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Abstract

The Riemann problem for two-dimensional gas dynamics with isentropic and polytropic gas is considered. The initial data is constant in each quadrant and chosen so that only a rarefaction wave, shock wave, or slip line connects two neighboring constant initial states. With this restriction the existence of sixteen (resp. fifteen) genuinely different wave combinations for isentropic (resp. polytropic) gas is proved. For each configuration the relations for the initial data and the symmetry properties of the solution are given. This paper corrects the conjectured classification presented in T. Zhang and Y. Zheng, SIAM J. Math. Anal. 21 (1990) 593–630.

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AMS(MOS) Subject Classification: primary 35L65, 35L67, 76N15; secondary 65M99

1. Introduction

The study of the Riemann problem for gas dynamics has a long tradition, starting with the work of Riemann himself in the last century. In the last twenty years the Riemann problem for one-dimensional gas dynamics has been studied and the results have been published in [6], [7] and [9] (they also contain further references). More recently the research was extended toward two-dimensional scalar conservation laws [2] – [4], [8], [10]. Riemann problems for two-dimensional gas dynamics were considered in [1] and [11].

Under certain assumptions T. Zhang and Y. Zheng [11] conjectured the existence of seventeen reasonable combinations of initial data (counting two subcases individually). Six of their configurations contain no slip lines. In this paper we analyze the same problem more thoroughly. We are able to prove that for isentropic gas one of these six configurations does not exist and one is centrally symmetric. For polytropic gas both cannot exist. Moreover, one of the remaining four configurations is always axially symmetric.

After exposing the problem in the following section we classify the Riemann problem according to the combination of the elementary waves in Section 3. There it is shown that only sixteen (resp. fifteen) genuinely different configurations for isentropic (resp. polytropic) gas exist compared to seventeen found by T. Zhang and Y. Zheng [11]. These numbers are based on the same method of counting. In particular, those combinations which can be obtained by coordinate transformations are not counted.

Numerical solutions for each configuration have recently been computed by the author in joint work with J. P. Collins and H. M. Glaz. The wave structures are analyzed and illustrated by contour plots in [5].

2. Problem definition

The Euler equations of inviscid compressible isentropic flow consist of the continuity equations for the conservation of mass and momentum. For polytropic gas we have an additional equation for the conservation of energy. The conservation form of these equations in Cartesian coordinates together with the equation of state is

$$U_t + F(U)_x + G(U)_y = 0 \tag{1}$$

where

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, \quad F = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \end{pmatrix}, \quad G = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \end{pmatrix}, \quad p = A\rho^\gamma \tag{2}$$

for isentropic gas and

$$U = \begin{pmatrix} \rho \\ \rho u \\ \rho v \\ \rho E \end{pmatrix}, \quad F(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p \\ \rho u v \\ u(\rho E + p) \end{pmatrix}, \quad G(U) = \begin{pmatrix} \rho v \\ \rho u v \\ \rho v^2 + p \\ v(\rho E + p) \end{pmatrix}, \tag{2'}$$

$$E = \frac{1}{(\gamma - 1)} \frac{p}{\rho} + \frac{u^2 + v^2}{2}$$

for polytropic gas. Here ρ is the density, u the x-velocity component, v the y-velocity component, p the pressure, E the energy, $\gamma > 1$ the ratio of specific heats of the gas, and $A > 0$ a constant.

The characteristic speeds of (1) in x- (or y-) direction, i. e. the eigenvalues of the Jacobian matrix $\nabla_U F$ (or $\nabla_U G$) are $\lambda_- = u - c$, $\lambda_0 = u$ and $\lambda_+ = u + c$ (or $\lambda_- = v - c$, $\lambda_0 = v$ and $\lambda_+ = v + c$). Here the sound speed c is defined by $c^2 = \gamma p/\rho$.

The Riemann problem in the (x, y)-plane is the initial value problem for (1) with initial data

$$(\rho, u, v)(x, y, 0) = (\rho_i, u_i, v_i), \qquad i = 1, \dots, 4 \tag{3}$$

for isentropic gas and

$$(p, \rho, u, v)(x, y, 0) = (p_i, \rho_i, u_i, v_i), \qquad i = 1, \dots, 4 \tag{3'}$$

for polytropic gas where i denotes the ith quadrant.

The solution is a function of the similarity variables $\xi = x/t$ and $\eta = y/t$ and is called pseudostationary flow. Far enough away from the origin the general solution consists of four planar waves, each parallel to one of the coordinate axes, between the four constant initial states. In general, a planar wave is formed by up to three elementary waves corresponding to the eigenvalues λ_-, λ_0 and λ_+: a backward rarefaction wave $\overleftarrow{R}$ or shock wave $\overleftarrow{S}$, a slip line (resp. a contact discontinuity) for isentropic (resp. polytropic) gas J, and a forward rarefaction wave $\overrightarrow{R}$ or shock wave $\overrightarrow{S}$. This study of the two-dimensional Riemann problem is restricted to situations where each planar wave consists of a single elementary wave. Thus the initial data has to be chosen so that only a rarefaction wave, shock wave, or slip line[1] connects two neighboring constant initial states.

Before we start with the classification of the two-dimensional Riemann problem, the formulas for the one-dimensional elementary waves between two constant states are briefly reviewed.

