JOURNAL OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 5, Number 3, July 1992

CLASSIFICATION OF THREE-DIMENSIONAL FLIPS

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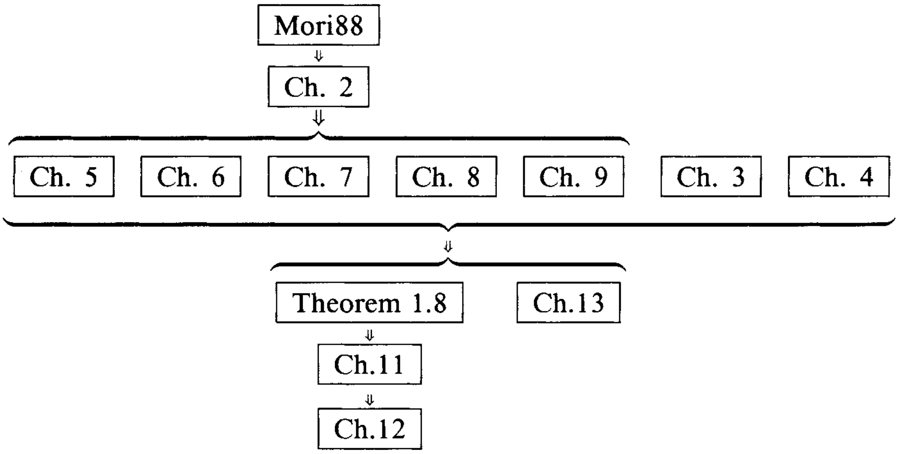
Dedicated to Professor Heisuke Hironaka on the occasion of his sixtieth birthday

CONTENTS

The following table shows the interdependece of the chapters and of [Mori88]. One important point is that Chapters 11 and 12 depend only on the statement of Theorem 1.8, not on its proof.

Received by the editors January 15, 1992.
1991 *Mathematics Subject Classification.* Primary 14E30, 14E35, 14E05, 32J17, 32J20, 14D22.
Partial financial support for Kollár was provided by the NSF under grant numbers DMS-8707320 and DMS-8946082 and by an A. P. Sloan Research Fellowship. Partial financial support for Mori was provided by the US-Japan Cooperative Science Program of the Japan Society for the Promotion of Science.



1. INTRODUCTION

In the past decade considerable attention was given to generalizing to dimension three the classical theory of minimal models of surfaces. This program was completed in [Mori88] and the following theorem was proved (for a general introduction, see [Kollár90]).

(1.1) **Theorem.** *Let X be a smooth projective three dimensional algebraic variety. There are two kinds of operations, called divisorial contractions and flips, such that repeated application of these operations transforms X into a variety X' which has the following properties:*

(1.1.1) *X' and X are birationally equivalent;*

(1.1.2) *In general X' is not smooth but has only very mild singularities (so-called terminal singularities);*

(1.1.3) *X' satisfies exactly one of the following alternatives:*

(1.1.3.1) *$K_{X'}$ is nef (i.e., it has nonnegative intersection with any curve C in X'), or*

(1.1.3.2) *There is a morphism $g : X' \to Z$ onto a lower dimensional variety such that $K_{X'}$ has negative intersection with every curve contained in a fiber of g.*

This X' is not unique, but only one of the alternatives can occur. Moreover, if (1.1.3.1) occurs then it is well understood how the different choices of X' are related to each other.

The importance of this theorem is that despite the fact that we introduce some singularities, the variety X' should be considered as much simpler than X. In fact X' is the simplest variety within the birational equivalence class of X. Thus if we want to study properties of X which are invariant under birational transformations then we should consider these properties on X'. This approach leads to the proof of several deep structure theorems (see, e.g., [Kollár91, Chapter 3] for a recent survey).

The aim of this article is to study the above process in families. To be more precise, assume that $\{X_t : t \in \Delta\}$ is a one-parameter flat family of smooth projective threefolds. Is it possible to perform the above series of operations such that at each step we have a flat family?

For a family of surfaces we have only one kind of operation, the contraction of a (-1)-curve. Deformation theory tells us that if C_0 is a (-1)-curve in X_0 then there is a flat family of (-1)-curves $\{C_t : t \in \Delta\}$ and the contraction gives a new flat family of smooth surfaces. Thus everything that we do in the central fiber can be done in a neighborhood as well.

In dimension three the situation is more complicated. The very first step was considered already in [Mori82]. It turns out that in the category of *algebraic spaces* a divisorial contraction of X_0 can be extended to a divisorial contraction of the family $\{X_t\}$. It is not clear that the same can be done in the category of schemes, let alone in the category of projective varieties. Since projectivity has a central role in the three-dimensional theory, this is a troubling prospect.

One of the main results of the article is that by choosing the sequence of contractions and flips with a little care, the above process can be performed in such a way that at each step we have a flat projective family of varieties. The most interesting part is, of course, the following.

(1.2) **Theorem.** *Let X/T be a flat family of smooth projective three-dimensional algebraic varieties over a scheme T. There are two kinds of operations, called (relative) divisorial contractions and (relative) flips, such that repeated application of these operations transforms X/T into X'/T which has the following properties:*

(1.2.1) *There is a rational map $X/T \cdots > X'/T$ which induces a birational equivalence on every fiber;*

(1.2.2) *In general X'/T is not smooth but every fiber has only very mild singularities (so-called terminal singularities);*

(1.2.3) *X'/T satisfies exactly one of the following alternatives:*

(1.2.3.1) *$K_{X'/T}$ is relatively nef (i.e., it has nonnegative intersection with any curve C that is contained in one of the fibers of X'/T), or*

(1.2.3.2) *There is an equidimensional scheme Z/T of relative dimension at most 2 and a surjective morphism $g : X'/T \to Z/T$ such that $K_{X'/T}$ has negative intersection with every curve contained in a fiber of g.*

This X' is not unique, but only one of the alternatives can occur. Moreover, if (1.2.3.1) occurs then it is well understood how the different choices of X' are related to each other.

These results can be used to investigate families of projective threefolds. In particular, one obtains the following results:

(1.3) **Theorem** (Deformation invariance of plurigenera). *Let $\{X_t : t \in T\}$ be a flat family of smooth projective threefolds. Assume that T is connected.*

Then $h^0(X_t, \mathscr{O}_{X_t}(nK_{X_t}))$ is independent of $t \in T$ for every $n \geq 0$.

(1.4) **Theorem** (Moduli space for threefolds of general type). *Let $\mathscr{M}$ be the functor "families of threefolds of general type modulo birational equivalence" (see (12.7.5) for a precise definition).*

Then there is a separated algebraic space $\mathbf{M}$ which coarsely represents $\mathscr{M}$. Every connected component of $\mathbf{M}$ is of finite type.

We are also able to handle complex analytic deformations of projective varieties:

(1.5) **Theorem.** *Let $g : X \to S$ be a proper smooth morphism of complex spaces. Assume that the fiber X_s is a projective threefold for some $s \in S$. Let*

$$X_s = X_s^0 \dashrightarrow \quad \cdots \quad \dashrightarrow X_s^n = X_s'$$

be any sequence of divisorial contractions and flips. Then there is an open neighborhood $s \in U \subset S$ such that the above sequence can be extended to a sequence of fiberwise bimeromorphic maps

$$X/U = X^0/U \dashrightarrow \quad \cdots \quad \dashrightarrow X^n/U = X'/U.$$

The fibers of X'/U have only terminal singularities. If $K_{X_s'}$ is nef then $K_{X'/U}$ is relatively nef. If there is a Fano contraction $g_s : X_s' \to Z_s$ then there is an equidimensional complex space Z/U of relative dimension at most 2 extending Z_s and a surjective morphism $g : X'/U \to Z/U$ such that $K_{X'/U}$ has negative intersection with every curve contained in a fiber of g.

This result has several consequences for possibly nonprojective deformations of projective threefolds:

(1.6) **Corollary.** *Let $g : X \to S$ be a proper smooth map of complex spaces. Assume that the fiber X_s is a projective threefold for some $s \in S$. Then there is an open neighborhood $s \in U \subset S$ such that:*

(1.6.1) *$h^0(X_u, \mathcal{O}_{X_u}(nK_{X_u}))$ is independent of $u \in U$ for every $n \geq 0$.*

(1.6.2) *If X_s is of general type then X_u is projective for every $u \in U$. (Note that in general g is not projective over U.)*

Most of the effort to prove (1.2) will be spent on understanding flips on a single threefold. This amounts to analyzing the following situation in great detail.

Let $f : X \to Y$ be a proper bimeromorphic morphism of complex spaces which satisfies the following conditions:

(i) X has only terminal singularities;

(ii) Y is normal with a distinguished point $Q \in Y$;

(iii) $f^{-1}(Q)$ consists of a single irreducible curve $C \subset X$;

(iv) The canonical class of X has negative intersection number with C.

In the above situation we say that $f : X \supset C \to Y \ni Q$ is an *extremal nbd.* We usually think of Y as being a germ around Q.

Extremal nbds come in two types. Both are of considerable interest in the study of birational transformations of threefolds. The two types are distinguished by the exceptional set of the map f. This can be either one- or two-dimensional.

If the exceptional set is one dimensional then it coincides with C. We will say that the extremal nbd is isolated. In this case K_Y is not $\mathbb{Q}$-Cartier.

If the exceptional set is a divisor then K_Y is $\mathbb{Q}$-Cartier; in fact, Y is terminal at Q. We will say that the extremal nbd is divisorial.

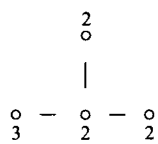
While these two cases are very different, the local computations along the curve C are very similar. Frequently it is very hard to tell which case occurs, even when an extremal nbd is given by explicit equations. In some sense the divisorial case can be considered as the degenerate version of the isolated contraction case, though at the moment we cannot attach any clear meaning to this statement.

Building on results of [Mori88] we prove two results about extremal nbds.

(1.7) **Theorem** (Reid's conjecture about general elephants). *Let $f : X \supset C \to Y \ni Q$ be an extremal nbd. Then the general member of $|-K_X|$ and the general member of $|-K_Y|$ have only DuVal singularities.*

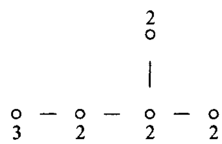
(1.8) **Theorem.** *Let $f : X \supset C \to Y \ni Q$ be an* isolated *extremal nbd. Let $t \in \mathscr{I}_Q \subset \mathscr{O}_Y$ be a general element of the ideal of Q and let $H' = (t = 0)$. Then H' is either a cyclic quotient singularity or one of the following singularities described by the dual graph of their minimal resolution.*

Triple points:



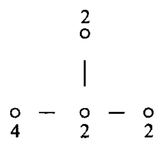
(*dihedral quotient*)

or



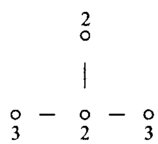
(*dihedral quotient*)

Quadruple points:



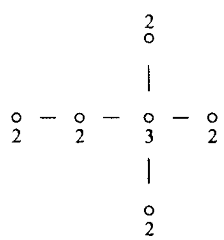
(*dihedral quotient*)

or



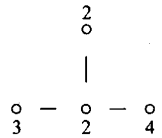
(*tetrahedral quotient*)

or



(*not a quotient singularity*)

Quintuple point:



(*icosahedral quotient*)

The basic idea of the proof of these results is the following method. Let $f : X \supset C \to Y \ni Q$ be an extremal nbd. Assume that we already have a member $D \in |-iK_X|$. Consider the following exact sequence:

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(-(i-1)K_X) \to \mathcal{O}_D(-(i-1)K_X|D) \to 0.$$

Since $H^1(X, \mathcal{O}_X(K_X)) = 0$, we know that every section of

$$H^0(D, \mathcal{O}_D(-(i-1)K_X|D))$$

lifts to a section of $H^0(X, \mathcal{O}_X(-(i-1)K_X))$. If we understand D sufficiently well then this way we get some information about the general section of $\mathcal{O}_X(-(i-1)K_X)$.

In principle one could start with a very large i, but in practice this is very difficult. Fortunately in [Mori88] it was shown that one can always find a D for $i = 1$ or 2.

Chapter 2 contains the proof of (1.7). In most cases this was already done in [Mori88, Chapter 9]. The proof of the remaining cases is similar. It uses the above observation and computations similar to those in [Mori88].

The easy part of (1.8) is done in Chapter 3. This is the case when the general member of $|-K_Y|$ has a type A singularity. In fact in this case one can forget about X and prove directly that the general hyperplane section of Y has a cyclic quotient singularity. This approach also points to a weakness of the method that we use. The general member H of $|\mathcal{O}_X|$ containing the curve C will sometimes have fairly complicated singularities while its image $H' \subset Y$ is simpler. This is so since the curve $C \subset H$ corresponds to a (-1)-curve in the resolution of H. Therefore it seems very natural to forget about X and work directly on Y. Unfortunately we do not know how to do this.

In Chapter 4 those extremal nbds that do not have points of index greater than 2 are considered. To complete some results of Chapter 2, divisorial nbds are also considered. The methods used are independent of [Mori88]. As an application we show that certain nbds with index four points cannot be isolated. Chapter 5 is devoted to the study extremal nbds with three singular points. Such nbds can never be isolated, thus we do not need to study them in greater detail.

Chapters 6–10 contain the rest of the proof of (1.8). The basic idea—as presented above—is very simple, but in practice it requires long computations and a thorough knowledge of the infinitesimal structure of the extremal nbd. In several cases our results are more complete than is strictly necessary for the proof of (1.8).

The proof that flips are continuous in families is given in Chapter 11. The main problem is for one-parameter families. Assume that $f : X_t \supset C_t \to Y_t \ni Q_t$ is a one-parameter family of extremal nbds. It is easy to see that the Y_t glue together into a four-dimensional space $\mathscr{Y}$. If H_0' is a general hyperplane section of Y_0 through Q_0 then we can view $\mathscr{Y}$ as the total space of a two-dimensional family of deformations of H_0'. Therefore we can hope to understand the canonical modification of $\mathscr{Y}$ if we understand sufficiently the deformation theory of H_0'.

In most cases H_0' is a quotient singularity, and exactly this aspect of their deformation theory was analyzed in [KSB88, §3]. In the remaining case H_0 is a rational quadruple point. Their deformation theory was recently analyzed in detail by [de Jong-van Straten88]. Using their analysis [Stevens91b] obtained the necessary results for quadruple points.

We would also like to point out that this method gives a new proof of the existence of flips using only the existence of flips in the semistable reduction case. At the moment, however, this proof is considerably longer than the original one.

Chapter 12 contains the proof of (1.2) Theorem and the proof of the applications (1.3–1.5). It also contains several auxiliary results that may be useful in different situations too.

Finally, in Chapter 13 flips are studied in more detail. We prove that all the cases not excluded so far do indeed occur and we determine the flip in the exceptional cases. We hope to discuss the flip for the two main series in a subsequent paper.

These computations show that the behavior of extremal nbds in families can be quite complicated. For example if $f : X_t \supset C_t \to Y_t \ni Q_t$ is a one-parameter family of extremal nbds and C_0 is an irreducible curve then it can easily happen that C_t is reducible for every $t \neq 0$. Since our procedure of flipping is to flip one curve at a time, this shows that the *procedure of flipping* is not continuous in families. Of course, we know that the end result is continuous.

We also give an example of an extremal nbd $X \supset C$ with its flip $X^+ \supset C^+$ such that the curve C has many irreducible components but the curve C^+ is irreducible (13.7.1).

Finally in an appendix we make a list of nonsemistable isolated extremal nbds and collect all the results about them that are scattered all over the article.

We believe that similar computations will yield a complete description of divisorial extremal nbds or extremal nbds with reducible central curve as well. However, the article is long enough already as it is, and therefore we restrict ourselves to treating the divisorial extremal nbd case only if not much extra work is required.

Some of the results of this article were announced in [Mori89,90] and [Kollár90].

Terminology. (T.1) By a *three-dimensional extremal curve neighborhood* we mean the germ of a three-dimensional complex space X along a compact curve C that satisfies the following properties:

(T.1.1) There is a germ of a complex space $Y \ni Q$ and a proper bimeromorphic morphism $f : X \supset C \to Y \ni Q$ such that $C = f^{-1}(Q)$.

(T.1.2) $-K_X$ is $\mathbb{Q}$-Cartier and f-ample.

(T.2) A three-dimensional extremal curve neighborhood $X \ni C$ is called *terminal* (resp. *canonical*) if X has terminal (resp. canonical) singularities.

(T.3) The curve C will be called the *central curve* of $X \supset C$.

(T.4) A three-dimensional extremal curve neighborhood $X \supset C$ is called *isolated* if $f : X \setminus C \to Y \setminus Q$ is an isomorphism. Otherwise it is called *divisorial*. If $X \supset C$ is divisorial then the exceptional set of f contains a divisor.

(T.5) In this paper the expression *extremal nbd* means a three-dimensional extremal curve neighborhood with terminal singularities and irreducible central curve.

(T.6) Let $g : U \to V$ be a proper bimeromorphic morphism of normal and irreducible complex spaces. Let $E \subset U$ be the exceptional set. Assume that $\dim E \leq \dim U - 2$ and that $-K_U$ is $\mathbb{Q}$-Cartier and g-ample.

By the *flip* of g (or, if no confusion is likely, by the flip of U) we mean a proper bimeromorphic morphism of normal and irreducible complex spaces $g^+ : U^+ \to V$ with exceptional set E^+ such that

(T.6.1) $\dim E^+ \leq \dim U^+ - 2$ and

(T.6.2) K_{U^+} is $\mathbb{Q}$-Cartier and g^+-ample.

In general the flip may not exist but it is unique if it does.

A superscript $^+$ will always refer to a flip.

(T.7) Let $k[x_1, \dots, x_n]$ be a polynomial ring and let G be an abelian group. A function $\alpha : \{x_1, \dots, x_n\} \mapsto G$ is called a G-weight. This will be abbreviated as G-wt or even wt if no confusion is likely. α can be multiplicatively extended to a map

$$\alpha : \{\text{all monomials in } x_1, \dots, x_n\} \to G.$$

If $f = \sum a_I x^I$ is a polynomial or powerseries in the variables $x_1, \dots, x_n$ then for $g \in G$ we define

$$f_{\alpha=g} = \sum_{\alpha(x^I)=g} a_I x^I,$$

which will be called the wt g part of f.

If G is ordered then we define

$$\alpha(f) = \min_{a_I \neq 0} \alpha(x^I).$$

We will use these notions in two cases. First, when $G \cong \mathbb{Z}_n$. This coincides with the terminology of [Mori88, 2.5]. Second, when $G = \mathbb{R}$ (or a subgroup of $\mathbb{R}$). The "order" defined in [Mori88, 2.5] is thus an $\mathbb{R}$-wt (or a $\mathbb{Z}$-wt) in the current terminology. We decided to change since here we need the wt function to blow up, and the generally accepted terminology is "weighted blow-up."

Classification of extremal nbds. (C.1) We will use the following notation for terminal singularities.

An index one terminal singularity is the same as an isolated cDV point. We will say that an index one terminal singularity has type cA, cD, or cE if the general hyperplane section is a DuVal singularity of type A, D, or E. A smooth point is considered to have type cA. We extend this terminology for higher index points as follows:

name	description	index
cA/n	$(xy + f(z^n, t) = 0)/\mathbb{Z}_n(1, -1, a, 0)$	$n \geq 1$
$cAx/2$	$(x^2 + y^2 + g(z, t) = 0)/\mathbb{Z}_2(0, 1, 1, 1)$ where $\operatorname{mult}_0 g \geq 4$.	2
$cAx/4$	$(x^2 + y^2 + h(z, t) = 0)/\mathbb{Z}_4(1, 3, 1, 2)$	4
cD/n	quotient of an index one cD point	$n = 1, 2, 3$
cE/n	quotient of an index one cE point	$n = 1, 2$

Thus, for instance, an index one *cA* type point is also called *cA*/1. We will frequently leave out the /n part of the notation if no confusion is likely or the index is not specified. This will be used most frequently with *cA*, which refers to any terminal singularity of type *cA*/n. Note that *cA* does not include *cAx*/n.

The following table shows the relationship between the notation of [Mori88] and the current notation.

germ of an extremal nbd		threefold singularity
$IA, IA^\vee$	$\to$	$cA/n, cAx/2, cD/n, cE/n$ $(n \geq 2)$
IC	$\to$	cA/n $(n \geq 2$, cyclic quotient)
$IIA, IIB, II^\vee$	$\leftrightarrow$	$cAx/4$
III	$\leftrightarrow$	index one point

In the above notation the letter A, D, or E also indicates the cover of the general member of $|-K|$. We reproduce the list of [Reid87, p. 393] (with a typographical error corrected).

name	cover of general elephant
cA/n	$A_{k-1} \xrightarrow{n:1} A_{kn-1}$
$cAx/2$	$A_{2k-1} \xrightarrow{2:1} D_{k+2}$
$cAx/4$	$A_{2k-2} \xrightarrow{4:1} D_{2k+1}$
$cD/2$	$D_{k+1} \xrightarrow{2:1} D_{2k}$
$cD/3$	$D_4 \xrightarrow{3:1} E_6$
$cE/2$	$E_6 \xrightarrow{2:1} E_7$

(C.2) Let $f : X \supset C \to Y \ni Q$ be an extremal nbd (isolated or divisorial). Let $E_X \subset X$ be a general member of $|-K_X|$ and let $E_Y = f(E_X) \subset Y$. Note that E_Y need not be a general member of $|-K_Y|$ if the nbd is divisorial.

(C.3) An extremal nbd $f : X \supset C \to Y \ni Q$ is said to be *semistable* if E_Y is a DuVal singularity of type A. (This definition is slightly more general than the one in [Kawamata88].)

(C.4) A semistable extremal nbd $X \supset C \to Y \ni Q$ is said to be of type $k1A$ if E_X has only one singular point. It is said to be of type $k2A$ if E_X has two singular points. (There are no other cases.)

(C.5) An extremal nbd $X \supset C \to Y \ni Q$ is said to be of type kAD if E_X has a point of type A_n $(n \geq 4)$ and a point of type D_{2m} $(m \geq 1)$. (D_2 is by definition two points of type A_1.) An extremal nbd $X \supset C \to Y \ni Q$ is said to be of type $k3A$ if E_X has three points of type A_1, A_2, and A_n $(n \geq 1)$.

(C.6) An extremal nbd $X \supset C \to Y \ni Q$ is said to be of type cD if it has exactly one singular point of index at least 2 and this is a cD type point. Thus the index is 2 or 3. An extremal nbd $X \supset C \to Y \ni Q$ is said to be of type cE if it has exactly one singular point of index at least 2 and this is a cE type point. Thus the index is 2. An extremal nbd $X \supset C \to Y \ni Q$ is said to be of type $cAx/2$ if it has exactly one singular point of index at least 2 and this is a $cAx/2$ type point. Thus the index is 2.

(C.7) An extremal nbd $X \supset C \to Y \ni Q$ is said to be of type IIA (resp. IIB, IC, $II^\vee$) if it has exactly one singular point P of index at least 2 and locally at P the extremal nbd $X \supset C \ni P$ is of type IIA (resp. IIB, IC, $II^\vee$) (cf. [Mori88, Appendix A]). (A type IA point does not describe an extremal nbd sufficiently. In the new terminology extremal nbds with IA points are: $k1A$, $k2A$, kAD, $k3A$, cD, cE, $cAx/2$.)

2. General members of $|-K|$

(2.1) **Definition.** Let E be a surface with a curve C (which may be empty) and let $\pi : M \to E$ be a resolution such that the exceptional curves for π and the irreducible components of $\pi^{-1}(C)$ form a divisor, say F, with normal crossing. We denote by $\Delta(M \to E \supset C)$ the dual graph of the divisor F. If $C = \varnothing$, (resp. π is the minimal resolution, $C = \varnothing$, and π is the minimal resolution) then we may simply write $\Delta(M \to E)$ (resp. $\Delta(E \supset C)$, $\Delta(E)$). An irreducible curve in F, say D, is denoted by its name D or by $\bullet$ (resp. $\circ$) if D is contained (resp. not contained) in the proper transform of C by π, and if D is proper then we attach the number $-(D^2)$ to the vertex. We may omit the numbers if it does not cause confusion as in (2.2).

(2.2) **Theorem.** *Let $f : X \supset C \to Y \ni Q$ be an extremal nbd. Then the general member E_X of $|-K_X|$ and $E_Y = f(E_X) \in |-K_Y|$ have only DuVal singularities. To be precise, the minimal resolution of E_Y dominates E_X and we have the list depending on the singularities of $X \supset C$. (In the text, k is the axial multiplicity of a certain point of X and different from the k in the labels like $k1A$).*

(2.2.1) Cases IA, $IA^\vee$, $IIA(+III)$: *In this case,* $E_X \not\supset C$;

(2.2.1.1) $cA(+III)$: $\Delta(E_X) = \Delta(E_Y)$ *is* A_{mk-1}

$(k1A)$
$$\underbrace{\circ - \cdots - \circ}_{mk-1}$$

(2.2.1.2) $cD/3(+III)$: $\Delta(E_X) = \Delta(E_Y)$ *is* E_6,

(2.2.1.3) $IIA(+III)$: $\Delta(E_X) = \Delta(E_Y)$ *is* D_{k+2},

where m and k are the index and the axial multiplicity of the non-Gorenstein point.

(2.2.1′) Cases *II*ˇ, *cAx/2, cD/2, cE/2*: *In this case* , $E_X \not\supset C$ *and* $X \supset C$ *is divisorial*;

(2.2.1′.1) *cAx/2*: $\Delta(E_X) = \Delta(E_Y)$ *is* D_4,

(2.2.1′.2) *cD/2*: $\Delta(E_X) = \Delta(E_Y)$ *is* D_{2k},

(2.2.1′.3) *cE/2*: $\Delta(E_X) = \Delta(E_Y)$ *is* E_7,

(2.2.1′.4) *II*ˇ: $\Delta(E_X) = \Delta(E_Y)$ *is* D_{k+2},

where k *is the axial multiplicity of the non-Gorenstein point.*

(2.2.2) Case *IC*: (E_Y, Q) *is* D_m *and* $\Delta(E_X \supset C)$ *is*

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    m−3
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where m , *the index of the IC point of* C , *is odd and* ≥ 5 .

(2.2.2′) Case *IIB*: *In this case,* $X \supset C$ *is divisorial.* (E_Y, Q) *is* E_6 *and* $\Delta(E_X \supset C)$ *is*

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          o
          |
o — o — o — o — •
```

(2.2.3) Case *exceptional* $IA + IA$: *The two IA points are an ordinary point of odd index* $m \geq 5$ *and a* cA *point of index* 2 *and axial multiplicity* k , *and we have* $(K_X \cdot C) = -1/2m$. (E_Y, Q) *is* D_{2k+m}, SingE_X *is* $A_{m-1} + D_{2k}$ $(A_{m-1} + A_1 + A_1$ *if* $k = 1)$ *and* $\Delta(E_X \supset C)$ *is*

(*kAD*)

```
                                      o
                                      |
o — ··· — o — • — o — ··· — o — o
\_________/
   m−1≥4
```

(2.2.3′) Case $IA + IA + III$: *In this case* , $X \supset C$ *is divisorial. The two IA points are both ordinary and of indices* 2 *and* m (*odd,* ≥ 3) . *Furthermore* (E_Y, Q) *is* D_{m+2} . *The graph* $\Delta(E_X \supset C)$ *is*

(*k*3*A*)

```
                  o
                  |
o — ··· — o — • — o
\_________/
    m−1
```

(2.2.4) Case *semistable* $IA + IA$: (E_Y, Q) *is* $A_{km+ln-1}$ *and* $\Delta(E_X \supset C)$ *is*

(*k*2*A*)

