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**CLASSIFICATION OF SINGULAR SOLUTIONS
OF A NONLINEAR HEAT EQUATION**

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CLASSIFICATION OF SINGULAR SOLUTIONS

OF A NONLINEAR HEAT EQUATION

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0. Introduction

In this paper we consider the problem of finding all nonnegative solutions of the nonlinear heat equation with absorption

$$(0.1) \quad u_t = \Delta(u^m) - u^p \quad \text{in } Q = \mathbb{R}^n \times (0, T)$$

for $n \geq 1$, $m, p > 1$, with initial conditions

$$(0.2) \quad u(x, 0) = 0 \quad \text{for } x \neq 0.$$

Here we mean by a solution a function $u(x, t)$ which is defined, nonnegative and continuous in $\overline{Q} \setminus \{(0, 0)\}$, satisfies (0.1) in the sense of distributions, (0.2) in the classical sense, and is uniformly bounded in x for every $t \in (0, T)$. The behaviour of $u(x, t)$ as $(x, t) \rightarrow (0, 0)$, $(x, t) \in Q$ is not prescribed so that u may exhibit a singularity at the origin. A first example of a solution of (0.1), (0.2) is the trivial solution $u \equiv 0$. Nontrivial solutions can be obtained by considering the problem

$$(P_c) \quad \begin{cases} u_t = \Delta(u^m) - u^p & \text{in } Q, \\ u(x, 0) = c\delta(x) & \text{in } \Omega \in \mathbb{R}^n, \end{cases}$$

where δ is Dirac's mass and $c \in \mathbb{R}^+$. A solution u_c of P_c is called a *fundamental solution (of value c)* for equation (0.1). Fundamental solutions satisfy the property

$$(0.3) \quad \lim_{t \rightarrow 0} \int_{|x| < r} u(x, t) dx = c$$

for every $r > 0$. Brezis, Peletier and Terman [BPT] found another type of solution of (0.1), (0.2) in the case $m = 1$, $1 < p < (n + 2)/n$ which has a stronger singularity at $(0, 0)$, i.e., such that

$$(0.4) \quad \lim_{t \rightarrow 0} \int_{|x| < r} u(x, t) dx = +\infty.$$

These solutions are called *very singular solutions*.

The existence of fundamental solutions has been established in the papers [KP1] precisely in the exponent range $1 < p < m + 2/n$. Peletier and Terman [PT] have shown the existence of a very singular solution of self-similar form,

$$(0.5) \quad W(x, t) = t^{-\alpha} f(|x|t^{-\gamma}),$$

with $\alpha = 1/(p - 1)$, $\gamma = (p - m)/2(p - 1)$ for $m < p < m + 2/n$. The uniqueness of such a solution has been established recently by Kamin and Veron [KV].

Here we give a complete classification of singular solutions of (0.1), (0.2).

Theorem. (i) If $1 < p \leq m$, there exists for every $c > 0$ a unique fundamental solution w_c which satisfies (0.3).

(ii) If $m < p < m + 2/n$, there exists for every $c > 0$ a unique fundamental solution w_c which satisfies (0.3) and in addition a unique very singular solution W , which satisfies (0.4).

(iii) If $p \geq m + 2/n$, no singular solutions exist.

An analogous classification has been established by Oswald [O] for $m = 1$, $p > 1$. In his case a very singular solution exists whenever fundamental solutions exist. This does not always happen in our setting. In fact the existence of very singular solutions is restricted to the parameter strip $0 < p - m < 2/n$. He also considers a more general situation where the spatial domain is bounded and the initial data do not necessarily vanish identically outside of the singular point. We have chosen not to consider this generality in order to be able to use the group invariance properties of the problem in \mathbb{R}^n which enables us to give much simpler proofs.

Let us recall that the transformation

$$(0.6) \quad \mathcal{T}u(x, t) = Ku(Lx, Tt)$$

maps solutions of (0.1) into solutions of $u_t = a\Delta u^m - bu^p$ if K, L, T satisfy

$$K^{p-1}T^{-1} = 1/b \quad , \quad K^{p-m}L^{-2} = a/b$$

for instance if $K = 1$, $T = b$, $L = (b/a)^{1/2}$. Therefore, we may set $a = 1$, $b = 1$ and consider only equation (0.1) without loss of generality.