Across a backward (resp. forward) rarefaction wave $\overleftarrow{R}$ (resp. $\overrightarrow{R}$) the corresponding Riemann invariants are constant. They are $w + \frac{2}{\gamma-1}c$ (resp. $w - \frac{2}{\gamma-1}c$) for isentropic gas and additionally the entropy s for polytropic gas, where w is the velocity. For a given left and right state (denoted by the indices l and r) we thus have

$$w_r - w_l = \frac{2\sqrt{\gamma}}{\gamma - 1}\left(\sqrt{\frac{p_l}{\rho_l}} - \sqrt{\frac{p_r}{\rho_r}}\right) =: \Phi_{lr} \qquad (\text{resp.} \quad w_l - w_r = \Phi_{lr})$$

for $\overleftarrow{R}$ (resp. $\overrightarrow{R}$). For polytropic gas we further find

$$p_l/p_r = (\rho_l/\rho_r)^\gamma$$

using the additional Riemann invariant s.

[1] In the following the use of the term *slip line* always denotes a slip line for isentropic gas but should be read to include the possibility of a contact discontinuity for polytropic gas.

The Rankine-Hugoniot conditions for the system of equations (1) give the relations between the states on each side of a shock wave S. From these conditions we can derive

$$(w_l - w_r)^2 = \frac{(p_l - p_r)(\rho_l - \rho_r)}{\rho_l \rho_r} =: \Psi_{lr}^2 \qquad (\Psi_{lr} > 0)$$

and additionally

$$\frac{\rho_l}{\rho_r} = \left(\frac{p_l}{p_r} + \frac{(\gamma-1)}{(\gamma+1)}\right) \Big/ \left(1 + \frac{(\gamma-1)}{(\gamma+1)}\frac{p_l}{p_r}\right) =: \Pi_{lr}$$

for polytropic gas.

The type of elementary wave is determined by the pressure and the velocity inequalities:

$$\begin{array}{lll} p_l < p_r, \quad w_l < w_r : \quad \overrightarrow{R} & \qquad & p_l > p_r, \quad w_l < w_r : \quad \overleftarrow{R} \\ p_l < p_r, \quad w_l > w_r : \quad \overleftarrow{S} & \qquad & p_l > p_r, \quad w_l > w_r : \quad \overrightarrow{S} \end{array}$$

Across a contact discontinuity J the (normal) velocity and the pressure are constant, but the density can jump arbitrarily. For isentropic gas a slip line J only occurs in two-dimensional flow when the density and the normal velocity are constant and the tangential velocity is discontinuous.

3. Classification

In the following we assume that the initial data (3) or (3') are chosen so that only one elementary wave connects two neighboring constant states. First we consider all combinations which involve rarefaction and shock waves exclusively. Thereafter all combinations involving slip lines are considered. For all possible configurations we give the relations which have to be satisfied by the initial data and the symmetry properties of the solution.

Configurations without slip lines.

Here we consider the configurations which only involve rarefaction and shock waves. Then there exist only three distinct relations between the pressure values in the four quadrants:

$$\begin{array}{ccc} p_2 < p_1 & \qquad p_2 < p_1 & \qquad p_2 < p_1 \\ \vee \quad\; \vee & \qquad \wedge \quad\; \vee & \qquad \vee \quad\; \vee \\ p_3 < p_4 & \qquad p_3 > p_4 & \qquad p_3 > p_4 \end{array}$$

The remaining relations can be derived from the above by coordinate transformations.

For the velocities four different relations are possible:

$$\begin{array}{cccc} u_2 = u_3 < u_4 = u_1 & u_2 = u_3 > u_4 = u_1 & u_2 = u_3 < u_4 = u_1 & u_2 = u_3 > u_4 = u_1 \\ \text{and} & \text{and} & \text{and} & \text{and} \\ v_1 = v_2 > v_3 = v_4 & v_1 = v_2 < v_3 = v_4 & v_1 = v_2 < v_3 = v_4 & v_1 = v_2 > v_3 = v_4 \end{array}$$

Altogether we get twelve configurations:

$$\begin{array}{lllll}
p_1 > p_2,\, p_4 > p_3: & \overrightarrow{R}_{21}\overrightarrow{R}_{32}\overrightarrow{R}_{34}\overrightarrow{R}_{41} & \overleftarrow{S}_{21}\overleftarrow{S}_{32}\overleftarrow{S}_{34}\overleftarrow{S}_{41} & \overrightarrow{R}_{21}\overleftarrow{S}_{32}\overrightarrow{R}_{34}\overleftarrow{S}_{41} & \overleftarrow{S}_{21}\overrightarrow{R}_{32}\overleftarrow{S}_{34}\overrightarrow{R}_{41} \\
p_1 > p_2,\, p_4 < p_3: & \overrightarrow{R}_{21}\overleftarrow{R}_{32}\overleftarrow{R}_{34}\overrightarrow{R}_{41} & \overleftarrow{S}_{21}\overrightarrow{S}_{32}\overrightarrow{S}_{34}\overleftarrow{S}_{41} & \overrightarrow{R}_{21}\overrightarrow{S}_{32}\overleftarrow{R}_{34}\overleftarrow{S}_{41} & \overleftarrow{S}_{21}\overleftarrow{R}_{32}\overrightarrow{S}_{34}\overrightarrow{R}_{41} \\
p_1 > p_2 > p_3 > p_4: & \overrightarrow{R}_{21}\overrightarrow{R}_{32}\overleftarrow{R}_{34}\overrightarrow{R}_{41} & \overleftarrow{S}_{21}\overleftarrow{S}_{32}\overrightarrow{S}_{34}\overleftarrow{S}_{41} & \overrightarrow{R}_{21}\overleftarrow{S}_{32}\overleftarrow{R}_{34}\overleftarrow{S}_{41} & \overleftarrow{S}_{21}\overrightarrow{R}_{32}\overrightarrow{S}_{34}\overrightarrow{R}_{41}
\end{array}$$

In this table and in the following, E_{ij} with $E \in \{J, \overleftarrow{R}, \overrightarrow{R}, \overleftarrow{S}, \overrightarrow{S}\}$ and $i, j \in \{1, 2, 3, 4\}$ denotes an elementary wave E between the ith and jth quadrant.