```
o — ··· — o — • — o — ··· — o ,
\_________/       \_________/
   km−1              ln−1
```

where m *and* k *are the index and the axial multiplicity of a singular point of* X *on* C *and* l *and* n *are those of the other singular point.*

(2.2.5) Gorenstein case $E_X \simeq E_Y$ *are smooth,* $E_X \not\supset C$, *and* $(E_X \cdot C) = 1$.

(2.3) **Definition.** The non-Gorenstein extremal nbds $X \supset C$ are divided into cases as follows. In the cases (2.2.1.1) (resp. (2.2.3), (2.2.3′), (2.2.4)), we say that $X \supset C$ is $k1A$ (resp. kAD, $k3A$, $k2A$) by listing the singularities of E_X (or equivalently, those of $E_X^\sharp$). In the rest of (2.2.1) and (2.2.2), we classify $X \supset C$ by its unique non-Gorenstein point.

(2.4) *Remark.* (2.4.1) For isolated extremal nbds X, E_Y is a general member of $|-K_Y|$ by $|-K_X| \simeq |-K_Y|$.

(2.4.2) The assertion that $|-K_X|$ has a DuVal member for extremal nbds $X \supset C \simeq \mathbb{P}^1$ is completed in this chapter (special case of Reid's general elephant conjecture).

(2.4.3) From the 5 cases of the table in [Mori88, (B)], our division comes out as follows:

(2.4.3.1) Case *IA, IIA, IA*ˇ, or *II*ˇ (and one *III* point): (2.2.1), (2.2.1′);

(2.4.3.2) Case *IC* or *IIB*: (2.2.2), (2.2.2′);

(2.4.3.3) Case two *IA* points of indices m, 2 (and one *III* point): (2.2.3), (2.2.3′), (2.2.4);

(2.4.3.4) Case two *IA* points of indices ≥ 3: (2.2.4);

(2.4.3.5) Case Gorenstein X: (2.2.5).

(2.4.4) In the cases (2.4.3.4) and (2.4.3.5), our (2.2) is proved in [Mori88, (9.9.3) and (B.2)] (cf. also [Mori88, §10]). In the case (2.4.3.1), our (2.2) is partly proved by [Mori88, (7.3)] (cf. also [Mori88, §10]) and [Reid87, (6.4.B)]; we still need to prove

(2.4.4.1) the nonexistence of type *III* points in case (2.2.1′),

(2.4.4.2) the divisoriality of $X \supset C$ in case (2.2.1′), and

(2.4.4.3) $\Delta(E_X)$ is D_4 in case (2.2.1′.1).

The assertions (2.4.4.1) and (2.4.4.2) follow from (4.5) and (4.7) and (2.4.4.3) is done in (4.8.5.7). The divisoriality of (2.2.2′) is done in (4.5). As for case (2.2.3′), the divisoriality is proved in Chapter 5. Thus it remains to treat the cases (2.4.3.2) and (2.4.3.3) in this chapter.

In our cases, we have a "good" member in $|-2K_X|$ by [Mori88, (7.3)(ii)]. Therefore the following is important in our proof.

(2.5) **Lemma.** *Let* $X \supset C$ *be an extremal nbd with* $D \in |-2K_X|$ *such that* $D \cap C = \{P\}$ *for some* $P \in C$. *Then the natural map*

$$H^0(X, \mathcal{O}_X(-K_X)) \to \mathcal{O}_D(-K_X)$$

is surjective, where $\mathcal{O}_D(-K_X)$ *is the stalk at* P.

Proof. From the short exact sequence

$$0 \to \mathcal{O}_X(K_X) \to \mathcal{O}_X(-K_X) \to \mathcal{O}_D(-K_X) \to 0,$$

we have

$$H^0(\mathcal{O}_X(-K_X)) \to H^0(\mathcal{O}_D(-K_X)) \to H^1(\mathcal{O}_X(K_X)),$$

where the last term is zero by the Grauert-Riemenschneider vanishing theorem. □

For the proof of (2.2), we will use the notation of [Mori88, (8.8) and (8.9)]. We start with a general

(2.6) **Lemma.** (2.6.1) *Let $a \geq 1$ and let $L_1, \dots, L_a$ and $M_1, \dots, M_a$ be ℓ-invertible $\mathcal{O}_C$-modules such that $\bigoplus_i L_i$ is ℓ-isomorphic to $\bigoplus_i M_i$. Then, after renumbering M_i's, we have ℓ-isomorphisms $L_i \simeq M_i$ for all i.*

(2.6.2) *Let L and M be locally ℓ-free $\mathcal{O}_C$-modules and*

$$(2.6.2.1) \qquad 0 \to L \to E \to M \to 0$$

an ℓ-exact sequence of $\mathcal{O}_C$-modules. If $H^1(C, L\tilde{\otimes}M^{\tilde{\otimes}(-1)}) = 0$ for the sheaf $L\tilde{\otimes}M^{\tilde{\otimes}(-1)}$ (forgetting the ℓ-structure), then the ℓ-sequence is ℓ-split.

Proof. (2.6.1) is standard. (2.6.2) is reduced to lifting the ℓ-homomorphism id_M to an ℓ-homomorphism $M \to E$ by considering $(2.6.2.1)\tilde{\otimes}M^{\tilde{\otimes}(-1)}$. This follows from the vanishing of H^1. □

(2.7) **Lemma.** *Let $X \supset C \simeq \mathbb{P}^1$ be an extremal nbd that is locally primitive. Then*

$$Cl^{sc}(X) \overset{\tilde{\otimes}\mathcal{O}_C}{\to} \mathrm{Pic}^{\ell}(C) \overset{ql_C}{\to} QL(C)$$

are isomorphisms (cf. [Mori88, (8.9.1)(ii)]*), where* $\mathrm{Pic}^{\ell}(C)$ *denotes the set of ℓ-isomorphism classes of ℓ-invertible $\mathcal{O}_C$-modules.*

Proof. The homomorphisms induce isomorphisms $\mathrm{Pic}\,X \simeq \mathrm{Pic}\,C \simeq \mathbb{Z}$ [Mori88, (1.3)] and their quotient isomorphisms $Cl^{sc}(X)/\mathrm{Pic}\,X \simeq \mathrm{Pic}^{\ell}(C)/\mathrm{Pic}(C) \simeq QL(C)/\mathbb{Z}$ by the local primitivity [Mori88, (1.8)]. □

(2.7.1) *Remark.* (2.7.1.1) We note that (2.7) applies to our cases (2.2.2) and (2.2.3), and we may identify these for simplicity of notation.

(2.7.1.2) If P is a primitive point of index m, then one can associate to $nP^{\sharp}$ ($n \in \mathbb{Z}$) a divisor $[n/m]P$ on C with ℓ-structure $\mathcal{O}_C([n/m]P)^{\sharp} \subset \mathcal{O}_{C^{\sharp}}(nP^{\sharp})$. This is compatible with the above identification.

(2.8) **Lemma.** *Assume that the canonical lifting $C^{\sharp}$ of C to the canonical covers of X at arbitrary non-Gorenstein points are smooth. Then every locally ℓ-free $\mathcal{O}_C$-module E is of the form $\widetilde{\bigoplus} L_i$ for some ℓ-invertible $\mathcal{O}_C$-modules L_i.*

Proof. We only treat $\mathcal{O}_C$-modules E of rank 2 since other cases are similar. Let X be Gorenstein outside of two points P and R, which are of indices m and n, respectively, where $m > 1$ and $n \geq 1$. Let L be a direct summand of E such that $\mathrm{rk}\,L = 1$ and $\deg L \geq \deg(E/L)$. Then L and E/L are ℓ-invertible sheaves by the induced ℓ-structures. Let $ql_C L = \deg L + a_1 P^{\sharp} + b_1 R^{\sharp}$ and $ql_C(E/L) = \deg(E/L) + a_2 P^{\sharp} + b_2 R^{\sharp}$, where $0 \leq a_1, a_2 < m$ and $0 \leq b_1, b_2 < n$. If $\deg L > \deg(E/L)$, then we have $\deg L\tilde{\otimes}(E/L)^{\tilde{\otimes}(-1)}$ by

$$\begin{aligned} ql_C(L\tilde{\otimes}(E/L)^{\tilde{\otimes}(-1)}) &= ql_C(L) - ql_C(E/L) \\ &= (\deg L - \deg(E/L) - 2) + (m + a_1 - a_2)P^{\sharp} \\ &\quad + (n + b_1 - b_2)R^{\sharp}. \end{aligned}$$

Thus $E \simeq L\tilde{\oplus}(E/L)$ if $\deg L > \deg(E/L)$ (2.6). If $\deg L = \deg(E/L)$, then we can choose L so that $a_1 \geq a_2$. Then $E \simeq L\tilde{\oplus}(E/L)$ from

$$ql_C(L\tilde{\otimes}(E/L)^{\tilde{\otimes}(-1)}) = -1 + (a_1 - a_2)P^\sharp + (n + b_1 - b_2)R^\sharp. \quad \square$$

Alternative proof. let $u = \text{l.c.m.}\{m, n\}$. We can take an u-sheeted cover $\mathbb{P}^1 \to C$ which ramifies at P, R. Then the ℓ-decomposition corresponds to a $\mathbb{Z}_u$-invariant decomposition of a locally free $\mathbb{Z}_u$-module on $\mathbb{P}^1$. $\square$

(2.9) When we want to prove the nonexistence of an isolated extremal nbd $X \supset C$ with certain condition (say A), it often helps to assume some genericity assumption. It is done in the following way. Let $X_t \supset C_t$ be a flat deformation of $X \supset C$ such that $X = X_0 \supset C = C_0$ and $X_t^\circ \supset C_t$ satisfies A if $|t| \ll 1$, where X_t° is the germ of X_t along C_t. If we show that $X_t \supset C_t$ is not an isolated extremal nbd for $t \neq 0$, then neither is $X \supset C$ [Mori88, (1b) and (10)]. There are two types of constructions for $X_t \supset C_t$.

(2.9.1) Given a point $P \in C$, we deform the equation of $(X^\sharp, P^\sharp)$ and extend it to the deformation of $X \supset C$.

L-deformations and L'-deformations are such examples. We will give an explicit construction for an example of the other type.

(2.9.2) **Lemma.** *Let $X \supset C$ be an extremal nbd with a point P of index m and J a C-laminal ideal of width w. Assume that the canonical cover at P is given as*

$$X^\sharp = (x_1, x_2, x_3, x_4; \phi) \supset C^\sharp = x_1\text{-axis}$$

and $J^\sharp = (x_2, x_3, x_4^w)$, where $x_1, \dots, x_4$ and ϕ are $\mathbb{Z}_m$-semi-invariants and $\phi \equiv x_1^r x_3 \mod J^\sharp I_{C^\sharp}$ for some $r > 0$. If $wt\phi \equiv wt x_4^w$ (resp. $wt x_2$, $wt x_1^i x_4^w$ for some $i > 0$) mod (m), then there is a flat deformation $X_t \supset C_t \ni P_t$ $(t \in \Delta$, a small disk) of $X \supset C \ni P$ such that:

(2.9.2.1) $\bigcup_t (X_t^\circ - U_t) = (X - U) \times \Delta \supset \bigcup_t (C_t - C_t \cap U_t) = (C - C \cap U) \times \Delta$ *and P_t is the only singular point of U_t on C_t, for a sufficiently small nbd $\bigcup_t U_t$ of $\bigcup_t P_t$ in $\bigcup_t X_t$.*

(2.9.2.2) *The trivial extension of J to $X_t^\circ - U_t$ extends to C_t-laminal ideal J_t such that $\bigcup_t \operatorname{Spec} \mathcal{O}_{X_t}/J_t = (\operatorname{Spec} \mathcal{O}_X/J) \times \Delta$, which is compatible with the identification of* (2.9.2.1).

(2.9.2.3) *The canonical cover $X_t^\sharp$ at P_t, $C_t^\sharp$, and $J_t^\sharp$ are given in exactly the same way as $X^\sharp$, $C^\sharp$, and $J^\sharp$ above except that the equation for $X_t^\sharp$ is $\phi_t = \phi + t x_4^w$ (resp. $\phi + t x_2$, $\phi + t x_1^i x_4^w$). Hence, (x_4, x_2, x_3) is a $(1, w, w)$-monomializing ℓ-basis of the second kind of $I_{C_t} \supset J_t$ at P_t (resp. P_t is ordinary and (x_4, x_3) is a $(1, w)$-monomializing ℓ-basis of $I_{C_t} \supset J_t$ at P_t, $x_1^i x_4^w$ appears in ϕ_t with a nonzero coefficient) for $t \neq 0$.*

This is similar to [Mori88, (9.7)(a)] and can be proved in the same way except that X_t is known to have only terminal singularities (cf. [Mori88, (10.7)]).

(2.10) **The case of** *IC*. Let P be the *IC* point of index m and $(y_1, y_2, y_4)/\mathbb{Z}_m(2, m-2, 1)$ be the coordinates for the canonical cover $P^\sharp \in$

$C^\sharp \subset X^\sharp$ given in [Mori88, (A.3)] so that $C^\sharp$ is parametrized by $(t^2, t^{m-2}, 0)$. In this case, P is the only singular point of X on C [Mori88, (B.1)]. Since $y_1^{m-2} - y_2^2$ and y_4 generate the defining ideal of $C^\sharp$, they form an ℓ-free ℓ-basis of $gr_C^1\mathcal{O}_X$. It is easy to see that $\Omega = dy_1 \wedge dy_2 \wedge dy_4$ is an ℓ-free ℓ-basis of $gr_C^0\omega_X$. Then $ql_C(\omega_X) = -P^\sharp$ and $D = \{y_1 = 0\}/\mathbb{Z}_m \in |-2K_X|$ by $(D \cdot C) = 2/m$. By

$$ql_C(gr_C^0(\omega^*)) = ql_C(\omega^*) = -ql_C(\omega) = P^\sharp,$$

one has

$$\deg(gr_C^0(\omega^*)) = TL(P^\sharp) = -U(-1) = -1$$

because

$$U(x) = \mathrm{Min}\{z \in \mathbb{Z} \mid mz - x \in 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+\}$$

[Mori88, (8.9.1)(iii)]. Thus $gr_C^0(\omega^*) = \omega^*/F_C^1(\omega^*) \simeq \mathcal{O}_C(-1)$ and $H^0(\mathcal{O}_X(-K_X)) = H^0(F_C^1(\omega^*))$. Hence a general section $s \in H^0(\mathcal{O}_X(-K_X))$ is written as $(\lambda \cdot y_4 + \mu \cdot (y_1^{m-2} - y_2^2))/\Omega$ near P, where $\lambda \in \mathcal{O}_X$ and $\mu \in \mathcal{O}_{X^\sharp}$ with $wt\mu \equiv 5 \mod (m)$. By (2.5), we see that $\lambda(0) \neq 0$. Hence s induces a section $\bar{s}$ of $gr_C^1(\omega^*) = F_C^1(\omega^*)/F_C^2(\omega^*)$ and $\bar{s}$ is a part of an ℓ-free ℓ-basis of $gr_C^1(\omega^*)$ at P. This induces an ℓ-exact sequence

(2.10.1) $$0 \to (a) \to gr_C^1(\omega^*) \to (b + 5P^\sharp) \to 0,$$

where $a, b \in \mathbb{Z}, a \geq 0$, and $(c + dP^\sharp)$ in general denotes the element of $Cl(X)$ corresponding to $c + dP^\sharp \in QL(C)$ by (2.7). This is because y_4/Ω and $(y_1^{m-2} - y_2^2)/\Omega$ have $wt \equiv 0, m-5 \mod (m)$, respectively. We claim an ℓ-isomorphism

(2.10.2) $$gr_C^1\mathcal{O} \simeq (4P^\sharp)\tilde{\oplus}(-1 + (m-1)P^\sharp).$$

First we recall that m is odd and $m \geq 5$ since P is an IC point. By $(2.10.1)\tilde{\otimes} gr_C^1\omega$, we have an ℓ-exact sequence

(2.10.3) $$0 \to ((a-1) + (m-1)P^\sharp) \to gr_C^1\mathcal{O} \to (b + 4P^\sharp) \to 0.$$

By $i_P(1) = 2$ [Mori88, (6.5)], we have $\deg gr_C^1\mathcal{O} = -1$. By

$$\deg((a-1) + (m-1)P^\sharp) = a - 1 \quad \text{and} \quad \deg(b + 4P^\sharp) = b$$

[Mori88, (8.9.1)(iii)], we have $a + b = 0$. Hence from

$$ql_C((a-1) + (m-1)P^\sharp - (b + 4P^\sharp)) = ql_C(2a - 1 + (m-5)P^\sharp) = 2a - 1 \geq -1,$$

we see that (2.10.3) is ℓ-split by (2.6). Since $H^1(C, gr_C^1\mathcal{O}) = 0$ by [CKM88, 14.5.8], we have $b \geq -1$ and hence $(a, b) = (0, 0)$ or $(1, -1)$. Whence (2.10.2) follows if $(a, b) = (0, 0)$ or $m = 5$. Assuming $(a, b) = (1, -1)$ and $m \geq 7$, we will derive a contradiction. By $(2.10.1)\tilde{\otimes}\omega_X^{\tilde{\otimes}2}$, we have an ℓ-exact sequence

$$0 \to (-1 + (m-2)P^\sharp) \to gr_C^1\omega \to (-1 + 3P^\sharp) \to 0.$$

By $H^1(X, \omega_X) = 0$ and $gr_C^0\omega \simeq \mathscr{O}_C(-1)$, we have $H^1(C, gr_C^1\omega) = 0$ whence $-1 \le \deg(-1+3P^\sharp) = TL(-1+3P^\sharp) = -2$. This is a contradiction and (2.10.2) is proved. Hence

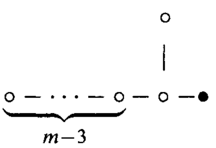
$$gr_C^1(\omega^*) \simeq (5P^\sharp)\tilde{\oplus}(0). \tag{2.10.4}$$

We claim that $\bar{s}$ is a nowhere vanishing section of the locally free sheaf $gr_C^1(\omega^*) \simeq \omega^*\tilde{\otimes} gr_C^1\mathscr{O}$. In case $m \ge 7$, we have $gr_C^1(\omega^*) \simeq \mathscr{O}_C \oplus \mathscr{O}_C$ or $\mathscr{O}_C \oplus \mathscr{O}_C(-1)$ by (2.10.4) and $\bar{s}(P) \ne 0 \in gr_C^1(\omega^*)\otimes \mathbb{C}(P)$ whence $\bar{s}$ is nowhere vanishing. In case $m = 5$, we have $gr_C^1(\omega^*) \simeq \mathscr{O}_C \oplus \mathscr{O}_C(1)$ and $\bar{s}(P) = (\lambda(0)\cdot y_4 + \mu(0)\cdot(y_1^{m-2} - y_2^2))/\Omega \in gr_C^1(\omega^*)\otimes\mathbb{C}(P)$ is a generic element because $\lambda(0)$ and $\mu(0)$ are independent constants by (2.5). Thus $\bar{s}$ is nowhere vanishing and the claim is proved. We study $E_X = \{s = 0\} \in |-K_X|$. Since $\bar{s}$ is a nowhere vanishing section of $gr_C^1(\omega^*) \simeq \omega^*\tilde{\otimes} gr_C^1\mathscr{O}$, E_X is smooth on $C - \{P\}$. The canonical cover $E_X^\sharp$ at P is defined by $y_4 + y_2(\cdots) + y_1(\cdots) = 0$. Thus $(E_X, P) = (y_1, y_2)/\mathbb{Z}_m(2, m-2)$ has only DuVal singularities, whence so is E_Y by $(K_X \cdot C) = 0$. For the precise result, we express $(E_X, P) = (x_1, x_2, x_3; x_1x_2 = x_3^m)$, where $x_1 = y_1^m$, $x_2 = y_2^m$, and $x_3 = y_1y_2$. The curve C is the image of $C^\sharp$, the locus of (t^2, t^{m-2}), where C is the locus of (s^2, s^{m-2}, s) in the embedding of (E_X, P), where $s = t^m$. Then it is easy to check.

(2.10.5) **Computation.** Let (E, P) be an A_{m-1}-singularity

$$(E, P) = (x_1, x_2, x_3; x_1x_2 = x_3^m)$$

and C be the locus of (s^2, s^{m-2}, s). Then $\Delta(E \supset C)$ is



Thus we are done in the case IC.

(2.11) **The case of** *IIB*. Let (X, P) be

$$(y_1, y_2, y_3, y_4; \phi)/\mathbb{Z}_4(3, 2, 1, 1; 0)$$

with C the (quotient of the) locus of $(t^3, t^2, 0, 0)$ [Mori88, (A.3)], where

$$\phi = (y_1^2 - y_2^3) + \psi$$

and $\psi \in (y_3, y_4)$ satisfies $wt\psi \equiv 2 \bmod (4)$ and $\psi(0, 0, y_3, y_4) \notin (y_3, y_4)^3$. The last condition comes from the classification of terminal singularities [Reid87, (6.1)(2)]. In this case, P is the only singular point of X on C [Mori88, (B.1)]. Since y_3 and y_4 generate the defining ideal of $C^\sharp$, they form an ℓ-free ℓ-basis of $gr_C^1\mathscr{O}_X$. By residue,

$$\Omega = \operatorname{Res}\frac{dy_1\wedge dy_2\wedge dy_3\wedge dy_4}{\phi} = \frac{dy_2\wedge dy_3\wedge dy_4}{\partial\phi/\partial y_1}$$

is an ℓ-free ℓ-basis of $gr_C^0\omega_X$ with $wt\Omega \equiv 1 \mod (4)$. We see $i_P(1) = 2$ as follows. Using the parametrization $(t^3, t^2, 0, 0)$ of $C^\sharp$ and ℓ-free ℓ-basis (y_3, y_4) of $gr_C^1\mathscr{O}_X$, we see the following on $C^\sharp \subset X^\sharp$:

$$gr_C^0\omega\,|_{\tilde{C}} = \mathscr{O}_{\tilde{C}}t^3\Omega\,|_{\tilde{C}} = \mathscr{O}_{\tilde{C}}tdt \wedge dy_3 \wedge dy_4,$$
$$\bigwedge^2(gr_C^1\mathscr{O}) \otimes \Omega_C^1\,|_{\tilde{C}} = \mathscr{O}_{\tilde{C}}(t^3y_3) \wedge (t^3y_4) \otimes d(t^4) = \mathscr{O}_{\tilde{C}}t^9y_3 \wedge y_4 \otimes dt.$$

Thus (cf. [Mori88, (2.2)])

$$\bigwedge^2(gr_C^1\mathscr{O}) \otimes \Omega_C^1 = t^8 gr_C^0\omega.$$

Hence $i_P(1) = 2$ as claimed because t^4 is the coordinate of C at P. Hence $\deg gr_C^0\omega = -1$ and $\deg gr_C^1\mathscr{O} = -1$ [Mori88, (2.3.2)]. Thus we see $gr_C^0\omega \simeq (-1 + 3P^\sharp)$ and $gr_C^1\mathscr{O} \simeq (3P^\sharp)\tilde{\oplus}(-1 + 3P^\sharp)$ with ℓ-structures using their ℓ-free ℓ-bases at P above. Let $D = \{y_2 = 0\}/\mathbb{Z}_4$. Then $D \in |-2K_X|$ by $(D \cdot C) = 1/2$. By

$$ql_C(gr_C^0(\omega^*)) = ql_C(\omega^*) = -ql_C(\omega) = P^\sharp,$$

one has

$$\deg(gr_C^0(\omega^*)) = TL(P^\sharp) = -U(-1) = -1$$

because

$$U(x) = \mathrm{Min}\{z \in \mathbb{Z} \mid 4z - x \in 2\mathbb{Z}_+ + 3\mathbb{Z}_+\}$$

[Mori88, (8.9.1)(iii)]. Thus $gr_C^0(\omega^*) \simeq \mathscr{O}_C(-1)$ and a generic section $s \in H^0(\mathscr{O}_X(-K_X))$ vanishes along C, i.e., $s \in H^0(F_C^0(\omega^*))$. Hence $s = (\lambda \cdot y_3 + \mu \cdot y_4)/\Omega$ for some λ and $\mu \in \mathscr{O}_X$. We see that $\lambda(0)$ and $\mu(0) \in \mathbb{C}$ are generic by (2.5). We study $E_X = \{s = 0\} \in |-K_X|$. We see that s induces a section $\bar{s}$ of

$$gr_C^1(\omega^*) \simeq (gr_C^0\omega)^{\tilde{\otimes}(-1)}\tilde{\otimes} gr_C^1\mathscr{O} \simeq (0)\tilde{\oplus}(1)$$

such that $\bar{s}(P)$ is generic in $gr_C^1(\omega^*) \otimes \mathbb{C}(P)$. Thus $\bar{s}$ is nowhere vanishing, whence $E_X \supset C$ and E_X is smooth on $C - \{P\}$. Eliminating y_4, we see that $(E_X, P) \simeq (y_1, y_2, y_3; \bar{\phi})/\mathbb{Z}_4(3, 2, 1)$ with C the locus of $(t^3, t^2, 0)$, where

$$\bar{\phi} = (y_1^2 - y_2^3) + y_3(cy_3 + \cdots) \in \mathbb{C}\{y_1, y_2, y_3\}$$

for some $c \in \mathbb{C}^*$ by independence of $\lambda(0)$ and $\mu(0)$. We claim that we may take

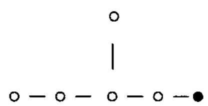
(2.11.1) $$\bar{\phi} = y_1^2 - y_2^3 + y_3^2$$

modulo multiplication by units and $\mathbb{Z}_m$-automorphisms fixing C. First by the Weierstrass preparation theorem, we may assume $\bar{\phi} = y_1^2 + f(y_2, y_3)y_1 + g(y_2, y_3)$ with $wtf \equiv 3$ and $wtg \equiv 2 \mod (4)$. Since $\bar{\phi}(t^3, t^2, 0) = 0$, we see $f \equiv 0$ and $g \equiv y_2^3 \mod (y_3)$. Hence we may assume $f = 0$, after replacing y_1 by $y_1 + f/2$. Since $wt(g - y_2^3)/y_3 \equiv 1$ and $wty_2 \equiv 2 \mod (4)$, we see that $g \equiv y_2^3 \mod (y_3^2)$. Thus we have (2.11.1) by $c \in \mathbb{C}^*$. Then it is easy to check (cf. [Reid87, (4.10)]).

(2.11.2) **Computation.** Let

$$(E, P) = (y_1, y_2, y_3; y_3^2 - y_2^3 + y_3^2)/\mathbb{Z}_4(3, 2, 1; 2)$$

and $C \subset E$ be the locus of $(t^3, t^2, 0)$. Then (E, P) is D_5 and $\Delta(E \supset C)$ is



Thus $|-K_X|$ has a Du Val member in case *IIB*.

(2.12) **The case of two** ***IA*** **points** P, R **with indices** m, 2 **and a** ***III*** **point** S. We know that $siz_P = 1$, m is odd, and $w_P(0) = (m-1)/2m$ [Mori88, (6.2)(ii)] and that $i_P(1) = i_R(1) = i_S(1) = 1$ and $gr_C^1\mathscr{O} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-1)$ [Mori88, (2.3)]. We start with the set-up.

(2.12.1) **Lemma.** *We can express*

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_m(1, (m+1)/2, -1, 0; 0) \supset (C, P) = y_1\text{-}axis/\mathbb{Z}_m,$$
$$(X, R) = (z_1, z_2, z_3, z_4; \beta)/\mathbb{Z}_2(1, 1, 1, 0; 0) \supset (C, R) = z_1\text{-}axis/\mathbb{Z}_2,$$
$$(X, S) = (w_1, w_2, w_3, w_4; \gamma) \supset (C, S) = w_1\text{-}axis,$$

using equations α, β, *and* γ *such that* $\alpha \equiv y_1y_3 \bmod (y_2, y_3)^2 + (y_4)$, $\beta \equiv z_1z_3 \bmod (z_2, z_3)^2 + (z_4)$, *and* $\gamma \equiv w_1w_3 \bmod (w_2, w_3, w_4)^2$.

Proof. We express $(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_m(a_1, a_2, -a_1, 0; 0)$ and C as the locus of $(t^{a_1}, t^{a_2}, 0, 0)$, where a_1 and a_2 are positive integers such that $(a_1a_2, m) = 1$. By $w_P(0) = (m-1)/2m$, we have $a_2 = (m+1)/2$ [Mori88, (4.9)(i)]. By $siz_P = 1 = U(a_1a_2)$, we have $a_1a_2 \le m$ and $a_1 = 1$. We need only to replace y_2 by $y_2 - y_1^{(m+1)/2}$ to get the assertion for (X, P). We can attain $\alpha \equiv y_1y_3$ because P is a cA point [Mori88, (B.1)(g)]. The rest is similar except for $\beta \equiv z_1z_3$ and $\gamma \equiv w_1w_3$, which follow from $i_R(1) = 1$ and $i_S(1) = 1$ and [Mori88, (2.16)]. □

We will improve the set-up in two steps.

(2.12.2) **Lemma.** *The point* P *is ordinary, that is,*

$$(X, P) = (y_1, y_2, y_3)/\mathbb{Z}_m(1, (m+1)/2, -1) \supset (C, P) = y_1\text{-}axis/\mathbb{Z}_m.$$

Proof. Assuming that P is not ordinary, we will derive a contradiction. By the assumption, we may assume $\alpha \equiv y_1y_3 \bmod (y_2, y_3, y_4)^2$. Applying L-deformation at R, we may assume that R is ordinary (2.9.1) and hence $\beta = z_4$. We see that $\{y_2, y_4\}$ and $\{z_2, z_3\}$ are the ℓ-free ℓ-bases of $gr_C^1\mathscr{O}$ at P and Q. By (2.12.1), we see $gr_C^0\omega \simeq (-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$ and $gr_C^0(\omega^*) \simeq (-1 + \frac{m+1}{2}P^\sharp + R^\sharp)$. Thus $H^0(\omega^*) = H^0(F_C^1(\omega^*))$. Let $D = \{y_1 = 0\}/\mathbb{Z}_m$. Then $D \in |-2K_X|$ by $(D \cdot C) = 1/m$. By (2.5), there exists $s \in H^0(F_C^1(\omega^*))$ inducing $(y_2 + y_1\mathscr{O}_X)/\Omega \in \mathscr{O}_D(-K_X)$. Thus s induces a global section $\overline{s}$ of

$gr^1_C(\omega^*) \simeq gr^1_C\mathcal{O}\tilde{\otimes}gr^0_C(\omega^*)$, which is a part of ℓ-free ℓ-basis at P. Hence we have an exact sequence

$$0 \to gr^0_C\omega \to gr^1_C\mathcal{O} \to (gr^1_C\mathcal{O}/gr^0_C\omega) \to 0.$$

It is split because $gr^0_C\omega \simeq \mathcal{O}(-1)$ and $gr^1_C\mathcal{O} \simeq \mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Then it is ℓ-split at R because ℓ-bases of $gr^0_C\omega$ and $gr^1_C\mathcal{O}$ have $wt \equiv 1 \mod (2)$. Hence $gr^1_C\mathcal{O}/gr^0_C\omega$ is an ℓ-invertible sheaf such that $ql_C(gr^1_C\mathcal{O}/gr^0_C\omega) = ql_C(gr^1_C\mathcal{O}) - ql_C(gr^0_C\omega) = -1 + R^\sharp$. Applying (2.6) to

$$ql_C(gr^0_C\omega) - ql_C(gr^1_C\mathcal{O}/gr^0_C\omega) = \frac{m-1}{2}P^\sharp,$$

we have an ℓ-splitting

$$gr^1_C\mathcal{O} \simeq \left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right)\tilde{\oplus}(-1 + R^\sharp). \qquad (2.12.2.1)$$

We may further assume that y_2, z_2, and w_2 (resp. y_4, z_3, and w_4) are the ℓ-free ℓ-bases of $(-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$ (resp. $(-1 + R^\sharp)$) at P, R, and S, by making coordinate changes to the ones in (2.12.1). Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathcal{O} = (-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$. Then $\{y_2, y_3, y_4^2\}$ form an ℓ-basis of J at P. By replacing y_3 by an element of the form $y_3 + y_4^2(\cdots)$ if necessary, we may assume $\alpha \equiv y_1y_3 + cy_4^2 \mod J^\sharp I_{C^\sharp}$ for some $c \in \mathbb{C}$. If $c \neq 0$ then $I \supset J$ is $(1, 2, 2)$-monomializable at P, and if $c = 0$ we may still assume that $I \supset J$ is $(1, 2, 2)$-monomializable at P by (2.9.2). In the same way, we may assume that $I \supset J$ is $(1, 2, 2)$-monomializable at S. At the ordinary point R, $I \supset J$ is $(1, 2)$-monomializable. Thus we have ℓ-isomorphisms

$$\begin{aligned}
gr^1(\mathcal{O}, J) &\simeq (-1 + R^\sharp),\\
gr^{2,0}(\mathcal{O}, J) &\simeq \left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right),\\
gr^{2,1}(\mathcal{O}, J) &\simeq gr^1(\mathcal{O}, J)^{\tilde{\otimes}2}\tilde{\otimes}(1 + P^\sharp) \simeq (P^\sharp),\\
gr^{3,0}(\mathcal{O}, J) &\simeq gr^{2,0}(\mathcal{O}, J)\tilde{\otimes}gr^1(\mathcal{O}, J) \simeq \left(-1 + \frac{m-1}{2}P^\sharp\right),\\
gr^{3,1}(\mathcal{O}, J) &\simeq gr^{2,1}(\mathcal{O}, J)\tilde{\otimes}gr^1(\mathcal{O}, J) \simeq (-1 + P^\sharp + R^\sharp)
\end{aligned}$$

by [Mori88, (8.12)]. Hence we have an ℓ-isomorphism and ℓ-exact sequences

$$gr^1(\omega, J) \simeq \left(-1 + \frac{m-1}{2}P^\sharp\right),$$

$$0 \to \left(-1 + \frac{m+1}{2}P^\sharp + R^\sharp\right) \to gr^2(\omega, J) \to (-1 + (m-1)P^\sharp) \to 0,$$

$$0 \to \left(-1 + \frac{m+1}{2}P^\sharp\right) \to gr^3(\omega, J) \to (-2 + (m-1)P^\sharp + R^\sharp) \to 0.$$

Thus $H^1(\omega/F^4(\omega, J)) \neq 0$, which is a contradiction. Thus (2.12.2) is proved. □

(2.12.3) **Lemma.** *The point R is ordinary, that is,*

$$(X, R) = (z_1, z_2, z_3)/\mathbb{Z}_2(1, 1, 1) \supset (C, P) = z_1\text{-}axis/\mathbb{Z}_2.$$

Proof. We will assume that R is not ordinary whence

$$\beta \equiv z_1 z_3 \mod (z_2, z_3, z_4)^2.$$

As in the proof of (2.12.2), we have a split exact sequence $0 \to gr_C^0\omega \to gr_C^1\mathscr{O} \to (gr_C^1\mathscr{O}/gr_C^0\omega) \to 0$ which is ℓ-split at P. Since ℓ-free ℓ-bases of $gr_C^0\omega$ and $gr_C^1\mathscr{O}$ at R have wt 1 and $\{0, 1\}$ mod (2), it is also ℓ-split at R. Thus we have ℓ-exact sequences

$$0 \to \left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right) \to gr_C^1\mathscr{O} \to (-1 + P^\sharp) \to 0,$$

$$0 \to (-1 + (m-1)P^\sharp) \to gr_C^1\omega \to \left(-2 + \frac{m+1}{2}P^\sharp + R^\sharp\right) \to 0.$$

Hence $H^1(\omega/F_C^2\omega) \neq 0$ and (2.12.3) is proved. □

(2.12.4) As in the argument for (2.12.2), we have an ℓ-isomorphism

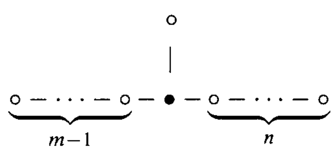
$$gr_C^1\mathscr{O} \simeq \left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right)\tilde{\oplus}(-1 + P^\sharp + R^\sharp).$$

Let J be the C-laminal ideal such that $J/F_C^2\mathscr{O} = (-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$. After an (equivariant) change of coordinates if necessary, we may assume that (y_2, z_2, w_2) (resp. (y_3, z_3, w_4)) are ℓ-free ℓ-bases of $(-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$ (resp. $(-1 + P^\sharp + R^\sharp)$), whence $J = (w_2, w_3, w_4^2)$ at S. Replacing w_3 by an element $\equiv w_3 \mod (w_2, w_4)^2$ if necessary, we may further assume

$$\gamma \equiv w_1 w_3 + c_1 w_4^2 + c_2 w_4 w_2 + c_3 w_2^2 \mod (w_3, w_2^2, w_2 w_4, w_4^2) \cdot I_C$$

for some $c_1, c_2, c_3 \in \mathbb{C}$. We note that $\gamma \equiv w_1 w_3 + c_1 w_4^2 \mod J \cdot I_C$.

(2.12.5) **Lemma.** *The general member E_X of $|-K_X|$ has singularities A_{m-1}, A_1, and A_n at P, R, and S, respectively, and is smooth elsewhere, and $\Delta(E_X \supset C)$ is*



where n is some integer ≥ 1. We have $n = 1$ if $c_1 \neq 0$ when $m \geq 5$ or if $(c_1, c_2, c_3) \neq (0, 0, 0)$ when $m = 3$.

Proof. We have an ℓ-isomorphism $gr_C^1(\omega^*) \simeq (0)\tilde{\oplus}(-1 + \frac{m+3}{2}P^\sharp)$. Let $D = \{y_1 = 0\}/\mathbb{Z}_m \in |-2K_X|$ as before. We treat the case $m \geq 5$. Then $gr_C^1\omega^* \simeq \mathscr{O}_C \oplus \mathscr{O}_C(-1)$ and $H^0(\mathscr{O}(-K_X)) = H^0(gr_C^2(\omega^*, J))$. The general section $s \in H^0(\mathscr{O}(-K_X))$ induces $(y_2 + \cdots)/\Omega$ up to some units whence induces a nonzero global section $\bar{s}$ of $gr_C^1\omega^*$. Hence $\bar{s}$ is nowhere vanishing and the defining equations of $E_X = \{s = 0\}$ are y_2, z_2, and w_2 mod $F_C^2\mathscr{O}$ up to units at P, R, and S. Then E_X is smooth outside of P, R, and S, $(E_X, P) \simeq (y_1, y_3)/\mathbb{Z}_m(1, -1)$, $(E_X, R) \simeq (z_1, z_3)/\mathbb{Z}_2(1, 1)$, and $(E_X, S) \simeq (w_1, w_3, w_4; \bar{\gamma})$, where

$$\bar{\gamma}(w_1, w_3, w_4) \equiv w_1w_3 + c_1w_4^2 \mathrm{mod}(w_3, w_4^2)(w_3, w_4).$$

We are done in case $m \geq 5$. In case $m = 3$, we can see that $gr_C^1\omega^* \simeq (0)\tilde{\oplus}(0)$ and $H^0(\mathscr{O}(-K_X)) \twoheadrightarrow H^0(gr_C^1\omega^*)$, and we get a similar assertion on E_X except that $\gamma \equiv w_1w_3 + (c_3t^2 + c_2t + c_1)w_4^2$ for some generic $t \in \mathbb{C}$. Thus we are done in case $m = 3$. □

(2.12.6) **Lemma.** *If $m \geq 5$, then $c_1 \neq 0$ and $n = 1$ in* (2.12.5).

Proof. Assume that $m \geq 5$ and $c_1 = 0$. By $w_1w_3 \in J \cdot I_C$, we have $w_3 \in F^3(\mathscr{O}, J)$ and $gr^2(\mathscr{O}, J) = \mathscr{O}_Cw_2 \oplus \mathscr{O}_Cw_4^2$ at S. Thus we have ℓ-isomorphisms

$$gr^1(\mathscr{O}, J) = (-1 + P^\sharp + R^\sharp),$$
$$gr^{2,0}(\mathscr{O}, J) = \left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right),$$
$$gr^{2,1}(\mathscr{O}, J) = gr^1(\mathscr{O}, J)^{\tilde{\otimes}2} \simeq (-1 + 2P^\sharp).$$

Thus we have

$$gr^1(\omega, J) \simeq \left(-1 + \frac{m+1}{2}P^\sharp\right),$$

$$0 \to \left(-2 + \frac{m+3}{2}P^\sharp + R^\sharp\right) \to gr^2(\omega, J) \to (-1 + (m-1)P^\sharp) \to 0,$$

and $H^0(\omega/F^3(\omega, J)) \neq 0$, which is a contradiction. □

(2.12.7) **Lemma.** *If $m = 3$, then $(c_1, c_2, c_3) \neq 0$ and $n = 1$ in* (2.12.5).

Proof. Assume that $(c_1, c_2, c_3) = 0$. Then $w_3 \in F_C^3\mathscr{O}$. Changing w_1 and w_3, we may further assume $\gamma = w_1w_3 + \delta(w_2, w_4)$ where δ is a power series in w_2 and w_4 of order $d \geq 3$. Then we have $\chi(\mathscr{O}/F_C^n\mathscr{O}) = O(n^2)$ by (2.18.8) below because $2 \cdot \mathrm{ldeg}_C((-1 + P^\sharp + R^\sharp)) + 1/d \leq 0$. This contradicts (2.12.9) below. □

(2.12.8) **Lemma.**

$$\chi(\mathscr{O}/F_C^n\mathscr{O}) \leq \frac{1}{6}n^3\left(\mathrm{ldeg}_C gr_C^1\mathscr{O} + \frac{1}{d}\right) + O(n^2).$$

Proof. An argument similar to the proof of [Mori88, (8.12)] shows

$$\chi(\mathscr{O}/F_C^n\mathscr{O}) \leq \frac{1}{6}n^3\mathrm{ldeg}_C gr_C^1\mathscr{O} + \mathrm{len}\frac{\mathscr{O}_{X,S}/(F_C^n\mathscr{O})_S}{\mathbb{C}\{w_1, w_2, w_4\}/(w_2, w_4)^n} + O(n^2),$$

where we used the ordinarity of P and R. By the equation $w_1 w_3 = g(w_2, w_4)$ above, it is easy to see that

$$(F_C^n \mathscr{O})_S = \sum_{i=1}^{[n/d]} (w_2, w_4)^{n-di} w_3^i + (w_3^{[n/d]+1}),$$

and hence

$$\operatorname{len} \frac{\mathscr{O}_{X,S}/(F_C^n \mathscr{O})_S}{\mathbb{C}\{w_1, w_2, w_4\}/(w_2, w_4)^n} \le \frac{n^3}{6d} + O(n^2). \quad \square$$

The following is a rather general lemma.

(2.12.9) **Lemma.** *Let $f : X \to (Y, Q)$ be a proper bimeromorphic morphism of an irreducible reduced 3-fold to a 3-fold singularity (Y, Q) such that $C = f^{-1}(Q)$ is 1-dimensional. Then for an arbitrary number $c > 0$, there exists a number $M(c)$ such that*

$$\chi(\mathscr{O}/K) \ge M(c) + c \cdot \operatorname{rk}(\mathscr{O}/K)$$

for an arbitrary ideal $K \subset \mathscr{O}_X$ defining C (as a set), where $\operatorname{rk}(\mathscr{O}/K)$ is the sum of $\operatorname{len}_\xi(\mathscr{O}/K)_\xi$ for the generic points of C.

Proof. Since $\dim C = 1$, let L be an f-ample line bundle on X. Since $f^* f_*(L^{-1}) = L^{-1}$ on $X - C$, we see that

$$J = L \otimes (\text{Image of } f^* f_*(L^{-1}) \to L^{-1})$$

is an ideal defining C such that $L^{\oplus n} \twoheadrightarrow J$ for some $n > 0$. Let

$$\mathscr{O}_X = J_0 \supset J_1 \supset \cdots \supset J_a = J$$

be ideals such that $\operatorname{Annih}(J_i/J_{i+1})$ is a prime ideal P_i and J_i/J_{i+1} is a torsion-free $\mathscr{O}/P_i$-module $(i = 0, \dots, a-1)$. Let $\lambda = a\mu + \nu$ $(\mu \in \mathbb{Z},\ \nu \in [0, a-1])$. Then

$$S^\mu(L^{\oplus n}) \otimes J_\nu \twoheadrightarrow J^{\{\lambda\}},$$

where $J^{\{\lambda\}} = J^\mu \cdot J_\nu$. Hence

$$S^\mu(\mathscr{O}^{\oplus n}) \otimes L^{\otimes \mu} \otimes J_\nu / J_{\nu+1} \twoheadrightarrow J^{\{\lambda\}}/J^{\{\lambda+1\}}.$$

Hence if $K + J^{\{\lambda\}}/K + J^{\{\lambda+1\}}$ is of rank n_λ along $\operatorname{Supp}(\mathscr{O}/P_\nu)$ of dimension 1, then there is an injection

$$(L^{\otimes \mu} \otimes J_\nu/J_{\nu+1})^{\oplus n_\lambda} \to K + J^{\{\lambda\}}/K + J^{\{\lambda+1\}}$$

whose cokernel is of finite length. There exists $\lambda(c)$, which is independent of K, such that if $\lambda = a\mu + \nu \ge \lambda(c)$ and $\operatorname{Supp}(\mathscr{O}/P_\nu)$ is a curve then $\chi(L^{\otimes \mu} \otimes J_\nu/J_{\nu+1}) \ge c$. Hence if $\lambda = a\mu + \nu \ge \lambda(c)$, then

$$\chi(K + J^{\{\lambda\}}/K + J^{\{\lambda+1\}}) \ge n_\lambda \cdot c, \tag{2.12.9.1}$$

where n_λ is as above if $\dim \operatorname{Supp}(\mathscr{O}/P_\nu) = 1$ and $n_\lambda = 0$ otherwise. Since $K \supset J^{\{\lambda\}}$ for $\lambda >> 0$, (2.12.9.1) implies

$$\chi(K + J^{\{\sigma\}}/K) \ge c \cdot \operatorname{rk}(K + J^{\{\sigma\}}/K)$$

for some $\sigma < \lambda(c)$. It remains to give a lower bound of $\chi(\mathscr{O}/K + J^{\{\sigma\}})$. For each $\sigma < \lambda(c)$, we choose one sequence of ideals

$$\mathscr{O}_X = I_0 \supset I_1 \supset \cdots \supset I_b = J^{\{\sigma\}}$$

such that $\operatorname{Annih}(I_i/I_{i+1}) = Q_i$ is a prime ideal and I_i/I_{i+1} is a torsion-free $\mathscr{O}/Q_i$-module of rank 1 $(i = 0, \ldots, b-1)$. If $\chi(I_i + K/I_{i+1} + K) < 0$, then $\dim \operatorname{Supp}\mathscr{O}/Q_i = 1$ and

$$I_i/I_{i+1} \xrightarrow{\sim} I_i + K/I_{i+1} + K.$$

Thus $\chi(I_i + K/I_{i+1} + K) \geq \min\{0, \chi(I_i/I_{i+1})\}$ and $\chi(\mathscr{O}/K + J^{\{\sigma\}}) \geq M(c) + c \cdot \operatorname{rk}(\mathscr{O}/K + J^{\{\sigma\}})$, where

$$M(c) = \operatorname{Min}_\sigma \left(\sum_{i=0}^{b-1} \min\{0, \chi(I_i/I_{i+1})\} - c \cdot \operatorname{rk}(\mathscr{O}/J^{\{\sigma\}}) \right),$$

where $\operatorname{Min}_\sigma$ is the minimum taken over all $\sigma < \lambda(c)$. Hence we have (2.19.2). □

It seems worthwhile to remark.

(2.12.10) **Corollary.** *The inequality in* [Mori88, (8.12.ii)] *is strict.*

Now (2.2.3′) is proved in the case *IA+IA+III* except the divisoriality, which will be treated in Chapter 5.

(2.13) **The case of two *IA* points P, R with indices m, 2.** We know that P is a cA point, $siz_P = 1$, $m \geq 3$ [Mori88, (B.1)]. We start with the set-up.

(2.13.1) **Lemma.** *We can write*

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_m(1, a, -1, 0; 0) \supset (C, P) = y_1\text{-}axis/\mathbb{Z}_m,$$
$$(X, R) = (z_1, z_2, z_3, z_4; \beta)/\mathbb{Z}_2(1, 1, 1, 0; 0) \supset (C, R) = z_1\text{-}axis/\mathbb{Z}_2$$

using equations α and β and an integer a such that $\alpha \equiv y_1y_3 \mod (y_2, y_3)^2 + (y_4)$, $m/2 < a < m$, and $(a, m) = 1$.

Proof. This is similar to (2.12.1). We only need to prove $a > m/2$, which follows from $1 > w_P(0) + w_R(0) = (m-a)/m + 1/2$ [Mori88, (4.9)]. □

We recall $\ell(P) = \operatorname{len}_{P^\sharp}(I^{\sharp(2)}/I^{\sharp 2})$ where $I^\sharp$ is the defining ideal of $C^\sharp$ in $(X^\sharp, P^\sharp)$ and $\ell(R)$ is defined similarly.

(2.13.2) **Lemma.** *We have $\ell(P) = 0$ or 1 and $i_P(1) = 1$.*

Proof. This follows from $\alpha \equiv y_1y_3$ and [Mori88, (2.16)]. □

(2.13.3) **Lemma.** *We have either*

(2.13.3.1) *$\ell(R) = 0$ or 1, $i_R(1) = 1$, and $gr_C^1\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$; or*

(2.13.3.2) *$\ell(R) = 2$, $i_R(1) = 2$, m is odd, P is ordinary, $a = (m+1)/2$, and $gr_C^1\mathscr{O} \simeq \mathscr{O}(-1) \oplus \mathscr{O}(-1)$.*

Furthermore in case (2.13.3.2), $X \supset C$ *has a small deformation* $X_t \supset C_t$ *so that* X_t *has three singular points on* C_t.

Proof. The assertion on $i_R(1)$ follows from the one on $\ell(R)$ by [Mori88, (2.16)]. We assume $\ell(R) \geq 2$ and denote it by r. Thus we may choose $\beta \equiv z_1^r z_i \bmod (z_2, z_3, z_4)^2$, where $i = 3$ (resp. 4) if $r \equiv 1$ (resp. 0) mod (2). If we extend the deformation $\beta + tz_1^{r-2}z_i = 0$ of (X, R) to a deformation $X_t \supset C_t \ni R_t$ of $X \supset C \ni R$, which is trivial outside of a small nbd of R, then X_t has two *IA* points and one *III* point on C_t and $\beta + tz_1^{r-2}z_i = 0$ is the equation for (X_t, R_t) (cf. [Mori88, (4.12.2)]). Hence $r = 2$ by (2.12.3). Since $X_t \supset C_t$ is a trivial deformation of $X \supset C$ in a nbd of P, we have the rest of the assertion by (2.12). □

First we treat the special case (2.13.3.2).

(2.13.4) **Lemma.** *The assertion* (2.2.3) *holds in the case* (2.13.3.2).

Proof. The argument is quite similar to (2.12). As in (2.12.4), we have an ℓ-isomorphism

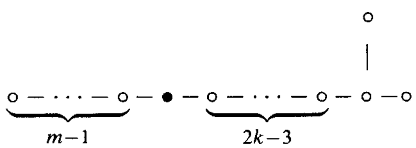
$$gr_C^1\mathcal{O} \simeq \left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right) \tilde{\oplus}(-1 + P^\sharp + R^\sharp),$$

and let J be the C-laminal ideal such that $J/F_C^2\mathcal{O} = (-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$. We may assume that (y_2, z_2) (resp. (y_3, z_3)) are ℓ-free ℓ-bases of $(-1 + \frac{m-1}{2}P^\sharp + R^\sharp)$ (resp. $(-1 + P^\sharp + R^\sharp)$), $J^\sharp = (z_2, z_4, z_3^2)$, and

$$\beta \equiv z_1^2 z_4 + c_1 z_3^2 + c_2 z_2 z_3 + c_3 z_2^2 \bmod (z_4, z_3^2, z_2 z_3, z_2^2) \cdot I_C$$

at R for some $c_1, c_2, c_3 \in \mathbb{C}$. We note that $\beta \equiv z_1^2 z_4 + c_1 z_3^2 \bmod J^\sharp I^\sharp$. The following (2.13.5) corresponds to (2.12.5). The fact that $(c_1, c_2, c_3) \neq (0, 0, 0)$ follows from the classification of terminal 3-fold singularities [Reid87, (6.1)]. The assertion that $c_1 \neq 0$ if $m \geq 5$ is proved in the same way as (2.12.6). Thus (2.13.4) is proved. □

(2.13.5) **Lemma.** *Assume that* $c_1 \neq 0$ *when* $m \geq 5$, *or* $(c_1, c_2, c_3) \neq (0, 0, 0)$ *when* $m = 3$. *Then for a general member* E_X *of* $|-K_X|$, $\Delta(E_X \supset C)$ *is*



where k (≥ 2) *is the axial multiplicity of* (X, R).

Proof. The only difference from (2.12.5) is the analysis of the singularity

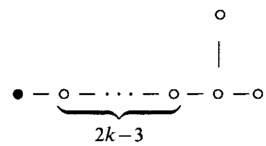
$$(E_X, R) \simeq (z_1, z_3, z_4; \overline{\beta})/\mathbb{Z}_2(1, 1, 0; 0)$$

where $\overline{\beta}$ satisfies $\overline{\beta} \equiv z_1^2 z_4 + z_3^2 \bmod (z_4, z_3^2)(z_4, z_3)$ and ord $\overline{\beta}(0, 0, z_4) = k < \infty$. It is easy to see that $\overline{\beta} = z_1^2 z_4 + z_3^2 + z_4^k$ modulo formal $\mathbb{Z}_m$-automorphisms in (z_1, z_3, z_4). Thus it is reduced to the following explicit computation (cf. [Reid87, (4.10)]). □

(2.13.6) **Computation.** Let

$$(E, R) = (z_1, z_3, z_4; z_1^2 z_4 + z_3^2 + z_4^k)/\mathbb{Z}_2(1, 1, 0; 0)$$

and $C = z_1$-axis $/\mathbb{Z}_2$, where $k \geq 2$. Then (E, R) is D_{2k} and $\Delta(E \supset C)$ is



(2.13.7) In the rest of this chapter, we assume the case (2.13.3.1) unless otherwise mentioned.

We choose an ℓ-splitting $gr_C^1\mathscr{O} \simeq L\tilde{\oplus}M$ (2.8) such that $\deg L = 0$ and $\deg M = -1$ (2.13.3.1). Let J be the C-laminal ideal of width 2 such that $J/F_C^2\mathscr{O} = L$. For an ℓ-invertible sheaf F with an ℓ-free ℓ-basis f at a point T of index n, we can give an equivalent definition of $\mathrm{qldeg}(F, T) \in [0, n)$ as $\mathrm{qldeg}(F, T) \equiv -wtf \mod (n)$. (This is because $(C^\sharp, P^\sharp)$ and $(C^\sharp, R^\sharp)$ are smooth.)

(2.13.8) **Theorem.** $\mathrm{qldeg}(M, R) = 1$.

Proof. We assume $\mathrm{qldeg}(M, R) = 0$. Then $M \simeq (-1 + iP^\sharp)$ for $i = 0, 1$, or $(m-a)$ since y_2, y_3, and y_4 form an ℓ-basis of $gr_C^1\mathscr{O}$ at P. By $ql_C(gr_C^0\omega) = -1 + (m-a)P^\sharp + R^\sharp$, we have

$$gr_C^1\omega \simeq gr_C^1\mathscr{O}\otimes gr_C^0\omega \simeq L\tilde{\otimes}gr_C^0\omega\tilde{\oplus}(-2 + (m-a+i)P^\sharp + R^\sharp).$$

By $m - a + i \leq 2m - 2a < m$, we have $H^1(gr_C^1\omega) \neq 0$. This is a contradiction to $H^1(\omega/F_C^2\omega) = 0$ because of $H^1(gr_C^0\omega) = 0$. □

(2.13.8.1) *Remark.* For comparison with [Mori88, (9)], it might be worthwhile to mention

$$\mathrm{qldeg}(M, R) = 1 \quad \text{iff } \ell(R) + q(R) = 1,$$
$$\mathrm{qldeg}(M, P) = m - a \quad \text{iff } \ell(P) + q(P) = 1.$$

(2.13.9) **Lemma.** *The case* (2.2.4) *holds if* $\mathrm{qldeg}(M, P) = m - a$.

Proof. We have an ℓ-isomorphism $M \simeq gr_C^0\omega$. We may assume that y_2 is an ℓ-free ℓ-basis of M at P. Let $D = \{y_1^{2a-m} = 0\}/\mathbb{Z}_m$ (note that $2a > m$). It is easy to see $D \in |-2K_X|$ by $(D \cdot C) = (2a - m)/m$. By $H^0(\mathscr{O}(-K_X)) = H^0(F_C^1(\omega^*))$, its general section s induces a section $\bar{s}$ of $gr_C^1\omega^* \simeq L\tilde{\otimes}(gr_C^0\omega)^{\tilde{\otimes}(-1)} \oplus (0)$. The projection of $\bar{s}$ to (0) is nonzero because y_2/Ω is an ℓ-free ℓ-basis of (0) at P and s induces an element of the form $y_2/\Omega + \cdots$ up to units, where Ω is an ℓ-free ℓ-basis of $gr_C^0\omega$ at P. Thus $\bar{s}$ is nowhere vanishing, whence $E_X = \{s = 0\}$ is smooth outside of P and R. The analysis of (E_X, P) and (E_X, R) is the same as [Mori88, (9.9.3)]. □

(2.13.10) **Lemma.** *We assume* qldeg$(M, P) \neq m - a$ *in the case* (2.13.3.1). *Let* k *be the axial multiplicity of* R. *Then* P *is ordinary,* m *is odd* ≥ 5, *and* $a = (m+1)/2$. *After changing coordinates, we may assume*

$$(X, P) = (y_1, y_2, y_3)/\mathbb{Z}_m(1, (m+1)/2, -1) \supset (C, P) = y_1\text{-}axis/\mathbb{Z}_m,$$
$$(X, R) = (z_1, z_2, z_3, z_4; \beta)/\mathbb{Z}_2(1, 1, 1, 0; 0) \supset (C, R) = z_1\text{-}axis/\mathbb{Z}_2;$$

y_2 *and* y_3 *are* ℓ*-free* ℓ*-bases of* L *and* M *at* P, *respectively;* z_3 *(resp.* z_4*) and* z_2 *are* ℓ*-free* ℓ*-bases of* L *and* M *at* R, *respectively,*