The following exponents will appear frequently in the sequel:

$$\alpha = \frac{1}{p-1} \quad , \quad \theta = \frac{2}{p-m} \quad , \quad \gamma = \frac{\alpha}{\theta} .$$

The results of this paper have been announced in the note [KPV].

1. Preliminaries

We begin with a lemma which will be convenient in characterizing the solutions of (0.1), (0.2).

Lemma 1.1. *Let u be a nontrivial solution of (0.1), (0.2). Then for any $r > 0$, there exists the limit*

$$(1.1) \quad \lim_{t \rightarrow 0} \int_{|x| < r} u(x, t) dx = c \quad ,$$

where $0 < c \leq \infty$, and c does not depend on r .

Proof: Fix $r > 0$. Then by the continuity of u in $Q \setminus \{(0,0)\}$, there exists a time $T_r > 0$ such that

$$u(x,t) < 1 \quad \text{for } |x| = r, \quad 0 \leq t \leq T_r.$$

We now write

$$\begin{aligned} \int_{B_r} u(x,t) dx &= \int_{B_r} [u(x,t) - 1]_+ dx + \int_{B_r} \{u(x,t) - [u(x,t) - 1]_+\} dx \\ &= I_1(t) + I_2(t), \end{aligned}$$

where $B_r = \{|x| < r\}$ and $[z]_+ = \max\{z, 0\}$.

Multiplication of (0.1) by $\text{sgn}^+(u - 1)$ and integration over $B_r \times [s, t]$, where $0 < s < t < T_r$ yields, after one integration by parts,

$$\int_{B_r} [u(x,t) - 1]_+ dx \leq \int_{B_r} [u(x,s) - 1]_+ dx.$$

Therefore, $I_1(t)$ is nondecreasing as t decreases and $\lim_{t \rightarrow 0} I_1(t)$ exists.

Since the function $u - [u - 1]_+$ is bounded, and converges to zero a.e. in B_r , $I_2(t) \rightarrow 0$ as $t \rightarrow 0$ by the dominated convergence theorem. Thus, both $I_1(t)$ and $I_2(t)$ tend to a nonnegative limit as $t \rightarrow 0$. Since $I_1(t)$ is nondecreasing as $t \rightarrow 0$, this limit must be positive, unless $u \leq 1$ in Q . However, the only bounded solution of (0.1), (0.2) is $u \equiv 0$, which was excluded. #

In the next lemma we collect some properties of fundamental solutions of (0.1), i.e., solutions of (0.1) which satisfy (0.2) and (0.3) with $0 < c < \infty$.

Lemma 1.2. For every $c \in \mathbb{R}^+$ and $1 < p < m + 2/n$ there exists a unique fundamental solution w_c of (0.1) in Q . If $0 < c_1 < c_2 < \infty$, then $w_{c_1} \leq w_{c_2}$ in Q .

Proof: The existence of a fundamental solution in the range of exponents considered has been established in [KP1] (with a definition of fundamental solution which is slightly different but equivalent). It is also shown there that every fundamental solution u which satisfies (0.3) for some c is bounded above by the fundamental solution E_c of the equation without absorption $u_t = \Delta(u^m)$, satisfying (0.3) with the same c . This implies in particular that $u(\cdot, t)$ has compact support and that $\text{supp}(u(\cdot, t))$ is contained in the ball with the origin as centre and radius

$$(1.1) \quad R(t) = C(m, n)t^{1/\beta n}, \quad \beta = m - 1 + 2/n.$$

In addition it was shown that if u and v are two fundamental solutions then

$$(1.2) \quad |u(x, t) - v(x, t)| = o(t^{-1/\beta}) \quad \text{as } t \rightarrow 0 ,$$

uniformly with respect to x . Thus, in view of (1.1), (1.2)

$$\int_{\mathbf{R}^n} |u(x, t) - v(x, t)| dx = o(t^{-1/\beta}) O(R(t)^n) = o(1) \quad \text{as } t \rightarrow 0 .$$

Fix $r > 0$ and $t_0 > 0$ so that $R(t_0) = r$. Then for any $t \in (0, t_0)$, $u(\cdot, t), v(\cdot, t) \in L^1(B_r) \cap L^\infty(B_r)$ and it follows from the contraction property in $L^1(B_r)$ [ACP] that

$$(1.3) \quad \|u(\cdot, t) - v(\cdot, t)\|_{L^1(B_r)} \leq \|u(\cdot, s) - v(\cdot, s)\|_{L^1(B_r)} \quad \text{if } 0 < s < t < t_0 .$$

Since the right hand side of (1.3) can be made arbitrarily small, we can conclude that $u = v$.

