# Classification of pivotal tensor categories with fusion rules related to $S O(4)$ 

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#### Abstract

In this paper we classify all semisimple tensor categories with the same fusion rules as $\operatorname{Rep}(S O(4))$, or one of the associated truncations. We show that such categories are explicitly classified by two non-zero complex numbers. Furthermore we show these tensor categories are always braided, and there exist exactly 8 braidings.


## 1 Introduction

In this note we continue the program to classify tensor categories with fusion rules the same as $\operatorname{Rep}(G)$ for $G$ a semisimple Lie group (or of the associated fusion categories). Classification is currently known for the majority of the classical Lie groups. The known results are for: $S U(2)$ [FK93], $S U(N)$ [KW93], $O(N)$ and $S p(N)$ TW05, and $S O(N)(N \neq 4)$ Cop20. The latter two results apply to ribbon categories, while the first two do not require any assumption of braiding and provide a classification for pivotal tensor categories. Our technique for $S O(4)$-type categories also does not require a braiding assumption.

The standard technique for attacking these classification problems is to identify the endomorphism algebras of tensor powers of the "vector representation" in an arbitrary tensor category with the same fusion rules of $\operatorname{Rep}(G)$, and to show that this algebra must agree with the known examples. In the case of $S U(N)$ this gives well-known quotients of the Hecke algebras Wen88, and in the $O(N)$ and $S O(N)$ cases we find quotients of BMW algebras BW89. For $S O(N)$ with $N \neq 4$ the endomorphism algebras also afford representations of the BMW algebra, but the image of the BMW algebra does not generate the endomorphism algebra for $S O(2 n)$ for $n>2$.

The gap at $S O(4)$ is due to the fact that the tensor square of the vector representation splits into four simples, rather than three (as is the case for every other $S O(N)$ with $N \geq 3$ ). This means that a braid element on $X^{\otimes 2}$ need not satisfy the cubic BMW skein relation, which was required for the method of Cop20.

There is another important distinction between $S O(4)$ and $S O(2 n)$ with $n>2$, which is that the root system for $S O(4)$ is not irreducible (its root system is the product $A_{1} \times A_{1}$ ). As we shall see, this manifests in categorifications of $S O(4)$ fusion rules being described by two independent parameters $q_{1}, q_{2}$, rather than a single parameter $q$.

In this paper we close this gap by studying a known $S O(4)$-type category and identifying the monoidal subcategory whose objects are tensor powers of the vector representation. This subcategory is essentially a planar algebra, and we describe it by generators and relations in a planar algebraic way, although we do not use that language. The planar algebras we describe can be seen as natural extensions of the Fuss-Catalan planar algebras [BJ97. We then show that the corresponding subcategory of any category with $S O$ (4)-type fusion rules must have the same presentation. We then obtain the classification of tensor categories with $S O(4)$ fusion rules from standard reconstruction arguments.

We say a tensor category has $S O(4)$ fusion rules if its Grothiendieck ring is isomorphic to $K(\operatorname{Rep}(S O(4)))$, or isomorphic to the Grothendieck ring of one of the associated fusion categories. We label these fusion rings by $K_{n_{1}, n_{2}}$ where $n_{i} \in \mathbb{N} \cup\{\infty\}$ (see Definition 2.2 for a precise definition). The fusion graph of $K_{n_{1}, n_{2}}$ for
the vector representation is given by (shown here with $n_{1}=5$ and $n_{2}=8$ ):


The classification of such categories is given in our main theorem.
Theorem 1.1. Let $\mathcal{C}$ be a pivotal tensor category with $K(\mathcal{C})=K_{n_{1}, n_{2}}$ where $n_{1}, n_{2} \in \mathbb{N}_{\geq 2} \cup \infty$. We have the following:

1. The category $\mathcal{C}$ is monoidally equivalent to $\mathcal{C}_{q_{1}, q_{2}}$ where $q_{1}, q_{2} \in \mathbb{C}^{\times}$with the order of $q_{i}^{2}$ equal to $n_{i}+1$ (or possibly $q_{i}^{2}=1$ if $n_{i}=\infty$ ). Further we have the monoidal equivalences

$$
\mathcal{C}_{q_{1}, q_{2}} \simeq \mathcal{C}_{q_{2}, q_{1}} \simeq \mathcal{C}_{q_{1}, q_{2}^{-1}} \simeq \mathcal{C}_{q_{1}^{-1}, q_{2}} \simeq \mathcal{C}_{-q_{1},-q_{2}}
$$

2. The category $\mathcal{C}_{q_{1}, q_{2}}$ is braided, and the possible braidings on these categories are parameterised by the set

$$
\left\{\left(s_{1}, s_{2}\right): s_{1}^{2}=-q_{1}^{ \pm 1} \quad \text { and } \quad s_{2}^{2}=-q_{2}^{ \pm 1}\right\} /\left\{\left(s_{1}, s_{2}\right)=\left(-s_{1},-s_{2}\right)\right\}
$$

When both $n_{1}, n_{2}>2$, these eight braidings are all distinct. If either $n_{1}$ or $n_{2}$ are equal to 2 , then four of these braidings are distinct. If both $n_{1}$ and $n_{2}$ are equal to 2 , then two of these braidings are distinct.
Constructions of these categories are given in Definition 2.1.
Remark 1.2. The above classification is up to equivalences which preserve the distinguished objects $f^{(1)} \boxtimes$ $f^{(1)}$ in the categories $\mathcal{C}_{q_{1}, q_{2}}$. The equivalences given in Theorem 1.1 are all the possible equivalences which preserve $f^{(1)} \boxtimes f^{(1)}$. There can exist additional equivalences between the categories $\mathcal{C}_{q_{1}, q_{2}}$ which don't preserve $f^{(1)} \boxtimes f^{(1)}$.

An illustrating example is seen in the case when $q_{2}^{2}$ is a root of unity of even order $n_{2}+1$ such that $\left[n_{2}\right]_{q_{2}}=-1$. For these parameters, we have that $\mathcal{C}_{q_{1}, q_{2}}$ is monoidally equivalent to $\mathcal{C}_{q_{1},-q_{2}}$ with the map sending $f^{(1)} \boxtimes f^{(1)}$ to $f^{(1)} \boxtimes f^{\left(n_{2}-1\right)}$.

This paper is outlined as follows.
In Section 2 we define the categories $\mathcal{C}_{q_{1}, q_{2}}$ which are examples of categories with $S O(4)$ fusion rules. We define what it means to give a semisimple presentation of a pivotal tensor category, and give such a presentation for the categories $\mathcal{C}_{q_{1}, q_{2}}$.

In Section 3 we use planar algebraic inspired techniques to completely presentation for an arbitrary pivotal tensor category with $S O(4)$ fusion rules. These techniques were inspired by similar results in [BJ00, Liu16]. The presentation we describe is exactly the same as the category $\mathcal{C}_{q_{1}, q_{2}}$, hence reconstruction techniques allow us to deduce that the arbitrary pivotal tensor category must be $\mathcal{C}_{q_{1}, q_{2}}$. Our methods to describe the arbitrary presentation rely heavily on the $S O(4)$ fusion rules for objects appearing in the tensor square, and the tensor cube, of the "vector representation". By working in the idempotent basis, we are able to use these fusion rules to pin down a large number of relations in our arbitrary category. The hard part of the argument is determining the Fourier transformation of our generators. By playing off the standard algebra multiplication in $\operatorname{End}(X \otimes X)$ against the special convolution algebra structure, we are able to fully pin down the Fourier transform, and finish our presentation.

