SINGULARITIES
BANACH CENTER PUBLICATIONS, VOLUME 20
PWN POLISH SCIENTIFIC PUBLISHERS
WARSAW 1988

CLASSIFICATION OF SINGULARITIES WITH COMPACT ABELIAN SYMMETRY

GORDON WASSERMANN

Department of Mathematics, Ruhr-University Bochum,
Bochum, Federal Republic of Germany

In this paper I wish to report on some results which, though by no means recent, have never previously been published in a generally available form, and may therefore still be of some interest despite their age.

In many applications of singularity theory, there is an underlying symmetry which results either from experimental constraints or from laws of nature, and symmetry has a significant effect on the mathematical description, in particular because phenomena which are ordinarily not stable or generic may become stable and generic, and thus amenable to treatment by singularity theory, when their symmetry is taken into account. For this reason it becomes interesting to classify singularities in the presence of symmetry.

Fortunately, although the details of what is stable or generic change when symmetry is present, the general framework of the "classical" Thom–Mather singularity theory remains valid (so long as the symmetry group is compact). In [6], V. Poènaru proved a "G-invariant" version of the Malgrange preparation theorem, and using this it is fairly easy to generalize most of the Thom–Mather theory to the symmetric case, as was done for the theory of unfoldings and of stable mappings by Poènaru in the same book. In his *Diplomarbeit* [4], M. Beer developed a symmetric version of the theory of finite determinacy; to a large extent, the results and their proofs directly mirror the non-symmetric case. Beer also gives a classification of singularities for the case of Z_2-symmetry. This was the starting point for the present work.

Abelian compact Lie groups have particularly simple irreducible representations. Using this fact it is possible to obtain a general description for abelian symmetry which enables one to treat all abelian Lie groups simultaneously when classifying symmetric singularities. In the present paper, we present a classification of symmetric germs, up to symmetric codimension 4, for an arbitrary abelian compact symmetry group. (As a prerequisite to this classification we present a G-invariant splitting lemma, which is valid also in the nonabelian case). In addition, we extend Arnold's notion of *simple germs* to the symmetric case and classify the simple germs with compact abelian symmetry.

Because of space restrictions we shall omit all computations and proofs. These may be found in full detail in [12].

We should impress upon the reader that we restrict to the abelian case only for the actual classification; the underlying symmetric singularity theory, including the parts we develop in this paper, is valid for symmetry by arbitrary *compact* Lie groups.

This paper relies heavily on the work of Poènaru [6] and Beer [4]; most of the basic results of symmetric singularity theory are due to them.

1. Preliminaries

DEFINITION 1.1. Let G be a compact Lie group acting linearly on the Euclidean spaces $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$. We shall denote by $\mathscr{E}_G(n, p)$ the set of germs, at $0 \in \boldsymbol{R}^n$, of smooth G-equivariant mappings $\boldsymbol{R}^n \to \boldsymbol{R}^p$ (f is *G-equivariant* if $f(gx) = gf(x)$ for all $g \in G$, x near 0 in $\boldsymbol{R}^n$), and we set $\mathfrak{m}_G(n, p) = \{f \in \mathscr{E}_G(n, p) \mid f(0) = 0\}$. If $p = 1$ and G acts trivially on $\boldsymbol{R}^p = \boldsymbol{R}$ (i.e. $gx = x$ for all $g \in G$, $x \in \boldsymbol{R}$), then we shall write simply $\mathscr{E}_G(n)$ for $\mathscr{E}_G(n, 1)$ and $\mathfrak{m}_G(n)$ instead of $\mathfrak{m}_G(n, 1)$; the elements of $\mathscr{E}_G(n)$ are called *G-invariant smooth function germs* at $0 \in \boldsymbol{R}^n$. With the usual multiplication of real-valued function germs, $\mathscr{E}_G(n)$ becomes a local ring, $\mathfrak{m}_G(n)$ is its unique maximal ideal, and $\mathscr{E}_G(n, p)$ and $\mathfrak{m}_G(n, p)$ are modules over $\mathscr{E}_G(n)$ in the obvious way.

If G acts trivially on *both* $\boldsymbol{R}^n$ and $\boldsymbol{R}^p$ (or if we choose to *forget* for a moment the action of G), then we shall omit the subscript G in this notation and we obtain the ring $\mathscr{E}(n)$, the ideal $\mathfrak{m}(n)$, and the modules $\mathscr{E}(n, p)$, $\mathfrak{m}(n, p)$ of classical singularity theory without symmetry.

The sets $\mathfrak{n}_G^k(n) = \mathfrak{m}(n)^k \cap \mathscr{E}_G(n)$ are ideals of $\mathscr{E}_G(n)$. These and the powers of $\mathfrak{m}_G(n)$ will be needed below.

To apply the usual algebraic methods of singularity theory it is important that the ideals and modules defined above be finitely generated over $\mathscr{E}_G(n)$. This is a consequence of

THEOREM 1.2 (Hilbert's finitude theorem). *Let G be a compact Lie group acting linearly on $\boldsymbol{R}^n$, and denote by $\boldsymbol{R}_G[X_1, \dots, X_n]$ the ring of G-invariant real polynomials on $\boldsymbol{R}^n$. Then there exist finitely many homogeneous G-invariant polynomials $\varrho_1, \dots, \varrho_k$ which generate $\boldsymbol{R}_G[X_1, \dots, X_n]$ as an $\boldsymbol{R}$-algebra.*

THEOREM 1.3 (Schwarz's finitude theorem ([8])). *Let G and $\varrho_1, \ldots, \varrho_k$ be as in Theorem 1.2, and let ϱ: $\boldsymbol{R}^n \to \boldsymbol{R}^k$ be the map $(x_1, \ldots, x_n) \mapsto (\varrho_1(x_1, \ldots, x_n), \ldots, \varrho_k(x_1, \ldots, x_n))$. Define ϱ^*: $\mathscr{E}(k) \to \mathscr{E}_G(n)$ by $\varrho^*(f) = f \circ \varrho$. Then ϱ^* is surjective.*

Proofs of these two theorems and of the fact that $\mathscr{E}_G(n, p)$ and $\mathfrak{m}_G(n, p)$ are finitely generated over $\mathscr{E}_G(n)$ can be found in [6].

A further consequence is the easy but important

LEMMA 1.4. *For every integer $k \geqslant 0$ there is an integer $r \geqslant 0$ such that $\mathfrak{m}_G^r(n) \subseteq \mathfrak{m}_G(n)^k$.*

We are interested in classifying symmetric germs up to the following equivalence:

DEFINITION 1.5. Let G be a compact Lie group acting linearly on $\boldsymbol{R}^n$. Set $L_G(n) = \{\phi \in \mathscr{E}_G(n, n) | \ \phi$ is non-singular at $0\}$; this is a group with composition of map-germs as the group operation.

Let f and h be germs in $\mathfrak{m}_G(n)$. We say f is *G-right equivalent* to $h (f \sim_G h)$ if there is a $\phi \in L_G(n)$ such that $f = h \circ \phi$.

In many cases it is possible, up to equivalence, to replace a germ by a finite portion of its Taylor series (i.e., a finite jet), which provides an important simplification:

DEFINITION 1.6. Let G be a compact Lie group acting linearly on $\boldsymbol{R}^n$, and let $f \in \mathfrak{m}_G(n)$. We say that f is *strongly k-determined* if for any $h \in \mathfrak{m}_G(n)$ such that $f - h \in \mathfrak{m}_G^{k+1}(n)$ we have $f \sim_G h$. We say that f is *G-k-determined* if for any $h \in \mathfrak{m}_G(n)$ such that $f - h \in \mathfrak{m}_G(n)^{k+1}$ we have $f \sim_G h$.

We say that f is *G-finitely determined* if for *some* k f is strongly k-determined (by virtue of Lemma 1.4. this is equivalent to requiring that f be G-k-determined for some k).

For $f \in \mathfrak{m}_G(n)$, the *strong k-determinacy* (resp. *G-k-determinacy*) of f is the smallest k for which f is strongly k-determined (resp. G-k-determined), or ∞ if f is not G-finitely determined. The determinacies are invariants of G-right equivalence.

An important technique of singularity theory is to study singular germs by studying the "tangent spaces" of their right-equivalence classes. What more or less plays the role of the tangent space is a certain ideal which can be associated to a germ, its *Jacobian ideal*:

DEFINITION 1.7. Let the Lie group G operate linearly on $\boldsymbol{R}^n$. Then the differential of the action of the elements of G defines an action of G on the tangent bundle $T\boldsymbol{R}^n$, such that the projection $T\boldsymbol{R}^n \to \boldsymbol{R}^n$ is equivariant. A vector field germ X on $\boldsymbol{R}^n$ will be said to be *G-equivariant* if it is equivariant as a map-germ $\boldsymbol{R}^n \to T\boldsymbol{R}^n$, with respect to the above actions of G.

Since $TR^n = R^n \times R^n$ is trivial, by projecting into the fibre R^n we may identify any vector field germ X on R^n with a map-germ $R^n \to R^n$, and X is G-equivariant if and only if this map-germ is G-equivariant. Thus the set of G-equivariant vector field germs on R^n may be identified with $\mathscr{E}_G(n, n)$.

If X is a vector field germ on R^n and $f \in \mathscr{E}_G(n)$, then $X(f)$ will denote the directional derivative of f in the direction of X.