The monotonicity of the functions w_c with respect to c follows from the maximum principle; for details we refer to [KP2], where this property was proved for $m = 1$. #

In Lemma 1.2 we found that the family of fundamental solutions $\{w_c : 0 < c < \infty\}$ is ordered. To complete the ordering we prove in the next lemma that for every $c > 0$, the fundamental solution w_c is bounded above by a very singular solution.

Lemma 1.3. *A very singular solution is always an upper bound for the family of fundamental solutions.*

Proof: Let u be a very singular solution and let $c > 0$. For $0 < t < \tau$, and τ small, one selects functions $\varphi_\tau(x)$ such that

$$0 \leq \varphi_\tau \leq u(\cdot, \tau) \quad , \quad \int_{\mathbf{R}^n} \varphi_\tau(x) dx = c .$$

Let u_τ be the solution of (0.1), with initial value φ_τ . Plainly, by the comparison principle,

$$(1.4) \quad u_\tau(x, t) \leq u(x, t + \tau) .$$

Because of (0.2) the functions φ_τ converge weakly to $c\delta$, where δ is the Dirac mass and hence (cf. [KP1]) u_τ converges to the fundamental solution w_c . Thus, if we let $\tau \rightarrow 0$ in (1.4) we obtain

$$w_c(x, t) \leq u(x, t) .$$

Since u was an arbitrary very singular solution and c an arbitrary positive number, the proof is complete. #

2. Case $1 < p \leq m$

We begin by recalling the existence of a flat solution

$$(2.1) \quad U(t) = c^* t^{-\alpha} \quad , \quad c^* = \alpha^\alpha = \left(\frac{1}{p-1} \right)^{\frac{1}{p-1}} \quad ,$$

which is an a priori upper bound for all solutions.

Lemma 2.1. *For every solution of (0.1), (0.2) we have*

$$(2.2) \quad u(x, t) \leq U(t) \quad \text{in } Q \quad .$$

Proof: We recall that we consider solutions which are bounded for $t \geq \tau > 0$. Therefore if we take $\tau > \delta > 0$ and consider

$$u_1(x, t) = u(x, t - \tau)$$

$$u_2(x, t) = U(t - \delta)$$

for δ close enough to 0 we will have $u_1(x, 0) \leq u_2(x, 0)$. Moreover, u_1 and u_2 are bounded solutions of (0.1), hence an easy application of the comparison principle cf. [BKP], gives $u_1 \leq u_2$ in Q . Now let $\tau, \delta \rightarrow 0$ to obtain (2.2). #

The fundamental solutions $\{w_c\}$ form a monotone family of nonnegative solutions of (0.1), (0.2) [KP2]. Since they are bounded above by (2.2), the limit

$$(2.3) \quad V(x, t) = \lim_{c \rightarrow \infty} w_c(x, t)$$

is again a solution of (0.1) and $V \leq U$. Moreover, passing to the limit in (0.3) we will get

$$(2.4) \quad \lim_{t \rightarrow 0} \int_{|x| \leq r} V(x, t) dx = +\infty \quad .$$

Therefore, if V satisfies the initial condition (0.2) it will be a very singular solution of our problem.

This will be the case for $p > m$ but it is not here. In fact we have

Proposition 2.2. *For $1 < p \leq m$*

$$(2.5) \quad \lim_{c \rightarrow \infty} w_c(x, t) = U(t) \quad .$$

Proof: i) We will first show that V is selfsimilar, namely, that it can be written in the form (0.5).

For this we consider the transformation

$$(2.6) \quad (Tu)(x, t) = ku(k^{(p-m)/2}x, k^{p-1}t) \quad .$$

If u is a solution of (0.1), (0.2), Tu is too. Taking $u = w_c$ it follows that Tw_c is again a fundamental solution and precisely

$$(2.7) \quad Tw_c = w_{ck^\mu} \quad , \quad \mu = 1 + \frac{n(m-p)}{2}$$

thanks to the uniqueness of fundamental solutions. Letting $c \rightarrow \infty$ in (2.7) we obtain $TV = V$, which can be rewritten as

$$(2.8) \quad V(x, t) = t^{-\alpha} g(\xi) \quad , \quad \xi = xt^{-\gamma} .$$

Moreover, V is radially symmetric, $V = V(|x|, t)$ and nonincreasing in $|x|$, since so were the w_c 's.