We finish the paper with Section 4 where we classify all the braidings on the monoidal categories $\mathcal{C}_{q_{1}, q_{2}}$. The key idea to classify these braidings is to consider the adjoint subcategory $\mathcal{C}_{q_{1}, q_{2}}^{\text {ad }}$, which we know is equivalent to a product of $S O(3)$ type categories. The braidings on the $S O(3)$ type categories are fully classified TW05, and we can leverage this information up via some technical computations to classify all braidings on the full category $\mathcal{C}_{q_{1}, q_{2}}$.

## 2 Preliminaries

We refer the reader to EGNO15 for the basics on tensor categories.

### 2.1 Tensor categories with $S O(4)$ fusion rules

In this subsection we present a family of pivotal tensor categories with $S O(4)$ fusion rules. We build these categories using Deligne products of $S U(2)$ categories.

Categories with $S U(2)$ fusion rules (and their truncations) are known as type $A$ categories. In the generic case there are infinitely many simple isotypes labeled $\mathbf{1}=X_{0}, X_{1}, X_{2}, \ldots$ and the fusion graph for multiplication by $X_{1}$ is


In the fusion case there are finitely many simples $1, X_{1}, X_{2}, \ldots, X_{n-1}$ and the fusion graph for multiplication by $X_{1}$ is the truncated graph


Fusion categories with these fusion rules are known as $A_{n}$ categories.
Type $A$ and $A_{n}$ categories are classified up to monoidal equivalence [FK93] by the dimension of the object $X_{1}$, which can be expressed as

$$
\begin{equation*}
\operatorname{dim}\left(X_{1}\right)=[2]_{q}=q+q^{-1} \tag{1}
\end{equation*}
$$

where $q$ is a non-zero complex number which is not a root of unity in the generic case, and is a primitive root of unity in the fusion case. These categories are spherical and there is a unique choice of spherical structure such that $X_{1}$ is symmetrically self-dual. We denote a type $A$ or $A_{n}$ category with parameter $q$ by $\mathcal{A}_{q}$. Note that $\mathcal{A}_{q}=\mathcal{A}_{q^{-1}}$.

The categories $\mathcal{A}_{q}$ are all braided. The $A$ and $A_{n}$ are classified up to braided equivalence (which fixes distinguished object $X_{1}$ ), by the two eigenvalues of the braid $\sigma_{X_{1}, X_{1}}$. These eigenvalues are $s$ and $-s^{-3}$ where $s$ is a solution to either $s^{2}=-q$ or $s^{2}=-q^{-1}$. Hence there are four distinct braidings on each of the monoidal categories $\mathcal{A}_{q}$.

With the categories $\mathcal{A}_{q}$ in hand, we can define the categories $\mathcal{C}_{q_{1}, q_{2}}$ which appear in our main theorem.
Definition 2.1. Let $\mathcal{C}_{q_{1}, q_{2}}$ denote the sub-tensor category of $\mathcal{A}_{q_{1}} \boxtimes \mathcal{A}_{q_{2}}$ generated by $X:=X_{1} \boxtimes Y_{1}$ (we use $X_{1}$, resp. $Y_{1}$, to denote the generating object of $\mathcal{A}_{q_{1}}$, resp. $\mathcal{A}_{q_{2}}$ ).

The categories $\mathcal{C}_{q_{1}, q_{2}}$ inherit 16 braidings from the four braidings on each of $\mathcal{A}_{q_{1}}$ and $\mathcal{A}_{q_{2}}$. These are parameterised by solutions to $s_{1}^{2}=-q_{1}^{ \pm 1}$ and $s_{2}^{2}=-q_{2}^{ \pm 1}$. The braided categories corresponding to the solutions $\left(s_{1}, s_{2}\right)$ and $\left(-s_{1},-s_{2}\right)$ are braided equivalent. Hence we get 8 distinct braidings on the categories $\mathcal{C}_{q_{1}, q_{2}}$.

We say a category has $S O(4)$ type fusion rules if its Grotheindeick ring is isomorphic to the Grotheindeick ring of a category $\mathcal{C}_{q_{1}, q_{2}}$.
Definition 2.2. For $n_{1}, n_{2} \in \mathbb{N} \cup\{\infty\}$ we define the fusion ring $K_{n_{1}, n_{2}}$ by

$$
K_{n_{1}, n_{2}}:=K\left(\mathcal{C}_{q_{1}, q_{2}}\right)
$$

where each $q_{i}$ is a non-zero complex number with order $2\left(n_{i}+1\right)$.

Let us expand on the fusion rules $K_{n_{1}, n_{2}}$ further, as an explicit description is useful later on in this paper. The simple elements of $K_{n_{1}, n_{2}}$ are those $X_{i} \boxtimes Y_{j}$ with $0 \leq i \leq n_{1}-1,0 \leq j \leq n_{2}-1$ and $i+j \in 2 \mathbb{Z}$.

From the fusion graphs we see that all the fusion rings are $\mathbb{Z}_{2}$-graded (since $\mathbf{1}$ only appears in even powers of $X)$. The adjoint subcategories have fusion rules of $S O(3) \times S O(3)$ type, an important fact we will use later.

### 2.2 Presentations for semisimple tensor categories.

We recall some basic facts regarding presentations of semisimple spherical tensor categories, before providing a presentation of the categories $\mathcal{C}_{q_{1}, q_{2}}$.

In this note a based tensor category will be a pair $(\mathcal{C}, X)$ where $X$ is a chosen tensor generator of a spherical tensor category $\mathcal{C}$. The $A_{n}$ categories are conventionally based by picking a simple object corresponding to the vector representation of $S U(2)$. Likewise, we consider any $S O(4)$-category based by a simple object $X$ corresponding to the vector rep of $S O(4)$.

Given a spherical tensor category $\mathcal{C}$, let $\mathcal{N}(\mathcal{C})$ denote the tensor ideal of negligible morphisms in $\mathcal{C}$. It is well-known that the quotient $\mathcal{C} / \mathcal{N}(\mathcal{C})$ is a semisimple spherical tensor category, called the semisimplification of $\mathcal{C}$ EO18.

A presentation of a (small) spherical based tensor category $(\mathcal{C}, X)$ is a set of morphisms $F$ between tensor powers of $X$, and a set of relations $R$ satisfied in $\mathcal{C}$ such that

$$
\mathcal{C} \cong \overline{\mathcal{C}(F) / \mathcal{R}}
$$

where $\mathcal{C}(F)$ is the free (based, strictly pivotal and strict monoidal) spherical $\mathbb{C}$-linear monoidal category generated by one object and the morphisms $F, \mathcal{R}$ is the smallest tensor ideal of $\mathcal{C}(F)$ containing $R$, and the notation $\overline{\mathcal{C}}$ denotes the Cauchy completion (additive and idempotent completion) of a category $\mathcal{C}$.

For instance, an $A_{n}$ category has a presentation with no generators and the relations

$$
\bigcirc=[2]_{q} \quad \text { and } \quad f^{(n)}=0
$$

where $q$ is a primitive $2(n+1)$-st root of 1 and $f^{(n)}$ denotes the $n$-th Jones-Wenzl projection. Note that here we have chosen a spherical strucure which makes the generating object symmetrically self-dual. This allows us to draw unorientated strands.

Definition 2.3. A semisimple presentation of a based semisimple spherical tensor category $\mathcal{C}$ is a set of morphisms $F$ and a set of relations $R$ satisfied in $\mathcal{C}$ such that

$$
\mathcal{C}^{\prime}=\overline{\mathcal{C}(F) / \mathcal{R}}
$$

is a tensor category (in particular its tensor unit is simple), and

$$
\mathcal{C} \cong \mathcal{C}^{\prime} / \mathcal{N}\left(\mathcal{C}^{\prime}\right)
$$

A semisimple presentation generally contains less information than a presentation (since we do not need to provide relations for the negligible ideal). For example, an $A_{n}$ category has a semisimple presentation with no generators and the single relation

$$
\bigcirc=[2]_{q}
$$

where $q$ is a primitive $2(n+1)$-st root of 1 . The relation $f^{(n)}=0$ is not necessary since the element $f^{(n)}$ gets sent to 0 when we quotient by negligibles.