Definition 1.8. Let the Lie group G operate linearly on R^n. If $f \in \mathscr{E}_G(n)$, we set

$$J_G(f) = \{X(f) \mid X \text{ is a } G\text{-equivariant vector field germ on } R^n\}.$$

It is easy to see that this is an ideal of $\mathscr{E}_G(n)$; it is called the *G-Jacobian ideal of f*.

If G is compact, then $J_G(f)$ is finitely generated over $\mathscr{E}_G(n)$ (essentially because $\mathscr{E}_G(n, n)$ is a finitely-generated module).

Since there are obviously infinitely many equivalence classes of singular germs, it will not be possible to classify them all, and so it makes sense to begin with the "most generic" and "least complicated" ones. To do this one needs a way of measuring genericity. Although there are several such measures, the most useful and in a sense most natural one is the *codimension*:

Definition 1.9. Let the Lie group G act linearly on R^n, and let $f \in \mathfrak{m}_G(n)$. We define the *G-codimension of f* to be

$$\operatorname{cod}_G(f) = \dim_R \mathfrak{m}_G(n)/(J_G(f) \cap \mathfrak{m}_G(n)).$$

Since $J_G(f)$ is more or less the tangent space of the G-right equivalence class of f, one may interpret $\operatorname{cod}_G(f)$ as the codimension of the G-right equivalence class (in a suitable space). In particular, the lower the G-codimension, the larger the G-right equivalence class, and so the more generic the germ.

We shall attempt to classify germs of low codimension, and it is a fortunate fact that this means that we may always assume we are dealing with polynomials:

Theorem 1.9 (Beer [4]). *Let G be a compact Lie group acting linearly on R^n, and let $f \in \mathfrak{m}_G(n)$. Then f is G-finitely determined if and only if* $\operatorname{cod}_G(f)$ *is finite.*

If $f \in \mathfrak{m}(n)$, we recall from the standard Thom–Mather theory that an *r-dimensional unfolding of f* is a smooth germ F defined on $R^n \times R^r = R^{n+r}$ near 0, i.e. a germ $F \in \mathscr{E}(n+r)$, such that for $x \in R^n$ we have $F(x, 0) = f(x)$, where 0 denotes the origin of R^r. In other words, F is the germ of an r-parameter family of functions which contains the germ f.

If the Lie group G acts linearly on R^n, then we may extend the action of G to $R^n \times R^r$, for any $r \geqslant 0$, by letting G act trivially on the second factor R^r, i.e., for $x \in \boldsymbol{R}^n$, $u \in \boldsymbol{R}^r$, and $g \in G$, we set $g(x, u) = (gx, u)$. An r-dimensional unfolding F of a germ $f \in \mathfrak{m}(n)$ will be said to be a *G-unfolding* if F is G-invariant, i.e. $F \in \mathscr{E}_G(n+r)$, with respect to the above action (f is then necessarily also G-invariant, i.e. in $\mathfrak{m}_G(n)$).

Now it is very easy to carry the standard Mather theory of unfoldings (as presented for example in [5] or [11]) over to the G-symmetric case and develop an analogous theory of G-unfoldings, simply by everywhere adding the requirement that all function germs be G-invariant and all mapping germs be G-equivariant with respect to the action of G defined above (which acts trivially on the unfolding parameters). In particular one can define, in an entirely analogous way to the standard case, G-morphisms and G-isomorphisms between G-unfoldings, and the concept of G-universal unfoldings of G-invariant germs, and one can prove results analogous to most of the standard ones by exactly the same methods. We shall not elaborate, except to mention the following main theorem, proved by Beer [4] and in a slightly different and slightly weaker form by Poènaru [6]:

THEOREM 1.10. *Let G be a compact Lie group acting linearly on $\boldsymbol{R}^n$, and let $f \in \mathfrak{m}_G(n)$.*

(i) *f has G-universal unfoldings if and only if f is G-finitely determined.*

(ii) *Any two G-universal unfoldings of f of the same unfolding dimension are G-isomorphic.*

(iii) *If f is G-finitely determined, then the minimal unfolding dimension of a G-universal unfolding of f is* $\operatorname{cod}_G(f)$. *Moreover, if $b_1, \ldots, b_r \in \mathfrak{m}_G(n)$ are representatives of a basis of $\mathfrak{m}_G(n)/(J_G(f) \cap \mathfrak{m}_G(n))$, then the r-dimensional G-unfolding*

$$F(x, u) = f(x) + u_1 b_1(x) + \ldots + u_r b_r(x) \qquad (x \in \boldsymbol{R}^n,\ u = (u_1, \ldots, u_r) \in \boldsymbol{R}^r)$$

is G-universal.

We see here that the codimension also tells us the minimal unfolding dimension of a universal unfolding.

Remark. It is an important restriction that above we have required all of the germs in the family of germs represented by an unfolding to have the same G-symmetry, and have required G to act trivially on the unfolding space. If instead one allows arbitrary actions of G on $\boldsymbol{R}^n \times \boldsymbol{R}^r$ extending a given action on $\boldsymbol{R}^n \times \{0\} \cong \boldsymbol{R}^n$, then one obtains entirely different universal unfoldings, namely, the minimal universal unfolding is the same as one has *without* symmetry, but it is in fact G-invariant with respect to a suitable natural G-action on the unfolding space; for details, see [9]. Essentially what happens here is that in the unfolding the symmetry of the unfolded germ is broken by partially transferring it to the unfolding space.

Unfoldings are important because they describe what happens when a singular germ is perturbed. This suggests another way of measuring the complexity of a germ – some germs, although quite non-generic (i.e. of high codimension), may still show very uncomplicated behaviour upon perturbation. Essentially one may use the dimension of the set of equivalence classes one encounters upon perturbation of a germ as a measure of complexity – this "dimension" is called the *modality* of the germ. We shall give precise definitions only for the case where the modality is 0 – germs of modality 0 are appropriately called *simple*. This concept was introduced, for the nonsymmetric case, by V. I. Arnold [2]. Our definition for the case of symmetry differs slightly from Arnold's but is equivalent to the analogue of his definition if the symmetry group is compact:

DEFINITION 1.11. Let G be a compact Lie group acting linearly on $\boldsymbol{R}^n$. Let $f \in \mathfrak{m}_G(n)$ and let $F \in \mathfrak{m}_G(n+r)$ be a G-unfolding of f. We shall say F *meets only finitely many G-right equivalence classes* if there is a G-invariant neighbourhood U of $0 \in \boldsymbol{R}^{n+r}$ and a G-invariant function $F'\colon U \to \boldsymbol{R}$, whose germ at 0 is F, such that for $u \in \boldsymbol{R}^r$ such that $(0, u) \in U$, the germs $F_u \in \mathfrak{m}_G(n)$, defined by $F_u(x) = F'(x, u) - F'(0, u)$ $(x \in \boldsymbol{R}^n)$, belong to only finitely many different G-right equivalence classes.

A germ $f \in \mathfrak{m}_G(n)$ is called G-simple if every G-unfolding of f meets only finitely many G-right equivalence classes.

Remark. The property of being G-simple is obviously invariant under G-right equivalence.

LEMMA 1.12 ([12, Lemma 1.35]). *Let G be a compact Lie group acting linearly on $\boldsymbol{R}^n$, and let $f \in \mathfrak{m}_G(n)$. If f is G-simple, then f is G-finitely determined.*

2. The splitting lemma

Let G be a compact Lie group acting linearly on an n-dimensional real vector space V. Then $V \cong \boldsymbol{R}^n$, and we may even choose suitable real coordinates $(x_1, \ldots, x_n)$ on V with respect to which G acts orthogonally. In particular, the quadratic form $x_1^2 + \ldots + x_n^2$ is then G-invariant. A G-invariant quadratic form q on V will be called *standard* if in some system of real linear coordinates on V with respect to which G acts orthogonally q takes on the above form.

In the following, we shall always assume that G is a compact Lie group acting linearly on $\boldsymbol{R}^n$. Then $\boldsymbol{R}^n$ decomposes as a direct sum of irreducible G-spaces; we choose such a decomposition $\boldsymbol{R}^n = V_1 \oplus \ldots \oplus V_k$.

As in the nonsymmetric case, an important first step in the problem of classifying singularities is the following *splitting lemma*:

LEMMA 2.1 (splitting lemma ([12])). *Let G and the V_i be as above, and let $f \in \mathfrak{m}_G(n)$ be singular at* 0 (i.e. $f \in \mathfrak{n}_G^2(n)$). *Then there is a subset A of*

$\{1, 2, \ldots, k\}$, *there is a smooth G-invariant real function germ* η *defined on* $W = \bigoplus_{i \in A} V_i$, *and in* $\mathfrak{m}_G^3$ *there, and for each* $j \notin A$ *there is a standard quadratic form* q_j *on* V_j *and a number* $\varepsilon_j = \pm 1$, *such that*

$$(*) \qquad f \sim_G \eta + \sum_{j \notin A} \varepsilon_j q_j.$$

Moreover, the G-representation type of the G-space W, the G-right equivalence class of η, *and the numbers and G-representation types of the* V_j, $j \notin A$ *for which* $\varepsilon_j = +1$ *and for which* $\varepsilon_j = -1$ *are uniquely determined by the G-right equivalence class of f; they also do not depend on the choice of the decomposition* $\mathbf{R}^n = V_1 \oplus \ldots \oplus V_k$.

A proof of this lemma can be found in [12, § 2], but for the existence part a more elegant proof is given in [9, Satz 4.2]. This lemma generalizes the *G-equivariant Morse lemma* proved in [3].

