Classification of Riemann surfaces — Source link

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CLASSIFICATION OF RIEMANN SURFACES

By Mitsuru OZAWA

§1. Introduction.

Does there exist a function harmonic or analytic and satisfying a prescribed condition on a given Riemann surface? This question is a principal leading idea in the recent function-theory, especially in the theory of classification, and it leads us to the various deep and important results as well as new notions. Moreover it brings to the various extremal problems to be solved. In this tendency several authors above all Finnish colleagues have contributed to the theory. Although this theory is gained in a great success, there remain many important and unsolved problems.

In the present paper we shall explain a method to classify the Riemann surface. It is directed by the following leading idea: Under what conditions does there exist a solution of a partial differential equation of elliptic type $\Delta u = Pu$ on a given Riemann surface? Although we have succeeded to establish a classification theory to a certain extent, there remain many problems unsolved, and we have found yet no application to the function-theory.

We first give a more precise explanation on our Riemann surface and differential equation to be considered.

Riemann surface F means here the one in the sense of Weyl-Radó. Its ideal boundary is denoted by Γ, and we restrict ourselves to the Riemann surface of infinite genus save when the contrary is explicitly mentioned.

Differential equation considered here is the following type:

$$(A) \quad \frac{\partial^2}{\partial x^2} u(x,y) + \frac{\partial^2}{\partial y^2} u(x,y) = P(x,y)\, u(x,y)$$

where $z = x + iy$ is a local uniformizing parameter at a point p on F, $P(x,y)$ is a real holomorphic function of (x, y) being positive except at a countable set of zero-points with accumulation points lying only on the ideal boundary Γ. We assume moreover that if we change the local parameter z to z', then $P(z)$ changes as follows:

$$P(z) = P(z')\left|\frac{dz'}{dz}\right|^2 .$$

For this type of differential equation (A) we can prove the existence of the so-called kernel function and Green functions under various boundary conditions, for instance, the Green function with ordinary, i.e. vanishing boundary values or the Neumann function, i.e. the one with vanishing normal derivative along the boundary, and also the solvability of the first boundary value problem. Moreover the maximum or minimum principle and Harnack's convergence theorem are valid with a slight modification. These facts will be shortly explained but the precise proofs will be omitted off. For the precise formulations one can refer to the S.Bergman's book [1] and S.Bergman-M.Schiffer [1,2]. For other ways of formulations cf. R.Courant-D.Hilbert [1], E.Picard [1,2], L.Lichtenstein [1,2], R. Nevanlinna [1] and D.Hilbert [1].

§2. Definitions and General Considerations.

Let $\{F_n\}_{n=0,1,2,\cdots}$ be an exhaustion of F in the ordinary sense, that is,

i) F_n is a compact connected analytic subregion of F, that is, F_n is connected and the closure of F_n, $\overline{F_n}$ say, is compact and its boundary consists of a finite number of simple closed analytic curves $\Gamma_n^{(\nu)}$ $(\nu = 1, \cdots, m)$, $\Gamma_n = \sum_{\nu=1}^{m} \Gamma_n^{(\nu)}$.
Especially F_0 is supposed to be a simply-connected compact analytic subregion of F;

ii) $\overline{F_n} \subset F_{n+1}$ $(n = 0, 1, \cdots)$;

iii) $\lim_{n\to\infty} F_n = F$.

Definition 2.1. Dirichlet integral $D(u)$ and mixed Dirichlet integral $D(u,v)$:

$$D_F(u,v)=\iint_F\left(\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\frac{\partial v}{\partial y}+Puv\right)dx\,dy,$$
$$D_F(u,u)=D_F(u).$$

By the Green's formula, we see that if $\Delta u = Pu$, then $D_F(u,v)=\int_\Gamma v\frac{\partial u}{\partial \nu}ds$ for any v, where $\partial/\partial\nu$ denotes the differentiation in the direction of outer normal.

Definition 2.2. If u is a solution of (A) and satisfy $D_{F_n}(u)<\infty$, then we call $u\in L^2(F_n)$.

Obviously $L^2(F_n)$ forms a real Hilbert space with respect to the inner product

$$(u,v)\equiv D_{F_n}(u,v),\quad u,v\in L^2(F_n).$$

Definition 2.3. Kernel function $K_n(z,z_0)$ of $L^2(F_n)$: If $K_n(z,z_0)\in L^2(F_n)$ has the reproducing property for any $L^2(F_n)$ -integrable solution u of (A):

$$u(z_0)=D_{F_n}\left(u(z),K_n(z,z_0)\right),$$

then we call $K_n(z,z_0)$ the reproducing kernel function of $L^2(F_n)$.

We explain here the existence of $K_n(z,z_0)$. According to N. Aronszajn [1], in a locally uniformly bounded Hilbert space, there exists one and only one reproducing kernel function $K_n(z,z_0)$ of $L^2(F_n)$, and hence we have only to prove the locally uniformly boundedness. Since Riemann surface is of locally Euclidean character, we may discuss the problem in a Euclidean small disc. Let $u(z)$ be an arbitrary solution of (A) with $D_{F_n}(u)<M$, then $\Delta(u^2)\geqq 0$. In fact, we have

$$\Delta(u^2)=2u\,\Delta u+2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial u}{\partial y}\right)^2$$
$$=2\left[Pu^2+\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right].$$

Let $B_{R,\varepsilon}$ denote a ring domain bounded by C_R ($|z-z_0|=R$) and C_ε ($|z-z_0|=\varepsilon$, $\varepsilon<R$), then we have

$$\int_{C_R}u^2\frac{\partial}{\partial\nu}\log\left(\frac{R}{|z-z_0|}\right)ds+\int_{C_\varepsilon}u^2\frac{\partial}{\partial\nu}\log\left(\frac{R}{|z-z_0|}\right)ds$$
$$=-\iint_{B_{R,\varepsilon}}\log\frac{R}{|z-z_0|}\,\Delta(u^2)\,dx\,dy$$
$$\leqq 0.$$

Letting ε tend to zero, we have

$$2\pi R\,u^2(z_0)\leqq\int_0^{2\pi}u^2R\,d\theta,$$

and integrating with respect to R from r to R, $0<r<R$, we have

$$\pi u^2(z_0)(R^2-r^2)\leqq\iint_{B_{R,\varepsilon}}u^2\,dx\,dy$$
$$\leqq\frac{1}{P_0}\iint_{B_{R,\varepsilon}}Pu^2\,dx\,dy\leqq\frac{1}{P_0}D_R(u)$$
$$<\frac{1}{P_0}\cdot M,$$

where $P_0=\inf_{r<|z-z_0|<R}P(z)$ and $P_0>0$ in view of the isolatedness of the zero-points of $P(z)$. Thus $u(z)$ is locally uniformly bounded, what leads to the existence of $K_n(z,z_0)$.

Definition 2.4. Green function of (A). Green function $g_n(z,z_0)$ (the ordinary one) is a fundamental solution of (A) in F_n and $g_n(z,z_0)\equiv 0$ on Γ_n, that is, $\Delta g_n=Pg_n$ for $z\,(\neq z_0)\in F_n$ and $g_n(z,z_0)-A(z,z_0)\log(1/|z-z_0|)$ is a holomorphic solution of (A) where $A(z,z_0)$ is also a holomorphic function satisfying $\lim_{z\to z_0}A(z,z_0)=1$, and moreover $g_n\equiv 0$ on Γ_n.

Once the existence of the kernel function and a fundamental solution having been established, we obtain the existence and the expression of Green function in the following way: Let $S(z,z_0)$ and $K_n(z,z_0)$ be a fundamental solution and the kernel function in F_n, respectively. If we put $\sigma(z_0,w)=D_{F_n}(K_n(z,z_0),S(z_0,w))$, then we see that

$$S(z,w)-\sigma(z,w)$$

is the desired Green function, a fact following from the consideration of uniqueness of g_n which is easily proved from the assumption. For the existence of a fundamental solution of (A) one can refer to the Hilbert's book [1] and E.Holmgren [1].

