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Classification of Semisimple Levi-Tanaka Algebras (*).

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Abstract. – *After showing that the partial complex structure is defined by an inner derivation, we give a complete classification of semisimple Levi-Tanaka algebra.*

The Levi-Tanaka algebras were introduced in [Tan70] to study the group of pseudoconformal automorphism of a real submanifold M of a complex manifold X. They are graded real Lie algebras $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$, provided of a partial complex structure, i.e. a complex structure J on the subspace $\mathfrak{g}_{-1}$ of elements of degree (-1), for which the inner derivations of degree 0 define complex linear maps on $\mathfrak{g}_{-1}$.

These algebras are defined as prolongations of the graded Lie algebra associated to the filtration of the real tangent space TM of M induced by the distribution $HM \subset TM$ of the holomorphic tangent vectors.

They are finite dimensional if and only if the Levi form is nondegenerate, and this is therefore a sufficient condition in order that the group of pseudoconformal automorphisms of the corresponding CR manifold be a finite dimensional Lie group.

In [MN] we continued the investigations of [Tan70], [Tan79] and [CM75], developing the algebraic theory of Levi-Tanaka algebras, especially to construct homogeneous CR manifolds which, for a given nondegenerate Levi form, and higher order Levi form, have the largest groups of pseudoconformal automorphisms.

We showed in [MN] that the Levi-Tanaka algebras admit a Levi-Mal'čev decomposition in which the semisimple part is the direct sum of two ideals, one of which is still a Levi-Tanaka algebra, while the other one is a semisimple algebra of 0-degree derivations of the radical.

It is therefore natural, as a first step toward a classification of homogeneous CR manifolds, to classify first all semisimple Levi-Tanaka algebras. In this paper we take up this question.

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The main tool is the observation that the partial complex structure is defined, in the semisimple case, by a 0-degree inner derivation: this means that the semisimple Levi-Tanaka algebras are of type (J), according to the definition given in [MN].

The classification is obtained by characterizing the weighted Satake's diagrams that are associated to a semisimple Levi-Tanaka algebra.

The paper is organized as follows. In the first two sections we discuss the general properties of the finite dimensional semisimple Levi-Tanaka algebras that are relevant to our problem. For this we rely on the results on real semisimple graded Lie algebras that can be found for instance in [Djo82], [KA88] and [Kan93]. After reducing the problem to that of classifying simple Levi-Tanaka algebras, we discuss separately simple graded Lie algebras of the complex and of the real type, first giving general criteria on their weighted Dynkin and Satake diagrams in order that they admit a structure of Levi-Tanaka algebra and next applying these criteria to the different classes, also giving matrix representations for the nonexceptional ones. At the end of the paper we give the tables relative to the simple Levi-Tanaka algebras corresponding to the exceptional Lie algebras of the complex type.

1. - Preliminaries.

A graded Lie algebra (abbreviated as GLA) is a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$, together with a direct sum decomposition $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ such that each $\mathfrak{g}_p$ is a finite dimensional $\mathbb{K}$-linear subspace of $\mathfrak{g}$ and $[\mathfrak{g}_p, \mathfrak{g}_q] \subset \mathfrak{g}_{p+q}$ for all integers p, q.

We note that $\mathfrak{g}_0$ is a subalgebra of $\mathfrak{g}$ and that by restriction of the adjoint representation we obtain natural linear representations ϱ_p of $\mathfrak{g}_0$ on $\mathfrak{g}_p$ for each integer p. We will denote by $\mathfrak{m}(\mathfrak{g})$ the nilpotent subalgebra $\bigoplus_{p<0} \mathfrak{g}_p$ of the GLA $\mathfrak{g}$.

A GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is said to be *transitive* if

$$0 \neq X \in \mathfrak{g}_p, \qquad p \geq 0 \Rightarrow [X, \mathfrak{g}_{-1}] \neq 0$$

and *fundamental* if $\mathfrak{g}_{-1}$ generates $\mathfrak{m}(\mathfrak{g})$. We say that $\mathfrak{g}$ is *nondegenerate* if for every $X \in \mathfrak{g}_{-1}$ there is $Y \in \mathfrak{g}_{-1}$ such that $[X, Y] \neq 0$.

The supremum μ of the set of integers p for which $\mathfrak{g}_{-p} \neq 0$ is called the *kind* of the GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$. Note that $\mu \geq 2$ for a nondegenerate GLA.

We assume in the following that the field $\mathbb{K}$ has characteristic 0. For a GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ over $\mathbb{K}$, the $\mathbb{K}$-linear map $e \colon \mathfrak{g} \to \mathfrak{g}$ defined by

$$e(X) = pX \quad \text{for } X \in \mathfrak{g}_p, \;\; p \in \mathbb{Z}$$

is a 0-degree derivation. If $e = \mathrm{ad}(E)$ for some $E \in \mathfrak{g}$ is an inner derivation, we call such an element $E \in \mathfrak{g}_0$ a *characteristic element* of the GLA $\mathfrak{g}$. Conversely, if a finite dimensional Lie algebra $\mathfrak{g}$ over $\mathbb{K}$ contains a semisimple element E such that $\mathrm{ad}(E)\colon \mathfrak{g} \to \mathfrak{g}$ has integral eigenvalues, then the subspaces $\mathfrak{g}_p = \{X \in \mathfrak{g} \mid [E, X] = pX\}$ (for $p \in \mathbb{Z}$) define a graduation of $\mathfrak{g}$ for which E is a characteristic element. The characteristic element is uniquely determined when the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$ is trivial. Note that, if there is a characteristic element for the GLA $\mathfrak{g}$, then every ideal $\mathfrak{a}$ of $\mathfrak{g}$ is also a GLA for the graduation induced from $\mathfrak{g}$.

A semisimple GLA $\mathfrak{g}$ contains a characteristic element E because all derivations of $\mathfrak{g}$ are inner and this element is unique because $\mathfrak{z}(\mathfrak{g})$ is trivial. In this case we will call E *the characteristic element of* $\mathfrak{g}$.

An isomorphism of GLA's is a map $\phi\colon \mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p \to \mathfrak{g}' = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}'_p$ between GLA's over the same field $\mathbb{K}$ such that: (i) ϕ is a Lie algebras isomorphism; (ii) $\phi(\mathfrak{g}_p) = \mathfrak{g}'_p$ for all $p \in \mathbb{Z}$. For semisimple GLA's having characteristic elements E, E' respectively condition (ii) is equivalent to $\phi(E) = E'$.

We collect some well known facts on semisimple GLA's in the following:

LEMMA 1.1. – *Let* $\mathfrak{g} = \bigoplus_{p=-\mu}^{\nu} \mathfrak{g}_p$ *be a real finite dimensional semisimple GLA of kind* μ. *Then*:

(*a*) *Each ideal of* $\mathfrak{g}$ *is, in a natural way, a GLA of kind less than or equal to* μ;

(*b*) *For the Killing form* κ *of* $\mathfrak{g}$, *we have*:

$$\kappa(\mathfrak{g}_p, \mathfrak{g}_q) \neq 0 \Leftrightarrow p + q = 0 .$$

In particular $\mathfrak{g}_p = 0$ *for* $p > \mu$ (*i.e.* $\mu = \nu$) *and the Killing form defines for each integer* p *a duality pairing between* $\mathfrak{g}_p$ *and* $\mathfrak{g}_{-p}$ *and therefore* $\dim_{\mathbb{R}} \mathfrak{g}_p = \dim_{\mathbb{R}} \mathfrak{g}_{-p}$ *for every* $p \in \mathbb{Z}$. *Moreover* $\mathfrak{g}_0$ *is reductive.*

If $\mu > 0$, *then* $\mathfrak{g}$ *is of the noncompact type, i.e.* κ *is not negative definite.*

(*c*) *If* $\mathfrak{g}$ *is simple, then it is fundamental if and only if*

$$\mathfrak{g}_{-1}^{(\mu)} = \underbrace{[\mathfrak{g}_{-1}, \dots, [\mathfrak{g}_{-1}, [\mathfrak{g}_{-1}, \mathfrak{g}_{-1}]] \dots]}_{\mu \text{ times}} \neq 0 ,$$

and in this case is nondegenerate and transitive if and only if $\mu > 1$.

For a proof we refer to [Tan70], [MN].

Let $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ be a real GLA. A *partial complex structure* on $\mathfrak{g}$ is an $\mathbb{R}$-linear map $J\colon \mathfrak{g}_{-1} \to \mathfrak{g}_{-1}$ which satisfies

$$\begin{cases} J^2 = -\mathrm{Id}_{\mathfrak{g}_{-1}}, \\ [JX, JY] = [X, Y] \qquad \forall X,\ Y \in \mathfrak{g}_{-1} . \end{cases} \tag{1}$$

A *Levi-Tanaka algebra* is a real GLA $\mathfrak{g} = \bigoplus_{p\in\mathbb{Z}} \mathfrak{g}_p$ endowed with a partial complex structure J for which the following holds:

(1) the kind μ of $\mathfrak{g}$ is finite;

(2) $\mathfrak{g}$ is fundamental;

(3) the adjoint representation gives an isomorphism between $\mathfrak{g}_0$ and the algebra of 0-degree derivations of $\mathfrak{m}(\mathfrak{g})$ whose restriction to $\mathfrak{g}_{-1}$ commutes with J;

(4) $\mathfrak{g}$ is the maximal transitive prolongation of the GLA $\mathfrak{m}(\mathfrak{g})\oplus\mathfrak{g}_0$.

A Levi-Tanaka algebra $\mathfrak{g}$ is finite dimensional if and only if it is nondegenerate (cf. [Tan70]). In particular a finite dimensional Levi-Tanaka algebra has kind $\mu\geq 2$.

We also note that for a finite dimensional simple GLA $\mathfrak{g}$ of kind $\mu\geq 2$, endowed with a partial complex structure, properties (1), (2) and (3) imply (4) (cf. [Tan70], [MN], where also different criteria for the semisimplicity of the prolongation are discussed).

An isomorphism of two Levi-Tanaka algebras $\mathfrak{g}=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}_p$ and $\mathfrak{g}'=\bigoplus_{p\in\mathbb{Z}}\mathfrak{g}'_p$, whose partial complex structures are denoted by J and J' respectively, is an isomorphism $\phi\colon\mathfrak{g}\to\mathfrak{g}'$ of GLA's such that $\phi(JX)=J'\phi(X)$ for every $X\in\mathfrak{g}_{-1}$.

2. - Real semisimple graded Lie algebras.

In this section we rehearse some general properties of real semisimple Lie algebras, for which we refer to [Djo82], [Sug59], [War72], and consider their bearing to semisimple Levi-Tanaka algebras.

Let $\mathfrak{s}$ be a semisimple real Lie algebra. Every Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ decomposes into a direct sum $\mathfrak{h}=\mathfrak{h}^+\oplus\mathfrak{h}^-$ where

$$\mathfrak{h}^+=\{X\in\mathfrak{h}\,|\,\mathrm{ad}_{\mathfrak{s}}(X)\text{ has purely imaginary eigenvalues}\},$$

$$\mathfrak{h}^-=\{X\in\mathfrak{h}\,|\,\mathrm{ad}_{\mathfrak{s}}(X)\text{ has real eigenvalues}\},$$

are called respectively the *toroidal* and *vectorial part* of $\mathfrak{h}$. A Cartan subalgebra $\mathfrak{h}$ whose toroidal part has minimal dimension is called *minimally compact* (*standard* in the sense of [Sug59]).

We have (cf. [Sug59, Cor. 2 to Th. 3] or [War72, Cor. 1.3.1.5]):

PROPOSITION 2.1. - *All minimally compact Cartan subalgebras of a semisimple real Lie algebra are conjugated by the action of the adjoint group.*

The complexification $\mathfrak{h}^{\mathbb{C}}$ of a minimally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$ is a Cartan subalgebra of the complexification $\mathfrak{s}^{\mathbb{C}}$ of $\mathfrak{s}$. Denote by $\mathcal{R}$ the relative root system. Consider the real form $\mathfrak{h}_u=\mathfrak{h}^-\oplus\sqrt{-1}\mathfrak{h}^+$ of $\mathfrak{h}^{\mathbb{C}}$. We identify the complexification of the dual $\check{\mathfrak{h}}_u$ of $\mathfrak{h}_u$ with the dual $\check{\mathfrak{h}}^{\mathbb{C}}$ of $\mathfrak{h}^{\mathbb{C}}$. Then $\mathcal{R}\subset\check{\mathfrak{h}}_u\subset\check{\mathfrak{h}}^{\mathbb{C}}$. Let σ be the conjugation on $\mathfrak{s}^{\mathbb{C}}$ defined by the real form $\mathfrak{s}$ and denote by σ the corresponding involution of $\mathfrak{h}^{\mathbb{C}}$ given by

$\check{\mathfrak{h}}^{\mathbb{C}} \ni \alpha \to \alpha^{\sigma} \in \check{\mathfrak{h}}^{\mathbb{C}}$, where

$$\alpha^{\sigma}(X) = \overline{\alpha(\sigma X)} \quad \forall X \in \mathfrak{h}^{\mathbb{C}}. \tag{2}$$

Then the root system $\mathcal{R}$ is σ-invariant.

In the following we will use $\mathfrak{s}$ to denote a semisimple real Lie algebra and $\mathfrak{g}$ for the GLA obtained by fixing a graduation on $\mathfrak{s}$.

We have (cf. [Djo82]):

LEMMA 2.2. – *The characteristic element E of a semisimple real GLA $\mathfrak{g} = \bigoplus_{p \in \mathbb{Z}} \mathfrak{g}_p$ is contained in a minimally compact Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ of $\mathfrak{g}$.*

We also note that every Cartan subalgebra $\mathfrak{h}$ of the semisimple GLA $\mathfrak{g}$ that contains the characteristic element E is contained in $\mathfrak{g}_0$ and that, in this case, E belongs to the intersection of the vectorial part $\mathfrak{h}^-$ of $\mathfrak{h}$ with the center $\mathfrak{z}(\mathfrak{g}_0)$ of the subalgebra $\mathfrak{g}_0$.

PROOF. – The Lemma is a consequence of the fact ([Sug59], Theorem 2) that the vectorial part of any Cartan subalgebra of $\mathfrak{g}$ is conjugated by an element of the adjoint group to a subspace of a minimally compact Cartan subalgebra of $\mathfrak{g}$. Then we reduce to the observation that the semisimple element E belongs to a Cartan subalgebra of $\mathfrak{g}$. ■

Let $\mathfrak{h}$ be a minimally compact Cartan subalgebra of the semisimple GLA $\mathfrak{g}$, containing its characteristic element E, and let $\mathcal{R}$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. We set:

$$\mathcal{R}_\bullet = \{\alpha \in \mathcal{R} \mid \alpha(H) = 0 \ \forall H \in \mathfrak{h}^-\};$$

$$\mathcal{R}_p = \{\alpha \in \mathcal{R} \mid \alpha(E) = p\} \quad \text{for } p \in \mathbb{Z};$$

$$\mathcal{R}_- = \{\alpha \in \mathcal{R} \mid \alpha(E) \leq 0\} = \bigcup_{p \leq 0} \mathcal{R}_p.$$

We have the inclusions: $\mathcal{R}_\bullet \subset \mathcal{R}_0 \subset \mathcal{R}_- \subset \mathcal{R}$.

We call the integer $|\alpha| = \alpha(E)$ the *degree* of the root $\alpha \in \mathcal{R}$.

The root systems $\mathcal{R}_\bullet$ and $\mathcal{R}_0$ are invariant under the action of the conjugation σ defined above. Indeed, $\alpha^{\sigma}(H) = \overline{\alpha(\sigma H)} = \overline{\alpha(H)} = \alpha(H)$ for every $H \in \mathfrak{h}^-$.

For every $\alpha \in \mathcal{R}$ we denote by $\mathfrak{g}_\alpha$ the relative eigenspace:

$$\mathfrak{g}_\alpha = \{X \in \mathfrak{g}^{\mathbb{C}} \mid [H, X] = \alpha(H)\, X \ \forall H \in \mathfrak{h}^{\mathbb{C}}\}.$$

We have:

$$\mathfrak{g}_0^{\mathbb{C}} = \mathfrak{h}^{\mathbb{C}} \oplus \bigoplus_{\alpha \in \mathcal{R}_0} \mathfrak{g}_\alpha, \qquad \mathfrak{g}_p^{\mathbb{C}} = \bigoplus_{\alpha \in \mathcal{R}_p} \mathfrak{g}_\alpha.$$

Assume that $\mathfrak{g}$ is a semisimple Levi-Tanaka algebra with partial complex structure $J \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g}_{-1})$. We define:

$$\mathfrak{g}_{-1}^{(1,0)} = \{X - \sqrt{-1}JX | X \in \mathfrak{g}_{-1}\} \subset \mathfrak{g}_{-1}^{\mathbb{C}},$$

$$\mathfrak{g}_{-1}^{(0,1)} = \{X + \sqrt{-1}JX | X \in \mathfrak{g}_{-1}\} = \sigma(\mathfrak{g}_{-1}^{(1,0)}) \subset \mathfrak{g}_{-1}^{\mathbb{C}}.$$

These are the eigenspaces corresponding respectively to the eigenvectors $\sqrt{-1}$ and $-\sqrt{-1}$ of J. Because $J \circ \varrho_{-1}(X) = \varrho_{-1}(X) \circ J$ for every $X \in \mathfrak{g}_0$, the subspaces $\mathfrak{g}_{-1}^{(1,0)}$ and $\mathfrak{g}_{-1}^{(0,1)}$ are invariant under $\varrho_{-1}(\mathfrak{g}_0)$. In particular each eigenspace $\mathfrak{g}_\alpha$ for $\alpha \in \mathcal{R}_{-1}$ is contained either in $\mathfrak{g}_{-1}^{(1,0)}$ or in $\mathfrak{g}_{-1}^{(0,1)}$ and is J-invariant. Moreover, $\mathfrak{g}_\alpha \subset \mathfrak{g}_{-1}^{(1,0)}$ if and only if $\mathfrak{g}_{\alpha^\sigma} = \sigma(\mathfrak{g}_\alpha) \subset \mathfrak{g}_{-1}^{(0,1)}$.

If X is an eigenvector corresponding to a root $\alpha \in \mathcal{R}$ with $|\alpha| \in \{-1, 0\}$, we define

$$\text{(3)} \qquad \operatorname{sgn}(X) = \begin{cases} +1 & \text{if } |\alpha| = -1 \text{ and } \mathfrak{g}_\alpha \subset \mathfrak{g}_p^{(1,0)}, \\ -1 & \text{if } |\alpha| = -1 \text{ and } \mathfrak{g}_\alpha \subset \mathfrak{g}_p^{(0,1)}, \\ 0 & \text{if } |\alpha| = 0. \end{cases}$$

Note that $JX = \operatorname{sgn}(X)\sqrt{-1}X$ when X is an eigenvector corresponding to a root α of degree (-1).