$$L = \left(\frac{m-1}{2}P^\sharp + R^\sharp\right) \left(resp.\ L \simeq \left(\frac{m-1}{2}P^\sharp\right)\right),$$
$$M = (-1 + P^\sharp + R^\sharp),$$

$I \supset J$ *has a* $(1, 2)$*-monomializing* ℓ*-basis* (y_3, y_2) *at* P, $I \supset J$ *has a* $(1, 2)$*-monomializing* ℓ*-basis* (z_2, z_3) *(resp. a* $(1, 2, 2)$*-monomializing* ℓ*-basis* (z_2, z_4, z_3) *of the second kind) at* R, $\beta = z_4$ *(resp.* $\beta \equiv z_1z_3 + z_2^2$ *mod* $(z_2^2, z_3, z_4)(z_2, z_3, z_4)$*) if* $k = 1$ *(resp.* $k \geq 2$*); and an* ℓ*-splitting*

$$gr^2(\mathcal{O}, J) \simeq (2P^\sharp)\tilde{\oplus}\left(-1 + \frac{m-1}{2}P^\sharp + R^\sharp\right).$$

Proof. Proof will be given in a few steps.

(2.13.10.1) Assuming that qldeg$(M, P) \neq m - a$ and that P is not ordinary, we will derive a contradiction. We may assume $\alpha \equiv y_1y_3 \mod (y_2, y_3, y_4)^2$ by (2.13.1). Thus y_2 and y_4 form an ℓ-free ℓ-basis of $gr_C^1\mathcal{O}$ at P, and we may assume that they are ℓ-free ℓ-bases of L and M, respectively, because qldeg$(M, P) \neq m - a$. Hence $M \simeq (-1 + R^\sharp)$. By the deformation $\alpha + ty_4^2$ (2.9.2), we may assume that $I \supset J$ has a $(1, 2, 2)$-monomializing ℓ-basis (y_4, y_2, y_3) at P. We may further assume that R is an ordinary point by (2.9.2). Hence $L \simeq ((m-a)P^\sharp + R^\sharp)$ and $gr^{2,1}(\mathcal{O}, J) \simeq M^{\tilde{\otimes}2}\tilde{\otimes}(P^\sharp) \simeq (-1+P^\sharp)$. Hence

$$gr^1(\omega, J) \simeq M\tilde{\otimes}gr_C^0\omega \simeq (-1 + (m-a)P^\sharp),$$
$$gr^{2,0}(\omega, J) \simeq L\tilde{\otimes}gr_C^0\omega \simeq ((2m - 2a)P^\sharp),$$
$$gr^{2,1}(\omega, J) \simeq gr^{2,1}(\mathcal{O}, J)\tilde{\otimes}gr_C^0\omega \simeq (-2 + (m - a + 1)P^\sharp + R^\sharp),$$
$$gr^{3,0}(\omega, J) \simeq gr^{2,0}(\omega, J)\tilde{\otimes}M \simeq (-1 + (2m - 2a)P^\sharp + R^\sharp),$$
$$gr^{3,1}(\omega, J) \simeq gr^{2,1}(\omega, J)\tilde{\otimes}M \simeq (-2 + (m - a + 1)P^\sharp).$$

Thus we have a contradiction $H^1(\omega/F^4(\omega, J)) \neq 0$ by $m - a + 1 \leq 2m - 2a < m$. Thus P is ordinary.

(2.13.10.2) We will prove that m is odd ≥ 5 and $a = (m+1)/2$. If $a = m - 1$, then qldeg$(M, P) = m - a$ by qldeg$(M, P) \equiv -wty_2 \equiv -wty_3$. This is impossible and thus $a \leq m - 2$. Whence $m \geq 5$ by $m - 2 \geq a > m/2$. As in (2.13.10.1), we may assume that R is ordinary by (2.9.2). Hence $L \simeq ((m-a)P^\sharp + R^\sharp)$ and $M \simeq (-1 + P^\sharp + R^\sharp)$ by qldeg$(M, P) \equiv -wty_2$ or $-wty_3$

mod (m). We have

$$gr^1(\omega, J) \simeq M\tilde{\otimes} gr^0_C\omega \simeq (-1+(m-a+1)P^\sharp),$$
$$gr^{2,0}(\omega, J) \simeq L\tilde{\otimes} gr^0_C\omega \simeq ((2m-2a)P^\sharp),$$
$$gr^{2,1}(\omega, J) \simeq M^{\tilde{\otimes}2}\tilde{\otimes} gr^0_C\omega \simeq (-2+(m-a+2)P^\sharp+R^\sharp),$$
$$gr^{3,0}(\omega, J) \simeq L\tilde{\otimes} M\tilde{\otimes} gr^0_C\omega \simeq (-1+(2m-2a+1)P^\sharp+R^\sharp),$$
$$gr^{3,1}(\omega, J) \simeq M^{\tilde{\otimes}3}\tilde{\otimes} gr^0_C\omega \simeq (-2+(m-a+3)P^\sharp).$$

By $m > 2m-2a \geq m-a+2$, we see that $H^1(\omega/F^4(\omega, J)) = 0$ only if $2m-2a+1 = m$, that is, $a = (m+1)/2$.

(2.13.10.3) Since $gr^1_C\mathscr{O}$ has an ℓ-free ℓ-basis $\{y_2, y_3\}$ at P and $\{z_2, z_3\}$ (resp. $\{z_2, z_4\}$) at R if $k=1$ (resp. $k \geq 2$), the assertions on L and M follow from $\operatorname{qldeg}(M, R) = 1$ (2.13.8). For the ℓ-bases of J, we only have to show that (z_2, z_4, z_3) is a $(1, 2, 2)$-monomializing ℓ-basis of $I \supset J$ of the second kind at R assuming $k \geq 2$ because $J^\sharp = (z_2^2, z_3)$ if $k=1$. Assume $k \geq 2$ (hence $J^\sharp = (z_2^2, z_3, z_4)$) and that $\beta \equiv z_1z_3 + cz_2^2 \mod J^\sharp I^\sharp$ for some $c \in \mathbb{C}$. If $c = 0$, then $z_3 \in F^3(\mathscr{O}, J)$ and $gr^{2,1}(\mathscr{O}, J) \simeq M^{\tilde{\otimes}2}$, whence

$$gr^{2,0}(\omega, J) \simeq L\tilde{\otimes} gr^0_C\omega \simeq (-1+(m-1)P^\sharp+R^\sharp),$$
$$gr^{2,1}(\omega, J) \simeq M^{\tilde{\otimes}2}\tilde{\otimes} gr^0_C\omega \simeq \left(-2+\frac{m+3}{2}P^\sharp+R^\sharp\right),$$

which implies a contradiction: $H^1(\omega/F^3(\omega, J)) \neq 0$. Thus $c \neq 0$ and the assertion on ℓ-basis is proved. In particular, the assertion on β follows.

(2.13.10.4) By $gr^{2,1}(\mathscr{O}, J) \simeq M^{\tilde{\otimes}2}$ if $k=1$ (resp. $M^{\tilde{\otimes}2}\tilde{\otimes}(R^\sharp)$ if $k \geq 2$), we have $gr^2(\mathscr{O}, J) \simeq \mathscr{O}_C \oplus \mathscr{O}_C(-1)$ as $\mathscr{O}_C$-modules. It is easy to see that the ℓ-free ℓ-basis at R of $gr^2(\mathscr{O}, J)/\mathscr{O}_C$ has $wt \not\equiv 0 \mod (2)$ by the argument for (2.13.8) and by $m \geq 5$. To determine the ℓ-splitting of $gr^2(\mathscr{O}, J)$ (cf. (2.9)), it is therefore enough to disprove the ℓ-isomorphism $gr^2(\mathscr{O}, J) \simeq (\frac{m-1}{2}P^\sharp)\tilde{\oplus}(-1+2P^\sharp+R^\sharp)$ when $m \geq 7$ (we note that this ℓ-splitting is what we want if $m=5$). Indeed, from this ℓ-splitting, we have

$$gr^2(\omega, J) \simeq (-1+(m-1)P^\sharp+R^\sharp)\tilde{\oplus}\left(-1+\frac{m+3}{2}P^\sharp\right),$$
$$gr^3(\omega, J) \simeq (0)\tilde{\oplus}\left(-2+\frac{m+5}{2}P^\sharp+R^\sharp\right),$$

which implies a contradiction $H^1(\omega/F^4(\omega, J)) \neq 0$. Thus we have

$$gr^2(\mathscr{O}, J) \simeq (2P^\sharp)\tilde{\oplus}\left(-1+\frac{m-1}{2}P^\sharp+R^\sharp\right). \quad \square$$

(2.13.11) **Lemma.** *We use the notation and assumptions of* (2.13.10). *We have that* $H^0(\mathscr{O}(-K_X)) = H^0(F^2(\omega^*, J))$ *and a general section* s *of* $H^0(\mathscr{O}(-K_X))$ *induces a section* $\bar{s}$ *of* $gr^2(\omega^*, J)$ *such that*

(2.13.11.1) $\bar{s}$ *generates* $L\tilde{\otimes} gr_C^0\omega^* \subset gr_C^1\omega^*$ *at* P, *and*

(2.13.11.2) *if* $m \geq 7$ *then* $\bar{s}$ *is a global generator of* (0) *in the* ℓ*-splitting of* (2.13.10)

$$gr^2(\omega^*, J) \simeq (0)\tilde{\oplus}\left(-1 + \frac{m+5}{2}P^\sharp + R^\sharp\right).$$

If $m = 5$, *we have the same assertion possibly after changing the* ℓ*-splitting of* $gr^2(\omega^*, J)$.

Proof. We see $H^0(\mathscr{O}(-K_X)) = H^0(F^2(\omega^*, J))$ by

$$H^0(gr^0(\omega^*, J)) = H^0(gr^1(\omega^*, J)) = 0 \quad (2.13.10).$$

Let $D = \{y_1 = 0\}/\mathbb{Z}_m \in |-2K_X|$ and let Ω be an ℓ-free ℓ-basis of $gr^0\omega$ at P. By (2.5), $y_2/\Omega \in \mathscr{O}_D(-K_X)$ lifts to a section of $H^0(F^2(\omega^*, J))$. Since y_2 is a part of an ℓ-free ℓ-basis of $gr^2(\mathscr{O}, J)$, we see that $\bar{s}$ is nonzero. If $m \geq 7$, then $\bar{s}$ must generate (0) because $H^0(C, (-1 + \frac{m+5}{2}P^\sharp + R^\sharp)) = 0$. If $m = 5$, then we see $H^0(\mathscr{O}(-K_X)) \twoheadrightarrow gr^2(\omega^*, J) \otimes \mathbb{C}(P)$ using $y_3^2/\Omega \in \mathscr{O}_D(-K_X)$. Then $\bar{s} \notin H^0(C, (R^\sharp))$ in the ℓ-splitting of $gr^2(\omega^*, J)$ and we have the same conclusion. □

(2.13.12) **Lemma.** *We assume the notation and assumptions of* (2.13.10). *Then the case* (2.2.3) *holds.*

Proof. Let $s \in H^0(\mathscr{O}(-K_X))$ be a general section. If $m = 5$, we change the ℓ-splitting of $gr^2(\mathscr{O}, J)$ for which (2.13.11) holds. Depending on the value of k, we treat two cases.

(2.13.12.1) Case $k = 1$. We claim that the image of $\bar{s}$ in $gr_C^1\omega^*$ generates $L\tilde{\otimes} gr_C^0\omega^* \simeq (1)$ $(\subset gr_C^1\omega^*)$ at P and R and vanishes at some point S $(\neq P, R)$. Indeed the generation at P is proved in (2.13.11). If $\bar{s}$ does not generate $L\tilde{\otimes} gr_C^0\omega^* = gr^{2,0}(\omega^*, J)$ at R, $\bar{s}$ is not a part of an ℓ-free ℓ-basis of $gr^{2,0}(\omega^*, J)$ at R because

$$\operatorname{qldeg}(gr^{2,1}(\omega^*, J), R) = \operatorname{qldeg}(M^{\tilde{\otimes}2}\tilde{\otimes} gr_C^0\omega^*, R) \neq 0.$$

It contradicts (2.13.11) and our claim is proved. Then it is easy to see that $E_X = \{s = 0\} \in |-K_X|$ is smooth outside of P, R, and S, $(E_X, P) \simeq (y_1, y_3)/\mathbb{Z}_m(1, -1)$, and $(E_X, R) \simeq (z_1, z_2)/\mathbb{Z}_2(1, 1)$. We choose coordinates at S so that $(X, S) = (w_1, w_2, w_3) \supset (C, S) = w_1$-axis and $J = (w_2, w_3^2)$ at S. Using a generator Ω of $\mathscr{O}(K_X)$ at S, we see $\Omega s \in (w_1w_2) + (w_2, w_3)^2$ because $\bar{s}$ vanishes at S to order 1. Since Ωs is a part of a free basis of $gr^2(\mathscr{O}, J)$ at S, we have $\Omega s \equiv fw_1w_2 + gw_3^2 \mod (w_2, w_3^2)(w_2, w_3)$ for some units f and g. Thus (E_X, R) is an A_1 point and we are done in case $k = 1$.

(2.13.12.2) Case $k \geq 2$. We see that the image of $\bar{s}$ in $gr_C^1\omega^*$ generates $L\tilde{\otimes} gr_C^0\omega^* \simeq (R^\sharp)$ outside of R by (2.13.11). Then $E_X = \{s = 0\} \in |-K_X|$ is smooth outside of P and R, $(E_X, P) \simeq (y_1, y_2)/\mathbb{Z}_m(1, -1)$. Using an ℓ-free ℓ-basis Ω of $\mathscr{O}(K_X)$ at R, we see that the image of $\bar{s}$ in $gr_C^1\omega^*$

is $z_1 z_4/\Omega$ at R. Since s is a part of an ℓ-free ℓ-basis of $gr^2(\omega^*, J)$ at R, we have $\Omega s \equiv z_1 z_4 + f z_3 \mod J^\sharp I^\sharp$ at R for some unit f. Eliminating z_3, we see that $(E_X, R) \simeq (z_1, z_2, z_3; \overline{\beta})/\mathbb{Z}_2(1, 1, 0; 0)$, where $\overline{\beta}$ satisfies $\overline{\beta} \equiv z_1^2 z_4 + z_2^2 \mod (z_2^2, z_4)(z_2, z_4)$ and $\operatorname{ord} \overline{\beta}(0, 0, z_4) = k$. Then we can apply (2.13.6). □

By (2.2) and (2.13), we see the following.

(2.13.13) **Theorem.** *Let $f: X \supset C \to Y \ni Q$ be an extremal nbd with $C \simeq \mathbb{P}^1$. Then the following are equivalent:*

(2.13.13.1) *$X \supset C$ is of type kAD.*

(2.13.13.2) *X has exactly two non-Gorenstein points on C and is smooth elsewhere and $(f(E_X), Q)$ is not a cyclic quotient singularity for a general member E_X of $|-K_X|$.*

(2.13.13.3) *$X \supset C$ is as described in* (2.13.3.2) *or* (2.13.10).

3. Some remarks about general DuVal elements

[Reid87] conjectured that if $X \supset C \to Y \ni P$ is an extremal nbd with the contraction map then the general member of the linear systems of Weil divisors $|-K_X|$ and $|-K_Y|$ have only DuVal singularities. He dubbed this member the "general elephant." In fact, he speculated that in even more general situations when contraction of an extremal face results in a singular point $Z \ni Q$, the general member of $|-K_Z|$ still has a DuVal singularity. He further hoped that this will be a key step toward establishing the existence of flips.

It seems that these conjectures and speculations are very close to being correct and they can serve as an important guideline toward understanding flipping singularities. In this chapter we present some of the evidence for the conjectures. The following theorem describes singularities with a DuVal general element.

(3.1) **Theorem.** *Let $Y \ni P$ be a threefold singularity. Let $P \in D \in |-K_Y|$ and assume that D has only DuVal singularities. Then*

(3.1.1) *There is a ramified double cover $p: Z \to Y$ such that Z has canonical singularities.*

(3.1.2) *For any Weil divisor W the symbolic power algebra*

$$\sum_{k=0}^{\infty} \mathcal{O}_Y(kW)$$

is a finitely generated $\mathcal{O}_Y$-algebra. In particular,

$$f_W: Y_{(W)} = \operatorname{Proj}_Y\Big(\sum_{k=0}^{\infty} \mathcal{O}_Y(kW)\Big) \to Y$$

is a proper map whose exceptional set consists of finitely many curves over P.

(3.1.3) *$Y_{(K_Y)}$ has only pseudoterminal singularities, and $Y_{(-K_Y)}$ has only canonical singularities.*

Proof. By [Shokurov91, 3.4] $K_Y + D$ is log terminal. Let $B \in |-2K_Y|$ be a general member. Let $p: Z \to Y$ be the double cover of Y ramified along B.

By the adjunction formula,

$$K_Z = p^*(K_Y + \tfrac{1}{2}B)$$

is Cartier and K_Z is log terminal (cf. the proof of [Shokurov91, 2.9]). Thus Z is canonical. This shows (3.1.1), which in turn implies (3.1.2) by [Kawamata88].

Let $Y' = Y_{(K_Y)}$ and let $D' \subset Y'$ be the proper transform of D. Then

$$K_{D'} = K_{Y'} + D'|D' = 0.$$

By assumption, D has only DuVal singularities, hence for any partial resolution (even for possibly nonnormal ones) $K_{D'} = 0$ iff D' is normal and is dominated by the minimal resolution. In particular, D' has only DuVal singularities. $K_{Y'}$ is $f_{(K_Y)}$-ample, thus D' contains the exceptional locus of $f_{(K_Y)}$. [Stevens88, §5] implies that Y' has only pseudoterminal singularities.

The same argument shows that $Y^* = Y_{(-K_Y)}$ has pseudoterminal singularities along the proper transform D^* of D. (Y^*, D^*) is log terminal and $D^* \in |-K_{Y^*}|$. Thus D^* contains all points where K_{Y^*} is not Cartier, hence Y^* is canonical outside D^*. □

(3.2) *Remarks.* (3.2.1) We proved in fact more: any non-Gorenstein singularity of $Y_{(-K_Y)}$ is pseudoterminal.

(3.2.2) In view of this result the natural set-up for extremal nbds might be to consider extremal nbds with canonical singularities. If in all cases $|-K_Y|$ has a general DuVal element then we have established a beautiful equivalence between isolated extremal nbds with canonical singularities and possibly reducible central curves and non-Gorenstein threefold singularities with a general DuVal element.

(3.3) **Theorem.** *Let $Y \ni P$ be a threefold singularity. Let $P \in D \in |-K_Y|$ and assume that D is a cyclic quotient (DuVal) singularity. Then*

(3.3.1) *The general hyperplane section $P \in H \subset Y$ has a cyclic quotient singularity at P;*

(3.3.2) *The pullback of H to $Y_{(K_Y)}$ has cyclic quotient singularities;*

(3.3.3) *The pullback of H to $Y_{(-K_Y)}$ has semi-log-canonical singularities* [KSB88, Chapter 4];

(3.3.4) *If $Y_n \to Y$ is the n-sheeted cyclic cover ramified along H then the general member of $|-K_{Y_n}|$ again has only cyclic quotient (DuVal) singularities.*

(3.4) **Corollary.** *Let $f : X \supset C \to Y \ni P$ be a semistable extremal nbd. Then the general hyperplane section $P \in H \subset Y$ has a cyclic quotient singularity.*

We start the proof with some lemmas:

(3.5.1) **Lemma.** *Let $x \in X$ be a three dimensional pseudoterminal singularity and let $x \in D \in |-K_X|$. Assume that D is a cyclic quotient (DuVal) singularity.*

(3.5.1.1) *Then in suitable local coordinates $x \in D \subset X$ can be written as*

$$[0 \in (z = 0) \subset (xy + f(z^m, t) = 0)]/\mathbb{Z}_m(1, -1, a, 0).$$

(3.5.1.2) *If $h \in \mathcal{O}_X$ is such that $h = t + zg$ for some powerseries g then in suitable local coordinates $(h = 0) \subset X$ has the form $(\overline{x}\overline{y} + \overline{f}(\overline{z}^m) = 0)$.*

Moreover we may assume that

$$x\text{-axis} = \overline{x}\text{-axis};$$
$$y\text{-axis} = \overline{y}\text{-axis}.$$

(3.5.1.3) *Let $X' \to X$ be the n-sheeted cover given by $t'^n = t$. Let $D' \subset X'$ be the pullback of D. Then in suitable local coordinates $x' \in D' \subset X'$ can be written as*

$$[0 \in (z' = 0) \subset (x'y' + f(z'^m, t') = 0)]/\mathbb{Z}_m(1, -1, a, 0).$$

Moreover we may assume that

$$x'\text{-axis} = \text{pullback of the } x\text{-axis};$$
$$y'\text{-axis} = \text{pullback of the } y\text{-axis}.$$

Proof. The first claim is clear from the list of pseudoterminal singularities [Hayakawa-Takeuchi87]. The other two are easy computations. □

(3.5.2) **Lemma.** *Let $0 \in S$ be a normal surface singularity. Let $g : S' \to S$ be a proper birational morphism which has the following properties*:

(3.5.2.1) *$g^{-1}(0) \subset S'$ is a chain of smooth rational curves intersecting transversally*;

(3.5.2.2) *If $s' \in S'$ is a singular point then in suitable local coordinates S' can be written as*

$$(xy + f(z^m) = 0)/\mathbb{Z}_m(1, -1, a) \qquad ((a, m) = 1),$$

where $g^{-1}(0) = (z = 0)$.

Then $0 \in S$ is a cyclic quotient singularity.

Proof of (3.3). Let Y' and D' be as in the proof of (3.1). By the proof of (3.1), D' is dominated by the minimal resolution of D. Therefore every singularity of D' is a cyclic quotient.

Look at the following exact sequence:

$$0 \to \omega_{Y'} \to \mathscr{O}_{Y'} \to \mathscr{O}_{D'} \to 0.$$

Since $R^1 g_* \omega_{Y'} = 0$, this gives a surjection

$$H^0(\mathscr{O}_{Y'}) \to H^0(\mathscr{O}_{D'}) \to 0.$$

Let $D \cong (uv - w^d = 0)$. Then the section $g^*(w)$ lifts to a section s of $\mathscr{O}_{Y'}$. By lifting generically, we may assume that $H' = (s = 0)$ is normal. Let $H = g(H')$. (3.5.1.2) and (3.5.2) imply that H' and H have only cyclic quotient singularities. This shows (3.3.1–3.3.2).

To show (3.3.4) we take the n-sheeted cover ramified along H'. The pullback D'_n of D' to Y_n is a member of $|-K_{Y_n}|$. (3.5.1.3) describes the local structure of D'_n and so by (3.5.2) $D_n = g_n(D'_n)$ is a cyclic quotient singularity.

To see (3.3.3) we consider $(Y_n)_{(-K_{Y_n})}$ and $(Y_{(-K_Y)})_n$ (the n-sheeted cover of $Y_{(-K_Y)}$ ramified along the proper transform of H). These are both small modifications of Y_n such that the anticanonical class is relatively ample. Therefore they are isomorphic. Since $(Y_n)_{(-K_{Y_n})}$ has canonical singularities by (3.1) the same holds for $(Y_{(-K_Y)})_n$. Now [KSB88,5.1] implies that the proper transform of H has only semi-log-canonical singularities. □

The proof of (3.3) implies the following result:

(3.6) **Corollary.** *Let $f : X \supset C \to Y \ni P$ be a semistable extremal nbd. Assume that the general $D \in |-K_X|$ contains C. Let $C \subset H \in |\mathcal{O}_X|$ be a sufficiently general member. Then at every point $P \in C$ one can choose local coordinates such that $P \in C \subset H \subset X$ is given as*

$$0 \in (x\text{-}axis) \subset (t = 0) \subset (xy - z^{dn} + tf(x, y, z, t) = 0)/\mathbb{Z}_n(1, -1, a, 0).$$

(3.7) *Remarks.* (3.7.1) It is possible that if $Y \ni P$ is a threefold singularity such that the general member of $|-K_Y|$ has a DuVal singularity then the general hyperplane section $P \in H \subset Y$ has a rational singularity at P. In fact, we should get a very limited class of rational surface singularities, though much larger than just quotients and quadruple points.

(3.7.2) It seems to be true—as illustrated by (3.1.3)—that the proper transform of H on $Y_{(K_Y)}$ is simpler than the proper transform on $Y_{(-K_Y)}$. Therefore it seems reasonable to try to prove the existence of a nice member H by finding its proper transform on $Y_{(K_Y)}$. In many cases this seems possible.

(3.8) **Example.** There is an interesting construction that can be used to create a slew of isolated extremal nbds (with canonical singularities in general) starting with one. It goes as follows:

Let $Y \ni P$ be a threefold singularity such that the general member of $|-K_Y|$ has a DuVal singularity. Then we construct Y' and D' as before. Let $C \subset D'$ be the exceptional curve. Now take a smooth curve Δ in D' which does not pass through any of the singular points. Blowing up Δ we get a new threefold $\overline{Y}$ and $\overline{D} \cong D'$ is the proper transform of D'. Clearly locally along $\overline{D}$, $\overline{D}$ is a member of $|-K_{\overline{Y}}|$. The proper transform $\overline{C}$ of C is contractible and this way we get a new threefold singularity which has a member of $|-K|$ isomorphic to D. This way we also get new examples of isolated extremal nbds (with canonical singularities in general).

Unfortunately it is very hard to understand what the new example will be like. It seems that in most cases it will have fairly complicated nonterminal canonical singularities.

One interesting special case is when we start with a terminal singularity as Y and pick any small modification as Y'. Thus we can get examples of extremal nbds without starting with one.

4. Index two nbds

The aim of this chapter is to give a fairly complete description of extremal nbds with index two points only. The methods are completely elementary. None of the machinery of [Mori88] is used. The classification will then be used to disprove the existence of certain types of nbds with index four points. In order to prove some results in Chapter 2 we also describe divisorial extremal nbds with index two points only. During the proof very little is gained by assuming that the central curve is irreducible, in fact, we need to understand some cases where it is not. Therefore we will consider the following general situation:

(4.1) **Cases to be considered.** In this chapter, $f : X \supset C \to Y \ni Q$ denotes a three dimensional extremal curve neighborhood as in (T.1). We assume furthermore that X has only points of index one and two. We do not assume that C is irreducible.

(4.2) **Theorem.** *Let $f : X \supset C \to Y \ni Q$ be as in* (4.1). *Assume that $X \supset C \to Y$ is isolated. Let $P \in C$ be a point of index 2. Then*

(4.2.1) *P is the only singular point and in appropriate coordinates $P^\sharp \in X^\sharp$ is given by the equation*

$$(x_1x_2 + p(x_3^2, x_4) = 0)/\mathbb{Z}_2(1, 1, 1, 0)$$

and $C^\sharp$ is the x_1-axis.

(4.2.2) *X^+ has at most one singular point with equation*

$$x_1x_2 + p(x_3, x_4) = 0$$

and C^+ is the x_1-axis. (*Same p as above but no group action and x_3 instead of x_3^2*).

(4.2.3) *Y is a rational triple point given by the 2×2-minors of*

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_5 & p(z_1, z_4) \end{pmatrix}.$$

(4.2.4) *C is irreducible.*

The proof uses a construction that will be used later in the divisorial case. Therefore we give it in the general setting.

(4.3) **Construction.** Let $f : X \supset C \to Y \ni Q$ be as in (4.1). Let C_i be the irreducible components of C. Since X has only points of index one and two, $m_i = -2K_X \cdot C_i$ is a positive integer. Let $E_i \subset X$ be the union of m_i disjoint discs transversal to C_i and let $E = \sum E_i$. Then $E \in |-2K_X|$, hence we can take the corresponding double cover $X' \to X$ ramified along E. X' has only index one terminal singularities. Let $E' \subset X'$ be the preimage of E. The natural map $E' \to E$ is an isomorphism. Let $D = f(E) \subset Y$ and let $Y' \to Y$ be the corresponding double cover ramified along D. We have a contraction map $f' : X' \to Y'$. By construction, $K_{X'}$ is trivial along the fibers of f'. Therefore Y' has a cDV point. (If f is divisorial, then Y' will have a double curve.) Thus we have the following diagram:

$$\begin{array}{ccc} E \subset X & \longleftarrow & E' \subset X' \\ {\scriptstyle f}\big\downarrow & & {\scriptstyle f'}\big\downarrow \\ D \subset Y & \longleftarrow & D' \subset Y' \end{array}$$

The double cover construction gives a $\mathbb{Z}_2$-action on $f' : X' \to Y'$ and the quotient is $f : X \to Y$. The fixed point set of the action on Y' is precisely

D'. Since Y' is a cDV point, it is a hypersurface in $\mathbb{C}^4$, thus it can be written down explicitly in principle. This will enable us to get equations for X and Y.

If the nbd is isolated then $E \to D$ is an isomorphism outside the origin, in fact, it will turn out to be an isomorphism. In particular, D is smooth. If f contracts an exceptional divisor $S \subset X$ then the general fiber F of S is a (-1)-curve, hence $E \cdot F = 2$. Therefore D will have a double curve along the image of S and will be smooth elsewhere. If E is chosen generically then D will have an ordinary double curve along the image of S (i.e., two branches intersecting generically transversally).

(4.4) *Proof of* (4.2). This will be done in several steps.

(4.4.1) *Claim.* Let the assumptions be as in (4.2). In suitable local coordinates (y_i) for $Y' \subset \mathbb{C}^4$, $\mathbb{Z}_2$ acts with wts (1,1,0,0) and $D' = (y_1 = y_2 = 0)$. Therefore C is irreducible.

Proof. We may assume that the coordinates are eigenvectors and $y_1, \dots, y_j$ are those with wts 1. Thus $D' = (y_1 = \cdots = y_j = 0) \cap Y'$. Hence $j = 1$ or 2. If $j = 1$ then D' is Cartier. On the other hand, since f' is an isomorphism outside the origin and E' is f'-ample, D' cannot be Cartier. Hence $j = 2$ and $(y_1 = y_2 = 0)$ must be contained in Y'. D' is irreducible and this implies that C is irreducible. □

(4.4.2) *Claim.* We can further change y_i such that $Y' = (y_1y_3 + y_2p(y_2^2, y_4) = 0)$. Wts stay as above.

Proof. Since $(y_1 = y_2 = 0) \subset Y'$, its equation can be written in the form $y_1g + y_2h = 0$. If $\mathrm{wt}(g) = \mathrm{wt}(h) = 1$ then $y_1g + y_2h \in (y_1, y_2)^2$, which implies that Y' is singular along $(y_1 = y_2 = 0)$. This is impossible. Thus $\mathrm{wt}(g) = \mathrm{wt}(h) = 0$. Since Y' is a double point, either g or h must contain a linear term. Say g contains y_j. By wt reasons $j = 3$ or 4. Now in the usual way we can normalize the equation in the required form. By wt reasons only even powers of y_2 can occur in p. □

(4.4.3) *End of proof of* (4.2). With this explicit equation we can easily compute everything. X is obtained by blowing up $(y_2 = y_3 = 0)$ and dividing by the group action. This gives us one singular point with the required equation. X^+ is obtained by blowing-up $(y_2 = y_1 = 0)$ and dividing by the group action.

To get equations for Y, the invariants of the $\mathbb{Z}_2$ action on $\mathbb{C}[y_1, y_2, y_3, y_4]$ are

$$z_1 = y_2^2 \qquad z_2 = y_1y_2 \qquad z_3 = y_3 \qquad z_4 = y_4 \qquad z_5 = y_1^2.$$

We get exactly the equations given by the minors of the above matrix. A hyperplane section given by $z_4 - cz_1 = 0$ (where c is a general constant) is easily seen to define a rational triple point. This completes the proof of (4.2). □

For future reference we note the following consequence of this proof:

(4.4.4) **Corollary.** *Assume* (4.1) *and the notation of* (4.3). *If D' is non-Cartier then C is irreducible and Y is a triple point. In particular, the nbd is isolated.*

Proof. By the proof of (4.4.1), if D' is not Cartier then D' is irreducible and smooth. C is also irreducible since D' is. As we saw at the end of (4.3), the nbd is isolated since D' is smooth. □

(4.4.5) **Corollary.** *Let* $X \supset C$ *be as in* (4.2). *Assume that* $X \ni P$ *is a cyclic quotient singularity. Then* $X \supset C$ *is unique up to analytic isomorphism.*

Proof. In this case the coefficient of x_4 in p is not zero, thus we can introduce a new coordinate $z_4' = p(z_1, z_4)$ and the equations for Y become the 2×2-minors of

$$\begin{pmatrix} z_1 & z_2 & z_3 \\ z_2 & z_5 & z_4' \end{pmatrix}. \quad \square$$

(4.4.6) **Corollary.** *An isolated extremal nbd* $\overline{X} \supset \overline{C}$ *cannot have a point of index* 4 *and a point of index* 2.

Proof. By [Mori88, 1.10] this nbd is imprimitive, in fact, it has a double cover $X \supset C$ which is étale outside the singular points. Above the index four point X has an index 2 point. Let $Q \in \overline{X}$ be the index two point. By (4.2) its double cover is smooth. If we write $Q \in \overline{X} \cong \mathbb{C}^3/\mathbb{Z}_2(1, 1, 1)$ then any plane through the origin is invariant under the $\mathbb{Z}_2$ action. We pick one such plane E, which has transversal intersection with the curve $C = \overline{C}^\sharp$. If we pick this E as the ramification divisor in the construction (4.3) then the fourfold cover $X' \to X \to \overline{X}$ will be Galois with Galois group $\mathbb{Z}_4$.

If $\overline{X} \supset \overline{C} \to \overline{Y}$ is the contraction map then from (4.2) we obtain that $\overline{Y}$ is isomorphic to the quotient of $Y' = (y_1y_3 + y_2p(y_2^2, y_4) = 0)$ by an action of $\mathbb{Z}_4$. Let μ be a fixed generator of $\mathbb{Z}_4$. Then μ^2 acts with wts $(2, 2, 0, 0)$ mod 4.

Since $Y \to \overline{Y}$ is étale outside the origin, the $\mathbb{Z}_4/\mathbb{Z}_2$ action on $D' = (y_1 = y_2 = 0)$ has no fixed points outside the origin. This implies that μ acts via $(y_3, y_4) \mapsto (-y_3, -y_4)$. μ acts on y_1 via multiplication by $\pm\sqrt{-1}$. We can assume that it acts via $\sqrt{-1}$. Since the equation of Z' has to be an eigenfunction, this implies that depending on the parity of the exponent of y_4 in p the action has wts

$$(1, 1, 2, 2) \bmod 4 \quad \text{or} \quad (1, 3, 2, 2) \bmod 4.$$

To obtain the extremal nbd $\overline{X}$ we blow up the plane $(y_2 = y_3 = 0)$ and we divide out by the action of $\mathbb{Z}_4$. Explicit computation gives that the index four point of $\overline{X} \supset \overline{C}$ is given as

$$(y_1y_3 + p(y_2^2, y_4) = 0)/\mathbb{Z}_4,$$

where the wts are

$$(1, 1, 1, 2) \bmod 4 \text{ or } (1, 3, 3, 2) \bmod 4.$$

A glance at the list of terminal singularities [Mori85] shows that these singularities are not terminal. □

It is worthwhile to remark that the above construction gives examples of isolated extremal nbds with log-terminal singularities of indices 4 and 2.

(4.5) **Theorem.** *An isolated extremal nbd cannot have any of the following types of singularities*:

(4.5.1) *A type* $II^\vee$ *point,*

(4.5.2) *A type IIB point.*

Proof. If the nbd $X \supset C$ has a point of type $II^\vee$ then it is imprimitive [Mori88, 6.11]. Its double cover has a point of index two and the central curve is reducible. By (4.2) there is no such isolated nbd.

If the nbd $X \supset C$ has a point of type *IIB* then it has an L-deformation [Mori88,4.7] to an isolated nbd that has a point of index 4 and a point of index 2. Thus (4.4.6) implies (4.5.2). □

Now we turn to the divisorial nbds of index two. First we study the configuration of the curves of C.

(4.6) **Proposition.** *Let* $X \supset C$ *be as in* (4.1). *If all points have index one then* C *is irreducible. If* P *is a point of index two then* P *is the only point of index two.* C *has at most three components, they all pass through* P *and they do not intersect elsewhere.*

Proof. We use the notation of (4.3). Let $X' \supset C' \to C$ be the double cover of C. Let $H_1 \subset Y'$ be a general hyperplane section and let $H_2 = f'^* H_1$. Since Y' is a *cDV* point and f' is crepant, we see that H_1 is a DuVal singularity and $H_2 \to H_1$ is dominated by the minimal resolution. Since $C' \subset H_2$, this implies that:

(4.6.1) Two components of C' intersect in at most one point;

(4.6.2) At most three components of C' intersect at any point;

(4.6.3) The components of C' are smooth and rational.

Let C_i be the components of C, and let C_i' be the preimage of C_i. C_i' is irreducible since the covering is locally irreducible at the points of E. $C_i' \to C_i$ is ramified at the points of $C \cap E$ and at the index two points. On each C_i there are precisely two ramification points by (4.6.3). Thus each C_i contains at most one index two point.

(4.6.1) implies that two components of C cannot intersect at an index one point. Thus there is at most one index two point, all components of C pass through P and they do not intersect elsewhere. By (4.6.2) there are at most three components. □

(4.7) **Theorem.** *Let* $X \supset C$ *be as in* (4.1). *Then we have one of the following cases. In each case we specify the type of the index two point, the minimal resolution of the general member of* $|\mathcal{O}_X|$, *and the general member of* $|\mathcal{O}_Y|$. *We use the following notational conventions*:

• *denotes the proper transforms of the components of* C. *These have selfintersection* (-1). *Minus the selfintersection of a curve is written under it. We do not indicate the selfintersection if it is* (-2) *(for* $\circ$*) or* (-1) *(for* $\bullet$*)*.

$$\underbrace{\underset{3}{\circ} - \cdots - \underset{3}{\circ}}_{m}$$

indicates that there are $(m-2)$ *curves with selfintersection* (-2) *in between.*

For $m = 1$ the above symbol denotes

$$\underset{4}{\circ}$$

List of possibilities:

(4.7.1) *Isolated nbds. The singularity has type cA and H_X is*

$$\bullet - \underbrace{\underset{3}{\circ} - \cdots - \underset{3}{\circ}}_{m} \quad \to \quad \underbrace{\circ - \cdots - \underset{3}{\circ}}_{m}$$

(4.7.2) *Index one points only. Then the nbd is divisorial and H_X is*

$$\bullet - \underbrace{\circ - \cdots - \circ}_{m} \quad \to \quad A_0$$

In the remaining cases there is exactly one index two point P and the nbd is divisorial.

(4.7.3) *P has type cA iff H_X has log-terminal singularities. We have the following cases*:

(4.7.3.1) *C has one component*:

(4.7.3.1.1)

$$\circ - \bullet - \underbrace{\underset{3}{\circ} - \cdots - \underset{3}{\circ}}_{m} \quad \to \quad A_1$$

(4.7.3.1.2)

$$\circ - \circ - \bullet - \underset{4}{\circ} \quad \to \quad A_0$$

(4.7.3.1.3)

$$\begin{array}{ccccc} \underset{3}{\circ} & - & \circ & - & \underset{3}{\circ} \\ & & | & & \\ & & \bullet & & \end{array} \quad \to \quad A_2$$

(4.7.3.1.4)

$$\begin{array}{ccccccc} \underset{3}{\circ} & - & \circ & - & \circ & - & \underset{3}{\circ} \\ & & | & & & & \\ & & \bullet & & & & \end{array} \quad \to \quad A_0$$

(4.7.3.2) *C has two components*:

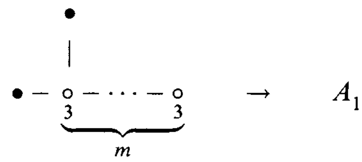
(4.7.3.2.1)

$$\bullet - \underbrace{\underset{3}{\circ} - \cdots - \underset{3}{\circ}}_{m} - \bullet \quad \to \quad A_m$$

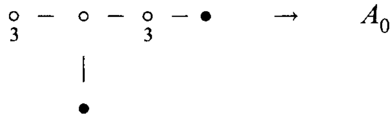
(4.7.3.2.2)

$$\circ - \bullet - \underbrace{\underset{3}{\circ} - \cdots - \underset{3}{\circ}}_{m} - \bullet \quad \to \quad A_0$$

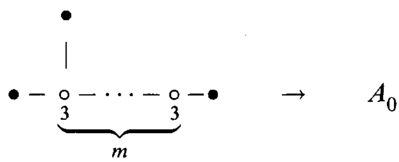
(4.7.3.2.3)



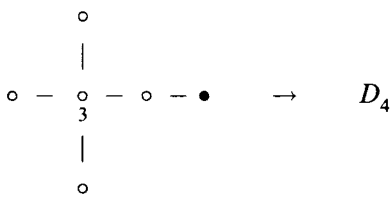
(4.7.3.2.4)



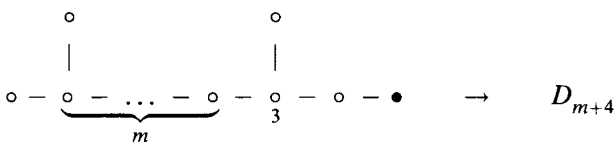
(4.7.3.3) *C has three components*:
(4.7.3.3.1)



(4.7.4) *If X has a type cAx point then H_X has a log-canonical singularity*:

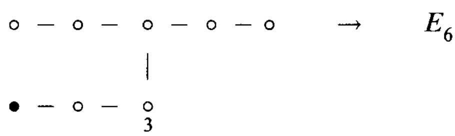


(4.7.5) *If X has a type cD point then H_X has a log-canonical singularity*:



where $m \geq 0$. For $m = 0$ this is the configuration of (4.7.4).

(4.7.6) *If X has a type cE point then either H_X has a log-canonical singularity as in* (4.7.5) *with $m \leq 1$; or H_X is not log-canonical and is given by*:



(4.8) *Proof*. We use the notation of (4.3). By (4.4.4) $D' \subset Y'$ is Cartier and the $\mathbb{Z}_2$-action is given by wts $(0,0,0,1)$. Thus for a suitable choice of coordinates we can write the equation of Y' as $sg(x, y, z, s) + h(x, y, z) = 0$ where $D' = (s = h = 0)$. By wt reasons only even powers of s occur. Thus we can write the equation in the form

$$s^2 g(x, y, z, s^2) + h(x, y, z) = 0. \tag{4.8.1}$$

The equation of Y is now given by

$$tg(x, y, z, t) + h(x, y, z) = 0 \qquad (t = s^2). \tag{4.8.2}$$

Since f' is crepant, Y' cannot be smooth, in particular, $\mathrm{mult}_0 h \geq 2$.

Knowing the general member of $|\mathcal{O}_Y|$ tells us very little about the general member of $|\mathcal{O}_X|$ in general. Therefore we will proceed in the following roundabout way. First we find the general $\mathbb{Z}_2$-invariant member of $|\mathcal{O}_{Y'}|$. Via pullback we will be able to determine the general $\mathbb{Z}_2$-invariant member of $|\mathcal{O}_{X'}|$. This is possible since f' is crepant and therefore very well behaved. Taking the quotient will then give the general member of $|\mathcal{O}_X|$. Let H_* denote the general member of $|\mathcal{O}_*|$ and let H'_* denote the general $\mathbb{Z}_2$-invariant member of $|\mathcal{O}_{*'}|$. We have the following diagram:

$$\begin{array}{ccc} H_X \subset X & \longleftarrow & H'_X \subset X' \\ {\scriptstyle f}\downarrow & & {\scriptstyle f'}\downarrow \\ H_Y \subset Y & \longleftarrow & H'_Y \subset Y' \end{array}$$

We will distinguish several cases.

(4.8.3) Case 1. $\mathrm{mult}_0 g = 0$.

The assumption means precisely that Y is smooth. We can change coordinates to bring the equation of Y to the form

$$s^2 + h(x, y, z) = 0. \tag{4.8.3.1}$$

(4.8.3.2) **Lemma.** *If $Y' = (s^2 + h(x, y, z) = 0)$ defines a cDV point then for a generic linear form l in three variables*

$$s^2 + h(x, y, z) = l(x, y, z) = 0$$

is a DuVal singularity.

Proof. By [KSB88, 6.9] $B_0 Y'$ has only rational hypersurface singularities. The proper transform of the linear system $|x, y, z|$ has one possible base point on $B_0 Y'$ at the point corresponding to the s-axis on the exceptional divisor. This, however, is not on $B_0 Y'$. Thus the proper transform of $l(x, y, z) = 0$ has only rational singularities. Since $s^2 + h(x, y, z) = l(x, y, z) = 0$ defines a double point, this implies that it is a DuVal singularity. □

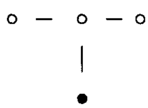
(4.8.3.3) **Proposition.** *If H'_Y has a DuVal singularity then H_X has only log-terminal singularities.*

Proof. Since f' is crepant, this implies that H'_X is dominated by the minimal resolution of H'_Y, in particular, it has only DuVal singularities. Thus any quotient of it has log-terminal singularities. □

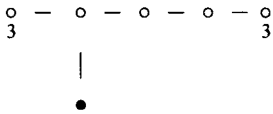
(4.8.3.4) *Cases where H'_Y has a DuVal singularity.* By (4.8.3.3) we need to enumerate those cases where H_X has log-terminal singularities only.

The only log-terminal points of index two are given by

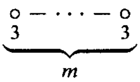
To this we have to attach the proper transforms of the components of C and then we can have the minimal resolutions of some DuVal singularities. Since H_Y has a DuVal singularity, the only combinatorial condition for the configuration is that repeated contraction of (-1)-curves gives the minimal resolution of a DuVal singularity (or the empty diagram if H_Y is smooth). To enumerate all cases note first that certain configurations cannot occur as subconfigurations. Two of these are



and



Now it is easy to see that if a (-1) curve is adjacent to a (-2)-curve inside



then we get one of the cases (4.7.3.1.3, 1.4, or 2.4). Otherwise all (-1)-curves are adjacent to (-3) or (-4)-curves. It is very easy to list all possibilities. □

(4.8.4) *Case* 2. $\operatorname{mult}_0 g > 0$.

The assumption means precisely that Y is singular. Also, this implies that $\operatorname{mult}_0 s^2 g \geq 3$. Y' has a double point, hence $\operatorname{mult}_0 h = 2$. Therefore, in suitable coordinates the equation of Y' becomes

$$s^2 g(x, y, s^2) + z^2 - h(x, y) = 0.$$

We can write $h = f^2 l$ where l has no multiple factors. The singular curve of D' is given by $s = z = f = 0$. The normalization of D' is given by

$$\overline{z}^2 - l(x, y) = 0 \quad \text{where } z = \overline{z} f(x, y).$$

By construction this normalization is E', which is smooth. Therefore $\operatorname{mult}_0 l < 2$. E' has two components if $\operatorname{mult}_0 l = 0$ and one if $\operatorname{mult}_0 l = 1$. Therefore we can write the equation of Y' in one of the following forms:

(4.8.4.1) $\quad s^2 g(x, y, s^2) + z^2 - f(x, y)^2 = 0$ if C is reducible,

(4.8.4.2) $\quad s^2 g(x, y, s^2) + z^2 - f(x, y)^2 l(x, y) = 0$ if C is irreducible.

The double curve of Y' is given by $s = z = f = 0$. We will see later (4.9.4) that X' is obtained from Y' by blowing up $s = z = f = 0$. Therefore, X' and X are explicitly computable in terms of the above equations. We will need the explicit computations only in the case when the double curve is smooth, therefore, we postpone it.

(4.8.4.3) *Case* 2.1. C reducible and $\text{mult}_0 f = 1$.

The equation then becomes $s^2 g(y, s^2) + z^2 - x^2 = 0$. H'_Y is given by $y = \text{const} \cdot s^2$ and it has a DuVal singularity. This case is covered by (4.8.3.4).

(4.8.4.4) *Case* 2.2. $\text{mult}_0 g = 1$.

We may assume that x appears in g with nonzero coefficient. Then H'_Y is given by $y = \text{const} \cdot s^2$ and it has a DuVal singularity of type D. This case is again covered by (4.8.3.4).

The remaining case is harder.

(4.8.5) *Case* 2.3. $\text{mult}_0 g > 1$.

The equation must contain a cubic term, which is therefore in $f^2 l$. In the reducible case this is only possible for $\text{mult}_0 f = 1$, which we treated already. Thus we may assume that C is irreducible and $\text{mult}_0 f = \text{mult}_0 l = 1$.

Depending on whether the linear terms of f and l are independent or not, we can bring the equation Φ of Y' to one of the following forms:

(4.8.5.1) $\Phi : s^2 g(x, y, s^2) + z^2 - y^2 x = 0$ if independent,

(4.8.5.2) $\Phi : s^2 g(x, y, s^2) + z^2 - y^2(y + x^n) = 0$ $(n \geq 2)$ if dependent.

The double curve is given by the equation $s = z = y = 0$. Blowing it up we obtain X'. We will need the chart that covers the index two point of X. This can be obtained by substituting $y = y's$ and $z = z's$. The equation of the index two point becomes

(4.8.5.3) $(g(x, y's, s^2) + z'^2 - y'^2 x = 0)/\mathbb{Z}_2(0, 1, 1, 1)$ if independent,

(4.8.5.4)

$$(g(x, y's, s^2) + z'^2 - y'^2(y's + x^n) = 0)/\mathbb{Z}_2(0, 1, 1, 1) \quad \text{if dependent.}$$

(4.8.5.5) *Notation.* For a monomial M the symbol $M \in g$ will mean that M appears in g with nonzero coefficient.

(4.8.5.6) **Proposition.** *Assume that we are in case* (2.3). *The following are equivalent*:

(i) H'_Y *has a DuVal singularity.*

(ii) *The index two point on* X *has type cA.*

(iii) $s^2 \in g$.