Obviously, exchanging the axes in the right column gives the neighboring one. Examination of the configurations in the last row shows that they are impossible.

For $\overrightarrow{R}_{21}\overrightarrow{R}_{32}\overleftarrow{R}_{34}\overrightarrow{R}_{41}$ we have $v_4 - v_1 = \Phi_{41}$, $v_3 - v_2 = \Phi_{32}$, $v_2 = v_1$, $v_3 = v_4$. This implies $\sqrt{p_2/\rho_2} - \sqrt{p_1/\rho_1} = \sqrt{p_3/\rho_3} - \sqrt{p_4/\rho_4}$ in contradiction to the pressure inequality.

For $\overleftarrow{S}_{21}\overleftarrow{S}_{32}\overrightarrow{S}_{34}\overleftarrow{S}_{41}$ and $\overrightarrow{R}_{21}\overleftarrow{S}_{32}\overleftarrow{R}_{34}\overleftarrow{S}_{41}$ we have $v_4 - v_1 = \Psi_{41}$, $v_3 - v_2 = \Psi_{32}$, $v_2 = v_1$, $v_3 = v_4$ yielding $\Psi_{41} = \Psi_{32}$. Since the pressure inequality gives $p_4 - p_1 < p_3 - p_2$ and $1/\rho_1 - 1/\rho_4 < 1/\rho_2 - 1/\rho_3$, this is a contradiction, too.

At this point six configurations are remaining which are examined individually in the following.

Configuration 1: $\overrightarrow{R}_{21}\overrightarrow{R}_{32}\overrightarrow{R}_{34}\overrightarrow{R}_{41}$

We have

$$p_1 > p_2,\, p_4 > p_3$$

and

$$\begin{array}{llll}
u_2 - u_1 = \Phi_{21}, & u_3 - u_4 = \Phi_{34}, & u_3 = u_2, & u_4 = u_1, \\
v_4 - v_1 = \Phi_{41}, & v_3 - v_2 = \Phi_{32}, & v_2 = v_1, & v_3 = v_4.
\end{array}$$

This gives the so-called compatibility condition $\Phi_{21} = \Phi_{34}$. For polytropic gas we have to include the following equations:

$$\rho_i/\rho_j = (p_i/p_j)^{1/\gamma} \quad \text{for} \quad (i,j) \in \{(2,1), (3,4), (3,2), (4,1)\}$$

Configuration 2: $\overrightarrow{R}_{21}\overleftarrow{R}_{32}\overleftarrow{R}_{34}\overrightarrow{R}_{41}$

We have

$$p_1 > p_2,\, p_4 < p_3$$

and

$$\begin{array}{llll}
u_2 - u_1 = \Phi_{21}, & u_4 - u_3 = \Phi_{34}, & u_3 = u_2, & u_4 = u_1, \\
v_4 - v_1 = \Phi_{41}, & v_2 - v_3 = \Phi_{32}, & v_2 = v_1, & v_3 = v_4
\end{array}$$

so that the compatibility conditions are $\Phi_{21} = -\Phi_{34}$ and $\Phi_{41} = -\Phi_{32}$. For polytropic gas we append the same equations as in Configuration 1.

Thus we must have $p_1 = p_3$ and $p_2 = p_4$ implying $u_1 - u_2 = v_1 - v_4$ and $u_4 - u_3 = v_2 - v_3$. Consequently the solutions are symmetric to $\eta - \xi = v_1 - u_1$ and $\xi + \eta = u_2 + v_2$.

Configuration 3: $\overleftarrow{S}_{21}\overleftarrow{S}_{32}\overleftarrow{S}_{34}\overleftarrow{S}_{41}$

We have

$$p_1 > p_2,\, p_4 > p_3 \tag{1}$$

and

$$\begin{aligned} u_2 - u_1 &= \Psi_{21}, & u_3 - u_4 &= \Psi_{34}, & u_3 &= u_2, & u_4 &= u_1, \\ v_4 - v_1 &= \Psi_{41}, & v_3 - v_2 &= \Psi_{32}, & v_2 &= v_1, & v_3 &= v_4. \end{aligned}$$

This gives the compatibility conditions

$$\Psi_{21} = \Psi_{34} \qquad \text{and} \qquad \Psi_{41} = \Psi_{32}. \tag{2}$$

For polytropic gas the following equations must be added:

$$\rho_i/\rho_j = \Pi_{ij} \qquad \text{for} \qquad (i,j) \in \{(2,1),\, (3,4),\, (3,2),\, (4,1)\} \tag{3}$$

Due to the compatibility conditions we have to choose $p_4 = p_2$ (which implies $\rho_4 = \rho_2$) according to the following theorem. Then the compatibility conditions (2) become a single equation and we have $u_2 - u_1 = v_4 - v_1$. Consequently the solutions are symmetric to $\eta - \xi = v_1 - u_1$.

Theorem 1. *The inequality (1), the compatibility conditions (2) and the additional equations (3) for polytropic gas can only be satisfied if $p_4 = p_2$. (For polytropic gas it is assumed that $1 < \gamma \leqslant 3$ holds.)*

Proof. For isentropic gas we apply the equation of state to the compatibility conditions (2) and set

$$\rho_1 = x\rho_3, \qquad \rho_2 = y\rho_3 \qquad \text{and} \qquad \rho_4 = z\rho_3$$

getting

$$(x^\gamma - y^\gamma)\left(\frac{1}{y} - \frac{1}{x}\right) = (z^\gamma - 1)\left(1 - \frac{1}{z}\right) \quad \text{and} \quad (x^\gamma - z^\gamma)\left(\frac{1}{z} - \frac{1}{x}\right) = (y^\gamma - 1)\left(1 - \frac{1}{y}\right).$$

We define $f(z,y)$ as the difference of the left and right hand side of the last equation (assuming x to be fixed). Now we have to prove that $f(y,z) = 0$ and $f(z,y) = 0$ for $x > y$, $z > 1$ only if $y = z$.