Obviously we have an inequality

$$\left|\frac{\partial}{\partial\nu}g_n(z,z_0)-\frac{\partial}{\partial\nu}g_n(z_1,z_0)\right|\leqq M\,|z-z_1|,$$

and, by means of the Green's formula, the Poisson integral representation

$$u(z_0)=-\int_{\Gamma_n}u(z)\,\frac{\partial g_n(z,z_0)}{\partial\nu}ds,$$

where $u(z_0)\in L^2(F_n)$. Thus we can conclude that the so-called Harnack's convergence theorem remains valid in a slight modified form in our case.

Definition 2.5. Neumann function of (A). Neumann function $N_n(z, z_0)$ is a fundamental solution of (A) in F_n and $\partial N_n(z,z_0)/\partial\nu \equiv 0$ on Γ_n.

The existence of $N_n(z, z_0)$ is easily proved. Furthermore, its explicit expression is given by

$$N_n(z, z_0) = 2\pi K_n(z, z_0) + g_n(z, z_0).$$

For our kernel function $K_n(z, z_0)$, we have

i) $K_n(z,z_0) = K_n(z_0, z)$ and $K_n(z_0, z_0) \gtrless 0$;

ii) If $n \leqq m$, then
$K_n(z_0, z_0) \geqq K_m(z_0, z_0)$, $z_0 \in F_n$;

iii) If $\{\varphi_\nu(z)\}$, $\nu = 1, 2, \cdots$ is a complete orthonormal system in $L^2(F_n)$, then

$$K_n(z, z_0) = \sum_{\nu=1}^{\infty} \varphi_\nu(z)\, \varphi_\nu(z_0).$$

These facts are obtained in more abstract way in the Hilbert space with a reproducing kernel function; cf. N.Aronszajn [1]. Especially in our case we have

iv) $K_n(z, z_0) \gtrless 0$.

This is proved by S.Bergman-M.Schiffer [1].

Here we shall explain two lemmata of which we shall often make use in later.

Lemma 2.1. If u and v are two solutions of (A) and $u \geqq v$ on Γ_n, then $u \geqq v$ on F_n.

Proof. Let $\psi = u - v$, then $\Delta\psi = P\psi$. Let D be a point-set satisfying $\psi < 0$, then a connected component D_1 of D is a subregion of F_n and $\Gamma_n \cap D = 0$. Now ψ is superharmonic on D_1, and hence $0 > \psi(z) \geqq \inf_{D_1} \psi(z) = \inf_{(D_1)} \psi(z) = 0$, where (D_1) is the boundary of D_1. Thus D_1 and hence D also is an empty set. Therefore $\psi(z) \gtrless 0$ on F_n. q.e.d.

Lemma 2.2. If u and v satisfy the equation $\Delta u = 0$ and $\Delta v = Pv$, respectively, and $u \geqq v \gtrless 0$ on Γ_n, then $u \gtrless v$ on F_n.

Proof. By Lemma 2.1, we have $v \geqq 0$ on F_n, and hence if we put $\psi = u - v$, then ψ is superharmonic on F_n and $\gtrless 0$ on Γ_n. Therefore we have $\psi \gtrless 0$ on F_n and hence $u \gtrless v$ on F_n.

To solve the first boundary value problem, we can also make use of the so-called Schwarz's alternating process and the method of successive approximation used by E.Picard [1,2]. Or more directly we can prove the solvability and expression named Poisson integral formula by making use of the kernel function or Green function. And the uniqueness of the solution can be deduced by Lemma 2.1.

The following Lemma is a precision of Lemma 2.2.

Lemma 2.3. Let u be a solution of (A) on F_n satisfying $u \equiv 1$ on Γ_n, then $0 \leqq u \leqq 1$ in F_n; the right hand inequality being of the strict sense, that is, the equality sign being excluded.

Proof. By Lemma 2.2, $0 \leqq u \leqq 1$ is obtained. If $u(z_0) = 1$ for a point $z_0 \in F_n$, $\notin \Gamma_n$, then $\frac{\partial u}{\partial x_0} = \frac{\partial u}{\partial y_0} = 0$ and $\Delta u(z_0) \leqq 0$.

On the other hand $\Delta u(z_0) = P(z_0)\, u(z_0) \geqq 0$, and thus we have $P(z_0) = 0$. Hence $u(z) \lneqq 1$ for any point $z \in F_n$ such that $P(z) > 0$. By the isolatedness of the zero-points of $P(z)$, we have $P(z) \gneqq 0$ for $0 < |z - z_0| \leqq \varepsilon$, where ε is a sufficiently small positive number. Thus, for such a point z, $u(z) \lneqq 1$. On the other hand, the subharmonicity of $u(z)$ implies that

$$u(z_0) \leqq \frac{1}{2\pi\varepsilon} \int_0^{2\pi} u(z)\, \varepsilon\, d\theta.$$

Hence $1 = u(z_0) \lneqq 1$, which is absurd. Thus we have the desired result.

Definition 2.6. Let $\omega_n(z, \Gamma_n, F_n - F_0)$ be the finite solution of (A) in $F_n - F_0$, being identically 1 on Γ_n and 0 on Γ_0.

Definition 2.7. Let $\Omega_n(z, \Gamma_n, F_n)$ be the finite solution of (A) in F_n, and be identically 1 on Γ_n.

By Lemma 2.1, we have, for $m \geqq n$,

$$1 \geqq \omega_n(z, \Gamma_n, F_n - F_0) \geqq \omega_m(z, \Gamma_m, F_m - F_0) \geqq 0,$$
$$z \in F_n - F_0,$$

and hence the limit

$$\lim_{n\to\infty} \omega_n(z, \Gamma_n, F_n - F_0) = \omega(z, \Gamma, F - F_0)$$

exists. Similarly, there exists

$$\lim_{n\to\infty} \Omega_n(z,\Gamma_n,F_n) = \Omega(z,\Gamma,F).$$

Definition 2.8. $F \in O_\omega$ means that $\omega(z,\Gamma,F-F_0) \equiv 0$.

Definition 2.9. $F \in O_\Omega$ means that $\Omega(z,\Gamma,F) \equiv 0$.

That these classes are determined independ of a special choise of F_0 and the process of exhaustion, can be proved analogously to the harmonic case.

Definition 2.10. If there is no non-constant bounded solution of (A) on F , then we call $F \in O_B$.

Definition 2.11. If there is no non-constant solution of (A) on F , having finite Dirichlet integral $D_F(u)$, then we write $F \in O_D$.

From Lemma 2.1, we have for $m \geq n$,

$$0 \leqq g_n(z,z_0) \leqq g_m(z,z_0) , \quad z \in F_n,$$

and hence the limiting function $\lim_{n\to\infty} g_n(z,z_0) = g(z,z_0)$.

Definition 2.12. (a) If there is a point $z \in F$, $z \neq z_0$, such that $g(z,z_0) = \infty$ for a fixed point $z_0 \in F$, then we write $F \in O_{G(z_0)}$.

(b) $F \in O_G$ means that there is at least one point z_0 , such that $F \in O_{G(z_0)}$; namely, $F \notin O_G$ means that there is no point z_0 , such that $F \in O_{G(z_0)}$.

Since $K_n(z_0,z_0)$ is non-negative and decreases as n increases, the limit $\lim_{n\to\infty} K_n(z_0,z_0) = K(z_0,z_0) \geqq 0$ exists.

Definition 2.13. $F \in O_K$ means that $K(z,z) \equiv 0$ for any point $z \in F$.

It is to be noticed that there is no constant solution of (A) except the vanishing one.

We shall often make use of also the quantities and definitions with respect to the harmonic case. To avoid the confusion we shall distinguish the harmonic case by the upper index (h) , for example, harmonic measure is denoted by $\omega^{(h)}$, Green function of $\Delta u = 0$ is denoted by $g^{(h)}$, etc.

§3. O_Ω, O_ω, O_B and O_G .

In this section we shall prove that

$$O_G \subset O_\omega = O_\Omega = O_B .$$

<u>Theorem 3.1</u>. If $F \in O_\Omega$, then $F \in O_B$, and vice versa.