The condition that $\overline{\mathcal{C}(F) / \mathcal{R}}$ (or equivalently $\mathcal{C}(F) / \mathcal{R}$ ) has a simple tensor unit is often summarized as "having enough relations to evaluate closed diagrams". The following well-known fact states that having enough relations to evaluate closed diagrams is a sufficient condition to produce a semisimple presentation.

Lemma 2.4. [BPMS12, Proposition 3.5] Suppose a semisimple spherical tensor category $\mathcal{C}$ is generated by morphisms $F$ and satisfies relations $R$ such that $\mathcal{C}(F) / \mathcal{R}$ has a simple tensor unit. Then $(F, R)$ is a semisimple presentation for $\mathcal{C}$.

We can outline our argument for classifying $S O(4)$-type categories:
Step 1. Provide a semisimple presentation for the categories $\mathcal{C}_{q_{1}, q_{2}}$ (the presentation depends on $q_{1}, q_{2}$ ).
Step 2. Given an arbitrary category $\mathcal{D}$ with $S O(4)$-type fusion rules, find parameters $q_{1}, q_{2}$ and morphisms in $\mathcal{D}$ which satisfy the relations for $\mathcal{C}_{q_{1}, q_{2}}$ from Step 1.

Step 3. Conclude that $\mathcal{D} \cong \mathcal{C}_{q_{1}, q_{2}}$, as follows. Let $\mathcal{C}^{\prime}=\overline{\mathcal{C}(F) / \mathcal{R}}$ where $(F, R)$ is the semisimple presentation of $\mathcal{C}_{q_{1}, q_{2}}$ from Step 1. Observe that Step 2 provides a tensor functor

$$
\Phi: \mathcal{C}^{\prime} \rightarrow \mathcal{D}
$$

The kernel of $\Phi$ is a tensor ideal of $\mathcal{C}^{\prime}$, which must be contained in $\mathcal{N}\left(\mathcal{C}^{\prime}\right)$ since $\mathcal{N}\left(\mathcal{C}^{\prime}\right)$ is the unique maximal tensor ideal of $\mathcal{C}^{\prime}$. Let $\operatorname{Im}(\Phi)$ denote the image of $\Phi$, a tensor subcategory of $\mathcal{D}$. If $X^{\otimes i}$ and $X^{\otimes j}$ are any objects of $\mathcal{C}^{\prime}$, then we may also consider them objects of $\mathcal{C}_{q_{1}, q_{2}}$ and $\mathcal{D}$ (through mild abuse of notation), and the previous two sentences give inequalities

$$
\operatorname{dim} \operatorname{Hom}_{\mathcal{C}_{q_{1}, q_{2}}}\left(X^{\otimes i}, X^{\otimes j}\right) \leq \operatorname{dim} \operatorname{Hom}_{\operatorname{Im}(\Phi)}\left(X^{\otimes i}, X^{\otimes j}\right) \leq \operatorname{dim} \operatorname{Hom}_{\mathcal{D}}\left(X^{\otimes i}, X^{\otimes j}\right)
$$

On the other hand,

$$
\operatorname{dim} \operatorname{Hom}_{\mathcal{D}}\left(X^{\otimes i}, X^{\otimes j}\right)=\operatorname{dim} \operatorname{Hom}_{\mathcal{C}_{q_{1}, q_{2}}}\left(X^{\otimes i}, X^{\otimes j}\right)
$$

since $\mathcal{D}$ and $\mathcal{C}_{q_{1}, q_{2}}$ have the same fusion rules. Hence both inequalities above are equalities and in particular $\operatorname{Im}(\Phi) \simeq \mathcal{D}$. Since $\mathcal{D}$ is semisimple, all negligible morphisms are zero so the kernel of $\Phi$ must be equal to $\mathcal{N}\left(\mathcal{C}^{\prime}\right)$. In conclusion, this shows $\mathcal{D} \simeq \mathcal{C}^{\prime} / \mathcal{N}\left(\mathcal{C}^{\prime}\right) \simeq \mathcal{C}_{q_{1}, q_{2}}$.

With the above ansatz in mind, let's give a semisimple presentation for the categories $\mathcal{C}_{q_{1}, q_{2}}$. To reduce clutter, we abbreviate the quantum numbers

$$
[n]_{q_{1}} \text { by }[n]_{1}, \text { and }[n]_{q_{2}} \text { by }[n]_{2} .
$$

Given a morphism $f \in \operatorname{End}\left(X^{\otimes 2}\right)$, we let $\rho(f)$ denote the Fourier transform, or one-click rotation of $f$ :


The second equality expresses that we assume our categories are strictly pivotal and that every object is self-dual (this is also equivalent to $\rho^{2}(f)=f$ ). Our presentation for $\mathcal{C}_{q_{1}, q_{2}}$ will use two generators $P$ and $Q$ in $\operatorname{End}\left(X^{\otimes 2}\right)$. They are defined by

$$
\begin{equation*}
P=\frac{1}{[2]_{2}} f^{(2)} \boxtimes \supseteq \text { and } Q=\frac{1}{[2]_{1}} \preceq \boxtimes f^{(2)} \tag{2}
\end{equation*}
$$

where $f^{(2)}$ denotes the second Jones-Wenzl projection in the respective factors. With these definitions, $P$ is the projection with image $X_{2} \boxtimes \mathbf{1} \subset X^{\otimes 2}$ and $Q$ is the projection with image $\mathbf{1} \boxtimes Y_{2} \subset X^{\otimes 2}$.

Lemma 2.5. The morphisms $P$ and $Q$ generate $\mathcal{C}_{q_{1}, q_{2}}$ as a spherical tensor category.

Proof. This has been proved in greater generality using planar algebra language by Liu Liu16, Corollary 3.2]. We provide a proof in our case for the reader's convenience. We will show that the simpler morphisms $g=| | \boxtimes \bigcup$ and $h=\bigcup \boxtimes| |$ generate $\mathcal{C}_{q_{1}, q_{2}}$. Since $P$ and $Q$ are related to $g$ and $h$ by the equations

$$
P=\frac{1}{[2]_{2}}\left(g-\frac{1}{[2]_{1}} \smile\right) \text { and } Q=\frac{1}{[2]_{1}}\left(h-\frac{1}{[2]_{2}} \smile\right) \text {, }
$$

the result will follow.
To show that $g$ and $h$ generate, it suffices to check they generate all the morphisms in the full tensor subcategory of $\mathcal{C}_{q_{1}, q_{2}}$ with objects $\mathbf{1}, X, X^{\otimes 2}, X^{\otimes 3}, \ldots$ (since $X$ tensor generates $\mathcal{C}_{q_{1}, q_{2}}$ ). Furthermore, $\mathcal{C}_{q_{1}, q_{2}}$ is $\mathbb{Z}_{2}$-graded, so by Frobenius reciprocity it's enough to show that $g$ and $h$ generate the endomorphism algebras $\operatorname{End}\left(X^{\otimes k}\right)$. We have

$$
\operatorname{End}\left(X^{\otimes k}\right) \cong \operatorname{End}_{\mathcal{A}_{q_{1}}}\left(X_{1}^{\otimes k}\right) \otimes_{\mathbb{C}} \operatorname{End}_{\mathcal{A}_{q_{2}}}\left(Y_{1}^{\otimes k}\right) .
$$

The subalgebra $\operatorname{End}_{\mathcal{A}_{q_{1}}}\left(X_{1}^{\otimes k}\right) \boxtimes \mathrm{id}_{k}$ is generated (as an algebra) by the cup/cap elements $g_{1}, g_{2}, \ldots, g_{k-1}$ where

$$
g_{i}=\mathrm{id}_{i-1} \otimes g \otimes \mathrm{id}_{k-i-1} .
$$

Similarly, $\operatorname{id}_{k} \boxtimes \operatorname{End}_{\mathcal{A}_{q_{2}}}\left(Y_{1}^{\otimes k}\right)$ is generated (as an algebra) by the corresponding $h_{i}$ 's. Hence $g$ and $h$ generate $\operatorname{End}\left(X^{\otimes k}\right)$ (as a Hom space in a spherical tensor category).