We know from [MN, Lemma 3.25], that the following is true:

LEMMA 2.3. – *Let $\mathfrak{g}$ be a semisimple Levi-Tanaka algebra and let $\mathcal{R}$ be the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to the complexification $\mathfrak{h}^{\mathbb{C}}$ of a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, contained in $\mathfrak{g}_0$. Then there is a fundamental system $\mathcal{B} = \{\alpha_1, \ldots, \alpha_l\}$ for $\mathcal{R}$ such that $|\alpha_i| \in \{-1, 0\}$ for every $i = 1, \ldots, l$.*

The following result will be important for the study of semisimple Levi-Tanaka algebras:

THEOREM 2.4. – *Let $\mathfrak{g} = \bigoplus_{-\mu \le p \le \mu} \mathfrak{g}_p$ be a finite dimensional semisimple Levi-Tanaka algebra, with partial complex structure $J \in \mathfrak{gl}_{\mathbb{R}}(\mathfrak{g}_{-1})$. Then there is a unique element $\tilde{J} \in \mathfrak{z}(\mathfrak{g}_0)$ such that*

$$\text{(4)} \qquad J = \varrho_{-1}(\tilde{J}).$$

PROOF. – Let $\mathfrak{h} \subset \mathfrak{g}_0$ be a minimally compact Cartan subalgebra of $\mathfrak{g}$ and $\mathcal{R}$ the root system of $\mathfrak{g}^{\mathbb{C}}$ with respect to $\mathfrak{h}^{\mathbb{C}}$. We associate to a fundamental root system $\mathcal{B} = \{\alpha_1, \ldots, \alpha_l\} \subset \mathcal{R}_{-1} \cup \mathcal{R}_0$ of $\mathcal{R}$ a Weyl basis

$$H_i \in \mathfrak{h}^{\mathbb{C}}, \qquad X_i \in \mathfrak{g}_{\alpha_i}, \qquad Y_i \in \mathfrak{g}_{-\alpha_i} \quad \text{for } 1 \le i \le l.$$

We note that, for every $1 \leqslant i \leqslant l$, $H_i \in \mathfrak{g}_0^{\mathbb{C}}$, $X_i \in \mathfrak{g}_{-1}^{\mathbb{C}} \cup \mathfrak{g}_0^{\mathbb{C}}$, $Y_i \in \mathfrak{g}_0^{\mathbb{C}} \cup \mathfrak{g}_1^{\mathbb{C}}$. Then a basis of $\mathfrak{m}(\mathfrak{g}^{\mathbb{C}})$ as a $\mathbb{C}$-vector space is provided by $\mathcal{X} = \{X_I | I \in \mathfrak{I}\}$, where $\mathfrak{I}$ is a set of k-uples $(i_1, \ldots, i_k)$ of multiindices, for k a positive integer, $1 \leqslant i_h \leqslant l$ for $h = 1, \ldots, k$, with $|\alpha_{i_1}| + \ldots + |\alpha_{i_k}| < 0$ and

$$X_I = [X_{i_k}, [\ldots [X_{i_2}, X_{i_1}] \ldots]] \quad \text{for } I = (i_1, \ldots, i_k) \in \mathfrak{I}.$$

Moreover, if $I = (i_1, \ldots, i_k)$, $I' = (i_1', \ldots, i_{k'}') \in \mathfrak{I}$, then either $[X_I, X_{I'}] = 0$ or $[X_I, X_{I'}] = c_{I,I'} X_{I''}$ for a nonzero complex number $c_{I,I'}$ and a $(k+k')$-uple $I'' = (i_1'', \ldots, i_{k+k'}'')$ which is a permutation of the $(k+k')$-uple (I, I'). In the last case we will denote the multiindex $I'' \in \mathfrak{I}$ by $I \cdot I'$.

To extend J to a $\mathbb{C}$-linear endomorphism J' of $\mathfrak{m}(\mathfrak{g}^{\mathbb{C}})$ it suffices to define it on the elements of the basis $\mathcal{X}$. To this aim we introduce the function $\phi \colon \mathfrak{I} \to \mathbb{Z}$ by

$$\phi(I) = \sum_{1 \leqslant i_k \leqslant k} \operatorname{sgn}(X_{i_h}) \quad \text{for } I = (i_1, \ldots, i_k) \in \mathfrak{I} \tag{5}$$

and set:

$$J'(X_I) = \phi(I)\sqrt{-1}\, X_I \quad \text{for } I \in \mathfrak{I}. \tag{6}$$

Since J commutes with the endomorphisms of $\varrho_{-1}(\mathfrak{g}_0^{\mathbb{C}})$, we have $J'(X) = JX$ for $X \in \mathfrak{g}_{-1}^{\mathbb{C}}$. Next we show that J' defines a zero degree derivation of the GLA $\mathfrak{m}(\mathfrak{g}^{\mathbb{C}})$. It suffices to verify that this is true for the elements of the basis $\mathcal{X}$. Let $I, I' \in \mathfrak{I}$. When $[X_I, X_{I'}] = 0$, we have

$$[J'(X_I), X_{I'}] = [X_I, J'(X_{I'})] = J'([X_I, X_{I'}]) = 0$$

and the Jacobi identity trivially holds. In the case where $[X_I, X_{I'}] \neq 0$, we have $I \cdot I' \in \mathfrak{I}$ and

$$J'([X_I, X_{I'}]) = J'(c_{I,I'} X_{I \cdot I'}) = c_{I,I'}\, \phi(I \cdot I')\sqrt{-1}\, X_{I \cdot I'} =$$

$$= c_{I,I'}(\phi(I) + \phi(I'))\sqrt{-1}\, X_{I \cdot I'} = [J'(X_1), X_{I'}] + [X_I, J'(X_{I'})].$$

Therefore J' is a zero degree derivation of $\mathfrak{m}(\mathfrak{g}^{\mathbb{C}})$ whose restriction to $\mathfrak{g}_{-1}^{\mathbb{C}}$ commutes with all elements of $\varrho_{-1}(\mathfrak{g}_0^{\mathbb{C}})$. Because $\mathfrak{g}^{\mathbb{C}}$ is a Levi-Tanaka algebra, there is an element $\tilde{J}$ in $\mathfrak{g}_0^{\mathbb{C}}$ such that

$$[\tilde{J}, X] = J'(X), \quad \forall X \in \mathfrak{m}(\mathfrak{g}^{\mathbb{C}}).$$

The real and imaginary parts of $\tilde{J}$ with respect to the real form $\mathfrak{g}$ are zero degree derivations of $\mathfrak{g}$. By construction, the imaginary part of $\tilde{J}$ defines the zero endomorphism on $\mathfrak{g}_{-1}$ and therefore is zero because $\mathfrak{g}$ is transitive. It follows that $\tilde{J}$ belongs to $\mathfrak{g}_0$. The proof is complete. ■

COROLLARY 2.5. – *Let* $\mathfrak{g} = \bigoplus_{-\mu \leqslant p \leqslant \mu} \mathfrak{g}_p$ *be a finite dimensional semisimple Levi-Tanaka algebra, with partial complex structure J. Then:*

1) $\dim_{\mathbb{R}} \mathfrak{z}(\mathfrak{g}_0) \geq 2$;

2) $\mathfrak{g}$ *contains a minimally compact Cartan subalgebra* $\mathfrak{h} \subset \mathfrak{g}_0$;

3) *every Cartan subalgebra* $\mathfrak{h}$ *of* $\mathfrak{g}$ *with* $\mathfrak{h} \subset \mathfrak{g}_0$ *contains both the characteristic element* E *and the unique element* $\tilde{J} \in \mathfrak{g}_0$ *such that* $J = \varrho_{-1}(\tilde{J})$; *moreover* $E \in \mathfrak{h}^- \cap \mathfrak{z}(\mathfrak{g}_0)$ *and* $\tilde{J} \in \mathfrak{h}^+ \cap \mathfrak{z}(\mathfrak{g}_0)$, *where* $\mathfrak{h}^-$ *and* $\mathfrak{h}^+$ *denote the vectorial part and the toroidal part of* $\mathfrak{h}$, *respectively.*

In the following we will use for simplicity the same symbol J to denote either the partial complex structure of a semisimple Levi-Tanaka algebra $\mathfrak{g}$ or the corresponding element of $\mathfrak{g}_0$ whose representation in $\mathfrak{g}_{-1}$ is the partial complex structure of $\mathfrak{g}$.

COROLLARY 2.6. – *Every ideal of a semisimple Levi-Tanaka algebra is a Levi-Tanaka algebra for the restriction of the partial complex structure.*

In particular it suffices to classify simple Levi-Tanaka algebras, as every semisimple Levi-Tanaka algebra is a direct sum of simple ideals which are Levi-Tanaka algebras for the restriction of the partial complex structure.

We list now some results, related to the classification of real semisimple GLA's, for which we refer to [Djo82], [KA88], [Kan93], [War72].

Let $\mathfrak{g} = \bigoplus_{-\mu \leq p \leq \mu} \mathfrak{g}_p$ be a semisimple real GLA of kind μ. Then there is a minimally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$ contained in $\mathfrak{g}_0$ and it contains the characteristic element E of $\mathfrak{g}$. We denote by $\mathcal{R}$ the root system of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{h}^{\mathbb{C}}$. It admits a fundamental system $\mathcal{B} = \{\alpha_1, \ldots, \alpha_l\}$ contained in $\mathcal{R}_-$, which is σ-fundamental in the sense precised in [War72, p. 23] (cf. condition (2) below).

We denote by $\Delta_{\mathfrak{g}}$ the *weighted Dynkin diagram* of $\mathfrak{g}^{\mathbb{C}}$, which is obtained from the Dynkin diagram of $\mathfrak{g}^{\mathbb{C}}$, in which the vertices are identified to the corresponding roots in $\mathcal{B}$, by attaching to each vertex α_i its degree $|\alpha_i|$. The *weighted Satake diagram* $\Sigma_{\mathfrak{g}}$ of $\mathfrak{g}$ is obtained from the weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ of $\mathfrak{g}^{\mathbb{C}}$ by the following procedure:

1) the vertices $\alpha \in \mathcal{B}_\bullet = \mathcal{B} \cap \mathcal{R}_\bullet$ are black and all other vertices are white. Note that black vertices have degree 0;

2) for every white vertex $\alpha \in \mathcal{B} - \mathcal{B}_\bullet$ there exists a unique white vertex $\alpha' \in \mathcal{B} - \mathcal{B}_\bullet$ such that $\alpha^\sigma - \alpha'$ is a linear combination of the black roots (cf. [War72, Lemma 1.1.3.2]). If $\alpha \neq \alpha'$, then we connect the pair $\{\alpha, \alpha'\}$ by a curved arrow. Note that roots connected by a curved arrow have the same degree.

Let $\mathfrak{s}$ be a semisimple real Lie algebra and $\mathfrak{g}$ a GLA obtained by fixing a graduation of $\mathfrak{s}$. The *weighted Satake diagram* of $\mathfrak{g}$ is obtained by attaching to each vertex α of the Satake diagram of $\mathfrak{s}$ its degree $|\alpha|$ with respect to $\mathfrak{g}$. This construction provides a partition of $\mathcal{B}$ into subsets $\mathcal{B}_p$, with $p \leq 0$, given by

$$\mathcal{B}_p = \mathcal{B} \cap \mathcal{R}_p = \{\alpha \in \mathcal{B} \mid |\alpha| = p\}. \tag{7}$$

Vice versa, every such partition of $\mathcal{B}$, indexed by the nonpositive integers, determines a unique graduation of $\mathfrak{s}$ and all graduations of $\mathfrak{s}$ are obtained in this way, up to isomorphisms of GLA's. We explicitly note that the fact that $\mathfrak{g}_{-1}$ generates $\mathfrak{m}(\mathfrak{g})$ is equivalent to have $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_{-1}$.

The classification of real semisimple GLA's is deduced from the following:

THEOREM 2.7. – *A necessary and sufficient condition in order that two real GLA's be isomorphic is that they have isomorphic weighted Satake diagrams.*

We also note that (cf. [War72, Theorem 2.4]):

PROPOSITION 2.8. – 1) *The kind μ of $\mathfrak{g}$ is equal to the absolute value of the degree of the highest root of $\mathfrak{g}^{\mathbb{C}}$.*

2) *Assume that the graduation of $\mathfrak{g}$ is associated to a partition $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_{-1}$ of the fundamental roots of $\mathfrak{g}$. Then $\mathfrak{g}_0$ is the direct sum of a semisimple Lie algebra, whose Satake diagram is obtained from $\Sigma_{\mathfrak{g}}$ by deleting all vertices in $\mathcal{B}_{-1}$ and all rods and arrows issuing from them, and its center $\mathfrak{z}(\mathfrak{g}_0)$, whose dimension equals the number of elements of $\mathcal{B}_{-1}$.*

We say that a real simple Lie algebra is *of the complex type* if its Satake diagram is disconnected, i.e. if it is obtained from a simple complex Lie algebra by change of the base field; we say that a simple real Lie algebra is *of the real type* if it has a connected Satake diagram, i.e. if its complexification is also simple.

We note that for simple Lie algebras $\mathfrak{s}$ of the complex type the Satake diagram is obtained by taking two copies $D'_{\mathfrak{s}}$ and $D''_{\mathfrak{s}}$ of its Dynkin diagram $D_{\mathfrak{s}}$, painting white all vertices and connecting by a curved arrow each root in $D'_{\mathfrak{s}}$ to the same root in $D''_{\mathfrak{s}}$. Thus for simple real Lie algebras of the complex type the Dynkin diagram already contains all information.

We collect some elementary facts on the Satake diagram of a Levi-Tanaka algebra in the following

THEOREM 2.9. – *Let $\mathfrak{g}$ be a finite dimensional semisimple Levi-Tanaka algebra, and $\Sigma_{\mathfrak{g}}$ its weighted Satake diagram related to a Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}_0$ of $\mathfrak{g}$. Then:*

(i) *the degree of the vertices in $\mathcal{B}$ are either (-1) or 0, so that $\mathcal{B} = \mathcal{B}_{-1} \cup \mathcal{B}_0$; the set $\mathcal{B}_{-1}$ of fundamental roots of degree (-1) contains at least two elements;*

(ii) *each vertex α in $\mathcal{B}_{-1}$ is connected to another vertex α' in $\mathcal{B}_{-1}$ by a curved arrow.*

PROOF. – The first claim in (i) follows from the fact that $\mathfrak{g}_{-1}$ generates the subalgebra $\mathfrak{m}(\mathfrak{g}) = \bigoplus_{p<0} \mathfrak{g}_p$, the second from (ii) and the fact that the kind μ of a finite dimensional Levi-Tanaka algebra is $\geqslant 2$.

We prove (ii) by contradiction. Assume that $\alpha \in \mathcal{B}_{-1}$ is not connected to any other root in $\mathcal{B}_{-1}$ by a curved arrow. Then $\alpha^\sigma = \alpha + \gamma$, where γ is a linear combination with integral coefficients of roots of $\mathcal{B}_0$. It follows that J acts as the multiplication by the same $\eta = \pm\sqrt{-1}$ both on $\mathfrak{g}_\alpha$ and on $\mathfrak{g}_{\alpha^\sigma}$. But we already noticed that this cannot be the case because $\mathfrak{g}_\alpha$ and $\mathfrak{g}_{-\alpha}$ are contained in eigenspaces of J corresponding to opposite eigenvalues. ■

For the sake of conciseness we will call in the following *LT-admissible* the weighted Satake diagrams of semisimple Levi-Tanaka algebras.

We also call *admissible* a weighted Satake diagram satisfying conditions (i) and (ii) in Theorem 2.9.

3. – The weighted Satake diagrams of simple Levi-Tanaka algebras of the complex type.

First we investigate the Levi-Tanaka structures that can be defined on a simple real Lie algebra $\mathfrak{s}$ of the complex type.

We fix a minimally compact Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{s}$. Note that $\mathfrak{h}$ is simply a Cartan subalgebra of $\mathfrak{s}$ considered as a complex Lie algebra. Let $\mathcal{R}$ be the corresponding root system (attached to $\mathfrak{s}$ and $\mathfrak{h}$ considered as complex Lie algebras), $\mathcal{B} = \{\alpha_1, \ldots, \alpha_l\}$ a fundamental root system for $\mathcal{R}$ and $D_{\mathfrak{s}}$ the associated Dynkin diagram. For simplicity we will call *connected* a subset Y of $\mathcal{B}$ if its points are the vertices of a connected subset of the graph $D_{\mathfrak{s}}$.

Up to equivalence, all admissible gradings of $\mathfrak{s}$ are obtained from a partition $\{\mathcal{B}_0, \mathcal{B}_{-1}\}$ of $\mathcal{B}$ into a set $\mathcal{B}_{-1}$ of fundamental roots of degree (-1) and a set $\mathcal{B}_0$ of fundamental roots of degree 0. Let $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\}$, with $1 \leq i_1 < \ldots < i_\nu \leq l$, be the set of roots of degree (-1).

Every root $\alpha \in \mathcal{R}$ is a linear combination with integral coefficients

$$\alpha = \sum_{i=1}^{l} k_i(\alpha)\, \alpha_i, \qquad k_i(\alpha) \in \mathbb{Z} \tag{8}$$

of the roots in $\mathcal{B}$. We associate to the root α its degree

$$|\alpha| = \sum_{i=1}^{l} k_i(\alpha)\, |\alpha_i| \tag{9}$$

where $|\alpha_i|$ equals (-1) if $\alpha_i \in \mathcal{B}_{-1}$ and 0 if $\alpha_i \in \mathcal{B}_0$. We recall that

$$Y(\alpha) = \{\alpha_i \,|\, k_i(\alpha) \neq 0\} \tag{10}$$

is a connected subset of $D_{\mathfrak{s}}$ and, for every connected $Y \subset D_{\mathfrak{s}}$,

$$\sum_{\alpha_i \in Y} \alpha_i \in \mathcal{R} \tag{11}$$

(cf. Corollary 3 to Proposition 19, Ch. VI, § 1 in [Bou81]).

We denote by $\mathfrak{g}$ the graded Lie algebra obtained from $\mathfrak{s}$ by the partition $\{\mathcal{B}_{-1}, \mathcal{B}_0\}$ of $\mathcal{B}$: the subspace $\mathfrak{g}_p$ of homogeneous elements of degree p is defined by:

$$\mathfrak{g}_0 = \mathfrak{h} \oplus \bigoplus_{\alpha \in \mathcal{R}_0} \mathfrak{g}^{\alpha}, \qquad \mathfrak{g}_p = \bigoplus_{\alpha \in \mathcal{R}_p} \mathfrak{g}^{\alpha} \quad \text{for } p \neq 0 .$$

To define on the so obtained graded Lie algebra $\mathfrak{g}$ a partial complex structure satisfying (1), by Theorem 2.4, we need to find an element $J \in \mathfrak{z}(\mathfrak{g}_0) \subset \mathfrak{h}$ such that

$$\begin{cases} \varrho_{-1}(J)^2 = -\mathrm{Id}_{\mathfrak{g}_{-1}}, \\ [J, \mathfrak{g}_{-2}] = 0 . \end{cases} \tag{12}$$

We note that J must satisfy

$$\langle J, \alpha_i \rangle = \begin{cases} 0 & \text{if } \alpha_i \in \mathcal{B}_0 , \\ \pm\sqrt{-1} & \text{if } \alpha_i \in \mathcal{B}_{-1} \end{cases} \tag{13}$$

because the eigenspaces of the roots α_i are contained in the eigenspaces of $\mathrm{ad}_{\mathfrak{g}}(J)$.

Then from (12) we obtain:

(*a*) *if* Y *is a connected subset of* $D_{\mathfrak{s}}$ *and* $Y \cap \mathcal{B}_{-1} = \{\alpha_i, \alpha_j\}$, *with* $i<j$, *then*

$$\langle J, \alpha_i \rangle = -\langle J, \alpha_j \rangle . \tag{14}$$

Let indeed Y be a connected subset of $D_{\mathfrak{s}}$ containing exactly two elements α_i, α_j of $\mathcal{B}_{-1}$. Decompose Y into two disjoint connected subsets Y_1, Y_2 such that $\{\alpha_i\} = Y_1 \cap \mathcal{B}_{-1}$ and $\{\alpha_j\} = Y_2 \cap \mathcal{B}_{-1}$. Then $\beta_1 = \sum_{\alpha_h \in Y_1} \alpha_h$ and $\beta_2 = \sum_{\alpha_h \in Y_2} \alpha_h$ are roots of degree (-1) and

$$\langle J, \beta_1 \rangle = \langle J, \alpha_i \rangle, \quad \langle J, \beta_2 \rangle = \langle J, \alpha_j \rangle .$$

Since $\beta_1 + \beta_2 \in \mathcal{R}$, we obtain for nonzero eigenvectors X_{β_1}, X_{β_2} of β_1, β_2 respectively:

$$0 \neq [X_{\beta_1}, X_{\beta_2}] = [JX_{\beta_1}, JX_{\beta_2}] = \langle J, \alpha_i \rangle \langle J, \alpha_j \rangle [X_{\beta_1}, X_{\beta_2}] .$$

This shows that condition (*a*) is necessary.