Proof. If $u(z)$ is a bounded solution of (A) vanishing not identically, then, by Lemma 2.1,

$$m\Omega_n(z,\Gamma_n,F_n) \leqq u(z) \leqq M\Omega_n(z,\Gamma_n,F_n),$$

where $M = \sup_F u(z)$, $m = \inf_F u(z)$. Letting n tend to ∞ , we have

$$m\Omega(z,\Gamma,F) \leqq u(z) \leqq M\Omega(z,\Gamma,F) .$$

If $F \in O_\Omega$, then $\Omega(z,\Gamma,F)$ and therefore $u(z)$ is identically zero. This contradicts $u(z) \not\equiv 0$.

Conversely, if we assume $F \notin O_\Omega$, then $\Omega(z,\Gamma,F)$ is a bounded solution of (A) and $\not\equiv 0$.

<u>Theorem 3.2</u>. If $F \in O_G$, then $F \in O_\omega$.

Proof. In order to prove the theorem, we need some lemmata:

Lemma 3.1. $F \in O_\omega^{(h)} \longrightarrow F \in O_\omega$.

Lemma 3.2. $F \in O_G \longrightarrow F \in O_G^{(h)}$.

Lemma 3.3. $F \in O_\omega^{(h)} \rightleftarrows F \in O_G^{(h)}$.

Proof of Lemmata 3.1 and 3.2. By the Lemma 2.2, we have

$$\omega_n(z,\Gamma_n,F_n-F_0) \leq \omega_n^{(h)}(z,\Gamma_n,F_n-F_0)$$

and

$$g_n(z,z_0) \leq g_n^{(h)}(z,z_0) ,$$

from which we obtain the desired result.

Lemma 3.3 is a well-known result.

Thus we have $O_G \subset O_G^{(h)} = O_\omega^{(h)} \subset O_\omega$. q.e.d.

By Lemma 3.2, we deduce the following property: If the boundary of contains a continuum, then $F \notin O_G$.

Remark. The Definition 2.12 (a) is very artificial in the sense that the fact $F \notin O_{G(z_0)}$ may depend upon the choice of the point $z_{0(k)}$. In the harmonic case, $F \notin O^{(k)}_{G(z_0)}$ is equivalent to $F \notin O^{(k)}_G$, but in our case, we can not yet conclude whether $F \notin O_{G(z_0)}$ is equivalent to $F \notin O_G$ or not. Here we shall explain a sufficient condition in order that $F \notin O_{G(z_0)}$ implies $F \notin O_G$.

If $\Omega(z) \geqq \varepsilon > 0$ for all points z on F, then $F \notin O_{G(z_0)}$ implies $F \notin O_G$.

Proof. Let Γ_0 surround a compact connected analytic subregion F_0, such that $z_0,\ z_0' \in F_0$. Then we can choose two points p_m, p_m' on Γ_0, such that

$$g_m(p_m, z_0) = \frac{2\pi \Omega_m(z_0)}{-\int_{\Gamma_0} \frac{\partial}{\partial\nu}\omega_m(z)\, ds}$$

and

$$g_m(p_m', z_0') = \frac{2\pi \Omega_m(z_0')}{-\int_{\Gamma_0} \frac{\partial}{\partial\nu}\omega_m(z)\, ds},$$

because, by Green's formula, we have

$$-\int_{\Gamma_0} g_m(z,z_0)\frac{\partial}{\partial\nu}\omega_m(z)\, ds = 2\pi\,\Omega_m(z_0).$$

Thus we have

$$\frac{g_m(p_m, z_0)}{\Omega_m(z_0)} = \frac{g_m(p_m', z_0')}{\Omega_m(z_0')},$$

from which we get the desired result.

<u>Theorem 3.3.</u> If $F \in O_\Omega$, then $F \in O_\omega$ and vice versa.

Proof. Necessity. By Lemma 2.1, we have

$$\Omega_n(z,\Gamma_n,F_n) \geqq \omega_n(z,\Gamma_n,F_n-F_0),\quad z\in F_n-F_0,$$

thus, by $n\to\infty$, we conclude the necessity.

In order to prove the sufficiency, we need some preparatory lemmata.

Lemma 3.4. If $F \notin O_\Omega$ and $\in O_\omega$, then $F \in O_G$.

Lemma 3.5. If $F \notin O_\Omega$, then $\sup_F \Omega(z,F) = 1$.

Lemma 3.6. If $F \in O_\omega^{(k)}$, then $F \in O_\Omega$.

Proof of Lemma 3.4. By Green's formula, we have

$$g_m(p_m, z_0) = \frac{2\pi \Omega_m(z_0)}{-\int_{\Gamma_0} \frac{\partial}{\partial\nu}\omega_m(z)\, ds}.$$

If $F \in O_\omega$ and $\notin O_\Omega$, then $\omega_m \to 0$, and hence $\partial\omega_m/\partial\nu \to 0$ on Γ_0. We may assume that $\Omega_m(z_0) \neq 0$ and $\Omega(z_0) \neq 0$, and thus $g_m(p_m, z_0) \to \infty$. Therefore $g(z, z_0) = \infty$ for a point $z \neq z_0$.

Proof of Lemma 3.5. Let $\sup_F \Omega(z,F) = \lambda$, then $\lambda \leqq 1$ is evident, and thus we shall prove $\lambda \geqq 1$. For $z\in F_n$, we shall consider the function $\varphi_n(z) = \Omega_n(z,F_n)\sup_F \Omega(z,F) - \Omega(z,F)$ then $\Delta\varphi_n(z) = P\varphi_n(z)$ for $z\in F_n$, and $\varphi_n(z) \geqq 0$ on Γ_n. Therefore, from Lemma 2.1, we have $\varphi_n(z) \geqq 0$ on F_n, and hence

$$\Omega_n(z,F_n)\sup_F \Omega(z,F) \geqq \Omega(z,F)$$

on F_n. Letting $n\to\infty$,

$$\lambda\Omega(z,F) \geqq \Omega(z,F)\ (\geqq 0, \neq 0),$$

from which $\lambda \geqq 1$. q.e.d.

Since $\Omega(z,F)$ is a bounded non-negative solution of (A) on F, we have

$$\sup_F \Omega(z,F) = \sup_{F-R} \Omega(z,F),$$

where R is an arbitrary compact analytic subregion of F.

Proof of Lemma 3.6. Suppose that F does not belong to O_Ω, but belongs to $O_\omega^{(k)}$, then we have, by Lemma 3.5, $\sup_F \Omega(z,F) = \sup_{F-F_0} \Omega(z,F) = 1$, and $\sup_{F-F_0} \omega^{(k)}(z,F-F_0) = 0$.

Let $\varphi_n(z) = \Omega_n(z,F_n) - \omega_n^{(k)}(z,F_n-F_0)$ for $z\in F_n - F_0$, then $\Delta\varphi_n(z) = P\Omega_n(z,F_n) \geqq 0$, and hence $\sup_{\Gamma_n+\Gamma_0}\varphi_n(z) = \sup_{F_n-F_0}\varphi_n(z)$. On the other hand, $\varphi_n(z) = 0$ on Γ_n and $\geqq 0$ on Γ_0, but $\neq 0$ on Γ_0. Thus

$$\sup_{F_n-F_0}\varphi_n(z) = \sup_{\Gamma_0}\varphi_n(z) = \sup_{\Gamma_0}\Omega_n(z,F_n),$$

or

$$\Omega_n(z,F_n) \leqq \sup_{\Gamma_0}\Omega_n(z,F_n) + \omega_n^{(k)}(z,F_n-F_0).$$

Letting $n \to \infty$, we have

$$\Omega(z,F) \leq \sup_{\Gamma_0} \Omega(z,F) + \omega^{(k)}(z, F-F_0),$$

and therefore

$$\sup_{F} \Omega(z,F) = \sup_{F-F_0} \Omega(z,F)$$
$$\leq \sup_{\Gamma_0} \Omega(z,F) + \sup_{F-F_0} \omega^{(k)}(z, F-F_0).$$

Considering our first assumptions, we have

$$1 \leq \sup_{\Gamma_0} \Omega(z,F),$$

which contradicts Lemma 2.3 and the monotone decreasing property of $\Omega_n(z,F)$, that is, $0 \leq \Omega_n(z,F_n) \lneqq 1$ and, for $m \geq n$, $\Omega_m(z,F_m) \leq \Omega_n(z,F_n)$ and hence $\sup_{\Gamma_0} \Omega(z,F) \lneqq 1$, which is absurd.

q.e.d.