Proof. If H'_Y has a DuVal singularity then H_X has log-terminal singularities. By [KSB88,3.10] if a Cartier divisor on a terminal singularity is log terminal then the terminal singularity has type cA. Thus (i) implies (ii). (ii) $\Rightarrow$ (iii) can be read off from (4.8.5.3-4.8.5.4). (Note that $x^2 \in g$ does not imply that it is cA.)

Assume (iii). If we assign $\mathbb{Q}$-wts to the variables by

$$\alpha(x, y, z, s) = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{4}),$$

then every monomial in the equations (4.8.5.1-4.8.5.2) will have α-wt at least one and

$$\Phi_{\alpha=1} = as^4 + z^2 - y^2 x \quad \text{respectively, } as^4 + z^2 - y^3$$

and $a \neq 0$ by (iii). Thus H'_Y is a deformation of the DuVal singularity E_6. □

(4.8.5.7) **Proposition.** *The index two point on X has type cAx iff $s^2 \notin g$ and $x^2 \in g$. If this is the case then the general member of $|-K_X|$ is D_4.*

Proof. The first statement can be read off from (4.8.5.3-4.8.5.4). Looking at (4.8.5.3-4.8.5.4) again we see that the hyperplane section $s = \text{const} \cdot y'$ has an A_3 singularity. By [Reid87, p. 393] this implies the second claim. □

For future reference we note the following.

(4.8.5.8) **Lemma.** (i) *The surface singularities $(z^2 - y^3 - y^2s^2 = 0)$ and $(z^2 - y^3s - y^2s^2 = 0)$ are semi-log-canonical* [KSB88, Chapter 4].

(ii) *Any small normal deformation of a Gorenstein semi-log-canonical double point is either a DuVal singularity, or a cusp or a simple elliptic singularity. In the last two cases, if E denotes the reduced exceptional curve of the minimal resolution then $k = -E^2$ is* 1 *or* 2 [KSB88, 5.6].

(iii) *Any small deformation of a simple elliptic singularity with $k = 1$ (these have equation $z^2 + y^3 + s^6 + ay^2s^2 = 0$; $(4a^3 + 27 \neq 0)$) is either simple elliptic or a DuVal singularity.*

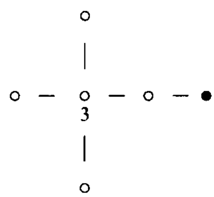
Proof. The first statement is an easy computation. The proof of the second is outlined in [KSB88,5.6]. The third one is again an easy computation. □

(4.8.6) *Case* 2.3.1. Independent linear forms, $s^2 \notin g$.

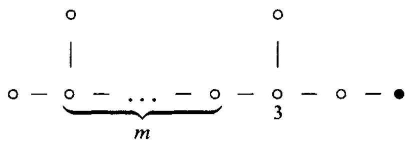
Y' can be viewed as the total space of a deformation of the pinch point $z^2 - y^2x = 0$. Substituting $x = y + s^2$ gives a nonnormal surface singularity $S = (z^2 - y^3 - y^2s^2 = 0)$ and H'_Y is a small deformation of S. Therefore H'_Y is either log-canonical or DuVal. The latter is impossible by (4.8.5.6). From the classification of [Kawamata80] we see that the only possibilities are simple elliptic with selfintersection (-3) or a cusp with exactly one curve with selfintersection (-3). H'_X is obtained from H'_Y by blowing up the origin. The deformation from S to H'_Y is equimultiple, therefore, $H'_X = B_0H'_Y$ is a flat deformation of B_0S. B_0S has only one singularity above the origin given by the equation $(z'^2 - y'^3s - y'^2s^2 = 0)$. Thus H'_X has a single log-canonical singularity at the origin of the new chart. We complete the description of this case using the following

(4.8.6.1) **Lemma.** *Let $X \supset C$ be an extremal nbhd. Assume that C is irreducible. Let $H \subset X$ be a normal member of $|\mathscr{O}_X|$ containing C. Assume that H has a singular point, which is an index two $\mathbb{Z}_2$-quotient of a simple elliptic or cusp singularity with $k \leq 2$. Then the minimal resolution of H is given by one of the following diagrams:*

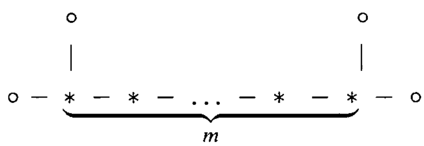
For the simple elliptic case:



For the cusp case $(m > 0)$:



Proof. The $\mathbb{Z}_2$ quotients of cusps and of simple elliptic singularities are described in [Kawamata80].There is no required $\mathbb{Z}_2$-quotient if $k = 1$. For $k = 2$ we get the following possibilities:



where among the $*$ there is exactly one curve with selfintersection (-3), the rest have (-2). In the relative canonical divisor the curves $*$ appear with coefficient (-1), the curves $\circ$ with coefficient $(-\frac{1}{2})$. Thus the proper transform of C is adjacent to one of the curves $\circ$, call it B. There is a unique curve $*$ adjacent to B. If this curve has selfintersection (-2) then repeated contraction of (-1)-curves leads to a contradiction. Thus this curve has selfintersection (-3) and repeated contraction of (-1)-curves gives the D_{m+3} configuration. Now it is clear that we cannot have any other singularities on H. □

(4.8.7) *Case* 2.3.2. Dependent linear forms, $s^2 \notin g$. Equation (4.8.5.4) defines a cDV point, hence $\operatorname{mult}_0 g(x, y's, s^2) \leq 3$. The possible terms that can occur of degree at most three are x^2, xs^2, $xy's$, and x^3. We have to consider separately the cases when we have one of the first three possibilities or x^3.

(4.8.7.1) **Proposition.** *The index two point on X has type cAx or cD iff $x^2 \in g$ or $xy \in g$ or $xs^2 \in g$. In these cases H_X has a log-canonical singularity.*

Proof. It is clear from (4.8.5.4) that X has type cAx or cD iff $x^2 \in g$ or $xy \in g$ or $xs^2 \in g$. To see that H_X has a log-canonical singularity we assign $\mathbb{Q}$-wts to the variables by

$$\alpha(x, y, z, s) = (\tfrac{1}{3}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}).$$

Then the α-wt of every monomial in the equation (4.8.5.2) is at least one and

$$\Phi_{\alpha=1} = z^2 - y^3 + ax^2s^2 + bxs^4 + cxys^2 + ds^6 + ey^2s^2 + fys^4$$

where at least one of a, b, c is not zero. For a sufficiently general u we take the hypersurface section $x = us^2$. This gives us the equation

$$z^2 - y^3 + y^2s^2[e] + ys^4[uc + f] + s^6[u^2a + ub + d].$$

We claim that this defines a simple elliptic singularity with $k = 1$. To see this we blow up the origin and introduce new coordinates $z = z''s, y = y''s$ to obtain

$$z''^2 - y''^3 s + y''^2 s^2[e] + y'' s^3[uc + f] + s^4[u^2a + ub + d].$$

A routine discriminant computation gives that the homogeneous quartic in (y'', s) has no multiple roots for general u if at least one of a, b, c is not zero. Therefore this equation defines a simple elliptic singularity with $k = 2$. By (4.8.6.1) we get the case (4.7.4).

At this point we should note that one can get a simple elliptic singularity even when $a = b = c = 0$. These will correspond to some extremal nbds with a cE type point and to some with nonterminal singularities. □

(4.8.7.2) **Proposition.** *If $x^3 \in g$ but $x^2 \notin g$, $xy \notin g$, $xs^2 \notin g$, then H'_Y is a small deformation of $S = (z^2 + y^3 + s^8 = 0)$. The deformation is equivariant with respect to the group action $(x, y, s) \to (x, y, -s)$.*

Proof. We assign $\mathbb{Q}$-wts by the formula

$$\alpha(x, y, z, s) = (\tfrac{2}{9}, \tfrac{1}{3}, \tfrac{1}{2}, \tfrac{1}{6}),$$

then every monomial in g will have α-wt at least one and

$$\Phi_{\alpha=1} = z^2 - y^3 + ax^3s^2 \quad \text{where } a \neq 0.$$

The substitution $x = s^2$ gives the singularity S. □

(4.8.7.3) **Computation.** *Let us consider a small deformation of $S = (z^2 + y^3 + s^8 = 0)$, equivariant with respect to the group action $(x, y, s) \to (x, y, -s)$. The following is the list of all possible nearby fibers:*

(i) $z^2 + y^3 + s^8 + ays^6 = 0$ (*Equisingular deformation*; E_{14} *in* [AGV85, pp. 184–185]),

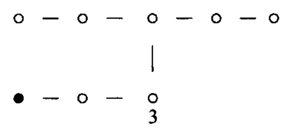
(ii) $z^2 + y^3 + s^8 + y^2s^2 = 0$ (*Cusp with* $k = 1$; $T_{2,3,8}$ *in* [AGV85, pp. 184–185]),

(iii) $z^2 + y^3 + s^6 + ay^2s^2 = 0$ $(4a^3 + 27 \neq 0)$ (*Simple elliptic with* $k = 1$; J_{10} *in* [AGV85, pp. 184–185]),

(iv) *certain DuVal singularities.*

Proof. One can use general results about deformations of unimodal singularities [AGV85] but the easiest method is to do it from scratch. □

(4.8.7.4) **Computation.** *Consider a singularity $z^2 + y^3 + s^8 + ays^6 = 0$ with $\mathbb{Z}_2$-action $(x, y, s) \to (x, y, -s)$. Blow up the origin and take the quotient to get a surface germ H. The minimal resolution of H is given by the following diagram where • indicates the proper transform of the exceptional curve of the* *blow-up*



This completes the proof of (4.7). □

It should be noted that for divisorial contractions the description given by (4.7) is not entirely satisfactory. One could try to describe them using Y and the image of the exceptional surface. This approach also has theoretical advantages.

(4.9) **Theorem.** *Let X, Y be normal algebraic varieties. Assume that K_X and K_Y are $\mathbb{Q}$-Cartier and that X and Y are smooth in codimension two. Let $f : X \to Y$ be a proper birational morphism. Assume that $-K_X$ is f-ample and that the dimension of every fiber is at most one. Then*

(4.9.1) *The exceptional set is a $\mathbb{Q}$-Cartier divisor; call it E (with reduced structure);*

(4.9.2) *$B = f(E)$ has pure codimension two;*

(4.9.3) *$f_*(\mathscr{O}_X(-mE)) = I_B^{(m)}$ (I_B is the ideal sheaf of B, $I_B^{(m)}$ denotes symbolic power);*

(4.9.4) *$X \cong \operatorname{Proj}_Y \sum_{m=0}^{\infty} I_B^{(m)}$.*

Proof. There is a codimension three closed subset $S \subset Y$ such that $Y - S$ and $X - f^{-1}(S)$ are both smooth. Therefore

$$X - f^{-1}(S) \cong B_{B-S}(Y - S).$$

If E denotes the divisorial part of the exceptional set with reduced structure then we get that

$$K_{X-f^{-1}(S)} \cong f^*(K_{Y-S}) \otimes \mathscr{O}(E - f^{-1}(S)).$$

Since $f^{-1}(S)$ has codimension at least two in X this implies that

$$K_X^{[n]} \cong f^*(K_Y^{[n]}) \otimes \mathscr{O}(nE),$$

where n is a common multiple of the indices of X and Y. In particular, E is $\mathbb{Q}$-Cartier and $-E$ is f-ample. Therefore E is the whole exceptional set. This shows (4.9.1). (4.9.2) is clear. To see (4.9.3) first note that

$$f_*(\mathscr{O}_X(-mE))|Y - S = (I_B|Y - S)^m = I_B^{(m)}|Y - S.$$

Let $i : Y - S \to Y$ be the injection. By the definition of symbolic powers,

$$i_*(I_B|Y - S)^m = I_B^{(m)}.$$

Therefore (4.9.3) is established once we show that

$$i_*(f_*(\mathscr{O}_X(-mE))|Y - S) = f_*(\mathscr{O}_X(-mE)).$$

This follows from the general principle:

(4.9.5) **Proposition.** *Let $g : U \to V$ be a proper morphism. Let $S \subset V$ be a closed subset such that the codimension of $f^{-1}(S)$ in U is everywhere at* *least two. Let $\mathscr{F}$ be a sheaf on U that satisfies Serre's condition S_2 (e.g., U is normal and $\mathscr{F}$ is reflexive). Then $g_*\mathscr{F} = i_*(g_*\mathscr{F}|V-S)$, where $i: V-S \to V$ is the injection.*

Finally (4.9.4) is essentially a reformulation of (4.9.3). □

(4.9.6) **Corollary.** *Let the assumptions be as in* (4.9). *Then $B \subset Y$ uniquely determines X. This applies, in particular, if X and Y are threefolds with terminal singularities and $f: X \to Y$ is the contraction of a divisorial extremal nbd.*

(4.9.7) **Corollary.** *Let X, Y be threefolds with terminal singularities. Let f: $X \to Y$ be a proper birational morphism such that $-K_X$ is f-ample and that the dimension of every fiber is at most one. Let T be the spectrum of a complete local ring and let $\overline{X}/T$ be a flat deformation of X. Then*

(4.9.7.1) *The morphism f extends to a morphism $\overline{F}: \overline{X} \to \overline{Y}$ where $\overline{Y}/T$ is flat; This defines $\overline{E}$ and $\overline{B}$;*

(4.9.7.2) $\mathscr{O}_{\overline{X}}(-m\overline{E}) \otimes \mathscr{O}_X \cong \mathscr{O}_X(-mE)$;

(4.9.7.3) $I_{\overline{B}}^{(m)} \otimes \mathscr{O}_Y \cong I_B^{(m)}$;

(4.9.7.4) $(\overline{B} \subset \overline{Y})$ *is a flat deformation of* $(B \subset Y)$.

Proof. Since $R^1 f_*\mathscr{O}_X = 0$, (4.9.7.1) follows from [Wahl76] (cf. (11.4)).

Next we claim that the sheaves $\mathscr{O}_X(-mE)$ are all S_3. This is a local question. Let $g: Z \to X$ be the index one cover (with group G) around a point of X. Then $E' = f^{-1}(E)$ is a Cartier divisor and

$$\mathscr{O}_X(-mE) = \text{(G-invariant part of) } g_*(\mathscr{O}_Z(-mE')).$$

Therefore $\mathscr{O}_X(-mE)$ is a direct summand of the S_3 sheaf $g_*(\mathscr{O}_Z(-mE')) \cong g_*(\mathscr{O}_Z)$.

This implies (4.9.7.2) using (12.1.8). Since $\mathscr{O}_X(-mE)$ is f-ample, $R^1 f_*\mathscr{O}_X(-mE) = 0$ [KMM87, 1-2-3], hence again by [Wah76l] we obtain (4.9.7.3). Finally (4.9.7.4) is just a reformulation of earlier statements. □

(4.10) **Alternate description of index two divisorial nbds.** By the previous results we can also describe index two divisorial nbds by specifying the pair $B \subset Y$. Here Y is a cDV point, thus easily understandable via equations. The curve B is unknown at the moment. Let us note first that we cannot expect that B is easy to describe.

(4.10.1) **Proposition.** *Let $f: X \to Y$ be a divisorial contraction and let $B \subset Y$ be the image of the exceptional divisor. If B is a complete intersection (inside Y) then Y is smooth, B is planar, and X has only index one points.*

Proof. If Y is defined by $p = q = 0$ then by (4.9.4), X is obtained by blowing up the ideal (p, q). In particular, X has only complete intersection points, hence X has only index one points. Explicit computation of the blow-up shows that $B_{(p,q)}Y$ is singular along the preimage of the origin unless the required conditions are satisfied. □

(4.10.2). Now let us consider index two nbds in more detail. In the cases when Y is not smooth, the description provided during the proof of (4.7) determines the curve B explicitly. As in (4.8.4) we get the following equations for Y:

$$Y = (tg(x, y, t) + z^2 - f(x, y)^2 = 0) \qquad B = (t = z = f = 0)$$

or

$$Y = (tg(x, y, t) + z^2 - f(x, y)^2 l(x, y) = 0) \qquad B = (t = z = f = 0).$$

Thus we need to consider the case when Y is smooth. Then the equation of D is given by $f(x, y, z) = 0$ and B is the double curve of D.

(4.10.3) **Lemma.** *With the above notation,* $\mathrm{mult}_0 f = 3$.

Proof. Since $s^2 - f = 0$ defines a cDV point, $\mathrm{mult}_0 f \le 3$. B is not empty so $\mathrm{mult}_0 f > 1$. If $\mathrm{mult}_0 f = 2$ then in suitable coordinates $f = z^2 - h(x, y)$ and the double curve of $f = 0$ is contained in $z = 0$. Therefore B is planar, hence a complete intersection. This is impossible by (4.10.1). □

(4.10.4) **Computation.** In the cases when Y is smooth, the following are the general hyperplane sections of Y':

(4.7.3.1.2): E_6;
(4.7.3.1.4): E_8;
(4.7.3.2.2): D_*;
(4.7.3.2.4): E_7;
(4.7.3.3.1): D_*.

(4.10.5) **Normal form of f for D reducible.** (4.7.3.3.1): D has three components, so f is the product of three factors, all smooth at the origin by (4.10.3). $s^2 + f$ defines a compound D_* point, thus at least two of the linear terms of the three factors are independent. We can write f in the form

$$f = xyh(x, y, z) \quad \text{where } \mathrm{mult}_0 h = 1.$$

(4.7.3.2.2): D has two components, so f is the product of two factors. One of them is smooth at the origin by (4.10.3). We can choose that to be x. $s^2 + f$ defines a compound D_* point, thus the quadratic term of the other factor is not a multiple of x^2. We can write f in the form

$$f = x(y^2 - p(x, z)) \quad \text{where } \mathrm{mult}_0 p \ge 2.$$

The double curve contains $x = y^2 - p = 0$, which is planar. There must be another component, coming from the double curve of $y^2 - p = 0$. Let $p = g(x, z)^2 h(x, y)$ where h has no multiple factors. Then $y = g = 0$ is the double curve of $y^2 - p = 0$. The normalization of $y^2 - p = 0$ is $y'^2 - h = 0$. This has to be smooth. Therefore we can write f as

$$f = x(y^2 - g(x, z)^2 h(x, z)) \quad \text{where } \mathrm{mult}_0 g \ge 1, \ \mathrm{mult}_0 h = 1.$$

(4.7.3.2.4): D has two components, so f is the product of two factors. One of them is smooth at the origin by (4.10.3). We can choose that to be x. $s^2 + f$

defines a compound E_7 point, thus the quadratic term of the other factor is a multiple of x^2. Hence we can write f in the form

$$f = x(x^2 + 2g(y, z)x + p(y, z)).$$

The double curve contains $x = p = 0$, which is planar. There must be another component, coming from the double curve of $x^2 + 2g(y, z)x + p(y, z) = 0$. Let $p^2 - g^2 = q(x, z)^2 h(x, y)$, where h has no multiple factors. Then $y + g = q = 0$ is the double curve of $x^2 + 2g(y, z)x + p(y, z) = 0$. Since $z^2 + f$ defines a cE_7 point, $\mathrm{mult}_0 p \le 3$. Therefore $\mathrm{mult}_0 q = \mathrm{mult}_0 h = 1$. We may assume that $q = y$ to get the normal form

$$f = x(x^2 + 2g(y, z)x + g(y, z)^2 + y^2 h(y, z)) \quad \text{where } \mathrm{mult}_0 h = 1.$$

(4.10.6) **Normal form of f for D irreducible.** These cases seem much harder and we do not have complete results. For (4.7.3.1.2) B has a triple point and for (4.7.3.1.4) a quadruple point. These can be computed using the methods of (13.6.2). Moreover, for (4.7.3.1.2) we seem to get every triple point with irreducible tangent cone.

Concerning (4.7.3.1.4) we give only two examples:

(4.10.7) **Examples.** (4.10.7.1) The monomial curve $B = \mathrm{im}[t \mapsto (t^4, t^9, t^{15})]$ is the double curve of the surface

$$D = (z^3 - 3x^3y^2z + y^5 + x^9y = 0).$$

The normalization of D is smooth.

(4.10.7.2) The monomial curve $t \mapsto (t^4, t^5, t^7)$ cannot be the double curve of any surface triple point.

Proof of (4.10.7). All the partials of $z^3 - 3x^3y^2z + y^5 + x^9y$ vanish along B, thus B is contained in the double locus of D. Next view D as a family of curve singularities parametrized by x. For $x = 0$ we have $z^3 + y^5 = 0$, this has $p_a = 4$. In the general fiber we have at least 4 nodes. Since p_a is upper semicontinuous [Teissier80], there are no other singularities in the general fiber. Thus p_a is constant, hence we have simultaneous normalization in the family. Therefore the normalization of D is smooth.

For the second part we claim that, in fact, no triple point is contained in the symbolic square of the ideal of the curve. If we give $\mathbb{Z}$-weights to the variables by $\alpha(x, y, z) = (4, 5, 7)$ then it is sufficient to consider weighted homogeneous elements of the symbolic square of α-degree at most 21. Now a quick computation gives the result. □

Finally we should address the question whether the above examples do lead to an extremal contraction $f : X \to Y$. The positive answer is supplied by the following:

(4.10.8) **Proposition.** *Let $Y = \mathbb{C}^3$ and let $D = f(x, y, z)$ be a surface germ and let $B = \mathrm{Sing}\, D$. Assume that B is a curve and that B is an ordinary double curve on D outside the origin. Assume furthermore that*

(i) *the normalization of D is smooth,*

(ii) *B is not planar, and*

(iii) *$s^2 - f = 0$ defines a cDV point.*

Then there is an extremal nbd $f : X \to Y$ such that B is the image of the exceptional divisor. X has a single point of index two and no other points of index larger than one.

Proof. The singular locus of $Y' = (s^2 - f = 0)$ is the curve B inside the $s = 0$ plane and along it Y' has generically A_1 singularities. $s \mapsto -s$ gives a $\mathbb{Z}_2$-action on Y' whose fixed point set is $D' \cong D$. The resolution of the DuVal locus given in [Reid83, 2.6] gives a crepant morphism $f' : X' \to Y'$, which is relatively projective, the exceptional divisor $S' \subset X'$ is $\mathbb{Q}$-Cartier, and the $\mathbb{Z}_2$-action lifts to X'. Moreover X' has isolated cDV singularities. To see this we have to exclude the possibility that there is a singular curve $F \subset X'$ which maps to a point $Q \in Y'$. However, blowing up this curve yields a crepant exceptional divisor dominating a point of Y'. This is impossible since Y' is cDV.

Now take $X = X'/\mathbb{Z}_2$ with the natural map $f : X \to Y$. Every singular point of X is the $\mathbb{Z}_2$-quotient of a cDV point. We want to prove that they are all terminal. We need to check the fixed points of the $\mathbb{Z}_2$-action only. Let σ be a generator of $\omega_{Y'}$. We can assume that under the $\mathbb{Z}_2$-action σ is anti-invariant. Since $f'^*(\sigma)$ is a generator of $\omega_{X'}$ locally everywhere, at an isolated fixed point the quotient has index two. Such points are canonical by [KSB88, 6.12], and by the list of [Hayakawa-Takeuchi87] they are also terminal.

The nonisolated fixed points of the $\mathbb{Z}_2$-action an X' are on the proper transform E' of D'. E' is $\mathbb{Q}$-Cartier since $f'^*(D') = E' + 2S'$. E' lies on a hypersurface singularity and, therefore, it is also Cartier. E' is smooth outside the origin since we blew up the double curve. This implies that E' is normal, hence smooth by the assumption (i). Therefore X' is smooth along E' and X is smooth along the image of E'. □

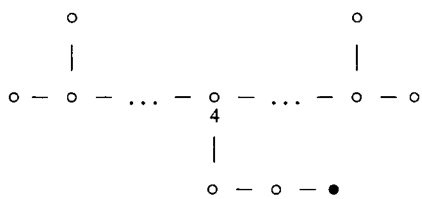
(4.11) Next we use the previous results to describe extremal nbds of type $II^\vee$. We already saw in (4.5) that such nbds are always divisorial. We will use the double cover construction of (4.3) in the following setting.

(4.11.1) *Notation.* Let $X \supset C \to Y \ni Q$ be an extremal nbd of type $II^\vee$. By [Mori88, 6.1] the nbd is imprimitive, thus there is a double cover $(\overline{X} \supset \overline{C}) \to (X \supset C)$, which is étale outside the singular point. Note that $\overline{C}$ is reducible. If $E \subset X$ is a transversal disc, let $\overline{E}$ be its preimage. Now we can take the double cover of $\overline{X}$ ramified along $\overline{E}$ to obtain $f' : X' \supset C' \to Y'$. By construction X'/X is Galois with Galois group $\mathbb{Z}_4$. Since $\overline{X}$ is an extremal nbd with a single index two point, we already described it somewhere in (4.8). Thus we have to identify Y' and the group action to describe $X \supset C \to Y \ni Q$.

(4.11.2) **Theorem.** *Let $X \supset C \to Y \ni Q$ be an extremal nbd with a type $II^\vee$ point. Let Y' be as above. Then in suitable coordinates*

$$Y' = (s^2 g(x, s^2) + z^2 - y^2 = 0) \qquad B' = (y = z = t = 0)$$
$$Y = Y'/\mathbb{Z}_4 \qquad wt(x, y, z, s) = (2, 2, 0, 1).$$

The minimal resolution of the general member of $|\mathscr{O}_X|$ is given by the diagram:



Proof. Consider first the possibility that $\overline{Y}$ is smooth. Then Y' is given by an equation $s^2 - f(x, y, z) = 0$, where by (4.10.3) $\operatorname{mult}_0 f = 3$. $z = f = 0$ defines D', which in our case has two components. Also, the $\mathbb{Z}_4$-action interchanges the two components of D', hence f has even multiplicity, a contradiction. Thus $\overline{Y}$ is singular and (since $\overline{C}$ is reducible) we can write its equation in the form

$$(s^2 g(x, y, s^2) + z^2 - f(x, y)^2 = 0).$$

We have to extend the $\mathbb{Z}_2$-action $(0, 0, 0, 1)$ to a $\mathbb{Z}_4$ action. Let the $\mathbb{Z}_4$-action be given by wts (a, b, c, d). We know that

$$(2a, 2b, 2c, 2d) = (0, 0, 0, 2) \mod 4.$$

Since the action is free, at most one of a, b, c, d can be zero. On the other hand, the quotient is an index two singularity. The list of those tells us that exactly one of a, b, c is zero. Thus $\operatorname{wt}(s) = 1$ and among a, b, c one is 0, the other two are 2.

Assume first that $\operatorname{mult}_0 f \geq 2$. Then g contains a linear term (4.8.5), say x. Since $0 = \operatorname{wt}(z^2) = \operatorname{wt}(s^2 g) = 2 + \operatorname{wt}(x)$, this implies that $\operatorname{wt}(x) = 2$.

Now compute the blow up of the singular set $s = z = f = 0$. The chart where we get the index four point is given by

$$g(x, y, s^2) + Z'^2 - F'^2 = 0,$$
$$f(x, y) - sF' = 0,$$

where $z = Z's$ and

$$\operatorname{wt}(x, y, s, F', Z') = (2, \operatorname{wt}(y), 1, \operatorname{wt}(f) - 1, \operatorname{wt}(z) - 1).$$

g contains a linear term, thus we can use the first equation to eliminate x and the second equation becomes

$$f(\phi(y, s, Z'^2, F'^2), s^2) - sF' = 0. \tag{4.11.3}$$

By definition of $II^\vee$ the wt of the above equation has to be two. Thus $\operatorname{wt}(f) = \operatorname{wt}(sF') = 2$.

The $\mathbb{Z}_4$-action interchanges the two components of D' given by $(z - f = 0)$ and $(z + f = 0)$. Therefore $\operatorname{wt}(z) = 0$ hence $\operatorname{wt}(y) = 2$ and

$$\operatorname{wt}(x, y, s, F', Z') = (2, 2, 1, 1, 3).$$

Since (4.11.3) defines a $II^\vee$ point, Z'^2 must appear in the equation with nonzero coefficient. This is only possible if f contains x with nonzero coefficient. This contradicts the starting assumption that $\operatorname{mult}_0 f \geq 2$.

Thus $\mathrm{mult}_0 f = 1$ and we can write the equation as

$$Y' = (s^2 g(x, s^2) + z^2 - y^2 = 0).$$

The $\mathbb{Z}_4$-action interchanges the two components of D' given by $(z - y = 0)$ and $(z + y = 0)$. Therefore $\mathrm{wt}(z) \neq \mathrm{wt}(y)$. z and y are symmetric, thus we can assume that wts are $(2, 2, 0, 1)$. This shows the first part of the theorem.

To see the second part we take $H'_Y = (z - ax^2 - bs^4 = 0)$ where a, b are general constants. We claim that this gives the required resolution for H_X. To get X' we have to blow up $s = z = y = 0$. The important chart is given by $y = y's$, $z = z's$ and we get the equations

$$g(x, s^2) + z'^2 - y'^2 = 0 \quad \text{(equation for } X'),$$
$$z's' - ax^2 - bs'^4 = 0 \quad \text{(equation for } H'_X)$$

and

$$\mathrm{wt}(x, y', z', s) = (2, 1, 3, 1).$$

By [Mori85, 12.3] $\mathrm{mult}_0 g \geq 4$. The two equations give a singularity whose resolution we want to determine. This is made possible by the following observation:

If we blow up the origin then we get a chart $x = x''s$, $y' = y''s$, $z' = z''s$, $s = s''$ and equations

$$\overline{g}(x'', s'') + z''^2 - y''^2 = 0,$$
$$z'' - ax''^2 - bs''^2 = 0.$$

From here we can eliminate z'' to get

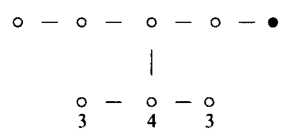
$$y''^2 = (ax''^2 + bs''^2)^2 + \overline{g}(x'', s'') \tag{4.11.4}$$

$\mathrm{mult}_0 \overline{g} \geq 4$ since $\mathrm{mult}_0 g \geq 4$. Now (4.11.4) defines a simple elliptic or a cusp singularity, and their resolution is well understood. Everything else is a routine computation. □

(4.12) **Example.** For completeness sake we give an example of an extremal nbd of type IIB. [Mori88, Appendix B] gives the local description. We will try to put together the simplest case as follows:

$$X^\sharp = (x^2 - y^3 + z^2 - t^2 = 0),$$
$$C^\sharp = (x^2 - y^3 = z = t = 0),$$
$$H^\sharp = (xz + yt^2 = 0),$$
$$\mathrm{wt}(x, y, z, t) = (3, 2, 1, 1).$$

Explicit computation of the minimal resolution of H gives the diagram



This contracts to an A_2 point and as in (13.8) leads to an example of an extremal nbd with a type IIB point. □

5. Nbds with three singular points

The following is the main theorem of this chapter.

(5.1) **Theorem.** *Let* $f : X \supset C \to Y \ni Q$ *be an extremal nbd with three singular points. Then* f *is divisorial.*

This (5.1) completes the proof of (2.2.3′) as remarked in (2.4). We have the following description by (2.12.1–2.12.3).

(5.2). Under the notation and assumptions of (5.1), let P, R, and S be the singular points of X with index $P \geq$ index $R \geq$ index S. Then we can express

$$(X, O) = (x_1, x_2, x_3) \supset (C, O) = x_1\text{-axis},$$
$$(X, P) = (y_1, y_2, y_3)/\mathbb{Z}_m(1, (m+1)/2, -1) \supset (C, P) = y_1\text{-axis}/\mathbb{Z}_m,$$
$$(X, R) = (z_1, z_2, z_3)/\mathbb{Z}_2(1, 1, 1) \supset (C, R) = z_1\text{- axis}/\mathbb{Z}_2,$$
$$(X, S) = (w_1, w_2, w_3, w_4; \gamma) \supset (C, S) = w_1\text{- axis}$$

using an odd number m (≥ 3) and equation $\gamma \equiv w_1 w_3 \mod (w_2, w_3, w_4)^2$, where $O \in C - \{P, R, S\}$ is an arbitrary point chosen for simplicity of the subsequent computation.

Unless otherwise mentioned, we will fix the meaning of these symbols above and $P^\sharp$ and $R^\sharp$ in (2.12) throughout this chapter.

(5.3) *Remark.* (5.3.1) By (2.12.5)–(2.12.7), we know that $(f(D), Q)$ is a D_{m+2}-point and (D, P) is an A_{m-1}-point for a general member D of $|-K_X|$ through C as explained in (2.12.7).

(5.3.2) By (1.10) and (8.9.1) of [Mori88], we see

$$Cl^{sc}(X) \overset{\tilde{\otimes}\mathcal{O}_C}{\to} \{\ell\text{- invertible } \mathcal{O}_C\text{-modules}\}/\text{isomorphisms}$$
$$\overset{ql_C}{\to} QL(C) \simeq \mathbb{Z}.$$

The proof of (5.1) consists of several steps. The first step is to write down ℓ-splittings for $gr_C^1\mathcal{O}$ and $gr_C^2\mathcal{O}$ explicitly so that we can write down infinitesimal thickenings of C in subsequent arguments.

(5.4) **Proposition.** *Under the notation and assumptions of* (5.1), *we have the* ℓ*-isomorphism*

$$gr_C^1\mathcal{O} \simeq \mathcal{O}_C(K_X)\tilde{\oplus}\mathcal{O}_C((m-2)K_X) \tag{5.4.1}$$

and

$$ql_C\mathcal{O}_C(K_X) = -1 + \frac{m-1}{2}P^\sharp + R^\sharp,$$
$$ql_C\mathcal{O}_C((m-2)K_X) = -1 + P^\sharp + R^\sharp.$$

Furthermore we have an ℓ-splitting

$$gr^1_C\mathscr{O} = \mathscr{O}_C(-D)\tilde{\oplus}\mathscr{O}_C(-D'), \tag{5.4.2}$$

where D and D' are general members of $|-K_X|$ and $|-(m-2)K_X|$, respectively, and we are using the notation $\mathscr{O}_C(E) = \mathscr{O}_C\tilde{\otimes}\mathscr{O}_X(E)$ for an ℓ-invertible $\mathscr{O}_X$-module $\mathscr{O}_X(E)$.

Proof. The ℓ-isomorphism (5.4.1) is given in (2.12.4). From the ℓ-isomorphisms

$$gr^0_C(\omega_X^{\tilde{\otimes}(-1)}) \simeq (-1 + \frac{m+1}{2}P^\sharp + R^\sharp),$$
$$gr^0_C(\omega_X^{\tilde{\otimes}(-m+2)}) \simeq (-1 + (m-1)P^\sharp + R^\sharp)$$

given in (2.12), we see C is contained in the base loci of $|-K_X|$ and $|-(m-2)K_X|$, that is,

$$H^0(\mathscr{O}_X(-K_X)) = H^0(F^1_C\mathscr{O}_X(-K_X)),$$
$$H^0(\mathscr{O}_X((2-m)K_X) = H^0(F^1_C\mathscr{O}_X((2-m)K_X)).$$

Let $(E, P) = \{y_1 = 0\}/\mathbb{Z}_m$. Then $E \in |-2K_X|$ by $(E \cdot C) = 1/m$. From the ℓ-exact sequence

$$0 \to \omega_X \to \mathscr{O}_X(-K_X) \to \mathscr{O}_E(-K_X) \to 0$$

and $H^1(\omega_X) = 0$, we see the surjection

$$H^0(F^1_C\mathscr{O}_X(-K_X)) \twoheadrightarrow \mathscr{O}_E(-K_X). \tag{5.4.3}$$

We claim that the natural ℓ-surjection

$$F^1_C\mathscr{O}_X(-K_X) \twoheadrightarrow gr^1_C\mathscr{O}\tilde{\otimes}\mathscr{O}(-K_X) = \mathscr{O}_C\tilde{\oplus}\mathscr{O}_C((m-3)K_X) \tag{5.4.4}$$

induces a surjection

$$H^0(F^1_C\mathscr{O}_X(-K_X)) \twoheadrightarrow H^0(\mathscr{O}_C) \oplus H^0(\mathscr{O}_C((m-3)K_X). \tag{5.4.5}$$

To see (5.4.5), we first assume $m > 3$. Then $y_2/dy_1 \wedge dy_2 \wedge dy_3 \in \mathscr{O}_E(-K_X)$ lifts to $s \in H^0(F^1_C\mathscr{O}_X(-K_X))$ (5.4.3). Then the image $\bar{s}$ of s by (5.4.4) generates the first factor because $\bar{s}$ generates the trivial $\mathscr{O}_C$ mod I_E at P. So (5.4.5) is a surjection. If $m = 3$, we lift another $y_3/dy_1 \wedge dy_2 \wedge dy_3$ together, and (5.4.5) is again surjective. Thus if $m = 3$, (5.4) is done. We assume $m > 3$ for (5.4). Then just like (5.4.3), we see the surjection

$$H^0(F^1_C\mathscr{O}_X(-(m-2)K_X)) \twoheadrightarrow \mathscr{O}_E(-(m-2)K_X) \tag{5.4.6}$$

because $H^1(\mathscr{O}_X((4-m)K_X)) = 0$ by $4 - m < 0$. We consider the ℓ-surjection

$$\begin{aligned} F^1_C\mathscr{O}_X(-(m-2)K_X) &\twoheadrightarrow gr^1_C\mathscr{O}\tilde{\otimes}\mathscr{O}_X(-(m-2)K_X) \\ &= \mathscr{O}_C((3-m)K_X)\tilde{\oplus}\mathscr{O}_C \\ &\twoheadrightarrow \mathscr{O}_C \text{ (second projecton).} \end{aligned} \tag{5.4.7}$$

We claim that (5.4.7) induces a surjection

$$(5.4.8) \qquad H^0(F^1_C\mathscr{O}_X(-(m-2)K_X)) \twoheadrightarrow H^0(\mathscr{O}_C).$$

Indeed, similarly to (5.4.5), one can see it by lifting $y_3/(dy_1 \wedge dy_2 \wedge dy_3)^{m-2}$ by (5.4.6) to $s' \in H^0(F^1_C\mathscr{O}_X(-(m-2)K_X))$. Now (5.4.5) and (5.4.8) prove (5.4.2). □

(5.5) **Proposition.** *Under the notation and assumptions of* (5.4), *we have the nonzero vector space* $H^0(X, \mathscr{O}_X(D'-D)) \neq 0$. *Let* α *be a general element of* $H^0(X, \mathscr{O}_X(D'-D))$. *Then the homomorphism* $\alpha : \mathscr{O}_X(-D') \to \mathscr{O}_X(-D)$ *is an* ℓ*-injection, which is an* ℓ*-isomorphism outside* $P^\sharp$. *Furthermore the induced* ℓ*-injection* $\overline{\alpha} : \mathscr{O}_C(-D') \to \mathscr{O}_C(-D)$ *is an isomorphism* $\mathscr{O}_C(-1) \simeq \mathscr{O}_C(-1)$ *of invertible sheaves if we forget* ℓ*-structures.*

Proof. This follows from the ℓ-isomorphism

$$\mathscr{O}_C(D'-D) \simeq \left(\frac{m-3}{2}P^\sharp\right),$$

which was given in the proof of (5.4). □

We will fix the meaning of D, D', α, and $\overline{\alpha}$ above for the rest of this chapter.

(5.6) **Proposition.** *Let the notation and assumptions be as in* (5.5). *By making coordinate changes to the coordinates in* (5.2), *we may assume that*

(5.6.1) (x_2, y_2, z_2, w_2) *(resp.* (x_3, y_3, z_3, w_4)*) forms* ℓ*-free* ℓ*-bases of the* ℓ*-invertible sheaf* $\mathscr{O}(-D)$ *(resp.* $\mathscr{O}(-D')$*) at* O, P, R, *and* S, *respectively;*

(5.6.2) $\alpha : \mathscr{O}(-D') \to \mathscr{O}(-D)$ *sends* (x_3, y_3, z_3, w_4) *to* $(x_2, y_1^{(m-3)/2}y_2, z_2, w_2)$;

(5.6.3) $\gamma = w_1w_3 - G(w_2, w_4)$ *for some* $G \in (w_2, w_4)^2\mathbb{C}\{w_2, w_4\}$, *where* γ *is the equation in* (5.2).

Proof. The first assertion follows from (5.4). Then α sends

$$(x_3, y_3, z_3, w_4) \quad \text{to} \quad (u_xx_2, u_yy_1^{(m-3)/2}y_2, u_zz_2, u_ww_2)$$

for some units u_x, u_y, u_z, and u_w at O, P, R, and S. Then we replace y_2 (resp. z_2, w_2, x_2) by $u_y^{-1}y_2$ (resp. $u_z^{-1}z_2, u_w^{-1}w_2, u_x^{-1}x_2$) at $P^\sharp$ (resp. $R^\sharp, S, O$), which attains (5.6.2). By making a coordinate change at S leaving w_2 and w_4 fixed, we may attain (5.6.3) because (X, S) is an isolated singularity and $\gamma \equiv w_1w_3 \mod (w_2, w_3, w_4)^2$ by (5.2). □

The following deformation procedure allows us to make $G \in (w_2, w_4)^2 \times \mathbb{C}\{w_2, w_4\}$ in (5.6) general keeping other properties.

(5.7) **Lemma.** *Let the notation and assumptions be as in* (5.6). *Let* δ *be an arbitrary power series in* $(w_2, w_4)^2\mathbb{C}\{w_2, w_4\}$. *Let* $(X_t, S_t) \supset (C_t, S_t)$ *be the deformation of* $(X, S) \supset (C, S)$ *given by*

$$\{\gamma + t\delta = 0\} \supset w_1\text{-axis} \qquad (t \in \Delta, \text{ a small disk}).$$

Let $X_t \supset C_t$ *be the twisted extension of* $(X_t, S_t) \supset (C_t, S_t)$ *by* (w_2, w_4). *Let* D, D', *and* α *be trivially extended outside a small nbd of* S *and extended to* (X_t, S_t) *by* $\{w_2 = 0\}$, $\{w_4 = 0\}$, *and* w_2/w_4, *which are consistent. If* $X \supset C$ *is isolated, then a nearby nbd* $X_t^\circ \supset C_t$ *is an isolated extremal nbd satisfying* (5.2), (5.4), (5.5), *and* (5.6) *except that* G *is replaced by* $G - t\delta$.

This is similar to (2.9) and we omit the proof since it is similar to that of (2.9).

From now on in this chapter, we assume that f is isolated because of (5.7) (cf. (5.24)). Thus we may assume the following.

(5.8) **Proposition.** *Let* G_ν *be the homogeneous part of degree* ν *of* G *in* (5.6). *Then for any* n, *we may assume that* $G_2, G_3, \dots, G_n$ *are all general homogeneous polynomials* (*by replacing* X *with its nearby extremal nbd*).

(5.9) Due to (5.8), we will assume that G_2 and G_3 are general in the rest of this chapter.

(5.10) **Proposition.** *Under the notation and assumptions of* (5.6) *and* (5.9), *we have an* ℓ*-splitting*

$$gr_C^2\mathscr{O} = \mathscr{O}_C(-2D)\tilde{\oplus}\mathscr{O}_C(-D-D')\tilde{\oplus}N$$

such that

$$ql_C\mathscr{O}_C(-2D) = 2ql_C\mathscr{O}_C(-D) = -1 + (m-1)P^\sharp,$$
$$ql_C\mathscr{O}_C(-D-D') = ql_C\mathscr{O}_C(-D) + ql_C\mathscr{O}_C(-D') = -1 + \frac{m+1}{2}P^\sharp,$$
$$ql_C(N) = 2P^\sharp,$$

for some ℓ*-invertible subsheaf* N. *Such an* N *is unique and is given by*

$$N = G_2(\alpha, 1)\cdot\mathscr{O}_C(-2D')(S).$$

In particular, $(G_2(x_2, x_3), G_2(y_1^{(m-3)/2}y_2, y_3), G_2(z_2, z_3), w_3)$ *forms* ℓ*-free* ℓ*-bases of* N *at* O, P, R, *and* S, *respectively.*

Proof. By (5.2), we see an ℓ-exact sequence

$$0 \to \tilde{S}^2 gr_C^1\mathscr{O} \to gr_C^2\mathscr{O} \to \mathbb{C}_S \cdot w_3 \to 0.$$

By $\alpha^i\mathscr{O}_C(-2D') = \mathscr{O}_C(-iD-(2-i)D') \subset \tilde{S}^2 gr_C^1\mathscr{O}$ for $i \in [0, 2]$, we see an ℓ-splitting

$$\tilde{S}^2 gr_C^1\mathscr{O} = \mathscr{O}_C(-2D)\tilde{\oplus}\mathscr{O}_C(-D-D')\tilde{\oplus}G_2(\alpha, 1)\cdot\mathscr{O}_C(-2D'),$$

because G_2 is general. In the stalk of $gr_C^2\mathscr{O}$ at S, we have

$$w_1w_3 = G_2(w_2, w_4) = G_2(\alpha, 1)\cdot w_4^2,$$

whence $G_2(\alpha, 1)\cdot\mathscr{O}_C(-2D')(S) \subset gr_C^2\mathscr{O}$. The uniqueness is obvious because $\mathscr{O}_C(-2D) \simeq \mathscr{O}_C(-D-D') \simeq \mathscr{O}_C(-1)$ and $N \simeq \mathscr{O}_C$ if we ignore ℓ-structures. □

(5.11) *Remark*. In view of the definition of $ql_C(F)$ [Mori88, (8.9.1)] for a locally ℓ-free $\mathscr{O}_C$-module on $C \simeq \mathbb{P}^1$, the following are easy to see.

(5.11.1) If $u : F \to G$ is an (ℓ-)injection of locally ℓ-free $\mathscr{O}_C$-modules that is generically an isomorphism, then $ql_C(F) \leq ql_C(G)$, that is , $ql_C(G) - ql_C(F) \in \mathbb{Z}_+ P^\sharp + \mathbb{Z}_+ R^\sharp$. Furthermore u is an ℓ-isomorphism iff $ql_C(F) = ql_C(G)$.

(5.11.2) If $0 \to F \to G \to H \to 0$ is an ℓ-exact sequence of locally ℓ-free $\mathscr{O}_C$- modules, then $ql_C(G) = ql_C(F) + ql_C(H)$.

(5.12) **Proposition.** *Under the notation and assumptions of* (5.10), *let* J_3 *be the ideal such that* $F_C^2\mathscr{O} \supset J_3 \supset F_C^3\mathscr{O}$ *and* $J_3/F_C^3\mathscr{O} = N$. *Then we have an* ℓ*-exact sequence*

$$0 \to F_C^3\mathscr{O}/F_C^1 J_3 \to gr_C^0 J_3 \to N \to 0 \tag{5.12.1}$$

and ℓ*-isomorphisms*

$$\begin{aligned} &F_C^3\mathscr{O}/F_C^1 J_3 \simeq \mathscr{O}_C(-3D)\tilde{\oplus}\mathscr{O}_C(-2D - D'), \\ &F_C^1 J_3/F_C^4\mathscr{O} \simeq N\tilde{\otimes} gr_C^1\mathscr{O}, \end{aligned} \tag{5.12.2}$$

where

$$\begin{aligned} ql_C\mathscr{O}_C(-3D) &= -1 + \frac{m-3}{2}P^\sharp + R^\sharp, \\ ql_C\mathscr{O}_C(-2D - D') &= -1 + R^\sharp. \end{aligned}$$

Proof. The kernel of the natural ℓ-surjection $J_3 \to N$ is $F_C^3\mathscr{O}$, which proves the first ℓ-exact sequence. By $F_C^2\mathscr{O} \supset J_3 \supset F_C^3\mathscr{O}$, we have $F_C^3\mathscr{O} \supset F_C^1 J_3 \supset F_C^4\mathscr{O}$. Since $J_3\tilde{\otimes} I_C \to F_C^1 J_3/F_C^4\mathscr{O} \subset gr_C^3\mathscr{O}$ factors through $N\tilde{\otimes} gr_C^1\mathscr{O}$, we have an ℓ-homomorphism $u : N\tilde{\otimes} gr_C^1\mathscr{O} \to F_C^1 J_3/F_C^4\mathscr{O}$. Let v be the natural ℓ-homomorphism $\mathscr{O}_C(-3D)\tilde{\oplus}\mathscr{O}_C(-2D - D') \to F_C^3\mathscr{O}/F_C^1 J_3$. It is easy to see that u and v are injections by

$$J_3 = (G_2(x_2, x_3)) + (x_2, x_3)^3 \quad \text{at } O.$$

We see that the natural ℓ-isomorphism $\tilde{\alpha}_3 : \tilde{S}^3 gr_C^1\mathscr{O} \to gr_C^3\mathscr{O}$ is an ℓ-isomorphism outside S by (5.2). Since S is an ordinary double point, we have

$$\mathrm{len}_S \mathrm{Coker}\, \tilde{\alpha}_3 = i_S(3) = [3^2/4] = 2$$

[Mori88, (4.9.ii)]. Thus

$$ql_C(gr_C^3\mathscr{O}) = ql_C(\tilde{S}^3 gr_C^1\mathscr{O}) + 2. \tag{5.12.3}$$

By (5.11.1) applied to u and (5.10), we have

$$\begin{aligned} ql_C(F_C^1 J_3/F_C^4\mathscr{O}) &\geq ql_C(N\tilde{\otimes} gr_C^1\mathscr{O}) \\ &= ql_C(\mathscr{O}_C(-D - 2D')\tilde{\oplus}\mathscr{O}_C(-3D')) + 2. \end{aligned} \tag{5.12.4}$$

By (5.11.1) applied to v, we have

$$ql_C(F_C^3\mathscr{O}/F_C^1 J_3) \geq ql_C(\mathscr{O}_C(-3D)\tilde{\oplus}\mathscr{O}_C(-2D - D')). \tag{5.12.5}$$

Since (5.12.4)+(5.12.5) reduces to the equality (5.12.3) by (5.11.2), we see that (5.12.4) and (5.12.5) are both equalities, whence u and v are ℓ-isomorphisms (5.11.1). □

(5.13) **Proposition.** *If we ignore ℓ-structures, then* (5.12.1) *is split and has a unique splitting submodule, say,* $N_1 \simeq \mathscr{O}_C$. *The sheaf* N_1 *has the following generators:*

$$\begin{aligned}
&O : G_2(x_2, x_3) + (\cdots)x_2^3 + (\cdots)x_2^2x_3,\\
&P : y_1^2G_2(y_1^{(m-3)/2}y_2, y_3) + u_P\cdot(b_0y_1^{(m-3)/2}y_2^3 + b_1y_2^2y_3),\\
&R : G_2(z_2, z_3) + (\cdots)z_1z_2^3 + (\cdots)z_1z_2^2z_3,\\
&S : w_3 + (\cdots)w_2^3 + (\cdots)w_2^2w_4,
\end{aligned}$$

where $u_P \in \mathscr{O}_{C,P}$ *is a unit and* $b_0, b_1 \in \mathbb{C}^*$ *are general with respect to the coefficients of* G_2.

Proof. By $ql_C\mathscr{O}_C(-2D') = -1+2P^\sharp$ and $ql_C\mathscr{O}_C(-2D-D') = -1+R^\sharp$, we have

$$-2D' - 2E_P \sim -2D - D' - E_R$$

in a small enough nbd X of C, where $E_P = \{y_1 = 0\}/\mathbb{Z}_m$ and $E_R = \{z_1 = 0\}/\mathbb{Z}_2$ are considered Weil divisors on X. Let β be a meromorphic function on X such that $\beta\cdot\mathscr{O}_X(-2D'-2E_P) = \mathscr{O}_X(-2D-D'-E_R)$ and β sends bases

$$(x_3^2, y_1^2y_3^2, z_3^2, w_4^2) \quad \text{to} \quad (u_Ox_2^2, u_Py_2^2y_3, u_Rz_1z_2^2z_3, u_Sw_2^2w_4)$$

for some units u_O, u_P, u_R, and u_S of $\mathscr{O}_{X,O}, \mathscr{O}_{X,P}, \mathscr{O}_{X,R}$, and $\mathscr{O}_{X,S}$, respectively. We may assume $u_S(S) = 1$ by multiplying β by a constant. Since G_2 and G_3 are general, we can find $b_0, b_1, c_0, c_1 \in \mathbb{C}^*$, which are general with respect to G_2 and