We find that

$$z\left(1 - \frac{z}{x}\right) f(y,z) + z(1-z) f(z,y)$$

is a quadratic polynomial in z with the roots

$$z_1(y) = \frac{xy}{y(x+1) - x} \qquad \text{and} \qquad z_2(y) = \frac{x(x^\gamma - 1)}{y^\gamma(x-1) + x^\gamma - x}.$$

By construction these roots are the only candidates for the roots of $f(y,z)$ and $f(z,y)$. After replacing z by z_1 in $f(y,z)$ and $f(z,y)$ we multiply them by a positive factor which has no influence on the roots of a function:

$$\begin{aligned} g_1(y) &= \frac{(y(x+1)-x)^\gamma}{y^\gamma(y^{-1}-x^{-1})} f(y,z_1) = \frac{(y(x+1)-x)^\gamma}{y^\gamma(1-y^{-1})} f(z_1,y) \\ &= \left(\frac{1+x^\gamma}{y^\gamma}-1\right)(y(x+1)-x)^\gamma - x^\gamma \end{aligned}$$

Since $g_1(1)=g_1(x)=0$ and $g_1(y)$ has its extremum for $y^{\gamma+1}=x(x^\gamma+1)/(x+1)$, there exists no root $y\in(1,x)$ of $g_1(y)$ and consequently of $f(y,z)$ or $f(z,y)$.

Now we define

$$\begin{aligned} g_2(y) &= (y^\gamma(x-1)+x^\gamma-x)^\gamma\,[f(y,z_2)+f(z_2,y)] \\ &= (y^\gamma(x-1)+x^\gamma-x)^\gamma\left(\frac{1}{y}(x^\gamma-1)-x^{\gamma-1}+1\right) - x^\gamma(x^\gamma-1)^\gamma\left(1-\frac{1}{x}\right). \end{aligned}$$

The common roots of $f(y,z_2)$ and $f(z_2,y)$ are a subset of the roots of $g_2(y)$. As before we have $g_2(1)=g_2(x)=0$ and we compute the first derivative of $g_2(y)$:

$$\begin{aligned} g_2'(y) &= \gamma^2 y^{\gamma-1}(x-1)(y^\gamma(x-1)+x^\gamma-x)^{\gamma-1}\left(\frac{1}{y}(x^\gamma-1)-x^{\gamma-1}+1\right) \\ &\quad - (y^\gamma(x-1)+x^\gamma-x)^\gamma\frac{1}{y^2}(x^\gamma-1) \end{aligned}$$

In order to show that $g_2'(y)$ has at most two roots in $(1,x)$ we examine

$$\widehat{g}_2(y) = y^2(y^\gamma(x-1)+x^\gamma-x)^{1-\gamma}g_2'(y)$$

which has the same roots as $g_2'(y)$ in $(1,x)$. $\widehat{g}_2(y)$ is extremal at $y=(1-\gamma^{-1})(x^\gamma-1)/(x^{\gamma-1}-1)$. Thus $g_2'(y)$ has not more than two roots and consequently $g_2(y)$ has at most one root in $(1,x)$.

On the other hand, $f(1,1)f(x,x)<0$ and $f(y,y)$ is smooth so that a root of $f(y,y)$ does exist in $(1,x)$. This must be the unique root of $g_2(y)$ in $(1,x)$, showing that $y=z$ is required as claimed. Moreover, for given x we can compute y by solving $y=z_2(y)$.

For polytropic gas we rewrite the compatibility conditions (2) to get

$$\frac{\rho_3}{\rho_2}(p_1-p_2)\left(1-\frac{\rho_2}{\rho_1}\right) = (p_4-p_3)\left(1-\frac{\rho_3}{\rho_4}\right)$$

and

$$\frac{\rho_3}{\rho_4}(p_1-p_4)\left(1-\frac{\rho_4}{\rho_1}\right) = (p_2-p_3)\left(1-\frac{\rho_3}{\rho_2}\right).$$

Then we use the additional equations (3) to eliminate ρ_i, $i = 1, \ldots, 4$ in these equations. Setting

$$p_1 = xp_3, \qquad p_2 = yp_3, \qquad p_4 = zp_3 \qquad \text{and} \qquad \varepsilon = (\gamma - 1)/(\gamma + 1)$$

the compatibility conditions are equivalent to

$$(x - y)^2(1 + \varepsilon y)(z + \varepsilon) = (z - 1)^2(y + \varepsilon)(x + \varepsilon y)$$

and

$$(x - z)^2(1 + \varepsilon z)(y + \varepsilon) = (y - 1)^2(z + \varepsilon)(x + \varepsilon z).$$

We define $f(z, y)$ as the difference of the left and right hand side of the last equation (assuming x to be fixed). Now we have to prove that $f(y, z) = 0$ and $f(z, y) = 0$ for $x > y$, $z > 1$ only if $y = z$.