Now that we know $P$ and $Q$ generate $\mathcal{C}_{q_{1}, q_{2}}$, we can give a semisimple presentation generated by these two elements. This presentation is closely related to the Fuss-Catalan algebras of [BJ97]. By choosing spherical structures on the categories $\mathcal{A}_{q_{1}}$ and $\mathcal{A}_{q_{2}}$, we can ensure that $\mathcal{C}_{q_{1}, q_{2}}$ is generated by a symmetrially self-dual object.

Proposition 2.6. For $q_{1}, q_{2}$ non-zero complex numbers, the pivotal category $\mathcal{C}_{q_{1}, q_{2}}$ is tensor generated by the symmetrically self-dual object $X=X_{1} \boxtimes Y_{1}$, and has a semisimple presentation with two generators $P, Q \in E n d\left(X^{\otimes 2}\right)$ and the following relations:
(a) $\bigcirc=[2]_{1}[2]_{2}$
(b) $P^{2}=P, Q^{2}=Q$ and $P Q=Q P=0$
(c) Fourier equation:

$$
\rho(P)=\frac{-1}{[2]_{1}[2]_{2}}| |+\frac{1}{[2]_{2}^{2}} 乞+\frac{[2]_{1}}{[2]_{2}} Q .
$$

(d) Bubble popping:

$$
\begin{aligned}
& \frac{\square}{{ }^{P}}=\frac{\square \downarrow}{{ }^{P}}=\frac{\square}{Q}=\frac{\square \downarrow}{Q}=0
\end{aligned}
$$

(e) Triangle popping:



Furthermore, for $\left(s_{1}, s_{2}\right)$ solutions to $s_{1}^{2}=-q_{1}^{ \pm 1}$ and $s_{2}^{2}=-q_{2}^{ \pm 1}$, we have a braiding on $\mathcal{C}_{q_{1}, q_{2}}$ defined by

$$
==s_{1} s_{2}| |+\frac{\frac{q_{1} s_{1}^{2}}{q_{1}^{2}+1}+\frac{q_{2} s_{2}^{2}}{q_{2}^{2}+1}+1}{s_{1} s_{2}} \asymp+\frac{\left(q_{2}^{2}+1\right) s_{1}}{q_{2} s_{2}} P+\frac{\left(q_{1}^{2}+1\right) s_{2}}{q_{1} s_{1}} Q
$$

Remark 2.7. Note that the Fourier equation (c) implies

$$
\rho(Q)=\frac{-1}{[2]_{1}[2]_{2}} \left\lvert\,+\frac{1}{[2]_{1}^{2}} \frown+\frac{[2]_{2}}{[2]_{1}} P .\right.
$$

Proof. By Lemma 2.5, the morphisms $P$ and $Q$ generate the category. Checking that they satisfy the given relations we leave as an exercise in type $A$ skein theory.

We must check that we have enough relations to describe the category $\mathcal{C}_{q_{1}, q_{2}}$. By Lemma 2.4 , it suffices to show that we can use the provided relations to evaluate any closed planar diagram made from $P$ 's and $Q$ 's to a scalar. We can represent such a diagram as a planar 4 -valent graph with vertices labeled by $P$, $Q, \rho(P)$ or $\rho(Q)$. A simple modification of MPS17, Lemma 6.15] shows that a planar 4-valent graph must contain either a loop, bigon, or triangle. We prove by induction on the number of vertices that the diagram can be reduced to a scalar using the relations. If there are no vertices, then relation (a) reduces any diagram (made of cups/caps) to a scalar. For the inductive step, note that if the graph contains any self-loops then the bubble popping relations allow one to reduce the number of vertices. If there are no self-loops, then the graph must contain a bigon or a triangle. The relations (b) and (c) imply that any diagram with a bigon can be reduced to a sum of diagrams with fewer vertices. Finally, any triangle can be reduced in a similar way using the triangle popping relations and relation (c) (possibly after applying a 2 or 4 -click rotation to the triangle).

The braidings described in the final statement come from the known braidings on $\mathcal{A}_{q_{1}} \boxtimes \mathcal{A}_{q_{2}}$.

## 3 Monoidal Classification

In this section we classify pivotal categories $\mathcal{C}$ with $K(\mathcal{C}) \cong K_{n_{1}, n_{2}}$. We may identify the Grothendieck ring of $\mathcal{C}$ with that of $\mathcal{C}_{q_{1}, q_{2}}$ thus use the symbols $X_{a} \boxtimes Y_{b}$ to denote simple objects in $\mathcal{C}$.

The subcategories tensor generated by $X_{2} \boxtimes 1$ and $1 \boxtimes Y_{2}$ have $S O(3)$-type fusion rules. A result of Etingof and Ostrik ( EO 18 , Thms. A.1, A. 3 and Remark A.4) states that any pivotal category with $S O(3)$ type fusion rules is monoidally equivalent to $\operatorname{Rep}\left(S O(3)_{q}\right) \cong \mathcal{A}_{q}^{\text {ad }}$ where $q$ is not a root of unity or $q^{2}= \pm 1$
(if there are infinitely many simples) or $q$ is an appropriate root of unity in the fusion case. ${ }^{1}$

$$
\left\langle X_{2} \boxtimes \mathbf{1}\right\rangle \cong \mathcal{A}_{q_{1}}^{\text {ad }} \text { and }\left\langle\mathbf{1} \boxtimes Y_{2}\right\rangle \cong \mathcal{A}_{q_{2}}^{\text {ad }}
$$

where

- In the $K_{\infty, \infty}$ case, $q_{1}$ and $q_{2}$ are not roots of unity and/or $q_{1}^{2}=1$ and/or $q_{2}^{2}=1$.
- In the $K_{n_{1}, n_{2}}$ case, $q_{1}^{2}$ is a primitive $\left(n_{1}+1\right)$-st root of 1 and $q_{2}^{2}$ is a primitive $\left(n_{2}+1\right)$-st root of 1 .
- In the $K_{n_{1}, \infty}$ case, $q_{1}^{2}$ a primitive $\left(n_{1}+1\right)$-st root of unity, and $q_{2}$ is not a root of $1\left(\right.$ or $\left.q_{2}^{2}=1\right)$.

In particular we have

$$
\begin{equation*}
\operatorname{dim}\left(X_{1} \boxtimes \mathbf{1}\right)=[3]_{1} \text { and } \operatorname{dim}\left(\mathbf{1} \boxtimes Y_{1}\right)=[3]_{2} \tag{3}
\end{equation*}
$$

again using the convention $[n]_{i}=[n]_{q_{i}}$. Since $\mathcal{A}_{q_{i}}^{\text {ad }} \simeq \mathcal{A}_{-q_{i}}^{\text {ad }}$ we are free to replace $q_{i}$ by $-q_{i}$.
Lemma 3.1. By possible replacing $q_{1}$ with $-q_{1}$ and/or modifying the spherical structure, we may assume $X=X_{1} \boxtimes Y_{1}$ is symmetrically self-dual and

$$
\operatorname{dim}(X)=[2]_{1}[2]_{2}
$$

Proof. By changing the pivotal structure by an element of $\operatorname{Hom}\left(\mathbb{Z}_{2} \rightarrow \mathbb{C}^{\times}\right)$we can assume that $X$ is symmetrically self-dual.

