$$\alpha_Q^{-1} - \alpha_R^{-1} = \frac{[2]_1}{[2]_2} (\alpha_P - \alpha_R).$$

The last two equations yield

$$[2]_1^2(2 - \epsilon_P \epsilon_R(q_2^2 + q_2^{-2})) = [2]_2^2(2 - \epsilon_Q \epsilon_R(q_1^2 + q_1^{-2})).$$

Solving this equation shows four cases:

$$\begin{split} \epsilon_P &= & \epsilon_Q &= & -\epsilon_R & \text{for all } q_1 \text{ and } q_2, \\ \epsilon_P &= & \epsilon_Q &= & \epsilon_R & \text{for } q_1 &= \pm q_2^{\pm 1}, \\ \epsilon_P &= & -\epsilon_Q &= & \epsilon_R & \text{for } q_1^2 &= -1, \text{ or } q_2^4 &= -1, \\ \epsilon_P &= & -\epsilon_Q &= & -\epsilon_R & \text{for } q_1^4 &= -1, \text{ or } q_2^2 &= -1. \end{split}$$

Immediately we can disregard the latter two cases, due to Remark 3.2. In the second case we can use the third equation to find

$$\alpha_1^2 = \begin{cases} \pm 1 & \text{if } q_2 = \pm q_1^{-1} \\ \mp q_1^{-6} & \text{if } q_2 = \pm q_1. \end{cases}$$

However we can now consider the first equation which tells us that either $q_2^4 = -1$ or q_2 is a primitive 6-th root of unity, both of which have already been dealt with in Remark 3.2.

Finally we have the first case. Again we use the third equation to find

$$\alpha_1^2 = \frac{\epsilon_P}{q_1^3 q_2^3}.$$

Comparing this to the first equation shows that $\epsilon_P = -1$. Hence we have two possible solutions for the braiding, corresponding to the two square roots of $\frac{-1}{q_1^3 q_2^3}$. These two braidings exist as they are realised in Proposition 2.6.

Putting everything together, we have classified all braidings on the categories C_{q_1,q_2} . This completes the proof of part 2 of Theorem 1.1.

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