The resultant $r(z)$ of $f(y, z)$ and $f(z, y)$ has the following roots:

$$z_0 = -\varepsilon, \qquad z_1 = x, \qquad z_2 = 1,$$

$$z_3^{\pm} = \frac{1}{2(1 + 2\varepsilon)} \left(\varepsilon(x + 1) \pm \sqrt{\varepsilon^2(x + 1)^2 + 4(1 + 2\varepsilon)x} \right),$$

$$z_4^{\pm} = \frac{1}{2\varepsilon(2\varepsilon^2 + \varepsilon x - x)} \Big(-2\varepsilon^3 + (1 + \varepsilon - 6\varepsilon^2 + 2\varepsilon^3)x + (1 - 3\varepsilon + 2\varepsilon^2)x^2 \pm \sqrt{x - 1}\, \sqrt{2\varepsilon^3(x - 1) + (1 - \varepsilon - 2\varepsilon^2)x}\, \sqrt{-(1 + 3\varepsilon - 2\varepsilon^2 - 2\varepsilon^3 x^{-1})x - (1 - \varepsilon)x^2} \Big)$$

By the definition of the resultant, the common roots of $f(y, z)$ and $f(z, y)$ are a subset of the roots of $r(z)$. Obviously z_0, z_1, z_2 and z_3^- are not in the interval $(1, x)$. Under the assumption for γ we have $0 < \varepsilon \leqslant 1/2$. Then $z_4^{\pm}$ is well defined and its discriminant is negative because $x - 1$ and $2\varepsilon^3(x - 1) + (1 - \varepsilon - 2\varepsilon^2)x$ are positive and $-(1 + 3\varepsilon - 2\varepsilon^2 - 2\varepsilon^3 x^{-1})x - (1 - \varepsilon)x^2$ is negative since

$$1 + 3\varepsilon - 2\varepsilon^2 - 2\varepsilon^3 x^{-1} \geqslant 1 + 3\varepsilon - 2\varepsilon - 2\varepsilon = 1 - \varepsilon > 0.$$

Thus $z_4^{\pm}$ is complex and z_3^+ is the only positive root of $r(z)$. It is easy to verify that $z_3^+ \in (1, x)$ and that $f(z_3^+, z_3^+) = 0$.

□

Configuration 4: $\overleftarrow{S}_{21} \overrightarrow{S}_{32} \overrightarrow{S}_{34} \overleftarrow{S}_{41}$

We have

$$p_1 > p_2,\; p_4 < p_3$$

and the same equations and compatibility conditions as in Configuration 3.

Necessarily we must have $p_1 = p_3$ and $p_2 = p_4$ (which implies $\rho_1 = \rho_3$ and $\rho_2 = \rho_4$) yielding $u_2 - u_1 = v_4 - v_1$ and $u_3 - u_4 = v_3 - v_2$. Consequently the solutions are symmetric to $\eta - \xi = v_1 - u_1$ and $\xi + \eta = u_2 + v_2$.

Configuration 5: $\overrightarrow{R}_{21}\overleftarrow{S}_{32}\overrightarrow{R}_{34}\overleftarrow{S}_{41}$
We have

$$p_1 > p_2,\, p_4 > p_3 \tag{4}$$

and

$$\begin{aligned} u_2 - u_1 &= \Phi_{21}, & u_3 - u_4 &= \Phi_{34}, & u_3 &= u_2, & u_4 &= u_1,\\ v_4 - v_1 &= \Psi_{41}, & v_3 - v_2 &= \Psi_{32}, & v_2 &= v_1, & v_3 &= v_4. \end{aligned}$$

This gives the compatibility conditions

$$\Phi_{21} = \Phi_{34} \qquad \text{and} \qquad \Psi_{41} = \Psi_{32}. \tag{5}$$

For polytropic gas the equations
(6)

$$\rho_3/\rho_2 = \Pi_{32}, \qquad \rho_4/\rho_1 = \Pi_{41}, \qquad \rho_2/\rho_1 = (p_2/p_1)^{1/\gamma} \qquad \text{and} \qquad \rho_3/\rho_4 = (p_3/p_4)^{1/\gamma}$$

are added.

We can prove that this configuration is impossible.

Theorem 2. *There exist no p_i, $i = 1, \ldots, 4$ satisfying the inequality (4), the compatibility conditions (5) and the additional equations (6) for polytropic gas.*

Proof. For isentropic gas we apply the equation of state to the compatibility conditions (5) getting

$$\sqrt{\rho_1^{\gamma-1}} - \sqrt{\rho_2^{\gamma-1}} = \sqrt{\rho_4^{\gamma-1}} - \sqrt{\rho_3^{\gamma-1}} \qquad \text{and} \qquad \frac{(\rho_1^\gamma - \rho_4^\gamma)(\rho_1 - \rho_4)}{\rho_1\rho_4} = \frac{(\rho_2^\gamma - \rho_3^\gamma)(\rho_2 - \rho_3)}{\rho_2\rho_3}.$$

Now we show that for any p_i, $i = 1, \ldots, 4$ satisfying the inequality (4) and the first compatibility condition the second is violated. Defining

$$\delta = 2/(\gamma - 1) \qquad \text{and} \qquad R_i = \rho_i^{1/\delta}, \quad i = 1, \ldots, 4,$$

the first compatibility condition is equivalent to $R_1 - R_2 = R_4 - R_3$. Introducing $\Delta = R_1 - R_2 > 0$ we have

$$R_1 = R_2 + \Delta \qquad \text{and} \qquad R_4 = R_3 + \Delta. \tag{7}$$

Then the second compatibility condition can be written as

$$\frac{\left[(R_2+\Delta)^{\gamma\delta} - (R_3+\Delta)^{\gamma\delta}\right]\left[(R_2+\Delta)^{\delta} - (R_3+\Delta)^{\delta}\right]}{(R_2+\Delta)^\delta (R_3+\Delta)^\delta} = \frac{\left[R_2^{\gamma\delta} - R_3^{\gamma\delta}\right]\left[R_2^\delta - R_3^\delta\right]}{R_2^\delta R_3^\delta}.$$