Sufficiency proof of Theorem 3.3. Suppose that $F \in O_\omega$ and $F \notin O_\Omega$, then by Lemma 3.4 $F \in O_G$ and hence, by Lemmata 3.2 and 3.3, $F \in O_\omega^{(k)}$.

Thus $F \in O_\Omega$ by Lemma 3.6, which is a contradiction. q.e.d.

§4. Subregions and O_B.

Let G be a non-compact connected subregion, and its relative boundary C consist of a finite number of analytic curves. Supposing that $\{F_n\}$, $n=0,1,\cdots$ is an exhaustion of F, we introduce the notations: $G_n = F_n \cap G$, $\gamma_n = \Gamma_n \cap G$ and $C_n = F_n \cap C$.

Definition 4.1. Let $\omega_n(z,G_n)$, $\Omega_n(z,G_n)$ and $\omega_n'(z,G_n)$ be the solutions of (A) on G_n, satisfying the following boundary conditions:

$$\omega_n(z,G_n) = \begin{cases} 0 & \text{on } C_n, \\ 1 & \text{on } \gamma_n; \end{cases}$$

$$\Omega_n(z,G_n) = 1 \quad \text{on } C_n + \gamma_n;$$

and

$$\omega_n'(z,G_n) = \begin{cases} 0 & \text{on } \gamma_n, \\ 1 & \text{on } C_n, \end{cases}$$

respectively.

Evidently we have $\Omega_n(z,G_n) = \omega_n(z,G_n) + \omega_n'(z,G_n)$, and, from Lemma 2.1, we have the following monotoneity: for $m \geq n$,

$$\omega_m(z,G_m) \leq \omega_n(z,G_n),$$

$$\Omega_m(z,G_m) \leq \Omega_n(z,G_n)$$

and

$$\omega_m'(z,G_m) \geq \omega_n'(z,G_n)$$

Hence, from the uniform boundedness of these quantities, the limits

$$\lim_{n\to\infty} \omega_n(z,G_n) = \omega(z,G),$$

$$\lim_{n\to\infty} \Omega_n(z,G_n) = \Omega(z,G).$$

and

$$\lim_{n\to\infty} \omega_n'(z,G_n) = \omega'(z,G)$$

exist, and moreover we have $\Omega(z,G) = \omega(z,G) + \omega'(z,G)$.

Definition 4.2. $G \in SO_\omega$ means that $\omega(z,G) \equiv 0$.

Evidently $G \in SO_\omega$ is equivalent to $\Omega(z,G) \equiv \omega'(z,G)$.

If $u(z)$ is a bounded non-constant solution of (A), being continuous on $G + C$ and $u(z) = k$ (const.) on C, then we get

$$k\Omega(z,G) - M\omega(z,G) \leq u(z)$$
$$\leq k\Omega(z,G) + M\omega(z,G),$$

where $M = \sup_G |u(z) - k\Omega(z,G)|$. This is easily verified by means of Lemma 2.1 for G_n. Let n tend to ∞.

If $\omega(z,G) \equiv 0$, then $u(z) = k\Omega(z,G)$. Even if $\omega(z,G) \equiv 0$, there is at least a bounded non-constant solution $u(z)$ of (A), satisfying the condition $u(z) \equiv$ const. $(\neq 0)$ on C, but, indeed, essentially only one, that is, $\Omega(z,G)$. If $\omega(z,G) \not\equiv 0$, then there is at least two bounded non-constant solutions $u(z)$ of (A), satisfying the condition $u(z) =$ const. $(\neq 0)$ on C, and being linearly independent, for example, $k\Omega(z,G)$ and $k\Omega(z,G) + M\omega(z,G)$.

<u>Theorem 4.1.</u> $G \in SO_\omega$ is equivalent that there is only one linearly independent solution $\Omega(z,G)$ of (A) bounded non-constant in G and satisfying the condition $\Omega(z,G) = 1$ on C. $G \in SO_\omega$ is equi-

valent that there is no bounded non-constant solution of (A) vanishing identically on C.

Theorem 4.2. In order to $F \in O_B$ it is necessary and sufficient that there is at least a non-compact connected subregion G which does not belong to SO_ω.

Proof. Sufficiency. Let $u_n(z)$ be a finite solution of (A) on F_n such that

$$u_n = \begin{cases} 1 & \text{on } \gamma_n, \\ 0 & \text{on } \Gamma_n - \gamma_n, \end{cases}$$

then $0 \leqq u_n(z) \leqq 1$ on F_n, by Lemma 2.1.

$u_n(z)$ being uniformly bounded, we can select a subsequence $\{u_{n_\nu}(z)\}$ of $\{u_n(z)\}$ which converges to a bounded solution $u(z)$ of (A) on G. For the sake of simplicity we shall retain the original suffices.

By Lemma 2.1, we have $u_n(z) \geqq \omega_n(z, G_n)$ on G_n. Let n tend to ∞, we have $u(z) \geqq \omega(z, G)$. By the assumption $G \notin SO_\omega$, $\omega(z, G) \not\equiv 0$. Hence there exists a point z_0 on G, at which $\omega(z_0, G) > 0$. Thus $u(z_0) > 0$. Hence $u(z) \not\equiv 0$ and is not constant.

Necessity. By the definition of $F \notin O_B$, there is a bounded non-constant solution $u(z)$ of (A) on F. Then there exists a point z_0 on F at which $u(z_0) \neq 0$.

(a) If $u(z_0) > 0$, then we take G as the point-set on which $u(z) > u(z_0)$.

(b) If $u(z_0) < 0$, then we take G as the point-set on which $u(z) < u(z_0)$.

We shall treat here only the case (a), since the case (b) is similar.

If G has a compact connected component G_ν, $u(z)$ is subharmonic in G_ν, thus by Lemma 2.1 $\sup\limits_{C_\nu} u(z) = \sup\limits_{G_\nu} u(z) > 0$, where C_ν is the boundary of G_ν. On the other hand $\sup\limits_{C_\nu} u(z) = u(z_0)$, which is absurd. Thus G must have a non-compact connected subregion. Obviously we have $u(z) \not\equiv u(z_0) \cdot \Omega(z, G)$ on G, and $u(z)$ is a bounded solution of (A) on G, with constant boundary value $u(z_0)$ on C. By Theorem 4.1, $G \notin SO_\omega$. q.e.d.

Theorem 4.3. (i) If G is a connected subregion of F, having Jordan relative boundary C, and $u(z)$ is a solution of (A) continuous on $G + C$. Suppose that $F \in O_B$ and $u(z)$ is bounded, non-negative and non-constant on G, then the maximum principle holds, that is,

$$\sup_G u(z) = \sup_C u(z).$$

(ii) If $u(z)$ is non-positive and the above assumptions remain valid, then the minimum principle holds, that is,

$$\inf_G u(z) = \inf_C u(z).$$

Proof. We shall prove the first part of the theorem, as remaining part can be proved similarly.

If G is a compact subregion, the theorem is evident. Thus we shall confine ourselves to a non-compact subregion. If we suppose that $\sup\limits_C u(z) \neq \sup\limits_G u(z)$, then there is a positive number M, such that

$$0 \leqq \sup_C u(z) < M < \sup_G u(z).$$

Let H be a point-set on which $u(z) > M$ holds, and H_ν be a connected component of H. H_ν is a connected subregion with a non-compact closure. This fact is shown by Lemma 2.1. Now $u(z) > M$ in H_ν and $\not\equiv M\,\Omega(z, H_\nu)$, since $M\,\Omega(z, H_\nu) \leqq M$ in H_ν. Hence, by Theorem 4.1, $\omega(z, H) \not\equiv 0$. By Theorem 4.2, $F \notin O_B$, which is absurd.