The fusion rules for $\mathcal{C}$ dictate

$$
\begin{equation*}
X^{\otimes 2} \cong \mathbf{1} \oplus X_{2} \boxtimes \mathbf{1} \oplus \mathbf{1} \boxtimes Y_{2} \oplus X_{2} \boxtimes Y_{2} \tag{4}
\end{equation*}
$$

Taking dimensions we find

$$
\operatorname{dim}(X)^{2}=1+[3]_{1}+[3]_{2}+[3]_{1}[3]_{2}
$$

Hence

$$
\operatorname{dim}(X)= \pm[2]_{1}[2]_{2}
$$

By possibly replacing $q_{1}$ with $-q_{1}$ we can ensure that $\operatorname{dim}(X)=[2]_{1}[2]_{2}$
Remark 3.2. We note some small degenerate cases, which will allows us to restrict $q_{1}$ and $q_{2}$. If either of $n_{1}$ or $n_{2}$ is equal to 2 , then $K_{n_{1}, n_{2}}$ has either type $A$ or type $A_{n}$ fusion rules. Classification is already known in these cases [FK93], so we can assume both $q_{1}^{2}$ and $q_{2}^{2}$ have orders larger than three. If both $n_{1}$ and $n_{2}$ are equal to 3 , then $K_{n_{1}, n_{2}}$ is a Tambara-Yamagami fusion ring with group $G=\mathbb{Z}_{2} \times \mathbb{Z}_{2}$, which is another case where classification is known TY98. Hence we can assume that the order of $q_{2}^{2}$ is greater than four.

### 3.1 Planar calculations

We now wish to obtain a semisimple presentation for the category $\mathcal{C}$. To do this we first need to find generators. Using the fusion rule Eq. (4) we can define morphisms $P$ and $Q$ :

Definition 3.3. Let $P$ and $Q$ in $\operatorname{End}_{\mathcal{C}}\left(X^{\otimes 2}\right)$ denote the minimal idempotents with images isomorphic to $X_{2} \boxtimes 1$ and $1 \boxtimes Y_{2}$, respectively.

Note that $\{P, Q,| | \bigvee\}$ forms a basis for $\operatorname{End}_{\mathcal{C}}\left(X^{\otimes 2}\right)$.
Our goal will be to show that $P$ and $Q$ generate a category with the same semisimple presentation as $\mathcal{C}_{q_{1}, q_{2}}$. As mentioned previously, this will show that $\mathcal{C}$ must be monoidally equivalent to $\mathcal{C}_{q_{1}, q_{2}}$.

Note that relation (a) is true from our choice of normalization and (b) follows from the fact $P$ and $Q$ are orthogonal idempotents. We show the rest of the relations hold in a series of lemmas.

[^0]Lemma 3.4. The bubble popping relations are satisfied in $\mathcal{C}$.
Proof. If we cap off $P$ or $Q$ on the top or bottom, we must get 0 since $P$ and $Q$ are projections onto nontrivial objects of $\mathcal{C}$. Capping the sides of $P$ or $Q$ must result in a scalar times the identity of $X$, and taking traces yields the result.

Lemma 3.5. The triangle popping relations are satisfied in $\mathcal{C}$.
Proof. The relations that include both $P$ and $Q$ follow from the fusion rules. Let us prove the triangle relation involving three $P$ 's. By the fusion rules, $\operatorname{Hom}_{\mathcal{C}}\left(P \otimes X^{\otimes 2}, P\right)$ is 2-dimensional if $n_{1}>3$, and 1-dimensional if $n_{1}=3$, and spanned by the following diagrams:


By turning the lower right strand upwards, it is seen that these diagrams are linearly independent if $n_{1}>3$ (as $P \otimes X \nsupseteq X$ ). Therefore the triangle with 3 P's is a linear combination of these two diagrams. By precomposing with $\operatorname{id}_{X^{\otimes 2}} \otimes P$ and $\operatorname{id}_{X^{\otimes 2}} \otimes \frown$, the coefficients are determined and give the triangle popping relation.

The case of a triangle with three $Q$ 's is very similar.
The trickiest relation to prove is the Fourier transform equation (c). In order to do this we need to study the convolution algebra of $\operatorname{End}_{\mathcal{C}}\left(X^{\otimes 2}\right)$. This is the algebra obtained by taking horizontal multiplication, which is denoted by $*$. Note that we have $x * y=\rho(\rho(x) \rho(y))$.

First observe we can compute the structure coefficients of the convolution algebra of $\operatorname{End}_{\mathcal{C}}\left(X^{\otimes 2}\right)$ in the $\{P, Q, \preceq,| |\}$ basis. The convolution of anything with $\curvearrowleft$ or $\mid$ |is easy to figure out, so it suffices to compute $P \star P, P \star Q$ and $Q \star Q$. At this point it's useful to name another element of $\operatorname{End}\left(X^{\otimes 2}\right)$ :
Definition 3.6. Denote by $R$ the minimal projection of $\operatorname{End}\left(X^{\otimes 2}\right)$ of type $X_{2} \boxtimes Y_{2}$.
The element $R$ has the expression

$$
R=| |-\frac{1}{[2]_{1}[2]_{2}} \supseteq-P-Q
$$

since the set $\left\{\frac{1}{[2]_{1}[2]_{2}} \bigcup, P, Q, R\right\}$ form a complete set of minimal idempotents of $\operatorname{End}_{\mathcal{C}}(X \otimes X)$.
Lemma 3.7. We have the following relations in $\mathcal{C}$ :

$$
\begin{aligned}
P \star P & =\frac{[3]_{1}}{[2]_{1}^{2}[2]_{2}^{2}} \frown+\frac{[3]_{1}-1}{[2]_{1}[2]_{2}} P \\
Q \star Q & =\frac{[3]_{2}}{[2]_{1}^{2}[2]_{2}^{2}} \frown+\frac{[3]_{2}-1}{[2]_{1}[2]_{2}} Q \\
P \star Q & =\frac{1}{[2]_{1}[2]_{2}} R=\frac{1}{[2]_{1}[2]_{2}}\left(| |-\frac{1}{[2]_{1}[2]_{2}} 乞-P-Q\right) .
\end{aligned}
$$

Proof. First consider the diagram for $P \star P$. It must be contained in the span of $\smile$ and $P$ (this follows from the fusion rules). The coefficients are determined by applying caps to the bottom and side (and using $\operatorname{dim}(P)=[3]_{1}$ and $\left.\operatorname{dim}(X)=[2]_{1}[2]_{2}\right)$.

The equation for $Q \star Q$ is verified similarly. To derive the equation for $P \star Q$, note that $P \otimes Q$ is a minimal idempotent of $\operatorname{End}_{\mathcal{C}}\left(X^{\otimes 4}\right)$ whose image is a simple object of type $X_{2} \boxtimes Y_{2}$. Since $X_{2} \boxtimes Y_{2}$ appears with multiplicity 1 in $X \otimes X$ and $P \star Q$ factors through $P \otimes Q$, we see that $P \star Q$ is a scalar multiple of $R$. The scalar is computed by taking traces.