- ii) Let us now prove that $g(\xi) \equiv c^*$ for $p < m$. The inequality $g(\xi) \leq c^*$ is a consequence of the fact that $V(x, t) \leq U(t)$. To prove the converse inequality we use the asymptotic behaviour for the Cauchy-Dirichlet problem of [BNP]. We consider the problem

$$(P_D) \begin{cases} u_t = \Delta u^m - u^p & \text{in } S = B_r(0) \times (0, \infty) \\ u(x, t) = 0 & \text{for } |x| = r, t > 0 \\ u(x, 0) = u_0(x) & \text{for } |x| < r, \end{cases}$$

where $r > 0$ is chosen so that $V(\cdot, t_0) > 0$ in $B_r(0)$ for some t_0 and $u_0 \in C(\overline{B_r(0)})$ is positive in $B_r(0)$, vanishes on the boundary $|x| = r$ and $u(x, 0) \leq V(x, t_0)$. By the results of [BNP] we have

$$(2.9) \quad \lim_{t \rightarrow \infty} u(x, t)t^\alpha = c^* \quad , \quad |x| < r .$$

Since $u(x, 0) \leq V(x, t_0)$ the maximum principle implies that $u(x, t) \leq V(x, t + t_0)$ in S . Hence

$$(2.10) \quad \liminf_{t \rightarrow \infty} g(xt^{-\gamma}) \geq c^*$$

for $|x| < r$. Since $-\gamma > 0$ if $1 < p < m$ we conclude that $g(\infty) = c^*$ and by the monotonicity of g , $g \geq c^*$.

- iii) The case $p = m$ is somewhat different, since we do not have the asymptotic behaviour (2.9) for the solutions of problem (P_D) . We will prove below that this asymptotic convergence is nevertheless true for the nonnegative solutions $u \not\equiv 0$ of the Cauchy problem. Once this is established we make a comparison with the solution of (P_D) to obtain as in (2.9), (2.10) with $\gamma = 0$

$$(2.11) \quad \liminf_{t \rightarrow \infty} g(xt^{-\gamma}) = g(x) \geq c^*$$

for every x such that $V(\cdot, t_0) > 0$ in $B_r(0)$ for $r > |x|$. Now it is well known, cf. [BKP], that for $p = m$ nonnegative solutions of (0.1) are not localized, i.e., if $u_0 \geq 0$, $u_0 \not\equiv 0$ then the sets $\Omega(t) = \{(x, t) : u(x, t) > 0\}$ form a nondecreasing family and

$$(2.12) \quad \bigcup_{t>0} \Omega(t) = \mathbb{R}^n .$$

Thus for any fixed $x_0 \in \mathbb{R}^n$ there exists t_0 large enough such that (2.11) above holds true for x_0 .

The proof of Proposition 2.2 will be complete after we establish the asymptotic behaviour.

Lemma 2.3. *If u is a nontrivial, nonnegative solution of (0.1) with $p = m$, then (2.9) holds for every $x \in \mathbb{R}^n$.*

Proof: The behaviour of the solutions of the Cauchy-Dirichlet problem in a bounded spatial domain Ω is considered in [BNP], where it is proved that for every nonnegative, nontrivial solution u of (0.1) $u(x, t)t^\alpha$ converges towards the positive solution $f = f(x)$ of the problem

$$(2.13.a) \quad \Delta(f^m) + \alpha f - f^m = 0 \quad \text{in } \Omega ,$$

$$(2.13.b) \quad f = 0 \quad \text{on } \partial\Omega .$$

Let us take $\Omega = B_R(0)$, $R > 0$ and write $f = f_R(x)$. f_R is unique, positive, radially symmetric and nonincreasing with respect to $|x|$. Moreover, the family $\{f_R\}$ is nondecreasing in R (apply the maximum principle in B_R to f_R and $f_{R'}$ with $R' > R$). It is also bounded above by c^* by (2.2).