The important point is that the so-called *remainder germ* η above inherits many of the properties of f:

LEMMA 2.2. *Let G, f,* η, *W and the* V_i *be as in Lemma* 2.1. *Then*

(a) $\mathrm{cod}_G(f) = \mathrm{cod}_G(\eta)$.

(b) *If f has a degenerate singularity at* 0, *then f and* η *have the same strong determinacy, and they have the same G-determinacy except in the one very special case that* η *is G-*1*-determined and G acts trivially on some* V_i, $i \notin A$ *(then f is G-*2*-determined).*

(c) *Let* $p = \dim_{\mathbf{R}} W$, *and let* $b_1, \ldots, b_r \in \mathfrak{m}_G(p)$. *Let* π *be the projection* $\mathbf{R}^n \to W$. *Then* $b_1, \ldots, b_r$ *represent a basis of* $\mathfrak{m}_G(p)/(J_G(\eta) \cap \mathfrak{m}_G(p))$ *if and only if* $b_1 \circ \pi, \ldots, b_r \circ \pi$ *represent a basis of* $\mathfrak{m}_G(n)/(J_G(f) \cap \mathfrak{m}_G(n))$. *(By virtue of Theorem* 1.10 (iii), *this gives a correspondence between the minimal universal unfoldings of f and of* η*).*

(d) *f is G-simple if and only if* η *is G-simple.*

Remark on part (b) above: if f has a non-degenerate singularity at 0, then $W = \{0\}$, $\eta = 0$, and η is trivially 0-determined in both senses, which f will not be unless $n = 0$.

By virtue of Lemmas 2.1 and 2.2, it is only necessary to classify germs belonging to $\mathfrak{m}_G^3$ (and any singular germ may be reduced to this case by splitting off a non-degenerate quadratic form). One reason this is important is that for germs in $\mathfrak{m}_G^3$ the minimal codimension of the germs grows strictly with the number of variables (i.e., if one places a bound on the G-codimension, then the dimension of the G-spaces to be considered is also bounded):

LEMMA 2.3. *Let G be a compact Lie group acting linearly on* $\mathbf{R}^n$, *and let* $\mathbf{R}^n = V_1 \oplus \ldots \oplus V_k$ *be a decomposition of* $\mathbf{R}^n$ *into irreducible G-subspaces. For* *each G-representation type ϱ let r_ϱ be the number of V_i of type ϱ. Then for any $f \in \mathfrak{m}_G^3(n)$,*

$$\operatorname{cod}_G(f) \geq \sum_{\varrho} \binom{r_\varrho + 1}{2}$$

In applications, the splitting lemma often permits a significant reduction in the number of variables which need to be considered in the mathematical model under study.

3. The abelian case

If G is a compact *abelian* Lie group, and if V is an irreducible real G-space, then as is well known, V is either of dimension one or two over $\mathbf{R}$. Moreover, in the one-dimensional case each element of G acts on V either as the identity or as negation $v \mapsto -v$, i.e., the action of G factors through the standard action of $\mathbf{Z}_2$ on V via a homomorphism $\lambda\colon G \to \mathbf{Z}_2$. In the 2-dimensional case, there is a real isomorphism of V to $\mathbf{C}$ with respect to which the action of G on V factors through the standard action of the circle group S^1 on $\mathbf{C}$ (complex multiplication) via a homomorphism $\lambda\colon G \to S^1$. We shall explain how this greatly simplifies the task of classifying G-invariant singularities when G is abelian.

Suppose we have an arbitrary linear action of a compact abelian Lie group G on $\mathbf{R}^n$. Then we may decompose $\mathbf{R}^n$ into irreducible G-subspaces and we may factor the action of G on each of these through a homomorphism of G into $\mathbf{Z}_2$ or S^1 as above. Let us write down such a decomposition in a systematic way, whereby we shall gather together irreducible summands of the same representation type into the so-called *primary components* of the representation, and we shall write down first the primary component on which G acts trivially, then the primary components of the different one-dimensional representation types, and last the primary components of the different two-dimensional representation types. Then up to a real isomorphism we may write

$$\mathbf{R}^n \cong \mathbf{R}^r \times \mathbf{R}^{s_1} \times \ldots \times \mathbf{R}^{s_k} \times \mathbf{C}^{t_1} \times \ldots \times \mathbf{C}^{t_l},$$

where G acts trivially on $\mathbf{R}^r$, each $\mathbf{R}^{s_i}$ is primary of one-dimensional representation type and G acts on it via a homomorphism $\lambda_i\colon G \to \mathbf{Z}_2$ (i.e., $g \in G$ acts by multiplication with $\lambda_i(g) = \pm 1$), and each $\mathbf{C}^{t_j}$ is primary of two-dimensional representation type and G acts on it via a homomorphism μ_j: $G \to S^1$ (i.e., $g \in G$ acts on $\mathbf{C}^{t_j}$ by coordinatewise complex multiplication with $\mu_j(g)$).

If we order the primary components above so that $s_1 \leqslant \ldots \leqslant s_k$ and $t_1 \leqslant \ldots \leqslant t_l$ then the symbol $(r \mid s_1, \ldots, s_k \mid t_1, \ldots, t_l)$ is uniquely determined by the G-isomorphism type of the original action of G on $\boldsymbol{R}^n$; we shall call this symbol the *rank* of the G-action.

We may combine the homomorphisms λ_i and μ_j above into one homomorphism λ: $G \to (\boldsymbol{Z}_2)^k \times (S^1)^l$, $\lambda(g) = (\lambda_1(g), \ldots, \lambda_k(g), \mu_1(g), \ldots, \mu_l(g))$; we shall say G *acts via* λ.

Although these remarks are quite trivial, they have a very important consequence: obviously, the ring $\mathscr{E}_G(n)$ and its structure, and the G-equivariant map-germs from $\boldsymbol{R}^n$ to any other G-space, do not depend directly on G and its action; *they depend only on the image of* λ *in* $(\boldsymbol{Z}_2)^k \times (S^1)^l$ (which we shall call the *effective group* of the G-action). In other words, for the classification of germs with compact abelian symmetry, it is enough to consider only closed subgroups H of the groups $(\boldsymbol{Z}_2)^k \times (S^1)^l$ acting in the standard way described on the spaces $\boldsymbol{R}^r \times \boldsymbol{R}^{s_1} \times \ldots \times \boldsymbol{R}^{s_k} \times \boldsymbol{C}^{t_1} \times \ldots \times \boldsymbol{C}^{t_l}$. It is merely necessary to impose a few "non-degeneracy" conditions on H, so obvious that we need not specify them here, to insure that the factors $\boldsymbol{R}^{s_1}$ are really of *different* and non-trivial representation types, and that the factors $\boldsymbol{C}^{t_j}$ are of different and of essentially two-dimensional representation types. Moreover, although neither the homomorphism λ above nor its image $H = \lambda(G)$ in $(\boldsymbol{Z}_2)^k \times (S^1)^l$ are uniquely determined by the original action of G on $\boldsymbol{R}^n$, there are equally simple and obvious criteria to determine when the actions of two different subgroups H and H' of $(\boldsymbol{Z}_2)^k \times (S^1)^l$ acting in the standard way are isomorphic, i.e., can be associated to the same original action of G on $\boldsymbol{R}^n$.

Not only have we greatly reduced the list of groups which we must consider, it turns out that it is very easy to enumerate the relevant subgroups H of $(\boldsymbol{Z}_2)^k \times (S^1)^l$ by virtue of the Pontryagin Duality Theorem [7, Theorem 40], which gives a correspondence between the closed subgroups H of $(\boldsymbol{Z}_2)^k \times (S^1)^l$ and the subgroup of characters of $(\boldsymbol{Z}_2)^k \times (S^1)^l$ (or of $(S^1)^{k+l} \supseteq (\boldsymbol{Z}_2)^k \times (S^1)^l$) which vanish on H. It seems slightly more convenient to work in the character group of $(S^1)^{k+l}$, which is $\boldsymbol{Z}^{k+l}$. Subgroups R of $\boldsymbol{Z}^{k+l}$ are easily classified by classifying their bases; moreover the condition that H is in fact contained in $(\boldsymbol{Z}_2)^k \times (S^1)^l$, and the non-degeneracy conditions on H mentioned in the previous paragraph, translate immediately to obvious conditions on the corresponding group of characters $R \subseteq \boldsymbol{Z}^{k+l}$.

There is a further very important benefit from the description of abelian symmetry given above, which greatly simplifies calculations in the abelian case: namely, the structure of $\mathscr{E}_G(n)$ (and also of $\mathscr{E}_G(n, n)$, and hence indirectly of the G-Jacobian ideals of germs) can immediately be determined by inspection. Let us elaborate briefly:

We shall take coordinates on $\boldsymbol{R}^n \cong \boldsymbol{R}^r \times \boldsymbol{R}^{s_1} \times \ldots \times \boldsymbol{R}^{s_k} \times \boldsymbol{C}^{t_1} \times \ldots \times \boldsymbol{C}^{t_l}$ as follows: on the factors $\boldsymbol{R}^r$ and $\boldsymbol{R}^{s_i}$ we shall take the standard real coordinates, but on the factors $\boldsymbol{C}^{t_j}$ we shall take the standard *complex* coordinates $z_{j,v}$, and we shall write real polynomials on $\boldsymbol{R}^n$ as *real-valued* polynomials in the real coordinates of $\boldsymbol{R}^r$ and the $\boldsymbol{R}^{s_i}$ and in the complex coordinates $z_{j,v}$ *and their conjugates* $\bar{z}_{j,v}$. Then because the subgroups H of $(\boldsymbol{Z}_2)^k \times (S^1)^l$ which we must consider act essentially by complex multiplication, a polynomial in the above mixed coordinates is H-invariant if and only if each of its monomials is invariant. Moreover, to determine if a monomial is invariant, we need merely add up, for each of the spaces $\boldsymbol{R}^{s_i}$ or $\boldsymbol{C}^{t_j}$, the exponents in the monomial of the coordinates belonging to this space (counting the exponents of the complex conjugate coordinates as negative). The monomial is H-invariant exactly when the resulting $k+l$-tuple of integers belongs to the group $R \subseteq \boldsymbol{Z}^{k+l}$ of characters of $(S^1)^{k+l}$ vanishing on H.