We define $f(\Delta)$ as the difference of the left and right hand side of the last equation. Obviously $f(0)=0$. Now we have to prove that $f(\Delta)$ does not vanish for any positive R_2, R_3 and Δ. We differentiate f with respect to Δ. Using $\gamma\delta=\delta+2$ and (7) this yields

$$\frac{\partial f}{\partial\Delta}=\frac{1}{(R_1R_4)^{\delta+1}}\{(\delta+2)R_1R_4\left[R_1^{\delta+1}-R_4^{\delta+1}\right]\left[R_1^{\delta}-R_4^{\delta}\right] \\ -\delta\left[R_1^{\delta+2}-R_4^{\delta+2}\right]\left[R_1^{\delta+1}-R_4^{\delta+1}\right]\}.$$

Introducing $R=R_1/R_4=(R_2+\Delta)/(R_3+\Delta)>1$, this derivative becomes

$$\frac{\partial f}{\partial\Delta}=\frac{R_4}{R^{\delta+1}}\left(R^{\delta+1}-1\right)\underbrace{\left\{(\delta+2)R\left(R^{\delta}-1\right)-\delta\left(R^{\delta+2}-1\right)\right\}}_{=h(R)}.$$

In order to show that $\partial f/\partial\Delta$ is negative for all positive R_2, R_3 and Δ it is equivalent to show that the function $h(R)$ defined above is negative for all R strictly greater than one. Since $h(1)=0$ we differentiate h with respect to R and get

$$h'(R)=(\delta+2)\left\{(\delta+1)R^{\delta}-1-\delta R^{\delta+1}\right\}.$$

Observing that $h'(1)=0$ we compute $h''(R)$ to find

$$h''(R)=(\delta+2)(\delta+1)\delta R^{\delta-1}\{1-R\}.$$

Obviously $h''(R)$ is negative for R strictly greater than one. Thus we have proved that $f(\Delta)$ is negative for all positive R_2, R_3 and Δ.

For polytropic gas we use the additional equations (6) to derive the following equation which is independent of ρ_i, $i=1,\ldots,4$:

$$(p_2/p_3)^{1/\gamma}\,\Pi_{32}=(p_1/p_4)^{1/\gamma}\,\Pi_{41}$$

Introducing

$$f(P)=\left(\frac{1}{P}\right)^{1/\gamma}\left(P+\frac{(\gamma-1)}{(\gamma+1)}\right)\Big/\left(1+\frac{(\gamma-1)}{(\gamma+1)}P\right)$$

this can be written as

$$f(p_3/p_2)=f(p_4/p_1).$$

Since $f(P)$ is strictly decreasing for $P\in(0,\infty)$, the last equation implies that

(8) $$p_3/p_2=p_4/p_1 \qquad \text{which is equivalent to} \qquad p_3/p_4=p_2/p_1.$$

Again using the additional equations we eliminate ρ_i, $i=1,\ldots,4$ in the first compatibility condition and get

$$\sqrt{\Pi_{41}}\left(1-(p_2/p_1)^{\frac{\gamma-1}{2\gamma}}\right)=\sqrt{p_4/p_1}\left(1-(p_3/p_4)^{\frac{\gamma-1}{2\gamma}}\right).$$

With our previous result (8) it follows that this is equivalent to $\Pi_{41} = p_4/p_1$. The only admissible solution $p_1 = p_4$ violates the inequality (4).

□

Configuration 6: $\overrightarrow{R}_{21}\overrightarrow{S}_{32}\overleftarrow{R}_{34}\overleftarrow{S}_{41}$

We have

$$p_1 > p_2,\, p_4 < p_3 \tag{9}$$

and

$$\begin{aligned} u_2 - u_1 &= \Phi_{21}, & u_4 - u_3 &= \Phi_{34}, & u_3 &= u_2, & u_4 &= u_1, \\ v_4 - v_1 &= \Psi_{41}, & v_3 - v_2 &= \Psi_{32}, & v_2 &= v_1, & v_3 &= v_4, \end{aligned}$$

so that the compatibility conditions are

$$\Phi_{21} = -\Phi_{34} \qquad \text{and} \qquad \Psi_{41} = \Psi_{32}. \tag{10}$$

For polytropic gas we have the same additional equations as in Configuration 5 and this configuration is impossible, too.

Theorem 3. *For polytropic gas there exist no p_i, $i = 1, \ldots, 4$ satisfying the inequality (9), the compatibility conditions (10) and the additional equations (6) for polytropic gas.*

Proof. As in the preceding proof we use the additional equations (6) to derive that $p_3/p_2 = p_4/p_1$. This is in contradiction to the inequality (9) which implies that $p_3/p_2 > 1 > p_4/p_1$.

□

For isentropic gas the following theorem states that we have to choose $p_3 = p_1$ and $p_4 = p_2$ (which implies $\rho_3 = \rho_1$ and $\rho_4 = \rho_2$). Thus the solutions are symmetric with respect to the point $(\xi, \eta) = \left(\frac{1}{2}(u_1 + u_2), \frac{1}{2}(v_1 + v_3)\right)$.