The limit

$$(2.14) \quad F(x) = \lim_{R \rightarrow \infty} f_R(x)$$

is again a radially symmetric, positive solution of (2.13.a) in \mathbb{R}^n , it is nonincreasing in $|x|$, and it is bounded above by c^* . Suppose that $F \not\equiv c^*$. Then

$$(2.15) \quad \lim_{|x| \rightarrow \infty} F(x) = c < c^*$$

If $c > 0$ we have

$$(2.16) \quad \Delta(F^m) = F^m - \alpha F \longrightarrow c(c^{m-1} - \alpha) \equiv A < 0$$

as $|x| \rightarrow \infty$, which is a contradiction with the fact that $G = F^m$ is bounded, since we have for all large r

$$(2.17) \quad (r^{n-1}G'(r))' \approx -Ar^{n-1} ,$$

hence $G'(r) \approx -Ar/n$ and $G(r) \approx -Ar^2/(2n)$. On the other hand, if $c = 0$ we argue as follows. We have

$$\Delta F^m = \frac{m}{m-1} F \Delta(F^{m-1}) + \frac{m}{(m-1)^2} F^{\frac{2-m}{m-1}} |\nabla(F^{m-1})|^2 = F^m - \alpha F,$$

so that

$$\Delta(F^{m-1}) \leq \frac{m-1}{m} (F^{m-1} - \alpha) \longrightarrow -\frac{\alpha(m-1)}{m} < 0$$

and we arrive at a contradiction with the boundedness of F^{m-1} . #

From Lemma 1.3 it follows that there exists no very singular solution for $1 < p \leq m$.

3. $p > m$

For the range of exponents $p > m$ there exist a priori, interior bounds for the solutions of (0.1). In this paper we use only one of them, but since they are interesting in their own right, we shall present all three of them.

Lemma 3.1. *Let u be a solution of (0.1) in a cylindrical domain $S = \Omega \times (0, T)$ where $\Omega \subset \mathbb{R}^n$. If $p > m > 1$ then there exists $c = c(m, p, n) > 0$ such that*

$$(3.1) \quad u(x, t) \leq c(m, p, n) \left(\frac{1}{d(x)^\theta} + \frac{1}{t^\alpha} \right) \quad \text{in } S,$$

where $\alpha = 1/(p-1)$, $\theta = 2/(p-m)$ and $d(x)$ denotes the distance from $x \in \Omega$ to the boundary $\partial\Omega$.

Proof: Let $x_0 \in \Omega$ and $0 < t_1 < T$ and let $k = d^2(x_0)/t_1$. We consider in $S_1 = \{(x, t) : |x - x_0|^2 < kt, 0 < t \leq t_1\}$ the function

$$(3.2) \quad U(x, t) = \frac{C}{(kt - r^2)^\theta},$$

with $r = |x - x_0|$ and C a constant to be chosen later. We want to show that for C large enough $U \geq u$ in S_1 . In fact $U = \infty$ on the parabolic boundary of S_1 . Moreover, if $\omega = kt - r^2$ we have

$$(3.3) \quad \begin{aligned} L(U) \equiv U_t - \Delta U^m + U^p &= -kC\theta\omega^{-(\theta+1)} - 2mn\theta C^m \omega^{-(\theta m+1)} \\ &\quad - 4\theta m(\theta m + 1)C^m r^2 \omega^{-(m\theta+2)} + C^p \omega^{-\theta p} \end{aligned}$$

so that $LU \geq 0$ if the following inequalities hold

$$(3.4) \quad \begin{aligned} \frac{1}{3}C^{p-m} &\geq 4\theta m(\theta m + 1)r^2 \\ \frac{1}{3}C^{p-m} &\geq 2mn\omega \\ \frac{1}{3}C^{p-1} &\geq k\theta\omega^{\theta(m-1)+1} \end{aligned}$$

Since $r^2, \omega \leq d^2(x_0) = kt_1$, (3.4) is satisfied if

$$(3.5) \quad C = c(m, n, p) [d(x_0)^\theta + k^\alpha d(x_0)^{2(\theta-\alpha)}] .$$

It follows from the maximum principle applied to u and U in S_1 that $u \leq U$, hence

$$u(x_0, t_1) \leq U(x_0, t_1) = \frac{C}{(kt_1)^\theta} = c \left[\frac{1}{d(x_0)^\theta} + \frac{1}{t_1^\alpha} \right] \quad \#$$

Estimate (3.1) can be improved if we know that the initial data are bounded.

Lemma 3.2. *Let u be a solution of (0.1) as in Lemma 3.1 and assume that $u(x, 0) \leq M$ in Ω .*

Then

$$(3.7) \quad u(x, t) \leq c \frac{1}{d(x)^\theta} + M$$

where c depends on m, n, p .