Note that this criterium works even if we allow negative exponents. This allows us to derive immediately a corresponding criterium for H-equivariant mappings: a monomial mapping f into one of the factors $\boldsymbol{R}$ or $\boldsymbol{C}$ of the above decomposition of $\boldsymbol{R}^n$ is clearly equivariant if and only if the monomial obtained by dividing f by the coordinate corresponding to the target factor is invariant. Again, a polynomial mapping is equivariant if and only if each of its monomials is.

To summarize: to handle all abelian symmetries it is sufficient to consider the very comfortable situation of closed subgroups of torus groups acting on $\boldsymbol{R}^n$ essentially by complex multiplication. Such groups are easily classified by the free abelian groups of characters vanishing on them, which also immediately describe the invariant and equivariant polynomials. This so much simplifies the task of computation that it is possible to obtain the *general* classification, for all abelian symmetries, of germs of low G-codimension and G-simple germs given in the next section.

4. The classification

In this section we shall give a complete classification, for all compact abelian Lie groups G, of the germs in $\mathfrak{m}_G(n)$ of G-codimension $\leqslant 4$ and of the G-simple germs. In presenting the classification lists we shall of course apply the simplifications and notations introduced in Section 3; in particular, the lists mention only the *effective groups* H of the actions, which are closed subgroups of $(\boldsymbol{Z}_2)^k \times (S^1)^l$, and it is to be understood that each class of symmetric germs mentioned in the list can also arise for any Lie group G which admits the given group H as a quotient.

The lists include the following information: (1) a running number to identify the class; (2) the *rank* of the group action (as defined in Section 3), and, below the dimensions of the primary invariant subspaces in the rank symbol, the names of the variables which will be used to denote the real or complex coordinates on this subspace; (3) the effective group $H \subseteq (\boldsymbol{Z}_2)^k \times (S^1)^l$ (in defining H, we shall often write "$\boldsymbol{Z}_p$" to denote the cyclic subgroup of S^1 consisting of the p-th roots of unity); (4) a polynomial normal form f for the class of germs; (5) the G-codimension of f; (6) representatives of an $\boldsymbol{R}$-basis of $\mathfrak{m}_G(n)/J_G(f)$ (as explained in Section 1, this gives the G-universal unfolding of f); and finally (7), (8) the strong and G-determinacies of f.

The polynomial normal forms listed usually contain variable exponents, sometimes also variable coefficients, and describe an entire *family* of classes, whose codimensions depend on the variable exponents so that the lists include many germs of codimension > 4; the method of computation by which the normal forms are derived automatically yields normal forms of this generality at no extra cost. However, the list is *complete* only for G-codimension $\leqslant 4$. The variable coefficients, if present, may either simply represent a variable sign ± 1, in which case they are usually denoted by Greek letters ε, δ, or they may vary continuously on some open subset of $\boldsymbol{R}$, in which case they are denoted by lowercase Roman letters a, b, c from the beginning of the alphabet. In the lists, the permissible range of values for the variable exponents and coefficients is given with the normal forms, and is chosen so that *different permissible values of the variable exponents and coefficients define different G-right equivalence classes.* (In particular, the variable coefficients with a continuous range are so-called *moduli* of the normal forms.) Nor do the different normal forms listed "overlap" – *all* of the germs listed are *inequivalent to each other.*

Some of the conditions given in the lists for the permissible range of variable coefficients, in particular, the conditions stating that the coefficient of a monomial is ± 1 or real > 0, are conditions which may be *assumed* to hold up to G-right equivalence; choosing non-zero values violating these conditions merely violates uniqueness, and yields germs belonging to the same class as some germ in the permissible range. However, all conditions on the variable *exponents*, and all other conditions on the variable coefficients, in particular those involving the $\neq$ relation, *must* hold, else the data on codimension, determinacy etc. will be incorrect and the germ will belong to a different entry in the list (or will not belong to the list at all). (The one exception to these remarks, in the separate $\boldsymbol{Z}_2$-classification list below, is clearly marked.)

Note finally that since the complex polynomial normal forms given represent real germs and hence are real-valued, any variable coefficient of a self-conjugate monomial (such as $z^k \bar{z}^k$) must automatically be real; this condition is therefore not stated explicitly in the lists.

Obviously in this short space we cannot go into any details about the proof of a classification of this magnitude. However, we shall give a few brief comments on the method of proof in the following section.

In the lists below, we may of course assume that the effective group acts non-trivially on $\boldsymbol{R}^n$, since the classification for the non-symmetric case is well known (Thom's list of the seven elementary catastrophes, Arnold's 1972 classification of simple germs [2]).

The previously mentioned *Diplomarbeit* [4] of M. Beer contains a classification of the germs of codimension $\leqslant 4$ for the simplest case of symmetry, the case $G = Z_2$. As this thesis was never published, we include Beer's results here in a separate list, (and therefore exclude the case $G = Z_2$ in the main list):

THEOREM 4.1 (Beer [4, Satz 5.6]). *Let the group Z_2 act on R^n in the standard way, namely, so that the non-zero element of Z_2 negates the last s coordinates for some $s \leqslant n$, and suppose the action is non-trivial* (i.e. $s > 0$). *The rank of this action is* $(n-s|s|0)$.

Suppose $f \in \mathfrak{m}^3_{Z_2}(n)$ and $\mathrm{cod}_{Z_2}(f) \leqslant 4$. Then f is Z_2-right equivalent to a unique germ in the following list:

Number	rank	germ	Z_2-cod	R-basis for $\mathfrak{m}_{Z_2}(n)/J_{Z_2}(f)$	determinacy strong	determinacy Z_2
(i)$_k$	$(0\|1\|0)$ x	$\pm x^{2k}$, $k \geqslant 2$	$k-1$	$x^{2j}(1 \leqslant j \leqslant k-1)$	$2k$	k
(ii)$_k$	$(1\|1\|0)$ $x\ y$	$xy^2 \pm x^k$, $k \geqslant 3$	$k-1$	$x^j(1 \leqslant j \leqslant k-1)$	k	k
(iii)$_k$		$x^3 \pm y^{2k}$, $k \geqslant 2$	$2k-1$	$y^{2i}(1 \leqslant i \leqslant k-1)$, $xy^{2j}(0 \leqslant j \leqslant k-1)$	$2k$	$\max(k, 3)$
(iv)	$(0\|2\|0)$ x, y	$(x^2-y^2)(x^2-ay^2)$, a **must be** > 0, $\neq 1$, but up to right-equivalence may be **assumed** to be > 1	4	x^2, y^2, xy, x^2y^2	4	2
(v)		$(x^2+y^2)(x^2-ay^2)$, a **must be** > 0	4	x^2, y^2, xy, x^2y^2	4	2
(vi)		$\varepsilon(x^2+y^2)(x^2+ay^2)$, $\varepsilon = \pm 1$, a **must be** > 0, $\neq 1$, but up to right-equivalence may be **assumed** to be > 1	4	x^2, y^2, xy, x^2y^2	4	2

Our main result is:

THEOREM 4.2. *Let G be a compact abelian Lie group acting linearly and non-trivially on R^n. Suppose the effective group of the G-action (as defined in Section 3) is not Z_2 (the Z_2 case is covered by Theorem 4.1). Let $f \in \mathfrak{m}^3_G(n)$ and suppose $\mathrm{cod}_G(f) \leqslant 4$. Then f is G-right equivalent to a unique germ in the following table:*