From this observation, since any two roots in $\mathcal{B}$ can be joined by a segment in $D_{\mathfrak{s}}$, we deduce that J is uniquely determined by the value it assumes on one of the roots in $\mathcal{B}_{-1}$. In particular, for each admissible structure of graded Lie algebra $\mathfrak{g}$ of $\mathfrak{s}$, either there is no partial complex structure J satisfying (1), or there are two such structures, one being conjugated to the other.

Moreover, condition (14) does not restrict the possibility of defining the partial complex structure J, unless $D_{\mathfrak{s}}$ contains ramification points. In the last case, there are two possibilities:

(*b*) *either the ramification point* α_i *of* $D_{\mathfrak{s}}$ *belongs to* $\mathcal{B}_{-1}$, *or at most two of the branches issuing from* α_i *contain elements of* $\mathcal{B}_{-1}$.

Assuming that the conditions (*a*) and (*b*) are fulfilled by a J defined on the elements of $\mathcal{B}$, then a necessary and sufficient condition in order that J extends to an $\mathbb{R}$-linear map defining a complex structure on $\mathfrak{g}_{-1}$ satisfying (1) is that:

(*c*) *there are no roots* $\alpha \in \mathcal{R}$ *with* $|\alpha| = -2$ *and* $k_i(\alpha) = 2$ *for some* $\alpha_i \in \mathcal{B}_{-1}$.

Indeed for $\alpha \in \mathcal{R}$ with $|\alpha| = -2$ there are the two possibilities:

(*) $Y(\alpha) \cap \mathcal{B}_{-1} = \{\alpha_i, \alpha_j\}$ contains exactly two roots and $k_i(\alpha) = k_j(\alpha) = 1$;

(**) $Y(\alpha) \cap \mathcal{B}_{-1} = \{\alpha_i\}$ contains only one root and $k_i(\alpha) = 2$.

We first extend J to a $\mathbb{C}$-linear map on $\mathfrak{g}_0 \oplus \mathfrak{g}_{-1}$ by setting $J = 0$ on $\mathfrak{g}_0$ and $JX_\beta = \langle J, \alpha_i \rangle X_\beta$ if $\{\alpha_i\} = Y(\beta) \cap \mathcal{B}_{-1}$.

A nonzero eigenvector X_α corresponding to a root $\alpha \in \mathcal{R}$ of degree (-2) is obtained as the Lie product $[X_\beta, X_\gamma]$ of two eigenvectors corresponding to roots $\beta, \gamma \in \mathcal{R}$ with $|\beta| = |\gamma| = -1$. If the root α satisfies (**), then J acts on X_β and X_γ as the multiplication by the same $\eta = \pm\sqrt{-1}$ and hence, if (1) is satisfied,

$$X_\alpha = [X_\beta, X_\gamma] = [JX_\beta, JX_\gamma] = -[X_\beta, X_\gamma] \Rightarrow X_\alpha = 0\,,$$

gives a contradiction.

Vice versa, if all roots $\alpha \in \mathcal{R}$ with $|\alpha| = -2$ satisfy (*), then each of these roots can be represented as $\alpha = \beta + \gamma$ for two roots $\beta, \gamma \in \mathcal{R}$ with $|\beta| = |\gamma| = -1$. Then $Y(\beta) \cap \mathcal{B}_{-1} = \{\alpha_i\}$ and $Y(\gamma) \cap \mathcal{B}_{-1} = \{\alpha_j\}$, where α_i, α_j are the edges of a segment in $D_{\mathfrak{s}}$ which does not contain any other root in $\mathcal{B}_{-1}$. By condition (*a*), J acts on an eigenvector X_β of β as the multiplication by η and on an eigenvector X_γ of γ as the multiplication by $-\eta$, with $\eta = \pm\sqrt{-1}$. Then

$$X_\alpha = [X_\beta, X_\gamma] = [JX_\beta, JX_\gamma]$$

and by $\mathbb{C}$-linearity condition (1) is satisfied in this case.

We summarize the discussion above by:

THEOREM 3.1. – *A necessary and sufficient condition in order that a weighted Dynkin diagram* $\Delta_{\mathfrak{s}}$ *with vertices* $\mathcal{B} = \{\alpha_1, \ldots, \alpha_l\}$, *associated to a graduation* $\mathfrak{g}$ *of a simple Lie algebra* $\mathfrak{s}$ *of the complex type correspond to a LT-admissible Satake diagram is that the following three conditions be satisfied:*

1) *the graduation of* $\mathfrak{g}$ *is defined by a partition of* $\mathcal{B}$ *into a subset* $\mathcal{B}_{-1}$ *of roots of degree* (-1) *and a subset* $\mathcal{B}_0$ *of roots of degree* 0 *and* $\mathcal{B}_{-1}$ *contains at least two elements;*

2) *if* $\alpha_i \in \mathcal{B}$ *is a ramification point of* $\Delta_{\mathfrak{s}}$, *then either* $\alpha_i \in \mathcal{B}_{-1}$ *or at most two of the branches issuing from* α_i *contain vertices in* $\mathcal{B}_{-1}$;

3) *there are no roots* $\alpha = \sum_{i=1}^{l} k_i(\alpha)\, \alpha_i \in \mathcal{R}_-$ *with* $|\alpha| = -2$ *and* $k_i(\alpha) = 2$ *for some* $\alpha_i \in \mathcal{B}_{-1}$.

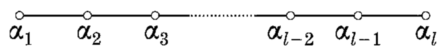
To every LT-admissible weighted Satake diagram of a simple Lie algebra of the complex type correspond exactly two partial complex structures on the corresponding graded Lie algebra $\mathfrak{g}$, *one conjugated to the other.*

Using this theorem, we can give the complete list of LT-admissible Satake diagrams for simple Lie algebras of the complex type. We refer to [Bou81], Ch. VI, for all relevant information on the root systems. Note that the issue reduces to finding the partitions $\{\mathcal{B}_{-1}, \mathcal{B}_0\}$ of $\mathcal{B}$ into a subset $\mathcal{B}_{-1}$ of roots of degree (-1) and a subset $\mathcal{B}_0$ of roots of degree 0 leading to LT-admissible Satake diagrams. We will say in this case that $\mathcal{B}_{-1}$ is *LT-admissible.*

While considering the classification of a simple Levi-Tanaka algebra $\mathfrak{g}$ of the complex type, we must take into account the fact that an automorphism of its weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ induces either an automorphism or an antiautomorphism (i.e. changing J into $(-J)$ of $\mathfrak{g}$.

3.1. *Simple Levi-Tanaka algebras of the complex type* A_l $(l \geqslant 1)$.

The basic roots $\alpha_1, \dots, \alpha_l$ are organized in the Dynkin diagram:



Since the Dynkin diagram has no ramification points and there are no roots in $\mathcal{R}_-$ with coefficients $k_i \geqslant 2$, every $\mathcal{B}_{-1}$ containing at least two elements is LT-admissible. Note that we need $l \geqslant 2$.

Since the highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + \dots + \alpha_l$, the kind μ equals the number ν of roots in $\mathcal{B}_{-1}$.

We note that the isomorphism of the Dynkin diagram $s\colon \mathcal{B} \ni \alpha_i \to \alpha_{l+1-i} \in \mathcal{B}$ yields isomorphisms of the Levi-Tanaka algebras corresponding to $\mathcal{B}_{-1}$ and $s(\mathcal{B}_{-1})$. Therefore, up to equivalence, the weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type A_l are parametrized by

(15) $$\mathcal{B}_{-1} = \{\alpha_1, \dots, \alpha_{i_\nu}\} \quad \text{with } \nu \geqslant 2$$

and, to take into account the isomorphism s, we impose that

(16) $$(i_1, \dots, i_\nu) \leqslant (l+1-i_\nu, \dots, l+1-i_1) \quad \text{for the lexicographic order}.$$

To each $\mathcal{B}_{-1}$ in (15) there correspond two nonisomorphic Levi-Tanaka algebras when $(i_1, \dots, i_\nu) < (l+1-i_\nu, \dots, l+1-i_1)$ or when $(i_1, \dots, i_\nu) = (l+1-i_\nu, \dots, l+1-i_1)$

and ν is odd. We obtain, up to isomorphisms, only one structure of Levi-Tanaka algebra when $(i_1, \dots, i_\nu) = (l+1-i_\nu, \dots, l+1-i_1)$ and ν is even.

Let $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ with $\nu \geq 2$, $1 \leq i_1 < \dots < i_\nu \leq l$ and let us define

$$d_0 = i_1, \qquad d_1 = i_2 - i_1, \dots, d_{\nu-1} = i_\nu - i_{\nu-1}, \qquad d_\nu = l+1-i_\nu .$$

Let $\mathfrak{g} = \bigoplus_{-\nu \leq p \leq \nu} \mathfrak{g}_p$ be a Levi-Tanaka algebra associated to $\mathcal{B}_{-1}$. We know from Lemma 1.1 that $\mathfrak{g}_0$ is reductive. Its center $\mathfrak{z}(\mathfrak{g}_0)$ has dimension ν and the semisimple part is isomorphic to the direct sum:

$$\bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}) .$$

Moreover we obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \left(\sum_{i=0}^{\nu} d_i^2 \right) - 1 , \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm p} = \sum_{i=p}^{\nu} d_i d_{i-p} \qquad \text{for } p = 1, \dots, \nu . \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$, it is convenient to write every matrix X in $\mathfrak{sl}(l+1, \mathbb{C})$ as

$$X = (x_{ij})_{0 \leq i, j \leq \nu} \qquad \text{with} \qquad x_{ij} \in \mathfrak{M}(d_i \times d_j, \mathbb{C}) .$$

Then the characteristic element E of $\mathfrak{g}$ is given by the matrix

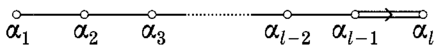
$$\begin{pmatrix} e_0 I_{d_0} & 0 & \cdots & 0 \\ 0 & e_1 I_{d_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & e_\nu I_{d_\nu} \end{pmatrix} \qquad \text{where } e_j = \frac{j d_0 + (j-1) d_1 + \dots + (j-\nu) d_\nu}{l+1}$$

while the partial complex structure is represented by plus or minus the matrix

$$\begin{pmatrix} \eta_1 I_{d_0} & 0 & \cdots & 0 \\ 0 & \eta_{-1} I_{d_1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \eta_{(-1)^\nu} I_{d_\nu} \end{pmatrix} \qquad \text{where} \begin{cases} \eta_1 = \sqrt{-1}\, \dfrac{d_1 + d_3 + \dots}{l+1} , \\ \eta_{-1} = -\sqrt{-1}\, \dfrac{d_0 + d_2 + \dots}{l+1} . \end{cases}$$

3.2. *Simple Levi-Tanaka algebras of the complex type B_l ($l \geqslant 2$).*

The basic roots $\alpha_1, \ldots, \alpha_l$ are organized in the Dynkin diagram:



Also in this case there are no ramification points. The positive roots which have a coefficient larger than or equal to 2 are those of the form:

$$\sum_{i \leqslant k < j} \alpha_i + 2\left(\sum_{j \leqslant k \leqslant l} \alpha_k\right) \quad \text{with} \quad 1 \leqslant i < j \leqslant l .$$

Hence, if $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\}$ with $1 \leqslant i_1 < \ldots < i_\nu \leqslant l$, the necessary and sufficient condition for the existence of a Levi-Tanaka structure on $\mathfrak{g}$ is that

$$i_\nu = i_{\nu-1} + 1 .$$

The highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + 2(\alpha_2 + \ldots + \alpha_l)$ and therefore for a LT-admissible $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}\}$ the kind μ of the corresponding Levi-Tanaka algebras equals $2\nu - 1$ or 2ν according to either α_1 belongs or does not belong to $\mathcal{B}_{-1}$.

There are no automorphisms of the Dynkin diagram of B_l and therefore we distinguish the weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type B_l into two classes, that parametrize up to equivalence these algebras:

(17) $$\mathcal{B}_{-1} = \{\alpha_1, \alpha_{i_2}, \ldots, \alpha_{i_{\nu-1}}, \alpha_{i_{\nu-1}+1}\}$$

$$\text{with } \nu \geqslant 2, \quad 1 < i_2 < \ldots < i_{\nu-1} < l, \quad \text{of kind } \mu = 2\nu - 1$$

and

(18) $$\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_{\nu-1}} \alpha_{i_{\nu-1}+1}\}$$

$$\text{with } \nu \geqslant 2, \quad 1 < i_1 < \ldots < i_{\nu-1} < l, \quad \text{of kind } \mu = 2\nu .$$

Let us set:

(19) $$d_0 = l - i_{\nu-1} - 1, \quad d_1 = 1, \ldots, d_h = i_{\nu-h+1} - i_{\nu-h}, \ldots, d_\nu = i_1 .$$

It is also convenient to set $d_h = 0$ if $h \neq 0, 1, \ldots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ has complex dimension ν. The subalgebra $\mathfrak{g}_0$ is the direct sum of its center and a semisimple part which is isomorphic to the direct sum

$$\begin{cases} \bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}) \oplus \mathfrak{so}(2d_0 + 1, \mathbb{C}) & \text{if } d_0 > 0 , \\ \bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}) & \text{if } d_0 = 0 . \end{cases}$$

Then we obtain

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 1 + \sum_{i=2}^{\nu} d_i^2 + 2d_0^2 + d_0 \,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = \sum_{i=2}^{\nu-2p-1} d_i d_{2p+1+i} + \sum_{i=2}^{p} d_i d_{2p+1-i} + \\ \qquad\qquad + d_{2p} + (2d_0+1)\, d_{2p+1} + d_{2p+2} \,, \qquad \text{for } p \geqslant 0 \,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = \sum_{i=2}^{\nu-2p} d_i d_{2p+i} + \sum_{i=2}^{p-1} d_i d_{2p-i} + \dfrac{d_p(d_p-1)}{2} + \\ \qquad\qquad + d_{2p-1} + (2d_0+1)\, d_{2p} + d_{2p+1}, \qquad \text{for } p > 0 \,. \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$ we introduce the $(2l+1)\times(2l+1)$ symmetric matrix

$$B = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & I_{d_\nu} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & I_{d_{\nu-1}} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & I_{d_1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_{2d_0+1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_{d_1} & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We can identify $\mathfrak{so}(2l+1, \mathbb{C})$ to the space of matrices $X \in \mathfrak{sl}(2l+1, \mathbb{C})$ such that

$${}^t XB + BX = 0 \,.$$

We write these matrices in the form

$$X = \begin{pmatrix} x_{-i,\,-j} & x_{-i,\,0} & x_{-i,\,j} \\ x_{0,\,-j} & x_{0,\,0} & x_{0,\,j} \\ x_{i,\,-j} & x_{i,\,0} & x_{i,\,j} \end{pmatrix}_{1 \leqslant i,\, j \leqslant \nu} .$$

They are characterized by:

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}) & \text{for } |i|, |j| > 0, \\ x_{0,j} \in \mathfrak{M}((2d_0+1) \times d_{|j|}, \mathbb{C}) & \text{for } |j| > 0, \\ x_{i,0} \in \mathfrak{M}((d_{|i|} \times (2d_0+1), \mathbb{C}) & \text{for } |i| > 0, \\ x_{0,0} \in \mathfrak{M}((2d_0+1) \times (2d_0+1), \mathbb{C}), & \\ {}^t x_{i,j} = -x_{-j,-i} & \text{for } |i|, |j| = 0, \dots, \nu. \end{cases}$$

We note that $x_{-1,1} = x_{1,-1} = 0$ because $d_1 = 1$. The characteristic element E of $\mathfrak{g}$ is represented by the diagonal matrix:

$$\begin{bmatrix} -\nu I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (1-\nu) I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & -1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0_{2d_0+1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (\nu-1) I_{d_{\nu-1}} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \nu I_{d_\nu} \end{bmatrix}$$

i.e. $E = (e_{i,j})$ with $e_{i,i} = i I_{d_{|i|}}$ for $i = -\nu, \dots, 0, \dots, \nu$ and $e_{i,j} = 0$ for $i \neq j$; the element J which defines the partial complex structure of $\mathfrak{g}$ is defined by plus or minus the matrix

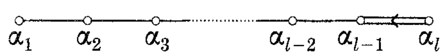
$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \sqrt{-1}\, I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0_{2d_0+1} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}\, I_{d_3} & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

i.e. $J = (\eta_{i,j})$ with

$$\begin{cases} \eta_{2h+1,2h+1} = \sqrt{-1}\, I_{d_{2h+1}} & \text{for } h = -1, \dots, -\left[\dfrac{\nu+1}{2}\right], \\ \eta_{2h+1,2h+1} = -\sqrt{-1}\, I_{d_{2h+1}} & \text{for } h = 0, \dots, \left[\dfrac{\nu-1}{2}\right], \\ \eta_{i,j} = 0 & \text{if } i \neq j \text{ or } i = j \text{ even}. \end{cases}$$

3.3. *Simple Levi-Tanaka algebras of the complex type* C_l $(l \geqslant 3)$.

The basic roots $\alpha_1, \dots, \alpha_l$ are organized in the Dynkin diagram:



Since there are no ramification points, we only need to check that $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ with $1 \leqslant i_1 < \dots < i_\nu \leqslant l$ satisfies condition (3) of Theorem 3.1. The positive roots which have a coefficient larger than or equal to two are:

$$\sum_{i \leqslant k < j} \alpha_k + 2\left(\sum_{j \leqslant k < l} \alpha_k\right) + \alpha_l \quad \text{for } 1 \leqslant i < j < l$$

and

$$2\left(\sum_{i \leqslant k < l} \alpha_k\right) + \alpha_l \quad \text{for } 1 \leqslant i < l\,.$$

Thus we must have

$$\alpha_l \in \mathcal{B}_{-1} \tag{20}$$

and this condition, together with $\nu \geqslant 2$, is also sufficient in order that there exist Levi-Tanaka structures corresponding to $\mathcal{B}_{-1}$.

The highest root in $\mathcal{R}_-$ is $\delta = 2(\alpha_1 + \dots + \alpha_{l-1}) + \alpha_l$ and therefore the Levi-Tanaka algebras corresponding to a LT-admissible $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ have kind $\mu = 2\nu - 1$.

Since there are no automorphisms of the Dynkin diagram of C_l, the weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type C_l are parametrized by

(21) $\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_{\nu-1}}, \alpha_l\}$ with $\nu \geq 2$, $1 \leq i_1 < \ldots < i_{\nu-1} < l$ and $\mu = 2\nu - 1$.

Let us set:

(22) $d_1 = l - i_{\nu-1}, \quad d_2 = i_{\nu-1} - i_{\nu-2}, \ldots, d_h = i_{\nu-h+1} - i_{\nu-h}, \ldots, d_\nu = i_1$.