Proof: The argument is similar to the one in Lemma 3.1. Now we take a time-independent comparison function

$$(3.8) \quad U = \frac{C}{(R^2 - r^2)^\theta}$$

with $R = d(x_0)$, $r = |x - x_0|$ and domain $S_1 = B_R(x_0) \times (0, T)$. We will have

$$(3.9) \quad LU \equiv U_t - \Delta U^m + U^p \geq 0$$

if

$$(3.10) \quad C \geq c(m, p, n) R^\theta .$$

On the other hand $U(x, 0) \geq u(x, 0)$ if

$$(3.11) \quad CR^{-2\theta} \geq M .$$

(3.10), (3.11) are satisfied if we take

$$(3.12) \quad C = c(m, n, p) R^\theta + MR^{2\theta}$$

then $u \leq U$ in S_1 , so that

$$(3.13) \quad u(x_0, t) \leq U(x_0) = \frac{c}{R^\theta} + M \quad \#$$

For the solutions considered in this paper, a slightly different estimate gives boundness in $Q^\tau = \mathbb{R}^n \times (\tau, \infty)$, $\tau > 0$.

Lemma 3.3. *Let u be a solution of (0.1) in Q with $u(x, 0) = 0$ for every $x \neq 0$. Then there exists $c = c(m, p, n) > 0$ such that*

$$(3.14) \quad u(x, t) \leq \frac{c}{|x|^\theta + t^\alpha} .$$

Proof: i) Given $x_0 \neq 0$ and $t_1 \in (0, T)$ we consider the function

$$(3.15) \quad U = \frac{C}{(R^2 + t - r^2)^\theta} , \quad r = |x - x_0| ,$$

in the domain $S_1 = \{(x, t) : |x - x_0|^2 < R^2 + t, 0 < t < t_1\}$. We assume that $0 < R < |x_0|$ and $C > 0$. These precise values will be determined later. Comparison on the parabolic boundary of S_1 gives $u \leq U$. On the other hand $LU \equiv U_t - \Delta U^m + U^p \geq 0$ if for instance

$$(3.16) \quad C = c_1(R^2 + t_1)^{\theta/2} + c_2(R^2 + t_1)^{\theta - \alpha} ,$$

where c_1, c_2 depend only on m, n, p . We conclude that $u \leq U$ in S_1 , so that letting $R \rightarrow |x_0|$ we get

$$(3.17) \quad u(x_0, t_1) \leq c_1(|x_0|^2 + t_1)^{-\theta/2} + c_2(|x_0|^2 + t_1)^{-\alpha} .$$

ii) The second term in (3.17) can be eliminated by means of a rescaling argument. Given a solution u we consider the rescaled solution \tilde{u} given by

$$(3.18) \quad u(x, t) = R^{-\theta} \tilde{u} \left(\frac{x}{R}, \frac{t}{R^{1/\gamma}} \right)$$

and apply (3.17) to \tilde{u} at the point $y = \frac{x}{R}$ with $R = |x|$. We have

$$u(x, t) \leq R^{-\theta} \left\{ c_1 \left(1 + \frac{t}{R^{1/\gamma}} \right)^{-\theta/2} + c_2 \left(1 + \frac{t}{R^{1/\gamma}} \right)^{-\alpha} \right\}$$

since $\alpha < \theta/2$, $1 + tR^{-1/\gamma} > 1$ and $\gamma\theta = \alpha$ we get

$$(3.19) \quad \begin{aligned} u(x, t) &\leq R^{-\theta} \cdot 2c_2(1 + tR^{-1/\gamma})^{-\alpha} = 2c_2(R^{1/\gamma} + t)^{-\alpha} \\ &\leq c_3(R^\theta + t^\alpha)^{-1} , \end{aligned}$$

which finishes the proof. #

These bounds fail for $1 < p \leq m$, as explicit examples show. Thus if $p < m$ the function

$$(3.20) \quad u(x, t) = c|x|^\lambda$$

is a stationary solution of (0.1) if $\lambda = 2/(m - p) = -\theta > 0$ and $c = c(m, p, n)$ is suitably chosen.

For $p = m$ a stationary solution is

$$(3.21) \quad u = c \exp(x_1/m) .$$

Both (3.20), (3.21) fail to be uniformly bounded for $t > 0$.

An interesting consequence of estimates (3.7), (3.14) for the solutions of problem (0.1), (0.2) is the fact that we can control a priori the support of the solutions.