running number	rank	effective group	germ	G-codim	$\boldsymbol{R}$-basis for $m_G(n)/J_G(f)$	determinacies (s[trong], G)
$1_{k,m}$	(0|0|1) z	$\boldsymbol{Z}_m$, $m \geqslant 5$	$\pm z^k \bar{z}^k + az^m + a\bar{z}^m + \sum_{j=1}^{k-2} b^j (z^m + \bar{z}^m) z^j \bar{z}^j$, $2 \leqslant k < m/2$, a, b_j real, $a > 0$	$2k-2$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant 2k-2)$	s: $m+2k-4$, G: $2k-2$
$2_{l,m}$		$\boldsymbol{Z}_m$, $m \geqslant 3$	$\pm z^2 \bar{z}^2 + az^{lm} + a\bar{z}^{lm}$, $l \geqslant 2$, a real > 0	$2l$	$z\bar{z}$, $z^{jm} + \bar{z}^{jm}$ $(1 \leqslant j \leqslant l)$, $iz^{rm} - i\bar{z}^{rm}$ $(1 \leqslant r \leqslant l-1)$	s: lm, G: l
3_m		$\boldsymbol{Z}_m$, m odd, $m \geqslant 3$	$z^m + \bar{z}^m + \sum_{j=(m+1)/2}^{m-2} a_j z^j \bar{z}^j$	$m-2$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant m-2)$	s: max $(2m-4, m)$, G: $m-2$
4_m		$\boldsymbol{Z}_m$, m even, $m = 2k$, $k \geqslant 2$	$z^m + \bar{z}^m + az^k \bar{z}^k + \sum_{j=1}^{k-2} b_j z^{k+j} \bar{z}^{k+j}$, $a \neq \pm 2$	$m-2$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant m-2)$	s: $2m-4$, G: $m-2$
$5_{l,m}$			$z^m + \bar{z}^m \pm 2z^k \bar{z}^k + \sum_{j=l}^{l+k-2} b_j z^j \bar{z}^j$, $l > k$, $b_l \neq 0$	$l+k-2$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant l+k-2)$	s: $2l+m-4$, G: $l+k-2$
6_k		S^1	$\pm z^k \bar{z}^k$, $k \geqslant 2$	$k-1$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant k-1)$	s: $2k$, G: k
7	(1|0|1) x z	$\boldsymbol{Z}_3$	$z^3 + \bar{z}^3 + axz\bar{z} + x^3$, $a \neq -3$	3	x, $z\bar{z}$, $xz\bar{z}$	s: 3, G: 3
8_k			$z^3 + \bar{z}^3 - 3xz\bar{z} + x^3 + ax^k$, $k \geqslant 4$, $a \neq 0$	k	$x^j (1 \leqslant j \leqslant k)$	s: k, G: k
$9_{k,l,m}$		$\boldsymbol{Z}_m$, $m \geqslant 3$	$xz\bar{z} + z^{km} + \bar{z}^{km} + ax^l$, $km \geqslant 3$, $l \geqslant 3$, if $km = 3$ then $l > 3$; $a \neq 0$	$2k+l-2$	x^j $(1 \leqslant j \leqslant l)$, $z^{rm} + \bar{z}^{rm}$ $(1 \leqslant r \leqslant k-1)$, $iz^{sm} - i\bar{z}^{sm}$ $(1 \leqslant s \leqslant k-1)$	s: max(l, km), G: max(l, k)
10_k		S^1	$xz\bar{z} \pm x^k$, $k \geqslant 3$	$k-1$	$x^j (1 \leqslant j \leqslant k-1)$	s: k, G: k
11_k			$x^3 \pm z^k \bar{z}^k$, $k \geqslant 2$	$2k-1$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant k-1)$, $xz^r \bar{z}^r$ $(0 \leqslant r \leqslant k-1)$	s: $2k$, G: max$(k, 3)$

running number	rank	effective group	germ	G-codim	**R**-basis for $\mathfrak{m}_G(n)/J_G(f)$	determinacies (s[trong], G)
12	$(0\|1, 1\|0)$ $x\ y$	$\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$	$\varepsilon x^4 + ax^2y^2 + \delta y^4$, $\varepsilon = \pm 1$, $\delta = \pm 1$, $a^2 \neq 4\delta\varepsilon$	3	x^2, y^2, x^2y^2	s: 4, G: 2
13_k			$\varepsilon(x^4 + 2\delta x^2y^2 + y^4) + ay^{2k}$, $\varepsilon = \pm 1$, $\delta = \pm 1$, $k \geqslant 3$, $a \neq 0$	$k+1$	x^2, y^{2j} $(1 \leqslant j \leqslant k)$	s: $2k$, G: k
$14_{k,l}$			$\varepsilon x^{2k} + \delta x^2y^2 + ay^{2l}$, $\varepsilon = \pm 1, \delta = \pm 1, a \neq 0$; $k \geqslant 2$, $l \geqslant 2$, $k+l > 4$	$k+l-1$	x^{2j} $(1 \leqslant j \leqslant k-1)$, y^{2r} $(1 \leqslant r \leqslant l)$	s: $\max(2k, 2l)$, G: $\max(k, l)$
$15_{k,l}$	$(0\|1\|1)$ $x\ z$	$\{(\alpha, \beta) \in \boldsymbol{Z}_2 \times S^1$: $\beta^4 = 1$, $\alpha = \beta^2\}$ $(\cong \boldsymbol{Z}_4)$	$xz^2 + x\bar{z}^2 \pm z^k\bar{z}^k + ax^{2l}$, $k \geqslant 2$, $l \geqslant 2$, $a \neq 0$	$k+l-1$	$z^j\bar{z}^j$ $(1 \leqslant j \leqslant k-1)$, x^{2r} $(1 \leqslant r \leqslant l)$	s: $\max(2k, 2l)$, G: $\max(k, l)$
$16_{k,l}$		$\boldsymbol{Z}_2 \times \boldsymbol{Z}_3$	$z^3 + \bar{z}^3 + \varepsilon x^{2k}z\bar{z} + \sum_{j=l}^{p} a_jx^{2j}$, $\varepsilon = \pm 1$, $k \geqslant 1$, $l > k$, $p = \min(2l-k-1, l+2k-1)$; $a_l \neq 0$, and if $l = 3k$ then $a_l \neq -\varepsilon/27$	min $(2l-1, 3k+l-1)$ $(= p+k)$	$x^{2j}z\bar{z}$ $(0 \leqslant j \leqslant k-1)$, x^{2r} $(1 \leqslant r \leqslant p)$	s: min $(4l-2k-2, 2l+4k-2)$ $(= 2p)$, G: min $(2l-k-1, l+2k-1)$ $(= p)$
17_k			$z^3 + \bar{z}^3 \pm x^{2k}$, $k \geqslant 2$	$2k-1$	$x^{2j}z\bar{z}$ $(0 \leqslant j \leqslant k-1)$, x^{2r} $(1 \leqslant r \leqslant k-1)$	s: $2k$, G: k
18		$\boldsymbol{Z}_2 \times S^1$	$\varepsilon x^4 + ax^2z\bar{z} + \delta z^2\bar{z}^2$, $\varepsilon = \pm 1$, $\delta = \pm 1$, $a^2 \neq 4\delta\varepsilon$	3	x^2, $z\bar{z}$, $x^2z\bar{z}$	s: 4, G: 2
19_k			$\varepsilon(x^4 + 2\delta x^2z\bar{z} + z^2\bar{z}^2) + az^k\bar{z}^k$, $\varepsilon = \pm 1$, $\delta = \pm 1$, $k \geqslant 3$, $a \neq 0$	$k+1$	x^2, $z^j\bar{z}^j$ $(1 \leqslant j \leqslant k)$	s: $2k$, G: k
$20_{k,l}$			$\varepsilon x^{2k} + \delta x^2z\bar{z} + az^l\bar{z}^l$, $\varepsilon = \pm 1$, $\delta = \pm 1$, $a \neq 0$, $k \geqslant 2$, $l \geqslant 2$, $k+l > 4$	$k+l-1$	x^{2j} $(1 \leqslant j \leqslant k-1)$, $z^r\bar{z}^r$ $(1 \leqslant r \leqslant l)$	s: $\max(2k, 2l)$ G: $\max(k, l)$
21	$(0\|0\|1, 1)$ $z\ w$	$\boldsymbol{Z}_3 \times \boldsymbol{Z}_3$	$z^3 + \bar{z}^3 + w^3 + \bar{w}^3 + az\bar{z}w\bar{w}$	3	$z\bar{z}$, $w\bar{w}$, $z\bar{z}w\bar{w}$	s: 4 G: 2

running number	rank	effective group	germ	G-codim	$\boldsymbol{R}$-basis for $\mathfrak{m}_G(n)/J_G(f)$	determinacies (s[trong], G)
$22_{k,m}$	$(0\|0\|1, 1)$ z w	$\{(\alpha, \beta) \in S^1 \times S^1$: $\beta^{2m} = 1$, $\alpha = \beta^2\}$ $(\cong \boldsymbol{Z}_{2m})$, $m \geqslant 5$	$z\bar{w}^2 + \bar{z}w^2 + az^k\bar{z}^k + z^m + \bar{z}^m + \sum_{j=1}^{k-1} b^j(z^m + \bar{z}^m)z^j\bar{z}^j$, $2 \leqslant k < m/2$, $a \neq 0$, h_j real	$2k$	$w\bar{w}$, $z^j\bar{z}^k$ $(1 \leqslant j \leqslant 2k-1)$	s: $m+2k-2$, G: $2k-1$
23_m		$\{(\alpha, \beta) \in S^1 \times S^1$: $\beta^{2m} = 1$, $\alpha = \beta^2\}$ $(\cong \boldsymbol{Z}_{2m})$, m odd, $m \geqslant 3$	$z\bar{w}^2 + \bar{z}w^2 + z^m + \bar{z}^m + \sum_{j=(m+1)/2}^{m-1} a_j z^j\bar{z}^j$	m	$w\bar{w}$, $z^j\bar{z}^j$ $(1 \leqslant j \leqslant m-1)$	s: $2m-2$, G: $m-1$
24_m		$\{(\alpha, \beta)$, $\in S^1 \times S^1$: $\beta^{2m} = 1$, $\alpha = \beta^2\}$ $(\cong \boldsymbol{Z}_{2m})$, m even, $m = 2k$, $k \geqslant 2$	$z\bar{w}^2 + \bar{z}w^2 + z^m + \bar{z}^m + az^k\bar{z}^k + \sum_{j=1}^{k-1} b_j z^{k+j}\bar{z}^{k+j}$, $a \neq \pm 2$	m	$w\bar{w}$, $z^j\bar{z}^j$ $(1 \leqslant j \leqslant m-1)$	s: $2m-2$, G: $m-1$
$25_{l,m}$		$\{(\alpha, \beta) \in S^1 \times S^1$: $\beta^m = 1$, $\alpha = \beta^2\}$ $(\cong \boldsymbol{Z}_m)$, m odd, $m = 2k+1$, $k \geqslant 3$	$z\bar{w}^2 + \bar{z}w^2 + z^k w + \bar{z}^k\bar{w} + \sum_{j=l}^{2l-1} a_j z^j\bar{z}^j$, $2 \leqslant l < k$, $a_l \neq 0$	$2l$	$w\bar{w}$, $z^j\bar{z}^j$ $(1 \leqslant j \leqslant 2l-1)$	s: $m+2l-4$, G: $2l-1$
26_m		$\{(\alpha, \beta) \in S^1 \times S^1$: $\beta^m = 1$, $\alpha = \beta^2\}$ $(\cong \boldsymbol{Z}_m)$, m odd, $m = 2k+1$, $k \geqslant 2$	$z\bar{w}^2 + \bar{z}w^2 + z^k w + \bar{z}^k\bar{w} + \sum_{j=k}^{2k-2} a_j z^j\bar{z}^j$	$2k-1$ $(=m-2)$	$w\bar{w}$, $z^j\bar{z}^j$ $(1 \leqslant j \leqslant 2k-2)$	s: $4k-4$ $(=2m-6)$, G: $2k-2$ $(=m-3)$