Theorem 4.3'. If $F - G$ is compact, then the converse of the Theorem 4.3 holds.

Proof. Let $u(z)$ be a bounded non-negative solution of (A) on F. If we suppose $\sup\limits_G u(z) = \sup\limits_C u(z)$, then $\sup\limits_G u(z)$ is attained on C. Thus $u(z)$, as a function defined on F, has its maximum on $F - G$. This is contradictory. Thus $u(z) \equiv 0$, that is, $F \in O_B$.

Theorem 4.4. Let G be a non-compact connected subregion, having a Jordan relative boundary C. Then $G \notin SO_\omega$ is a necessary and sufficient condition in order that the maximum principle does not hold on G.

Proof. Sufficiency was already proved in the Theorem 4.3.

Necessity. Suppose that $G \notin SO_\omega$ and the maximum principle holds. Then $\omega(z, G)$ is a bounded non-constant solution of (A) on G. Thus

$$\sup_G \omega(z,G) = \sup_C \omega(z,G) = 0,$$

and hence $\omega(z,G) \equiv 0$ on G, which is absurd.

§5. Subregions, O_D and O_K.

Theorem 5.1. $O_D \rightleftarrows O_K$.

Proof. We first show that $O_K \to O_D$. Suppose that $F \notin O_D$, then there is a function $\varphi(z) \in L^2(F)$ and $\varphi \not\equiv 0$. Let

$$u(z) = \begin{cases} \varphi(z), & z \in F_n, \\ 0, & z \in F - F_n, \end{cases}$$

then we may assume that $u(z_0) \neq 0$, where z_0 is a fixed point on F_n. By the reproducing property of the kernel and the Schwarz's inequality, we have

$$0 \lneq |u(z_0)|^2 \leqq D_{F_n}(K_n(z,z_0)) D_{F_n}(u(z))$$

$$= K_n(z_0,z_0) D_{F_n}(u(z)).$$

Let n tend to ∞, then we have

$$0 \lneq |\varphi(z_0)|^2 \leqq K(z_0,z_0) D_F(\varphi(z)).$$

Thus $K(z_0,z_0) \neq 0$, which shows $F \notin O_K$.

We next show that $O_D \to O_K$. Let $F \notin O_K$, then we can choose a point $z_0 \in F$ such that $K(z_0,z_0) \neq 0, \neq \infty$, that is, we have $\infty > M > K_n(z_0,z_0) > \varepsilon > 0$ for a sufficiently large integer n, M and ε being independent of n. Now the limit $\lim_{n\to\infty} K_n(z_0,z_0) = K(z_0,z_0)$ exists. Evidently $D_{F_n}(K_n(z_0,z_0)) = K_n(z_0,z_0) < M$. Thus $K(z,z_0) \in L^2(F)$.

Corollary. Let E be a compact set on F_1, being itself a compact subregion of F, and $\varphi(z)$ be an arbitrary Dirichlet-finite solution of (A) in $F_1 - E$. Then a necessary and sufficient condition in order that $\varphi(z)$ is prolongable onto E in the sense of (A) is that $K_{F_1-E}(z,z) = K_{F_1}(z,z)$.

For later usages we shall explain an extremal property of the solution of (A).

Let R be a compact connected subregion of F, surrounded by compact analytic curves C. If $u(z)$ satisfies the following properties, then we write $u(z) \in \mathcal{G}_R$:

1) $u(z)$ is a continuous differentiable function in R,

2) $u(z)$ has a given boundary function $\varphi(z)$ being continuously differentiable on C except only at a finite number of points.

Lemma 5.1. If $v(z)$ is a solution of (A) with the boundary value $\varphi(z)$, then $D_R(v) \leqq D_R(u)$ for any $u \in \mathcal{G}_R$.

Proof. We put $u(z) = v(z) + \Phi(z)$, then $\Phi(z)$ vanishes everywhere on C, and is continuously differentiable on $G + C$, except only at a finite number of boundary points. Thus we have from the Green's formula

$$D_R(v,\Phi) = \int_C \Phi \frac{\partial v}{\partial \nu} ds = 0.$$

Hence we have $D_R(u) = D_R(v) + D_R(\Phi) \geqq D_R(v)$. q.e.d.

Let G be a non-compact connected subregion of F with an analytic, but not necessarily compact, relative boundary C. Notations G_m, C_m, γ_m and others are the same as those defined in the preceding section.

Lemma 5.2. There exists a fundamental solution $\hat{N}_m(z,x)$ of (A), such that $\hat{N}_m(z,x) = 0$ on C_m and $\frac{\partial}{\partial\nu}\hat{N}_m(z,x) = 0$ on γ_m.

Proof. We may assume that G_m is any canonical domain, for example, γ_m consists of a finite number of segments on the real axis, because $P(z)$ has the covariance character and hence Dirichlet integral, Green function, etc. with regard to the differential equation (A) remain invariant by conformal mapping.

Let $\tilde{G}_m$ and $\tilde{z}$ be the inversions of G_m and z with respect to γ_m, respectively. And then we identify G_m and $\tilde{G}_m$ by all corresponding segments γ_m, such a domain is denoted by $\tilde{\tilde{G}}_m$, that is, $\tilde{\tilde{G}}_m = G_m \cup \tilde{G}_m \cup \gamma_m$. Moreover we define that $P(\tilde{z}) = P(z)$ for $\tilde{z} \in \tilde{G}_m$ when we construct the following solution of (A):

$$\hat{N}_m(z,x) = g_{\bar{G}_m}(z,x) + g_{\bar{G}_m}(z,\tilde{x}), \quad x \in G_m.$$

Then $\hat{N}_m(z,x)$ is a desired solution of (A). In fact, we have evidently

$$g_{\bar{G}_m}(z,x) = g_{\bar{G}_m}(\tilde{z},\tilde{x}),$$

and hence

$$\hat{N}_m(z,x) = \hat{N}_m(\tilde{z},\tilde{x}) = \hat{N}_m(\tilde{z},x).$$

Thus

$$\frac{\partial}{\partial y}\hat{N}_m(z,x) = 0, \quad I_m z = y \text{ on } \gamma_m.$$

On the other hand, we have $\hat{N}_m(z,x) = 0$ on C_m and $\hat{N}_m(z,x)$ is a fundamental solution of (A) on G_m.

Uniqueness can be proved as follows:

Let $\hat{L}$ and $\hat{N}$ be two desired functions, then $\hat{L} - \hat{N} = \varphi$ is a finite solution of (A) satisfying the condition: $\varphi = 0$ on C_m and $\frac{\partial}{\partial \nu}\varphi = 0$ on γ_m. By Green's formula we have

$$D_{G_m}(\varphi) = -\int_{C_m+\gamma_m}\varphi\frac{\partial}{\partial\nu}\varphi\, ds - \iint_{G_m}\varphi[\Delta\varphi - P\varphi]\,dx\,dy$$

$$= 0.$$

Thus $\varphi \equiv 0$ on G_m, that is, $\hat{L} \equiv \hat{N}$. q.e.d.

Let $g_m(z,x)$ be the Green function of (A) on G_m, then $\lim_{m\to\infty} g_m(z,x) \neq \infty$ for $z(\neq x) \in G$ was proved in § 3. This will be made use of in later Lemma 5.4.

By the Green's formula we have $D_{G_m}(u, g_m) = 0$ and $D_{G_m}(u(z), \hat{N}_m(z,x)) = 2\pi\, u(x)$, where $u(z)$ is an arbitrary Dirichlet finite solution of (A) on $\bar{G}_m$ with continuous boundary values, such that $u(z) = 0$ on C_m. This family is denoted by $L_0^2(G_m)$. That the function

$$\hat{K}_m(z,x) = \frac{1}{2\pi}\left(\hat{N}_m(z,x) - g_m(z,x)\right)$$

is the reproducing kernel function of $L_0^2(G_m)$ is an immediate consequence of the above two relations.