$$G_3(w_2, w_4) = G_2(w_2, w_4)\cdot(c_0w_2 + c_1w_4) + (b_0w_2^3 + b_1w_2^2w_4).$$

Using α in (5.6), we see $\alpha\beta\cdot\mathscr{O}(-2D'-2E_P) \subset \mathscr{O}_X(-3D)$, whence

$$\{b_0\alpha\beta + b_1\beta + G_2(\alpha, 1)\}\cdot\mathscr{O}_X(-2D'-2E_P) \subset J_3.$$

We denote its image in $gr_C^0(J_3)$ by M, and we note that $M \simeq \mathscr{O}_C(-1)$.

First we work in a neighborhood of S. Since $w_3 \in J_3$ (5.10) and $I_C^3 \subset J_3$, we see that

$$(F_C^1J_3)_S \supset w_3(w_2, w_3, w_4) + (w_2, w_3, w_4)^4.$$

Since $\gamma = 0$, $G_2 \in J_3$ (5.10), and $G_\nu \in F_C^1J_3$ $(\forall\nu \geq 4)$, we see

$$G_2(w_2, w_4) + (b_0w_2^3 + b_1w_2^2w_4) \equiv w_1w_3 \bmod F_C^1J_3.$$

We see that M is generated at S by

$$\begin{aligned}
\{b_0\alpha\beta + b_1\beta + G_2(\alpha, 1)\}\cdot w_4^2 &= u_S\cdot(b_0w_2^3 + b_1w_2^2w_4) + G_2(w_2, w_4)\\
&\equiv w_1\cdot\{w_3 + (\cdots)w_2^3 + (\cdots)w_2^2w_4\} \bmod F_C^1J_3.
\end{aligned}$$

Hence $M(S) \subset gr_C^0 J_3$ is the splitting subsheaf N_1.

Then, at P, we see that

$$\{b_0\alpha\beta + b_1\beta + G_2(\alpha, 1)\}y_1^2 y_3^2 = u_P(b_0 y_1^{(m-3)/2} y_2^3 + b_1 y_2^2 y_3) + y_1^2 G_2(y_1^{(m-3)/2} y_2, y_3)$$

is a generator of N_1. □

(5.14) By (5.3.1), the singularity $(f(D), Q)$ is a D_{m+2}-point and (D, P) is an A_{m-1}-point, where D is as in (5.4). From this it is easy to see the following.

(5.15) **Lemma-Definition.** *There exist sections $\tilde{s}_1, \tilde{s}_2 \in H^0(\mathcal{O}_D)$ such that the multiplicity of C in $\{\tilde{s}_1 = 0\}$ is 2 and $\tilde{s}_2 = y_3^m \cdot (unit)$ on the germ (D, P). These lift to sections s_1 and $s_2 \in H^0(\mathcal{O}_X)$ by $H^0(\mathcal{O}_X) \twoheadrightarrow H^0(\mathcal{O}_D)$ because $H^1(\mathcal{O}(-D)) = H^1(\omega_X) = 0$. We will fix the meaning of s_1 and s_2 in the rest of this chapter.*

(5.16) **Proposition.** (5.16.1) *We have $s_1 \in H^0(J_3)$ and its image $\bar{s}_1$ in $gr_C^0 J_3$ is a global generator of N_1 in* (5.13).

(5.16.2) *We have $s_2 \in H^0(F_C^1 J_3)$ and, in the Taylor expansion of s_2 in $(D^\sharp, P^\sharp)$, y_3^m appears with a nonzero coefficient.*

Proof. We have $H^0(F_C^1\mathcal{O}) = H^0(F_C^2\mathcal{O})$ (5.4) and $H^0(F_C^2\mathcal{O}) = H^0(J_3)$ (5.10). Hence $s_1, s_2 \in H^0(J_3)$. Since $s_1|_D$ vanishes to order 2 along C, we see that $s_1 \notin H^0(F_C^3\mathcal{O})$. Thus $\bar{s}_1 \in H^0(N_1)$ is a global generator of $N_1 \simeq \mathcal{O}_C$ by (5.13). If $s_2 \notin H^0(F_C^1 J_3)$, then its image $\bar{s}_2$ in $gr_C^0 J_3$ is also a global generator of N_1 for the same reason. Then one can see, at O, that $s_2|_D = s_2(x_1, 0, x_3) = x_3^2 \cdot$ (unit). This contradicts the choice of s_2 in (5.2.1). Thus $s_2 \in H^0(F_C^1 J_3)$, and the rest is obvious from the choice of s_2. □

(5.17) **Proposition.** *Let $s \in H^0(\mathcal{O}_X)$ be a general linear combination of s_1 and s_2. Then we have the congruence relations* (*up to multiplication by unit*) *at the following points.*

$$O : s \equiv G_2(x_2, x_3) \bmod F_C^3\mathcal{O},$$
$$P : s \equiv y_1^2 G_2(y_2, y_3) \bmod F_C^3\mathcal{O},$$
$$R : s \equiv G_2(z_2, z_3) \bmod F_C^3\mathcal{O},$$
$$S : s \equiv w_3 \bmod F_C^3\mathcal{O}.$$

This follows from (5.13) and (5.16.1).

(5.18) To study s at P, we will make a weighted blow-up at P. To simplify the notation, we make the coordinate change $y_i \mapsto u_P^{-1} y_i$ $(i = 1, 2, 3)$. Let σ and τ be the $\mathbb{Z}$-wts (cf. (T.7))

$$\sigma(y_1, y_2, y_3) = \left(1, \frac{m+1}{2}, m-1\right) \qquad \tau(y_1, y_2, y_3) = \left(m-1, \frac{m-1}{2}, 1\right).$$

(5.19) **Proposition.** *We have $\sigma(s) = 2m$ and $\tau(s) = m$.*

(5.19.1) *If* $m \geq 5$, *then* $s_{\sigma=2m}$ (*up to multiplication by constant*) *is a general linear combination of all the monomials of* σ-*wt* $2m$ *in* $(y_2, y_3)^2$ *and* $s_{\tau=m}$ *is squarefree.*

(5.19.2) *If* $m = 3$, *then we have* (*up to multiplication by constant*)

$$s_{\sigma=2m} = y_1^2 G_2(y_2, y_3) + H(y_2, y_3),$$

where H *is a homogeneous cubic polynomial which is squarefree and prime to* G_2.

Proof. We have $\sigma(s) = 2m$ by (5.17) and $\tau(s) = m$ by (5.16.2). First we assume $m \geq 5$. Then $y_1^2 y_3^2$, $y_1^{(m+1)/2} y_2 y_3$, $y_1^{m-1} y_2^2$, $y_2^2 y_3$, $y_1^{(m-3)/2} y_2^3$ are the only monomials in $(y_2, y_3)^2$ with σ-wt $2m$. Since all the elements in $(F_C^4 \mathscr{O})_P$ have σ-wt $> 2m$, we see that all the elements in $(F_C^1 J_3)_P$ have σ-wt $> 2m$ by (5.12) and $m \geq 5$. Thus the first part of (5.19.1) follows by (5.13) and (5.16.1). Furthermore $y_2^2 y_3$, $y_2 y_3^{(m+1)/2}$, y_3^m are the only monomials with τ-wt m in $(y_2, y_3)^2$. Thus the second part of (5.19.1) follows from (5.13), and (5.16.2) by Bertini's theorem. When $m = 3$, (5.19.2) follows also from (5.13), and (5.16.2) by Bertini's theorem. □

(5.20) Let $H = \{s = 0\}$, $\pi : \overline{H} \to (H, P)$ be the σ-blow-up (cf. (10.3)), $\overline{C}$ the proper transform of C by π, and $E = \pi^{-1}(P)_{red}$.

(5.21) **Lemma.** (5.21.1) $E \subset \mathbb{P}(1, \frac{m+1}{2}, m-1)$ *is isomorphic to a plane nodal cubic* (*i.e.*, $\mathbb{P}^1$ *with one node*) *with singularity at* $P_1 = (1:0:0)$, *and*

$$(\overline{H}, P_1) \simeq (x, y, z; yz) \supset (E, P_1) = \{x = 0\}$$

with $\overline{C} = x$-*axis.*

(5.21.2) *If* $m \geq 5$, *then the singular locus of* $\overline{H}$ *consists of* $\overline{C}$ *and the two points* $P_2 = (0:1:0)$ *and* $P_3 = (0:0:1)$ *such that*

$$(\overline{H}, P_2) \simeq (x, y)/\mathbb{Z}_{(m+1)/2}(1, 1) \supset E = x\text{-axis}/\mathbb{Z}_{(m+1)/2},$$

$$(\overline{H}, P_3) \simeq (u_1, u_2, u_3; \phi)/\mathbb{Z}_{m-1}(m-2, \frac{m-3}{2}, 1; m-3) \supset E$$
$$= \{u_3 = 0\}/\mathbb{Z}_{m-1},$$

where $\rho(\phi) = m - 3$ *and* $\phi_{\rho=m-3}$ *is squarefree for* $\mathbb{Z}$-*wt* $\rho(u_1, u_2, u_3) = (m-2, \frac{m-3}{2}, 1)$.

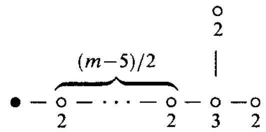
(5.21.3) *If* $m = 3$, *then the singular locus of* H *consists of* $\overline{C}$ *and three* A_1-*points* $\in E - \{P_1\}$.

Outline of proof. It is easy to see that $E \cap U_1$ is a cubic curve with exactly one node (cf. (10.3)). Then the rest follows from (5.19) by direct computation. We only make two comments. The assertion on ρ follows from that on τ in (5.19.1). When $m = 3$, the three singular points of $\overline{H}$ come from the singular locus $D_+(y_1)$ of the ambient space of the σ-blow-up. □

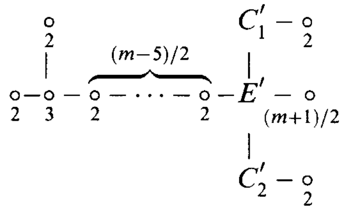
(5.22) **Computation.** Assume $m \geq 5$. Under the notation of (5.21),

$$\Delta((\overline{H}, P_3) \supset (E, P_3))$$

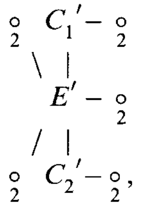
is



(5.23) **Conclusion.** Since $\overline{H}$ has nodes everywhere along $\overline{C}$, the inverse image of C in the minimal resolution H' of $\overline{H}$ is $C_1' \coprod C_2'$, where $C_1' \simeq C_2' \simeq \mathbb{P}^1$ and the proper transform E' of E fits in $\Delta(\overline{H} \supset \overline{C})$ as follows



$(m \geq 5)$,



$(m = 3)$,

(cf. (10.5)). Using $(2mD|_H \cdot C) = 1$, we can show $(C_1')^2 = (C_2')^2 = -1$ and $(E')^2 = -5$ if $m \geq 5$ (-6 if $m = 3$) by computing the pullback of $2mD|_H$ on H'.

Hence $(f(H), Q)$ is a D_4-point and f is divisorial. Thus (5.1) is proved. □

(5.24) *Remark.* (5.24.1) Since small deformations of an extremal nbd are not proved to be extremal nbds, the argument in this chapter only shows the divisoriality of f.

(5.24.2) One can define "formal" extremal nbd that are formal schemes X along $C \simeq \mathbb{P}^1$ in the usual way. Then it does not seem hard to prove that small deformations of a formal extremal nbd is extremal. Then our argument shows further that $(f(H), Q)$ is a D_4-point for a sufficiently general formal extremal nbd.

6. General members of $|\mathscr{O}_X|_C$; isolated $cD/3$ case

We consider the following set-up in this chapter unless otherwise mentioned explicitly.

(6.1) Let $f : X \supset C \to Y \ni Q$ be an isolated extremal nbd with only one non-Gorenstein point P such that $X \supset C$ has a IA point at P and (X, P) is a $cD/3$ point. Let H_X be a general member of $|\mathscr{O}_X|$ through C and let $H_Y = f(H_X)$. Let $\Delta_X = \Delta(H_X \supset C)$ and $\Delta_Y = \Delta(H_Y)$. Let

$$(X, P) \simeq (x, y, z, u; u^2 + z^3 + g(x, y) + \cdots)/\mathbb{Z}_3(1, 1, 2, 0; 0)$$

with nonzero homogeneous cubic part g in x and y [Reid87, (6.1)] or [Mori85, Theorem 23]. Then g, up to linear transformations in x and y does not depend on the choice of such coordinates. If g is squarefree (resp. has simple and double factors, is the cube of a linear factor), we say that P is a *simple* (resp. *double*, *triple*) cD point.

Our main results in this chapter are the following.

(6.2) **Theorem.** *Under the notation and assumptions of* (6.1), *assume that* $i_P(1) = 1$. *Then we have the following*:

(6.2.1) X *is smooth outside of IA point* P, *which is a simple or double* cD *point with* $\ell(P) = 2$ *and we have an* ℓ-*isomorphism*

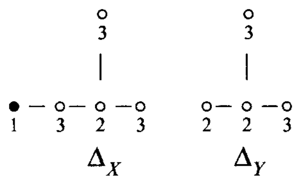
(6.2.1.1) $$gr_C^1\mathscr{O} \simeq (P^\sharp)\tilde{\oplus}(0).$$

(6.2.2) $2C = D \cdot D''$ *for general members* $D \in |K_X|$ *and* $D'' \in |2K_X|$.

(6.2.3) H_X *is normal, and* Δ_X *and* Δ_Y *consist of smooth rational curves intersecting transversely and their configurations are as follows.*

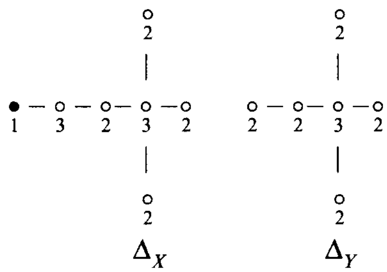
Case of simple cD *point* P.

(6.2.3.1)



Case of double cD *point* P.

(6.2.3.2)



(6.2.4) *Let* $X \supset C$ *be a germ of a 3-fold along* $C \simeq \mathbb{P}^1$ *which need not be an extremal nbd. If* $X \supset C$ *has the properties in* (6.2.1), *then it is an isolated extremal nbd of type* cD. (*An example is given in* (6.11).)

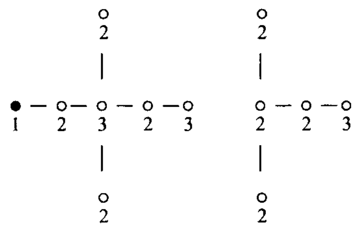
(6.3) **Theorem.** *Under the notation and assumptions of* (6.1), *assume* $i_P(1) > 1$. *Then* $i_P(1) = 2$ *and we have the following*:

(6.3.1) X *is smooth outside of IA point* P, *which is a double* cD *point with* $\ell(P) = 3$ *or 4, and we have an* ℓ-*isomorphism*

(6.3.1.1) $$gr_C^1\mathscr{O} \simeq \begin{cases} (P^\sharp)\tilde{\oplus}(-1+2P^\sharp) & \text{if } \ell(P) = 3, \\ (0)\tilde{\oplus}(-1+2P^\sharp) & \text{if } \ell(P) = 4. \end{cases}$$

(6.3.2) $4C = D \cdot D'$ *for general members* D *and* D' *of* $|K_X|$.

(6.3.3) *H_X is normal, and Δ_X and Δ_Y consist of smooth rational curves intersecting transversely and their configurations are as follows.*



Δ_X Δ_Y

(6.3.4) *Let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. If $X \supset C$ has the the properties in* (6.3.1), *then it is an isolated extremal nbd of type cD.* (*Examples are given in* (6.17) *and* (6.21).)

We need the following lemma for the proof of (6.2.4), (6.3.4).

(6.4) **Proposition.** *Let $X \supset C$ be the germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume that X has only terminal singularities on C and that the proper transform $C^\sharp$ of C to the canonical cover $(X^\sharp, P^\sharp)$ of (X, P) is smooth at an arbitrary singular point P. Let $I^\sharp$ be the defining ideal of $C^\sharp$ in $(X^\sharp, P^\sharp)$. Then using the notation $\ell(P) = \mathrm{length}_{P^\sharp}(I^{\sharp(2)}/I^{\sharp 2})$, we have*

$$(6.4.1) \qquad (K_X \cdot C) = \mathrm{qldeg}(gr^1\mathcal{O}) - 2 + \sum_P \left(1 + \frac{\ell(P)-1}{m_P}\right),$$

where P runs over all the singular points of X on C and m_P is the index of P.

Proof. We may give the homomorphism α_1 in [Mori88, (2.2)] the ℓ-structure at P by

$$\alpha_1{}^\sharp : \bigwedge^2 (I^\sharp/I^{\sharp(2)}) \otimes \Omega_{C^\sharp} \to gr^0_{C^\sharp}\omega_{X^\sharp},$$

where $\alpha_1{}^\sharp(e \wedge f) \otimes g dh = g de \wedge df \wedge dh$ and Ω^1_C is given the ℓ-structure $\Omega^1_C \subset \Omega^1_{C^\sharp}$ at P. This defines an ℓ-homomorphism

$$\tilde{\alpha}_1 : \tilde{\bigwedge}^2 (gr^1\mathcal{O}) \tilde{\otimes} \Omega^1_C \to gr^0_C\omega,$$

which is an isomorphism outside of singular points. At P, we have

$$(X^\sharp, P^\sharp) = (x, y, z, u; \phi) \supset C^\sharp = x\text{-axis},$$

for some $\phi \equiv x^{\ell(P)} y \mod (y, z, u)^2$. Then $z \wedge u$ (resp. dx) is an ℓ-free ℓ-basis of $\tilde{\bigwedge}^2(gr^1\mathcal{O})$ (resp. Ω^1_C) at P. Furthermore $gr^0_C\omega$ has an ℓ-free ℓ-basis

$$\mathrm{Res}_{C^\sharp} \frac{dx \wedge dy \wedge dz \wedge du}{\phi} = \pm \frac{dx \wedge dz \wedge du}{\phi_y} = \pm \frac{dx \wedge dz \wedge du}{x^{\ell(P)}}.$$

Thus $\operatorname{Im}(\alpha_1{}^\sharp)$ is generated by $dx \wedge dz \wedge du$ and $\operatorname{len}\operatorname{Coker}\alpha_1{}^\sharp = \ell(P) - 1$, which proves our claim (6.4.1) because

$$\operatorname{qldeg}\Omega^1_C = -2 + \sum_P (m_P - 1)/m_P. \quad \square$$

(6.5) Let us express the cD point as

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_3(1, 1, 2, 0; 0) \supset y_1\text{-axis}/\mathbb{Z}_3,$$

using an equation α such that $\alpha \equiv y_1^{\ell(P)} y_i \mod (y_2, y_3, y_4)^2$ with $i = 2$ (resp. 3, 4) if $\ell(P) \equiv 2$ (resp. 1, 0) mod 3 [Mori88, (2.16)].

Then

(6.6) **Lemma.** *P is the only singular point of X on C.*

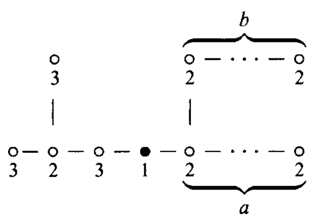
Proof. It is enough to derive a contradiction assuming that X has a type III point R on C with $i_R(1) = 1$ and that X is smooth outside of P and R (6.1). By deformation method (2.9), we may assume $\ell(P) = 2$ [Mori88, (4.12.2)]. Then y_3 and y_4 form an ℓ-free ℓ-basis of $gr^1_C\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$. Hence, after a possible coordinates change, we claim an ℓ-isomorphism

$$gr^1_C\mathscr{O} \simeq (0)\tilde{\oplus}(-1 + P^\sharp), \tag{6.6.1}$$

where y_4 (resp. y_3) is an ℓ-free ℓ-basis of (0) (resp. $(-1 + P^\sharp)$) at P. Indeed if otherwise, we have $gr^1_C\mathscr{O} \simeq (P^\sharp)\tilde{\oplus}(-1)$ (2.8), which implies $gr^1_C\omega \simeq gr^0_C\omega\tilde{\otimes}gr^1_C\mathscr{O} \simeq (0)\tilde{\oplus}(-2 + 2P^\sharp)$ and $H^1(\omega/F^2_C\omega) \neq 0$. This is a contradiction and (6.6.1) is proved. Let J be the C-laminal ideal of width 2 such that $J/F^2_C\mathscr{O} = (0)$ in the decomposition (6.6.1). Since P is a cD point, we know that y_4^2 and y_3^3 as well as $y_1^2 y_2$ appear with nonzero coefficients in the Taylor expansion of α. By deformation $\alpha + ty_2^3 = 0$ of (X, P) (2.9), we may further assume that P is a simple cD point. Let $D = \{y_1 = 0\} \in |-K_X|$ and let $s \in H^0(\mathscr{O}_X)$ be a lifting of $y_4 \in \mathscr{O}_D$ by $H^0(\mathscr{O}_X) \twoheadrightarrow \mathscr{O}_D$. Then s induces a section $\bar{s}$ of $gr^1_C\mathscr{O}$, which is a part of the ℓ-free ℓ-basis of $gr^1_C\mathscr{O}$. Thus $\bar{s}$ is nowhere vanishing (6.6.1) and $E_X = \{s = 0\} \in |\mathscr{O}_X|$ is smooth outside of P and R. Since $s \equiv y_4 \cdot (\text{unit}) \mod F^2_C\mathscr{O}$, it is easy to see that

$$(E_X, P) = (y_1, y_2, y_3; \bar{\alpha})/\mathbb{Z}_3(1, 1, 2; 0),$$

where $\bar{\alpha} \equiv y_3^3 + g(y_1, y_2) \mod (y_1, y_2, y_3)^4$ with a squarefree cubic part g in y_1 and y_2. By the computation (6.7.1), Δ_X is



for some $a \geq b$ such that $a \geq 1$, where A_{a+b} at the right-hand side comes from (H_X, R). Since Δ_X forms an exceptional set, we see $a = 1$ and $b = 0$

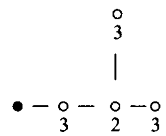
and that Δ_Y is A_2. This means (H_Y, Q) and hence (Y, Q) is Gorenstein. This is a contradiction and (6.6) is proved. □

(6.7) **Computation.** Let (D, P) be a normal surface singularity

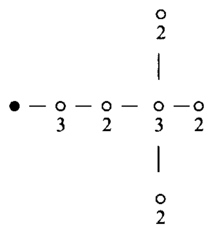
$$(D, P) = (y_1, y_2, y_3; \alpha)/\mathbb{Z}_3(1, 1, 2; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3.$$

Let σ be the $\mathbb{Z}$-wt $\sigma(y_1, y_2, y_3) = (1, 1, 2)$ (T.7). Then in each of the following cases, $\Delta(D \supset C)$ consists of smooth rational curves and C' intersects transversely with configuration as listed.

(6.7.1) $\alpha_{\sigma=3} = \alpha_{\sigma=3}(y_1, y_2, 0)$ is squarefree and $\alpha_{\sigma=6}(0, 0, 1) \neq 0$

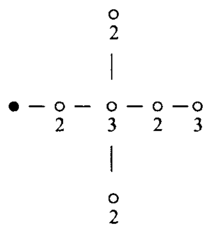


(6.7.2) $\alpha_{\sigma=3} = y_1^2 y_2$ and $\alpha_{\sigma=6}(0, y_2, y_3)$ is squarefree



where $\bullet - \underset{3}{\circ} - \underset{2}{\circ}$ intersects the central $\mathbb{P}^1$ ($\underset{3}{\circ}$) at ∞, and three $\underset{2}{\circ}$ at the three roots of $\alpha_{\sigma=6}(0, 1, y_3) = 0$ with respect to a certain coordinate system of the central $\mathbb{P}^1$.

(6.7.3) $\alpha_{\sigma=3} = y_1 y_2^2$ and $\alpha_{\sigma=6}(y_1, 0, y_3)$ is square-free



where $\underset{2}{\circ} - \underset{3}{\circ}$ intersects the central $\mathbb{P}^1$ ($\underset{3}{\circ}$) at ∞, $\bullet - \underset{2}{\circ}$ at 0, and two $\underset{2}{\circ}$ at the two nonzero roots of $\alpha_{\sigma=6}(1, 0, y_3) = 0$ with respect to a certain coordinate system of the central $\mathbb{P}^1$.

A similar argument shows

(6.8) **Lemma.** *If* $i_P(1) = 1$, *then* (6.2.1) *holds.*

Proof. By $i_P(1) = 1$, we see $\ell(P) = 2$ [Mori88, (2.16)]. Thus $y_1^2 y_2$ appears in α and P is not a triple cD point. Assume that $gr_C^1 \mathscr{O} \not\simeq \mathscr{O} \oplus \mathscr{O}$. Then

$gr_C^1\mathscr{O} \simeq \mathscr{O}(1) \oplus \mathscr{O}(-1)$ and $gr_C^1\mathscr{O} \simeq (1)\tilde{\oplus}(-1 + P^\sharp)$ as in (6.6.1). A general global section s of $\mathscr{O}_X$ vanishing along C induces a section $\bar{s}$ of $gr_C^1\mathscr{O}$ which lies on the first factor (1) and vanishes at a point $R(\neq P)$ to order 1. Then $E_X = \{s = 0\} \in |\mathscr{O}_X|$ has a point of type A at R and the analysis at P is the same as that in (6.6). Hence the same computation for (6.6) induces a contradiction. Thus $gr_C^1\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}$ and it has an ℓ-free ℓ-basis $\{y_3, y_4\}$ at P, whence (6.2.1) follows. □

We first prove (6.2) in two steps.

(6.9) **Lemma.** *Let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which has the properties in* (6.2.1). *Let J be the C-laminal ideal of width 2 such that $J/F_C^2\mathscr{O} = (P^\sharp)$ in the ℓ-splitting* (6.2.1.1). *We will use the coordinates in* (6.5) *and assume that y_3 (resp. y_4) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp. (0)) in the ℓ-splitting* (6.2.1.1) *by modifying them. Then*

(6.9.1) *We have an ℓ-splitting*

$$gr^2(\mathscr{O}, J) = (P^\sharp)\tilde{\oplus}(2P^\sharp) \tag{6.9.1.1}$$

such that $gr^{2,1}(\mathscr{O}, J) = (2P^\sharp)$ and $gr^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$. By changing coordinates if necessary we may assume further that y_3 (resp. y_2) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp. $(2P^\sharp)$) in the ℓ-splitting (6.9.1.1).

(6.9.2) *$X \supset C$ is an isolated extremal nbd and hence* (6.2.4) *holds.*

(6.9.3) *Let $D \in |K_X|$ and $D'' \in |2K_X|$ be general members. Then $D \cap D''$ is defined by J. In particular, $D \cdot D'' = 2C$ and* (6.2.2) *holds.*

Proof. We note $I^\sharp = (y_2, y_3, y_4)$ and $J^\sharp = (y_2, y_3, y_4^2)$ at $P^\sharp$. Since (X, P) is a cD point, we may assume

$$\alpha \equiv y_4^2 + y_1^2 y_2 \bmod I^\sharp J^\sharp$$

by a change of coordinates because $y_1 y_3$ (resp. y_4^2) appears with zero (resp. nonzero) coefficient in the Taylor expansion of α. Thus (y_4, y_3, y_2) is a $(1, 2, 2)$-monomializing ℓ-basis of $I \supset J$ at P of the second kind and $J^\sharp = (y_2, y_3)$. We see $gr^1(\mathscr{O}, J) \simeq (0)$, $gr^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$, and $gr^{2,1}(\mathscr{O}, J) \simeq gr^1(\mathscr{O}, J)^{\tilde{\otimes}2}\tilde{\otimes}(2P^\sharp) \simeq (2P^\sharp)$ [Mori88, (8.10)]. Hence we get (6.9.1.1) by (2.6), and (6.9.1) follows. From (6.9.1.1) follows that $gr_C^0F^6(\mathscr{O}, J)$ is ample on C by $gr_C^0F^6(\mathscr{O}, J) \simeq gr^6(\mathscr{O}, J) \supset \tilde{S}^3 gr^2(\mathscr{O}, J) \simeq \mathscr{O}(1)\oplus\mathscr{O}(1)\oplus\mathscr{O}(1)\oplus(2)$. Thus C is contractible in X and $(K_X \cdot C) = -1/3$ (6.4), whence (6.9.2) follows. We also see that

$$H^1(gr^n(\mathscr{O}(iK_X), J)) = 0 \quad \forall n \geq 0, \forall i \leq 3.$$

Hence

$$H^0(\mathscr{O}(iK_X)) \twoheadrightarrow H^0(gr^2(\mathscr{O}(iK_X), J)) \quad \text{for } i = 1, 2$$

by $H^0(gr^j(\mathscr{O}(iK_X), J)) = 0$ $(j = 0, 1)$. Let $s_i \in H^0(\mathscr{O}(iK_X))$ be such that its image globally generates the ith factor (0) in $(6.9.1.1)\tilde{\otimes}\mathscr{O}(iK_X)$ for $i = 1, 2$. Let $D_i = \{s_i = 0\} \in |iK_X|$ and I_i be the defining ideal of D_i. Then

$J = I_1 + I_2 + F^3(\mathscr{O}, J)$ and $J^\sharp = I_1{}^\sharp + I_2{}^\sharp + J^\sharp I^\sharp$. Thus $J^\sharp = I_1{}^\sharp + I_2{}^\sharp$ and we also see that $J = I_1 + I_2$ outside of P because $F^3(\mathscr{O}, J) = IJ$ outside of P. Hence $J = I_1 + I_2$ and we are done. □

(6.10) **Lemma.** *Under the notation and assumptions of* (6.1), *assume that* $i_P(1) = 1$. *Then* (6.2.3) *holds.*

Proof. Let the notation and assumptions be as in (6.9). A general section s of $H^0(\mathscr{O}_X)$ vanishing on C is of the form $s = ay_4 + \cdots + by_2y_3 + cy_2^3 + \cdots$ with general $a, b, c \in \mathbb{C}$ by (6.8) and (6.9.1.1). Thus it induces a global section $\bar{s}$ of $gr_C^1\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}$ which is nowhere vanishing since $\bar{s}$ is a part of basis of $gr_C^1\mathscr{O}$ at P, whence $H_X = \{s = 0\} \in |\mathscr{O}|$ is smooth outside of P. At P, we have $y_4 = \gamma(y_1, y_2, y_3) = \cdots + a'y_2y_3 + b'y_2^3 + \cdots$ with general $a', b' \in \mathbb{C}$ and

$$(H_X, P) = (y_1, y_2, y_3; \beta)/\mathbb{Z}_3(1, 1, 2; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3,$$

where $\beta = \alpha(y_1, y_2, y_3, \gamma)$. Let τ be the $\mathbb{Z}$-wt $\tau(y_1, y_2, y_3, y_4) = (1, 1, 2, 3)$ (T.7). Since y_4 does not appear in α, we see that

$$\beta_{\tau=3} = \alpha_{\tau=3}(y_1, y_2, y_3, 0) \qquad \beta_{\tau=6} = \alpha_{\tau=6}(y_1, y_2, y_3, \gamma_{\tau=3}).$$

If P is a simple cD point, then y_1y_3 and y_2y_3 do not appear in α and y_3^3 appears in α, whence $\beta_{\tau=3} = \alpha_{\tau=3}(y_1, y_2, 0)$ and $\beta_{\tau=6}(0, 0, 1) \neq 0$. Thus we can apply (6.7.1). To get (6.2.3), we only need to mention that $(\bullet^2) = -1$ follows from $(C \cdot K_{H_X}) = (C \cdot K_X) < 0$. Assume now that P is a double cD point. By changing coordinates, we may further assume $\alpha_{\tau=3} = y_1^2y_2$ and we see that y_4^2 and y_3^3 appear in $\alpha_{\tau=6}$ (say, with coefficient 1 for simplicity). If

$$\beta_{\tau=6}(0, y_2, y_3) = \alpha_{\tau=6}(0, y_2, y_3, a'y_2y_3 + b'y_2^3) = b'^2y_2^6 + y_3^3 + \cdots$$

is not squarefree for general a' and b', it has a multiple factor that is a polynomial in a', b', and y_i's. Thus $\beta_{\tau=6}(0, 1, y_3) = b'^2 + y_3^3 + \cdots$ is a square because y_2 is not a factor of $\beta_{\tau=6}(0, y_2, y_3)$, which is impossible because of y_3^3. Thus we can apply (6.7.2) and the rest is the same as the simple case. □

(6.11) **Example.** Let $Z \supset C$ be a germ of a smooth 3-fold along $C \simeq \mathbb{P}^1$ such that $N_{C/Z} \simeq \mathscr{O}_C \oplus \mathscr{O}_C$. Let $P \in C$ and let (z_1, z_2, z_3) be coordinates of (Z, P) such that $(C, P) = z_1$-axis. Let $(X, P) \supset (C, P)$ be a cD point as in (6.5) with $\alpha \equiv y_1^2y_2 \mod (y_2, y_3, y_4)^2$. For suitable ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, (y_1^3, y_4, y_1y_3) form coordinates for $U = (X, P) \cap \{\varepsilon_1 < |y_1^3| < \varepsilon_2\}$ by the implicit function theorem. Thus $z_1 = y_1^3$, $z_2 = y_4$, and $z_3 = y_1y_3$ patch (X, P) and $Z - (Z, P) \cap \{|z_1| \leq \varepsilon_1\}$ along U. Thus $X \supset C$ is an isolated extremal nbd of type cD by (6.2).

Thus (6.2) is proved, and we now treat the case $i_P(1) \geq 2$.

(6.12) **Lemma.** *If* $i_P(1) \geq 2$, *then* $i_P(1) = 2$ *and* $\ell(P) = 3$ *or* 4.

Proof. By [Mori88, (2.16)], we have $i_P(1) = 1 + [\ell(P)/3]$. It is enough to disprove $\ell(P) \geq 5$. Let i be such that $\alpha \equiv y_1^{\ell(P)} y_i \mod (y_2, y_3, y_4)^2$ as in (6.5). By deformation $\alpha + t y_1^{\ell(P)-3} y_i = 0$ of (X, P) (2.9), we get an isolated extremal nbd $X' \supset C$ with a cD point of index 3 and a III point on C [Mori88, (4.12.2)] because $\ell(P) - 3 \geq 2$. This contradicts (6.6) and we are done. □

We treat the case $\ell(P) = 3$ first.

(6.13) **Lemma.** *If* $\ell(P) = 3$, *then* (6.3.1) *holds.*

Proof. By $\alpha \equiv y_1^3 y_4 \mod (y_2, y_3, y_4)^4$, we see that y_2 and y_3 form an ℓ-free ℓ-basis of $gr_C^1\mathscr{O}$ at P.

To prove (6.3.1.1) first, we will derive a contradiction assuming that $gr_C^1\mathscr{O} \not\simeq (P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp)$. Then we have $gr_C^1\mathscr{O} \simeq (2P^\sharp)\tilde{\oplus}(-1 + P^\sharp)$. We may further assume that y_2 (resp. y_3) is an ℓ-free ℓ-basis of $(2P^\sharp)$ (resp. $(-1 + P^\sharp)$) after a change of coordinates. Let J be the C-laminal ideal of width 2 such that $J/F_C^2\mathscr{O} = (2P^\sharp)$. We note $J^\sharp = (y_2, y_3^2, y_4)$ at $P^\sharp$. Since we are going to derive a contradiction, we may assume that $y_1^2 y_3^2$ appears in α by deformation $\alpha + t y_1^2 y_3^2$ (2.9.2). Replacing y_3 by $y_3\cdot$(invariant unit), we may assume $\alpha \equiv y_1^3 y_4 + y_1^2 y_3^2 \mod J^\sharp I^\sharp$, where $I^\sharp = (y_2, y_3, y_4)$. Then $u = y_1 y_4 + y_3^2$ generates the torsion part $\simeq \mathbb{C}\{y_1\}/(y_1^2)$ of $J^\sharp/J^\sharp I^\sharp$, whence $F^3(\mathscr{O}, J)^\sharp = (u) + J^\sharp I^\sharp$. Hence $gr^2(\mathscr{O}, J)^\sharp = \mathscr{O}_{C^\sharp} y_2 \oplus \mathscr{O}_{C^\sharp} y_4$ and $y_3^2 \equiv -y_1 y_4 \mod F^3(\mathscr{O}, J)^\sharp$. Thus $gr^{2,1}(\mathscr{O}, J) \simeq gr^1(\mathscr{O}, J)^{\tilde{\otimes} 2}\tilde{\otimes}(P^\sharp) \simeq (-1)$ because J is a nested c.i. outside of P. We note that $gr^1(\mathscr{O}, J) \simeq (-1 + P^\sharp)$ and $gr^{2,0}(\mathscr{O}, J) \simeq (2P^\sharp)$ with ℓ-free ℓ-bases y_3 and y_2 at P, respectively. We claim that the following natural map is an ℓ-isomorphism

(6.13.1) $$gr^{3,i}(\mathscr{O}, J) \simeq gr^{2,i}(\mathscr{O}, J)\tilde{\otimes} gr^1(\mathscr{O}, J)\tilde{\otimes}(iP^\sharp) \qquad (i = 0, 1).$$

Since J is a nested c.i. outside of P, we only need to check (6.13.1) at P. We see that

$$\begin{aligned} F^3(\mathscr{O}, J)^\sharp &= I^\sharp J^\sharp + (u) \\ &\equiv (y_2 y_3, y_3^3, y_3 y_4) + (u) \mod F^4(\mathscr{O}, J)^\sharp \\ &\equiv (y_2 y_3, y_3 y_4, u) \mod F^4(\mathscr{O}, J)^\sharp \end{aligned}$$

because $u y_3 = y_3^3 + y_1 y_3 y_4 \in F^4(\mathscr{O}, J)^\sharp$. By $\alpha \equiv y_1^2 u \mod J^\sharp I^\sharp$, we have

$$\alpha \equiv y_1^2 u + g y_1^3 y_2 y_3 + h y_3^3 \mod F^4(\mathscr{O}, J)^\sharp$$

for some g and a unit $h \in \mathbb{C}\{y_1, y_2, y_3, y_4\}$ because P is a cD point. Thus $y_3^3 \equiv -y_1^2(u + g y_1 y_2 y_3) \mod F^4(\mathscr{O}, J)^\sharp$ and $gr^3(\mathscr{O}, J)^\sharp = \mathscr{O}_{C^\sharp} y_2 y_3 \oplus \mathscr{O}_{C^\sharp} u$ at $P^\sharp$. Furthermore from $y_3^3 \equiv -y_1 y_3 y_4$ above, we see that $y_3 y_4 \equiv y_1(u + g y_1 y_2 y_3)$

in $gr^3(\mathscr{O}, J)^\sharp$. Thus (6.13.1) is proved. Hence we have ℓ-isomorphisms

$$\begin{aligned} gr^1(\omega, J) &\simeq gr^1(\mathscr{O}, J)\tilde{\otimes} gr^0_C\omega \simeq (-1), \\ gr^{2,0}(\omega, J) &\simeq gr^{2,0}(\mathscr{O}, J)\tilde{\otimes} gr^0_C\omega \simeq (P^\sharp), \\ gr^{2,1}(\omega, J) &\simeq gr^{2,1}(\mathscr{O}, J)\tilde{\otimes} gr^0_C\omega \simeq (-2+2P^\sharp), \\ gr^{3,0}(\omega, J) &\simeq gr^{2,0}(\mathscr{O}, J)\tilde{\otimes} gr^1(\mathscr{O}, J)\tilde{\otimes} gr^0_C\omega \simeq (-1+2P^\sharp), \\ gr^{3,1}(\omega, J) &\simeq gr^{2,1}(\mathscr{O}, J)\tilde{\otimes} gr^1(\mathscr{O}, J)\tilde{\otimes} gr^0_C\omega\tilde{\otimes}(P^\sharp) \simeq (-2+P^\sharp). \end{aligned}$$

Thus $H^1(\omega/F^4(\omega, J)) \neq 0$, which is a contradiction, and (6.3.1.1) is proved.

By changing coordinates if necessary, we assume that y_3 (resp. y_2) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp. $(-1+2P^\sharp)$) in (6.3.1.1) in addition to (6.5). We note $J^\sharp = (y_2^2, y_3, y_4)$ at $P^\sharp$. We will derive a contradiction assuming that $y_1y_2^2$ does not appear in α. Then we may further assume $\alpha \equiv y_1^3y_4 \mod I^\sharp J^\sharp$ by replacing y_4 by $y_4 + y_1y_2^2\cdot(\cdots)$. Hence $y_4 \in F^3(\mathscr{O}, J)^\sharp$ and we have ℓ-isomorphisms $gr^{2,1}(\mathscr{O}, J) \simeq gr^1(\mathscr{O}, J)^{\tilde{\otimes}2} \simeq (-1+P^\sharp)$ and $gr^2(\mathscr{O}, J) \simeq gr^{2,0}(\mathscr{O}, J)\tilde{\oplus} gr^{2,1}(\mathscr{O}, J) \simeq (P^\sharp)\tilde{\oplus}(-1+P^\sharp)$ by (2.6). By changing coordinates, we may assume that y_3 (resp. y_2^2) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp. $(-1+P^\sharp)$) in the ℓ-splitting. Let K be the ideal such that $J \supset K \supset F^3(\mathscr{O}, J)$ and $K/F^3(\mathscr{O}, J) = (P^\sharp)$ in the above ℓ-splitting. By [Mori88, (8.14)], K is a C-laminal ideal of width 3 and a nested c.i. outside of P. By $K^\sharp \supset (y_2^3, y_3, y_4)$, we have $K^\sharp = (y_2^3, y_3, y_4)$ at $P^\sharp$. By [Mori88, (8.14.1)], $gr^1(\mathscr{O}, K) = gr^1(\mathscr{O}, J) \simeq (-1+2P^\sharp)$ and $gr^{3,0}(\mathscr{O}, K) = gr^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$. We claim that (y_2, y_3, y_4) is a $(1, 3, 3)$-monomializing ℓ-basis of $I \supset J$ of the second kind at P. Indeed by $\alpha \equiv y_1^3y_4 \mod I^\sharp J^\sharp$, we have $\alpha \equiv y_1^3y_4 + gy_2^3 \mod I^\sharp K^\sharp$ for some unit $g \in \mathbb{C}\{y_1, y_2, y_3, y_4\}$ because P is a cD point. Thus the claim is proved. Hence by [Mori88, (8.12)], we have $gr^2(\mathscr{O}, K) \simeq gr^1(\mathscr{O}, K)^{\tilde{\otimes}2} \simeq (-1+P^\sharp)$ and $gr^{3,1}(\mathscr{O}, K) \simeq gr^1(\mathscr{O}, K)^{\tilde{\otimes}3}\tilde{\otimes}(3P^\sharp) \simeq (0)$. Hence

$$gr^3(\mathscr{O}, K) \simeq gr^{3,0}(\mathscr{O}, K)\tilde{\oplus} gr^{3,1}(\mathscr{O}, K) \simeq (P^\sharp)\tilde{\oplus}(0)$$

by (2.6). Then as in the argument for (6.9.2), we can see that $\operatorname{Spec}(\mathscr{O}_X/K) = D\cdot D'$ for some $D \in |\mathscr{O}_X|$ and $D' \in |K_X|$. This means $\operatorname{Spec}(\mathscr{O}_X/K)$ moves in D', which is a contradiction. Hence P is a double cD point and (6.3.1) holds. □

(6.14) **Lemma.** *Let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ that has the properties in* (6.3.1) *and that $\ell(P) = 3$. Let J be the C-laminal ideal of width* 2 *such that $J/F^2_C\mathscr{O} = (P^\sharp)$ in the ℓ-splitting* (6.3.1.1). *We will use the coordinates in* (6.5) *and assume that y_3 (resp. y_2) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp. $(-1+2P^\sharp)$) in the ℓ-splitting* (6.3.1.1) *by modifying them. Then we have an ℓ-splitting*

$$gr^2(\mathscr{O}, J) = (P^\sharp)\tilde{\oplus}(0) \tag{6.14.1}$$

such that $gr^{2,1}(\mathscr{O}, J) = (0)$ *and* $gr^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$. *By changing coordinates if necessary we may assume further that* y_3 *(resp.* y_4*) is an* ℓ*-free* ℓ*-basis of* $(P^\sharp)$ *(resp.* (0)*) in the* ℓ*-splitting* (6.14.1).

Proof. Since P is a double cD point, $y_1y_2^2$ appears in α. Then $\alpha \equiv y_1^3y_4 + y_1y_2^2$ mod $I^\sharp J^\sharp$ after a change of coordinates. Hence $y_2^2 \equiv -y_1^2 y_4$ mod $F^3(\mathscr{O}, J)$ and $gr^{2,1}(\mathscr{O}, J)^\sharp = \mathscr{O}_{C^\sharp} y_4$ at $P^\sharp$ as in the argument for (6.13). Thus

$$gr^{2,1}(\mathscr{O}, J) \simeq gr^1(\mathscr{O}, J)^{\tilde{\otimes} 2} \tilde{\otimes} (2P^\sharp) \simeq (0),$$

which implies (6.14.1) by $H^1(C, gr^{2,1}(\mathscr{O}, J) \tilde{\otimes} gr^{2,0}(\mathscr{O}, J)^{\tilde{\otimes}(-1)}) = 0$. □

(6.15) **Lemma.** *Let the notation and assumptions be as in* (6.14). *Then* $X \supset C$ *is an isolated extremal nbd of type* cD *with* $\ell(P) = 3$, *whence* (6.3.4) *holds. Furthermore the assertions* (6.3.2) *and* (6.3.3) *hold.*

Proof. We treat as X the formal completion of X along C until C is proved to be contractible in (6.15.1), after which the assertions on the completion and the original X are equivalent by comparison theorems. By $gr^1(\mathscr{O}, J) \simeq (-1 + 2P^\sharp)$ and (6.14.1), we see that $H^0(\mathscr{O}(K_X)) = H^0(F^2(\omega, J))$ and $H^0(\mathscr{O}(K_X)) \twoheadrightarrow H^0(gr^2(\omega, J))$ as in the argument for (6.9.3). Similarly we see $H^1(\mathscr{O}_X) = 0$. We note that the same argument does not lead to a contradiction as in the proof of (6.14) because $J^\sharp$ is not a c.i. ideal.

(6.15.1) In this paragraph, we will prove that C is contractible (whence (6.3.4)) and (6.3.2). Since $gr(\omega, J)^2 \simeq (-P^\sharp) \tilde{\oplus} (0)$, a general global section $s \in H^0(\mathscr{O}(K_X))$ induces a global section $\bar{s}$ of $gr^2(\omega, J)$ which is a global basis of (0). So $D = \{s = 0\} \in |K_X|$ is smooth outside of P. We claim that D is a normal surface with only rational singularities and

(6.15.1.1) $\qquad 4C \sim K_X|_D$ as Weil divisors on D.

Since P is a double cD point, we may further assume that

$$\alpha \equiv y_1^3 y_4 + y_1 y_2^2 \bmod J^\sharp I^\sharp$$

after a change of coordinates. Since $s \equiv \Omega y_3$ mod $F^3(\omega, J)^\sharp$ at $P^\sharp$ for some ℓ-free ℓ-basis Ω of ω at P, $D^\sharp$ is defined in $X^\sharp$ by $y_3 = \gamma(y_1, y_2, y_4)$ for some $\gamma \in \mathbb{C}\{y_1, y_2, y_4\} \cap F^3(\mathscr{O}, J)$. Hence

$$(D, P) \simeq (y_1, y_2, y_4; \beta)/\mathbb{Z}_3(1, 1, 0; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3,$$

where $\beta = \alpha(y_1, y_2, \gamma, y_4)$. Using the notation of (6.16), we see

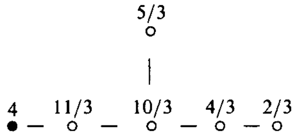
$$\beta_{\rho=3} = \alpha_{\rho=3}(y_1, y_2, \gamma_{\rho=2}, y_4) = \alpha_{\rho=3}(y_1, y_2, 0, 0) = y_1 y_2^2$$

because P is a cD point. Since $\gamma|_{C^\sharp} \equiv 0$, $\gamma_{\rho=2}$ is divisible by y_2 whence $\gamma_{\rho=2}(y_1, 0, y_4) = 0$. Hence

$$\beta_{\rho=6}(y_1, 0, y_4) = \alpha_{\rho=6}(y_1, 0, \gamma_{\rho=5}(y_1, 0, y_4), y_4) = \alpha_{\rho=6}(y_1, 0, 0, y_4)$$

since $y_1 y_3$ does not appear in α. Then $\beta_{\rho=6}(y_1, 0, y_4) = y_1^3 y_4 + c y_4^2$ for some $c \in \mathbb{C}^*$ because y_4^2 appears in α. In particular, $(D^\sharp, P^\sharp)$ is a point of type D

and we can apply (6.16) to $(D, P) \supset C$. Then it is easy to see that the pull-up of the $\mathbb{Q}$-Cartier Weil divisor $4C$ of D is given by



where the numbers denote the multiplicities. Since $(\bullet^2) = -1$ by $(K_D \cdot C) = 2(K_X \cdot C) < 0$ (6.4), we see $(\mathcal{O}_D(4C) \cdot C) = -1/3$. Since $(D^\sharp, P^\sharp)$ is a point of type D, $4F^\sharp$ is Cartier for every Weil divisor $F^\sharp$ on $(D^\sharp, P^\sharp)$. Thus $\mathcal{O}_D(4C)$ is an ℓ-invertible $\mathcal{O}_D$-module at P. Hence $\mathcal{O}_D(4C) \simeq \mathcal{O}_D(iK_X)$ near P for some i and we see that $i \equiv 1 \mod (3)$ by $(\mathcal{O}_D(4C) \cdot C) \equiv (\mathcal{O}_D(K_X) \cdot C) \mod \mathbb{Z}$. Hence $K_X|_D - 4C$ is a Cartier divisor and we have (6.15.1.1) by $(\mathcal{O}_D(4C) \cdot C) = (K_X \cdot C) = -1/3$ (6.4). Thus from the exact sequence

$$0 \to \mathcal{O}_X \to \mathcal{O}_X(K_X) \to \mathcal{O}_D(4C) \to 0,$$

we have a surjection $H^0(\mathcal{O}_X(K_X)) \twoheadrightarrow H^0(\mathcal{O}_D(4C)) = H^0(\mathcal{O}_D)$ by $H^1(\mathcal{O}_X) = 0$. Hence for another general $D' \in |K_X|$, we see $D \cdot D' = 4C$. Thus (6.3.2) follows for the completion. In particular, (6.3.4) holds. Hence (6.3.2) holds as explained above.

(6.15.2) Let u be a general global section of $\mathcal{O}_X$ vanishing on C. Then $u = \lambda y_1 y_3 + \mu y_4 + \cdots$ for general $\lambda, \mu \in \mathbb{C}$ by (6.14.1). The divisor $H_X = \{u = 0\} \in |\mathcal{O}_X|$ is smooth outside of P because the image of u globally generates $gr^{2,0}(\mathcal{O}, J) \simeq \mathcal{O}_C \subset gr^1_C\mathcal{O}$. Then

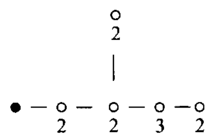
$$(H_X, P) = (y_1, y_2, y_3; \overline{\alpha})/\mathbb{Z}_3(1, 1, 2; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3,$$

where $\overline{\alpha} = \alpha(y_1, y_2, y_3, \nu y_1 y_3 + \cdots)$ with general $\nu \in \mathbb{C}$. Since $\alpha = cy_4^2 + y_1^3 y_4 + dy_3^3 + y_1 y_2^2 + \cdots$ for some $c, d \in \mathbb{C}^*$ as in (6.15.1), it is easy to see that (6.7.3) applies. The rest is the same as (6.10) and (6.3.3) follows. □

(6.16) **Computation.** Let (D, P) be a normal surface singularity

$$(D, P) = (y_1, y_2, y_4; \alpha)/\mathbb{Z}_3(1, 1, 0; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3.$$

Let ρ be the $\mathbb{Z}$-wt $\rho(y_1, y_2, y_4) = (1, 1, 3)$ (T.7). Assume that $\alpha_{\rho=3} = y_1 y_2^2$ and that $\alpha_{\rho=6}(y_1, 0, y_4)$ is squarefree. Then $\Delta(D \supset C)$ consists of smooth rational curves and C' intersecting transversely, and its configuration is



(6.17) **Example.** Let $Z \supset C$ be a germ of a smooth 3-fold along $C \simeq \mathbb{P}^1$ such that $N^*_{C/Z} \simeq \mathcal{O}_C \oplus \mathcal{O}(-1)$. Let $P \in C$ and let (z_1, z_2, z_3) be coordinates of (Z, P) such that $(C, P) = z_1$-axis and z_2 (resp. z_3) is a generator of $\mathcal{O}_C$

(resp. $\mathscr{O}(-1)$) in the splitting of $N^*_{C/Z}$. Let $(X, P) \supset (C, P)$ be a cD point as in (6.5) with $\alpha \equiv y_1^3 y_4 \mod (y_2, y_3, y_4)^2$. As in (6.11), $z_1 = y_1^3$, $z_2 = y_1 y_3$, and $z_3 = y_1^2 y_2$ patch (X, P) and $Z - (Z, P) \cap \{|z_1| \le \varepsilon_1\}$ for some $\varepsilon_1 > 0$, and we have an isolated extremal nbd $X \supset C$ of type cD such that $\ell(P) = 3$.

Now (6.3) is proved in the case $\ell(P) = 3$.

(6.18) **Lemma.** *Assume* $\ell(P) = 4$. *Then* (6.3.1) *holds.*

Proof. Since $\alpha \equiv y_1^4 y_3 \mod (y_2, y_3, y_4)^2$, y_2 and y_4 form an ℓ-free ℓ-basis of $gr_C^1\mathscr{O}$ at P. By deformation $\alpha + t y_1^3 y_4 = 0$ of (X, P) (2.9.2), $X_t^\circ \supset C_t \ni P_t$ is a cD point of index 3 and $\ell(P) = 3$. Since $y_1^2 y_2$ appears in $\alpha + t y_1^3 y_4$ by (6.14), so does $y_1^2 y_2$ in α. In particular, P is a double cD point. If (6.3.1.1) does not hold, then we have $gr_C^1\mathscr{O} \simeq (2P^\sharp)\tilde{\oplus}(-1)$ (2.8) and $gr_C^1\omega \simeq (P^\sharp)\tilde{\oplus}(-2 + 2P^\sharp)$. This implies a contradiction $H^1(\omega/F_C^2\omega) \neq 0$, whence (6.3.1.1) and hence (6.3.1) hold.

(6.19) **Lemma.** *Let* $X \supset C$ *be a germ of a* 3*-fold along* $C \simeq \mathbb{P}^1$ *that has the properties in* (6.3.1) *and that* $\ell(P) = 4$. *By changing the embedding* $(-1 + 2P^\sharp) \subset gr_C^1\mathscr{O}$ *in* (6.3.1.1) *and by changing the coordinates in* (6.5), *we may assume further that* y_4 (*resp.* y_2) *is an* ℓ*-free* ℓ*-basis of* (0) (*resp.* $(-1 + 2P^\sharp)$) *in the* ℓ*-splitting* (6.3.1.1) *and*

$$(6.19.1) \qquad y_1^4 y_3 + y_4^2 + y_1 y_2^2 \equiv 0 \mod F_C^3\mathscr{O}^\sharp.$$

Proof. We may assume that y_4 and y_2 are ℓ-free ℓ-bases of the components (0) and $(-1 + 2P^\sharp)$ in (6.3.1.1). By changing the embedding $(-1 + 2P^\sharp) \subset gr_C^1\mathscr{O}$, we can find a unit u at P such that y_4 and $y_2 + cuy_1y_4$ are also ℓ-free ℓ-bases of the components (0) and $(-1 + 2P^\sharp)$ of some ℓ-splitting like (6.3.1.1) for each $c \in \mathbb{C}$. Replacing y_4 by $y_4\cdot$(unit), we may assume $u = 1$. By $\alpha \equiv y_1^4 y_3 \mod (y_2, y_3, y_4)^4$, we see $y_3 \in F_C^2\mathscr{O}^\sharp$ and $\alpha \equiv y_1^4 y_3 + p \cdot y_4^2 + q \cdot y_1^2 y_2 y_4 + r \cdot y_1 y_2^2 \mod F_C^3\mathscr{O}^\sharp$ for some $p, q, r \in \mathscr{O}_{C,P}$ such that $p(0)r(0) \neq 0$. Replacing y_2 by $y_2 - q(0)y_1y_4/2r(0)$, we may assume $q = 0$ as explained above. Multilying y_2 and y_4 by units, we attain (6.19.1). □

Under the notation and assumptions of (6.19), let L (resp. M) be the component (0) (resp. $(-1 + 2P^\sharp)$) in the ℓ- splitting (6.3.1.1). Let $E \subset gr_C^2\mathscr{O}$ be the saturation of $L^2 + M^2$ and let K be the ideal such that $F_C^2\mathscr{O} \supset K \supset F_C^3\mathscr{O}$ such that $K/F_C^3\mathscr{O} = E$. Then

(6.20) **Lemma.** *Under the above notation and assumptions, we have*

(6.20.1) (6.3.4) *holds.*

(6.20.2) $\mathrm{Spec}(\mathscr{O}_X/K) = D \cap D'$ *for two general members* D *and* $D' \in |K_X|$. *In particular,* (6.3.2) *holds.*

(6.20.3) (6.3.3) *holds.*