running number	rank	effective group	germ	G-codim	$\boldsymbol{R}$-basis for $\mathfrak{m}_G(n)/J_G(f)$	determinacies (s[trong], G)
$27_{k,l}$	$(0\|0\|1, 1)$ $z\ w$	$\boldsymbol{Z}_3 \times S^1$	$z^3+\bar{z}^3+\varepsilon z\bar{z}w^k\bar{w}^k$ $+\sum_{j=l}^{p} a_j w^j\bar{w}^j$, $\varepsilon=\pm 1$, $k \geqslant 1$, $l>k$, $p=\min(2l-k-1,$ $l+2k-1)$; $a_l \neq 0$, and if $l=3k$ then $a_l \neq -\varepsilon/27$	min $(2l-1,$ $3k+l-1)$ $(=p+k)$	$z\bar{z}w^j\bar{w}^j$ $(0 \leqslant j \leqslant k-1)$, $w^r\bar{w}^r$ $(1 \leqslant r \leqslant p)$	s: min $(4l-2k-2,$ $2l+4k-2)$ $(=2p)$, G: min $(2l-k-1,$ $l+2k-1)$ $(=p)$
28_k			$z^3+\bar{z}^3 \pm w^k\bar{w}^k$, $k \geqslant 2$	$2k-1$	$z\bar{z}w^j\bar{w}^j$ $(0 \leqslant j \leqslant k-1)$, $w^r\bar{w}^r$ $(1 \leqslant r \leqslant k-1)$	s: $2k$, G: k
$29_{k,r}$		$\{(\alpha, \beta) \in S^1 \times S^1:$ $\alpha^k=\beta^r\}$ $1 \leqslant k \leqslant r$, $k+r \geqslant 5$	$z^k\bar{w}^r+\bar{z}^k w^r+az^2\bar{z}^2$ $+bz\bar{z}w\bar{w}+cw^2\bar{w}^2$, $a=\pm 1$, $c \neq 0$, $b^2 \neq 4ac$	4	$z\bar{z}$, $w\bar{w}$, $z\bar{z}w\bar{w}$, $w^2\bar{w}^2$	s: $k+r$, G: 2
30		$\{(\alpha, \beta) \in S^1 \times S^1:$ $\alpha^2=\beta^2\}$ $(\cong \boldsymbol{Z}_2 \times S^1)$	$z^2\bar{w}^2+\bar{z}^2w^2+az^2\bar{z}^2$ $+bz\bar{z}w\bar{w}+cw^2\bar{w}^2$, $a=\pm 1$, $c \neq 0$, $(b+2)^2 \neq 4ac$, $(b-2)^2 \neq 4ac$	4	$z\bar{z}$, $w\bar{w}$, $z\bar{z}w\bar{w}$, $w^2\bar{w}^2$	s: 4, G: 2
31		$\{(\alpha, \beta) \in S^1 \times S^1:$ $\alpha=\beta^3\}$ $(\cong S^1)$	$z\bar{w}^3+\bar{z}w^3+az^2\bar{z}^2$ $+bz\bar{z}w\bar{w}+cw^2\bar{w}^2$, $a=\pm 1$, $-8ab^2c^2$ $+36abc-27a+b^4c$ $-b^3+16c^3 \neq 0$	4	$z\bar{z}$, $w\bar{w}$, $z\bar{z}w\bar{w}$, $w^2\bar{w}^2$	s: 4, G: 2
32_k		$\{(\alpha, \beta) \in S^1 \times S^1:$ $\alpha=\beta^2\}$ $(\cong S^1)$	$z\bar{w}^2+\bar{z}w^2 \pm z^k\bar{z}^k$, $k \geqslant 2$	k	$w\bar{w}$, $z^j\bar{z}^j$ $(1 \leqslant j \leqslant k-1)$	s: $2k$, G: k
33		$S^1 \times S^1$	$\varepsilon z^2\bar{z}^2+az\bar{z}w\bar{w}$ $+\delta w^2\bar{w}^2$, $\varepsilon=\pm 1$, $\delta=\pm 1$, $a^2 \neq 4\delta\varepsilon$	3	$z\bar{z}$, $w\bar{w}$, $z\bar{z}w\bar{w}$	s: 4, G: 2
34_k			$\varepsilon(z^2\bar{z}^2+2\delta z\bar{z}w\bar{w}$ $+w^2\bar{w}^2)+aw^k\bar{w}^k$, $\varepsilon=\pm 1$, $\delta=\pm 1$, $k \geqslant 3$, $a \neq 0$	$k+1$	$z\bar{z}$, $w^j\bar{w}^j$ $(1 \leqslant j \leqslant k)$	s: $2k$, G: k

running number	rank	effective group	germ	G-codim	$\boldsymbol{R}$-basis for $\mathfrak{n}_G(n)/J_G(f)$	determinacies (s[trong], G)
$35_{k,l}$	$(0\|0\|1, 1)$ $z\ w$	$S^1\times S^1$	$\varepsilon z^k\bar{z}^k+\delta z\bar{z}w\bar{w}+aw^l\bar{w}^l$, $\varepsilon=\pm1$, $\delta=\pm1$, $a\neq0$; $k\geqslant2$, $l\geqslant2$, $k+l>4$	$k+l-1$	$z^j\bar{z}^j$ $(1\leqslant j\leqslant k-1)$, $w^r\bar{w}^r$ $(1\leqslant r\leqslant l)$	s: max $(2k, 2l)$, G: max(k, l)
36_k	$(1\|1, 1\|0)$ $x\ y\ t$	$\boldsymbol{Z}_2\times\boldsymbol{Z}_2$	$x^3+\varepsilon xy^2+\delta xt^2+ay^{2k}$ $\varepsilon=\pm1$, $\delta=\pm1$, $k\geqslant2$, $a\neq0$	$k+2$	x, t^2, y^{2j} $(1\leqslant j\leqslant k)$	s: $2k$, G: max$(k, 3)$
37_k	$(1\|1\|1)$ $x\ y\ z$	$\boldsymbol{Z}_2\times S^1$	$x^3+\varepsilon xy^2+\delta xz\bar{z}+ay^{2k}$, $\varepsilon=\pm1$, $\delta=\pm1$, $k\geqslant2$, $a\neq0$	$k+2$	x, $z\bar{z}$, y^{2j} $(1\leqslant j\leqslant k)$	s: $2k$, G: max$(k, 3)$
38_k	$(1\|0\|1, 1)$ $x\ \ z\ w$	$S^1\times S^1$	$x^3+\varepsilon xz\bar{z}+\delta xw\bar{w}+az^k\bar{z}^k$, $\varepsilon=\pm1$, $\delta=\pm1$, $k\geqslant2$, $a\neq0$	$k+2$	x, $w\bar{w}$, $z^j\bar{z}^j$ $(1\leqslant j\leqslant k)$	s: $2k$, G: max$(k, 3)$
39		$\{(\alpha, \beta)\in S^1\times S^1: \alpha=\beta^2\}$ $(\cong S^1)$	$x^3+\varepsilon xz\bar{z}+axw\bar{w}+z\bar{w}^2+\bar{z}w^2$, $\varepsilon=\pm1$, $a^2\neq-4\varepsilon$	4	x, x^2, $w\bar{w}$, $xw\bar{w}$	s: 3, G: 3
$40_{k,l,m}$	$(0\|1, 1, 1\|0)$ $x\ y\ t$	$\{(\alpha, \beta, \gamma)\in\boldsymbol{Z}_2\times\boldsymbol{Z}_2\times\boldsymbol{Z}_2: \alpha\beta\gamma=1\}$ $(\cong\boldsymbol{Z}_2\times\boldsymbol{Z}_2)$	$xyt+\varepsilon x^{2k}+\delta y^{2l}+at^{2m}$, $k\geqslant2$, $l\geqslant2$, $m\geqslant2$; $\varepsilon=\pm1$, $\delta=\pm1$, $a\neq0$	$k+l+m-2$	x^{2j} $(1\leqslant j\leqslant k-1)$, y^{2r} $(1\leqslant r\leqslant l-1)$, t^{2s} $(1\leqslant s\leqslant m)$	s: max $(2k, 2l, 2m)$, G: max(k, l, m)
$41_{k,l,m}$	$(0\|0\|1, 1, 1)$ $z\ w\ v$	$\{(\alpha, \beta, \gamma)\in S^1\times S^1\times S^1: \alpha\beta\gamma=1\}$ $(\cong S^1\times S^1)$	$zwv+\bar{z}\bar{w}\bar{v}+\varepsilon z^k\bar{z}^k+\delta w^l\bar{w}^l+av^m\bar{v}^m$, $k\geqslant2$, $l\geqslant2$, $m\geqslant2$; $\varepsilon=\pm1$, $\delta=\pm1$, $a\neq0$	$k+l+m-2$	$z^j\bar{z}^j$ $(1\leqslant j\leqslant k-1)$, $w^r\bar{w}^r$ $(1\leqslant r\leqslant l-1)$, $v^s\bar{v}^s$ $(1\leqslant s\leqslant m)$	s: max $(2k, 2l, 2m)$, G: max(k, l, m)