Lemma 5.3. $\hat{K}_m(z,x) \geq 0$ on G_m for a fixed x.

Proof. From the manner of construction of $\hat{N}_m(z,x)$, this proposition is evident, but we shall explain another proof, being similar to that in a Bergman-Schiffer's paper [2]. Suppose that, for a fixed $x \in G_m$, $\hat{N}_m(z,x)$ is negative somewhere in G_m. Let G_m^- be a point-set on which $\hat{N}_m < 0$, and b^- be its boundary; evidently G_m^- is a subregion of G_m, b^- consists either of smooth arcs of $C_m + \gamma_m$ on which $\hat{N}_m \frac{\partial}{\partial\nu}\hat{N}_m = 0$ or of smooth level curves $\hat{N}_m(z,x) = 0$. Then $D_{G_m^-}(\hat{N}_m(z,x))$ exists, because the pole x of $\hat{N}_m(z,x)$ does not lie in G_m^-, and evidently > 0. By the Green's formula, we have

$$0 \lneq D_{G_m^-}(\hat{N}_m(z,x))$$

$$= -\int_{b^-}\hat{N}_m(z,x)\frac{\partial}{\partial\nu}\hat{N}_m(z,x)\,ds$$

$$= 0,$$

which is absurd. Thus $\hat{N}_m(z,x) \geq 0$ on $\gamma_m + C_m + G_m$. On the other hand, $g_m(z,x) = 0$ on $\gamma_m + C$, and therefore $\hat{N}_m(z,x) - g_m(z,x) \geq 0$ on $\gamma_m + C_m$, and hence, by Lemma 2.1, we have

$$\hat{K}_m(z,x) = \frac{1}{2\pi}\left(\hat{N}_m(z,x) - g_m(z,x)\right)$$

$$\geq 0$$

on $G_m + C_m + \gamma_m$.

Lemma 5.4. Let G be a non-compact connected subregion of F with an analytic relative boundary C. If there exists a non-constant solution $u(z)$ of (A) on the closure $\bar{G}$ of G, such that $D_G(u) < \infty$ and $u(z) = 0$ on C, then there exists a non-constant bounded non-negative and Dirichlet-finite solution $v(z)$ of (A) on $G + C$, such that $v(z) = 0$ on C.

Proof. This is an analogue of Mori's Lemma [1], stating the same fact as in the harmonic case. Let $u(z)$ belong to $L_0^2(G)$, and be non-constant. Evidently $u(z) \in L_0^2(G_m)$. Thus $D_{G_m}(u(z), \hat{K}_m(z,x)) = u(x)$. Moreover $\hat{K}_m(z,x) \in L_0^2(G_m)$, therefore we have

$$D_{G_m}(\hat{K}_m(z,x), \hat{K}_m(z,x)) = \hat{K}_m(x,x).$$

For $m \geq n$, we obtain

$$D_{G_n}(\hat{K}_m(z,x), \hat{K}_n(z,x)) = \hat{K}_m(x,x),$$

and hence

$$0 \leq D_{G_n}(\hat{K}_m - \hat{K}_n)$$
$$= D_{G_n}(\hat{K}_m) - 2D_{G_n}(\hat{K}_m, \hat{K}_n) + D_{G_n}(\hat{K}_n)$$
$$\leq D_{G_m}(\hat{K}_m) - 2D_{G_n}(\hat{K}_m, \hat{K}_n) + \hat{K}_n(x,x)$$
$$= \hat{K}_m(x,x) - 2\hat{K}_m(x,x) + \hat{K}_n(x,x)$$
$$= \hat{K}_n(x,x) - \hat{K}_m(x,x).$$

Thus, for $m \geq n$,

$$0 \leq \hat{K}_m(x,x) \leq \hat{K}_n(x,x),$$

and hence

$$\lim_{n\to\infty} \hat{K}_n(x,x) = \hat{K}(x,x), \ (\geq 0)$$

exists. Thus

$$\lim_{m>n\to\infty} D_{G_n}(\hat{K}_m - K_n) = 0$$

and simultaneously

$$\lim_{m>n\to\infty} D_{G_n}(u(z), \hat{K}_m(z,x) - \hat{K}_n(z,x)) = 0.$$

Thus

$$\lim_{n\to\infty} D_{G_n}(u(z), \hat{K}_n(z,x))$$

exists and is equal to $u(x)$. If we choose a point x such that $u(x) \neq 0$, then we have

$$|u(x)|^2 \leq \hat{K}_n(x,x)\, D_{G_n}(u) \leq \hat{K}_n(x,x)\, D(u).$$

Thus $\infty > \hat{K}(x,x) > 0$. From this and the Schwarz's inequality, we see that $\lim_{n\to\infty} \hat{K}_n(z,x)$ exists on a compact subregion of G. Moreover, from

$$D_{G_n}(\hat{K}_n(z,x)) = \hat{K}_n(x,x),$$

we see that $D_G(\hat{K}(z,x)) < \infty$, and hence $\hat{K}(z,x) \in L_0^2(G)$ and $\neq 0$, and moreover is bounded uniformly in G in a wider sense.

Remaining part of the Lemma is that $K(z,x)$ is a bounded solution of (A) on G. To prove this we remark that $g_m(z,x) \neq \infty$ for all points $z\,(\neq x) \in G$. Let Γ_0 be an analytic compact curve in G, surrounding a compact domain G_0, in which the point x lies. $\hat{N}_m(z,x)$ is of non-negative value on γ_m and is also on $G_m - G_0$. On the other hand, on γ_m

$$\frac{\partial}{\partial\nu} \hat{N}_m(z,x) = 0,$$

thus

$$\sup_{G_m - G_0} \hat{N}_m(z,x) > \max_{\gamma_m} \hat{N}_m(z,x),$$

because, if $\hat{N}_m(z,x)$ has its $\sup_{G_m - G_0}$ on γ_m, then $\hat{N}_m(z,x)$, as a function on $\bar{\bar{G}}_m - \bar{\bar{G}}_0$, has a maximum at an inner point on $\bar{\bar{G}}_m - \bar{\bar{G}}_0$, which contradicts the subharmonicity of $\hat{N}_m(z,x)$, where $\bar{\bar{G}}_0$ is the union of two domains G_0 and $\tilde{G}_0$, $\tilde{G}_0$ being the inversion domain of G_0 with respect to γ_m. Since $\hat{N}_m(z,x) \equiv 0$ on C_m, from the subharmonicity of $\hat{N}_m(z,x)$ on $G_m - G_0$ we have

$$\sup_{G_m - G_0} \hat{N}_m(z,x) = \max_{\Gamma_0} \hat{N}_m(z,x).$$

Thus, for $z \in G_m - G_0$,

$$\lim_{m\to\infty} \hat{N}_m(z,x) \leq \lim_{m\to\infty} \sup_{\Gamma_0} \hat{N}_m(z,x)$$
$$= \lim_{m\to\infty} \sup_{\Gamma_0} \left(2\pi \hat{K}_m(z,x) - g_m(z,x)\right)$$
$$< \infty.$$

Thus $\hat{N}(z,x)$ is bounded on $G - G_0$. On the other hand, $\hat{N}(z,x) - g(z,x)$ is bounded on G_0. Therefore $\hat{K}(z,x)$ is bounded on G_0 and $G - G_0$, and hence on G. q.e.d.

<u>Theorem 5.1.</u> Let G be a non-compact connected subregion of F with an analytic relative boundary C. If there exists a non-constant solution $U(z)$ of (A) on $G + C$, such that $U = 0$ on C and $D_G(U) < \infty$, then $F \notin O_D$. Conversely, if $F \notin O_D$, then we can find such a domain G and a solution $U(z)$ of (A).

Proof. By the Lemma 5.4, we may assume that $0 \leq U(z) \leq 1$ in G, and $\sup_G U(z) = 1$. Let $V(z)$ be a solution of (A), such that $V(z) = U(z)$ on G, $= 0$ on $F - G$. $V(z)$ is continuous on F and has piecewise continuous partial derivatives, and $D_F(V) = D_G(U) < +\infty$.