Proof. We have $gr_C^2\mathscr{O} \simeq S^2 gr_C^1\mathscr{O}$ outside of P, whence $E = L^{\otimes 2} \oplus M^{\otimes 2}$ outside of P. Since $F_C^2\mathscr{O}^\sharp = (y_3, y_2^2, y_2y_4, y_4^2)$ by $\alpha \equiv y_1^4 y_3 \mod (y_2, y_3, y_4)^2$, we see

$$gr_C^2\mathcal{O}^\sharp = \mathcal{O}_{C^\sharp}y_3 \oplus \mathcal{O}_{C^\sharp}y_2y_4 \oplus \mathcal{O}_{C^\sharp}y_2^2$$

and $E^\sharp = \mathcal{O}_{C^\sharp}y_3 \oplus \mathcal{O}_{C^\sharp}y_2^2$ by (6.19.1). Thus $gr_C^2\mathcal{O}/E \simeq L\tilde{\otimes}M$. We also see that $L^{\sharp 2}(P^\sharp) \subset E^\sharp$ by $y_4^2 = -y_1(y_2^2 + y_1^3y_3)$ and that $E^\sharp/L^{\sharp 2}(P^\sharp) \simeq M^{\sharp 2}(3P^\sharp)$ by $y_2^2 \equiv -y_1^3y_3 \mod \mathcal{O}_{C^\sharp}y_4^2/y_1$. This induces an ℓ-exact sequence

$$0 \to L^{\tilde{\otimes}2}(P^\sharp) \to E \to M^{\tilde{\otimes}2}(3P^\sharp) \to 0$$

which is ℓ-split by $L^{\tilde{\otimes}2}(P^\sharp) \simeq M^{\tilde{\otimes}2}(3P^\sharp) \simeq (P^\sharp)$ and (2.6). We will see $K^\sharp = (y_3, y_2^2)$ using $E^\sharp = \mathcal{O}_{C^\sharp}y_3 \oplus \mathcal{O}_{C^\sharp}y_2^2$. Indeed by $(y_3, y_2^2) \subset K^\sharp$, it is enough to see that $y_2 = y_3 = 0$ defines $2C^\sharp$ in $X^\sharp$ because $\mathcal{O}/K$ is of length 4 at the generic point of C. Since y_4^2 appears in α and $\alpha - y_1^4y_3 \in (y_2, y_3, y_4)^2$, we see $y_4^2 \in (y_2, y_3)\mathcal{O}_{X^\sharp}$. Thus we have $K^\sharp = (y_3, y_2^2)$ as claimed. Thus K is locally a c.i. ideal outside of P and $K^\sharp$ is a c.i. ideal at $P^\sharp$. In particular, $gr_C^0K \simeq E \simeq (P^\sharp)\tilde{\oplus}(P^\sharp)$. Thus C is contractible as in the argument for (6.9), and (6.3.4) follows from (6.4). We also see that $\mathcal{O}/K$ has a filtration whose subquotients are $\mathcal{O}_C \simeq (0)$, $L \simeq (0)$, $M \simeq (-1 + 2P^\sharp)$, and $L\tilde{\otimes}M \simeq (-1 + 2P^\sharp)$. Then as in the argument for (6.9), one can see that $H^0(\mathcal{O}(K_X)) = H^0(\mathcal{O}(K_X)\tilde{\otimes}K)$, that $H^0(\mathcal{O}(K_X))$ generates $gr_C^0\omega\tilde{\otimes}gr_C^0K \simeq (0)\tilde{\oplus}(0)$, and that $K = I_1 + I_2$ for defining ideals I_1 and I_2 of general members D and D' of $|K_X|$. Thus (6.20.2) is proved. Because of the above filtration, we also see that a general section s of $\mathcal{O}_X$ vanishing on C is of the form $s = \lambda y_4 + \mu y_1y_3 + \cdots$ at P with general $\lambda, \mu \in \mathbb{C}$. Since its image into $gr_C^1\mathcal{O} = (0)\tilde{\oplus}(-1 + 2P^\sharp)$ globally generates (0), $H_X = \{s = 0\} \in |\mathcal{O}_X|$ is smooth outside of P. At P we have

$$(H_X, P) = (y_1, y_2, y_3; \overline{\alpha})/\mathbb{Z}_3(1, 1, 2; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3,$$

where $\overline{\alpha} = \alpha(y_1, y_2, y_3, \nu y_1y_3 + \cdots)$ with general $\nu \in \mathbb{C}$. Since $\alpha = cy_4^2 + 0 \cdot y_1^3y_4 + y_1^4y_3 + dy_3^3 + ey_1y_2^2 + \cdots$ for some $c, d, e \in \mathbb{C}^*$ (6.19), it is easy to see that (6.7.3) applies. The rest is the same as (6.10), and (6.3.3) follows. □

(6.21) **Example.** Let $Z \supset C$ be a germ of a smooth 3-fold along $C \simeq \mathbb{P}^1$ such that $N_{C/Z}{}^* \simeq \mathcal{O}_C \oplus \mathcal{O}(-1)$. Let $P \in C$ and let (z_1, z_2, z_3) be coordinates of (Z, P) such that $(C, P) = z_1$-axis and z_2 (resp. z_3) is a generator of $\mathcal{O}_C$ (resp. $\mathcal{O}(-1)$) in the splitting of $N_{C/Z}^*$. Let $(X, P) \supset (C, P)$ be a cD point as in (6.5) with $\alpha \equiv y_1^4y_3 \mod (y_2, y_3, y_4)^2$. As in (6.11), $z_1 = y_1^3$, $z_2 = y_4$ and $z_3 = y_1^2y_2$ patch (X, P) and $Z - (Z, P) \cap \{|z_1| \le \varepsilon_1\}$ for some $\varepsilon_1 > 0$, and we have an isolated extremal nbd $X \supset C$ of type cD such that $\ell(P) = 4$.

Thus the proof of (6.3) is completed.

The following lemma will be needed in the proof of the more general (13.11).

(6.22) **Lemma.** *Let $X \supset C \ni P$ be an isolated extremal nbd of $cD/3$ type. Let H be a general member of $|\mathcal{O}_X|$ through C and let H_0 be another member such* *that* $\Delta(H_0 \supset C)$ *is equal to one of the two configurations for* Δ_X *in* (6.2.3.1–6.2.3.2). *Then* $\Delta(H \supset C) = \Delta(H_0 \supset C)$.

Proof. Let us use the coordinates (6.5) for (X, P). We may also assume that (H, P) is defined by $y_4 = \gamma(y_1, y_2, y_3)$ in (X, P) such that $\alpha(y_1, y_2, y_3, \gamma)$ satisfies (6,9.1) or (6.9.2) (cf. (6.10)). Since (H_0, P) has the same configuration as one of (6.7), (H_0, P) is isomorphic to one of (6.7.1–6.7.2) by [Laufer73] (cf. (13.8.2) for details). In particular (H_0, P) is also defined in (X, P) by an equation $y_4 = \delta(y_1, y_2, y_3)$ for some δ. Then we note

$$\alpha_{\sigma=3}(y_1, y_2, y_3, \gamma) = \alpha_{\sigma=3}(y_1, y_2, y_3, \delta)$$

for $\sigma = (1, 1, 2)$ because α does not have the term y_4. Since (6.7.1) is distinguished from (6.7.2) by the squarefreeness of the $\sigma = 3$ part of the equation, we see $\Delta(H \supset C) = \Delta(H_0 \supset C)$.

7. General members of $|\mathscr{O}_X|_C$; isolated *IIA* case

We consider the following set-up in this chapter unless otherwise mentioned explicitly.

(7.1) Let $f : X \supset C \to Y \ni Q$ be an isolated extremal nbd with only one non-Gorenstein point P such that $X \supset C$ has a *IIA* point at P. Let H_X be a general member of $|\mathscr{O}_X|$ through C and let $H_Y = f(H_X)$. Let $\Delta_X = \Delta(H_X \supset C)$ and $\Delta_Y = \Delta(H_Y)$.

Our main results in this chapter are the following.

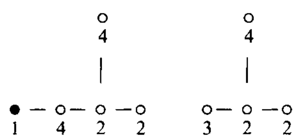
(7.2) **Theorem.** *Under the notation and assumptions of* (7.1), *assume that* $i_P(1) = 1$ *and* $gr_C^1\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}$. *Then we have the following*:

(7.2.1) X *is smooth outside of IIA point* P *with* $\ell(P) = 1$ *and we have an* ℓ*-isomorphism*

$$gr_C^1\mathscr{O} \simeq (P^\sharp)\tilde{\oplus}(2P^\sharp). \qquad (7.2.1.1)$$

(7.2.2) $2C = D'' \cdot D'''$ *for general* $D'' \in |2K_X|$ *and* $D''' \in |3K_X|$.

(7.2.3) H_X *is normal, and* Δ_X *and* Δ_Y *consist of smooth rational curves intersecting transversely and their configurations are as follows.*



Δ_X Δ_Y

(7.2.4) *Let* $X \supset C$ *be a the germ of a* 3*-fold along* $C \simeq \mathbb{P}^1$ *that need not be an extremal nbd. If* $X \supset C$ *has the properties in* (7.2.1), *then it is an isolated extremal nbd of type IIA such that* $i_P(1) = 1$ *and* $gr_C^1\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}$. (*An example is given in* (7.6.4).)

(7.3) **Theorem.** *Under the notation and assumptions of* (7.1), *assume* $i_P(1) = 1$ *and* $gr_C^1\mathscr{O} \not\simeq \mathscr{O} \oplus \mathscr{O}$. *Then we have the following*:

(7.3.1) *Outside of IIA point* P *with* $\ell(P) = 1$, X *has exactly* i *singular point* ($i = 0$ *or* 1), *which is of type III if any. Furthermore we have* ℓ*-isomorphisms*

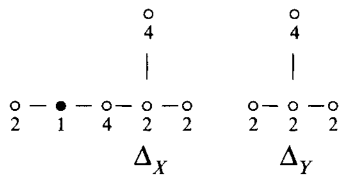
$$gr_C^1\mathscr{O} \simeq (1 - i + P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp), \tag{7.3.1.1}$$

$$gr^2(\mathscr{O}, J) \simeq (P^\sharp)\tilde{\oplus}(0), \tag{7.3.1.2}$$

where J *is the* C*-laminal ideal of width* 2 *with* $J/F_C^2\mathscr{O} = (1-i+P^\sharp)$ *in* (7.3.1.1).

(7.3.2) $2kC = D \cdot D''$ *for general* $D \in |K_X|$ *and* $D'' \in |2K_X|$, *where* k *is the axial multiplicity at* P.

(7.3.3) H_X *is normal, and* Δ_X *and* Δ_Y *consist of smooth rational curves intersecting transversely and their configurations are as follows.*



(7.3.4) *Let* $X \supset C$ *be the germ of a 3-fold along* $C \simeq \mathbb{P}^1$ *which need not be an extremal nbd. If* $X \supset C$ *has the the properties in* (7.3.1), *then it is an isolated extremal nbd of type IIA such that* $i_P(1) = 1$ *and* $gr_C^1\mathscr{O} \not\simeq \mathscr{O} \oplus \mathscr{O}$. (*Examples are given in* (7.9.4).)

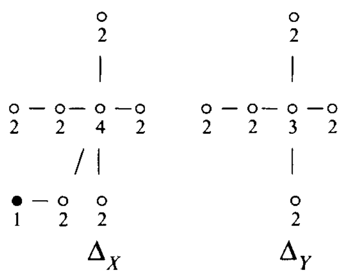
(7.4) **Theorem.** *Under the notation and assumptions of* (7.1), *assume* $i_P(1) \geq 2$. *Then* $i_P(1) = 2$ *and we have the following.*

(7.4.1) X *is smooth outside of IIA point* P *with* $\ell(P) = 3$ *or* 4 *and we have an* ℓ*-isomorphism*

$$gr_C^1\mathscr{O} \simeq \begin{cases} (2P^\sharp)\tilde{\oplus}(-1 + 3P^\sharp) & \text{if } \ell(P) = 3, \\ (P^\sharp)\tilde{\oplus}(-1 + 3P^\sharp) & \text{if } \ell(P) = 4. \end{cases} \tag{7.4.1.1}$$

(7.4.2) $2C = D \cdot D''$ *for general* $D \in |K_X|$ *and* $D'' \in |2K_X|$.

(7.4.3) H_X *is normal, and* Δ_X *and* Δ_Y *consist of smooth rational curves intersecting transversely and their configurations are as follows.*



(7.4.4) *Let* $X \supset C$ *be a the germ of a 3-fold along* $C \simeq \mathbb{P}^1$ *which need not be an extremal nbd. If* $X \supset C$ *has the the properties in* (7.4.1), *then it is an* *isolated extremal nbd of type IIA such that* $i_P(1) = 2$. (*Examples are given in* (7.12.5).)

(7.5) Let us express the IIA point as

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_4(1, 1, 3, 2; 2) \supset C = y_1\text{-axis}/\mathbb{Z}_4,$$

using an equation α such that $\alpha \equiv y_1^{\ell(P)} y_i \mod (y_2, y_3, y_4)^2$ with $i = 2$ (resp. 3, 4) if $\ell(P) \equiv 1$ (resp. 3, 0) mod 4 [Mori88, (2.16)]. We note that $\ell(P) \not\equiv 2$ mod 4 because of the lack of a variable with $\mathrm{wt} \equiv 0 \mod 4$.

(7.6) *Proof of* (7.2). By $i_P(1) = 1$ and $\deg gr_C^1 \mathcal{O} = 0$, we see that X is smooth outside of P [Mori88, (2.3.1), (2.15)]. By $i_P(1) = 1$, we also have $\ell(P) = 1$ [Mori88, (2.16)] and hence $\alpha \equiv y_1 y_2$ (7.5). Then y_3 and y_4 form an ℓ-free ℓ-basis of $gr_C^1 \mathcal{O} \simeq \mathcal{O} \oplus \mathcal{O}$. Hence, after a possible coordinates change, we have an ℓ-isomorphism

$$gr_C^1 \mathcal{O} \simeq (P^\sharp) \tilde{\oplus} (2P^\sharp), \tag{7.6.1}$$

where y_3 (resp. y_4) is an ℓ-free ℓ-basis of $(P^\sharp)$ (resp. $(2P^\sharp)$) at P. Thus (7.2.1) is proved. Let J be the C-laminal ideal such that $I \supset J \supset I^{(2)}$ and $J/I^{(2)} = (2P^\sharp)$ in (7.6.1). We write $I^\sharp = (y_2, y_3, y_4)$ and $J^\sharp = (y_2, y_3^2, y_4)$ at $P^\sharp$. Since y_3^2 must appear in α by the description of IIA points, we may assume

$$\alpha \equiv y_3^2 + y_1 y_2 \mod I^\sharp J^\sharp$$

by changing y_3 by $\lambda \cdot y_3$ $(\lambda \in \mathbb{C}^*)$. Thus (y_3, y_4, y_2) is a (1,2,2)-monomializing ℓ-basis of $I \supset J$ at P of second kind and $J^\sharp = (y_2, y_4)$. We see ℓ-isomorphisms $gr^1(\mathcal{O}, J) \simeq (P^\sharp)$, $gr^{2,0}(\mathcal{O}, J) \simeq (2P^\sharp)$, and $gr^{2,1}(\mathcal{O}, J) \simeq gr^1(\mathcal{O}, J)^{\tilde{\otimes} 2} \tilde{\otimes} (P^\sharp) \simeq (3P^\sharp)$ [Mori88, (8.10)]. Hence we have an ℓ-splitting

$$gr^2(\mathcal{O}, J) \simeq (2P^\sharp) \tilde{\oplus} (3P^\sharp), \tag{7.6.2}$$

where y_4 (resp. y_2) is an ℓ-free ℓ-basis of $(2P^\sharp)$ (resp. $(3P^\sharp)$) after a possible change of coordinates. From (7.6.2), one sees $H^1(C, gr^i(\omega^{\tilde{\otimes} j}, J)) = 0$ for all $i \geq 0$ and $j \leq 3$, whence

$$H^1(C, F^i(\omega^{\tilde{\otimes} j}, J)) = 0 \quad \text{for all } i \geq 0 \text{ and } j \leq 3 \tag{7.6.3}$$

by the contractibility of C. From (7.6.2) follows

$$H^0(\omega^{\tilde{\otimes} j} / F^2(\omega^{\tilde{\otimes} j}, J)) = 0 \quad \text{for } j = 2, 3.$$

Thus the induced homomorphism

$$H^0(\omega^{\tilde{\otimes} j}) \to H^0(gr^2(\omega^{\tilde{\otimes} j}, J)) = H^0(((2 - j)P^\sharp) \tilde{\oplus} ((3 - j)P^\sharp))$$

is a surjection for $j = 2, 3$. Let $D'' \in |2K_X|$ and $D''' \in |3K_X|$ be general members. Then from the above, it is easy to see that the natural map $I_{D''} \tilde{\oplus} I_{D'''} \to gr^2(\omega^{\tilde{\otimes} j}, J)$ is an ℓ-surjection, where the symbol I_Z denotes the defining ideal of a subscheme Z. Hence $J = I_{D''} + I_{D'''}$ since $J^\sharp$ is a c.i. ideal. Thus (7.2.2) is proved. By (7.6.3) with $j = 0$, we have a surjection

$$H^0(F^i(\mathscr{O}, J)) \twoheadrightarrow H^0(gr^i(\mathscr{O}, J)) \quad \text{for all } i.$$

Then a general member H_X of $|\mathscr{O}_X|$ containing C is defined at P by an equation β in which all of y_1y_3, y_2y_3, and y_4^2 appear. This is because y_1y_3 (resp. y_2y_3, y_4^2) is a part of a basis (at P) of $F^1(\mathscr{O}, J)$ (resp. $F^3(\mathscr{O}, J)$, $F^4(\mathscr{O}, J)$), which is generated by global sections. Hence we can apply (7.7.1). We note that $(\bullet)^2 = -1$ follows from $(C\cdot K_{H_X}) = (C\cdot K_X) < 0$. Thus (7.2.3) follows. We now prove (7.2.4). Let us assume only (7.2.1). By (6.4), we see $(K_X\cdot C) = -1/4 < 0$. We only have to prove the contractibility of C by (7.2.1.1), which follows from the ampleness of

$$gr_C^4\mathscr{O} \supset \tilde{S}^4 gr_C^1\mathscr{O} \simeq \mathscr{O}(1)^{\oplus 4} \oplus \mathscr{O}(2).$$

Thus (7.2.4) is proved. □

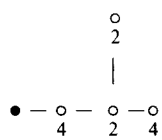
(7.6.4) **Example.** Let $Z \supset C$ be a germ of a smooth 3-fold along $C \simeq \mathbb{P}^1$ such that $N_{C/Z} \simeq \mathscr{O}_C \oplus \mathscr{O}_C$. Let $P \in C$ and let (z_1, z_2, z_3) be coordinates of (Z, P) such that $(C, P) = z_1$-axis. Let $(X, P) \supset (C, P)$ be a *IIA* point as in (7.5) with $\alpha \equiv y_1y_2 \mod (y_2, y_3, y_4)^2$. For suitable ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, $(y_1^4, y_1^2y_4, y_1y_3)$ form coordinates for $U = (X, P) \cap \{\varepsilon_1 < |y_1^4| < \varepsilon_2\}$ by the implicit function theorem. Thus $z_1 = y_1^4$, $z_2 = y_1^2y_4$, and $z_3 = y_1y_3$ patch (X, P) and $Z - (Z, P) \cap \{|z_1| \le \varepsilon_1\}$ along U. This $X \supset C$ is an isolated extremal nbd of type *IIA* by (7.2).

(7.7) **Computation.** Let (D, P) be a normal surface singularity

$$(D, P) = (y_1, y_2, y_3, y_4; \alpha, \beta)/\mathbb{Z}_4(1, 1, 3, 2; 2, 0) \supset C = y_1\text{-axis}/\mathbb{Z}_4.$$

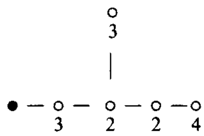
Let σ be the $\mathbb{Z}$-wt $\sigma(y_1, y_2, y_3, y_4) = (1, 1, 3, 2)$ (T.7). We have the configuration of $\Delta(D \supset C)$ in each of the following cases.

(7.7.1) $\alpha_{\sigma=2} = y_1y_2$ and the coefficient of y_3^2 in α is nonzero, and the coefficients of y_1y_3, y_2y_3 and y_4^2 in β are nonzero. Then $\Delta(D \supset C)$ consists of smooth rational curves and C' intersecting transversely with the following configuration.



(7.7.2) $\alpha_{\sigma=2} = y_1y_2$ and the coefficient of y_3^2 in α is nonzero, the coefficient of y_1y_3 in β is zero, and the coefficients of y_2y_3, $y_1^2y_4$, and y_4^2 in β are nonzero. Then $\Delta(D \supset C)$ consists of smooth rational curves and C'

intersecting transversely with the following configuration.



(7.8) In this paragraph, we will prove (7.3.1). By (7.5) and [Mori88, (2.16)], we have $\ell(P) = 1$, $\alpha \equiv y_1 y_2$, and that y_3 and y_4 form an ℓ-free ℓ-basis of $gr^1_C\mathscr{O}$ at P. We note that X has at most one singular point outside of P, which is a type III point (say, R) such that $i_R(1) = 1$ [Mori88, (A.3) and (B.1)]. Thus X has exactly i singular point ($i = 0$, or 1), which is of type III if any. We also have $\deg gr^1_C\mathscr{O} = -i$ under the notation of (7.3.1).

(7.8.1) We first prove (7.3.1.1) when $i = 0$. By (7.8), we have $gr^1_C\mathscr{O} \simeq \mathscr{O}(1) \oplus \mathscr{O}(-1)$ because $H^1(gr^1_C\mathscr{O}) = 0$. In view of (2.8), it is enough for (7.3.1.1) to show that $X \supset C$ is not isolated (a contradiction) assuming an ℓ-isomorphism

(7.8.1.1) $$gr^1_C\mathscr{O} \simeq (1 + 2P^\sharp)\tilde{\oplus}(-1 + P^\sharp),$$

where y_4 and y_3 form ℓ-free ℓ-bases of $(1+2P^\sharp)$ and $(-1+P^\sharp)$. Let J be the ideal such that $I \supset J \supset I^{(2)}$ and $J/I^{(2)} = (1 + 2P^\sharp)$. Then $J^\sharp = (y_4, y_2, y_3^2)$. Since y_3^2 must appear in α by the description of IIA points, we may assume

$$\alpha \equiv y_3^2 + y_1 y_2 \mod I^\sharp J^\sharp$$

by changing y_3 by $\lambda \cdot y_3$ ($\lambda \in \mathbb{C}^*$). Thus (y_3, y_4, y_2) is a (1,2,2)-monomializing ℓ-basis of $I \supset J$ at P of the second kind and $J^\sharp = (y_2, y_4)$. By [Mori88, (8.10)], we see ℓ-isomorphisms $gr^1(\mathscr{O}, J) \simeq (-1 + P^\sharp)$, $gr^{2,0}(\mathscr{O}, J) \simeq (1 + 2P^\sharp)$, and $gr^{2,1}(\mathscr{O}, J) \simeq gr^1(\mathscr{O}, J)^{\tilde{\otimes}2}\tilde{\otimes}(P^\sharp) \simeq (-2 + 3P^\sharp)$ and an ℓ-exact sequence

(7.8.1.2) $$0 \to (-2 + 3P^\sharp) \to gr^2(\mathscr{O}, J) \to (1 + 2P^\sharp) \to 0.$$

Since we are going to show that our X is not isolated, we may replace X with its nearby deformation keeping our hypotheses including (7.8.1.1). To be specific, we may assume the most general possibility under (7.8.1.2):

(7.8.1.3) $$gr^2(\mathscr{O}, J) \simeq (2P^\sharp)\tilde{\oplus}(-1 + 3P^\sharp).$$

This is because the twisted extension $X_t^\circ \supset C_t$ of the trivial deformation $(X, P) \times \mathbb{C}^1_t \supset (C, P) \times \mathbb{C}^1_t$ by $(y_1^2 y_4 + t y_1^{-1} y_2, y_1 y_3)$ gives the general case (cf. [Mori88, (1b.8)]) for general t with $|t| \ll 1$.

The idea of our proof is to show that a general member of $|\mathscr{O}_Y|$ containing Q has only a rational double point at Q since it implies a contradiction that (Y, Q) is Gorenstein. We will begin by finding an auxiliary normal member $E \in |-K_X|$ containing C as well as the usual transversal $D \in |-K_X|$.

(7.8.1.4) Let $D = \{y_1 = 0\}/\mathbb{Z}_4 \in |-K_X|$. We note

(7.8.1.5) $$F^2(\mathscr{O}_X(D), J)/F^2(\mathscr{O}_X, J) \simeq (gr^2(\mathscr{O}_X(D), J))\tilde{\otimes}\mathscr{O}_D \simeq \mathbb{C}\cdot(y_2/y_1).$$

We claim the surjection

(7.8.1.6) $$H^0(X, F^2(\mathscr{O}_X(D), J)) \twoheadrightarrow F^2(\mathscr{O}_X(D), J)/F^2(\mathscr{O}_X, J).$$

Indeed, we have the equality $H^1(X, F^2(\mathscr{O}_X, J)) = 0$ from $H^j(gr^1(\mathscr{O}_X, J)) = 0$ (all j) and $H^1(X, F^1(\mathscr{O}_X, J)) = H^1(X, I) = 0$, which follows from the exact sequence $0 \to I \to \mathscr{O}_X \to \mathscr{O}_C \to 0$. Thus (7.8.1.6) follows.

(7.8.1.7) By (7.8.1.6), a general global section s of $F^2(\mathscr{O}_X(D), J)$ restricts to $\lambda\cdot y_2/y_1 \in F^2(\mathscr{O}_X(D), J)/F^2(\mathscr{O}_X, J)$ for some $\lambda \in \mathbb{C}^*$. This s is of the form $(y_2 + y_1(\cdots))/y_1$, and it induces a global section $\bar{s}$ of $gr^2(\mathscr{O}(D), J)$, which is a part of an ℓ-free ℓ-basis at P. By (7.8.1.3), $\bar{s}$ is a nowhere vanishing section of $gr^2(\mathscr{O}(D), J)$ and we have an ℓ-isomorphism

$$gr^2(\mathscr{O}_X(-D), J) \simeq (3P^\sharp)\tilde{\oplus}\mathscr{O}_C\bar{s}.$$

Let $E \in |-K_X|$ be the divisor defined by $s = 0$. By the above ℓ-isomorphism, we see two ℓ-isomorphisms

$$gr_C^1(\mathscr{O}_E) \simeq gr^1(\mathscr{O}, J) \simeq (-1 + P^\sharp),$$
$$gr_C^2(\mathscr{O}_E) \simeq \{gr^2(\mathscr{O}(D), J)/\mathscr{O}_C\bar{s}\}\tilde{\otimes}\mathscr{O}(-D) \simeq (2P^\sharp).$$

Since the image of $\bar{s}$ in $gr_C^1\mathscr{O}$ is nonzero, E is smooth at general points of C. By the construction of E, E is smooth outside of C, whence E is normal. Since $J^\sharp = (sy_1, y_4)$, $2C^\sharp$ is a Cartier divisor of $E^\sharp$ defined by $y_4 = 0$. Thus $\mathscr{O}_E(2C)$ is an ℓ-invertible $\mathscr{O}_E$-module and $\mathscr{O}_E(-2C)\tilde{\otimes}\mathscr{O}_C \simeq (2P^\sharp)$. By $H^1(\mathscr{O}_E) = 0$, we see $\operatorname{Pic} E \simeq \operatorname{Pic} C \simeq \mathbb{Z}$. Thus $2D \cap E + 2C \sim 0$ on E and $y_1^2 y_4 = 0$ is its equation in (E, P). Since $H^0(\mathscr{O}_X) \simeq H^0(\mathscr{O}_E)$, we have a global section $s_1 \in H^0(\mathscr{O}_X)$ such that $s_1 \equiv (\text{unit})\cdot y_1^2 y_4 \mod (sy_1)$. Also by the surjection $H^0(\mathscr{O}_X) \twoheadrightarrow \mathscr{O}_D$, we have global sections $s_2, s_3 \in H^0(\mathscr{O}_X)$ such that $s_2 \equiv y_4^2$, $s_3 \equiv y_2 y_3 \mod (y_1)$. The natural map $H^0(I) \otimes \mathscr{O}_C \to gr_C^1\mathscr{O}$ factors through $H^0(I) \otimes \mathscr{O}_C \to (1 + 2P^\sharp)$ (7.8.1.1), which is a surjection at P. Thus a general member H_X through C of $|\mathscr{O}_X|$ is normal and has exactly one point of type A outside of P as singularities. Now we study (H_X, P). If we replace y_2 by sy_1 (we note $sy_1 \equiv y_2 \mod (y_1)$), then the equation of H_X satisfies (7.7.2), because of the sections s_1, s_2, and s_3. As in (7.6), we can compute $\Delta(f(H_X))$. In our case, it is A_1 since C is contractible. This is a contradiction as mentioned earlier. This completes (7.8.1).

(7.8.2) We prove (7.3.1.2) when $i = 0$. As in (7.8.1), we may assume that y_3 and y_4 form ℓ-free ℓ-bases of $(1 + P^\sharp)$ and $(-1 + 2P^\sharp)$ in (7.3.1.1). We have ℓ-isomorphisms $gr^1(\mathscr{O}, J) \simeq (-1 + 2P^\sharp)$ and $gr^{2,0}(\mathscr{O}, J) \simeq (1 + P^\sharp)$. We note that y_4^2 does not appear in α because $wt y_4^2 \not\equiv wt\alpha$. Thus by $I^\sharp = (y_2, y_3, y_4)$ and $J^\sharp = (y_2, y_3, y_4^2)$, we may further assume $\alpha \equiv y_1 y_2 \mod I^\sharp J^\sharp$ after changing coordinates. Hence $y_2 \in F^3(\mathscr{O}, J)$ and we have an ℓ-isomorphism $gr^{2,1}(\mathscr{O}, J) \simeq gr^1(\mathscr{O}, J)^{\tilde{\otimes}2} \simeq (-1)$, which has an ℓ-free ℓ-basis y_4^2. Thus we have an ℓ-exact sequence

$$(7.8.2.1) \qquad 0 \to (-1) \to gr^2(\mathscr{O}, J) \to (1 + P^\sharp) \to 0.$$

In view of (2.8), we note that (7.3.1.2) is the most general case under (7.8.2.1). It is enough for (7.3.1.2) to derive a contradiction assuming an ℓ-isomorphism

$$(7.8.2.2) \qquad gr^2(\mathscr{O}, J) \simeq (1)\tilde{\oplus}(-1 + P^\sharp),$$

which is the second most general case. This is because the twisted extension $X_t^\circ \supset C_t$ of the trivial deformation $(X, P) \times \mathbb{C}_t^1 \supset (C, P) \times \mathbb{C}_t^1$ by $(y_1y_3 + ty_4^2, y_4)$ satisfies (7.8.2.2) if (7.8.2.1) is ℓ-split for $X \supset C$ (cf. [Mori88, (1b.8)]).

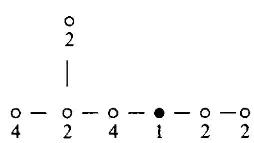
The idea of our proof is to show that $(f(H_X), Q)$ is a rational double point for a general member $H_X \in |\mathscr{O}_X|$ containing C.

Let $D = \{y_1 = 0\}/\mathbb{Z}_4 \in |-K_X|$. Because of (7.8.2.1), the ℓ-summand (1) in (7.8.2.2) is generated at P by an element $u \in \mathscr{O}_{X,P}$ such that $u \equiv y_4^2 + y_3(y_1 + \cdots)$ after replacing y_3 by $\lambda \cdot y_3$ for some $\lambda \in \mathbb{C}^*$. Since $H^0(\mathscr{O}_X) \to \mathscr{O}_D$ is a surjection, there is a section $s_1 \in H^0(\mathscr{O}_X)$ such that $s_1 \equiv u \bmod (y_1)$. Since $s_1(P) = 0$, we see $s_1 \in H^0(I)$ and $s_1 \in H^0(J)$ by $H^0(gr^1(\mathscr{O}, J)) = 0$. Thus s_1 induces a section (unit)$\cdot u$ of $gr^2(\mathscr{O}, J)$ at P. In particular, y_1y_3 and y_4^2 appear in s_1. Let $s_2 \in H^0(\mathscr{O}_X)$ be a section extending $y_2y_3 \in \mathscr{O}_D$. We see that $s_2 \in H^0(J)$ and that y_2y_3 appears in s_2. Let $s \in H^0(I)$ be a general section. Then, as we saw above, $s \in H^0(J)$ and the induced section $\bar{s}$ of $gr^2(\mathscr{O}, J)$ is a basis of the ℓ-summand (1) of (7.8.2.2) at P. Thus its image in $gr_C^1\mathscr{O}$ has exactly one simple zero outside P. Hence $H_X = \{s = 0\} \in |\mathscr{O}_X|$ has exactly one singular point outside of P, which we call R. As for (H_X, P), we can apply (7.7.1) by the above.

We claim that (H_X, R) is an A_k point for some $k \geq 2$. Since (7.8.2.2) is the splitting for (7.8.2.1), we can take coordinates (z_1, z_2, z_3) of (X, R) such that $(C, R) = z_1$-axis and such that z_3 is a basis for both of $(P^\sharp)$ in (7.3.1.1) at R and (1) in (7.8.2.2) at R. At R, we have $I = (z_2, z_3)$ and $J = (z_3, z_2^2)$. Whence

$$(7.8.2.3) \qquad s \equiv (\text{unit}) \cdot z_1z_3 \bmod (z_2^3, z_2z_3, z_3^2),$$

which proves our claim. Since C is contractible, it is easy to see that (H_X, R) is an A_2 point and $\Delta(H_X)$ consists of smooth rational curves intersecting transversely with the following configuration.



This means that $\Delta(f(H_X))$ is A_1, which is a contradiction as mentioned earlier. This completes (7.8.2).

(7.8.3) We will treat the case $i = 1$ for (7.3.1) until the end of (7.8). Let R be the type III point, which we express as

$$(X, R) = (z_1, z_2, z_3, z_4; \gamma) \supset (C, R) = z_1\text{-axis},$$

using an equation γ such that $\gamma \equiv z_1 z_2 \mod (z_2, z_3, z_4)^2$.

(7.8.4) In view of (2.8) and (7.8), it is enough for (7.3.1.1) to derive a contradiction assuming an ℓ-isomorphism

$$gr_C^1\mathscr{O} \simeq (-1 + P^\sharp)\tilde{\oplus}(2P^\sharp),$$

where y_3 and y_4 (resp. z_3 and z_4) form ℓ-free ℓ-bases (resp. free bases) of $(-1 + P^\sharp)$ and $(2P^\sharp)$ at P (resp. R).

The idea of our proof is to construct a nearby deformation of $X \supset C$ that does not satisfy (7.3.1.1) in case $i = 0$. Since it was proved in (7.8.1), we will have a contradiction as required.

Let $(X_t, R) \supset (C_t, R)$ be the deformation of $(X, R) \supset (C, R)$ given by the equation $\gamma + tz_4 = 0$. Let $X_t^\circ \supset C_t$ be its twisted extension by (z_3, z_4). We now work on $X_t^\circ \supset C_t$ for sufficiently small $t \in \mathbb{C}^*$, which we denote by $\overline{X} \supset \overline{C}$. We also use the same y_j and z_k in the same sense for the new $\overline{X} \supset \overline{C}$.

The main point is that $\overline{X}$ is now smooth outside of P and we have $t \cdot z_4 + z_1 z_2 \in (z_2, z_3, z_4)^2$. Thus we have $z_4 \equiv (\text{unit}) \cdot z_1 z_2$ in $gr_{\overline{C}}^1\mathscr{O}$ at R. Hence from $gr_{\overline{C}}^1\mathscr{O} \supset gr_C^1\mathscr{O}$, we get an ℓ-isomorphism

$$gr_{\overline{C}}^1\mathscr{O} \simeq (-1 + P^\sharp)\tilde{\oplus}(1 + 2P^\sharp).$$

This contradicts (7.8.1) as mentioned earlier and (7.3.1.1) is now proved.

(7.8.5) It remains to prove (7.3.1.2). Without loss of generality, we will assume that y_3 and y_4 (resp. z_3 and z_4) form an ℓ-free ℓ-bases (resp. free bases) of $(P^\sharp)$ and $(-1 + 2P^\sharp)$ at P (resp. R). We note that J now defined by (7.3.1.1) in (7.3.1) satisfies $gr^1(\mathscr{O}, J) \simeq (-1 + 2P^\sharp)$, $gr^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$, and the equality at P: $gr^{2,1}(\mathscr{O}, J) = gr^1(\mathscr{O}, J)^{\tilde{\otimes}2}$, which is proved by the same argument as (7.8.2). If z_4^2 appears (resp. does not appear) in γ, then we have the equality at R: $gr^{2,1}(\mathscr{O}, J) = gr^1(\mathscr{O}, J)^{\otimes 2}(R)$ (resp. $gr^1(\mathscr{O}, J)^{\otimes 2}$). We first finish the proof of (7.3.1.2) assuming that z_4^2 appears in γ. In this case, we have an ℓ-isomorphism $gr^{2,1}(\mathscr{O}, J) \simeq (0)$ and the ℓ-exact sequence

$$0 \to (0) \to gr^2(\mathscr{O}, J) \to (P^\sharp) \to 0$$

is ℓ-split, whence (7.3.1.2) follows.

Now we will derive a contradiction assuming that z_4^2 does not appear in γ, that is, $gr^{2,1}(\mathscr{O}, J) = gr^1(\mathscr{O}, J)^{\otimes 2}$ at R. In this case, we have an ℓ-isomorphism $gr^{2,1}(\mathscr{O}, J) \simeq (-1)$. Then we have

(7.8.5.1) $$gr^2(\mathscr{O}, J) \simeq (0)\tilde{\oplus}(-1 + P^\sharp),$$

because otherwise we will have $gr^2(\mathscr{O}, J) \simeq (P^\sharp)\tilde{\oplus}(-1)$, which implies a contradiction $H^1(\omega/F^3(\omega, J)) \neq 0$. By changing coordinates, we may further assume that z_3 is also a basis of (0) in (7.8.5.1). The rest of the argument is very similar to (7.8.2). To be specific, we can similarly prove that a general section $s \in H^0(I)$ defines a surface H_X which is smooth outside of $\{P, R\}$, all of y_1y_3, y_2y_3 and y_4^2 appear in s at P, and z_3 appears in s at R. Like (7.8.2.3), we have

$$s \equiv (\text{unit}) \cdot z_3 \bmod (z_2, z_3, z_4)^2.$$

Since z_4^2 does not appear in γ, it is easy to see (H_X, R) is an A_k point for some $k \geq 2$. The rest of the argument is the same as (7.8.2). This completes (7.8).

(7.9) In this paragraph, we will prove the rest of (7.3). To prove (7.3.4), we treat formal 3-folds X along C satisfying the conditions in (7.3.1) for a while. We will still use the notation (7.5) at P.

(7.9.1) **Proposition.** *Let $X \supset C$ be a formal 3-fold along $C \simeq \mathbb{P}^1$ that need not be an extremal nbd. Assume also that $X \supset C$ satisfies the conditions in* (7.3.1). *Then $X \supset C$ satisfies the condition in* (7.3.2).

(7.9.1.1) Since J is a C-laminal ideal of width 2, we have an inclusion for arbitrary $n \geq 1$:

$$gr^n(\mathscr{O}, J) \supset (\tilde{S}^{[n/2]} gr^2(\mathscr{O}, J))\tilde{\otimes} gr^{n-2[n/2]}(\mathscr{O}, J).$$

By (7.3.1.2) and $gr^1(\mathscr{O}, J) \simeq (-1 + 2P^\sharp)$ (7.3.1.1), we see that all the ℓ-summands of $gr^n(\mathscr{O}, J)$ have qldeg $\geq$ qldeg$(-1+2P^\sharp) = -1/2$ for all $n \geq 3$. Thus, by (6.4), we have ℓ-isomorphisms

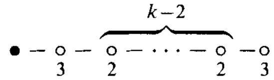
$$\begin{aligned} gr^0(\omega, J) &\simeq (-1 + 3P^\sharp), \\ gr^1(\omega, J) &\simeq (-1 + P^\sharp), \\ gr^2(\omega, J) &\simeq (0)\tilde{\oplus}(-1 + 3P^\sharp), \end{aligned}$$

and we see that all the ℓ-summands of $gr^n(\omega, J)$ have qldeg $\geq -3/4$ for all $n \geq 3$. Hence we see that $H^0(X, \omega) \simeq H^0(X, F^2(\omega, J))$ and a surjection $H^0(X, F^2(\omega, J)) \twoheadrightarrow H^0(C, gr^2(\omega, J)) \simeq \mathbb{C}$ and a vanishing $H^1(X, \omega_X) = 0$. Let s be a general global section of ω_X and $E \in |K_X|$ be the member defined by $s = 0$. We will study the singularities of E.

(7.9.1.2) **Lemma.** *The term y_3 appears in the equation of $E^\sharp$ at P and we have an isomorphism*

$$(E, P) = (y_1, y_2, y_4; y_1y_2 + y_4^k)/\mathbb{Z}_4(1, 1, 2) \supset C = y_1\text{-}axis/\mathbb{Z}_4,$$

where k is the axial multiplicity of X at P. Furthermore, $\Delta((E, P) \supset (C, P))$ consists of smooth rational curves and C' intersecting transversely with the following configuration:



Proof. As in (7.8), we see that ℓ-free ℓ-bases of the first and the second factors of $gr^2(\mathscr{O}, J)$ are of the forms $y_3\cdot(\text{unit}) + \cdots$ and $y_4\cdot(\text{unit}) + \cdots$, respectively. Since s generates the first factor of the ℓ-splitting of $gr^2(\omega, J)$ in (7.9.1.1), the first assertion of (7.9.1.2) follows. Since the equation of $E^\sharp$ has $\text{wt} \equiv 3 \mod 4$, no powers of y_4 appear in it. It is easy to compute the configuration and we have (7.9.1.2). □

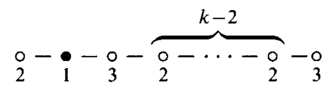
(7.9.1.3) **Lemma.** *If X does not have a type III point on C (i.e., $i = 0$ in (7.3.1)), then E has exactly one singular point (say, R) outside of P. Furthermore (E, R) is a type A point.*

Proof. By the natural surjection $gr^2(\mathscr{O}, J) \twoheadrightarrow gr^{2,0}(\mathscr{O}, J) \simeq (1 + P^\sharp)$, we see $gr^{2,1}(\mathscr{O}, J) \simeq (-1)$ by (7.3.1.2). Hence the image $\bar{s}$ of s in $(1 + P^\sharp)\tilde{\otimes}\omega$ is nonzero and generates it at P. Thus $\bar{s}$ vanishes at exactly one point R outside of P, and R is a simple zero. This proves (7.9.1.3). □

(7.9.1.4) **Lemma.** *If X has a type III point R on C (i.e., $i = 1$ in (7.3.1)), then E is smooth outside of $\{P, R\}$. Furthermore (E, R) is a type A point.*

Proof. We see ℓ-isomorphisms $gr^{2,0}(\mathscr{O}, J) \simeq (P^\sharp)$ and $gr^{2,1}(\mathscr{O}, J) \simeq (0)$ as in the proof of (7.9.1.3). Thus the image $\bar{s}$ of s globally generates $gr^{2,0}(\mathscr{O}, J)\tilde{\otimes}\omega \simeq (0)$. Hence using the notation of (7.8.3) at R, we see that z_3 or z_4 appears in the equation of (E, R) in (X, R). Since (X, R) is defined by $z_1 z_2 + \cdots = 0$ in $(\mathbb{C}^4, 0)$, we see that (E, R) is a type A point. □

(7.9.1.5) **Lemma.** *The point (E, R) is always of type A_1, and $\Delta(E \supset C)$ consists of smooth rational curves and C' intersecting transversely with the following configuration.*



Furthermore, we have $K_X|_E \not\sim k \cdot C$ and $2K_X|_E \sim 2k \cdot C$ among Weil divisors on E.

Proof. By $(K_E \cdot C) = (2K_X \cdot C) = -1/2 < 0$ (6.4), we have $(\bullet^2) = -1$. Putting together the results of the above lemmas, we see that the configuration of (7.9.1.5) is the only possibility for C to be contractible to a non-Gorenstein point (Y, Q). Hence (E, R) is of type A_1. From the configuration, it is easy to compute $(C^2) = -1/4k$. Since $k\cdot C^\sharp = (y_2)$ in $(E^\sharp, P^\sharp)$, we see that $\mathscr{O}_E(2k\cdot C)$ is ℓ-invertible at P and $\text{qldeg}(\mathscr{O}_E(k\cdot C), P) = 1$ and $\text{qldeg}(\mathscr{O}_X(2K_X), P) = 3$. Thus we have (7.9.1.5) by $(kC \cdot C) = (K_X \cdot C)$. □

(7.9.1.6) **Lemma.** *If D is a general member of $|2K_X|$, then $D \cdot E = 2k \cdot C$.*

Proof. Since $\mathscr{O}(K_X)$ is an ℓ-invertible $\mathscr{O}_X$-module, we have an ℓ-exact sequence

$$0 \to \mathscr{O}_X(K_X) \to \mathscr{O}_X(2K_X) \to \mathscr{O}_E(2k \cdot C) \to 0.$$

Since $(C^2) < 0$, we have $H^0(\mathscr{O}_E(2k \cdot C)) = \mathbb{C}$. By $H^1(X, \mathscr{O}_X(K_X)) = 0$ above (7.9.1.1), we have a surjection $H^0(X, \mathscr{O}_X(2K_X)) \twoheadrightarrow H^0(E, \mathscr{O}_E(2k \cdot C)) = \mathbb{C}$. Hence we have (7.9.1.6). □

Thus the proof of (7.9.1) is completed.

(7.9.2) **Lemma.** (7.3.4) *holds.*

Proof. Let $X \supset C$ be a 3-fold satisfying the conditions in (7.3.1). By (7.9.1), C is contractible on the completion of X, whence contractible on X. Hence $X \supset C$ is an extremal nbd and (7.3.4) holds. □

(7.9.3) **Proposition.** (7.3.3) *holds.*

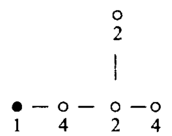
Let $X \supset C$ be an extremal nbd satisfying the conditions in (7.3.1). We will prove (7.3.3) in several steps.

As in (7.9.1.1), we have $H^1(gr^n(\mathcal{O}, J)) = 0$ for all n. Thus

$$H^1(F^n(\mathcal{O}, J)/F^m(\mathcal{O}, J)) = 0$$

for all $n \geq m$ and $H^1(F^n(\mathcal{O}, J)) = 0$ for all n because C is contractible. Hence we have a surjection $H^0(F^2(\mathcal{O}, J)) \twoheadrightarrow H^0(gr^2(\mathcal{O}, J))$. Let s be a general element of $H^0(I) = H^0(F^2(\mathcal{O}, J))$ and $H_X \in |\mathcal{O}_X|$ be the member defined by $s = 0$. We will study the singularities of H_X as in (7.9.1.2)–(7.9.1.5).

(7.9.3.1) **Lemma.** *The terms* y_1y_3, y_2y_3, *and* y_4^2 *appear in* s *at* P *and* $\Delta((H_X, P) \supset (C, P))$ *consists of smooth rational curves and* C' *intersecting transversely with the following configuration.*



Proof. As in (7.8), the ℓ-free ℓ-basis of the first factor $(P^\sharp)$ of $gr^2(\mathcal{O}, J)$ is of the form $y_3\cdot(\text{unit}) + \cdots$. Hence y_1y_3 appears in s at P. Let $D = \{y_1 = 0\}/\mathbb{Z}_4 \in |-K_X|$. Since $H^0(\mathcal{O}_X) \twoheadrightarrow \mathcal{O}_D$, the elements y_2y_3 and y_4^2 of $\mathcal{O}_D$ extend to global sections s_1 and s_2 of $\mathcal{O}_X$. Since $s_1(P) = s_2(P) = 0$, we see $s_1, s_2 \in H^0(I)$. Since y_2y_3 and y_4^2 appear in s_1 and s_2 at P, they also appear in s at P. The rest follows from (7.7.1). □

Since the rest is the same as (7.9.1.3)–(7.9.1.5), we only list the corresponding statements.

(7.9.3.2) **Lemma.** *If X does not have a type III point on C (i.e., $i = 0$ in* (7.3.1)), *then H_X has exactly one singular point (say, R) outside of P. Furthermore (H_X, R) is a type A point.*

(7.9.3.3) **Lemma.** *If X has a type III point R on C (i.e., $i = 1$ in* (7.3.1)), *then H_X is smooth outside of $\{P, R\}$. Furthermore (H_X, R) is a type A point.*

(7.9.3.4) **Lemma.** *The point (H_X, R) is always of type A_1 and $\Delta(H_X \supset C)$ is as in* (7.3.3).

Thus the proofs of (7.9.3) and hence (7.3) are completed.

(7.9.4) **Example.** Let $i = 0$ or 1. Let $Z \supset C$ be a germ of a smooth 3-fold along $C \simeq \mathbb{P}^1$ such that $N^*{}_{C/Z} \simeq \mathcal{O}_C(1-i) \oplus \mathcal{O}_C(-1)$. Let P and $R \in C$ be two distinct points and let (z_1, z_2, z_3) (resp. (u_1, u_2, u_3)) be coordinates of

(Z, P) (resp. (Z, R)) such that $(C, P) = z_1$-axis (resp. $(C, R) = u_1$-axis) and z_2 and z_3 (resp. u_2 and u_3) are generators of the first and the second summands of $N^*{}_{C/Z}$ above, respectively. Let

$$(X, R) = (x_1, x_2, x_3, x_4; x_1x_2 + x_3^2 + x_4^2) \supset C = x_1\text{-axis}.$$

Let $(Y, P) \supset (C, P)$ be a IIA point as in (7.5) with

$$\alpha \equiv y_1y_2 \bmod (y_2, y_3, y_4)^2.$$

For suitable ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, $(y_1^4, y_1^2y_4, y_1y_3)$ form coordinates for $U = (Y, P) \cap \{\varepsilon_1 < |y_1^4| < \varepsilon_2\}$ by the implicit function theorem. We treat two cases.

(7.9.4.1) Case $i = 0$. In this case, $z_1 = y_1^4$, $z_2 = y_1y_3 + y_1^{-4}y_4^2$, and $z_3 = y_1^2y_4$ patch (Y, P) and $Z - (Z, P) \cap \{|z_1| \leq \varepsilon_1\}$ along U. Then this $Y \supset C$ is an isolated extremal nbd of type IIA satisfying (7.3.1.1) and (7.3.1.2) by (7.2).

(7.9.4.2) Case $i = 1$. In this case, $z_1 = y_1^4$, $z_2 = y_1y_3$, and $z_3 = y_1^2y_4$ patch (Y, P) and V, which is the complement of two closed neighbourhoods of P and R in Z, and $u_1 = x_1$, $u_2 = x_3$, and $u_3 = x_4$ patch (X, R) and V. They patch together to an isolated extremal nbd of type IIA satisfying (7.3.1.1) and (7.3.1.2) by (7.2).

(7.10) In this paragraph, we will prove (7.4.1) in several steps. We use the notation of (7.5) at P. We note that X has at most one singular point outside of P, which is a type III point, say R, such that $i_R(1) = 1$ as we mentioned in (7.8). Thus we have $\deg gr_C^1\mathscr{O} = -1 - i$, where i is the number of singular points of X outside of P.

We first prove (7.4.1.1) in the case where $\ell(P) = 3$ and $\deg gr_C^1\mathscr{O} = -1$. By (7.5), we have $gr_C^1\mathscr{O} \simeq \mathscr{O} \oplus \mathscr{O}(-1)$ by $H^1(gr_C^1\mathscr{O}) = 0$ and see that y_2 and y_4 form an ℓ-free ℓ-basis of $gr_C^1\mathscr{O}$ at P. In view of (2.8), it is enough to prove the following.

(7.10.1) **Lemma.** *Let $X \supset C$ be an extremal nbd with a type IIA point P of $\ell(P) = 3$ and with an ℓ-isomorphism*

$$gr_C^1\mathscr{O} \simeq (3P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp)$$

such that y_2 and y_4 are ℓ-free ℓ-bases of $(3P^\sharp)$ and $(-1 + 2P^\sharp)$ under the notation of (7.5). Then X is not isolated.

(7.10.1.1) The proof will be done in several steps. Let J be the ideal such that $I \supset J \supset I^{(2)}$ and $J/I^{(2)} = (3P^\sharp)$ in the ℓ-isomorphism of (7.10.1). Then $J^\sharp = (y_2, y_3, y_4^2)$. Since y_4^2 does not appear in α, we have $\alpha \equiv y_1^3y_3 + y_1^2y_4^2 \cdot \gamma(y_1^4)$ mod $I^\sharp J^\sharp$ for some $\gamma(T) \in \mathbb{C}\{T\}$. Since we are going to show that X is not isolated, we may replace X by its nearby deformation, which satisfies the ℓ-isomorphism in (7.10.1).

(7.10.1.2) Let σ be the $\mathbb{Z}$-wt $\sigma(y_1, y_2, y_3, y_4) = (1, 1, 3, 2)$. Then $\sigma(\alpha) \equiv wt\alpha \equiv 2$ mod 4 and $\sigma(\alpha) = 6$ since none of y_1y_2 and y_4 appear in α. Therefore $\alpha_{\sigma=6}(y_1, 0, y_3, y_4) = 0$ at best defines a simple elliptic singularity.

Lemma. *We may assume that the axial multiplicity of* (X, P) *is* 3, $\gamma(0) \neq 0$, *and* $\alpha_{\sigma=6}(y_1, 0, y_3, y_4) = 0$ *defines a simple elliptic singularity of degree* 1. *Furthermore, on the smooth elliptic curve* $\alpha_{\sigma=6}(y_1, 0, y_3, y_4) = 0$ *in* $\mathbb{P}(1, 3, 2)$, $y_1 y_3 = 0$ *defines a divisor* $2(1:0:0) + (0:1:a_1) + (1:0:a_2)$ *of degree* 4 *for some* $a_1, a_2 \in \mathbb{C}^*$.

Proof. By the description of IIA points, we see that $y_1^3 y_3$ and y_3^2 appear in $\alpha_{\sigma=6}$. Let $(X_t, P) \supset (C, P) = y_1\text{-axis}/\mathbb{Z}_4$ be the deformation of $(X, P) \supset (C, P)$ inside $(\mathbb{C}^4, 0)$ defined by

$$\alpha_t \equiv \alpha + t(y_1^2 y_4^2 + y_4^3) = 0.$$

Then P is a IIA point of $X_t \supset C$ with $\ell(P) = 3$ and axial multiplicity 3. It is easy to see that $(\alpha_t)_{\sigma=6}(y_1, 0, y_3, y_4) = 0$ defines a simple elliptic singularity of degree 1 for general t. The twisted extension $X_t^\circ \supset C_t$ of $(X_t, P) \supset (C, P)$ by $(y_1^3 y_2, y_1^2 y_4)$ satisfies the ℓ-isomorphism in (7.10.1) since X_t and X can be identified modulo $I^{(2)}$. The last assertion is a simple computation, because the terms y_1^6 and $y_1^4 y_4$ are missing in α_t. □

(7.10.1.3) By $\gamma(0) \neq 0$ (7.10.1.2) and $y_1 y_3 + y_4^2 \cdot \gamma(y_1^4) \equiv 0$ in $gr^2(\mathcal{O}, J)$, we see that $gr^{2,1}(\mathcal{O}, J) \simeq gr^1(\mathcal{O}, J)^{\tilde{\otimes} 2} \tilde{\otimes} (P^\sharp) \simeq (-1 + P^\sharp)$ with ℓ-free ℓ-basis y_3 at P. Thus we have an ℓ-exact sequence

$$0 \to (-1 + P^\sharp) \to gr^2(\mathcal{O}, J) \to (3P^\sharp) \to 0.$$

Lemma. *We may further assume an* ℓ*-isomorphism*

$$gr^2(\mathcal{O}, J) \simeq (P^\sharp) \tilde{\oplus} (-1 + 3P^\sharp).$$