We set also $d_h = 0$ if $h \neq 1, \ldots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ of the subalgebra $\mathfrak{g}_0$ has complex dimension ν and $\mathfrak{g}_0$ is the direct sum of its center and a semisimple Lie algebra isomorphic to

$$\bigoplus_{d_1 > 1} \mathfrak{sl}(d_i, \mathbb{C}).$$

Then we obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \sum_{i=1}^{s} d_i^2, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = \sum_{h=2p+2}^{\nu} d_h d_{h-2p-1} + \sum_{h=p+2}^{\nu} d_h d_{2p+2-h} + \dfrac{d_{p+1}(d_{p+1}+1)}{2} & \text{for } p \geq 0, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = \sum_{h=2p+1}^{\nu} d_h d_{h-2p} + \sum_{h=p+1}^{\nu} d_h d_{2p+1-h} & \text{for } p > 0. \end{cases}$$

To describe a matrix representation of a Levi-Tanaka algebra of the complex type C_l, we consider the matrix:

$$A = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & I_{d_\nu} \\ 0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & I_{d_{\nu-1}} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & I_{d_2} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_{d_1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & -I_{d_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & -I_{d_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & -I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ -I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

We identify $\mathfrak{sp}(l, \mathbb{C})$ to the space of complex $(2l) \times (2l)$ matrices X such that ${}^tXA + AX = 0$.

Denote by S_ν the set of indexes $\{1/2, 3/2, \dots, (2\nu-1)/2\}$. Then we represent the matrices X in $\mathfrak{sp}(l, \mathbb{C})$ by

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,j} \\ x_{i,-j} & x_{i,j} \end{pmatrix}_{i,j \in S_\nu}$$

where

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_h \times d_k, \mathbb{C}) & \text{for } i = \pm \dfrac{2h-1}{2}, \quad j = \pm \dfrac{2k-1}{2}, \\ {}^t x_{i,j} = -\sigma(i)\,\sigma(j)\, x_{-j,-i} & \text{for } i, j \in S_\nu \cup -S_\nu, \end{cases}$$

and $\sigma(a)$ denotes the sign of the rational number a.

The characteristic element of $\mathfrak{g}$ is represented by the matrix E:

$$\begin{pmatrix}
\frac{1-2\nu}{2} I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{3-2\nu}{2} I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & -\frac{3}{2} I_{d_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & -\frac{1}{2} I_{d_1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \frac{1}{2} I_{d_1} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \frac{3}{2} I_{d_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{2\nu-3}{2} I_{d_{\nu-1}} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{2\nu-1}{2} I_{d_\nu}
\end{pmatrix}$$

i.e. $E = (e_{i,j})_{i,j \in S_\nu \cup -S_\nu}$ with

$$\begin{cases} e_{i,i} = i I_{d_h} & \text{for } i = \pm \dfrac{2h-1}{2}, \\ e_{i,j} = 0 & \text{for } i \neq j. \end{cases}$$

The partial complex structure of $\mathfrak{g}$ is given by plus or minus the matrix J:

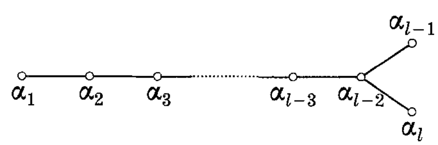
$$\begin{pmatrix}
\frac{(-1)^{\nu}}{2\sqrt{-1}}I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & \frac{(-1)^{\nu-1}}{2\sqrt{-1}}I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & \frac{-\sqrt{-1}}{2}I_{d_2} & 0 & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & \frac{\sqrt{-1}}{2}I_{d_1} & 0 & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & \frac{-\sqrt{-1}}{2}I_{d_1} & 0 & \cdots & 0 & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & \frac{\sqrt{-1}}{2}I_{d_2} & \cdots & 0 & 0 \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & \frac{-(-1)^{\nu-1}}{2\sqrt{-1}}I_{d_{\nu-1}} & 0 \\
0 & 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 & \frac{-(-1)^{\nu}}{2\sqrt{-1}}I_{d_\nu}
\end{pmatrix}$$

i.e. $J = (\eta_{i,j})_{i,j \in S_\nu \cup -S_\nu}$ with

$$\begin{cases} \eta_{i,i} = \sigma(i)\,\dfrac{(-1)^h\sqrt{-1}}{2} I_{d_h} & \text{for } i = \pm\dfrac{2h-1}{2}, \\ \eta_{i,j} = 0 & \text{for } i \neq j. \end{cases}$$

3.4. *Simple Levi-Tanaka algebras of the complex type D_l $(l \geq 4)$.*

The basic roots $\alpha_1, \dots, \alpha_l$ are organized in the Dynkin diagram:



In this case α_{l-2} is a ramification point. If $\alpha_{l-2} \in \mathcal{B}_0$, then, according to condition (2) of Theorem 3.1, only two branches issued from α_{l-2} may contain elements of $\mathcal{B}_{-1}$.

Then we consider the five cases:

$(D_l I)$ $\mathcal{B}_{-1} = \{\alpha_{l-1}, \alpha_l\}$;

$(D_l II)$ $\alpha_{l-2}, \alpha_{l-1}, \alpha_l \in \mathcal{B}_{-1}$;

$(D_l III)$ $\alpha_{l-1} \in \mathcal{B}_0, \ \alpha_l \in \mathcal{B}_{-1}$;

$(D_l III')$ $\alpha_{l-1} \in \mathcal{B}_{-1}, \ \alpha_l \in \mathcal{B}_0$;

$(D_l IV)$ $\alpha_{l-1}, \alpha_l \in \mathcal{B}_0$.

We note that the cases $D_l III$ and $D_l III'$ are interchanged by the automorphism of the Dynkin diagram of D_l which leaves α_i fixed for $i \leq l-2$ and exchanges α_{l-1} with α_l. Therefore, in order to give a classification of the LT-admissible weighted Dynkin diagrams of Levi-Tanaka algebras of the complex type D_l it will suffice to consider the four cases $D_l I$, $D_l II$, $D_l III$ and $D_l IV$. Moreover, for $l = 4$, all permutations of the roots that leave α_2 fixed are automorphisms of the Dynkin diagram of D_4. Therefore we need to consider only the first three cases when $l = 4$.

The positive roots in $\mathcal{R}$ having a coefficient larger than or equal to 2 are given by:

$$\alpha = \sum_{i \leq k < j} \alpha_k + 2\left(\sum_{j \leq k \leq l-2} \alpha_k\right) + \alpha_{l-1} + \alpha_l \quad \text{for } 1 \leq i < j \leq l-2 .$$

Thus condition (3) of Theorem 3.1 is always satisfied by $D_l I$, $D_l II$ and $D_l III$.

In case $D_l IV$, if $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ with $1 \leq i_1 < \dots < i_\nu \leq l-2$, the necessary and sufficient condition in order that $\mathcal{B}_{-1}$ be LT-admissible is that $\nu \geq 2$ and $i_\nu = i_{\nu-1} + 1$.

Since the highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + 2(\alpha_2 + \dots + \alpha_{l-2}) + \alpha_{l-1} + \alpha_l$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_\nu}\}$ is equal to $2\nu - \nu'$ where $0 \leq \nu' \leq 3$ is the number of elements of $\mathcal{B}_{-1} \cap \{\alpha_1, \alpha_{l-1}, \alpha_l\}$.

$D_l I$: $\mathcal{B}_{-1} = \{\alpha_{l-1}, \alpha_l\}$.

In this case $\mathfrak{g}_0$ is the direct sum of its center $\mathfrak{z}(\mathfrak{g}_0)$, which has dimension 2, and of a simple Lie algebra isomorphic to $\mathfrak{sl}(l-1, \mathbb{C})$. The corresponding Levi-Tanaka algebra has kind 2 and we have

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = l^2 - 2l + 2 , \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 2(l-1) , \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = \dfrac{(l-1)(l-2)}{2} . \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$ we introduce the matrix

$$B = \begin{pmatrix} 0 & 0 & I_{l-1} \\ 0 & I_2 & 0 \\ I_{l_1} & 0 & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2l, \mathbb{C})$ to the space of $(2l) \times (2l)$ matrices

$$X = \begin{pmatrix} x_{-1,-1} & x_{-1,0} & x_{-1,1} \\ x_{0,-1} & x_{0,0} & x_{0,1} \\ x_{1,-1} & x_{1,0} & x_{1,1} \end{pmatrix}$$

with

$$\begin{cases} x_{\pm 1, \pm 1} \in \mathfrak{M}((l-1) \times (l-1), \mathbb{C}), \\ x_{-1,0},\ x_{1,0} \in \mathfrak{M}((l-1) \times 2, \mathbb{C}), \\ x_{0,-1},\ x_{0,1} \in \mathfrak{M}(2 \times (l-1), \mathbb{C}), \\ {}^t x_{i,j} = -x_{-j,-i} \qquad \text{for } i, j = -1, 0, 1 . \end{cases}$$

The characteristic element E of $\mathfrak{g}$ is represented by the matrix:

$$\begin{pmatrix} -I_{l-1} & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & I_{l-1} \end{pmatrix}$$

and the partial complex structure J is defined by plus or minus the matrix:

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Note that for the case of D_4, all choice of $\mathcal{B}_{-1}$ equal to $\{\alpha_1, \alpha_3\}$, $\{\alpha_1, \alpha_4\}$, $\{\alpha_3, \alpha_4\}$ give equivalent LT-admissible weighted Dynkin diagrams of the type $D_4 I$, leading to Levi-Tanaka algebras of kind $\mu = 2$ with

$$\dim_{\mathbb{C}} \mathfrak{z}(\mathfrak{g}_0) = 2\,, \qquad \mathfrak{g}_0 \simeq \mathfrak{z}(\mathfrak{g}_0) \oplus \mathfrak{sl}(3, \mathbb{C})$$

and with

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 10\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 6\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 3\,. \end{cases}$$

$D_l II$: $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_{\nu-3}}, \alpha_{l-2}, \alpha_{l-1}, \alpha_l\}$ with $1 \leq i_1 < \dots < i_{\nu-3} < l-2$.

We set $\nu - 1 = s$ and define

$$d_0 = 2\,, \quad d_1 = 1\,, \quad d_2 = l - 2 - i_{\nu-3}, \dots, d_h = i_{\nu-h} - i_{\nu-h-1}, \dots, d_s = i_1\,.$$

It is convenient to set $d_h = 0$ for $h \neq 0, 1, \dots, s$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ of $\mathfrak{g}_0$ has complex dimension ν and $\mathfrak{g}_0$ is isomorphic to the direct sum of $\mathfrak{z}(\mathfrak{g}_0)$ and the semisimple Lie algebra:

$$\bigoplus_{\substack{i>1 \\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C})\,.$$

Therefore we obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \sum_{h=2}^{s} d_h^2 + 2\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = 2 d_{2p+1} + \sum_{h=1}^{s-2p-1} d_h d_{h+2p+1} + \sum_{h=1}^{p} d_h d_{2p+1-h} & \text{for } p \geq 0\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = 2 d_{2p} + \sum_{h=1}^{s-2p} d_h d_{h+2p} + \sum_{h=1}^{p-1} d_h d_{2p-h} + \dfrac{d_p(d_p-1)}{2} & \text{for } p > 0\,. \end{cases}$$

To give a matrix representation of the corresponding Levi-Tanaka algebras $\mathfrak{g}$, we introduce the $(2l) \times (2l)$ matrix

$$B = \begin{pmatrix} 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & I_{d_s} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & I_{d_2} & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & I_2 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & I_{d_2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ I_{d_s} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2l, \mathbb{C})$ with the space of $(2l)\times(2l)$ matrices X such that ${}^tXB + BX = 0$. It is convenient to write these matrices X in the form:

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,0} & x_{-i,j} \\ x_{0,-j} & x_{0,0} & x_{0,j} \\ x_{i,-j} & x_{i,0} & x_{i,j} \end{pmatrix}_{i,j=1,\dots,s}$$

with

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}), \\ {}^t x_{i,j} = -x_{-j,-i} \quad \text{for } i, j = -s, \dots, 0, \dots, s. \end{cases}$$

We note that the condition $d_2 = 1$ implies that $x_{-1,1} = x_{1,-1} = 0$.

Then the characteristic element E of $\mathfrak{g}$ is represented by the matrix:

$$\begin{bmatrix} -sI_{d_s} & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & -2I_{d_2} & 0 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0_2 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 0 & 2I_{d_2} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & 0 & \cdots & sI_{d_s} \end{bmatrix}$$

i.e. by the matrix $(e_{i,j})$ with $e_{i,i} = iI_{d_{|i|}}$ for $i = 0, \pm 1, \dots, \pm s$ and $e_{i,j} = 0$ for $i \neq j$; the partial complex structure is defined by the element J of $\mathfrak{z}(\mathfrak{g}_0)$ associated to plus or minus the matrix:

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \sqrt{-1}\,I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0_{d_2} & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0_2 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0_{d_2} & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}\,I_{d_3} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

i.e. by the matrix $(\eta_{i,j})$ with $\eta_{i,i} = (\sigma(i)/\sqrt{-1}) I_{d_i}$ if i is an odd integer, $\eta_{i,j} = 0$ otherwise.

When $l = 4$ and $\mathcal{B}_{-1} = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$, the algebra $\mathfrak{g}_0$ has complex dimension 4 and it is abelian, the kind of the corresponding Levi-Tanaka algebras is 5 and

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 4\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 4\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 3\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 3\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 4} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 5} = 1\,. \end{cases}$$

We rewrite explicitly the matrices corresponding to the elements E and J in this case. They are respectively:

$$\begin{bmatrix} -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 \end{bmatrix} \text{ and } \begin{bmatrix} \sqrt{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} \end{bmatrix}.$$

All the weighted Dynkin diagrams obtained from the choice of $\mathcal{B}_{-1}$ equal to $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_4\}$, $\{\alpha_2, \alpha_3, \alpha_4\}$ are isomorphic. In this case the subalgebra $\mathfrak{g}_0$ has center $\mathfrak{z}(\mathfrak{g}_0)$ of complex dimension 3 and it is the direct sum of its center and a simple Lie algebra isomorphic to $\mathfrak{sl}(2, \mathbb{C})$; the kind of $\mathfrak{g}$ is 4 and the complex dimensions of $\mathfrak{g}_0$, $\mathfrak{g}_{\pm 1}$, $\mathfrak{g}_{\pm 2}$, $\mathfrak{g}_{\pm 4}$ are respectively 6, 4, 4, 2, 1.

The matrices corresponding to the elements E and J are in this case:

$$\begin{bmatrix} -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

$D_l III$: $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_{\nu-1}}, \alpha_l\}$ with $\nu \geq 2$, $1 \leq i_1 < \dots < i_{\nu-1} < l-1$.

We set

$$d_1 = l - i_{\nu-1}, \quad d_2 = i_{\nu-1} - i_{\nu-2}, \dots, d_h = i_{\nu-h+1} - i_{\nu-h}, \dots, d_\nu = i_1 \tag{23}$$

and $d_h = 0$ for $h \neq 1, \dots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ has dimension ν and $\mathfrak{g}_0$ is the direct sum of its center and of a semisimple Lie algebra isomorphic to

$$\bigoplus_{d_i > 1} \mathfrak{sl}(d_i, \mathbb{C}).$$

We obtain

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \sum_{i=1}^{k} d_i^2, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = \sum_{h=1}^{\nu-2p-1} d_h d_{h+2p+1} + \sum_{h=1}^{p} d_h d_{2p+2-h} + \dfrac{d_{p+1}(d_{p+1}-1)}{2} & \text{for } p \geq 0, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = \sum_{h=1}^{\nu-2p} d_h d_{h+2p} + \sum_{h=1}^{p} d_h d_{2p+1-h} & \text{for } p > 0. \end{cases}$$

To give a matrix representation of $\mathfrak{g}$, we introduce the matrix

$$B = \begin{pmatrix} 0 & \cdots & 0 & 0 & \cdots & I_{d_\nu} \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & I_{d_1} & \cdots & 0 \\ 0 & \cdots & I_{d_1} & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \cdots & \vdots \\ I_{d_\nu} & \cdots & 0 & 0 & \cdots & 0 \end{pmatrix}$$

and identify $\mathfrak{so}(2l, \mathbb{C})$ to the space of $(2l) \times (2l)$ complex matrices X such that ${}^tXB + BX = 0$. The matrices X are better written as block matrices indexed by the set of half odd numbers $S_\nu \cup -S_\nu$ where $S_\nu = \{1/2, \dots, (2h-1)/2, \dots, (2\nu-1)/2\}$. It is convenient to introduce the following notation: for every integer $r \neq 0$ we set

$$\hat{r} = \begin{cases} \dfrac{2r-1}{2} & \text{for } r > 0, \\ \dfrac{2r+1}{2} & \text{for } r < 0. \end{cases}$$

Then we write the matrices X in the form

$$X = \begin{pmatrix} x_{-\hat{i}, -\hat{j}} & x_{-\hat{i}, \hat{j}} \\ x_{\hat{i}, -\hat{j}} & x_{\hat{i}, \hat{j}} \end{pmatrix}_{i, j, \dots, \nu}$$

with

$$\begin{cases} x_{\hat{i},\hat{j}} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}), \\ {}^t x_{\hat{i},\hat{j}} = -x_{-\hat{j},-\hat{i}} \quad \text{for } i, j = \pm 1, \dots, \pm\nu . \end{cases}$$

The characteristic element E of $\mathfrak{g}$ is represented by the matrix

$$\begin{pmatrix} -\hat{\nu} I_{d_\nu} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & \cdots & 0 \\ 0 & \cdots & -(1/2) I_{d_1} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & (1/2) I_{d_1} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ I_{d_\nu} & \cdots & 0 & 0 & \cdots & \hat{\nu} I_{d_\nu} \end{pmatrix}$$

i.e. by the matrix $(e_{\hat{i},\hat{j}})$ with $e_{\hat{i},\hat{i}} = \hat{i} I_{d_{|i|}}$ and $e_{\hat{i},\hat{j}} = 0$ when $i \neq j$.

The partial complex structure of $\mathfrak{g}$ is defined by the element J in $\mathfrak{z}(\mathfrak{g}_0)$ corresponding to plus or minus the matrix

$$\begin{pmatrix} \frac{(-1)^\nu}{2\sqrt{-1}} I_{d_\nu} & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & \frac{\sqrt{-1}}{2} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & -\frac{\sqrt{-1}}{2} & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & \cdots & \frac{(-1)^\nu \sqrt{-1}}{2} I_{d_\nu} \end{pmatrix}$$

i.e. by the matrix $(\eta_{\hat{i},\hat{j}})$ where $\eta_{\hat{i},\hat{i}} = ((\sigma(i)(-1)^i \sqrt{-1})/2) I_{d_{|i|}}$ and $\eta_{\hat{i},\hat{j}} = 0$ for $i \neq j$.

When $l = 4$, and $\mathcal{B}_{-1}$ contains three elements, the discussion reduces to the case $D_4 II$. The construction above yields an equivalent matrix representation for the choice of $\mathcal{B}_{-1}$ equal to $\{\alpha_1, \alpha_2, \alpha_3\}$, $\{\alpha_1, \alpha_2, \alpha_4\}$, $\{\alpha_2, \alpha_3, \alpha_4\}$.

Consider now the equivalent weighted Dynkin diagrams corresponding to the choice of $\mathcal{B}_{-1}$ respectively equal to $\{\alpha_1, \alpha_2\}$, $\{\alpha_2, \alpha_3\}$ and $\{\alpha_2, \alpha_4\}$.