Remark 3.8. The above lemma shows that the structure constants of the convolution algebra in the $P, Q,| | \smile$ basis depend only on $q_{1}$ and $q_{2}$.

Now that we know the multiplication structure on the convolution algebra, it is routine to compute the minimal idempotents.

Lemma 3.9. A complete set of minimal idempotents for the convolution algebra $\left(E n d_{\mathcal{C}}(X \otimes X), \star\right)$ is given by

$$
\begin{aligned}
&\left\{\frac{1}{[2]_{1}[2]_{2}}|\mid,\right. \\
& \frac{-1}{[2]_{1}[2]_{2}}\left|\left\lvert\,+\frac{1}{[2]_{2}^{2}} \supseteq+\frac{[2]_{1}}{[2]_{2}} Q\right.,\right. \\
& \frac{-1}{[2]_{1}[2]_{2}}\left|\left\lvert\,+\frac{1}{[2]_{1}^{2}} \supseteq+\frac{[2]_{2}}{[2]_{1}} P\right.,\right. \\
& \frac{1}{[2]_{1}[2]_{2}}\left|\left\lvert\,+\left(1-\frac{1}{[2]_{1}^{2}}-\frac{1}{[2]_{2}^{2}}\right) \smile-\frac{[2]_{2}}{[2]_{1}} P-\frac{[2]_{1}}{[2]_{2}} Q\right.\right\} .
\end{aligned}
$$

Proof. This can be checked directly using the structure constants given in the previous lemma.
As the Fourier transform preserves minimal idempotents, we can now pin down the Fourier transform of $P$ (and hence $Q$ ) to one of two possibilities.

Lemma 3.10. We have have two possibilies for the Fourier transform of P. Either

$$
\rho(P)=\frac{-1}{[2]_{1}[2]_{2}}| |+\frac{1}{[2]_{2}^{2}} \supseteq+\frac{[2]_{1}}{[2]_{2}} Q
$$

or

$$
\rho(P)=\frac{-1}{[2]_{1}[2]_{2}}| |+\frac{1}{[2]_{1}^{2}} \supseteq+\frac{[2]_{2}}{[2]_{1}} P
$$

with the latter case only occuring when $q_{1}= \pm q_{2}$.
Proof. The Fourier transform $\rho$ intertwines the standard product and convolution product in End $\mathcal{C}_{\mathcal{C}}(X \otimes X)$, so $\rho(P)$ must be a minimal idempotent with respect to the convolution product. Hence it must belong to the set listed in the previous lemma. A simple computation shows that

$$
\left.\rho\left(\frac{1}{[2]_{1}[2]_{2}} \supseteq\right)=\frac{1}{[2]_{1}[2]_{2}} \right\rvert\,
$$

in the space $\operatorname{End}_{\mathcal{C}}(X \otimes X)$, and thus

$$
\begin{aligned}
\rho(P) \in\{ & \left\{\frac { - 1 } { [ 2 ] _ { 1 } [ 2 ] _ { 2 } } \left|\left\lvert\,+\frac{1}{[2]_{2}^{2}} \frown+\frac{[2]_{1}}{[2]_{2}} Q\right.,\right.\right. \\
& \frac{-1}{[2]_{1}[2]_{2}}\left|\left\lvert\,+\frac{1}{[2]_{1}^{2}} \frown+\frac{[2]_{2}}{[2]_{1}} P\right.,\right. \\
& \frac{1}{[2]_{1}[2]_{2}}\left|\left\lvert\,+\left(1-\frac{1}{[2]_{1}^{2}}-\frac{1}{[2]_{2}^{2}}\right) \frown-\frac{[2]_{2}}{[2]_{1}} P-\frac{[2]_{1}}{[2]_{2}} Q\right.\right\} .
\end{aligned}
$$

We want to rule out the third listed solution. Indeed, if $\rho(P)$ was equal to that solution then taking traces gives

$$
[3]_{1}=[3]_{1}[3]_{2},
$$

which implies $[3]_{2}=1$, a contradiction to Remark 3.2 .
In a similar fashion, if $\rho(P)$ was equal to the second solution, then taking traces shows $[3]_{1}=[3]_{2}$. This can only happen if $q_{1}= \pm q_{2}^{ \pm 1}$.

Finally, by considering fusion of depth three objects, we can deduce the Fourier transform equation (c):
Lemma 3.11. In $\mathcal{C}$ we have the equation

$$
\rho(P)=\frac{-1}{[2]_{1}[2]_{2}}| |+\frac{1}{[2]_{2}^{2}} \supseteq+\frac{[2]_{1}}{[2]_{2}} Q .
$$

Proof. It suffices to prove that the second solution for $\rho(P)$ and $\rho(Q)$ in the previous lemma is not possible. So assume for contradiction that

$$
\rho(P)=\frac{-1}{[2]_{1}[2]_{2}}| |+\frac{1}{[2]_{1}^{2}} \supseteq+\frac{[2]_{2}}{[2]_{1}} P
$$

To find a contradiction, consider $(Q \otimes 1)(1 \otimes P)(Q \otimes 1)$. Note that $Q \otimes 1$ is a sum of two minimal idempotents, one a projection onto a simple isomorphic to $X$ and the other a projection onto a simple of type $X_{1} \boxtimes Y_{3}$. Since $X_{1} \boxtimes Y_{3}$ does not occur in the image of $1 \boxtimes P$, we have $(Q \otimes 1)(1 \otimes P)(Q \otimes 1)$ must be a scalar times the projection onto $X$. Taking traces, this proves that


On the other hand, we have:


In the second equality we used our assumption about $\rho(P)$ and also the triangle popping relation to remove a triangle with two $Q$ 's and a $P$. These two expressions for $(Q \otimes 1)(1 \otimes P)(Q \otimes 1)$ can only be equal if $n_{2}=3$. However by Remark 3.2 we can assume $n_{2}>3$.

Remark 3.12. We remark that there do exist categories satisfying the relations of Proposition 2.6, except with the different Fourier transformation

$$
\rho(P)=\frac{-1}{[2]_{1}[2]_{2}} \left\lvert\,+\frac{1}{[2]_{1}^{2}} \supseteq+\frac{[2]_{2}}{[2]_{1}} P .\right.
$$

This category is constructed as follows.
If $q_{1}=q_{2}^{ \pm 1}$, then the category $\mathcal{C}_{q_{1}, q_{2}}$ has an order two monoidal auto-equivalence, which is the restriction of the swap auto-equivalence on $\mathcal{A}_{q_{1}} \boxtimes \mathcal{A}_{q_{2}}$. This auto-equivalence exchanges the minimal idempotents $P$ and $Q$. We claim that the subcategory of $\mathcal{C}_{q_{1}, q_{2}} \rtimes \mathbb{Z}_{2}$ generated by the object $X$ in the non-trivial grading gives the desired category. We leave the proof of this fact to an interested reader.

Note that this subcategory of $\mathcal{C}_{q_{1}, q_{2}} \rtimes \mathbb{Z}_{2}$ does not have $S O(4)$-type fusion rules. This differing of fusion rules can first be seen in the third tensor power of $X$, which explains why we have to consider 3 box relations in order to prove Lemma 3.11

Putting everything together, we have given a semisimple presentation for a subcategory of $\mathcal{C}$ which is equivalent to the semisimple presentation of $\mathcal{C}_{q_{1}, q_{2}}$ for some $q_{1}, q_{2} \in \mathbb{C}^{\times}$. As explained in the preliminaries, this implies that $\mathcal{C}$ is equivalent to $\mathcal{C}_{q_{1}, q_{2}}$ as a pivotal tensor category.