Proof. The twisted extension of the trivial deformation $(X, P) \times \mathbb{C}_t^1 \supset (C, P) \times \mathbb{C}_t^1$ by $(y_1^3 y_2 + t y_1 y_3, y_1^2 y_4)$ satisfies the ℓ-isomorphism in (7.10.1) and the above ℓ-exact sequence is not ℓ-split for X_t with general t. Hence we have the ℓ-isomorphism by (2.8). □

(7.10.1.4) **Lemma.** *There is an element* $s \in H^0(I)$ *such that* $y_1 y_3$ *appears in* s *at* P *and* y_4^2 *does not appear in* s *at* P.

Proof. Let $A = \{y_1 = 0\}/\mathbb{Z}_4 \in |-K_X|$. Then from the ℓ-exact sequence

$$0 \to I \to I \tilde{\otimes} \mathcal{O}_X(A) \to I \tilde{\otimes} \mathcal{O}_A(A) \to 0,$$

we have a surjection $H^0(I \tilde{\otimes} \mathcal{O}(A)) \twoheadrightarrow I \tilde{\otimes} \mathcal{O}_A(A)$. Then y_2/y_1 lifts to a section $s \in H^0(I \tilde{\otimes} \mathcal{O}_X(A))$. Since $H^0(gr^1(\mathcal{O}(-K_X), J)) = 0$, we have

$$s \in H^0(J \tilde{\otimes} \mathcal{O}_X(A)).$$

Furthermore its image s' in $gr^2(\mathscr{O}(A), J) \simeq (2P^\sharp)\tilde{\oplus}(0)$ is nowhere vanishing outside of P since s restricts to y_2/y_1 at P. Let $B = \{s = 0\} \in |A|$.

We claim that B has exactly one singular point, say R, outside of P, and that R is an A_1-point. Let s'' be the image of s in $gr_C^1(\mathscr{O}(A)) = (1)\tilde{\oplus}(-1 + 3P^\sharp)$. Then s'' generates (1) at P, whence it has exactly one zero R of order 1 outside of P. Let z_1, z_2, z_3 be the coordinates of (X, R) such that $(C, Q) = z_1$-axis, z_2 (resp. z_3) is the basis of $(3P^\sharp)$ (resp. $(-1 + 2P^\sharp)$) at R in the ℓ-isomorphism of (7.10.1). Then $s'' = (\text{unit}) \cdot z_1 z_2$ at R and $s' = (\text{unit}) \cdot z_1 z_2 + r \cdot z_3^2$ for some $r \in \mathscr{O}_{X,R}$ since $J = (z_2, z_3^2)$ at R. Since s' does not vanish at R, we see that r is a unit at R. This means that R is an A_1-point of B as claimed above.

We claim that (B, P) is a simple elliptic singularity of degree 4. We note that the equation of $B^\sharp$ in $X^\sharp$ is $y_1 s = y_2 + \cdots \in J^\sharp$ or $y_2 - \delta(y_1, y_2, y_3) \in J^\sharp$ such that $\sigma(\delta) \geq 5$ because $\sigma(\delta) \equiv 1 \mod 4$ and $\sigma(\delta) \neq 1$, where σ is as given in (7.10.1.2). Thus $B^\sharp$ is defined in $y_1 y_3 y_4$-space by $\overline{\alpha}(y_1, y_3, y_4) = \alpha(y_1, \delta, y_3, y_4) = 0$, and we see that $\sigma(\delta) = 6$ and $\overline{\alpha}_{\sigma=6} = \alpha_{\sigma=6}(y_1, 0, y_3, y_4)$ since $y_1 y_2$ does not appear in α. Hence $(B^\sharp, P^\sharp)$ is a simple elliptic singularity of degree 1 by (7.10.1.2) and the claim follows.

Thus we have 3 curves on the minimal resolution $\rho : B' \to B$; the proper transform C' of C, a smooth elliptic curve P' over P, and a smooth rational curve R' over R. They form a linear chain and $\Delta(B \supset C)$ is as follows:

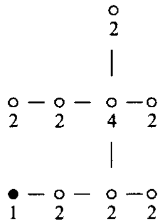
$$\underset{4}{P'} - \underset{1}{C'} - \underset{2}{R'}.$$

Let us consider the divisor $(y_1 y_3)$ on (B, P). Since $y_1 y_3 + y_4^2 \cdot \gamma(y_1^4) \equiv 0$ in $gr^2(\mathscr{O}, J)$, C' has multiplicity 2 in $(y_1 y_3)$. By (7.10.1.2), we see that $\rho^*((y_1 y_3)) = R' + T_1' + T_2' + 2C'$ for some divisors T_1' and T_2' such that T_1', T_2', and C' are disjoint from each other and $(T_1' \cdot R') = (T_2' \cdot R') - 1$. Thus the divisor $F' = R' + T_1' + T_2' + 2C' + R'$ descends to a Cartier divisor F on B such that $(F \cdot C) = 0$. Since we have $\operatorname{Pic} B \simeq \operatorname{Pic} C$ by $H^1(\mathscr{O}_B) = H^1(\mathscr{O}_X) = 0$, we have $F \sim 0$. Thus a global defining equation $\overline{s}$ of F in B lifts to an element $s \in H^0(I)$. Since $\overline{s} = (\text{unit}) \cdot y_1 y_3$ near P, we have (7.10.1.4). □

(7.10.1.5) **Lemma.** *There are elements* $s_1, s_2 \in H^0(I)$ *such that* $s_1 \equiv y_4^2$ *and* $s_2 \equiv y_2 y_3 \bmod (y_1)$ *near* $P^\sharp$.

Proof. With A in (7.10.1.4), we only have to lift elements $y_4^2, y_2 y_3 \in \mathscr{O}_A$ to those in $H^0(\mathscr{O}_X)$ by $H^0(\mathscr{O}_X) \twoheadrightarrow \mathscr{O}_A$. □

(7.10.1.6) Let H_X be a general member of $|\mathscr{O}_X|$ through C defined by a section $s \in H^0(J)$. Since $y_1 y_3$ appears in s at P, s generates the first factor $\mathscr{O}$ of $gr^2(\mathscr{O}, J) \simeq \mathscr{O} \oplus \mathscr{O}(-1)$ whence the first factor $\mathscr{O}$ of $gr_C^1 \mathscr{O}$ by (7.10.1.3). Thus H_X is smooth outside of P. We can now apply (7.11) to compute $\Delta(H_X \supset C)$, which is as follows.



This contracts to D_5, which means that (Y, Q) is Gorenstein. Thus X is not isolated. This completes the proof of (7.10.1).

(7.10.2) **Lemma.** *Let X be an extremal nbd with a type IIA point P with $i_P(1) = 2$ and a type III point R. Then X is not isolated.*

Proof. It is enough to prove that a nearby extremal nbd of X is not isolated. Deforming (X, P) by $\alpha + ty_1^3y_3 = 0$, we may assume $\ell(P) = 3$. Since $\deg gr_C^1\mathscr{O} = -2$, we have $gr_C^1\mathscr{O} \simeq \mathscr{O}(-1)^{\oplus 2}$ by $H^1(gr_C^1\mathscr{O}) = 0$. We may therefore assume an ℓ-isomorphism

$$gr_C^1\mathscr{O} \simeq (-1 + 3P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp), \tag{7.10.2.1}$$

such that y_2 and y_4 are ℓ-free ℓ-bases of $(-1 + 3P^\sharp)$ and $(-1 + 2P^\sharp)$. We will take coordinates for (X, R)

$$(X, R) = (z_1, z_2, z_3, z_4; \beta) \supset (C, R) = z_1\text{-axis},$$

where $\beta \equiv z_1z_3 \mod (z_2, z_3, z_4)^2$ and z_2 and z_4 are bases of $(-1 + 3P^\sharp)$ and $(-1 + 2P^\sharp)$ in (7.10.2.1) at R.

Let $(X_t, R) \supset (C, R)$ be the deformation of $(X, R) \supset (C, R)$ in z-space given by $\beta + tz_2 = 0$. Let $X_t \supset C_t$ be its twisted extension by (z_2, z_4). In

$$gr_{C_t}^1\mathscr{O} \otimes \mathscr{O}_{C_t, R} = \mathscr{O}_{C_t, R}z_3 \oplus \mathscr{O}_{C_t, R}z_4,$$

we see $\mathscr{O}_{C_t}z_2 = \mathscr{O}_{C_t}z_1 \cdot z_3$. Thus comparing $gr_C^1\mathscr{O}$ and $gr_{C_t}^1\mathscr{O}$, we get an ℓ-isomorphism

$$gr_{C_t}^1\mathscr{O} \simeq (3P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp).$$

By (7.10.1), X_t is not isolated. □

We will prove (7.4.2) in the case $\ell(P) = 4$. In this case, $\alpha \equiv y_1^4y_4 \mod (y_2, y_3, y_4)^2$ (7.5) and y_2 and y_3 form an ℓ-free ℓ-basis of $gr_C^1\mathscr{O}$ at P. Thus it is enough to prove the following.

(7.10.3) **Lemma.** *Let X be an extremal nbd with a type IIA point P of $\ell(P) = 4$ and with an ℓ-isomorphism*

$$gr_C^1\mathscr{O} \simeq (3P^\sharp)\tilde{\oplus}(-1 + P^\sharp). \tag{7.10.3.1}$$

Then X is not isolated.

Proof. By (7.10.3.1), X is smooth outside of P. We may assume that y_2 and y_3 are ℓ-free ℓ-bases of $(3P^\sharp)$ and $(-1 + P^\sharp)$. Let $(X_t, P) \supset (C, P)$ be a deformation given by $\alpha + ty_1^3 y_3 = 0$. Then P is a type *IIA* point of X_t with $\ell(P) = 3$. Let $X_t \supset C_t$ be the twisted extension of $(X_t, P) \supset (C, P)$ by (y_2, y_3). Comparing $gr_C^1\mathscr{O}$ and $gr_{C_t}^1\mathscr{O}$, we see $gr_{C_t}^1\mathscr{O} \simeq (3P^\sharp)\tilde{\oplus}(-1+2P^\sharp)$. Thus X is not isolated by (7.10.1). □

The following is the last step for the proof of (7.4.1).

(7.10.4) **Lemma.** *Let X be an extremal nbd with a type IIA point P of $\ell(P) \geq 5$. Then X is not isolated.*

Proof. If $\ell(P) \geq 5$, then the deformation $(X_t, P) \supset (C, P)$ given by $\alpha + ty_1^5 y_2 = 0$ has a type *IIA* point P with $\ell(P) = 5$. Hence we may assume $\ell(P) = 5$. Then we may assume $\alpha \equiv y_1^5 y_2 \mod (y_2, y_3, y_4)^2$ after a change of coordinates. Thus y_3 and y_4 form an ℓ-free ℓ-basis of $gr_C^1\mathscr{O}$ at P. There are two possibilities for $gr_C^1\mathscr{O}$. If it is in the special case $(2P^\sharp)\tilde{\oplus}(-1+P^\sharp)$ with ℓ-free ℓ-basis u for $(2P^\sharp)$ (resp. v for $(-1+P^\sharp)$), then the twisted extension of the trivial deformation $(X, P) \times \mathbb{C}_t^1 \supset (C, P) \times \mathbb{C}_t^1$ by $(y_1^2 u + ty_1 v, y_1 v)$ will be in the general case $(P^\sharp)\tilde{\oplus}(-1+2P^\sharp)$ for small enough $t \neq 0$. Therefore we may assume the ℓ-isomorphism $gr_C^1\mathscr{O} \simeq (P^\sharp)\tilde{\oplus}(-1+2P^\sharp)$ such that y_3 and y_4 are ℓ-free ℓ-bases of $(P^\sharp)$ and $(-1+2P^\sharp)$. Let $(X_t, P) \supset (C, P)$ be the deformation of $(X, P) \supset (C, P)$ in $\mathbb{C}^4$ given by $\alpha + ty_1^3 y_3 = 0$. Then $(X_t, P) \supset (C, P)$ has a type *IIA* point P with $\ell(P) = 3$ and its twisted extension $X_t \supset C_t$ by $(y_1 y_3, y_1^2 y_4)$ satisfies $gr_{C_t}^1\mathscr{O} \simeq (3P^\sharp)\tilde{\oplus}(-1+2P^\sharp)$ for small enough $t \neq 0$. Then X is not isolated by (7.10.1). □

This completes the proof of (7.4.1).

(7.11) **Computation.** Let (C, P) be a normal surface singularity

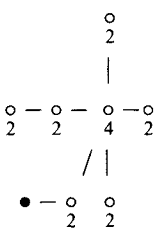
$$(D, P) = (y_1, y_2, y_3, y_4; \alpha, \beta)/\mathbb{Z}_4(1, 1, 3, 2; 2, 0) \supset C = y_1\text{-axis}/\mathbb{Z}_4.$$

Let σ be the $\mathbb{Z}$-wt $\sigma(y_1, y_2, y_3, y_4) = (1, 1, 3, 2)$. Assume that $\alpha_{\sigma=2} = y_2^2$, and let

$$\alpha_{\sigma=6} = y_3^2 + a'y_1^3 y_3 + b'y_1^4 y_4 + c'y_1 y_3 y_4 + \cdots,$$
$$\beta_{\sigma=4} = y_4^2 + ay_1 y_3 + by_2 y_3 + cy_1^2 y_4 + dy_1^3 y_2 + \cdots,$$

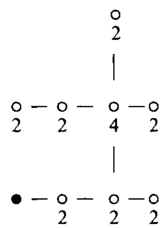
where $a, b, c, d, a', b', c' \in \mathbb{C}$. Then in each of the following cases, $\Delta(D \supset C)$ consists of smooth rational curves and C' intersecting transversely with configuration as listed.

(7.11.1) $a \neq 0$, and the equations $y^2 + a'y + b'x + c'xy = 0$ and $x^2 + ay + cx = 0$ have 4 distinct roots $x = 0$ and (say) $\alpha_1, \alpha_2, \alpha_3$ after the elimination of y.



In the diagram, $\bullet - \underset{2}{\circ}$ intersects the central $\mathbb{P}^1$ $(\underset{4}{\circ})$ at 0, $\underset{2}{\circ} - \underset{2}{\circ}$ at ∞, and three $\underset{2}{\circ}$ at α_1, α_2, and α_3, with respect to a certain coordinate system of the central $\mathbb{P}^1$.

(7.11.2) $aa'd \neq 0$, $b' = c = 0$, and the equation $x^2 + ac'x - aa' = 0$ has two distinct roots (say) α_1 and α_2.



In the diagram, $\bullet - \underset{2}{\circ} - \underset{2}{\circ} - \underset{2}{\circ}$ intersects the central $\mathbb{P}^1$ $(\underset{4}{\circ})$ at 0, $\underset{2}{\circ} - \underset{2}{\circ}$ at ∞, and the two $\underset{2}{\circ}$ at α_1 and α_2, with respect to a certain coordinate system of the central $\mathbb{P}^1$.

(7.12) In this paragraph, we prove the rest of (7.4). To prove (7.4.4), we consider formal 3-folds X along C satisfying the conditions in (7.4.1) for a while. We will still use the notation (7.5) at P.

(7.12.1) **Lemma.** *Let $X \supset C$ be a formal 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume also that $X \supset C$ satisfies the conditions in* (7.4.1). *Let J be the ideal such that $I \supset J \supset I^{(2)}$ and $J/I^{(2)} = ((5 - \ell(P))P^\sharp)$ under the notation of* (7.4.1.1). *Then we have ℓ-isomorphisms*

$$gr^1(\mathcal{O}, J) \simeq (-1 + 3P^\sharp),$$
$$gr^2(\mathcal{O}, J) \simeq (P^\sharp)\tilde{\oplus}(2P^\sharp),$$

and we can choose coordinates at P such that (y_2, y_4, y_3) (resp. (y_2, y_3, y_4)) is a $(1, 2, 2)$-monomializing ℓ-basis of $I \supset J$ of the second kind at P. In particular, $J^\sharp = (y_3, y_4)$.

Proof. If $\ell(P) = 3$ (resp. 4), we see $\alpha \equiv y_1^3 y_3$ (resp. $y_1^4 y_4$) mod $(y_2, y_3, y_4)^2$ and we may assume that y_4 (resp. y_3) and y_2 are ℓ-free ℓ-bases of the first and the second ℓ-summands of $gr_C^1\mathcal{O}$ given in (7.4.1.1). In particular, $J^\sharp = (y_2^2, y_3, y_4)$ in either case. Since none of y_1^2 or $y_1 y_2$ appears in α, y_2^2 appears in α by the description of IIA points. Thus

$$\alpha \equiv y_1^3 y_3 - (\text{unit}) \cdot y_2^2 (\text{resp. } y_1^4 y_4 - (\text{unit}) \cdot y_2^2) \bmod I^\sharp \cdot J^\sharp.$$

Hence we have the assertion on the monomializing ℓ-basis. From the above, we have

$$(\text{unit}) \cdot y_2^2 = y_1^3 y_3 (\text{resp. } y_1^4 y_4) \text{ in } gr^2(\mathcal{O}, J).$$

Whence we have an ℓ-isomorphism

$$gr^{2,1}(\mathcal{O}, J) \simeq gr^1(\mathcal{O}, J)^{\tilde{\otimes} 2} \tilde{\otimes} (\ell(P)P^\sharp) \simeq ((\ell(P) - 2)P^\sharp).$$

Thus we have $gr^2(\mathscr{O}, J) \simeq (P^\sharp)\tilde{\oplus}(2P^\sharp)$. □

(7.12.2) **Lemma.** *Under the notation and assumptions of* (7.12.1), *we have* $\mathrm{Spec}(\mathscr{O}/J) = D \cdot D''$ *for general members* $D \in |K_X|$ *and* $D'' \in |2K_X|$. *In particular,* (7.4.2) *holds.*

We omit the proof, because it follows from (7.12.1) by the argument in (7.6).

(7.12.3) **Lemma.** (7.4.4) *holds.*

Proof. Since (7.12.2) applies to the completion $X^\wedge$ of X along C, we have $2C = D \cdot D''$ such that $(D \cdot C), (D'' \cdot C) < 0$ in $X^\wedge$. This implies that C is formally contractible in X and hence contractible. Thus $X \supset C$ is an isolated extremal nbd by (2.6). □

(7.12.4) **Lemma.** (7.4.3) *holds.*

Proof. Since the argument is the same as (7.6), we will only sketch the proof. Let H_X be a general member of $|\mathscr{O}_X|$ through C defined by $s \in H^0(I)$. By (7.12.1), we see that y_1y_3 and $y_1^2y_4$ (resp. $y_1^2y_3^2$, $y_1^3y_3y_4$ and y_4^2) form a free basis at P of globally generated $gr^2(\mathscr{O}, J)$ (resp. $gr^4(\mathscr{O}, J)$). As in (7.6), we get a surjection $H^0(F^i(\mathscr{O}, J)) \twoheadrightarrow H^0(gr^i(\mathscr{O}, J))$ for each i. Hence y_4^2, y_1y_3, and $y_1^2y_4$ appear in s at P with independent coefficients. This first means that the image $\bar{s}$ of s in $gr_C^1\mathscr{O}$ is nowhere vanishing outside of P, whence H_X is smooth outside of P. It also means that we can apply (7.11.1) to (H_X, P). The rest is the same as (7.6). □

(7.12.5) **Example.** Let $Z \supset C$ be a germ of a smooth 3-fold along $C \simeq \mathbb{P}^1$ such that $N_{C/Z}^* \simeq \mathscr{O}_C \oplus \mathscr{O}_C(-1)$. Let $P \in C$ and let (z_1, z_2, z_3) be coordinates of (Z, P) such that $(C, P) = z_1$-axis and z_2 (resp. z_3) is a generator of the first (resp. second) summand of $N_{C/Z}^*$. Let $(X, P) \supset (C, P)$ be a IIA point as in (7.5) with $\alpha \equiv y_1^3y_3$ (resp. $y_1^4y_4$) mod $(y_2, y_3, y_4)^2$. For suitable ε_1 and ε_2 such that $0 < \varepsilon_1 < \varepsilon_2 \ll 1$, $(y_1^4, y_1^3y_2, y_1^2y_4)$ (resp. $(y_1^4, y_1^3y_2, y_1y_3)$) form coordinates for $U = (X, P) \cap \{\varepsilon_1 < |y_1^4| < \varepsilon_2\}$ by the implicit function theorem. Thus $z_1 = y_1^4$, $z_2 = y_1^2y_4$ and $z_3 = y_1^3y_2$ (resp. $z_1 = y_1^4$, $z_2 = y_1y_3$, and $z_3 = y_1^3y_2$) patch (X, P) and $Z - (Z, P) \cap \{|z_1| \le \varepsilon_1\}$ along U. This $X \supset C$ is an isolated extremal nbd with a type IIA P of $\ell(P) = 3$ (resp. 4) satisfying (7.4.1.1) by (7.2).

8. General members of $|\mathscr{O}_X|_C$; IC case

We consider the following set up in this chapter unless otherwise mentioned explicitly.

(8.1) Let $f : X \supset C \to Y \ni Q$ be an extremal nbd with only one singular point P such that $X \supset C$ has an IC point at P. Let H_X be a general member of $|\mathscr{O}_X|$ through C and let $H_Y = f(H_X)$. Let $\Delta_X = \Delta(H_X \supset C)$ and $\Delta_Y = \Delta(H_Y)$.

(8.2) Let

$$(X, P) = (y_1, y_2, y_4)/\mathbb{Z}_m(2, m-2, 1) \supset C = (\text{locus of } (t^2, t^{m-2}, 0))/\mathbb{Z}_m$$

with odd index $m \geq 5$. We have an ℓ-splitting

$$gr_C^1\mathcal{O} = (4P^\sharp)\tilde{\oplus}(-1 + (m-1)P^\sharp)$$

by (2.10.2), and hence the unique $(4P^\sharp)$ in $gr_C^1\mathcal{O}$. Since y_4 and $y_1^{m-2} - y_2^2$ form an ℓ-free ℓ-basis of $gr_C^1\mathcal{O}$ at P, $(4P^\sharp)$ has an ℓ-free ℓ-basis

$$\lambda_1 y_1^{(m-5)/2} y_4 + \mu_1(y_1^{m-2} - y_2^2)$$

for some λ_1 and $\mu_1 \in \mathcal{O}_{C,P}$. We remark that it is easy to see that whether $\lambda_1(P) \neq 0$ does not depend on the choice of coordinates.

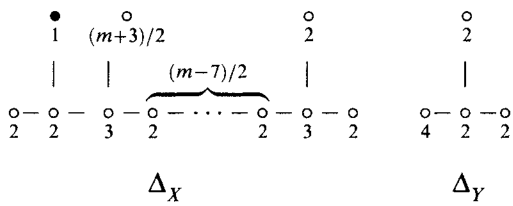
Our main result in this chapter is the following.

(8.3) **Theorem.** *Under the notation and assumptions of* (8.2), *we have the following*:

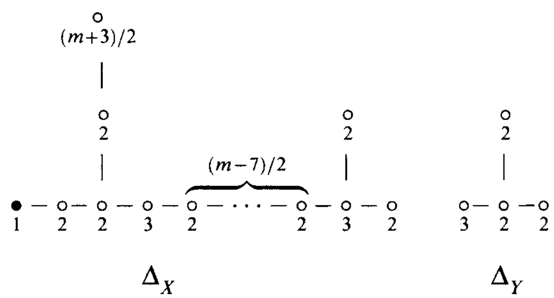
(8.3.1) H_X *is normal,* Δ_X *and* Δ_Y *consist of smooth rational curves intersecting transversely.*

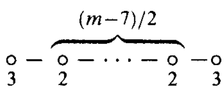
(8.3.2) Δ_X *and* Δ_Y *are as follows.*

(8.3.2.1) *Case* $\lambda_1(P) \neq 0$.



(8.3.2.2) *Case* $\lambda_1(P) = 0$.



where  *reduces to* $\underset{4}{\circ}$ *if* $m = 5$.

(8.3.3) **Corollary.** *There exist no divisorial extremal nbds of type IC.*

(8.4) *Remark.* By (2.10), we can choose coordinates y_1, y_2, y_4 so that there is a normal member $E \in |-K_X|$ with the singularities as described in (2.2.2) and that $E = \{y_4 = 0\}/\mathbb{Z}_m$ in a neighborhood of P. In particular, we have an ℓ-splitting

$$gr^1_C \mathscr{O} = (4P^\sharp)\tilde{\oplus}\mathscr{O}_C(-E). \tag{8.4.1}$$

We will prove (8.3) in several steps.

(8.5) Let $I = I_C$. Let J be the C-laminal ideal such that $I \supset J \supset F^2_C\mathscr{O}$ and $J/F^2_C\mathscr{O} = (4P^\sharp)$ in (8.4.1). Since J is locally a nested c.i. on $C - \{P\}$ and (y_4, u) is a (1,2)-monomializing ℓ-basis of $I \supset J$ at P with $u = \lambda_1 y_1^{(m-5)/2} y_4 + \mu_1(y_1^{m-2} - y_2^2)$ as in (8.2), we have an ℓ-exact sequence

$$0 \to \mathscr{O}_C(-2E) \to gr^0_C J \to (4P^\sharp) \to 0 \tag{8.5.1}$$

and an ℓ-isomorphism $\mathscr{O}_C(-2E) \simeq (-1 + (m-2)P^\sharp)$. Thus we have $gr^0_C J \simeq \mathscr{O} \oplus \mathscr{O}(-1)$ as $\mathscr{O}_C$-modules. The unique $\mathscr{O}$ in $gr^0_C J$ is generated near P by

$$y_1^2 u + \alpha y_2 y_4^2 \bmod F^3(\mathscr{O}, J) \tag{8.5.2}$$

for some $\alpha \in \mathscr{O}_{C,P}$. We note the following.

(8.5.3) **Lemma.**

$$F^3(\mathscr{O}, J)^\sharp \subset ((y_1^{m-2} - y_2^2)^2, (y_1^{m-2} - y_2^2)y_4, \lambda_1 y_1^{(m-5)/2} y_4^2, y_4^3).$$

Proof. We have $F^3(\mathscr{O}, J)^\sharp{}_{P^\sharp} \subset (u^2, uy_4, y_4^3)$. By

$$u \in (y_1^{m-2} - y_2^2, \lambda_1 y_1^{(m-5)/2} y_4),$$

we have (8.5.3). □

(8.6) **Lemma.** *The ℓ-exact sequence* (8.5.1) *is ℓ-split iff $\alpha(P) = 0$.*

Proof. If $\alpha(P) = 0$, then $\alpha = y_1 y_2 \cdot \alpha'$ for some $\alpha' \in \mathscr{O}_{C,P}$. Then $\alpha y_2 = \alpha' \cdot y_1 y_2^2 = \alpha' \cdot y_1^{m-1}$ and the element (8.5.2) is divisible by y_1^2 in $\mathscr{O}_{X^\sharp, P^\sharp}$, whence (8.5.1) is ℓ-split. If (8.5.1) is ℓ-split, then the unique $\mathscr{O}$ in $gr^0_C J$ must be the ℓ-splitting submodule $(4P^\sharp)$ and (8.5.2) is divisible by y_1^2. Now we have $\alpha(P) = 0$ by $\alpha y_2 \in y_1^2 \mathscr{O}_{C^\sharp, P^\sharp}$. □

(8.7) **Proposition.** *If $m \geq 7$, then $\alpha(P) \neq 0$.*

Proof. Assume that $\alpha(P) = 0$, that is, (8.5.1) is ℓ-split. Then $gr^0_C J$ contains a unique $(4P^\sharp)$. Let K be the C-laminal ideal such that $J \supset K \supset F^1_C J$ and $K/F^1_C J = (4P^\sharp)$. By [Mori88, (8.14)], K is locally a nested c.i. on $C - \{P\}$ and (1,3)-monomializable at P, and we have ℓ-isomorphisms

$$gr^i(\mathscr{O}, K) \simeq (-1 + (m-i)P^\sharp) \qquad (i = 1, 2) \tag{8.7.1}$$

and an ℓ-exact sequence

$$0 \to (-1 + (m-3)P^\sharp) \to gr^3(\mathscr{O}, K) \to (4P^\sharp) \to 0. \tag{8.7.2}$$

By (8.7.1) $\tilde{\otimes}\omega_X$, we see

$$gr^i(\omega_X, K) \simeq (-1 + (m-i-1)P^\sharp) \quad \text{and} \quad H^1(gr^i(\omega_X, K)) = 0 \qquad \text{for } i = 1, 2$$

by $m-2$, $m-3 \in 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$. Hence by $H^1(\omega_X) = 0$ and the standard exact sequences

$$0 \to F^{i+1}(\omega_X, K) \to F^i(\omega_X, K) \to gr^i(\omega_X, K) \to 0,$$

we have $H^1(F^3(\mathscr{O}, K)) = 0$. Hence $H^1(gr^3(\mathscr{O}, K)) = 0$ since C is a 1-dimensional fiber of proper f. Now by (8.7.2) $\tilde{\otimes}\omega_X$, we have

$$0 \to (-2 + (2m-4)P^\sharp) \to gr^3(\omega_X, K) \to (-1 + (m+3)P^\sharp) \to 0.$$

We note $(-1+(m+3)P^\sharp) \simeq \mathscr{O}(-1)$ as $\mathscr{O}_C$- modules because $3 \notin 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$ by $m \geq 7$. We similarly note that $(-2 + (2m-4)P^\sharp) \simeq \mathscr{O}(-2)$ by $m-4 \notin 2\mathbb{Z}_+ + (m-2)\mathbb{Z}_+$. Hence $H^1(gr^3(\omega_X, K)) \neq 0$, which is a contradiction. □

(8.8) **Proposition.** (8.8.1) *$\mathscr{O}_E(-C)$ is an ℓ-invertible $\mathscr{O}_E$-module with an ℓ-free ℓ-basis $y_1^{m-2} - y_2^2$ at P and an ℓ-isomorphism.*

$$\mathscr{O}_C \tilde{\otimes} \mathscr{O}_E(-C) \simeq (4P^\sharp).$$

(8.8.2) $H^0(\mathscr{O}_E(-\nu C)) \twoheadrightarrow H^0(\mathscr{O}_C \tilde{\otimes} \mathscr{O}_E(-\nu C))$ *for all* $\nu \geq 0$.

(8.8.3) *There are sections* s_1 *and* $s_2 \in H^0(I)$ *such that*

$$s_1 \equiv \textit{(unit)} \cdot y_1^2(y_1^{m-2} - y_2^2) \bmod y_4 \quad \textit{near } P,$$
$$s_2 \equiv \textit{(unit)} \cdot y_2(y_1^{m-2} - y_2^2)^{(m-1)/2} \bmod y_4 \quad \textit{near } P.$$

(8.8.4) $H^0(I) \twoheadrightarrow H^0(gr_C^0 J) = H^0(I/F^3(\mathscr{O}, J)) \simeq \mathbb{C}$.

Proof. (8.8.1) follows from the construction of E. Hence $H^1(\mathscr{O}_C \tilde{\otimes} \mathscr{O}_E(-\nu C)) = 0$ for all $\nu \geq 0$ and $H^1(\mathscr{O}_E(-\nu C)) = 0$ since C is a fiber of proper f. Thus we have (8.8.2). (8.8.3) follows from (8.8.2), and (8.8.4) follows from (8.8.3) by $H^0(gr_C^0 J) \simeq \mathbb{C}$. □

(8.9) By (8.7), there are four cases to treat.

(8.9.1) Case $m \geq 7$, $\alpha(P) \neq 0$.
(8.9.2) Case $m = 5$, $\lambda_1(P) \neq 0$.
(8.9.3) Case $m = 5$, $\lambda_1(P) = 0$, $\alpha(P) \neq 0$.
(8.9.4) Case $m = 5$, $\lambda_1(P) = 0$, $\alpha(P) = 0$.
We first prove (8.3) in the easy cases.

(8.10) *Proof of* (8.3). Cases (8.9.1) and (8.9.3). By (8.5.2) and (8.8), a general section $s \in H^0(I)$ satisfies

$$s \equiv (\text{unit}) \cdot \{y_1^2 u + \alpha y_2 y_4^2\} \bmod F^3(\mathscr{O}, J) \quad \text{at } P,$$

where $\alpha(P) \neq 0$ by assumption. Let us take s_2 given in (8.8.3). We claim that s_2 belongs to $H^0(F^3(\mathscr{O}, J))$. Indeed it is obvious that $s_2 \notin \mathbb{C}s + F^3(\mathscr{O}, J)$ near P. Hence by $H^0(I/F^3(\mathscr{O}, J)) = \mathbb{C}s$, we have $s_2 \in H^0(F^3(\mathscr{O}, J))$ as claimed. By (8.5.3), we see that the coefficient of $y_2 y_4^2$ (resp. y_2^m) in the Taylor expansion of s_2 at $P^\sharp$ is 0 (resp. nonzero) because $m \geq 7$ or $\lambda_1(P) = 0$. We now analyze the set $H = \{s = 0\}$. By Bertini's theorem, H is smooth outside C. Since $\mathscr{O}s$ is the unique $\mathscr{O}$ in $gr^1_C\mathscr{O} \simeq \mathscr{O}\oplus\mathscr{O}(-1)$, H is smooth on $C-\{P\}$. To study (H, P), we will apply (10.7). Indeed if $\lambda_1(P) = 0$, then $\mu_1(P) \neq 0$ by the construction (8.2). Thus (10.7.1) holds by (8.5.3). The existence of s_2 ensures (10.7.2). Since $m \geq 7$ or $\lambda_1(P) = 0$, we can now apply (10.7). It only remains to check $(\bullet^2) = -1$ in (8.3.2). Since

$$(C \cdot K_H)_H = (C \cdot K_X + H)_X = -\frac{1}{m} < 0,$$

the proper transform $\bullet$ of C in the minimal resolution H' of H satisfies $(\bullet^2) = -1$. □

(8.11) *Proof of* (8.3). Case (8.9.2). The argument is the same as (8.10) except that we need to check (10.7.3) when we apply (10.7).

We note $\mu_1(P) \neq 0$ by (8.4.1). For $D = \{y_1 = 0\}/\mathbb{Z}_m \in |-2K_X|$,we have a surjection $H^0(\mathscr{O}(-K_X)) \twoheadrightarrow \mathscr{O}_D(-K_X)$ by $H^1(\omega_X) = 0$. Let $\phi \in H^0(\mathscr{O}(-K_X))$ be a section sent to

$$\{u - \lambda_1(P)y_4\}/dy_1 \wedge dy_2 \wedge dy_4 \in \mathscr{O}_D(-K_X).$$

Thus the image of ϕ under the homomorphism

$$I\tilde{\otimes}\mathscr{O}_X(-K_X) \twoheadrightarrow gr^1_C\mathscr{O}_X(-K_X) = (1)\tilde{\oplus}(0) \twoheadrightarrow (0)$$

is nonzero by $\lambda_1(P) \neq 0$. Hence $E' = \{\phi = 0\} \in |-K_X|$ is smooth outside $\{P\}$ and we may choose ϕ so that E' is furthermore normal by Bertini's theorem. We have an ℓ-splitting

$$gr^1_C\mathscr{O} = (4P^\sharp)\tilde{\oplus}\mathscr{O}_C(-E').$$

By the construction of E', we see that $(E', P) = \{v = 0\}/\mathbb{Z}_m$ where $v = y_1^3 - y_2^2 + \lambda_1' \cdot y_4$ for some $\lambda_1' \in \mathscr{O}_{C,P}$ such that $\lambda_1'(P) = 0$. As in (8.8), we see that $\mathscr{O}_{E'}(-C)$ is an ℓ-invertible $\mathscr{O}_{E'}$-module with an ℓ-free ℓ-basis u at P and an ℓ-isomorphism

$$\mathscr{O}_C\tilde{\otimes}\mathscr{O}_{E'}(-C) \simeq (4P^\sharp).$$

We similarly see

$$H^0(\mathscr{O}_{E'}(-\nu C)) \twoheadrightarrow H^0(\mathscr{O}_C\tilde{\otimes}\mathscr{O}_{E'}(-\nu C)) \quad \text{for all } \nu \geq 0.$$

We note that y_1^2u, y_2u^2, y_1u^3, $y_2^2u^4$, u^5 are bases of $\mathscr{O}_C\tilde{\otimes}\mathscr{O}_{E'}(-\nu C)$ at P for $\nu = 1, \dots, 5$, respectively. Thus for arbitrary $a_1, \dots, a_5 \in \mathbb{C}$, there exist $s \in H^0(I)$ such that

$$s \equiv a_1y_1^2u + \cdots + a_5u^5 \bmod (v, u^6).$$

Hence by $(v, u^6) \subset (y_1^3, y_2^2, y_4^6)$, there exist $s' \in H^0(I)$ such that

$$s' \equiv a_1y_1^2y_4 + a_2y_2y_4^2 + a_3y_1y_4^3 + a_5y_4^5 \bmod (y_1^3, y_2^2, y_4^6).$$

By this, it is easy to check (10.7.3). The rest is the same as (8.10). □

Now (8.3) follows if we prove the following.

(8.12) **Proposition.** *The case* (8.9.4) *does not occur.*

Proof. We assume that we are in case (8.9.4). Since $\lambda_1(P) = 0$, we have $\mu_1(P) \neq 0$ by (8.2). By $\alpha(P) = 0$, we have $\alpha y_2 = \lambda_2 y_1^{m-1}$ for some $\lambda_2 \in \mathscr{O}_{C,P}$ as in (8.6). Thus a general section $s \in H^0(I)$ satisfies, near P, the following:

(8.12.1) $$s \equiv (\text{unit}) \cdot y_1^2(u + \lambda_2 y_1^{m-3} y_4^2) \bmod F^3(\mathscr{O}, J).$$

Let $H = \{s = 0\} \subset X$. As in (8.10), H is normal and has the exceptional curve C. We will construct a flat deformation $\pi : \mathscr{H} \to T = (\mathbb{C}, 0)$ of H with $\mathscr{C} = C \times T \subset \mathscr{H}$ such that $\mathscr{H}_0 = H \supset \mathscr{C}_0 = C$ as follows, where $\mathscr{H}_t = \pi^{-1}(t)$ and $\mathscr{C}_t = C \times t$ $(t \in \mathbb{C})$.

Construction. Let s_t be a local section at P:

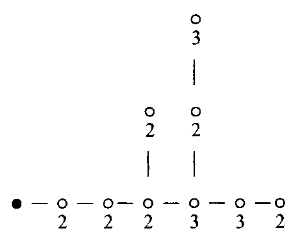
$$s_t = s + t(y_1 y_4^3 + y_2 y_4 u) \qquad (t: \text{ parameter}).$$

We note $s_t \equiv s \bmod F^3(\mathscr{O}, J)$. Let $(X_t, P_t) \supset (C_t, P_t)$ be the trivial deformation of $(X, P) \supset (C, P)$ with parameter t. Let $\mathscr{X} \supset \mathscr{C}$ be the twisted extension of $(X_t, P_t) \supset (C_t, P_t)$ by $(s_t, y_1^2 y_4)$ with parameter t (cf. [Mori88, (1b.8)]). By the construction, there is a section $\tilde{s} \in H^0(\mathscr{O}_{\mathscr{X}})$ such that

$$\tilde{s}|_{\mathscr{X}_t} = \begin{cases} s & \text{outside of a nbd of } P_t, \\ s_t & \text{in a nbd of } P_t. \end{cases}$$

We now set $\mathscr{H} = \{\tilde{s} = 0\} \subset \mathscr{X}$.

Since C is analytically contractible in H, so is $\mathscr{C}_t$ in $\mathscr{H}_t$ for sufficiently small t. We will derive a contradiction by showing that $\mathscr{C}_t$ is not analytically contractible in our $\mathscr{H}_t$. Since H is a normal surface, so is $\mathscr{H}_t$. We know that the images of s and s_t (by $s_t \equiv s \bmod F^3(\mathscr{O}, J)$) in $gr_C^1 \mathscr{O}$ are nowhere vanishing on $C - \{P\}$. Thus P_t is the only singular point of $\mathscr{H}_t$ on $\mathscr{C}_t$. By (8.12.1) and the definition of s_t, we can apply (10.8) and $\Delta(\mathscr{H}_t \supset \mathscr{C}_t)$ is as follows.



Since the deformation $(\mathscr{H}_t, P_t)$ is induced by the deformation of the canonical cover of (H, P), $5K_{\mathscr{H}}$ is a Cartier divisor and we see $(K_{\mathscr{H}} \cdot \mathscr{C}_t) = -1/5 < 0$ as in (8.10). Thus follows $(\bullet^2) = -1$. Then it is easy to see that $\Delta(\mathscr{H}_t \supset \mathscr{C}_t)$ contracts to $\underset{0}{\circ}$, which is not contractible. This is a contradiction and thus (8.9.4) is disproved. □

(8.13) *Remark.* Except for the usual vanishing of the cohomologies of the extremal nbd $X \supset C$, we have used in this chapter that C is contractible in H (8.12) to treat the case (8.9.4). Thus the birationality of the morphism $f: X \to Y$ is used here.

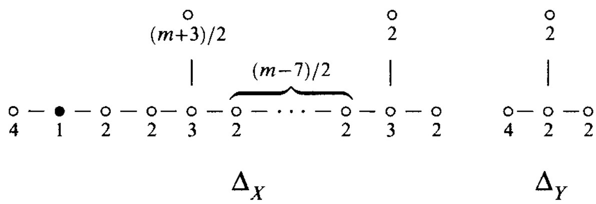
9. General members of $|\mathscr{O}_X|_C$; isolated kAD case

(9.1) Let $f: X \supset C \to Y \ni Q$ be an extremal nbd of type (kAD) with singular points P and R of indices m (≥ 3) and 2. Let H_X be a general member of $|\mathscr{O}_X|$ through C and let $H_Y = f(H_X)$. Let $\Delta_X = \Delta(H_X \supset C)$ and $\Delta_Y = \Delta(H_Y)$.
Our main result of this section is the following.

(9.2) **Theorem.** *If $X \supset C$ is an isolated extremal nbd, then we have the following:*

(9.2.1) *H_X is normal, Δ_X and Δ_Y consist of smooth rational curves intersecting transversely.*

(9.2.2) *Δ_X and Δ_Y are as follows*



where $\underset{3}{\circ} - \overbrace{\underset{2}{\circ} - \cdots - \underset{2}{\circ}}^{(m-7)/2} - \underset{3}{\circ}$ *reduces to* $\underset{4}{\circ}$ *if $m = 5$. To be precise,* $\circ$ *at the left end of Δ_X lies over R and the rest of* $\circ$*'s in Δ_X over P.*

We note first the following.

(9.3) **Proposition.** *If $X \supset C$ of type (kAD) is isolated, then it is as described in* (2.13.10).

Proof. In view of (2.13.13.3), it is enough to show that $X \supset C$ as described in (2.13.3.2) is not isolated. Indeed if such an $X \supset C$ is isolated, then so are arbitrary nearby nbds $X_t^\circ \supset C_t$. By (2.13.3), there are nearby nbds of type ($k3A$), which are divisorial by (5.1). This is a contradiction and we are done. □

We restate (2.13.10) with a slight modification.

(9.4) **Proposition** (Set up). (9.4.1) *We have*

$$(X, P) = (y_1, y_2, y_3)/\mathbb{Z}_m(1, (m+1)/2, -1) \supset (C, P) = y_1\text{-}axis/\mathbb{Z}_m,$$
$$(X, R) = (z_1, z_2, z_3, z_4; \gamma)/\mathbb{Z}_2(1, 1, 1, 0; 0) \supset (C, R) = z_1\text{-}axis/\mathbb{Z}_2,$$

where m is an odd integer ≥ 5 and $\gamma \equiv z_1 z_3 - z_2^2 \bmod (z_4) + z_3 I$.

(9.4.2) *We have an ℓ-splitting*

$$gr_C^1\mathscr{O} = L\tilde{\oplus}(-1 + P^\sharp + R^\sharp),$$

where $L = ((m-1)/2P^\sharp + R^\sharp)$ or $((m-1)/2P^\sharp)$ according as (X, R) is a quotient singularity or not.

Furthermore the ideal $I = I_C$ *contains a* C*-laminal ideal* J *of width* 2 *such that* $J/I^{(2)} \simeq \mathscr{O}$, $(J^\sharp, P^\sharp) = (y_2, y_3^2)$, $(J^\sharp, R^\sharp) = (z_3, z_4)$, *and an* ℓ*-isomorphism and an* ℓ*-splitting*

$$I/J \simeq (-1 + P^\sharp + R^\sharp) \quad \text{with } \ell\text{-bases } (y_3, z_2) \text{ at } P, R,$$
$$gr_C^0 J = (2P^\sharp)\tilde{\oplus}(-1 + \frac{m-1}{2}P^\sharp + R^\sharp),$$

where $(2P^\sharp)$ *(resp.* $(-1 + (m-1)/2P^\sharp + Q^\sharp)$*) has* ℓ*-bases*

$$(y_1^{(m-5)/2}y_2 + y_3^2, z_4) \text{ (resp. } (y_2, z_3)) \quad \text{if } m \geq 7,$$
$$(y_2, z_4) \text{ (resp. } (\mu y_1^m y_2 + y_3^2, z_3)) \quad \text{if } m = 5$$

at P *and* R, *and* $\mu \in \mathscr{O}_{C,P}$.

Proof. Let k be the axial multiplicity of R. We treat the case $k > 1$ first. In view of (2.13.10), we may take $\gamma \equiv z_1 z_3 - z_2^2 \mod (z_3, z_4)I$ and need to check the last part on ℓ-bases. Since $(2P^\sharp) \to L = ((m-1)/2P^\sharp)$ in (2.13.10) induces an isomorphism of invertible sheaves, $(2P^\sharp)$ has ℓ-bases $(y_1^{(m-5)/2}y_2 + \alpha y_3^2, z_4 + \beta z_1 z_3)$ at P and R for some $\alpha \in \mathscr{O}_{C,P}$ and $\beta \in \mathscr{O}_{C,R}$. We note that α is a unit if $m \geq 7$ since $y_1^{(m-5)/2}y_2 + \alpha y_3^2$ is not divisible by y_1. At P, we can make a coordinate change $y_3 \mapsto (\text{unit}) \cdot y_3$ if $m \geq 7$ ($y_2 \mapsto y_2 + (\cdots)y_3^2$ if $m = 5$) so that $(2P^\sharp)$ has the claimed ℓ-basis at P. At R, a coordinate change $z_4 \mapsto z_4 + (\cdots)z_1 z_3$ will make z_4 an ℓ-basis of $(2P^\sharp)$ at R keeping $\gamma \equiv z_1 z_3 - z_2^2 \mod (z_3, z_4)I$. If $m \geq 7$, then the standard choice $gr^{2,1}(\mathscr{O}, J)$ of $(-1 + (m-1)/2P^\sharp + R^\sharp)$ in the ℓ-splitting of $gr_C^0 J$ has the claimed ℓ-bases by [Mori88, (8.11.1.ii)]. If $m = 5$, then among ∞^1 $(-1 + (m-1)/2P^\sharp + R^\sharp)$'s in the ℓ-splitting of $gr_C^0 J$ we will choose one with an ℓ-basis $\mu y_1^m y_2 + y_3^2$ at P for some $\mu \in \mathscr{O}_{C,P}$. A coordinate change $z_3 \mapsto z_3 + (\cdots)z_1 z_4$ at R will attain the assertion on ℓ-bases keeping the other conditions.

We now assume $k = 1$. If we use the expression

$$(X, R) = (z_1, z_2, z_3)/\mathbb{Z}_2(1, 1, 1) \supset (C, R) = z_1\text{-axis}/\mathbb{Z}_2,$$

the previous argument provides us with an ℓ-basis $z_2^2 - \alpha z_1 z_3$ of $(2P^\sharp)$ at P with unit $\alpha \in \mathscr{O}_{C,R}$, because $(2P^\sharp) \to L = ((m-1)/2P^\sharp + R^\sharp)$ induces an isomorphism of invertible sheaves. By a coordinate change $z_1 \mapsto (\text{unit}) \cdot z_1$ at R, we may assume $\alpha = 1$ and will set $z_4 = z_2^2 - z_1 z_3$. The rest is easy. □

(9.5) **Proposition.** *There is a member* $E \in |-K_X|$ *such that*:

(9.5.1) E *is a normal surface smooth outside of* $\{P, R\}$;

(9.5.2) $\mathscr{O}_C(-E)$ $(= \mathscr{O}_C\tilde{\otimes}\mathscr{O}(-E))$ *is equal to* $(-1 + (m-1)/2P^\sharp + R^\sharp)$ *in* (9.4.2);

(9.5.3) *$\mathcal{O}_E(-2C) = J/\mathcal{O}(-E)$ and it is an ℓ-invertible $\mathcal{O}_E$-module with ℓ-isomorphisms*

$$\mathcal{O}_C \tilde{\otimes} \mathcal{O}_E(-iC) = \begin{cases} (iP^\sharp) & i \text{ even}, \\ (-1 + iP^\sharp + R^\sharp) & i \text{ odd}; \end{cases}$$

(9.5.4) $H^0(\mathcal{O}_E(-\nu C)) \twoheadrightarrow H^0(\mathcal{O}_C \tilde{\otimes} \mathcal{O}_E(-\nu C))$ *for all* $\nu \geq 0$,

(9.5.5) *We may change coordinates in* (9.4) *and we can assume* $(E, P) = \{y_2 = 0\}/\mathbb{Z}_m$ *if* $m \geq 7$ *and* $(E, R) = \{z_3 = 0\}/\mathbb{Z}_2$ *in addition to* (9.4).

Proof. Let $D = \{y_1 = 0\}/\mathbb{Z}_m \in |-2K_X|$. We have a surjection $H^0(\mathcal{O}(-K_X)) \to \mathcal{O}_D(-K_X)$ by $H^1(\omega_X) = 0$. Since $gr^i(\mathcal{O}(-K_X), J) \simeq \mathcal{O}_C(-1)$ for $i = 0$ and 1 by (9.4), we have $H^0(F^2(\mathcal{O}(-K_X), J)) = H^0(\mathcal{O}(-K_X))$. Thus the above surjection factors through

$$\begin{aligned} gr^2(\mathcal{O}(-K_X), J) &= gr_C^0 J \tilde{\otimes} \mathcal{O}(-K_X) \\ &= (-1 + \frac{m+5}{2} P^\sharp + R^\sharp) \tilde{\oplus} (0). \end{aligned}$$

Let $\phi \in H^0(F^2(\mathcal{O}(-K_X), J))$ be an element sent to $y_2/dy_1 \wedge dy_2 \wedge dy_3$ mod $y_1 \in \mathcal{O}_D(-K_X)$ $((\mu y_1^m y_2 + y_3^2)/dy_1 \wedge dy_2 \wedge dy_3$ mod y_1 if $m = 5)$ and $E = \{\phi = 0\}$. Since $F^3(\mathcal{O}(-K_X), J)$ is generated by global sections outside C, we may assume that E is smooth outside C. By construction, (9.5.2) is obvious and hence (9.5.1) follows. It is easy to see $(2C, P) = \{y_1^{(m-5)/2} y_2 + y_3^2 = 0\}/\mathbb{Z}_m$ $(\{y_2 = 0\}/\mathbb{Z}_5$ if $m = 5)$ in (E, P) and $(2C, R) = \{z_4 = 0\}/\mathbb{Z}_2$ in (E, R). Thus (9.5.3) follows. (9.5.4) follows from (9.5.3), and (9.5.5) is obvious. □

We will construct C-laminal ideals $J = J_2 \supset J_3 \supset J_4 \supset J_5$ successively.

(9.6) **Proposition.** *Let $J_2 = J$ and J_3 be such that $J_2 \supset J_3 \supset F_C^1 J_2$ and $J_3/F_C^1 J_2 = (2P^\sharp)$ given in* (9.4). *Then*

(9.6.1) *$I \supset J_3$ has ℓ-bases*

$$\begin{gathered} (1, 3, 2)\text{-monomializing } (y_3, y_1^{(m-5)/2} y_2 + y_3^2, y_2) \quad \text{at } P \text{ if } m \geq 7, \\ (1, 3)\text{-monomializing } (y_3, y_2) \quad \text{at } P \text{ if } m = 5, \\ (1, 3, 2)\text{-monomializing } (z_2, z_4, z_3) \quad \text{at } R; \end{gathered}$$

(9.6.2) *We have ℓ-isomorphisms*

$$gr^i(\mathcal{O}, J_3) \simeq \begin{cases} (-1 + P^\sharp + R^\sharp) & \text{if } i = 1, \\ (-1 + \frac{m-1}{2} P^\sharp + R^\sharp) & \text{if } i = 2, \end{cases}$$

$$gr^{3,0}(\mathcal{O}, J_3) \simeq (2P^\sharp),$$

$$gr^{3,1}(\mathcal{O}, J_3) \simeq (-1 + \frac{m+1}{2} P^\sharp) \simeq gr^{2,1}(\mathcal{O}, J_2) \tilde{\otimes} gr^1(\mathcal{O}, J_2);$$

(9.6.3) *The induced ℓ-exact sequence*

$$0 \to gr^{3,1}(\mathcal{O}, J_3) \to gr^3(\mathcal{O}, J_3) \to gr^{3,0}(\mathcal{O}, J_3) \to 0$$

is ℓ-split.

Indeed (9.6.3.1) follows from the definition of monomializing ℓ-bases, (9.6.3.2) from [Mori88, (8.11.1)], and (9.6.3.3) from (9.6.3.2) via

$$H^1(gr^{3,1}(\mathcal{O}, J_3)\tilde{\otimes} gr^{3,0}(\mathcal{O}, J_3)^{\tilde{\otimes}(-1)}) = 0.$$

The following (9.7) and (9.8) can be seen similarly.

(9.7) **Proposition.** *Let J_4 be such that $J_3 \supset J_4 \supset F_C^1 J_3$ and $J_4/F_C^1 J_3$ is the unique subsheaf $(2P^\sharp)$ of $gr^3(\mathcal{O}, J_3)$ given by* (9.6.2). *Then we have ℓ-isomorphism*

$$gr^{4,0}(\mathcal{O}, J_4) \simeq (2P^\sharp),$$
$$gr^{4,1}(\mathcal{O}, J_4) \simeq (-1 + (m-1)P^\sharp) \simeq gr^2(\mathcal{O}, J_3)^{\tilde{\otimes} 2}$$

and the induced ℓ-exact sequence

$$0 \to gr^{4,1}(\mathcal{O}, J_4) \to gr^4(\mathcal{O}, J_4) \to gr^{4,0}(\mathcal{O}, J_4) \to 0$$

is ℓ-exact.

(9.8) **Proposition.** *Let J_5 be such that $J_4 \supset J_5 \supset F_C^1 J_4$ and $J_5/F_C^1 J_4$ is the unique subsheaf $(2P^\sharp)$ of $gr^4(\mathcal{O}, J_4)$ given by* (9.7). *Then*

(9.8.1) *The monomializing ℓ-bases of $I \supset J_3$ at P given in* (9.6) *lift to the* $(1, 5, 2)$- *and* $(1, 5)$-*monomializing ℓ-bases of $I \supset J_5$ at P*

$$(y_3, y_1^{(m-5)/2} y_2 + y_3^2 + \alpha_3 y_1^{(m-3)/2} y_3 + \alpha_4 y_1^{m-3} y_2^2, y_2) \quad \textit{if } m \geq 7,$$
$$(y_3, y_2) \quad \textit{if } m = 5$$

for some α_3, $\alpha_4 \in \mathcal{O}_{C,P}$ by [Mori88, (8.15.1) and (8.16)] (*modulo a coordinate change*

$$y_2 \mapsto y_2 \cdot (\textit{unit}) + y_1 y_3^3(\cdots) + y_1^2 y_3^4(\cdots)$$

if $m = 5$, which keeps the earlier conditions satisfied);

(9.8.2) *We have ℓ-isomorphisms*

$$gr^i(\mathcal{O}, J_5) \simeq \begin{cases} (\frac{i}{2}(-1 + \frac{m-1}{2}P^\sharp + R^\sharp) & \textit{if } i = 2, 4, \\ (-1 + P^\sharp + R^\sharp + \frac{i-1}{2}(-1 + \frac{m-1}{2}P^\sharp + R^\sharp) & \textit{if } i = 1, 3, \end{cases}$$
$$gr^{5,0}(\mathcal{O}, J_5) \simeq (2P^\sharp),$$
$$gr^{5,1}(\mathcal{O}, J_5) \simeq (-1 + R^\sharp)$$

by [Mori88, (8.11.1)].

Under these notation and assumptions, we have the following.

(9.9) **Lemma.** *At $P^\sharp$, we have*

$$F^6(\mathcal{O}, J_5)^\sharp \begin{cases} \subset (y_1 y_2 y_3, y_1 y_2^2, y_3^3, y_2^3, y_2 y_3^2) & \textit{if } m \geq 7, \\ = (y_2 y_3, y_2^2, y_3^6) & \textit{if } m = 5. \end{cases}$$

Proof. Let $u = y_1^{(m-5)/2} y_2 + y_3^2 + \alpha_3 y_1^{(m-3)/2} y_2 y_3 + \alpha_4 y_1^{m-3} y_2^2$, assuming $m \geq 7$. Then $u \in (y_1 y_2, y_3^2)$. By $F^6(\mathcal{O}, J_5)^\sharp = (u^2, u y_3, u y_2, y_2^3)$ [Mori88, (8.11)], we see the assertion. The case $m = 5$ follows from [Mori88, (8.10)]. □

(9.10) **Proposition.** (9.10.1) *There are sections s_1 and $s_2 \in H^0(I)$ satisfying the following relations near P:*

$$s_1 \equiv \begin{cases} (unit)\cdot y_1^2 y_3^2 \bmod y_2 & \text{if } m \geq 7, \\ (unit)\cdot y_1^2 y_2 \bmod \phi & \text{if } m = 5, \end{cases}$$
$$s_2 \equiv \begin{cases} (unit)\cdot y_3^m \bmod y_2 & \text{if } m \geq 7, \\ (unit)\cdot y_2^2 y_3 \bmod \phi & \text{if } m = 5, \end{cases}$$

where ϕ is the equation for E at $P^\sharp$ given in (9.5) *and* $\phi \equiv \mu y_1^m y_2 + y_3^2 \bmod (y_2^2, y_2y_3, y_3^3)$.

(9.10.2) *The coefficient of $y_1^2y_3^2$ (resp. $y_1^2y_2$) in the Taylor expansion of s_1 at P is non-zero if $m \geq 7$ (resp. $m = 5$),*

(9.10.3) *The coefficient of $y_1^2y_3^2$ (resp. $y_1^2y_2$) is zero and the one of y_3^m (resp. $y_2^2y_3$) is non-zero in the Taylor expansion of s_2 at P if $m \geq 7$ (resp. $m = 5$),*

(9.10.4) $H^0(I) \twoheadrightarrow H^0(gr^5(\mathscr{O}, J_5)) = H^0(\mathscr{O}/F^6(\mathscr{O}, J_5)) \simeq \mathbb{C}$.

Proof. We note that y_3^2 (resp. y_2) is the ℓ-free ℓ-basis of $\mathscr{O}_C \tilde{\otimes} \mathscr{O}_E(-2C)$ at P if $m \geq 7$ (resp. $m = 5$) by (9.4.2) and (9.5.3). We have a surjection $H^0(\mathscr{O}_X) \twoheadrightarrow H^0(\mathscr{O}_E)$ by $H^1(\omega_X) = 0$. Hence the assertion on s_1 follows from (9.5.4). The assertion on s_2 is proved similarly. If $m \geq 7$, then (9.10.2) and (9.10.3) are obvious. If $m = 5$, then it is easy to see that the coefficients of $y_1^2y_2$ and $y_2^2y_3$ are zero in the Taylor expansion at P of an arbitrary $\mathbb{Z}_m$-invariant element of $\phi\mathscr{O}_{X^\sharp, P^\sharp}$. Whence (9.10.2) and (9.10.3) follow. By (9.8.2), we see $H^0(gr^i(\mathscr{O}, J_5)) = 0$ for $i \in [1, 4]$ and $H^0(gr^5(\mathscr{O}, J_5)) \simeq \mathbb{C}$. Then $H^0(I) \to H^0(\mathscr{O}/F^6(\mathscr{O}, J_5))$ is non-zero by (9.10.2), and we are done. □