We obtain Levi-Tanaka algebras of kind 3 with $\mathfrak{z}(\mathfrak{g}_0)$ of complex dimension two and $\mathfrak{g}_0$ equal to the direct sum of its center and a semisimple algebra isomorphic to

$\mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$. We have:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 8\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 5\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 4\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 1\,. \end{cases}$$

In this case $\mathfrak{g}$ is isomorphic to the Lie subalgebra of $\mathfrak{gl}(8, \mathbb{C})$ whose elements are the matrices:

$$\begin{pmatrix} x_{-\hat{2},\,-\hat{2}} & x_{-\hat{2},\,-\hat{1}} & x_{-\hat{2},\,\hat{1}} & x_{-\hat{2},\,\hat{2}} \\ x_{-\hat{1},\,-\hat{2}} & x_{-\hat{1},\,-\hat{1}} & x_{-\hat{1},\,\hat{1}} & x_{-\hat{1},\,\hat{2}} \\ x_{\hat{1},\,-\hat{2}} & x_{\hat{1},\,-\hat{1}} & x_{\hat{1},\,\hat{1}} & x_{\hat{1},\,\hat{2}} \\ x_{\hat{2},\,-\hat{2}} & x_{\hat{2},\,-\hat{1}} & x_{\hat{2},\,\hat{1}} & x_{\hat{2},\,\hat{2}} \end{pmatrix}$$

where all entries $x_{\hat{i},\hat{j}}$ are 2×2 complex matrices and ${}^t x_{\hat{i},\hat{j}} = -x_{-\hat{j},\,-\hat{i}}$ for $i, j = -2, -1, 1, 2$. The characteristic element E and the partial complex structure J are associated respectively to the matrices:

$$\begin{pmatrix} -3/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -3/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 3/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3/2 \end{pmatrix}$$

and

$$\pm \begin{pmatrix} -\sqrt{-1}/2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\sqrt{-1}/2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \sqrt{-1}/2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{-1}/2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -\sqrt{-1}/2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}/2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1}/2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \sqrt{-1}/2 \end{pmatrix}.$$

$D_l IV$: $\mathcal{B}_{-1} = \{\alpha_{i_1}, \dots, \alpha_{i_{\nu-1}}, \alpha_{i_{\nu-1}+1}\}$ with $\nu \geq 2$, $1 \leq i_1 < \dots < i_{\nu-1} < l-2$.

We set

$$d_0 = 2(l-1-i_{\nu-1}), \quad d_1 = 1, \quad d_2 = i_{\nu-1} - i_{\nu-2}, \dots, d_{\nu-1} = i_2 - i_1, \quad d_\nu = i_1 .$$

We also set $d_h = 0$ for $h \neq 1, \dots, \nu$.

The center $\mathfrak{z}(\mathfrak{g}_0)$ has complex dimension ν and the semisimple Lie algebra $\mathfrak{g}_0/\mathfrak{z}(\mathfrak{g}_0)$ is isomorphic to

$$\bigoplus_{\substack{i>1\\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C}) \oplus \mathfrak{so}(d_0, \mathbb{C}) .$$

We obtain:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = \dfrac{d_0(d_0-1)}{2} + \sum\limits_{i=2}^{\nu} d_i^2 + 1 , \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm(2p+1)} = d_{2p} + d_0 d_{2p+1} + d_{2p+2} + \sum\limits_{h=2}^{\nu-2p-1} d_h d_{h+2p+1} + \sum\limits_{h=2}^{p} d_h d_{2p+1-h} \\ \qquad\qquad \text{for } p \geq 0 , \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2p} = d_{2p-1} + d_0 d_{2p} + d_{2p+1} + \sum\limits_{h=2}^{\nu-2p} d_h d_{h+2p} + \sum\limits_{h=2}^{p-1} d_h d_{2p-h} + \dfrac{d_p(d_p-1)}{2} \\ \qquad\qquad \text{for } p > 0 . \end{cases}$$

To obtain a matrix representation of the Levi-Tanaka algebra $\mathfrak{g}$, we introduce the symmetric $(2l) \times (2l)$ matrix:

$$B = \begin{bmatrix} 0 & \cdots & 0 & 0 & 0 & \cdots & I_{d_\nu} \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ 0 & \cdots & 0 & I_{d_0} & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 0 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots \\ I_{d_\nu} & \cdots & 0 & 0 & 0 & \cdots & 0 \end{bmatrix}$$

and identify $\mathfrak{so}(2l, \mathbb{C})$ to the Lie algebra of the complex $(2l) \times (2l)$ matrices X such that ${}^t XB + BX = 0$. We write the matrices X in the form:

$$X = \begin{pmatrix} x_{-i,-j} & x_{-i,0} & x_{-i,j} \\ x_{0,-j} & x_{0,0} & x_{0,j} \\ x_{i,-j} & x_{i,0} & x_{i,j} \end{pmatrix}_{i,j=1,\dots,\nu}$$

with

$$\begin{cases} x_{i,\,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}), \\ {}^t x_{i,\,j} = -x_{-j,\,-i} \quad \text{for } i,\, j = 0,\, \pm 1, \dots, \pm\nu\,. \end{cases}$$

Note that $x_{-1,\,1} = x_{1,\,-1} = 0$ because $d_1 = 1$. Then the characteristic element of $\mathfrak{g}$ corresponds to the matrix:

$$\begin{bmatrix} -\nu I_{d_\nu} & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & -1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0_{d_0} & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & \nu I_{d_\nu} \end{bmatrix}$$

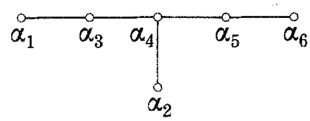
i.e. to the matrix $(e_{i,\,j})$ with $e_{i,\,i} = i I_{d_{|i|}}$ and $e_{i,\,j} = 0$ for $i \neq j$. The element $J \in \mathfrak{z}(\mathfrak{g}_0)$ that defines the partial complex structure of $\mathfrak{g}$ is associated to plus or minus the matrix

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \vdots & \sqrt{-1}\, I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & \sqrt{-1} & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0_{d_0} & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & -\sqrt{-1} & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \vdots \\ \vdots & 0 & 0 & 0 & 0 & 0 & 0 & -\sqrt{-1}\, I_{d_3} & \vdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

i.e. to the matrix $(\eta_{i,\,j})$ with $\eta_{i,\,j} = (\sigma(i)/\sqrt{-1})\, I_{d_{|i|}}$ if $i = j$ is an odd integer, and $\eta_{i,\,j} = 0$ otherwise.

3.5. *Simple Levi-Tanaka algebras of the complex type* E_6.

The basic roots $\alpha_1, \dots, \alpha_6$ of the complex Lie algebra E_6 are organized in the Dynkin diagram:



The root α_4 is a ramification point for the Dynkin diagram and therefore, according to condition (2) of Theorem 3.1, we must restrict to the following cases:

(i) $|\alpha_4| = -1$;

(ii) $\alpha_2 \in \mathscr{B}_{-1} \subset \{\alpha_1, \alpha_3, \alpha_2\}$;

(iii) $\alpha_2 \in \mathscr{B}_{-1} \subset \{\alpha_2, \alpha_5, \alpha_6\}$;

(iv) $\mathscr{B}_{-1} \subset \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}$.

The check condiction (3) of Theorem 3.1, we only need to consider positive roots having at least one coefficient equal to 2 and at least one coefficient equal to 0. These roots are:

$$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$$

$$\alpha_1 + \alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6$$

$$\alpha_1 + \alpha_2 + 2\alpha_3 + 2\alpha_4 + \alpha_5$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 .$$

In case (i) the LT-admissible $\mathscr{B}_{-1}$ are therefore given by:

all $\mathscr{B}_{-1}$ containg α_4 and at least one of the roots $\alpha_2, \alpha_3, \alpha_5$.

In case (ii) and (iii) all $\mathscr{B}_{-1}$, containing at least two elements are LT-admissible.

Finally, in case (iv), the necessary and sufficient condition for $\mathscr{B}_{-1}$ to be LT-admissible is that it equals one of the following:

$$\begin{array}{ll} \{\alpha_1, \alpha_3\}, & \{\alpha_1, \alpha_3, \alpha_5\}, \\ \{\alpha_1, \alpha_6\}, & \{\alpha_1, \alpha_3, \alpha_6\}, \\ \{\alpha_3, \alpha_5\}, & \{\alpha_1, \alpha_5, \alpha_6\}, \\ \{\alpha_5, \alpha_6\}, & \{\alpha_3, \alpha_5, \alpha_6\}, \\ \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}. & \end{array} \tag{24}$$

Since the highest root in $\mathcal{R}_-$ is $\delta = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1}$ is obtained by the formula:

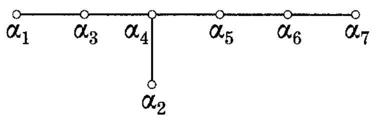
$$-\mu = |\alpha_1| + 2|\alpha_2| + 2|\alpha_3| + 3|\alpha_4| + 2|\alpha_5| + |\alpha_6|$$

and therefore we have $2 \leq \mu \leq 11$.

The Dynkin diagram of E_6 has the only non trivial automorphism that leaves α_2 and α_4 fixed and exchanges α_1 with α_6 and α_3 with α_5. There are therefore up to isomorphisms 26 unisomorphic weighted Dynkin diagrams associated to simple Levi-Tanaka algebras of the complex type E_6, corresponding to 49 nonisomorphic Levi-Tanaka algebras. We will give the complete list in the appendix, also indicating their kind, the complex dimension of each subspace $\mathfrak{g}_p$ and the structure of the reductive subalgebra $\mathfrak{g}_0$.

3.6. *Simple Levi-Tanaka algebras of the complex type* E_7.

The basic roots $\alpha_1, \ldots, \alpha_7$ of the complex Lie algebra E_7 are organized in the Dynkin diagram:



The root α_4 is a ramification point for the Dynkin diagram. The roots α in $\mathcal{R}_-$ for which we have $k_i(\alpha) = 2$ and $k_j(\alpha) = 0$ for some $1 \leq i, j \leq 7$ are all roots from E_6 which have a coefficient $k_i(\alpha)$ equal to two (we have in this case $k_7(\alpha) = 0$) and the roots

$$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7,$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7,$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7.$$

In particular the LT-admissible $\mathcal{B}_{-1}$ which do not contain α_7 are obtained from the $\mathcal{B}_{-1}$ which are LT-admissible for E_6. When $\alpha_7 \in \mathcal{B}_{-1}$, we observe that:

1) the only admissible $\mathcal{B}_{-1}$ of the form $\{\alpha_i, \alpha_7\}$ are $\{\alpha_1, \alpha_7\}$ and $\{\alpha_6, \alpha_7\}$, because the root system of E_6 contains roots α with $k_i(\alpha) = 2$ for every $i \neq 1, 6$;

2) if $\mathcal{B}_{-1}$ contains α_7 and at least other two roots α_i with $1 \leq i \leq 6$, then the necessary and sufficient condition in order that $\mathcal{B}_{-1}$ be LT-admissible is that $\mathcal{B}_{-1} \setminus \{\alpha_7\}$ be LT-admissible for E_6 and, when $\alpha_4 \in \mathcal{B}_0$, we need also that $\mathcal{B}_{-1}$ be contained in the union of only two of the three branches issued from α_4.

Since the highest root in $\mathcal{R}_-$ is $\delta = 2\alpha_1 + 2\alpha_2 + 3\alpha_3 + 4\alpha_4 + 3\alpha_5 + 2\alpha_6 + \alpha_7$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1}$ is obtained by the formula:

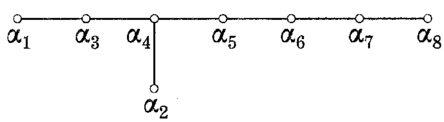
$$-\mu = 2|\alpha_1| + 2|\alpha_2| + 3|\alpha_3| + 4|\alpha_4| + 3|\alpha_5| + 2|\alpha_6| + |\alpha_7|$$

and therefore we have $3 \leq \mu \leq 17$.

Taking into account that there are no nontrivial automorphisms of the Dynkin diagram of E_7, we conclude that there are 84 nonequivalent weighted Dynkin diagrams, each one corresponding to two nonisomorphic simple Levi-Tanaka algebras $\mathfrak{g}$ of the complex type E_7. We list them in the appendix, also indicating their kind, the complex dimension of the subspaces $\mathfrak{g}_p$ and the structure of the reductive subalgebra $\mathfrak{g}_0$.

3.7. *Simple Levi-Tanaka algebras of the complex type* E_8.

The basic roots $\alpha_1, \ldots, \alpha_8$ of the complex Lie algebra E_8 are organized in the Dynkin diagram:



Again the root α_4 is of ramification. The roots $\alpha \in \mathcal{R}_-$ for which we have $k_i(\alpha) = 2$ and $k_j(\alpha) = 0$ for some $1 \leq i, j \leq 8$ are the roots of E_7 (where we take $k_8(\alpha) = 0$) for which $k_i(\alpha) = 2$ for some $1 \leq i \leq 7$ and the roots

$$\alpha_2 + \alpha_3 + 2\alpha_4 + \alpha_5 + \alpha_6 + \alpha_7 + \alpha_8,$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + \alpha_6 + \alpha_7 + \alpha_8,$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + \alpha_7 + \alpha_8,$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 + 2\alpha_5 + 2\alpha_6 + 2\alpha_7 + \alpha_8.$$

We note that for all these four roots $k_8(\alpha) = 1$. We conclude that when $\mathcal{B}_{-1}$ does not contain α_8, a necessary and sufficient condition in order that it be LT-admissible is that it was admissible for E_7. When $\alpha_8 \in \mathcal{B}_{-1}$, we note that:

1) the only LT-admissible $\mathcal{B}_{-1}$ of the form $\mathcal{B}_{-1} = \{\alpha_i, \alpha_8\}$ with $1 \leq i \leq 7$ is $\{\alpha_7, \alpha_8\}$ because for every $1 \leq i \leq 6$ the root system of E_7 contains some root α with $k_i(\alpha) = 2$;

2) if $\alpha_8 \in \mathcal{B}_{-1}$ and $\mathcal{B}_{-1}$ contains at least three elements, then it is LT-admissible if and only if $\mathcal{B}_{-1} \setminus \{\alpha_8\}$ is LT-admissible for E_7 and is all contained in two of the three branches issued from α_4 in case $\alpha_4 \in \mathcal{B}_0$.

Since the highest root in $\mathcal{R}_-$ is $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 6\alpha_4 + 5\alpha_5 + 4\alpha_6 + 3\alpha_7 + 2\alpha_8$, the kind μ of a Levi-Tanaka algebra associated to a LT-admissible $\mathcal{B}_{-1}$ is obtained by the formula:

$$-\mu = 2|\alpha_1| + 3|\alpha_2| + 4|\alpha_3| + 6|\alpha_4| + 5|\alpha_5| + 4|\alpha_6| + 3|\alpha_7| + 2|\alpha_8|$$

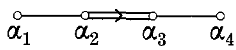
and therefore we have $5 \leq \mu \leq 29$ (with the one exception of 28).

Since there are no nontrivial automorphisms of the Dynkin diagram of E_8, we conclude that there are exactly 165 nonisomorphic weighted Dynkin diagrams corresponding to simple Levi-Tanaka algebras of the complex type E_8 (each corresponding to two nonisomorphic conjugated Levi-Tanaka algebras).

We list in the appendix all LT-admissible choices of $\mathcal{B}_{-1}$, giving for each one the kind of the corresponding Levi-Tanaka algebra $\mathfrak{g}$, the complex dimension of the $\mathfrak{g}_p$'s and describing the structure of the reductive subalgebra $\mathfrak{g}_0$.

3.8. *Simple Levi-Tanaka algebras of the complex type F_4.*

The basic roots $\alpha_1, \dots, \alpha_4$ of the complex Lie algebra F_4 are organized in the Dynkin diagram:



There are no ramification points, so that we have only to care for the positive roots having at least one coefficient equal to 2 and at least one coefficient equal to 0. The list of these roots is:

$$\alpha_2 + 2\alpha_3,$$
$$\alpha_1 + \alpha_2 + 2\alpha_3,$$
$$\alpha_2 + 2\alpha_3 + \alpha_4,$$
$$\alpha_1 + 2\alpha_2 + 2\alpha_3,$$
$$\alpha_2 + 2\alpha_3 + 2\alpha_4.$$

Again we can compute the kind μ of the Levi-Tanaka algebras associated to F_4, using the highest root $\delta = 2\alpha_1 + 3\alpha_2 + 4\alpha_3 + 2\alpha_4 \in \mathcal{R}_-$, by the formula

$$-\mu = 2|\alpha_1| + 3|\alpha_2| + 4|\alpha_3| + 2|\alpha_4|.$$

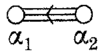
The possible values of μ are 5, 7, 9, 11. There exists a Levi-Tanaka structure on $\mathfrak{g}$ if and only if $\mathcal{B}_{-1}$ is equal to one of the following sets:

$$\{\alpha_1, \alpha_2\}, \quad \mu = 5; \qquad \{\alpha_1, \alpha_2, \alpha_4\}, \quad \mu = 7;$$
$$\{\alpha_2, \alpha_3\}, \quad \mu = 7; \qquad \{\alpha_2, \alpha_3, \alpha_4\}, \quad \mu = 9;$$
$$\{\alpha_1, \alpha_2, \alpha_3\}, \quad \mu = 9; \qquad \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}, \quad \mu = 11.$$

In the appendix we will list for each LT-admissible $\mathcal{B}_{-1}$ the kind of the corresponding Levi-Tanaka algebra $\mathfrak{g}$, together with the complex dimensions of the subspaces $\mathfrak{g}_p$ and the structure of the reductive subalgebra $\mathfrak{g}_0$.

3.9. *Simple Levi-Tanaka algebras of the complex type G_2.*

The basic roots α_1, α_2 of the root system G_2 are organized in the Dynkin diagram:



There are no ramification points and the only possible choice $\mathcal{B}_{-1} = \{\alpha_1, \alpha_2\} = \mathcal{B}$ satisfies condition (3) of Theorem 3.1, so that there is a Levi-Tanaka structure of the corresponding graded Lie algebra $\mathfrak{g}$. Since there are no nontrivial automorphisms of the Dynkin diagram, there are exactly two nonisomorphic Levi-Tanaka algebras of the complex type G_2. They have kind 5 and we have:

$$\begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 2\,, & \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{C})\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 2\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 4} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 5} = 1\,. \end{cases}$$

4. - The weighted Satake diagrams of simple Levi-Tanaka algebras of the real type.

We turn now to investigate the possibility of defining a Levi-Tanaka structure on a real simple Lie algebra of the real type. If $\mathfrak{g}$ is a Levi-Tanaka algebra, then the complexification $\mathfrak{g}^{\mathbb{C}}$ of $\mathfrak{g}$ is also a Levi-Tanaka algebra for the complexification of the partial complex structure. Therefore, if $\mathfrak{g}$ is a simple graded Lie algebra of the real type, admitting a structure of Levi-Tanaka algebra, it follows from Theorem 3.1 that there are only two possible partial complex structures on $\mathfrak{g}$, one being the opposite of the other. Moreover, the fact that $\mathfrak{g}^{\mathbb{C}}$ admits the structure of a Levi-Tanaka algebra is a necessary condition in order that $\mathfrak{g}$ could be made into a Levi-Tanaka algebra.

Let $\mathfrak{g}$ be a simple graded Lie algebra of the real type. By Theorem 2.9 in order that its weighted Satake diagram $\Sigma_{\mathfrak{g}}$ be LT-admissible, it must contain curved arrows and the graduation of $\mathfrak{g}$ will be determined by a partition $\{\mathcal{B}_0, \mathcal{B}_{-1}\}$ of its vertices, where the set $\mathcal{B}_{-1}$ of the roots having degree (-1) is a nonempty union of pairs of distinct white roots joined by a curved arrow.

If $\mathfrak{g}$ is a simple Levi-Tanaka algebra of the real type, the underlying weighted Dynkin diagram $\Delta_{\mathfrak{g}}$ corresponds to one which is associated to a Levi-Tanaka algebra of the complex type. Therefore we can use the results obtained in the case of simple graded Lie algebras of the complex type to obtain the classification of those of the real type.