The methods which are needed to obtain the above classification can also be applied, with a few fairly obvious additional ideas, to obtain the (considerably shorter) classification of the G-simple germs with abelian symmetry:

THEOREM 4.3. *Let G be a compact abelian Lie group acting linearly and non-trivially on $\boldsymbol{R}^n$. Let $f\in\mathfrak{n}_G^3(n)$ be G-simple. Then f is G-right equivalent to a unique germ in the following table*:

number	rank	effective group	germ	G-cod	$\mathbb{R}$-basis for $\mathfrak{m}_G(n)/J_G(f)$	determinacies strong	determinacies G
$(1)_k$	$(0\|1\|0)$ x	Z_2	$\pm x^{2k}$, $k \geqslant 2$	$k-1$	x^{2j} $(1 \leqslant j \leqslant k-1)$	$2k$	k
$(2)_k$	$(0\|0\|1)$ z	S^1	$\pm z^k \bar{z}^k$, $k \geqslant 2$	$k-1$	$z^j \bar{z}^j$ $(1 \leqslant j \leqslant k-1)$	$2k$	k
(3)		Z_3	$z^3 + \bar{z}^3$	1	$z\bar{z}$	3	1
$(4)_k$	$(1\|1\|0)$ $x\ y$	Z_2	$xy^2 \pm x^k$, $k \geqslant 3$	$k-1$	x^j $(1 \leqslant j \leqslant k-1)$	k	k
(5)			$x^3 \pm y^4$	3	x, y^2, xy^2	4	3
$(6)_k$	$(1\|0\|1)$ $x\ z$	S^1	$xz\bar{z} \pm x^k$, $k \geqslant 3$	$k-1$	x^j $(1 \leqslant j \leqslant k-1)$	k	k
(7)			$x^3 \pm z^2 \bar{z}^2$	3	x, $z\bar{z}$, $xz\bar{z}$	4	3
$(8)_k$	$(0\|0\|1, 1)$ $z\ w$	$\{(\alpha, \beta) \in S^1 \times S^1 : \alpha = \beta^2\}$ $(\cong S^1)$	$z\bar{w}^2 + \bar{z}w^2 \pm z^k \bar{z}^k$, $k \geqslant 2$	k	$w\bar{w}$, $z^j \bar{z}^j$ $(1 \leqslant j \leqslant k-1)$	$2k$	k

5. Some remarks on the proof of the classification theorems

Clearly, it is not possible to give even a very rough sketch of the proof of the classification theorems in the space available here. However, I would like to give the reader a brief general description of the method of proof, since similar procedures will find application in any problem of classifying germs of smooth functions. I should also like to present a very useful and flexible technical lemma which greatly eases the work of finding normal forms for germs.

Carrying out a classification of singularities is an interplay of two processes: (1) determining how degenerate the germ under consideration is (in most cases this means: estimating the codimension from below), in order to reject as quickly as possible the cases which exceed the degree of degeneracy to be covered by the classification; and (2) finding allowed equivalences which simplify as far as possible the form of the germ under consideration, both in order to make subsequent computations (for example, of codimension) easier, and to arrive at unique normal forms for the equivalence classes to be covered by the classification.

In our case, as was said above, the first process involves estimating the codimension of a singular germ f as simply and as well as possible, even if one has only incomplete knowledge of f. In general, one does this by computing the dimension not of $\mathfrak{m}_G(n)/J_G(f)$ (which will usually not be possible if f is not known exactly) but instead of $\mathfrak{m}_G(n)/(J_G(f)+A)$, where A is some linear subspace of $\mathfrak{m}_G(n)$ chosen so as to contain the possible contribution to $J_G(f)$ of those parts of the Taylor series of f which are not known exactly, while containing as few other elements of $\mathfrak{m}_G(n)$ as a clever choice of A can achieve. In this way general estimates of the codimension of large classes of germs can be obtained using only very little information about the germs (in fact, the less information one needs to put in about the germs, the more generally applicable (albeit the less accurate) the estimate of codimension will be), so that hopefully a large number of cases not to be covered by the classification can be rejected at an early stage and with minimal effort.

For example, if one chooses A to be $\mathfrak{n}_G^3(n)$, then without any special information at all about the germ $f \in \mathfrak{n}_G^3(n)$ or about the group G, one obtains the general estimate of codimension given by Lemma 2.3. If G is known to be abelian, then this estimate can be improved slightly: for each *two-dimensional* representation type ϱ present, the contribution $\binom{r_\varrho+1}{2}$ in the estimate of Lemma 2.3 can be increased to r_ϱ^2; for this, no *specific* knowledge of G or of $f \in \mathfrak{n}_G^3(n)$ is required. With only slight additional knowledge about G, namely, knowledge of the numbers of G-invariant cubic and quartic monomials of different forms, a set of much better general estimates of codimension can be obtained, using $A = \mathfrak{n}_G^4(n)$ or $\mathfrak{n}_G^5(n)$; still, no particular knowledge of f is required (or in some cases, very very little: in one case, for example, that the cubic terms in f have real coefficients). Of course, when one knows G exactly, and has some information about the Taylor series of f, then by a choice of A suited to the individual situation one can estimate the codimension very accurately, but the estimate applies only to that one case.

The second process mentioned above involves simplifying the form of the Taylor series of a germ by applying a suitable equivalence. As a somewhat naive first step, one may try applying a G-equivariant *linear* change of coordinates; at least it is easy to compute how the Taylor series will be transformed in this case, and such a linear coordinate change will generally be enough to remove some terms from the lowest-order homogeneous part of the Taylor series or to transform a coefficient known to be non-zero into ± 1. However, non-linear coordinate changes will be needed to simplify the higher-order part of the Taylor series, and then it becomes very unpleasant to compute how the Taylor series will be changed, especially for the higher-order terms which are usually not known exactly before the coordinate change is applied, and which will be affected in general by the entire lower-order part of the Taylor series in a complicated way.

Fortunately, there is a remedy for this situation. Part of it allows one to *linearize* the situation, essentially replacing the difficult computation of a composition of two Taylor series by the much easier computation of a G-Jacobian ideal. But what is more, just as the trick in estimating codimensions is to work *modulo* some suitably chosen linear subspace of $\mathfrak{m}_G(n)$, it is also possible to carry out this computation of the Jacobian ideal, and hence the construction of equivalences, *modulo some linear subspace A of* $\mathfrak{m}_G^k(n)$, which again may be suitably chosen to contain the effect of those parts of the Taylor series being simplified about which one lacks information, or which affect the transformed Taylor series in a complicated way. This is the essence of the following very useful *technical lemma*, which can be applied very flexibly and greatly simplifies the computation of normal forms:

TECHNICAL LEMMA 5.1. *Let G be a Lie group acting linearly on $\mathbf{R}^n$. Let A be a real vector subspace of $\mathscr{E}_G(n)$.*

Let f: $\mathbf{R}^n \times [0, 1] \to \mathbf{R}$ be a smooth function such that for every $t \in [0, 1]$, the germ at $0 \in \mathbf{R}^n$ of $x \mapsto f(x, t)$ (we shall denote this germ by f_t) belongs to $\mathscr{E}_G(n)$. We shall denote the germ at $0 \in \mathbf{R}^n$ of $x \mapsto \dfrac{\partial f}{\partial t}(x, t)$ by $\partial f_t/\partial t$.

Suppose there exists a smooth mapping X: $\mathbf{R}^n \times [0, 1] \to T\mathbf{R}^n$ such that for every $t \in [0, 1]$ the germ X_t at $0 \in \mathbf{R}^n$ of $x \mapsto X(x, t)$ is a G-equivariant vector field germ such that $\partial f_t/\partial t - X_t(f_t) \in A + \mathfrak{m}_G^k(n)$ and $X_t(A) \subseteq A + \mathfrak{m}_G^k(n)$ for every non-negative integer k.

Then for every non-negative integer k there exists a germ $\phi \in L_G(n)$ such that $f_0 \circ \phi - f_1 \in A + \mathfrak{m}_G^k(n)$ and such that $h \circ \phi \in A + \mathfrak{m}_G^k(n)$ for every $h \in A + \mathfrak{m}_G^k(n)$.

(Stated informally, the conclusion says that "f_0 is G-right equivalent to f_1 modulo A plus terms of arbitrarily high order".)