Let $u_n(z)$ be a solution of (A) on F_n, such that $u_n(z) = V(z)$ on Γ_n. Since $u_n(z)$ is uniformly bounded solution of (A), we may assume that $u_n(z)$ converges to a solution $u(z)$ uniformly in the wider sense on F.

For $m \geq n$, we have, by the Lemma 5.1,

$$D_{F_n}(u_m) \leq D_{F_m}(u_m) \leq D_{F_m}(V) \leq D_F(V).$$

Letting first m tend to ∞ and next n to ∞, we have

$$D_F(u) \leq D_F(V) < +\infty .$$

Let H be a subregion of G, on which $U(z) > 1/2$. H has a non-compact connected component H_ν. In H_ν, $1 \geq U(z) > 1/2$, thus $U(z) \not\equiv \frac{1}{2}\Omega(z, H_\nu)$. Hence, by Theorem 4.1, $H_\nu \notin SO_\omega$, that is, $\omega(z, H_\nu) \not\equiv 0$. Since $u_n(z) \geq 0$ in F_n and $\geq \frac{1}{2}$ on $\gamma_n(H) = H \cap \Gamma_n$, we have, by Lemma 2.1, on $H_\nu^{(n)} = H_\nu \cap F_n$,

$$u_n(z) \geq \frac{1}{2}\omega_n(z, H_\nu^{(n)}).$$

Hence, for $n \to \infty$, we have

$$u(z) \geq \frac{1}{2}\omega(z, H_\nu) \not\equiv 0.$$

Thus $u(z) \not\equiv 0$ and hence $u(z)$ is non-constant Dirichlet-finite solution of (A) on F, that is, $F \notin O_D$.

Conversely, if we assume $F \notin O_D$, then it happens only two cases: either 1) there exists at least two linearly independent solutions of (A), being non-constant and Dirichlet-finite on F, or 2) such a functional space is one dimensional.

If the case 1) happens, then we shall treat the problem in the following way. Let $u_1(z)$ and $u_2(z)$ be two linearly independent solutions of (A), being non-constant and Dirichlet-finite on F, then there are a number m and a point $z_0 \in F$, such that $u_1(z_0) = m\, u_2(z_0)$ and $u_1(z) \not\equiv m\, u_2(z)$. Let $U(z) = u_1(z) - m\, u_2(z)$, then $U(z)$ is a non-constant Dirichlet-finite solution of (A), being $U(z_0) = 0$. Without loss of generality, we may assume that $U(z) > 0$ for a point $z \in F$. Then, as the desired non-compact connected subregion G of F, we can take a connected component of a point-set on which $U(z) > U(z_0) = 0$. Then

$$D_G(U(z)) \leq D_F(U(z))$$

and $U(z) \equiv 0$ on C (boundary of G).

Suppose that $G + C = \overline{G}$ is compact, then we can find such a domain G_1 that $\overline{G}_1$ is compact and $G_1 \supset \overline{G}$ and that C does not touch the boundary C_1 of G_1. Obviously $U(z) \in L^2(G_1)$, and there exists the reproducing kernel function $K_{G_1}(z, z)$ defined in Definition 2.3 on G_1. Thus we have

$$U(z_1) = D_{G_1}(U(z), K_{G_1}(z, z_1)) \leq D_{G_1}(U(z))\ K_{G_1}(z_1, z_1).$$

Since $K_{G_1}(z_1, z_1)$ is uniformly bounded for $z_1 \in G$, $U(z_1)$ is a finite solution of (A) on G, being $u(z) \equiv 0$ on C. This implies that $U(z) \equiv 0$ on G, which is absurd. Thus G is non-compact connected subregion of F and $U(z)$ is the desired solution of (A) on G.

In the case 2), we have

$$K_F(z, \zeta) = \varphi(z)\ \varphi(\zeta),$$

where $\varphi(z) \in L^2(F)$, $D_F(\varphi) = 1$ and is non-constant. Suppose that $K_F(z, \zeta_1) \equiv m\, K_F(z, \zeta_2)$ for any $z \in F$ holds, even if we choose two points ζ_1 and $\zeta_2 (\in F)$ whatsoever, where m is a constant, then we have

$$\varphi(\zeta_1) = m\, \varphi(\zeta_2).$$

Since ζ_1 and ζ_2 are arbitrary, we have $m = \pm 1$, but $m = -1$ does not arise for $K_F(z, \zeta) \geq 0$. Thus $m = 1$, and hence $\varphi(z) \equiv$ constant, which is absurd. Thus there are two points ζ_1 and ζ_2 such that

$$K_F(z, \zeta_1) \not\equiv m\, K_F(z, \zeta_2)$$

and

$$K_F(z_0, \zeta_1) = m\, K_F(z_0, \zeta_2)$$

for a suitable point z_0 on F. Thus, if we put

$$U(z) = K_F(z, \zeta_1) - m\, K_F(z, \zeta_2),$$

and if we choose a non-compact connected subregion G of F which is a connected component of a point-set satisfying $U(z) > 0$, then we can conclude that G and $U(z)$ are the desired domain and function as in the case 1). q.e.d.

§6. O_B and O_D .

Theorem 6.1. If $D_F(1,1)=\iint_F P\,dx\,dy<\infty$ and $F\in O_K$, then $F\in O_\Omega$.

Proof. Obviously we have

$$\Omega_m(z,F_m)\equiv \Omega_m(z,\Gamma_m,F_m)$$
$$=-\frac{1}{2\pi}\int_{\Gamma_m}\frac{\partial}{\partial\nu}g_m(\zeta,z)\,ds_\zeta$$

$$K_m(\zeta,z)=\frac{1}{2\pi}\left(N_m(\zeta,z)-g_m(\zeta,z)\right)$$

$$\frac{\partial}{\partial\nu}N_m(\zeta,z)\equiv 0 \quad \text{on } \Gamma_m .$$

Hence we get

$$\Omega_m(z,F_m)=\int_{\Gamma_m}\frac{\partial}{\partial\nu}K_m(\zeta,z)\,ds_\zeta$$

$$=D_{F_m}\left(1,\ K_m(\zeta,z)\right)_\zeta .$$

Thus, by the Schwarz's inequality, we have

$$0\leqq \Omega_m(z,\ F_m)\leqq\left(K_m(z,z)\right)^{1/2}\left(D_{F_m}(1,1)\right)^{1/2}.$$

By the assumption of theorem, right hand side vanish when m tends to ∞. Thus, $F\in O_\Omega$. q.e.d.

Theorem 6.3. If $F\in O_B$, then $F\in O_D$.

Proof. Suppose that $F\in O_B$ and $F\in O_D$, then, by Theorem 5.1, there exists a pair $(G,U(z))$ such that G is non-compact connected analytic subregion of F and $u(z)$ is a non-constant Dirichet-finite solution of (A) and $u(z)\equiv 0$ on C. Then on G there exists a non-constant bounded Dirichlet-finite solution $v(z)$ of (A) being $V(z)\equiv 0$ on C by Lemma 5.4. Therefore, by Theorem 4.1, $G\notin SO_\omega$ and hence, by Theorem 4.2, $F\notin O_B$, which is absurd. q.e.d.

Alternative proof of Theorem 6.2.

Suppose that $F\notin O_D$, then there exists a non-constant Dirichlet-finite solution u of (A) on F_n. Without loss of generality we may assume that the point-set on which $u>0$ is not empty, since we may consider the function $-u$ if necessary. Let W be a compact subregion of F, on which $u>0$, and M be equal to $\sup_W u(z)>0$.