## 4 Braided Classification

In this section we classify all braidings on the fixed monoidal category $\mathcal{C}_{q_{1}, q_{2}}$. We will show that the eight braidings given in Definition 2.1 and described in Proposition 2.6 are the only braidings on $\mathcal{C}_{q_{1}, q_{2}}$.

We begin by considering the two distinguished subcategories $\mathcal{A}_{q_{1}}^{\text {ad }}$ and $\mathcal{A}_{q_{2}}^{\text {ad }}$. As these subcategories are equivalent to $S O(3)$ type categories, we know that if the order of $q_{1}$ is greater than 8 , then their braidings are classified by a choice of $q_{1}^{ \pm 1}$ and $q_{2}^{ \pm 1} \square^{2}$

The next lemma shows that the braidings on these subcategories determine the braiding on their product (which is the adjoint subcategory of $\mathcal{C}_{q_{1}, q_{2}}$ ).

Lemma 4.1. Let $\mathcal{C}$ and $\mathcal{D}$ be semisimple tensor categories. Then braidings on $\mathcal{C} \boxtimes \mathcal{D}$ are determined by braidings on $\mathcal{C}$ and $\mathcal{D}$, together with a bicharacter

$$
a: U(\mathcal{C}) \times U(\mathcal{D}) \rightarrow \mathbb{C}
$$

Proof. First we show how a braiding on $\mathcal{C} \boxtimes \mathcal{D}$ gives rise to braidings on $\mathcal{C}$ and $\mathcal{D}$ and a bicharacter. Clearly the braiding on the product gives braidings on the factors. Now suppose $X$ is an object of $\mathcal{C}$ and $Y$ an object of $\mathcal{D}$. Then the braiding

$$
c_{\mathbf{1} \boxtimes Y, X \boxtimes \mathbf{1},}: \mathbf{1} \boxtimes Y \otimes X \boxtimes \mathbf{1} \rightarrow X \boxtimes \mathbf{1} \otimes \mathbf{1} \boxtimes Y
$$

describes a morphism $a_{X, Y} \in \operatorname{End}_{\mathcal{C} \boxtimes \mathcal{D}}(X \boxtimes Y)$. The naturality of the braiding on $\mathcal{C} \boxtimes \mathcal{D}$ implies $a_{X, Y}$ is an automorphism of the identity functor of $\mathcal{C} \boxtimes \mathcal{D}$. If we fix one of the factors (say fix an object $X$ in $\mathcal{C}$ ) then the hexagon identity for the braiding implies $a_{X,-}$ is identified with a monoidal isomorphism of the identity functor of $\mathcal{D}$. In other words, the morphisms $a_{X, Y}$ for $X$ fixed are described by a character of $U(\mathcal{D})$. The same considerations hold when fixing an object $Y$ of $\mathcal{D}$, and the conclusion is that $a_{X, Y}$ may be identified with a bicharacter of $U(\mathcal{C}) \times U(\mathcal{D})$.

Now we show that braidings $c_{X_{1}, X_{2}}$ on $\mathcal{C}$ and $d_{Y_{1}, Y_{2}}$ on $\mathcal{D}$ together with a bicharacter $a$ uniquely determine a braiding on $\mathcal{C} \boxtimes \mathcal{D}$. Suppose $X_{1}, X_{2}$ are in $\mathcal{C}$ and $Y_{1}, Y_{2}$ are in $\mathcal{D}$. Then the braiding in $\mathcal{C} \boxtimes \mathcal{D}$ on $\left(X_{1} \boxtimes\right.$ $\left.Y_{1}\right) \otimes\left(X_{2} \boxtimes Y_{2}\right)$ factors as

$$
\left(c_{X_{1}, X_{2}} \boxtimes d_{Y_{1}, Y_{2}}\right) \circ\left(1 \otimes a_{X_{2}, Y_{1}} \otimes 1\right)
$$

which shows how the braiding on the product is completely determined by $c, d$ and $a$.
Corollary 4.2. There exist four distinct braidings on the subcategory

$$
\mathcal{C}_{q_{1}, q_{2}}^{a d}=\mathcal{A}_{q_{1}}^{a d} \boxtimes \mathcal{A}_{q_{2}}^{a d} .
$$

These are parameterised by the four choices of $q_{1}^{ \pm 1}$ and $q_{2}^{ \pm 1}$.
Proof. The universal grading group of $\mathcal{A}_{q}^{\text {ad }}$ is trivial, so by the previous lemma the braiding on $\mathcal{C}_{q_{1}, q_{2}}^{\text {ad }}$ is determined by the braidings on the factors. By the classification of braidings on $S O(3)$ type categories by Tuba and Wenzl TW05 there are exactly two braidings on $\mathcal{A}_{q}^{\text {ad }}$, parametrized by the choice of $q$ or $q^{-1}$.

Let us fix one of these four possible braidings. As the monoidal category $\mathcal{C}_{q_{1}, q_{2}}$ is determined up to $q_{1} \rightarrow q_{1}^{-1}$ and $q_{2} \rightarrow q_{2}^{-1}$, we can freely choose $q_{1}$ and $q_{2}$ so that this braiding corresponds to the choice $q_{1}^{+1}$ and $q_{2}^{+1}$ in the above lemma. In particular this gives us the following twists in $\mathcal{C}_{q_{1}, q_{2}}$ :

$$
\theta_{\mathbf{1}}=1, \quad \theta_{P}=q_{1}^{4}, \quad \theta_{Q}=q_{2}^{4}, \quad \text { and } \quad \theta_{R}=\left(q_{1} q_{2}\right)^{4}
$$

With these twists in hand, it is straightforward to determine all possible braidings on $\mathcal{C}_{q_{1}, q_{2}}$ compatible with the fixed braiding on $\mathcal{C}_{q_{1}, q_{2}}^{\mathrm{ad}}$.

[^1]Lemma 4.3. There exist two braidings on $\mathcal{C}_{q_{1}, q_{2}}$ which restrict to a fixed braiding on $\mathcal{C}_{q_{1}, q_{2}}^{a d}$.
Proof. For this proof it is more convenient to work in the idempotent basis of $\operatorname{End}_{\mathcal{C}_{q_{1}, q_{2}}}(X \otimes X)$. The braiding on $\mathcal{C}_{q_{1}, q_{2}}$ is determined by

$$
=\alpha_{\mathbf{1}} \frac{1}{[2]_{1}[2]_{2}} \asymp+\alpha_{P} P+\alpha_{Q} Q+\alpha_{R} R,
$$

where $\alpha_{\mathbf{1}}, \alpha_{P}, \alpha_{Q}, \alpha_{R} \in \mathbb{C}$. As we know the twists on $\mathbf{1}, P, Q$, and $R$ we can use the balancing equation to find

$$
1=\theta_{X}^{2} \alpha_{1}^{2}, \quad q_{1}^{4}=\theta_{X}^{2} \alpha_{P}^{2}, \quad q_{2}^{4}=\theta_{X}^{2} \alpha_{Q}^{2}, \quad \text { and } \quad\left(q_{1} q_{2}\right)^{4}=\theta_{X}^{2} \alpha_{R}^{2}
$$

This allows us to determine $\alpha_{P}, \alpha_{Q}$ and $\alpha_{R}$ in terms of $\alpha_{1}$, up to sign. For some $\epsilon_{P}, \epsilon_{Q}, \epsilon_{R} \in\{-1,1\}$ we have

$$
\alpha_{P}=\epsilon_{P} q_{1}^{2} \alpha_{\mathbf{1}}, \quad \alpha_{Q}=\epsilon_{Q} q_{2}^{2} \alpha_{\mathbf{1}}, \quad \text { and } \quad \alpha_{R}=\epsilon_{R}\left(q_{1} q_{2}\right)^{2} \alpha_{\mathbf{1}}
$$