The technical lemma is most often applied in the form of the following corollary, which, though it is a very simple application which does not use the full power of the lemma, will serve very well as an illustration of its use:

COROLLARY 5.2. *Let the Lie group G act linearly on $\mathbf{R}^n$ and let $p \in \mathfrak{m}_G(n)$ be a non-vanishing G-invariant homogeneous polynomial of degree k. Let f be a germ in $\mathfrak{m}_G(n)$ whose Taylor series begins with p (i.e. $f - p \in \mathfrak{m}_G^{k+1}(n)$). Let h be a homogeneous polynomial of degree $l > k$ and suppose $h \in J_G(p)$. Then we may find an $h_1 \in \mathfrak{m}_G^{l+1}(n)$ such that $f \sim_G f + h + h_1$.*

This is an application of Lemma 5.1 with $A = \mathfrak{m}_G^{l+1}(n)$ and $f_t = f + th$.

The technical lemma requires only a very easy computation to check its conditions (in fact, one can *choose* A so as to make this computation easy), and it enables one to kill or modify certain terms in the Taylor series of the germ under consideration. The price one pays for the ease with which the lemma can be applied is the introduction of new unwanted "error" terms (e.g.

h_1 in Corollary 5.2); however, these will generally be of higher degree (or of higher weighted degree with a suitable weighting) than the terms one has removed, so there is a real gain. Moreover, usually the technical lemma can be applied again to remove these new unwanted terms and replace them by an error of still higher degree; by repeated application of the technical lemma an entire class of terms can be killed in the Taylor series and replaced by terms of a known restricted type and an error of *arbitrarily* high degree. Finally, this residual error of arbitrarily high degree can safely be ignored, because the germs we wish to classify are always finitely determined, so that once the error is of sufficently high degree, it can be removed entirely.

As an illustration, suppose for simplicity that G acts trivially and that the Taylor series of $f \in \mathfrak{m}(2)$ has the form x^2y+terms of degree $\geqslant 4$. The Jacobian ideal of x^2y is generated by xy and x^2 and contains all monomials of degree $\geqslant 4$ divisible by x. One may conclude immediately by repeated application of Corollary 5.2 that if f is finitely determined, it is right-equivalent to a germ of the form $x^2y+h(y)$, where h depends only on the one variable y and is in $\mathfrak{m}(2)^4$. (Further simplification will then quickly yield a normal form belonging to the series D_k: $x^2y \pm y^{k-1}$.)

With these tools available (the codimension estimates and the technical lemma) it is now easy to describe the procedure for proving the theorems of Section 4. First, for Theorem 4.2:

1) The estimate of codimension given by Lemma 2.3 (or the improved version one has in the abelian case) shows that there are only a finite number of ranks for which germs of G-codimension $\leqslant 4$ can occur. Consider each of these in turn.

2) For each rank $(r|s_1, \ldots, s_k|t_1, \ldots, t_l)$ to be considered, classify all possible effective groups for this rank, using the Pontryagin duality theorem as described in Section 3. If the total rank $r+s_1+\ldots+s_k+t_1+\ldots+t_l$ is small, then germs of G-codimension $\leqslant 4$ will actually occur for most of these groups, and one continues with step 3) for each class of effective group. However, if the total rank is large (3 or 4) then there will usually be no germs of G-codimension $\leqslant 4$, as a glance at the classification list shows, and in fact, in spite of the very large number of ranks and groups which are possible when the total rank is 3 or 4, the refined estimates of codimension which take the structure of G and cubic and quartic terms into account can be applied to reject in one fell swoop almost all of these ranks and groups from consideration immediately, without needing to enumerate the groups or even the ranks individually. For the only 6 effective groups of total rank $\geqslant 3$ which survive, continue with step 3).

3) Write down the Taylor series of a germ in $\mathfrak{n}_G^3(n)$ in general form, with undetermined coefficients.

4) Divide the class being considered into several cases by making some

assumption about one of the lowest-order undetermined coefficients in the Taylor series (for example, by assuming that it is non-zero, or alternately, that it *is* zero).

5) On the basis of the assumption made in step 4, simplify the Taylor series up to right equivalence as much as possible, for example by applying a suitable linear change of coordinates to change a non-zero coefficient to some specific value, or by applying the technical lemma to kill some of the terms in the Taylor series.

6) On the basis of the form now achieved for the Taylor series, estimate the codimension of the germ as accurately as possible. If the estimate shows that the codimension is always > 4, then this case may now be concluded, and it will not appear in the classification list.

7) If the Taylor series still contains undetermined coefficients of arbitrarily high order, or undetermined coefficients which for certain values would permit a further simplification, then repeat from step 4). If the Taylor series has been simplified to polynomial form and no further simplification is possible, and if the codimension of the germs of this form is at least sometimes $\leqslant 4$, then a normal form for inclusion in the classification list has been found. The most difficult task remaining is to prove that this normal form specifies each class it covers *uniquely*, i.e., that different values of the undetermined coefficients and exponents remaining in the Taylor series yield inequivalent germs. The other information given in the classification table is usually quite easy to obtain.

8) After completing one case, go back to consider the other cases remaining by virtue of the choices made at step 4.

Remark: Steps 5 and 6 may sometimes be interchanged. If it is thought that the assumption made in step 4 will actually lead to germs of codimension $\leqslant 4$, it is better to carry out step 5 (simplification) first, but if it is suspected that the assumption taken in step 4 will make the codimension large (for example, assuming that a coefficient is 0 or some other "degenerate" value), then of course it is more efficient to perform step 6 first.

Only a few additional ideas are needed for the proof of Theorem 4.3. The main one is that if $f \in \mathfrak{n}_G^3(n)$ is G-simple, then since slight perturbations of f yield only finitely many different G-right equivalence classes, one of these must be "open" in $\mathfrak{n}_G^3(n)$, i.e., there must be germs in $\mathfrak{n}_G^3(n)$ which are right-equivalent to *any* sufficiently slight perturbation of themselves in $\mathfrak{n}_G^3(n)$. The G-codimension of such a germ will be $\leqslant \dim_{\mathbf{R}} \mathfrak{m}_G(n)/\mathfrak{n}_G^3(n)$, so in particular *there must exist germs h in $\mathfrak{n}_G^3(n)$ of at most this codimension.* Thus the estimates of codimension used in proving Theorem 4.2 can also be applied in determining which ranks and effective groups might possibly admit G-simple germs. However it is also necessary to consider the "openness" condition mentioned above directly (actually, in the form of an equivalent condition on a variant of the G-Jacobian ideal, which you will recall is a sort of tangent space to the G-right equivalence class) – this can be done by means of counts of G-invariant monomials in a fashion similar to the derivation of the more refined of the codimension estimates mentioned previously. By carrying out these counts carefully, one can show first that the codimension $\dim_{\mathbf{R}} \mathfrak{m}_G(n)/\mathfrak{n}_G^3(n)$ mentioned above must in any event be $\leqslant 4$, so the germs h mentioned above (but not necessarily the original G-simple germ f) must appear in the classification lists of Theorems 4.1 and 4.2; inspection of these lists for germs satisfying the codimension condition mentioned then reveals a very small number of ranks and effective groups for which G-simple germs can exist.

For each of these ranks and groups in turn, one now can determine the G-simple germs by applying to an initially unspecified germ f the procedure of steps 3)–8) above, whereby step 6) (estimate codimension) is to be replaced by

6′) count the G-right equivalence classes of slight perturbations of f, or check whether a perturbation of f can satisfy the openness and codimension conditions mentioned above, or whether there is an arbitrarily slight perturbation of f known to be non-simple (if f is G-simple, then all slight perturbations must be so as well).

Usually, counting the G-right equivalence classes near f can be done simply by inspection of the classification lists of Theorems 4.1 and 4.2, and of course, if the count can be carried out with the information currently available on f it determines whether or not f is G-simple.

References

[1] J. F. Adams, *Lectures on Lie Groups*, Benjamin, New York 1969.

[2] V. I. Arnold, *Normal forms of functions near degenerate critical points, the Weyl groups A_k, D_k, E_k and Lagrangian singularities*, Funktsional. Anal. Prilozhen. 6:4 (1972), 3–25 (English translation: Functional. Anal. Appl. 6 (1972), 254–272).

[3] —, *Wave front evolution and equivariant Morse lemma*, Comm. Pure Appl. Math. 29 (1976), 557–582.

[4] M. Beer, *Endliche Bestimmtheit und universelle Entfaltungen von Keimen mit Gruppenoperation*, Diplomarbeit, Regensburg 1976.

[5] Th. Bröcker and L. Lander, *Differentiable Germs and Catastrophes*, London Math. Soc. Lect. Note Ser. 17, Cambridge University Press, Cambridge 1975.

[6] V. Poènaru, *Singularités C^∞ en présence de symétrie*, Lecture Notes in Math. 510, Springer-Verlag, Berlin–Heidelberg–New York 1976.

[7] L. S. Pontryagin, *Topological Groups*, 2d ed., Gordon and Breach, New York 1966.

[8] G. Schwarz, *Smooth functions invariant under the action of a compact Lie group*, Topology 14 (1975), 63–68.

[9] P. Slodowy, *Einige Bemerkungen zur Entfaltung symmetrischer Funktionen*, Math. Z. 158 (1978), 157–170.

[10] R. Thom, *Topological models in biology*, Topology 8 (1969), 313–335.
[11] G. Wassermann, *Stability of Unfoldings*, Lecture Notes in Math. 393, Springer-Verlag, Berlin–Heidelberg–New York 1974.
[12] —, *Classification of singularities with compact abelian symmetry*, Regensburger Mathematische Schriften 1, Universität Regensburg, 1977.

Presented to the semester
Singularities
15 February–15 June, 1985