We construct a continuous piecewise continuously differentiable function f as follows:

$$\begin{aligned} f &= M && \text{if } z\in W_1 \text{ on which } u>M,\\ &= 0 && \text{if } z\in W_2 \text{ on which } u<0,\\ &= u && \text{if } z\in W_3 \text{ on which } 0\leqq u\leqq M.\end{aligned}$$

Then we have $D_F(f)\leqq D_F(u)<\infty$, and $D_F(f,u)\gneqq 0$, since

$$\infty>D_{W_1}(f,u)=M\iint_{W_1}Pu\,dx\,dy;$$

$$D_{W_2}(f,u)=0 \ ;$$

and

$$\infty>D_{W_3}(f,u)=D_{W_3}(u,u)$$
$$\geqq D_W(u,u)\gneqq 0 .$$

Let U_n be a continuous function such that $U_n\equiv f$ on $F-F_n$ and U_n is a solution of (A) on F_n, then $\{U_n\}$ is a uniformly bounded sequence, and hence we can select a subsequence of $\{U_n\}$ such that $\lim_{\nu\to\infty}U_{n_\nu}=U$ exists and U is a bounded solution of (A). For simplicity's sake we shall retain the original suffices. By Lemma 5.1, we have, for $m>n$,
$D_F(U_m)\leqq D_F(U_n)\leqq D_F(f)<\infty$.
Thus, $\lim_{n\to\infty}D_F(U_n)$ exists and is of finite value $D_F(U)$. On the other hand, $D_F(U_m,U_m-U_n)=0$ for $m>n$, thus

$$D_F(U_m-U_n)=D_F(U_n,U_n)-D_F(U_m,U_m),$$

from which we may conclude that U_n converges in a stronger sense. Since $D(u)<\infty$, we have the so-called weakly convergent property:

$$\lim_{n\to\infty}D_F(U_n,u)=D_F(U,u)<\infty .$$

On the other hand, we have

$$D_F(U_n,u)=D_{F_n}(U_n,u)+D_{F-F_n}(f,u)$$
$$=D_{F_n}(f,u)+D_{F-F_n}(f,u)$$
$$=D_F(f,u)\neq 0.$$

And hence we have

$$D_F(U,u)\neq 0.$$

Thus U is not identically zero. Thus we obtain the desired result:

$F \notin O_D$ implies $F \notin O_B$.

q.e.d.

Remark 1. We can arrange our results in the following schema:

$$O_G \subset O_G^{(k)} = O_\omega^{(k)} \subset O_\omega = O_\Omega = O_B \subset O_D = O_K.$$

2. For any admissible $P(z)$ we obtain the above schema, but it seems to us that we need a special choice of $P(z)$ in order to bring more precise results. If there exists a single-valued Dirichlet-finite non-constant harmonic function $k(z)$ on F, then we have $O_D = O_B$ if we choose $\frac{(\frac{\partial k}{\partial x})^2 + (\frac{\partial k}{\partial y})^2}{P(z)}$ as $P(z)$. Proof is easily performed by considering the fact:

$\iint_F \left((\frac{\partial k}{\partial x})^2 + (\frac{\partial k}{\partial y})^2 \right) dx\,dy < \infty$ is equivalent to $D_F(1,1) = \iint_F P\,dx\,dy < \infty$.

In this view-point we may say that a pair (F, P) consisting of a given Riemann surface F and an admissible function $P(z)$ has been classified, and thus we may write $O_T(P)$, where T is G or ω or K.

3. For a non-linear partial differential equation of elliptic type we shall discuss the results elsewhere. In this case the arguments and results must be modified.

4. If $P_1(z)$, $P_2(z)$ $\left(P_1(z) \leqq P_2(z)\right)$ are two admissible functions in our original case, then we have the following results:

i) $O_G(P_1) \supset O_G(P_2)$,

ii) $O_\omega(P_1) \subset O_\omega(P_2)$,

iii) $O_K(P_1) \subset O_K(P_2)$.

These results were obtained already by Bergman-Schiffer [2] without considerations of null-boundary. They considered only the case of planar schlicht compact analytic domains, but their proofs remain valid also in case of Riemann surfaces.

5. In a schlicht domain we may take $P(z)$ without the conformally covariant property: $P(z) = P(z')\left|\frac{dz'}{dz}\right|^2$, for example, constant $k (\neq 0)$. In these cases our arguments remain valid with some exceptions, but our main results, that is,

$$O_G \subset O_\omega = O_\Omega = O_B \subset O_D = O_K$$

also remain valid. Moreover we can define other several Hilbert spaces with the various different metrics, and we can define other several corresponding kernel functions and hence other sorts of null-set. Cf. M. Schiffer [1], Bergman-Schiffer [2].

§7. Riemann Surfaces of Finite Genus.

Lemma 7.1. Let E be a compact set on F_1, being compact subregion of F, and $\varphi(z)$ be an arbitrary bounded solution of (A) in $F_1 - E$. A necessary and sufficient condition in order that $\varphi(z)$ is prolongable on E in the sense of (A) is that $\omega(z, E, F_1) \equiv 0$.

Proof. Sufficiency. Let Γ be a finite number of Jordan curves, surrounding E, and belonging to F_1 with boundary C, and F_C be $F_1 - F_\Gamma$, where F_Γ is the domain bounded by Γ. Then $\omega(z, E, F_1) \equiv 0$ means that $\omega(z, \Gamma, F_c) < \varepsilon$ for any Γ sufficiently near to E. Let $\varphi_0(z)$ be a solution of (A) in F_1, coinciding with $\varphi(z)$ on C. If we introduce the function $\Omega(z, C, F_1)$, then we have, from Lemma 2.1, $|\varphi_0(z)| \leqq M\,\Omega(z, C, F_1)$, where $M = \sup_{F_1} |\varphi(z)|$.

Let $\Phi(z) = \varphi(z) - \varphi_0(z)$, then $|\Phi(z)| \leqq M + M\,\Omega(z) \leqq 2M$ in $\overline{F_C}$. Therefore if we put $U = \Phi(z) + 2M\,\omega(z, \Gamma, F_c)$, then $U \geqq 0$ in F_c; thus $\Phi(z) \geqq -2M\,\omega(z) \geqq -2M\varepsilon$ and hence $\Phi(z) \geqq 0$ in $F_1 - E$.

Similarly if we consider the function $V = \Phi - 2M\omega$, then $\Phi(z) \leqq 2M\,\omega(z)$ in $F_1 - F_\Gamma$, and hence $\Phi(z) \leqq 0$ in $F_1 - E$.

Therefore $\Phi(z) \equiv 0$ in $F_1 - E$. Thus, if we define $\varphi(z) = \varphi_0(z)$ in $\overline{E}$ and $\varphi(z) = \varphi(z)$ in $F - \overline{E}$, then $\varphi(z)$ is bounded and satisfies $\Delta\varphi = P\varphi$ in $\overline{E}$.

Necessity. There is at least a point z in $F_1 - E$, such that $\omega(z, E, F_1 - E) > 0$. On the other hand, $\omega(z, E, F_1 - E)$ is bounded in $F_1 - E$, and therefore, by the assumption, $\omega(z, E, F_1 - E)$ must be prolongable on E and of finite value. On the other hand, $\omega \equiv 0$ on C. Since F_1 is a compact subregion of F, $\omega \equiv 0$ on $\overline{F_1}$, which contradicts $\omega > 0$ q.e.d.

The following theorem yields another sufficiency proof of Theorem 3.3 in the case of Riemann surface of finite genus.

__Theorem 7.1.__ Let F be a Riemann surface of finite genus, and E a compact subset of F. Then $F-E \in O_\omega$ implies $F-E \in O_B$ and vice versa.

Proof. If $F-E \in O_\omega$ and if $u(z)$ is an arbitrary bounded non-constant solution of (A) on the neighbourhood F_1 of E, then from Theorem 7.1 we see that $u(z)$ is prolongable on E with regard to (A) and is of finite value on E. On the other hand, $\Omega(z, E) = \lim_{n\to\infty}\Omega(z, \Gamma_n, F_n)$ is a bounded solution of (A) on F_1, therefore $\Omega(z, E)$ is prolongable on E and is of finite value on E. Thus $\Omega(z, E)$ is a finite non-negative solution of (A) on the whole F, and hence from the subharmonicity of $\Omega(z, E)$, it reduces to a constant, that is, zero. This shows that $F-E \in O_B$.

The converse is immediately obtained by the necessity parts of Theorem 3.3 and Theorem 3.1. q.e.d.

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