Lemma 3.4. *Every nonnegative solution of (0.1), (0.2) with $p > m$ vanishes outside a set of the form*

$$(3.22) \quad \{(x, t) : |x| < Ct^\gamma\} \quad , \quad C = C(m, n, p) > 0 .$$

Proof: i) Consider a solution of (0.1), (0.2) defined in $\mathbb{R}^n \times (0, T)$ with $T > 1$. By (3.7) or (3.14), given $r > 0$, there exist $C = C(m, n, p, r)$ such that $u(x, t) \leq C$ for $|x| = r$ and $0 < t < T$. We now take a Barenblatt solution with a large mass $\hat{u}(x, t)$ so that $\hat{u}(x, t) \geq C$ for $|x| = r$ and $1 < t < T + 1$. Since $\hat{u}(x, t + 1)$ is a supersolution of (0.1) in $S = (\mathbb{R}^n \setminus B_r(0)) \times (0, T)$ ($L\hat{u} = \hat{u}_t - \Delta\hat{u}^m + \hat{u}^p = \hat{u}^p \geq 0$) and $\hat{u}(x, t + 1) \geq u(x, t)$ on the parabolic boundary of S we conclude that $u \leq \hat{u}$ in S . In particular $u(x, t)$ will have compact support for every $t \in (0, T)$. Specifically, for $t = 1$, we get

$$(3.23) \quad u(x, 1) = 0 \quad \text{if} \quad |x| \geq r_1 ,$$

r_1 does not depend on u .

ii) For an arbitrary time $t_1 \in (0, T)$, $T > 0$ we consider the transformation

$$(3.24) \quad u(x, t) = t_1^{-\alpha} \tilde{u} \left(\frac{x}{t_1^\gamma}, \frac{t}{t_1} \right) ,$$

i.e., (3.18) with $R = t_1^\gamma$. Since \tilde{u} is defined in $(0, T/t_1)$, $T/t_1 > 1$ we have $\tilde{u}(y, 1) = 0$ for $|y| \geq r_1$, i.e., $u(x, t_1) = 0$ for $|x| \geq t_1^\gamma r_1$. Put now $C = r_1$. #

4. The Case $p > m$. Proof of the Theorem

We consider the cases $m < p < m + (2/n)$ and $p \geq m + (2/n)$ in succession.

Case I. $m < p < m + (2/n)$.

Let u be a solution of (0.1), (0.2). By Lemma 1.1 it satisfies (0.3) for some $c \in (0, \infty]$. If $c < \infty$, then by Lemma 1.2 $u = w_c$, and if $c = \infty$ then $u = W$, where W is the unique very singular solution in this parameter range.

Case II. $p \geq m + (2/n)$.

We need a preliminary bound. Let

$$V(x, t) = t^{-\alpha} f(|x|t^{-\gamma}) ,$$

in which $f(\xi) = c^*$ when $0 \leq \xi \leq C(m, n, p)$ and $f(\xi) = 0$ when $\xi > C(m, n, p)$, where the constant $C(m, n, p)$ has been defined in Lemma 3.4.

Lemma 4.1. *Let $p > m$, and let u be a solution of (0.1), (0.2). Then*

$$u(x, t) \leq V(x, t) \quad \text{in } Q .$$

This bound is an immediate consequence of Lemmas 2.1 and 3.4.

We are now ready to complete the proof of the Theorem.

Lemma 4.2. *Let $p \geq m + (2/n)$. Then no singular solutions exist.*

Proof: Suppose w is a singular solution. Then by Lemma 4.1,

$$\begin{aligned} I(t) &= \int_{\mathbf{R}^n} w(x, t) dx \leq \int_{\mathbf{R}^n} t^{-\alpha} f(|x|t^{-\gamma}) dx \\ &= |S_1| t^{-\alpha+n\gamma} \int_0^\infty f(\xi) \xi^{n-1} d\xi \\ &= K t^{-\alpha+n\gamma} , \end{aligned}$$

where $K = c^* |S_1| n^{-1} (C(n, m, p))^n$.

Assume first that $p > m + (2/n)$. Then

$$-\alpha + n\gamma = -\frac{1}{p-1} + \frac{n(p-m)}{2(p-1)} = \frac{n(p-m)-2}{2(p-1)} > 0$$

and hence

$$\int_{\mathbf{R}^n} w(x, t) dx \longrightarrow 0 \quad \text{as } t \rightarrow 0 .$$

This contradicts Lemma 1.1.