Theorem 4. *For isentropic gas the inequality (9) and the compatibility conditions (10) can only be satisfied if $p_3 = p_1$ and $p_4 = p_2$.*

Proof. We apply the equation of state to the compatibility conditions (10) getting

$$\sqrt{\rho_1^{\gamma-1}} - \sqrt{\rho_2^{\gamma-1}} = \sqrt{\rho_3^{\gamma-1}} - \sqrt{\rho_4^{\gamma-1}} \qquad \text{and} \qquad \frac{(\rho_1^\gamma - \rho_4^\gamma)(\rho_1 - \rho_4)}{\rho_1 \rho_4} = \frac{(\rho_3^\gamma - \rho_2^\gamma)(\rho_3 - \rho_2)}{\rho_3 \rho_2}.$$

Defining

$$\delta = 2/(\gamma - 1) \qquad \text{and} \qquad R_i = \rho_i^{1/\delta}, \quad i = 1, \ldots, 4,$$

the first compatibility condition becomes $R_1 - R_2 = R_3 - R_4$ which is equivalent to $R_1 - R_3 = R_2 - R_4$. Introducing $\Delta = R_1 - R_3$ we have $R_1 = R_3 + \Delta$ and $R_4 = R_2 - \Delta$. Then the second compatibility condition can be written as

$$\frac{\left[(R_3 + \Delta)^{\gamma\delta} - (R_2 - \Delta)^{\gamma\delta}\right]\left[(R_3 + \Delta)^\delta - (R_2 - \Delta)^\delta\right]}{(R_3 + \Delta)^\delta (R_2 - \Delta)^\delta} = \frac{\left[R_3^{\gamma\delta} - R_2^{\gamma\delta}\right]\left[R_3^\delta - R_2^\delta\right]}{R_3^\delta R_2^\delta}.$$

We define $f(\Delta)$ as the difference of the left and right hand side of the last equation. Obviously $f(0) = 0$. Now we prove that $f(\Delta)$ is strictly increasing for any positive R_2, R_3 and $\Delta \in (-R_3, R_2)$. We differentiate f with respect to Δ. Using $\gamma\delta = \delta + 2$ and introducing $R = (R_3 + \Delta)/(R_2 - \Delta) = R_1/R_4 > 1$ this yields

$$\frac{\partial f}{\partial \Delta} = \frac{R_4}{R^{\delta+1}} \left(R^{\delta+1} + 1\right) \left\{(\delta + 2)R\left(R^{\delta} - 1\right) + \delta\left(R^{\delta+2} - 1\right)\right\} > 0.$$

Hence f is strictly increasing and $\Delta = 0$ is its unique root, i. e. $p_3 = p_1$ and $p_4 = p_2$ is necessary to satisfy the compatibility conditions.

□

Configurations involving slip lines.

Now we consider all combinations involving slip lines J. There are two genuinely different configurations with four J's, one where all the J's are moving clockwise and one where two J's are moving in the opposite direction.

Three J's imply $p_1 = p_2 = p_3 = p_4$ in contradiction to the pressure inequality of the fourth wave.

Two J's are either neighbors or not. In the first case we assume that the J's are between the third quadrant and its neighbors. Then we find the following eight configurations:

$$p_1 > p_2 = p_3 = p_4: \quad \overrightarrow{R}_{21}J_{32}J_{34}\overrightarrow{R}_{41} \quad \overleftarrow{S}_{21}J_{32}J_{34}\overleftarrow{S}_{41} \quad \overrightarrow{R}_{21}J_{32}J_{34}\overleftarrow{S}_{41} \quad \overleftarrow{S}_{21}J_{32}J_{34}\overrightarrow{R}_{41}$$

$$p_1 < p_2 = p_3 = p_4: \quad \overleftarrow{R}_{21}J_{32}J_{34}\overleftarrow{R}_{41} \quad \overrightarrow{S}_{21}J_{32}J_{34}\overrightarrow{S}_{41} \quad \overleftarrow{R}_{21}J_{32}J_{34}\overrightarrow{S}_{41} \quad \overrightarrow{S}_{21}J_{32}J_{34}\overleftarrow{R}_{41}$$

As before, exchanging the axes in the right column gives the neighboring one.

In the case where the J's are not neighbors, three different combinations are possible:

$$p_1 = p_2 > p_3 = p_4: \quad J_{21}\overrightarrow{R}_{32}J_{34}\overrightarrow{R}_{41} \quad J_{21}\overleftarrow{S}_{32}J_{34}\overleftarrow{S}_{41} \quad J_{21}\overleftarrow{S}_{32}J_{34}\overrightarrow{R}_{41}$$

A case with one J would imply $u_1 = u_2 = u_3 = u_4$ or $v_1 = v_2 = v_3 = v_4$ in contradiction to the existence of three shock and rarefaction waves.

In the following these eleven configurations involving slip lines are listed with their relations for the initial data and their symmetry properties. For polytropic gas we have to include additional equations. Namely, for a rarefaction or a shock wave between the ith and jth quadrant ($i, j \in \{1, \ldots, 4\}$) we add

$$\rho_i/\rho_j = (p_i/p_j)^{1/\gamma} \qquad \text{or} \qquad \rho_i/\rho_j = \Pi_{ij},$$

respectively.

Configuration A: $\overleftrightarrow{J_{21}J_{32}J_{34}J_{41}}$ (motion in opposite directions)

We have $p_1 = p_2 = p_3 = p_4 > 0$ and

$$u_1 = u_2 < u_3 = u_4, \qquad v_1 = v_4 < v_3 = v_2.$$

The solutions are symmetric with respect to the point $(\xi, \eta) = \left(\frac{1}{2}(u_1 + u_3), \frac{1}{2}(v_1 + v_2)\right)$ for isentropic gas.

Configuration B: $\overrightarrow{J_{21}J_{32}J_{34}J_{41}}$ (clockwise motion)

We have $p_1 = p_2 = p_3 = p_4 > 0$ and

$$u_1 = u_2 > u_3 = u_4, \qquad v_1 = v_4 < v_3 = v_2.$$

The solutions have the same symmetry properties as in Configuration A.

Configuration C: $\overrightarrow{R}_{21}J_{32}J_{34}\overrightarrow{R}_{41}$

We have $p_1 > p_2 = p_3 = p_4$ and

$$u_2 - u_1 = \Phi_{21}, \qquad u_3 = u_4 = u_1, \qquad v_4 - v_1 = \Phi_{41}, \qquad v_3 = v_2 = v_1.$$

The solutions are symmetric to $\eta - \xi = v_1 - u_1$.