Indeed we can use the following criterion:

THEOREM 4.1. – *Let $\mathfrak{g}$ be a simple graded Lie algebra of the real type. Then a necessary and sufficient condition in order that $\mathfrak{g}$ admits the structure of a Levi-Tanaka algebra is that for its Satake diagram $\Sigma_{\mathfrak{g}}$ the following conditions* (i) *and* (ii) *hold true*:

(i) $\mathcal{B} = \mathcal{B}_0 \cup \mathcal{B}_{-1}$ *and the set $\mathcal{B}_{-1}$ of vertices of weight (-1) is nonempty and consists of a disjoint union of pairs of white roots joined by a curved arrow*;

(ii) $\mathfrak{g}^{\mathbf{C}}$ *admits a structure of Levi-Tanaka algebra.*

PROOF. – A direct inspection of the Satake's diagrams of simple Lie algebras of the real type shows that, when (i) holds, we also have the following:

(iii) if $\alpha, \alpha' \in \mathcal{B}_{-1}$ are joined by a curved arrow, then the line joining α to α' in the Dynkin diagram $\Delta_{\mathfrak{g}}$ contains an even number of vertices in $\mathcal{B}_{-1}$.

Assume that $J \in \mathfrak{z}(\mathfrak{g}_0^{\mathbf{C}})$ defines a partial complex structure on $\mathfrak{g}^{\mathbf{C}}$ for which $\mathfrak{g}^{\mathbf{C}}$ is a simple Levi-Tanaka algebra of the complex type. To prove the theorem, it suffices to show that condition (iii) implies that $[J, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$.

Let $\alpha \in \mathcal{B}_{-1}$. Write an eigenvector of α in the form $X + \sqrt{-1}Y$ with $X, Y \in \mathfrak{g}_{-1}$. We have:

$$[J, X + \sqrt{-1}\,Y] = \eta X + \eta\sqrt{-1}\,Y \quad \text{with } \eta = \pm\sqrt{-1}\,.$$

If σ is the involution of $\mathfrak{g}^{\mathbf{C}}$ induced by the real form $\mathfrak{g}$, we obtain

$$\alpha^{\sigma} = \alpha' + \sum_{i=1}^{r} \beta_i\,, \qquad \text{with } \beta_1, \dots, \beta_r \in \mathcal{R}_{\bullet}$$

for the root α' joined to α by a curved arrow. It follows that $\operatorname{ad}(J)$ acts on $\mathfrak{g}^{\alpha'}$ and $\mathfrak{g}^{\alpha^{\sigma}}$ as the multiplication by the same factor $\pm\eta$. Since $X - \sqrt{-1}Y$ belongs to $\mathfrak{g}^{\alpha^{\sigma}}$, we obtain

$$[J, X] = \eta\sqrt{-1}\,Y\,, \quad [J, Y] = -\eta\sqrt{-1}\,X$$

because the line joining α to α' in the Dynkin diagram contains an even number of roots of $\mathcal{B}_{-1}$.

Because the real and imaginary parts of vectors in $\mathfrak{g}^{\alpha}$ for $\alpha \in \mathcal{B}_{-1}$, together with their images by $\operatorname{ad}(\mathfrak{g}_0)$, generate $\mathfrak{g}_{-1}$, we obtain that $[J, \mathfrak{g}_{-1}] \subset \mathfrak{g}_{-1}$. ∎

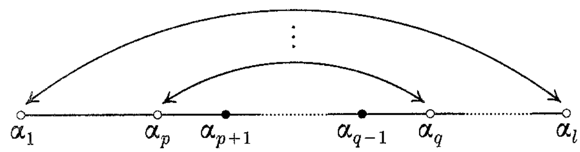
Hence the simple Levi-Tanaka algebras of the real type are, modulo isomorphisms, in a one to one correspondence with the Levi-Tanaka algebras of the complex type for which $\mathcal{B}_{-1}$ satisfies condition (i) above.

We can now proceed to classify, up to isomorphisms, all simple Levi-Tanaka algebras of the real type. We already know that their complexifications should be of the type A_l, D_l or E_6 (cf. [MN]).

4.1. *Simple Levi-Tanaka algebras of the real type A_l.*

There are only two types of Satake diagrams associated to A_l that contain curved arrows. They correspond respectively to the real Lie algebras $\mathfrak{su}(p, q)$ with $p < q$ and $p + q = l + 1$ and to the real Lie algebra $\mathfrak{su}(p, p)$ with $p \geq 2$ and $l = 2p - 1$. Accordingly, we divide the discussion of the case of Levi-Tanaka algebras of the real type A_l into two subcases.

Subtype $\mathfrak{su}(p, q)$, $1 \leq p < q$, $p + q = l + 1$. The Satake diagram is:



According to Theorem 4.1, the weighted Satake's diagrams associated to Levi-Tanaka algebras isomorphic to $\mathfrak{su}(p, q)$ are those corresponding to the choice of

$$\mathcal{B}_{-1} = \{\alpha_{i_1}, \ldots, \alpha_{i_\nu}, \alpha_{l-i_\nu+1}, \ldots, \alpha_{l-i_1+1}\} \quad \text{with } 1 \leq i_1 < \ldots < i_\nu \leq p .$$

The kind of these Levi-Tanaka algebras is 2ν, according to the discussion for the complex type A_l.

Let us set

$$d_0 = l + 1 - 2i_\nu , \quad d_1 = i_\nu - i_{\nu-1}, \ldots, d_h = i_{\nu-h+1} - i_{\nu-h}, \ldots, d_\nu = i_1 . \tag{25}$$

It is convenient to set also $d_h = 0$ for $h \neq 0, 1, \ldots, \nu$. We obtain then

$$\mathfrak{g}_0 \simeq \mathfrak{d}_{2\nu}(\mathbb{R}) \oplus \mathfrak{su}(p - i_\nu, l + 1 - p - i_\nu) \oplus \bigoplus_{\substack{i>0\\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C}) \tag{26}$$

and

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = d_0^2 + 2 \sum\limits_{i=1}^{\nu} d_i^2 - 1 \,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm(2r+1)} = 2 \sum\limits_{i=0}^{\nu-2r-1} d_i d_{2r+1+i} + 2 \sum\limits_{i=1}^{r} d_i d_{2r+1-i} & \text{for } r \geqslant 0 \,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2r} = 2 \sum\limits_{i=0}^{\nu-2r} d_i d_{2r+i} + 2 \sum\limits_{i=1}^{r-1} d_i d_{2r-i} + d_r^2 & \text{for } r > 0 \,. \end{cases}$$

To obtain a matrix representation of the Levi-Tanaka algebra $\mathfrak{g}$ associated to this choice of $\mathcal{B}_{-1}$, we first introduce the $d_0 \times d_0$ matrix

$$\tilde{B} = \begin{pmatrix} -I_{p-i_\nu} & 0 \\ 0 & I_{l+1-p-i_\nu} \end{pmatrix}$$

and consider then the $(l+1) \times (l+1)$ matrix

$$B = \begin{bmatrix} 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & I_{d_\nu} \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & I_{d_{\nu-1}} & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & I_{d_1} & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & \tilde{B} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & I_{d_1} & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \end{bmatrix} .$$

We identify $\mathfrak{g}$ to the Lie subalgebra of $\mathfrak{sl}(l+1, \mathbb{C})$ of the matrices X satisfying $X^* B + BX = 0$. It is convenient to write X as a block matrix:

$$X = \begin{pmatrix} x_{-i,\,-j} & x_{-i,\,0} & x_{-i,\,j} \\ x_{0,\,-j} & x_{0,\,0} & x_{0,\,j} \\ x_{i,\,-j} & x_{i,\,0} & x_{i,\,j} \end{pmatrix}_{i,\,j=1,\,\dots,\,\nu}$$

with

$$\begin{cases} x_{i,j} \in \mathfrak{M}(d_{|i|} \times d_{|j|}, \mathbb{C}) & \text{for } i, j = 0, \pm 1, \ldots, \pm \nu, \\ x_{i,j}^* = -x_{-j,-i} & \text{for } i, j = \pm 1, \ldots, \pm \nu, \\ x_{0,j}^* \tilde{B} + x_{-j,0} = 0 & \text{for } j = \pm 1, \ldots, \pm \nu, \\ x_{i,0}^* + \tilde{B} x_{0,-i} = 0 & \text{for } i = \pm 1, \ldots, \pm \nu, \\ x_{0,0}^* \tilde{B} + \tilde{B} x_{0,0} = 0. \end{cases}$$

The characteristic element E of $\mathfrak{g}$ is then associated to the matrix:

$$\begin{bmatrix} -\nu I_{d_\nu} & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & (1-\nu) I_{d_{\nu-1}} & \cdots & 0 & 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & -I_{d_1} & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0_{d_0} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 & I_{d_1} & \cdots & 0 & 0 \\ \vdots & \vdots & \cdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & (\nu-1) I_{d_{\nu-1}} & 0 \\ 0 & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 & \nu I_{d_\nu} \end{bmatrix}$$

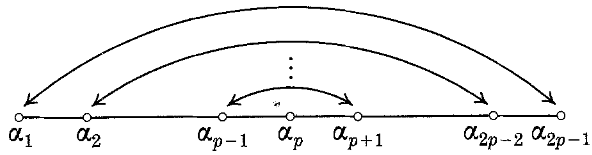
and the partial complex structure is defined by plus or minus the matrix:

$$\begin{bmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \cdots \\ \cdots & \eta_1 I_{d_3} & 0 & 0 & 0 & 0 & 0 & 0 & \cdots \\ \ddots & 0 & \eta_0 I_{d_2} & 0 & 0 & 0 & 0 & 0 & \ddots \\ \cdots & 0 & 0 & \eta_1 I_{d_1} & 0 & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & \eta_0 I_{d_0} & 0 & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & \eta_1 I_{d_1} & 0 & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & \eta_0 I_{d_2} & 0 & \cdots \\ \cdots & 0 & 0 & 0 & 0 & 0 & 0 & \eta_1 I_{d_3} & \cdots \\ \cdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

where

$$\begin{cases} \eta_0 = \sqrt{-1}\, \dfrac{2(d_1 + d_3 + \ldots)}{l+1}, \\ \eta_1 = -\sqrt{-1}\, \dfrac{d_0 + 2(d_2 + d_4 + \ldots)}{l+1}. \end{cases}$$

Subtype $\mathfrak{su}(p,p)$, $2p = l+1$, $p \geqslant 2$. The Satake diagram is:



Therefore, according to Theorem 4.1, the choices of $\mathcal{B}_{-1}$ corresponding to Levi-Tanaka algebras are:

$$\mathcal{B}_{-1} = \{\alpha_1, \dots, \alpha_{i_\nu}, \alpha_{2p-i_\nu}, \dots, \alpha_{2p-i_1}\} \quad \text{with } \nu \geqslant 1 \text{ and } 1 \leqslant i_1 < \dots < i_\nu \leqslant p-1 .$$

The corresponding Levi-Tanaka algebra $\mathfrak{g}$ has kind 2ν.

We set

$$(27) \qquad d_0 = 2(p - i_\nu), \quad d_1 = i_\nu - i_{\nu-1}, \dots, d_h = i_{\nu-h+1} - i_{\nu-h}, \dots, d_\nu = i_1$$

and also $d_h = 0$ for $h \neq 0, 1, \dots, \nu$.

We obtain

$$\mathfrak{g}_0 \simeq \mathfrak{d}_{2\nu}(\mathbb{R}) \oplus \mathfrak{su}(p-i_\nu, p-i_\nu) \oplus \bigoplus_{\substack{i>0\\ d_i>1}} \mathfrak{sl}(d_i, \mathbb{C})$$

and

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = d_0^2 + \sum_{i=1}^{\nu} d_i^2 - 1, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm(2r+1)} = 2 \sum_{i=0}^{\nu-2r-1} d_i d_{2r+1+i} + 2 \sum_{i=1}^{r} d_i d_{2r+1-i} & \text{for } r \geqslant 0, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2r} = 2 \sum_{i=0}^{\nu-2r} d_i d_{2r+i} + 2 \sum_{i=1}^{r-1} d_i d_{2r-i} + d_r^2 & \text{for } r > 0 . \end{cases}$$

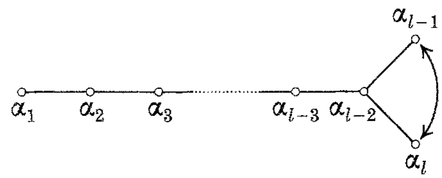
We obtain a matrix representation analogous to that of the case $\mathfrak{su}(p,q)$ taking instead of the matrix $\widetilde{B}$ used the new matrix

$$\widetilde{B} = \begin{pmatrix} 0 & I_{p-i_\nu} \\ I_{p-i_\nu} & 0 \end{pmatrix}.$$

4.2. *Simple Levi-Tanaka algebras of the real type D_l.*

There are only two types of Satake diagrams associated to D_l that contain curved arrows, which correspond to the real Lie algebras $\mathfrak{so}(l-1, l+1)$ and $\mathfrak{so}^*(2l)$ with $l = 2p+1$, respectively. We discuss the two cases separately.

Subtype $\mathfrak{so}(l-1, l+1)$, $l \geq 4$. The Satake diagram is:



Therefore the only possible choice is:

$$\mathcal{B}_{-1} = \{\alpha_{l-1}, \alpha_l\}$$

and it is LT-admissible. A Levi-Tanaka algebra $\mathfrak{g}$ associated to the corresponding weighted Satake diagram has kind two and we obtain:

$$\mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{sl}(l-1, \mathbb{R})$$

and

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = l^2 - 2l + 2 , \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 2(l-1) , \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = \dfrac{(l-1)(l-2)}{2} . \end{cases}$$

To obtain a matrix representation of $\mathfrak{g}$ we introduce the matrix

$$B = \begin{pmatrix} 0 & 0 & I_{l-1} \\ 0 & I_2 & 0 \\ I_{l-1} & 0 & 0 \end{pmatrix}$$

and identify $\mathfrak{g}$ to the subalgebra of $\mathfrak{gl}(l+1, \mathbb{R})$ of matrices X such that ${}^tXB + BX = 0$. We write these matrices in the form

$$X = \begin{pmatrix} x_{-1,-1} & x_{-1,0} & x_{-1,1} \\ x_{0,-1} & x_{0,0} & x_{0,1} \\ x_{1,-1} & x_{1,0} & x_{1,1} \end{pmatrix}$$

with

$$\begin{cases} x_{\pm 1, \pm 1} \in \mathfrak{M}((l-1) \times (l-1), \mathbb{R}) , \\ x_{\pm 1, 0} \in \mathfrak{M}((l-1) \times 2, \mathbb{R}) , \\ x_{0, \pm 1} \in \mathfrak{M}(2 \times (l-1), \mathbb{R}) , \\ x_{0,0} \in \mathfrak{M}(2 \times 2, \mathbb{R}) , \\ {}^t x_{i,j} = -x_{-j,-i} \quad \text{for } i, j = 0, \pm 1 . \end{cases}$$

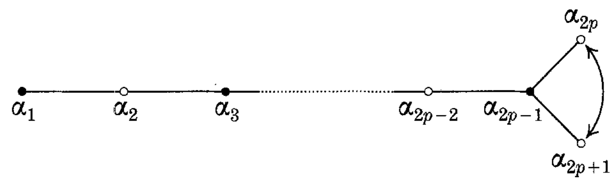
The characteristic element E is associated to the matrix

$$\begin{pmatrix} -I_{l-1} & 0 & 0 \\ 0 & 0_2 & 0 \\ 0 & 0 & I_{l-1} \end{pmatrix}$$

and the partial complex structure is defined by plus or minus the matrix

$$\begin{pmatrix} 0_{l-1} & 0 & 0 \\ 0 & \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} & 0 \\ 0 & 0 & 0_{l-1} \end{pmatrix}.$$

Subtype $\mathfrak{so}^*(2l)$, $l = 2p+1$, $p \geqslant 2$. The Satake diagram is:



Therefore the only possible choice of $\mathcal{B}_{-1}$ is:

$$\mathcal{B}_{-1} = \{\alpha_{2p}, \alpha_{2p+1}\}$$

and it is LT-admissible. If $\mathfrak{g}$ is a corresponding Levi-Tanaka algebra, then it has kind 2 and

$$\mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{su}^*(2l-4).$$

Moreover we obtain:

$$\begin{cases} \dim_{\mathbb{R}} \mathfrak{g}_0 = 4p^2 + 1 , \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 4p , \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = p(2p-1) . \end{cases}$$

Let us describe a matrix representation of $\mathfrak{g}$. For every positive integer h we denote by $\check{I}_{2h}$ the $(2h) \times (2h)$ matrix

$$\begin{pmatrix} 0 & -I_h \\ I_h & 0 \end{pmatrix}.$$

Then we introduce the two $(4p+2)\times(4p+2)$ matrices:

$$A=\begin{pmatrix}\check{I}_{2p} & 0 & 0\\ 0 & \check{I}_2 & 0\\ 0 & 0 & \check{I}_{2p}\end{pmatrix} \quad \text{and} \quad B=\begin{pmatrix}0 & 0 & I_{2p}\\ 0 & I_2 & 0\\ I_{2p} & 0 & 0\end{pmatrix}.$$

Then we identify $\mathfrak{g}$ to the space of complex $(4p+2)\times(4p+2)$ matrices X such that

$$\bar{X}A=AX \quad \text{and} \quad {}^tXB+BX=0\,.$$

Using the block notation

$$X=\begin{pmatrix}x_{-1,\,-1} & x_{-1,\,0} & x_{-1,\,1}\\ x_{0,\,-1} & x_{0,\,0} & x_{0,\,1}\\ x_{1,\,-1} & x_{1,\,0} & x_{1,\,1}\end{pmatrix}$$

with

$$\begin{cases} x_{\pm 1,\,\pm 1}\in \mathfrak{M}(2p\times 2p,\,\mathbb{C}),\\ x_{\pm 1,\,0}\in \mathfrak{M}(2p\times 2,\,\mathbb{C}),\\ x_{0,\,\pm 1}\in \mathfrak{M}(2\times 2p,\,\mathbb{C}),\\ x_{0,\,0}\in \mathfrak{M}(2\times 2,\,\mathbb{C}),\end{cases}$$

we obtain the relations

$$\begin{cases} \bar{x}_{i,\,j}\check{I}=\check{I}x_{i,\,j} & \text{for } i,\,j=0,\,\pm 1\,,\\ {}^tx_{i,\,j}=-x_{-j,\,-i} & \text{for } i,\,j=0,\,\pm 1\,.\end{cases}$$

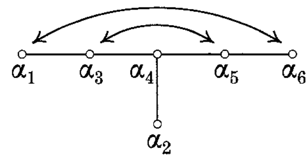
where $\check{I}$ is either $\check{I}_{2p}$ or $\check{I}_2$ according to the sizes of the matrices involved. The characteristic element and the partial complex structure correspond respectively to the matrices

$$\begin{pmatrix}-I_{2p} & 0 & 0\\ 0 & 0_2 & 0\\ 0 & 0 & I_{2p}\end{pmatrix} \quad \text{and} \quad \pm\begin{pmatrix}0_{2p} & 0 & 0\\ 0 & \begin{pmatrix}0 & -1\\ 1 & 0\end{pmatrix} & 0\\ 0 & 0 & 0_{2p}\end{pmatrix}.$$

4.3. *Simple Levi-Tanaka algebras of the real type E_6.*

There are only two types of Satake diagrams associated to E_6 that contain curved arrows: they are usually referred to as E_6II and E_6III. Accordingly, we divide the description into two parts.