Next, let $p = m + (2/n)$. Then

$$I(t) \leq K \quad \text{for } 0 < t < T$$

which means that neither a very singular solution, nor a fundamental solution with $c > K$ can exist.

We now complete the proof with the observation – formulated in the next lemma – that if there exists *one* value of $c > 0$, for say c_1 , which there exists a fundamental solution such that (0, 3) is satisfied, then necessarily, there exists for any $c \geq c_1$ a fundamental solution which satisfies (0.3). Since this is ruled out, there cannot exist any fundamental solution.

Lemma 4.3. *Suppose there exists a fundamental solution with initial mass $c_1 > 0$. Then there exists a fundamental solution for every initial mass $c \geq c_1$.*

Proof: We may suppose, without loss of generality, that $c_1 = 1$. Denoting the corresponding fundamental solution by u_1 , one easily verifies that for $a > 1$, the function

$$(4.1) \quad \tilde{u}(x, t) = a^{2/(m-1)} u_1 \left(\frac{x}{a}, t \right)$$

is a supersolution of (0.1) with initial trace $c\delta$, where $c = a^{n+(2/(m-1))} > 1$.

Consider for $\tau > 0$, the solutions u_τ of (0.1) such that

$$(4.2) \quad u_\tau(x, 0) = \tilde{u}(x, \tau).$$

Then, since \tilde{u} is a supersolution,

$$(4.3) \quad 0 \leq u_\tau(x, t) \leq \tilde{u}(x, t + \tau).$$

These bounds allow us to let $\tau \rightarrow 0$ and obtain in the limit a function u which satisfies (0.1) and (0.2), and thus, by Lemma 1.1, has initial trace $c'\delta$, where $c' \leq c$ by (4.3).

To prove that $c' = c$, we observe that by definition u_τ satisfies the identity

$$(4.4) \quad \iint_Q \left(u_\tau \frac{\partial \zeta}{\partial t} + u_\tau^m \Delta \zeta - u_\tau^p \zeta \right) dx dt = 0$$

for any test function $\zeta \in C_0^\infty(Q)$. Proceeding as in [KP1], we choose $\zeta(x, t) = \eta(t)\chi(x)$, where $\eta \in C_0^\infty(0, T)$ has the properties $0 \leq \eta \leq 1$ and

$$\eta(t) = \begin{cases} 0 & 0 < t < s, \\ 1 & 2s \leq t \leq T - 2s, \\ 0 & T - s < t < T, \end{cases}$$

where s is chosen positive and small, and χ is any function in $C_0^\infty(\mathbb{R}^n)$. Then, if we substitute ζ in (4.4) and let $s \rightarrow 0$, we obtain

$$(4.5) \quad \int_{\mathbb{R}^n} u_\tau(x, 0)\chi(x)dx - \int_{\mathbb{R}^n} u_\tau(x, T)\chi(x)dx = - \iint_Q u_\tau^m \Delta \chi(x) dx dt + \iint_Q u_\tau^p(x, t)\chi(x)dx.$$

Now choose χ so that for some R , to be selected shortly, $\chi(x) = 1$ if $|x| \leq R$ and $\chi(x) = 0$ if $|x| \geq 2R$. Recall that u_1 has compact support. Hence, by choosing R sufficiently large, we can ensure that the first integral on the right of (4.5) is equal to zero. The second integral is bounded, because it is also possible to derive an equality like (4.5) for u_1 . Hence, remembering the definition (4.2) of $u_\tau(x, 0)$, we obtain in view of (4.3)

$$(4.6) \quad \left| \int_{\mathbb{R}^n} \{ \tilde{u}(x, \tau) - u_\tau(x, T) \} \chi(x) dx \right| \leq \int_0^{T+\tau} dt \int_{\mathbb{R}^n} \tilde{u}^p(x, t) dx.$$

Letting $\tau \rightarrow 0$ we obtain

$$\left| c - \int_{\mathbb{R}^n} u(x, T)\chi(x)dx \right| \leq \int_0^T dt \int_{\mathbb{R}^n} \tilde{u}^p(x, t) dx$$

from which we conclude that

$$\int_{\mathbb{R}^n} u(x, T)\chi(x)dx \longrightarrow c \quad \text{as} \quad T \rightarrow 0,$$

i.e., $c' = c$, which concludes the proof. #

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