Configuration D: $\overleftarrow{R}_{21}J_{32}J_{34}\overleftarrow{R}_{41}$

We have $p_1 < p_2 = p_3 = p_4$ and

$$u_1 - u_2 = \Phi_{21}, \qquad u_3 = u_4 = u_1, \qquad v_1 - v_4 = \Phi_{41}, \qquad v_3 = v_2 = v_1.$$

The solutions are symmetric to $\eta - \xi = v_1 - u_1$.

Configuration E: $\overleftarrow{S}_{21}J_{32}J_{34}\overleftarrow{S}_{41}$

We have $p_1 > p_2 = p_3 = p_4$ and

$$u_2 - u_1 = \Psi_{21}, \qquad u_3 = u_4 = u_1, \qquad v_4 - v_1 = \Psi_{41}, \qquad v_3 = v_2 = v_1.$$

The solutions are symmetric to $\eta - \xi = v_1 - u_1$.

Configuration F: $\overrightarrow{S}_{21}J_{32}J_{34}\overrightarrow{S}_{41}$

We have $p_1 < p_2 = p_3 = p_4$ and the same equations as in Configuration E.

Configuration G: $\overrightarrow{R}_{21}J_{32}J_{34}\overleftarrow{S}_{41}$

We have $p_1 > p_2 = p_3 = p_4$ and

$$u_2 - u_1 = \Phi_{21}, \qquad u_3 = u_4 = u_1, \qquad v_4 - v_1 = \Psi_{41}, \qquad v_3 = v_2 = v_1.$$

Configuration H: $\overleftarrow{R}_{21}J_{32}J_{34}\overrightarrow{S}_{41}$

We have $p_1 < p_2 = p_3 = p_4$ and

$$u_1 - u_2 = \Phi_{21}, \qquad u_3 = u_4 = u_1, \qquad v_4 - v_1 = \Psi_{41}, \qquad v_3 = v_2 = v_1.$$

Configuration I: $J_{21}\overrightarrow{R}_{32}J_{34}\overrightarrow{R}_{41}$

We have $p_1 = p_2 > p_3 = p_4 > 0$ and

$$u_1 = u_2 = u_3 = u_4, \qquad v_4 - v_1 = \Phi_{41}, \qquad v_3 - v_2 = \Phi_{32}.$$

Configuration J: $J_{21}\overleftarrow{S}_{32}J_{34}\overleftarrow{S}_{41}$

We have $p_1 = p_2 > p_3 = p_4 > 0$ and

$$u_1 = u_2 = u_3 = u_4, \qquad v_4 - v_1 = \Psi_{41}, \qquad v_3 - v_2 = \Psi_{32}.$$

Configuration K: $J_{21}\overleftarrow{S}_{32}J_{34}\overrightarrow{R}_{41}$

We have $p_1 = p_2 > p_3 = p_4 > 0$ and

$$u_1 = u_2 = u_3 = u_4, \qquad v_4 - v_1 = \Phi_{41}, \qquad v_3 - v_2 = \Psi_{32}.$$

Conclusion.

Combining the results of this section we have sixteen (resp. fifteen) different configurations for the Riemann problem for isentropic (resp. polytropic) gas in two space dimensions:

$4\,R$:	$\overrightarrow{R}_{21}\,\overrightarrow{R}_{32}\,\overrightarrow{R}_{34}\,\overrightarrow{R}_{41}$	$\overrightarrow{R}_{21}\,\overleftarrow{R}_{32}\,\overleftarrow{R}_{34}\,\overrightarrow{R}_{41}$	
$4\,S$:	$\overleftarrow{S}_{21}\,\overleftarrow{S}_{32}\,\overleftarrow{S}_{34}\,\overleftarrow{S}_{41}$	$\overleftarrow{S}_{21}\,\overrightarrow{S}_{32}\,\overrightarrow{S}_{34}\,\overleftarrow{S}_{41}$	
$2\,R+2\,S$:		$\overrightarrow{R}_{21}\,\overrightarrow{S}_{32}\,\overleftarrow{R}_{34}\,\overleftarrow{S}_{41}$	(only for isentropic gas)
$4\,J$:	$\overleftrightarrow{J_{21}\;J_{32}\;J_{34}\;J_{41}}$	$\overrightarrow{J_{21}\;J_{32}\;J_{34}\;J_{41}}$	
$2\,J+2\,R$:	$\overrightarrow{R}_{21}\;J_{32}\;J_{34}\,\overrightarrow{R}_{41}$	$\overleftarrow{R}_{21}\;J_{32}\;J_{34}\,\overleftarrow{R}_{41}$	$J_{21}\,\overrightarrow{R}_{32}\;J_{34}\,\overrightarrow{R}_{41}$
$2\,J+2\,S$:	$\overleftarrow{S}_{21}\;J_{32}\;J_{34}\,\overleftarrow{S}_{41}$	$\overrightarrow{S}_{21}\;J_{32}\;J_{34}\,\overrightarrow{S}_{41}$	$J_{21}\,\overleftarrow{S}_{32}\;J_{34}\,\overleftarrow{S}_{41}$
$2\,J+R+S$:	$\overrightarrow{R}_{21}\;J_{32}\;J_{34}\,\overleftarrow{S}_{41}$	$\overleftarrow{R}_{21}\;J_{32}\;J_{34}\,\overrightarrow{S}_{41}$	$J_{21}\,\overleftarrow{S}_{32}\;J_{34}\,\overrightarrow{R}_{41}$

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