Subtype $E_6 II$. The Satake diagram of $E_6 II$ is:



The choice of $\mathcal{B}_{-1}$ corresponding to Levi-Tanaka algebras are:

$$\{\alpha_3, \alpha_5\}, \quad \{\alpha_1, \alpha_6\} \quad \text{and} \quad \{\alpha_1, \alpha_3, \alpha_5, \alpha_6\}.$$

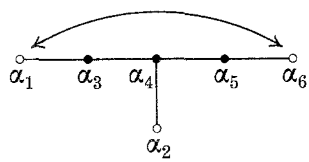
Due to the automorphisms of $E_6 II$, there is, modulo isomorphisms a unique Levi-Tanaka algebra corresponding to each LT-admissible choice of $\mathcal{B}_{-1}$. We list below the main features of each of these algebras:

$$(28) \qquad \begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_6\}, \qquad \mu = 2\,, \\ \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{so}(3, 5)\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 30\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 16\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 8\,, \end{cases}$$

$$(29) \qquad \begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_5\}, \qquad \mu = 4\,, \\ \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{R})\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 16\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 12\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 12\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 3} = 4\,, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 4} = 3\,, \end{cases}$$

(30)
$$\begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_3, \alpha_5 \alpha_6\}, \qquad \mu = 6, \\ \mathfrak{g}_0 \simeq \mathfrak{d}_4(\mathbb{R}) \oplus \mathfrak{sl}(3, \mathbb{R}), \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 12, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 8, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 9, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 3} = 6, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 4} = 5, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 5} = 2, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 6} = 3. \end{cases}$$

Subtype $E_6 III$. The Satake diagram of the root system $E_6 III$ is:



Therefore the only possible choice of $\mathcal{B}_{-1}$ is:

$$\mathcal{B}_{-1} = \{\alpha_1, \alpha_6\}.$$

It is LT-admissible and, due to the automorphisms of $E_6 III$, there is, modulo isomorphisms, a unique Levi-Tanaka algebra $\mathfrak{g}$ associated to it. Its main features are:

(31)
$$\begin{cases} \mathcal{B}_{-1} = \{\alpha_1, \alpha_6\}, \qquad \mu = 2, \\ \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{R}) \oplus \mathfrak{so}(1, 7), \\ \dim_{\mathbb{R}} \mathfrak{g}_0 = 30, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 1} = 16, \\ \dim_{\mathbb{R}} \mathfrak{g}_{\pm 2} = 8. \end{cases}$$

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE E_6(*).

	$-\lvert\alpha_i\rvert$	μ	$\dim_{\mathbb{C}}\mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11	$\mathfrak{g}_0$
01.	1 0 0 0 0 1	02	30 16 08 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})$
02.	1 1 0 0 0 0	03	26 15 10 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
03.	1 0 1 0 0 0	03	26 11 10 05 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
04.	0 1 1 0 0 0	04	20 12 12 04 01 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
05.	0 0 1 0 1 0	04	16 12 12 04 03 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
06.	1 0 1 0 0 1	04	18 11 10 05 04 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
07.	0 0 1 1 0 0	05	16 08 12 06 03 02 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
08.	0 1 0 1 0 0	05	18 10 09 09 01 01 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
09.	1 1 1 0 0 0	05	18 09 10 06 04 01 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
10.	1 0 1 0 1 0	05	14 10 09 07 03 03 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
11.	1 1 0 1 0 0	06	14 09 09 06 06 01 01 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
12.	1 0 1 1 0 0	06	14 08 07 09 03 03 02 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
13.	1 0 0 1 1 0	06	12 08 10 06 05 02 02 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
14.	1 0 1 0 1 1	06	12 08 09 06 05 02 03 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
15.	0 1 1 1 0 0	07	14 06 09 06 06 03 01 01 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
16.	0 0 1 1 1 0	07	12 06 08 08 04 04 01 02 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
17.	1 1 0 1 0 1	07	10 09 08 06 05 04 01 01 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
18.	1 0 1 1 0 1	07	10 08 07 07 05 03 02 02 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
19.	1 1 1 1 0 0	08	12 06 07 06 06 03 03 01 01 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
20.	1 1 0 1 1 0	08	10 07 07 07 04 05 02 01 01 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
21.	1 0 1 1 1 0	08	10 06 07 06 06 03 03 01 02 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
22.	0 1 1 1 1 0	09	10 06 05 08 04 04 04 01 01 01 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
23.	1 1 1 1 0 1	09	08 07 06 06 05 04 03 02 01 01 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
24.	1 0 1 1 1 1	09	08 06 06 06 05 04 03 02 01 02 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
25.	1 1 1 1 1 0	10	08 06 05 06 05 04 03 03 01 01 01 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
26.	1 1 1 1 1 1	11	06 06 05 05 05 04 03 03 02 01 01 01	$\mathfrak{d}_6(\mathbb{C})$

(*) In case 1), 5), 14) the two Levi-Tanaka algebras corresponding to the weighted Satake diagrams are isomorphic, in the other cases they are not isomorphic.

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE E_7.

	$-\|\alpha_i\|$	μ	$\dim_{\mathbb{C}}\mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11 ±12 ±13 ±14 ±15 ±16 ±17	$\mathfrak{g}_0$
01.	1 0 0 0 0 0 1	03	47 26 16 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(10, \mathbb{C})$
02.	0 0 0 0 0 1 1	03	47 17 16 10 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{so}(10, \mathbb{C})$
03.	1 1 0 0 0 0 0	04	37 21 20 06 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(6, \mathbb{C})$
04.	0 1 0 0 0 1 0	04	29 20 20 07 05 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})$
05.	1 0 0 0 0 1 1	05	31 17 16 09 08 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(8, \mathbb{C})$
06.	0 1 0 0 0 1 1	05	27 16 15 11 06 05 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})$
07.	1 0 1 0 0 0 0	05	37 16 15 15 01 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(6, \mathbb{C})$
08.	0 1 1 0 0 0 0	05	29 15 20 10 05 02 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})$
09.	0 1 0 0 1 0 0	05	25 16 18 12 04 04 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(4, \mathbb{C})\oplus\mathfrak{sl}(3, \mathbb{C})$
10.	0 0 0 0 1 1 0	05	29 12 20 10 05 05 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})$
11.	1 0 1 0 0 0 1	06	27 16 15 10 10 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})$
12.	0 1 0 0 1 0 1	06	21 14 16 10 09 03 04 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4, \mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})$
13.	0 0 0 0 1 1 1	06	27 12 11 15 05 05 05 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})$
14.	0 0 1 0 1 0 0	06	21 15 18 09 09 03 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})\oplus\mathfrak{sl}(3, \mathbb{C})\oplus\mathfrak{sl}(3, \mathbb{C})$
15.	0 1 0 1 0 0 0	06	25 13 12 18 04 04 03 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3, \mathbb{C})\oplus\mathfrak{sl}(4, \mathbb{C})$
16.	1 1 1 0 0 0 0	07	27 11 15 10 10 05 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5, \mathbb{C})$
17.	1 0 1 0 0 1 0	07	21 15 14 10 09 06 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4, \mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})$
18.	1 0 0 0 1 1 0	07	21 12 16 10 09 04 04 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4, \mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})$
19.	0 1 0 0 1 1 0	07	21 10 14 12 08 06 02 04 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4, \mathbb{C})\oplus\mathfrak{sl}(2, \mathbb{C})$

20.	0 0 1 0 1 0 1	07	17 14 15 10 08 06 03 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
21.	0 1 0 1 0 0 1	07	19 13 12 12 10 04 03 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
22.	0 0 1 1 0 0 0	07	23 10 16 12 06 08 01 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
23.	0 0 0 1 1 0 0	07	21 09 18 09 09 06 02 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
24.	1 0 1 0 0 1 1	08	19 12 14 09 09 05 06 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
25.	1 0 0 0 1 1 1	08	19 12 11 11 09 05 04 04 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
26.	0 1 0 0 1 1 1	08	19 10 11 10 10 05 05 02 04 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
27.	1 0 1 0 1 0 0	08	19 13 12 12 06 09 03 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
28.	0 0 1 0 1 1 0	08	17 11 12 13 06 08 03 03 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
29.	1 1 0 1 0 0 0	08	21 11 12 10 12 04 04 02 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
30.	0 1 0 1 0 1 0	08	17 11 12 09 12 05 04 02 03 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
31.	0 0 1 1 0 0 1	08	17 11 14 10 09 05 06 01 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
32.	0 0 0 1 1 0 1	08	17 10 13 12 06 08 04 02 03 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
33.	1 0 1 0 1 0 1	09	15 12 12 10 08 06 06 03 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
34.	0 0 1 0 1 1 1	09	15 11 10 10 09 06 05 03 03 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
35.	1 1 0 1 0 0 1	09	15 12 11 09 10 07 04 03 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
36.	0 1 0 1 0 1 1	09	15 10 11 09 09 07 05 03 02 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
37.	1 0 1 1 0 0 0	09	21 10 09 14 06 06 08 01 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
38.	0 1 1 1 0 0 0	09	21 07 12 08 12 06 04 04 01 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
39.	0 0 1 1 0 1 0	09	15 10 12 10 09 06 05 04 01 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
40.	1 0 0 1 1 0 0	09	17 09 14 09 09 06 06 02 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
41.	0 1 0 1 1 0 0	09	19 07 12 09 09 09 03 04 01 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$

42.	0 0 0 1 1 1 0	09	17 09 08 15 06 06 07 02 02 03 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
43.	1 0 1 0 1 1 0	10	15 09 12 09 09 05 07 03 03 01 01 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
44.	1 1 0 1 0 1 0	10	13 11 10 09 08 09 04 04 02 02 01 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
45.	1 0 1 1 0 0 1	10	15 11 09 11 08 06 05 06 01 01 01 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
46.	0 1 1 1 0 0 1	10	15 09 10 09 08 09 04 04 03 01 02 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
47.	0 0 1 1 0 1 1	10	13 09 12 08 09 06 06 03 04 01 02 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
48.	1 0 0 1 1 0 1	10	13 10 11 10 08 06 06 04 02 02 01 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
49.	0 1 0 1 1 0 1	10	15 08 10 09 09 06 07 03 03 01 03 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
50.	0 0 0 1 1 1 1	10	15 09 08 10 09 06 05 05 02 02 03 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
51.	0 0 1 1 1 0 0	10	17 07 10 12 06 09 03 06 02 01 02 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
52.	1 0 1 0 1 1 1	11	13 09 10 09 08 06 06 04 03 03 01 01 00 00 00 00 00 00	$\mathfrak{h}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
53.	1 1 0 1 0 1 1	11	11 10 10 08 08 07 06 04 03 02 02 01 00 00 00 00 00 00	$\mathfrak{h}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
54.	1 1 1 1 0 0 0	11	19 07 09 08 10 06 06 04 04 01 01 01 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
55.	1 0 1 1 0 1 0	11	13 10 09 09 09 06 05 05 04 01 01 01 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
56.	0 1 1 1 0 1 0	11	13 09 08 10 06 09 06 03 04 02 01 02 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
57.	1 1 0 1 1 0 0	11	15 08 09 10 06 09 06 03 04 01 02 01 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
58.	1 0 0 1 1 1 0	11	13 09 08 11 08 06 05 06 02 02 02 01 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
59.	0 1 0 1 1 1 0	11	15 07 08 09 09 06 06 05 03 02 01 03 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
60.	0 0 1 1 1 0 1	11	13 08 09 10 08 06 06 04 04 02 01 02 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
61.	1 1 1 1 0 0 1	12	13 09 08 08 08 07 06 04 04 03 01 01 01 00 00 00 00 00	$\mathfrak{h}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
62.	1 0 1 1 0 1 1	12	11 09 09 09 07 07 05 05 03 04 01 01 01 00 00 00 00 00	$\mathfrak{h}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
63.	0 1 1 1 0 1 1	12	11 08 09 08 07 07 06 05 03 03 02 01 02 00 00 00 00 00	$\mathfrak{h}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

64.	1 1 0 1 1 0 1	12	11 09 08 09 07 07 06 05 03 03 01 02 01 00 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
65.	1 0 0 1 1 1 1	12	11 09 08 08 09 06 05 05 04 02 02 02 01 00 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
66.	0 1 0 1 1 1 1	12	13 07 08 07 09 06 06 04 05 02 02 01 03 00 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
67.	1 0 1 1 1 0 0	12	15 07 09 08 09 06 06 03 06 02 01 01 01 00 00 00 00 00	$\mathfrak{o}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
68.	0 1 1 1 1 0 0	12	15 07 06 11 06 06 09 03 03 04 01 01 02 00 00 00 00 00	$\mathfrak{o}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
69.	0 0 1 1 1 1 0	12	13 07 08 08 10 05 06 04 05 02 02 01 02 00 00 00 00 00	$\mathfrak{o}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
70.	1 1 1 1 0 1 0	13	11 09 07 08 07 07 06 05 03 04 02 01 01 01 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
71.	1 1 0 1 1 1 0	13	11 08 07 08 08 06 06 05 04 03 02 01 02 01 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
72.	1 0 1 1 1 0 1	13	11 08 08 08 08 06 06 04 04 04 02 01 01 01 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
73.	0 1 1 1 1 0 1	13	11 08 06 09 07 06 06 06 03 03 03 01 01 02 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
74.	0 0 1 1 1 1 1	13	11 07 08 07 08 07 05 04 05 03 02 02 01 02 00 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
75.	1 1 1 1 0 1 1	14	09 08 08 07 07 06 06 05 04 03 03 02 01 01 01 00 00 00	$\mathfrak{o}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
76.	1 1 0 1 1 1 1	14	09 08 07 07 07 07 05 05 04 04 02 02 01 02 01 00 00 00	$\mathfrak{o}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
77.	1 1 1 1 1 0 0	14	13 07 06 08 07 06 06 06 03 03 04 01 01 01 01 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
78.	1 0 1 1 1 1 0	14	11 07 07 08 07 07 05 05 03 05 02 02 01 01 01 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
79.	0 1 1 1 1 1 0	14	11 07 06 07 08 06 05 06 04 03 03 02 01 01 02 00 00 00	$\mathfrak{o}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
80.	1 1 1 1 1 0 1	15	09 08 06 07 07 06 05 06 04 03 03 03 01 01 01 01 00 00	$\mathfrak{o}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
81.	1 0 1 1 1 1 1	15	09 07 07 07 07 06 06 04 04 04 03 02 02 01 01 01 00 00	$\mathfrak{o}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
82.	0 1 1 1 1 1 1	15	09 07 06 07 06 07 05 05 04 04 03 02 02 01 01 02 00 00	$\mathfrak{o}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
83.	1 1 1 1 1 1 0	16	09 07 06 06 07 06 05 05 05 03 03 03 02 01 01 01 01 00	$\mathfrak{o}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
84.	1 1 1 1 1 1 1	17	07 07 06 06 06 06 05 05 04 04 03 03 02 02 01 01 01 01	$\mathfrak{o}_7(\mathbb{C})$

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE E_8.

	$-\lvert \alpha_i \rvert$	μ	$\dim_{\mathbb{C}} \mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11 ±12 ±13 ±14 ±15 ±16 ±17 ±18 ±19 ±20 ±21 ±22 ±23 ±24 ±25 ±26 ±27 ±28 ±29	$\mathfrak{g}_0$
001.	1 1 0 0 0 0 0 0	05	50 28 35 21 08 07 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(7, \mathbb{C})$
002.	0 0 0 0 0 0 1 1	05	80 28 27 27 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{e}_6$
003.	1 0 1 0 0 0 0 0	06	50 22 20 36 07 07 07 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(7, \mathbb{C})$
004.	1 0 0 0 0 0 1 1	07	48 27 26 18 17 10 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{so}(10, \mathbb{C})$
005.	0 1 1 0 0 0 0 0	07	40 18 29 21 15 12 03 06 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(6, \mathbb{C})$
006.	0 1 0 0 0 1 0 0	07	34 25 30 18 16 10 05 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
007.	0 0 0 0 0 1 1 0	07	50 18 32 20 10 16 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{so}(10, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
008.	1 0 1 0 0 0 0 1	08	34 22 21 20 22 07 06 06 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(6, \mathbb{C})$
009.	0 1 0 0 1 0 0 0	08	32 20 24 24 10 16 06 04 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
010.	1 1 1 0 0 0 0 0	09	38 13 21 14 21 15 06 07 02 06 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(6, \mathbb{C})$
011.	1 0 1 0 0 0 1 0	09	30 21 20 15 22 12 07 05 05 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$

012.	0 1 0 0 0 1 0 1	09	30 22 25 17 16 11 10 05 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
013.	1 0 0 0 0 1 1 0	09	34 18 24 18 17 10 09 08 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
014.	0 0 0 0 0 1 1 1	09	48 18 17 26 10 10 16 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{so}(10,\mathbb{C})$
015.	0 0 1 0 1 0 0 0	09	28 18 24 15 19 12 09 06 03 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
016.	0 0 0 0 1 1 0 0	09	34 13 30 15 15 15 05 10 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
017.	0 1 0 1 0 0 0 0	09	34 16 15 30 10 10 15 03 03 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
018.	0 1 0 0 0 1 1 0	10	30 17 20 21 12 15 06 10 05 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
019.	1 0 1 0 0 1 0 0	10	26 19 18 15 16 18 07 07 04 04 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
020.	0 1 0 0 1 0 0 1	10	26 19 22 18 16 10 12 06 04 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
021.	0 0 1 1 0 0 0 0	10	32 12 20 19 11 20 05 10 04 02 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
022.	1 0 1 0 0 0 1 1	11	28 17 20 15 15 13 12 06 05 05 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
023.	1 0 0 0 0 1 1 1	11	32 18 17 17 17 10 09 09 08 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{so}(8,\mathbb{C})$
024.	0 1 0 0 1 0 1 0	11	24 16 20 16 17 10 11 08 06 04 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
025.	1 0 1 0 1 0 0 0	11	26 16 15 18 09 19 12 05 07 03 03 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$

026.	0 0 1 0 1 0 0 1	11	22 18 21 15 15 13 11 07 06 03 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
027.	1 0 0 0 1 1 0 0	11	26 13 22 15 15 12 12 07 05 06 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
028.	0 0 0 0 1 1 0 1	11	30 14 21 20 10 15 10 05 10 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
029.	1 1 0 1 0 0 0 0	11	30 13 15 15 20 10 10 10 06 03 02 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C})$
030.	0 1 0 1 0 0 0 1	11	26 17 15 21 16 10 09 12 03 03 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
031.	0 0 0 1 1 0 0 0	11	28 10 24 12 18 12 08 12 03 06 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_2(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
032.	0 1 0 0 0 1 1 1	12	28 17 16 16 16 11 10 06 10 05 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C})$
033.	1 0 1 0 0 1 0 1	12	22 17 18 14 14 13 13 07 06 04 04 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_4(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
034.	0 0 1 0 1 0 1 0	12	20 16 18 16 12 15 09 10 05 06 03 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
035.	0 1 0 0 1 1 0 0	12	26 11 18 18 12 15 06 13 04 06 04 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
036.	0 0 0 0 1 1 1 0	12	30 13 12 25 10 10 15 05 05 10 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
037.	0 1 0 1 0 0 1 0	12	22 16 15 15 19 10 09 09 09 03 03 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
038.	1 0 1 1 0 0 0 0	12	30 12 11 20 09 11 20 05 05 07 02 02 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C})$
039.	0 0 1 1 0 0 0 1	12	24 14 18 16 13 13 13 06 08 04 02 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{h}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$