To determine $\alpha_{\mathbf{1}}$ and the three signs, we solve for the inverse of the braiding being equal to its Fourier transform. This gives us the following equations:

$$
\begin{aligned}
\alpha_{R}^{-1} & =\frac{1}{[2]_{1}[2]_{2}}\left(\alpha_{\mathbf{1}}-\alpha_{P}-\alpha_{Q}+\alpha_{R}\right) \\
\frac{\alpha_{1}^{-1}-\alpha_{R}^{-1}}{[2]_{1}[2]_{2}} & =\frac{\alpha_{P}}{[2]_{2}^{2}}+\frac{\alpha_{Q}}{[2]_{1}^{2}}+\alpha_{R}\left(1-\frac{1}{[2]_{1}^{2}}-\frac{1}{[2]_{2}^{2}}\right) \\
\alpha_{P}^{-1}-\alpha_{R}^{-1} & =\frac{[2]_{2}}{[2]_{1}}\left(\alpha_{Q}-\alpha_{R}\right) \\
\alpha_{Q}^{-1}-\alpha_{R}^{-1} & =\frac{[2]_{1}}{[2]_{2}}\left(\alpha_{P}-\alpha_{R}\right) .
\end{aligned}
$$

The last two equations yield

$$
[2]_{1}^{2}\left(2-\epsilon_{P} \epsilon_{R}\left(q_{2}^{2}+q_{2}^{-2}\right)\right)=[2]_{2}^{2}\left(2-\epsilon_{Q} \epsilon_{R}\left(q_{1}^{2}+q_{1}^{-2}\right)\right)
$$

Solving this equation shows four cases:

$$
\begin{array}{lrrl}
\epsilon_{P}= & \epsilon_{Q}= & -\epsilon_{R} & \text { for all } q_{1} \text { and } q_{2}, \\
\epsilon_{P}= & \epsilon_{Q}= & \epsilon_{R} & \text { for } q_{1}= \pm q_{2}^{ \pm 1}, \\
\epsilon_{P}= & -\epsilon_{Q}= & \epsilon_{R} & \text { for } q_{1}^{2}=-1, \text { or } q_{2}^{4}=-1, \\
\epsilon_{P}= & -\epsilon_{Q}= & -\epsilon_{R} & \text { for } q_{1}^{4}=-1, \text { or } q_{2}^{2}=-1 .
\end{array}
$$

Immediately we can disregard the latter two cases, due to Remark 3.2. In the second case we can use the third equation to find

$$
\alpha_{\mathbf{1}}^{2}= \begin{cases} \pm 1 & \text { if } q_{2}= \pm q_{1}^{-1} \\ \mp q_{1}^{-6} & \text { if } q_{2}= \pm q_{1}\end{cases}
$$

However we can now consider the first equation which tells us that either $q_{2}^{4}=-1$ or $q_{2}$ is a primitive 6 -th root of unity, both of which have already been dealt with in Remark 3.2.

Finally we have the first case. Again we use the third equation to find

$$
\alpha_{1}^{2}=\frac{\epsilon_{P}}{q_{1}^{3} q_{2}^{3}}
$$

Comparing this to the first equation shows that $\epsilon_{P}=-1$. Hence we have two possible solutions for the braiding, corresponding to the two square roots of $\frac{-1}{q_{1}^{3} q_{2}^{3}}$. These two braidings exist as they are realised in Proposition 2.6

Putting everything together, we have classified all braidings on the categories $\mathcal{C}_{q_{1}, q_{2}}$. This completes the proof of part 2 of Theorem 1.1.

## References

[BJ97] Dietmar Bisch and Vaughan Jones. Algebras associated to intermediate subfactors. Invent. Math., 128(1):89-157, 1997.
[BJ00] Dietmar Bisch and Vaughan Jones. Singly generated planar algebras of small dimension. Duke Math. J., 101(1):41-75, 2000.
[BPMS12] Stephen Bigelow, Emily Peters, Scott Morrison, and Noah Snyder. Constructing the extended Haagerup planar algebra. Acta Math., 209(1):29-82, 2012.
[BW89] Joan S. Birman and Hans Wenzl. Braids, link polynomials and a new algebra. Trans. Amer. Math. Soc., 313(1):249-273, 1989.
[Cop20] Daniel Copeland. Classification of ribbon categories with the fusion rules of $S O(N)$. PhD thesis, 2020.
[EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. Tensor categories, volume 205 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 2015.
[EO18] P. Etingof and V. Ostrik. On semisimplification of tensor categories. January 2018. https: //arxiv.org/abs/1801.04409.
[FK93] Jürg Fröhlich and Thomas Kerler. Quantum groups, quantum categories and quantum field theory, volume 1542 of Lecture Notes in Mathematics. Springer-Verlag, Berlin, 1993.
[KW93] D. Kazhdan and H. Wenzl. Reconstructing monoidal categories. Adv. Soviet Math., 16:111-136, 1993.
[Liu16] Zhengwei Liu. Exchange relation planar algebras of small rank. Trans. Amer. Math. Soc., 368, 2016.
[MPS17] Scott Morrison, Emily Peters, and Noah Snyder. Categories generated by a trivalent vertex. Selecta Math. (N.S.), 23(2):817-868, 2017.
[TW05] I. Tuba and H. Wenzl. On braided tensor categories of type BCD. J. Reine Agnew. Math., 581:31-69, 2005.
[TY98] Daisuke Tambara and Shigeru Yamagami. Tensor categories with fusion rules of self-duality for finite abelian groups. J. Algebra, 209(2):692-707, 1998.
[Wen88] Hans Wenzl. Hecke algebras of type $A_{n}$ and subfactors. Invent. Math., 92(2):349-383, 1988.


[^0]:    ${ }^{1}$ This is only true when $q$ is not a primitive eight root of unity. When $q$ is a primitive eight root of unity we have $K\left(\operatorname{Rep}\left(S O(3)_{q}\right) \cong K\left(\operatorname{Vec}\left(\mathbb{Z}_{2}\right)\right)\right.$, and there is an additional category coming from the twisting of the associator. However, in our situation we have that $\left(X_{1} \boxtimes Y_{1}\right) \otimes\left(X_{1} \boxtimes Y_{1}\right)$ is an algebra object which contains the simple object $X_{2} \boxtimes \mathbf{1}$. Hence we can not have that $\left\langle X_{2} \boxtimes \mathbf{1}\right\rangle \simeq \operatorname{Vec}^{\omega}\left(\mathbb{Z}_{2}\right)$. Hence we can still assume that $\left\langle X_{2} \boxtimes \mathbf{1}\right\rangle \simeq \mathcal{A}_{q}^{\text {ad }}$.

[^1]:    ${ }^{2}$ In the case of $n_{i} \in\{3,5\}$, there exist additional Tannakian braidings on the categories $\mathcal{A}_{q_{i}}^{a d}$. We can repeat the analysis of this section for these special cases. We find that these Tannakian braidings can not lift to braidings of the categories $\mathcal{C}_{q_{1}, q_{2}}$. Furthermore, in the case of $n_{1}=3$, we have that only two of the braidings on the subcategory $\mathcal{A}_{q_{1}}^{a d} \boxtimes \mathcal{A}_{q_{2}}^{a d}$ lift to the category $\mathcal{C}_{q_{1}, q_{2}}$. However in this case each of these two braidings on $\mathcal{A}_{q_{1}}^{a d} \boxtimes \mathcal{A}_{q_{2}}^{a d}$ has four extensions to $\mathcal{C}_{q_{1}, q_{2}}$. Hence these special cases are still covered by Theorem 1.1. We leave the details to a motivated reader.