040.	1 0 1 0 0 1 1 0	13	22 13 18 13 13 12 13 08 07 05 04 04 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
041.	0 1 0 0 1 0 1 1	13	22 15 18 14 15 11 11 07 08 06 04 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
042.	1 0 1 0 1 0 0 1	13	20 16 15 15 12 12 13 10 05 06 03 03 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
043.	1 0 0 0 1 1 0 1	13	22 14 17 16 14 10 12 09 06 05 06 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
044.	0 0 1 0 1 1 0 0	13	22 12 15 19 08 16 09 09 09 03 06 03 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
045.	1 1 0 1 0 0 0 1	13	22 15 14 13 17 12 10 08 09 05 03 02 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
046.	0 1 1 1 0 0 0 0	13	30 08 15 10 19 11 10 10 05 10 02 03 01 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(5,\mathbb{C})$
047.	0 0 1 1 0 0 1 0	13	20 14 16 14 14 11 13 08 07 06 04 02 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
048.	0 1 0 1 0 1 0 0	13	22 13 15 12 18 12 09 07 10 06 03 03 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
049.	1 0 0 1 1 0 0 0	13	24 10 18 12 14 12 12 08 08 06 03 04 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
050.	0 0 0 1 1 0 0 1	13	22 12 19 15 12 15 08 09 09 03 06 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
051.	1 0 1 0 1 0 1 0	14	18 14 15 13 13 09 14 09 08 05 05 03 03 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
052.	0 0 1 0 1 0 1 1	14	18 15 17 13 14 10 12 08 08 05 06 03 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
053.	0 1 0 0 1 1 0 1	14	22 12 15 16 14 10 11 08 09 04 06 04 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

054.	1 0 0 0 1 1 1 0	14	22 13 12 17 14 10 09 12 06 05 05 06 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
055.	0 0 0 0 1 1 1 1	14	28 13 12 16 15 10 10 10 05 05 10 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C})$
056.	1 1 0 1 0 0 1 0	14	18 15 13 12 14 14 09 09 07 08 04 03 02 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
057.	0 1 0 1 0 0 1 1	14	20 14 15 15 13 13 09 09 06 09 03 03 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
058.	1 0 1 1 0 0 0 1	14	22 14 11 16 12 09 13 13 05 05 06 02 02 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
059.	0 0 1 1 0 1 0 0	14	20 12 14 14 12 13 09 12 05 08 04 04 02 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
060.	0 1 0 1 1 0 0 0	14	26 08 15 12 12 18 06 10 04 12 03 03 03 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
061.	0 0 0 1 1 0 1 0	14	20 12 14 18 09 14 11 06 10 06 03 06 01 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
062.	1 0 1 0 0 1 1 1	15	20 13 15 13 13 09 13 08 08 06 05 04 04 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
063.	0 1 0 0 1 1 1 0	15	22 11 12 14 16 09 10 08 10 05 04 06 04 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
064.	1 0 1 0 1 1 0 0	15	20 10 15 12 13 08 13 09 09 06 05 04 03 03 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
065.	0 0 1 0 1 1 0 1	15	18 13 13 16 12 10 11 09 08 07 03 06 03 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
066.	1 1 1 1 0 0 0 0	15	28 08 11 10 15 09 11 10 10 05 05 06 02 02 01 05 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(5, \mathbb{C})$
067.	0 1 1 1 0 0 0 1	15	22 11 13 11 14 13 09 10 07 06 08 02 03 01 04 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$

068.	1 0 1 1 0 0 1 0	15	18 14 11 13 13 09 10 13 08 05 05 05 02 02 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
069.	0 0 1 1 0 0 1 1	15	18 12 17 12 13 10 12 08 09 05 06 04 02 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
070.	1 1 0 1 0 1 0 0	15	18 13 12 12 11 15 09 09 07 07 07 03 03 02 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
071.	0 1 0 1 0 1 0 1	15	18 13 14 12 15 11 11 07 08 07 06 03 03 02 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
072.	1 0 0 1 1 0 0 1	15	18 12 15 13 12 11 12 08 08 07 05 03 04 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
073.	0 0 1 1 1 0 0 0	15	24 08 12 16 08 15 07 12 08 04 09 02 04 02 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
074.	0 0 0 1 1 1 0 0	15	22 10 09 21 09 09 15 06 06 11 03 03 06 01 01 03 00 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
075.	1 0 1 0 1 0 1 1	16	16 13 14 13 11 11 09 11 07 08 04 05 03 03 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
076.	1 0 0 0 1 1 1 1	16	20 13 12 12 15 10 09 09 09 05 05 05 06 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
077.	0 0 1 0 1 1 1 0	16	18 12 11 13 15 08 11 08 09 07 05 03 06 03 01 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
078.	1 1 0 1 0 0 1 1	16	16 13 14 11 13 11 11 08 08 06 07 04 03 02 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
079.	0 1 1 1 0 0 1 0	16	18 12 11 12 10 14 10 08 09 05 07 06 02 03 01 03 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
080.	0 1 0 1 0 1 1 0	16	18 11 13 12 12 13 09 09 06 09 04 06 03 03 02 01 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
081.	1 0 1 1 0 1 0 0	16	18 12 11 11 13 09 10 09 12 05 05 05 04 02 02 02 03 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$

082.	0 0 1 1 0 1 0 1	16	16 12 14 12 13 09 12 08 09 06 06 04 04 02 02 02 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
083.	1 1 0 1 1 0 0 0	16	22 09 11 13 08 14 12 06 10 04 08 06 02 03 02 01 04 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
084.	0 1 0 1 1 0 0 1	16	20 10 13 12 12 12 12 07 07 06 09 03 03 03 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
085.	1 0 0 1 1 0 1 0	16	16 12 12 14 11 10 11 10 06 08 06 04 03 04 01 02 02 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
086.	0 0 0 1 1 0 1 1	16	18 11 15 13 12 11 10 09 07 07 06 03 06 01 02 01 01 00 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
087.	0 1 0 0 1 1 1 1	17	20 11 12 11 14 11 09 07 10 06 05 04 06 04 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
088.	1 0 1 0 1 1 0 1	17	16 11 13 12 12 09 10 09 09 07 06 04 04 03 03 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
089.	1 1 1 1 0 0 0 1	17	20 11 10 10 12 11 09 09 10 07 05 05 05 02 02 01 04 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
090.	1 0 1 1 0 0 1 1	17	16 12 12 13 10 11 08 11 08 08 05 04 05 02 02 03 01 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
091.	1 1 0 1 0 1 0 1	17	14 13 12 11 11 12 10 09 07 07 06 06 03 03 02 02 02 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
092.	0 1 1 1 0 1 0 0	17	18 11 09 13 08 12 13 06 09 07 04 08 04 02 03 01 02 03 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
093.	0 0 1 1 0 1 1 0	17	16 10 14 10 13 09 11 08 09 06 07 04 04 04 02 02 01 02 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
094.	0 1 0 1 1 0 1 0	17	18 10 11 12 12 09 13 08 07 05 08 06 03 03 03 01 02 02 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
095.	1 0 1 1 1 0 0 0	17	22 08 11 10 12 10 09 07 12 08 04 05 05 03 02 02 01 04 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$

096.	0 0 1 1 1 0 0 1	17	18 10 11 14 10 11 09 10 08 07 05 07 02 04 02 01 03 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
097.	1 0 0 1 1 1 0 0	17	18 10 09 15 11 09 09 12 06 06 08 05 03 03 04 01 01 03 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
098.	0 0 0 1 1 1 0 1	17	18 11 09 16 12 09 10 10 06 07 08 03 03 06 01 01 02 01 00 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
099.	1 0 1 0 1 1 1 0	18	16 10 11 12 11 10 08 10 07 09 05 06 03 04 03 03 01 01 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
100.	0 0 1 0 1 1 1 1	18	16 12 11 11 12 12 07 09 08 08 05 05 03 06 03 01 01 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
101.	1 1 1 1 0 0 1 0	18	16 12 09 10 10 11 09 09 08 09 05 05 05 04 02 02 01 03 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
102.	0 1 1 1 0 0 1 1	18	16 10 13 10 11 11 09 10 07 07 06 05 06 02 03 01 03 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
103.	1 1 0 1 0 1 1 0	18	14 11 12 10 11 10 11 08 08 06 07 05 05 03 03 02 02 01 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
104.	0 1 0 1 0 1 1 1	18	16 11 12 11 12 10 11 07 08 07 06 04 06 03 03 02 01 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
105.	1 0 1 1 0 1 0 1	18	14 12 11 11 11 10 08 10 08 09 05 05 04 04 02 02 02 02 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
106.	1 1 0 1 1 0 0 1	18	16 11 10 12 09 11 11 09 07 07 05 07 05 02 03 02 01 03 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
107.	1 0 0 1 1 0 1 1	18	14 11 13 11 12 09 10 09 08 06 07 05 04 03 04 01 02 01 01 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
108.	0 1 1 1 1 0 0 0	18	22 08 07 14 08 08 15 07 06 10 04 04 09 02 02 03 01 01 04 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(4,\mathbb{C})$
109.	0 0 1 1 1 0 1 0	18	16 10 10 12 12 08 10 09 09 06 06 06 05 02 04 02 01 02 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

110.	0 1 0 1 1 1 0 0	18	20 08 09 12 12 09 09 12 06 06 04 10 03 03 03 03 01 01 03 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
111.	0 0 0 1 1 1 1 0	18	18 10 09 11 15 09 08 10 07 06 08 05 03 03 06 01 01 01 02 00 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
112.	1 1 1 1 0 1 0 0	19	16 11 08 10 09 10 09 10 06 09 07 04 05 05 03 02 02 01 02 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
113.	0 1 1 1 0 1 0 1	19	14 11 10 11 09 11 09 10 07 07 06 05 06 04 02 03 01 02 02 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
114.	1 0 1 1 0 1 1 0	19	14 10 11 11 09 11 07 10 07 09 06 05 05 03 04 02 02 02 01 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
115.	0 0 1 1 0 1 1 1	19	14 10 13 10 11 10 09 08 09 06 07 05 04 04 04 02 02 01 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
116.	1 1 0 1 1 0 1 0	19	14 11 09 11 10 09 10 10 07 07 05 06 06 04 02 03 02 01 02 02 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
117.	0 1 0 1 1 0 1 1	19	16 09 12 10 12 09 10 09 08 05 07 05 06 03 03 03 01 02 01 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
118.	1 0 1 1 1 0 0 1	19	16 10 10 10 11 09 10 06 10 08 07 04 05 04 03 02 02 01 03 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
119.	1 0 0 1 1 1 0 1	19	14 11 09 12 12 09 08 10 08 06 06 07 04 03 03 04 01 01 02 01 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$
120.	0 0 1 1 1 1 0 0	19	18 08 09 10 14 07 08 10 09 06 06 05 07 03 02 04 02 01 01 03 00 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_4(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus$ $\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
121.	1 0 1 0 1 1 1 1	20	14 10 11 10 11 09 10 06 09 07 07 05 05 03 04 03 03 01 01 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
122.	1 1 1 1 0 0 1 1	20	14 10 11 09 10 09 10 07 09 07 07 05 05 04 04 02 02 01 03 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(3,\mathbb{C})$
123.	1 1 0 1 0 1 1 1	20	12 11 11 10 10 10 09 09 07 07 06 06 04 05 03 03 02 02 01 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})\oplus\mathfrak{sl}(2,\mathbb{C})$

124.	0 1 1 1 0 1 1 0	20	14 09 11 09 10 09 09 10 07 07 06 05 06 04 04 02 03 01 02 01 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
125.	1 1 1 1 1 0 0 0	20	20 08 07 10 09 08 10 09 07 06 10 04 04 05 05 02 02 02 01 01 04 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(4, \mathbb{C})$
126.	0 1 1 1 1 0 0 1	20	16 10 07 12 09 08 11 09 07 07 07 04 05 07 02 02 03 01 01 03 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
127.	1 0 1 1 1 0 1 0	20	14 10 09 10 10 09 09 07 08 09 06 06 04 05 03 03 02 02 01 02 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
128.	0 0 1 1 1 0 1 1	20	14 09 11 11 10 10 08 08 09 07 05 07 04 05 02 04 02 01 02 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
129.	1 1 0 1 1 1 0 0	20	16 09 08 10 11 08 09 09 09 06 06 04 07 05 03 02 03 02 01 01 03 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
130.	0 1 0 1 1 1 0 1	20	16 09 09 10 12 09 09 08 10 05 05 06 07 03 03 03 03 01 01 02 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
131.	1 0 0 1 1 1 1 0	20	14 10 09 09 13 09 08 08 09 06 06 06 06 03 03 03 04 01 01 01 02 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
132.	0 0 0 1 1 1 1 1	20	16 10 09 11 10 12 08 08 07 07 07 05 05 03 03 06 01 01 01 01 01 00 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
133.	1 1 1 1 0 1 0 1	21	12 11 09 09 09 09 09 08 08 07 07 06 04 05 04 03 02 02 01 02 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
134.	1 0 1 1 0 1 1 1	21	12 10 10 11 09 09 09 07 08 08 06 06 05 04 03 04 02 02 02 01 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
135.	1 1 0 1 1 0 1 1	21	12 10 10 10 09 10 08 09 08 07 05 06 05 05 04 02 03 02 01 02 01 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
136.	0 1 1 1 1 0 1 0	21	14 10 07 10 10 08 08 10 08 06 07 05 04 06 05 02 02 03 01 01 02 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
137.	0 1 0 1 1 1 1 0	21	16 08 09 08 12 09 09 07 09 07 05 04 08 04 03 03 03 03 01 01 01 02 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$

138.	1 0 1 1 1 1 0 0	21	16 08 08 10 09 10 07 08 07 09 06 06 05 04 05 02 03 02 02 01 01 03 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
139.	0 0 1 1 1 1 0 1	21	14 09 09 09 12 09 08 06 11 06 06 05 06 05 03 02 04 02 01 01 02 01 00 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
140.	1 1 1 1 0 1 1 0	22	12 09 10 08 09 08 09 07 09 06 07 06 05 04 05 03 03 02 02 01 02 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
141.	0 1 1 1 0 1 1 1	22	12 09 10 10 08 10 08 08 08 07 06 05 06 04 04 04 02 03 01 02 01 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
142.	1 1 1 1 1 0 0 1	22	14 10 07 09 09 08 08 10 06 07 07 07 04 04 05 04 02 02 02 01 01 03 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
143.	1 0 1 1 1 0 1 1	22	12 09 10 09 10 08 09 07 07 08 07 05 06 04 04 03 03 02 02 01 02 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
144.	1 1 0 1 1 1 0 1	22	12 10 08 09 10 09 08 08 08 08 05 05 05 06 04 03 02 03 02 01 01 02 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
145.	1 0 0 1 1 1 1 1	22	12 10 09 09 10 10 08 08 07 07 06 06 05 05 03 03 03 04 01 01 01 01 01 00 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
146.	0 1 1 1 1 1 0 0	22	16 08 07 08 11 08 07 08 10 06 06 06 04 04 07 03 02 02 03 01 01 01 03 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
147.	0 0 1 1 1 1 1 0	22	14 08 09 08 10 11 07 06 09 08 05 06 04 07 03 03 02 04 02 01 01 01 02 00 00 00 00 00 00 00	$\mathfrak{d}_5(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus$ $\oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
148.	1 1 1 1 1 0 1 0	23	12 10 07 08 09 08 07 09 07 07 06 07 05 04 04 05 03 02 02 02 01 01 02 02 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
149.	0 1 1 1 1 0 1 1	23	12 09 08 10 08 09 08 08 07 08 06 05 05 05 04 05 02 02 03 01 01 02 01 01 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
150.	1 1 0 1 1 1 1 0	23	12 09 08 08 09 10 07 08 07 08 06 05 04 06 05 03 03 02 03 02 01 01 01 02 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
151.	0 1 0 1 1 1 1 1	23	14 08 09 08 10 09 09 07 08 06 07 04 06 05 04 03 03 03 03 01 01 01 01 01 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$

152.	1 0 1 1 1 1 0 1	23	12 09 08 09 09 09 08 07 06 09 06 06 05 05 04 04 02 03 02 02 01 01 02 01 00 00 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
153.	1 1 1 1 0 1 1 1	24	10 09 09 09 08 08 08 08 06 08 06 06 05 05 04 04 03 03 02 02 01 02 01 01 01 01 00 00 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
154.	1 1 1 1 1 1 0 0	24	14 08 07 07 09 08 07 07 08 07 06 06 06 04 04 04 05 02 02 02 02 01 01 01 03 03 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
155.	0 1 1 1 1 1 0 1	24	12 09 07 08 09 09 07 08 06 09 06 05 05 04 05 05 03 02 02 03 01 01 01 02 01 01 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
156.	1 0 1 1 1 1 1 0	24	12 08 08 08 09 08 09 06 06 08 07 05 06 04 05 04 03 02 03 02 02 01 01 01 02 02 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
157.	0 0 1 1 1 1 1 1	24	12 08 09 08 09 09 09 06 07 07 07 05 05 05 05 03 03 02 04 02 01 01 01 01 01 01 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
158.	1 1 1 1 1 0 1 1	25	10 09 08 08 08 08 07 08 07 06 07 06 05 05 04 04 04 03 02 02 02 01 01 02 01 01 00 00 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
159.	1 1 0 1 1 1 1 1	25	10 09 08 08 08 09 08 07 07 07 06 06 04 05 05 04 03 03 02 03 02 01 01 01 01 01 00 00 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
160.	0 1 1 1 1 1 1 0	25	12 08 07 08 07 10 07 07 06 08 07 05 05 04 04 06 03 03 02 02 03 01 01 01 01 01 00 00 00 00	$\mathfrak{d}_6(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
161.	1 1 1 1 1 1 0 1	26	10 09 07 07 08 08 07 07 07 06 07 06 05 05 04 04 04 04 02 02 02 02 01 01 01 01 01 00 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
162.	1 0 1 1 1 1 1 1	26	10 08 08 08 08 08 08 07 06 06 07 06 05 05 04 05 03 03 02 03 02 02 01 01 01 01 01 00 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
163.	1 1 1 1 1 1 1 0	27	10 08 07 07 07 08 07 07 06 06 07 06 05 05 04 04 04 04 03 02 02 02 02 01 01 01 01 02 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
164.	0 1 1 1 1 1 1 1	27	10 08 07 08 07 08 08 07 06 06 07 06 05 04 04 05 04 03 03 02 02 03 01 01 01 01 01 01 00 00	$\mathfrak{d}_7(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
165.	1 1 1 1 1 1 1 1	29	08 08 07 07 07 07 07 07 06 06 05 07 05 05 04 04 04 04 03 03 02 02 02 02 01 01 01 01 01 01	$\mathfrak{d}_8(\mathbb{C})$

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE F_4.

	$-\lvert\alpha_i\rvert$	μ	$\dim_{\mathbb{C}} \mathfrak{g}$ 0 ±1 ±2 ±3 ±4 ±5 ±6 ±7 ±8 ±9 ±10 ±11	$\mathfrak{g}_0$
1.	1 1 0 0	05	10 08 06 06 01 01 00 00 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(3, \mathbb{C})$
2.	1 1 0 1	07	06 06 05 05 03 03 01 01 00 00 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
3.	0 1 1 0	07	08 04 04 07 03 02 01 02 00 00 00 00	$\mathfrak{d}_2(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
4.	1 1 1 0	09	06 04 03 06 03 03 02 01 01 01 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
5.	0 1 1 1	09	06 04 03 04 03 04 02 01 01 02 00 00	$\mathfrak{d}_3(\mathbb{C}) \oplus \mathfrak{sl}(2, \mathbb{C})$
6.	1 1 1 1	11	04 04 03 03 03 04 02 02 01 01 01 01	$\mathfrak{d}_4(\mathbb{C})$

SIMPLE LEVI-TANAKA ALGEBRAS OF THE COMPLEX TYPE G_2.

$$\mathcal{B}_{-1} = \{\alpha_1, \alpha_2\}, \qquad \mu = 5, \qquad \begin{cases} \dim_{\mathbb{C}} \mathfrak{g}_0 = 2\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 1} = 2\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 2} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 3} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 4} = 1\,, \\ \dim_{\mathbb{C}} \mathfrak{g}_{\pm 5} = 1\,, \end{cases} \qquad \mathfrak{g}_0 \simeq \mathfrak{d}_2(\mathbb{C})\,.$$

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