(9.11) **Proposition.** *Let $s \in H^0(I)$ be a general section. Then*

(9.11.1) *$H = \{s = 0\}$ is a normal surface smooth outside $\{P, R\}$, and (H, R) is a rational singularity*

$$(H, R) \simeq (z_1, z_2, z_3; z_1z_3 + z_2^2)/\mathbb{Z}_2(1, 1, 1; 0) \supset (C, R) = z_1\text{-}axis/\mathbb{Z}_2;$$

(9.11.2) *$\Delta((H, R) \supset (C, R))$ is $\bullet - \underset{4}{\circ}$.*

Proof. Since $gr^5(\mathscr{O}, J_5) \to gr^2(\mathscr{O}, J_2)$ induces an ℓ-isomorphism of their subschemes $(2P^\sharp)$, the image $\bar{s}$ of s in $gr_C^1\mathscr{O}$ is nonvanishing outside $\{P, R\}$ and $s \equiv (\text{unit})\cdot z_4 \bmod F^3(\mathscr{O}, J_2)$ at R. At $R^\sharp$, we see $F^3(\mathscr{O}, J_2)^\sharp = (z_3, z_4)\cdot(z_2, z_3, z_4)$ by (9.4.2). Thus we have (9.11.1), and (9.11.2) follows from (9.11.1). □

(9.12) By (9.8.2), the standard ℓ-exact sequence for $gr^5(\mathscr{O}, J_5)$ takes the form

(9.12.1) $$0 \to (-1 + R^\sharp) \to gr^5(\mathscr{O}, J_5) \to (2P^\sharp) \to 0.$$

(9.13) **Proposition.** *If $m \geq 7$, then* (9.12.1) *is not ℓ-split.*

(9.13.1) *Remark.* The isolatedness of the nbd $X \supset C$ is not used in the proof of (9.13).

Proof. Assuming that (9.12.1) is ℓ-split, we will derive $H^1(\omega_X/F^8(\omega_X, J_5)) \neq 0$, which is a contradiction. By (9.8.2) and $gr^0_C\omega \simeq (-1+(m-1)/2P^\sharp + R^\sharp)$, we see $H^i(gr^j(\omega_X, J_5)) = 0$ for all $i = 0, 1$ and $1 \le j \le 4$. Similarly we see $H^i(gr^5(\omega_X, J_5)) = 0$ for $i = 0, 1$. For $gr^6(\mathscr{O}, J_5)$, we have an ℓ-exact sequence

$$0 \to (-1 + \frac{m-3}{2}P^\sharp + R^\sharp) \to gr^6(\mathscr{O}, J_5) \to (-1 + 3P^\sharp + R^\sharp) \to 0$$

by [Mori88, (8.11.1)]. Hence by $m \ge 7$, we see $H^i(gr^6(\omega_X, J_5)) = 0$ for $i = 0, 1$. Thus

$$H^1(\omega_X/F^8(\omega_X, J_5)) \simeq H^1(gr^7(\omega_X, J_5)).$$

We see an ℓ-isomorphism

$$gr^5(\mathscr{O}, J_5)\tilde{\otimes} gr^2(\mathscr{O}, J_5) \simeq gr^7(\mathscr{O}, J_5)$$

and $(9.12.1)\tilde{\otimes} gr^2(\mathscr{O}, J_5)$ is the standard ℓ-exact sequence for $gr^7(\mathscr{O}, J_5)$ [Mori88, (8.11.1)]. Since (9.12.1) is ℓ-split, we have an ℓ-isomorphism

$$gr^7(\omega_X, J_5) \simeq (-2 + (m-1)P^\sharp + R^\sharp)\tilde{\oplus}(P^\sharp).$$

This implies $H^1(gr^7(\omega_X, J_5)) \neq 0$ and we have the contradiction as claimed. □

(9.14) **Proposition.** *If* $m = 5$, *then* (9.12.1) *is not* ℓ*-split.*

(9.14.1) *Remark.* The proof actually shows that $X \supset C$ is not isolated if $m = 5$ and (9.12.1) is ℓ-split. The argument is very similar to the proof of (8.12).

Proof. We assume (9.12.1) is ℓ-split and $X \supset C$ is isolated. Then $gr^5(\mathscr{O}, J_5)$ contains $(2P^\sharp)$ and it has an ℓ-basis $y_2\cdot(\text{unit}) + \alpha_5 y_1^3 y_3^5$ at P for some $\alpha_5 \in \mathscr{O}_{C,P}$. Then a general section $s \in H^0(I)$ satisfies, near P, the following:

$$s \equiv (\text{unit})\cdot(y_1^2 y_2 + \alpha_5 y_1^5 y_3^5) \bmod F^6(\mathscr{O}, J_5)$$

for some $\alpha_5 \in \mathscr{O}_{X,P}$. Let $H = \{s = 0\} \subset X$. By (9.11), H is normal and has the exceptional curve C. We will construct a flat deformation $X_t \supset C_t$ of $X \supset C$.

Construction. Let s_t be a local section at P:

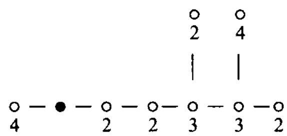
$$s_t = s + t(y_1 y_3^6 + y_2 y_3^3 + y_2^2 y_3) \qquad (t : \text{parameter}).$$

We note $s_t \equiv s \bmod F^6(\mathscr{O}, J_5)$. Let $(X_t, P_t) \supset (C_t, P_t)$ be the trivial deformation of $(X, P) \supset (C, P)$ with parameter $t \in \mathbb{C}$. Let $\mathscr{X} \supset \mathscr{C}$ be the twisted extension of $(X_t, P_t) \supset (C_t, P_t)$ by $(s_t, y_1 y_3)$ with parameter t (cf. [Mori88, (1b.8)]). By the construction, there is a section $\tilde{s} \in H^0(\mathscr{O}_{\mathscr{X}})$ such that

$$\tilde{s}\,|_{\mathscr{X}_t} = \begin{cases} s & \text{outside of a nbd of } P_t, \\ s_t & \text{in a nbd of } P_t. \end{cases}$$

We set $\mathscr{H} = \{\tilde{s} = 0\} \subset \mathscr{X}$. We denote by X_t, H_t, C_t the fibers of $\mathscr{X}$, $\mathscr{H}$, $\mathscr{C}$ over t.

Since H is a normal surface, so is H_t. We know that the images of s and s_t (by $s_t \equiv s \bmod F^6(\mathscr{O}, J_5)$) in $gr^5(\mathscr{O}, J_5)$ and $gr^1_C\mathscr{O}$ are nowhere vanishing on $C - \{P, R\}$. At R_t $(= R)$, the singularity of H_t is as described in (9.11). By the definition of s_t, we can apply (10.9) to study $(H_t, P_t) \supset (C_t, P_t)$, where our y_1, y_2, y_3 correspond to x_1, x_3, x_2 in (10.9). Then $\Delta(H_t \supset C_t)$ is as follows.



Since $5K_{\mathscr{X}}$ is a Cartier divisor, we see $(K_{\mathscr{X}} \cdot C_t) = -1/10 < 0$ by $(K_H \cdot C) = -1/10$. Thus $(\bullet^2) = -1$ and $H_t \supset C_t$ contracts to an A_1-point. Since $X \supset C$ is isolated, so is its nearby nbd $X_t^\circ \supset C_t$. Hence the contraction Y_t of $X_t \supset C_t$ is not Gorenstein. Since H_t is normal, the contraction of C_t in H_t is a hypersurface section of Y_t. This is a contradiction. □

We now finish the proof of (9.2).

(9.15) Since (9.12.1) is not ℓ-split, $gr^5(\mathscr{O}, J_5)$ has a unique subsheaf of $\mathscr{O}_C$ which is generated at P by

$$s_P = \begin{cases} y_1^{(m-1)/2}y_2 + y_1^2y_3^2 + \alpha_3 y_1^{(m+1)/2}y_3 + \alpha_4 y_1^{m-1}y_2^2 + (\text{unit})y_2^2y_3 & \text{if } m \geq 7, \\ y_1^2y_2 + (\text{unit})y_3^5 \quad \text{if } m = 5 \end{cases}$$

modulo $F^6(\mathscr{O}, J_5)$. Hence by (9.10.4), a general section $s \in H^0(I)$ satisfies $s \equiv (\text{unit}) \cdot s_P \bmod F^6(\mathscr{O}, J_5)$ near P and let $s_2 \in H^0(F^6(\mathscr{O}, J_5))$ be a section such that the coefficient of y_3^m (resp. $y_2^2y_3$) is nonzero in its Taylor expansion when $m \geq 7$ (resp. $m = 5$).

Let $H = \{s = 0\}$. Then we know the singularity of H outside $\{P\}$ by (9.11). We apply the case "$a_0 \neq 0$" of (10.7) to study $(H, P) \supset (C, P)$, where our y_1, y_2, y_3 correspond to x_1, x_3, x_2 in (10.7). Using s_2 and (9.9), we can check (10.7.2). If $m = 5$, we see $c = 0$ and $e \neq 0$ in (10.7), hence (10.7.3). Thus we have Δ_X as in (9.2.2), where $(\bullet^2) = -1$ follows by $(K_H \cdot C) < 0$. Thus (9.2) is proved.

10. Sample computations

In this chapter, we recall the notion of weighted projective space and exhibit several computations related to group quotients. There is nothing new in this chapter. The materials are contained only for convenience of reference.

(10.1) **Proposition-Definition.** *Let* $a_1, \dots, a_n > 0$ *be integers with the property that* g.c.d. $(a_1, \dots, a_n) = 1$. *Then we define the weighted projective space*

$$\mathbb{P}(a_1, \dots, a_n) = (\mathbb{C}^n - \{0\})/\mathbb{C}^*,$$

where $\xi \in \mathbb{C}^*$ *acts on* $(x_1, \dots, x_n)$ *by*

$$\xi(x_1, \dots, x_n) = (\xi^{a_1} x_1, \dots, \xi^{a_n} x_n).$$

Then

(10.1.1) *We have*

$$\mathbb{P}(a_1, \dots, a_n) \simeq \mathrm{Proj}\mathbb{C}[x_1, \dots, x_n],$$

where x_i *has weight* a_i $(i = 1, \dots, n)$;

(10.1.2) *Let* $U_i = D_+(x_i) = \{x \in \mathbb{P} \mid x_i \neq 0\}$; *then*

$$U_i = \mathrm{Spec}\,\mathbb{C}[y_1, \dots, \overset{i\text{th}}{1}, \dots, y_n]/\mathbb{Z}_{a_i}(-a_1, \dots, \overset{i\text{th}}{0}, \dots, -a_n),$$

where $\overline{x}_i = x_i^{1/a_i}$ *and* $y_j = x_j/\overline{x}_i^{a_j}$ $(j \neq i)$;

(10.1.3) *The sheaf* $\mathscr{O}_{\mathbb{P}}(r)$ *(by* (10.1.1)) *is locally free if* $a_i \mid r$ $(\forall i)$;

(10.1.4) *If* $D \in |\mathscr{O}_{\mathbb{P}}(s)|$ *for some* $s > 0$, *then we have*

$$(D \cdot \overbrace{\mathscr{O}(r) \cdots \mathscr{O}(r)}^{n-2 \text{ times}}) = \frac{s \cdot r^{n-2}}{a_1 \cdots a_n},$$

for all r.

The proof is left to the reader.

(10.2) Let

$$X = (x_1, \dots, x_n)/\mathbb{Z}_m(a_1, \dots, a_n),$$

where $x_1, \dots, x_n$ are variables and $m, a_1, \dots, a_n > 0$ are integers with the property that g.c.d. $(a_1, \dots, a_n) = 1$. Let

$$e = \frac{1}{m}(a_1, \dots, a_n),$$

$$e_i = (0, \dots, 0, \overset{i\text{th}}{1}, 0, \dots, 0) \in \mathbb{Q}^n \qquad (i \in [1, n]).$$

By the theory of torus embeddings, X corresponds to the lattice $\mathbb{Z}e_1 + \cdots + \mathbb{Z}e_n + \mathbb{Z}e$ and the cone $C(X) = \mathbb{Q}_+ e_1 + \cdots + \mathbb{Q}_+ e_n$ in $\mathbb{Q}^n$, where $\mathbb{Q}_+ = \{z \in \mathbb{Q} \mid z \geq 0\}$.

(10.3) **Proposition-Definition.** *Let* σ *be the* $\mathbb{Z}$*-wt given by* $\sigma(x_i) = a_i$ *(cf.* (T.7)). *Then the* σ*-blow-up* $\pi_\sigma \colon B_\sigma(X) \to X$ *of* X *is the proper birational morphism from a normal variety* $B_\sigma(X)$ *corresponding to the cone decomposition of* $C(X)$ *consisting of* $C_i = \sum_{j \neq i} \mathbb{Q}_+ e_j + \mathbb{Q}_+ e$ *for* $i = 1, \dots, n$ *(and their intersections). Then*

(10.3.1) *The open set* U_i *of* $B_\sigma(X)$ *corresponding to* C_i *is given by*

$$U_i = (y_1, \dots, y_n)/\mathbb{Z}_{a_i}(-a_1, \dots, \overset{i\text{th}}{m}, \dots, -a_n),$$

where $\overline{x}_i = x_i^{1/a_i}$, $y_i = \overline{x}_i^m$, *and* $y_j = x_j/\overline{x}_i^{a_j}$ $(j \neq i)$;

(10.3.2) *The exceptional set* $\Delta = \pi_\sigma^{-1}(0)_{red}$ *is a* $\mathbb{Q}$*-Cartier Weil divisor and*

$$\Delta \cap U_i = \{y_i = 0\}/\mathbb{Z}_{a_i} \qquad \Delta \simeq \mathbb{P}(a_1, \cdots, a_n) \qquad \mathscr{O}_\Delta(r\Delta) \simeq \mathscr{O}_{\mathbb{P}}(-mr)$$

for r *divisible by* Πa_i.

If H is a subvariety of X, then the proper transform of H by π_σ is the σ-blow-up of H.

Comments on proof. Since $\mathbb{Q}^n = \sum_{j\neq i} \mathbb{Q}e_j + \mathbb{Q}e$ for any i, the relation

$$-a_i e_i = me - \sum_{j\neq i} a_j e_i$$

implies (10.3.1) modulo a simple computation. The assertion (10.3.2) follows from (10.3.1) and (10.1). □

(10.4) Let

$$X = (x, y)/\mathbb{Z}_m(1, q),$$

for variables x, y and relatively prime integers m and q such that $1 \leq q < m$. Let $(u_0, v_0) = (0, m)$ and $(u_1, v_1) = (1, q) \in \mathbb{Z}^2$. The elements $(u_{i+1}, v_{i+1}) \in \mathbb{Z}^2$ and the integers $a_i \geq 2$ $(i \geq 1)$ are inductively defined by

$$(u_{i+1}, v_{i+1}) = a_i(u_i, v_i) - (u_{i-1}, v_{i-1}) \qquad (0 \leq v_{i+1} < v_i)$$

while $i \leq r$, where $r > 0$ is the integer such that $v_r > 0$ and $v_{r+1} = 0$.

(10.5) **Proposition** [Hirzebruch53]. (10.5.1) $(u_{r+1}, v_{r+1}) = (m, 0)$.

(10.5.2) *Let $\pi : Y \to X$ be the minimal resolution. Then π has exactly r exceptional curves $C_1, \ldots, C_r$. After appropriate renumbering, $C_0, C_1, \ldots, C_r$, and C_{r+1} form a linear chain, where C_0 and C_{r+1} are the proper transforms of x-axis $/\mathbb{Z}_m$ and y-axis $/\mathbb{Z}_m$. Furthermore $(C_i)^2 = -a_i$ for $i \in [1, r]$*

(10.5.3)
$$\frac{m}{q} = a_1 - \cfrac{1}{a_2 - \cfrac{1}{\ddots - \cfrac{}{a_r}}}$$

where $a_i \geq 2$ for all i.

(10.5.4) *For arbitrary $c, d \in \mathbb{Z}$ such that $c + qd \equiv 0 \bmod (m)$, the Cartier divisor $\{x^c y^d = 0\}$ is pulled back to*

$$\{\pi^*(x^c y^d) = 0\} = \sum_{i=0}^{r+1} (cu_i + dv_i) C_i.$$

We list a few computations which are used in Chapters 8 and 9.

(10.6) Let (H, P) be a normal surface singularity

$$(H, P) = (x_1, x_2, x_3; h)/\mathbb{Z}_m(2, m-2, 1; 0),$$
$$(Y_0, P) = (\text{the locus of } (t^2, t^{m-2}, 0))/\mathbb{Z}_m(2, m-2, 1),$$
$$(Y_1, P) = x_1\text{-axis}/\mathbb{Z}_m(2, m-2, 1),$$

where m is an odd integer ≥ 5. Let σ be the $\mathbb{Z}$-wt given by $\sigma(x_1, x_2, x_3) = (2, m-2, 1)$. We note $\sigma(h) \geq m$. We remark that x_1x_2, $x_2x_3^2$, and $(m+1)/2$ terms $x_1^{(m-1)/2-i}x_3^{2i+1}$ $(i = 0, \ldots, (m-1)/2)$ are all the monomials with σ-wt $= m$.

(10.7) **Computation.** *In the Taylor expansion of* h, *assume that* x_1x_2 (*resp.* $x_2x_3^2$) *appears with coefficient* 0 (*resp.* 1) *and let* a_0, b, c, d, *and* $e \in \mathbb{C}$ *be the coefficients of* $x_1^{(m-1)/2}x_3$, x_1^m, $x_1^2x_2^2$, $x_2^{(m+1)/2}x_3$, *and* x_2^m. *Assume that*

(10.7.1) $a_0 \neq 0$ *or* $bc \neq 0$, *and*

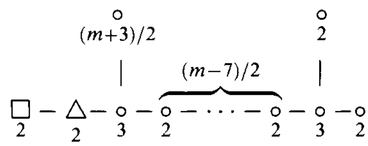
(10.7.2) $d^2 - 4e \neq 0$.

If $m = 5$ *and* $a_0 \neq 0$, *then assume furthermore*

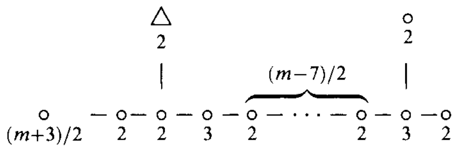
(10.7.3) $a_0Z_1Z_3 + Z_3^2 + cZ_1Z_2 + dZ_2Z_3 + eZ_2^2$ *is a nondegenerate quadratic form in* (Z_1, Z_2, Z_3).

Then (H, P) *is a rational singularity and the configuration of the exceptional curves in the minimal resolution* H' *of* (H, P) *is the following*:

(10.7.3.1) Case $a_0 \neq 0$.



(10.7.3.2) Case $a_0 = 0$.



where

(10.7.3.3) *The configuration* $\underset{3}{\circ} - \overbrace{\underset{2}{\circ} - \cdots - \underset{2}{\circ}}^{(m-7)/2} - \underset{3}{\circ}$ *reduces to* $\underset{4}{\circ}$ *if* $m = 5$;

(10.7.3.4) *If* $Y_0 \subset H$, *then* $b + c = 0$ *and the proper transform* Y_0' *in* H' *intersects only with* $\triangle$ *and* $(Y_0' \cdot \triangle) = 1$;

(10.7.3.5) *If* $Y_1 \subset H$, *then* $b = 0$ *and the proper transform* Y_1' *in* H' *intersects only with* $\square$ *and* $(Y_1' \cdot \square) = 1$.

(10.7.4) *Remark.* (10.7.4.1) Under the notation of (10.7), we have

$$h_{\sigma=m} = x_2x_3^2 + \sum_{i=0}^{(m-1)/2} a_i x_1^{(m-1)/2-i} x_3^{1+2i}$$

for some $a_i \in \mathbb{C}$. The σ-blow-up $\pi_\sigma : B_\sigma(H) \to H$ of H is the proper transform of H under

$$\pi_\sigma : B_\sigma(X) \to X, \quad \text{where } X = (x_1, x_2, x_3)/\mathbb{Z}_m(2, m-2, 1).$$

The exceptional set E of $B_\sigma(H) \to H$ is a curve $\subset \Delta = \mathbb{P}(2, m-2, 1)$ defined by the weighted homogeneous equation $h_{\sigma=m} = 0$.

(10.7.4.2) Assume that we are in case (10.7.3.1). Then E is a reduced curve with exactly two components; one (denoted by E') defined by $x_3 = 0$ corresponding to $\triangle$ and the other (denoted by E'') corresponding to $\circ$ with

$(m+3)/2$ in the configuration (10.7.3.1); $E' \cap E'' = \{(0:1:0)\}$, and $\operatorname{Sing} B_\sigma(H) = \{(1:0:0), (0:1:0)\} \subset \Delta$, and $(1:0:0)$ is an ordinary double point of $B_\sigma(H)$ corresponding to $\square$. These can be checked by a direct computation using (10.1–10.3), and a further computation at $(0:1:0)$ proves (10.7.3.1).

(10.7.4.3) Later in Chapter 13, we will consider a family H_t defined by $h + t(x_1x_2 - x_3^m) = 0$ (denoted by $h_t = 0$) with h in (10.7.3.1) and variable coefficient t. For generic t, it is easy to see that (H_t, P) is a rational singularity and the configuration of the exceptional curves in the minimal resolution H_t' of (H_t, P) is

(10.7.4.4)
$$\underset{2}{\square} - \underset{(m+5)/2}{\triangle} - \overbrace{\underset{2}{\circ} - \cdots - \underset{2}{\circ}}^{(m-5)/2} - \underset{3}{\circ}$$

Although H_t' does not fit in a flat family, one can construct a flat family of simultaneous blow-ups of H_t and see how some curves specialize as $t \to 0$ as follows. We note $\sigma(h) = \sigma(h_t) = m$ and

$$(h_t)_{\sigma=m} = h_{\sigma=m} + t(x_1x_2 - x_3^m).$$

Thus the σ-blow-up $B_\sigma(H_t) \to H_t$ exists in a flat family; and for a generic t the exceptional set $E_t \subset \Delta$ of $B_\sigma(H_t)$ is an irreducible curve corresponding to $\triangle$ in (10.7.4.4) and it passes through $(1:0:0)$ and $(0:1:0)$, $\operatorname{Sing} B_\sigma(H_t) = \{(1:0:0), (0:1:0)\}$, and $(1:0:0)$ is an ordinary double point corresponding to $\square$. These can also be checked similarly to (10.7.4.2).

In summary, we see that the blow-up $Y_t = B_{(1:0:0)}B_\sigma(H_t)$ of $B_\sigma(H_t)$ at $(1:0:0)$ deforms in a flat family; $H_0' \to Y_0$ is a morphism contracting exactly the exceptional curves marked $\bullet$ in

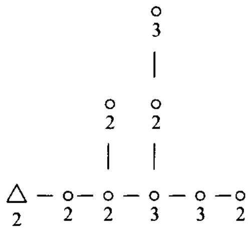
$$\begin{array}{ccccccccccc} & & & & \triangle & & & & \bullet & & \\ & & & & | & & & & | & & \\ \square & - & \triangle & - & \bullet & - & \overbrace{\bullet - \cdots - \bullet}^{(m-7)/2} & - & \bullet & - & \bullet \end{array}$$

and $H_t' \to Y_t$ with generic t contracts $\bullet$'s in

$$\square - \triangle - \overbrace{\bullet - \cdots - \bullet}^{(m-5)/2} - \bullet$$

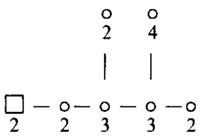
$\square$ (resp. $\triangle$) in Y_t specializes to the reduced curve which consists of $\square$ (resp. two $\triangle$) in Y_0.

(10.8) **Computation.** Under the notation of (10.6), assume that $m = 5$ and that the coefficients of x_1x_2, $x_2x_3^2$, $x_1^2x_3$ (resp. $x_1x_3^3$, x_1^5, $x_1^2x_2^2$, $x_2^3x_3$) are all 0 (resp. all nonzero) in the Taylor expansion of h. Then (H, P) is a rational singularity and the configuration of the exceptional curves in the minimal resolution H' of (H, P) is



where if $Y_0 \subset (H, P)$ then the proper transform Y_0' of Y_0 in H' intersects only with $\triangle$ and $(Y_0' \cdot \triangle) = 1$.

(10.9) **Computation.** Under the notation of (10.6), assume that $m = 5$ and that the coefficients of x_1x_2, x_1^5, $x_1^2x_2^2$, $x_1^3x_2^3$, x_2^5 (resp. $x_2x_3^2$, $x_1^2x_3$, $x_2^3x_3$, $x_1x_2^6$) are all 0 (resp. all nonzero) in the Taylor expansion of h. Then (H, P) is a rational singularity and the configuration of the exceptional curves in the minimal resolution H' of (H, P) is



where if $Y_1 \subset (H, P)$ then the proper transform Y_1' of Y_1 in H' intersects only with $\square$ and $(Y_1' \cdot \square) = 1$.

11. How to flip

The aim of this chapter is to give a somewhat new proof of the existence of flips. This proof will then work to show that flips are continuous in families. The result depends on viewing an extremal nbd as a one parameter family of surfaces and then understanding the deformation theory of certain surface singularities. For quotient singularities the theory was developed in [KSB88]. We recall the relevant facts.

(11.1) **Definition.** (11.1.1) A quotient singularity is called a T-singularity if it is a DuVal singularity or is analytically equivalent to

$$(xy - z^{dn} = 0)/\mathbb{Z}_n(1, -1, a) \qquad \text{where } (a, n) = 1.$$

In [KSB88, 3.11] it is explained how to recognize these singularities from the dual graph of their minimal resolution.

(11.1.2) Given a T-singularity $P \in U$ the deformations of the form

$$(xy - z^{dn} + tf(x, y, z, t) = 0)/\mathbb{Z}_n(1, -1, a, 0)$$

fill out a whole component of the deformation space. We call this the qG-component (for $\mathbb{Q}$-Gorenstein) and denote it by $\mathrm{Def}^{qG}(P \in U)$. Any qG-deformation of a T-singularity is again a T-singularity.

(11.1.3) If U is a complex space with only T-singularities then by $\operatorname{Def}^{qG} U$ we mean the closed subset of the deformation space $\operatorname{Def} U$ of U consisting of deformations that induce a qG-deformation everywhere locally. This makes sense since $\operatorname{Def}^{qG} U$ is the preimage of the product of the qG-components under the natural morphism

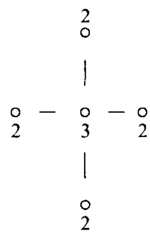
$$\operatorname{Def} U \to \prod_i \operatorname{Def}(P_i \in U).$$

A deformation of U is called a qG-deformation if it can be obtained from $\operatorname{Def}^{qG} U$ via base change.

(11.1.4) Let V be a quotient singularity. A P-modification $f : U \to V$ is a proper bimeromorphic morphism f such that U is normal with only T-singularities and $K_{U/V}$ is f-ample.

A large part of the theory in [KSB88, Chapter 3] can be generalized to more general rational singularities (see [Kollár91, Chapter 6]), though at the moment the scope of this generalization or even the correct definitions are unclear. Since we need to have this for a single class of singularities only, we make the necessary definitions only in this case. It should be clear that these definitions are adopted for temporary convenience only and they should be changed if used for any other purpose.

(11.2) **Definition.** (11.2.1) A nonquotient rational singularity is called a T-singularity if the dual graph of its minimal resolution has the form



These singularities are log-canonical and they are quotients of certain elliptic double points

$$(x^2 + g(y, z) = 0)/\mathbb{Z}_2(1, 1, 1)$$

where g is a homogeneous polynomial of degree four without multiple factors..

(11.2.2) Given a T-singularity $P \in U$ as above the deformations of the form

$$(x^2 + g(y, z) + tf(x, y, z, t) = 0)/\mathbb{Z}_2(1, 1, 1, 0)$$

fill out a whole component of the deformation space. We call this the qG-component (for $\mathbb{Q}$-Gorenstein) and denote it by $\operatorname{Def}^{qG}(P \in U)$. It has dimension 5. From the explicit description it is easy to see that any qG-deformation of a T-singularity is again a T-singularity.

(11.2.3) If U is a complex space with only T-singularities then by $\operatorname{Def}^{qG} U$ we mean the closed subset of the deformation space $\operatorname{Def} U$ of U consisting of deformations that induce a qG-deformation everywhere locally. This makes sense since $\operatorname{Def}^{qG} U$ is the preimage of the product of the qG-components under the natural morphism

$$\operatorname{Def} U \to \prod_i \operatorname{Def}(P_i \in U).$$

A deformation of U is called a qG-deformation if it can be obtained from $\operatorname{Def}^{qG} U$ via base change.

(11.2.4) Let V be any rational singularity. A P-modification $f : U \to V$ is a proper bimeromorphic morphism f such that U is normal with only T-singularities and $K_{U/V}$ is f-ample.

(11.3) **Comparing certain deformation spaces.** We will study the relationship between deformation spaces of certain analytic spaces. It is very convenient if these deformation spaces exist not only as formal schemes but as germs of analytic spaces. The existence is known in the following three cases:

(11.3.1) U is a compact complex space [Grauert74];

(11.3.2) $0 \in V$ is the germ of an isolated singularity [Grauert72];

(11.3.3) U is a proper modification of an isolated singularity [Bingener87]. More precisely, there is a proper morphism $f : U \to V$ where $0 \in V$ is a germ of an isolated singularity and f^{-1} is an isomorphism outside 0.

By $\operatorname{Def} U$ resp. $\operatorname{Def}(0 \in V)$ we denote either the germ or a suitable analytic representative of the versal deformation space of the corresponding objects. This will not lead to any confusion. Let $v : \mathscr{V} \to \operatorname{Def} V$ and $u : \mathscr{U} \to \operatorname{Def} U$ be the versal families.

(11.4) **Proposition.** *Use the same notation as above.* (11.4.1) *Let* $f : U \to V$ *be a proper morphism of complex spaces. Assume that* $f_*\mathscr{O}_U = \mathscr{O}_V$ *and* $R^1 f_*\mathscr{O}_U = 0$. *Assume furthermore that either* U *and* V *are proper or that* $f : U \to V$ *is as in* (11.3.3). *Then there are natural morphisms* F *and* $\mathscr{F}$ *that make the following diagram commutative*:

(11.4.1.1)
$$\begin{array}{ccc} \mathscr{U} & \xrightarrow{\mathscr{F}} & \mathscr{V} \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ \operatorname{Def} U & \xrightarrow{F} & \operatorname{Def} V \end{array}$$

Here v *and* u *are the projections* $F[U] = [V]$ *and* $\mathscr{F}|U = f$.

(11.4.2) *Let* $0 \in V$ *be the germ of an isolated singularity and let* $f : U \to V$ *be a proper morphism such that* f^{-1} *is an isomorphism outside* 0. *Assume that* $f^{-1}(0)$ *is a curve and that* U *has only finitely many singularities* $P_i \in U$. *Then the natural morphism*

$$\operatorname{Def} U \to \prod \operatorname{Def}(P_i \in U)$$

is smooth, in particular, surjective.

Proof. In the formal category the diagram (11.4.1.1) exists in both cases by [Wahl76].

To see that the diagram (11.4.1.1) exists in the category of analytic spaces consider first the case when U and V are proper. Let $W \subset U \times V$ be the graph of f. We want to find the graph of F that is a closed subset $\mathscr{W} \subset \mathscr{U} \times \mathscr{V}$ such that the first projection is an isomorphism and $\mathscr{W} \cap (U \times V) = W$. A component D of the relative Douady space of $\mathscr{U} \times \mathscr{V}/\operatorname{Def} U \times \operatorname{Def} V$ parametrizes graphs of morphisms and D is an analytic space. The projection morphism $[W] \in D \to 0 \in \operatorname{Def} U$ has a formal section given by the formal contraction morphism. Thus by [Artin68, 1.5] it has an analytic section. This gives (11.4.1.1) in the category of analytic spaces

If $f : U \to V$ is a proper modification of an isolated singularity as in (11.3.3) then the contraction morphism f extends to a contraction morphism $\mathscr{F} : \mathscr{U}/\operatorname{Def} U \to Z/\operatorname{Def} U$ by results of [Markoe-Rossi71, Siu71]. (Note that in general Z is not a deformation of V.) If I_U is the ideal sheaf of $U \subset \mathscr{U}$ then we get the exact sequence

$$\mathscr{F}_* \mathscr{O}_{\mathscr{U}} \to \mathscr{F}_* \mathscr{O}_U \to R^1 \mathscr{F}_* I_U.$$

Since $R^1 f_* \mathscr{O}_U = 0$, the theorem of formal functions gives that $R^1 \mathscr{F}_* I_U = 0$. Thus the central fiber of $Z/\operatorname{Def} U$ is isomorphic to V and $Z/\operatorname{Def} U$ is flat. Thus Z is the total space of a deformation of V. Thus we have a (nonunique) morphism $F : \operatorname{Def} U \to \operatorname{Def} V$ such that $\operatorname{Def} V \times_F \mathscr{U} \cong Z$. This is what we want.

To see (11.4.2) it is sufficient to note that the obstruction to globalize a deformation in $\prod \operatorname{Def}(P_i \in U)$ lies in $R^2 f_* T_U$. This is zero since f has only one-dimensional fibers. Therefore $\operatorname{Def} U \to \prod \operatorname{Def}(P_i \in U)$ is smooth of relative dimension $\dim(R^1 f_* T_U)$. □

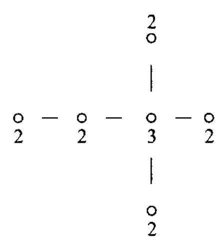
(11.5) Let $f : U \to V$ be a P-modification of a rational singularity V. Let $\mathscr{U}^{qG} \to \operatorname{Def}^{qG}(U)$ be the versal qG-deformation of U. By (11.4) we obtain a diagram

$$\begin{array}{ccc} \mathscr{U}^{qG} & \xrightarrow{\mathscr{F}^U} & \mathscr{V}^U \\ \downarrow & & \downarrow \\ \operatorname{Def}^{qG}(U) & \longrightarrow & Z^U \end{array}$$

By definition $\mathscr{F}^U : \mathscr{U}^{qG} \to \mathscr{V}^U$ is proper and $K_{\mathscr{U}^{qG}/\mathscr{V}^U}$ is $\mathbb{Q}$-Cartier and relatively ample. $\mathscr{F}^U$ is an isomorphism over the smooth locus of $\mathscr{V}^U/Z^U$ since a smooth surface has no modification with relatively ample canonical class.

(11.6) **Proposition.** *Let $P \in V$ be either a quotient singularity or a singularity with the following dual resolution graph:*

(11.6.1)



Let $0 \in Z$ *be the normalization of a component of* $\operatorname{Def}(P \in V)$ *and let* $v_Z : \mathscr{V}^Z \to Z$ *be the versal family of deformations over* Z. *Then there is a unique P-modification* $f : U \to V$ *such that*

$$(v_Z : \mathscr{V}^Z \to Z) \cong (\mathscr{V}^U \to Z^U).$$

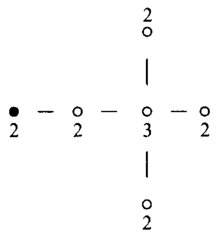
In particular, we get a proper morphism $\mathscr{F}^U : \mathscr{U}^{qG} \to \mathscr{V}^Z$ *such that:*

(11.6.2.1) $K_{\mathscr{U}^{qG}/\mathscr{V}^Z}$ *is* $\mathbb{Q}$*-Cartier and relatively ample;*

(11.6.2.2) $\mathscr{F}^U$ *has at most one dimensional fibers;*

(11.6.2.3) $\mathscr{F}^U$ *is an isomorphism above the smooth locus of* v_Z.

Proof. If $P \in V$ is a quotient singularity then this was proved in [KSB88, Chapter 3]. Any singularity as in (11.6.1) is a rational quadruple point. This case was treated in [Stevens91b]. According to his results, there are two P-modifications of this singularity. U_1 is the minimal DuVal resolution and U_2 is the resolution obtained by contracting all the curves except the one marked •.



□

This result easily implies the existence of flips in families:

(11.7) **Theorem.** *Let* $f_0 : X_0 \supset C_0 \to Y_0 \ni Q_0$ *be an extremal nbd. Let* $\mathscr{X} \to S$ *be a flat deformation of* X_0 *over the germ of a normal space* $0 \in S$. *Then*

(11.7.1) f_0 *extends to a contraction morphism* $F : \mathscr{X} \to \mathscr{Y}$;

(11.7.2) *The flip* $F^+ : \mathscr{X}^+ \to \mathscr{Y}$ *exists and commutes with any base change* $S' \to S$;

(11.7.3) *If* H' *is a general hypersurface section of* Y_0 *through* Q_0 *then* $\mathscr{X}^+$ *is obtained as the total space of a* qG*-deformation of a suitable P-modification* $H^+ \to H'$ *over the base space* $S \times \Delta$.

Proof. The contraction morphism F exists by (11.4.1). This gives us $\mathscr{Y}$, which is a flat deformation of Y_0 over S. By (1.8) the general surface section H' of Y_0 through Q_0 is either a quotient singularity or a singularity as in (11.2.6). Therefore we can view $\mathscr{Y}$ as a flat deformation of H' over $S \times \Delta$. This gives a morphism $m : S \times \Delta \to Z \subset \operatorname{Def} H'$ where Z is a component of $\operatorname{Def} H'$. Using the notation of (11.6) we get that

$$\mathscr{Y} \cong \mathscr{V}^Z \times_Z (S \times \Delta).$$

Now we construct the flip as

$$(F^+ : \mathscr{X}^+ \to \mathscr{Y}) \cong (\mathscr{F}^U \times m : \mathscr{U}^{qG} \times_Z (S \times \Delta) \to S \times \Delta).$$

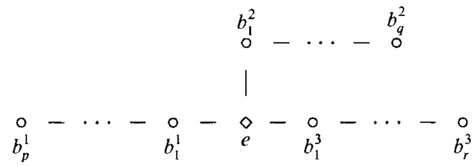
The fibers of F^+ are at most one-dimensional and, in fact, zero-dimensional over the smooth locus of $\mathscr{Y}/(S \times \Delta)$. Thus the exceptional set has codimension at least two. Furthermore, $K_{\mathscr{X}^+/\mathscr{Y}}$ is $\mathbb{Q}$-Cartier and relatively ample since these properties are preserved under base change. Therefore $\mathscr{X}^+$ is the flip of $\mathscr{X}$. The flip clearly commutes with base change and (11.7.3) directly follows from the construction. □

(11.8) **Guessing the P-modification.** Let $X \supset C$ be an extremal nbd. By (11.7.3) it is clear that the P-modification H^+ is one of the most important invariants of X^+. Knowing it tells us for instance the number of exceptional curves after flip and the indices of the singularities of X^+.

In [KSB88, 3.14] there is an algorithm to compute all P-modifications of a quotient singularity. Unfortunately this algorithm is rather tedious. Recently better algorithms were found by [Christophersen91, Stevens91a]. Frequently there are several P-modifications as natural candidates. Therefore it is important to have some additional information. We will prove for instance (13.5) that if $X \to Y$ is an extremal nbd then the exceptional curve of $X^+ \to Y$ is irreducible. This means that we only have to consider those P-modifications that have only one exceptional curve C. The computationally messy condition that $K_{U/V}$ be nef reduces in this case to computing the coefficient of C in the relative canonical class. $K_{U/V}$ is nef iff this coefficient is negative. The computations are especially easy for noncyclic quotient singularities.

Consider a noncyclic quotient singularity V with the following dual graph of the minimal resolution:

(11.8.1)

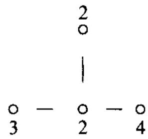


Let $U \to V$ be a P-modification with only one exceptional curve $C \subset U$ and let $U' \to U \to V$ be the minimal resolution of U. Let $C' \subset U'$ be the proper transform of C.

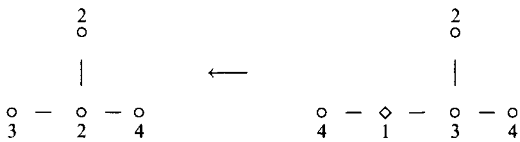
If U' is also the minimal resolution of V then C' is one of the curves in (11.8.1). After removing this curve we have only T-singularities, in particular, the remaining graph contains no vertex with degree three. In particular, C' is one of the curves adjacent to $\diamond$ or $\diamond$ itself. In all four cases it is easy to see if the complement has T-singularities.

If U' is not the minimal resolution of V then C' is a (-1)-curve on U'. Thus U' is obtained from the configuration of (11.8.1) by repeatedly blowing up intersection points of certain curves. At the end, after removing C' the remaining graph contains no vertex with degree three. Thus at each step we have to blow up an intersection point of $\diamond$ and another curve. This reduces the number of possibilities to a handful of cases.

(11.8.2) **Example.** We will need to study the icosahedral quotient singularity



Let U be a P-modification with only one exceptional curve. It is easy to check that U' cannot be dominated by the minimal resolution. Short computation gives that the only P-modification with one exceptional curve is obtained as follows:



Here $\diamond$ becomes the curve C'. Thus the flip has one index two point and one index three point.

(11.9) **Flips and local Picard groups.** Sometimes additional restrictions can be obtained by computing the rank of the local Picard group of Y in two ways. The computations rest on the following simple result:

(11.9.1) *Claim.* Let $0 \in V$ be an isolated analytic threefold singularity and let $f : U \to V$ be a bimeromorphic morphism. Assume that f is an isomorphism outside 0, U has only finitely many singular points $P_j \in U$, and $f^{-1}(0)$ is one-dimensional.

If the rank of the local Picard group of $0 \in V$ is finite then

$$\operatorname{rank}(\operatorname{Pic}(0 \in V)) = \sum \operatorname{rank}(\operatorname{Pic}(P_j \in U)) + \#(\text{irreducible components of } f^{-1}(0)). \quad \square$$

In order to apply this we need some information about the local Picard groups of terminal singularities. One easy result is the following:

(11.9.2) **Lemma.** *Assume that*

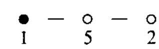
$$Y = (xy + f(z^m, t) = 0)/\mathbb{Z}_m(a, -a, b, 0) \qquad (ab, m) = 1.$$

defines an isolated singularity. If z^m appears with nonzero coefficient in $f(z^m, t)$ then Y is $\mathbb{Q}$-factorial.

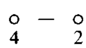
Proof. This follows directly from [Kollár91, 2.2.7]. $\square$

The following examples will be needed later.

(11.9.3) **Examples.** (11.9.3.1) Let $f : X \supset C \to Y \ni Q$ be an extremal nbd. Let $H' \subset Y$ be a hypersurface section through Q and let $H = f^* H'$. Assume that the configuration of compact curves on the minimal resolution of H is



where $\bullet$ is the proper transform of C. This contracts to the quadruple point

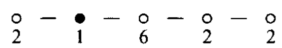


Claim. The flip of the above extremal nbd has an index two point.

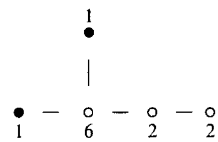
Proof. We view the nbd as a one-parameter family $H_t : t \in \Delta$ where $H_0 = H$. We can apply any base change $t = t'^m$. This way we get a nbd $X_m \to Y_m$ with a single singular point on X_m having equation $y_1 y_3 - y_2^3 + y_4 h(y_1, y_2, y_3, y_4) = 0/\mathbb{Z}_3(1, 1, 2, 0)$. By (11.9.2) this is always $\mathbb{Q}$-factorial. Therefore the rank of the local Picard group of Y_m is one.

The flip is given as a family $H_t^+ : t \in \Delta$ where H_0^+ is a P-modification of H'. It is easy to see that there are only two P-modifications. Assume that H_0^+ is the minimal DuVal resolution of H'. Then, after a base change, this family can be blown up to a deformation of the minimal resolution. Thus the rank of the local Picard group of Y_m is two for suitable m since we can have two exceptional curves. This is a contradiction. Thus the flip has to be given by the only other P-modification. This contracts the (-4)-curve, hence gives rise to an index two point. □

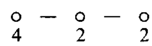
(11.9.3.2) Let $f : X \supset C \to Y \ni Q$ be an extremal nbd with a possibly reducible central curve. Let $H' \subset Y$ be a hypersurface section through Q and let $H = f^* H'$. Assume that the configuration of compact curves on the minimal resolution of H is one of the following:



or



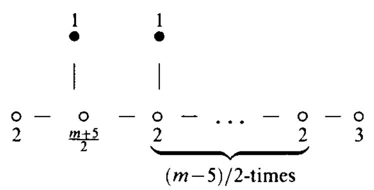
where • are the proper transforms of the f-exceptional curves. Both contract to



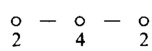
Claim. The flip of the above extremal nbd has an index two point.

Proof. Again we have only two P-modifications. We exclude the minimal DuVal resolution as before because after a base change this would give three exceptional curves. □

(11.9.3.3) Let $f : X \supset C \to Y \ni Q$ be an extremal nbd with a reducible central curve. Let $H' \subset Y$ be a hypersurface section through Q and let $H = f^* H'$. Assume that the configuration of compact curves on the minimal resolution of H is



where • denotes the proper transforms of the f-exceptional curves. Repeatedly contracting (-1)-curves we obtain



Claim. The flip of the above extremal nbd has an index two point.

Proof. Again we have only two P-modifications. The singular point on H has equation

$$(xy - z^m = 0)/\mathbb{Z}_m(2, m-2, 1).$$

Thus (11.9.2) again applies and we exclude the minimal DuVal resolution as before because after a base change this would give three exceptional curves. □

Finally for later reference we consider flops in families.

(11.10) **Theorem.** *Let $f_0 : X_0 \to Y_0$ be a proper morphism between normal threefolds. Assume that X_0 has only terminal singularities. Assume that f_0 contracts a curve $C_0 \subset X_0$ to a point $Q_0 \in Y_0$ and that K_{X_0} has zero intersection with any component of C_0. Let $X_S \to S$ be a flat deformation of X_0 over the germ of a complex space $0 \in S$. Then*

(11.10.1) *f_0 extends to a contraction morphism $F_S : X_S \to Y_S$;*

(11.10.2) *The flop $F_S^+ : X_S^+ \to Y_S$ exists and commutes with any base change $S' \to S$.*

Proof. By [Kollár91, 2.2] the flop of X_0 is independent of the choice of an f_0-ample divisor.

Let U be a miniversal deformation space of X_0 and let $X_U \to U$ be the corresponding deformation. By (11.4.2) U is smooth. By (11.4.1) there is a contraction morphism $F_U : X_U \to Y_U$ which induces f_0 on X_0. Therefore $Y_U \to U$ is a flat deformation of Y_0. By the classification of terminal singularities we can represent Y_U in the form of a hypersurface quotient

$$(x^2 + f(y, z, t, u_1, \dots, u_k) = 0)/\mathbb{Z}_n$$

or

$$(xy + f(z, t, u_1, \dots, u_k) = 0)/\mathbb{Z}_n,$$

where the coordinates are eigenfunctions of the group action and the u_i are coordinates on U. The construction of the flop given in [Kollár91, 2.2] is unchanged if we replace the single coordinate t used there with a collection of coordinates $(t, u_1, \dots, u_k)$. In particular, the flop $F_U^+ : X_U^+ \to Y_U$ exists.

There is a morphism $g : S \to U$ such that $X_S/S \cong g^* X_U/S$. Let $Y_S = g^* Y_U$, $X_S^+ = g^* X_U^+$ and let $F_S^+ : X_S^+ \to Y_S$ be the natural morphism. On every fiber, X_S^+ is the flop of X_S. Since the flop is unique, X_S^+ is the flop of X_S. □

(11.11) **Corollary.** *Use the same notation as in* (11.10). *Let S_{gen} be a generic point of S and let X_{gen} be the fiber of X_S over S_{gen}. Let $\{C_0^i\}$ be the irreducible components of C_0. Then for every i, there is a curve $C_{\text{gen}}^i \subset X_{\text{gen}}$ such that C_{gen}^i specializes to a multiple of C_0^i.*

Proof. By replacing X_S with an analytic neighborhood of C_0^i we may assume that $C_0 = C_0^i$. Thus we only need to prove that $X_{\text{gen}} \to Y_{\text{gen}}$ is not finite. Assume the contrary. Let $T = \operatorname{Spec} \mathbb{C}[[t]]$. After a sufficiently general base change, $T \to S$ we get $f_T : X_T \to Y_T$, which is an isomorphism outside Q_0.

Let $D_T \subset X$ be a general hyperplane section through a general point of C_0^i. Therefore $f_T(D_T)$ is Cartier outside Q_0 but not $\mathbb{Q}$-Cartier at Q_0. Y_T is a quotient of a four-dimensional hypersurface singularity, hence parafactorial [Grothendieck68, XI.3.13]. This is a contradiction. □

12. Applications

(12.1) **Factoriality and deformations.** In this section we consider the behavior of factoriality and $\mathbb{Q}$-factoriality under flat deformations.

(12.1.1) **Lemma.** *Let $y \in Y$ be the germ of an analytic space. Let $f : X \to Y$ be a proper morphism such that $f_*\mathcal{O}_X = \mathcal{O}_Y$. Assume that X is smooth and that $f^{-1}(y) = \bigcup D_i$ is a divisor with normal crossings only. Then*

(12.1.1.1) *$R^1 f_*\mathcal{O}_X = 0$ implies that $H^1(\bigcup D_i, \mathbb{Z}) = 0$.*

(12.1.1.2) *$R^1 f_*\mathcal{O}_X = R^2 f_*\mathcal{O}_X = 0$ implies that $\mathrm{Pic}(X) \to H^2(\bigcup D_i, \mathbb{Z})$ is an isomorphism and $H_2(\bigcup D_i, \mathbb{Q})$ is generated by algebraic cycles.*

Proof. Consider the exponential sequence on X and apply f_*. We obtain the exact sequence

$$\begin{array}{ccccccccc} 0 & \longrightarrow & f_*\mathbb{Z}_X & \longrightarrow & f_*\mathcal{O}_X & \xrightarrow{\exp} & f_*\mathcal{O}_X^* & \longrightarrow & \\ & \longrightarrow & R^1 f_*\mathbb{Z}_X & \longrightarrow & R^1 f_*\mathcal{O}_X & \longrightarrow & R^1 f_*\mathcal{O}_X^* & \longrightarrow & \\ & \longrightarrow & R^2 f_*\mathbb{Z}_X & \longrightarrow & R^2 f_*\mathcal{O}_X & \longrightarrow & R^2 f_*\mathcal{O}_X^* & & \end{array}$$

Since Y is a germ, exp is surjective, thus $R^1 f_*\mathcal{O}_X = 0$ implies that $H^1(\bigcup D_i, \mathbb{Z}) = R^1 f_*\mathbb{Z}_X = 0$.

If $R^1 f_*\mathcal{O}_X = R^2 f_*\mathcal{O}_X = 0$, then

$$\mathrm{Pic}(X) = R^1 f_*\mathcal{O}_X^* \to R^2 f_*\mathbb{Z}_X = H^2(\bigcup D_i, \mathbb{Z})$$

is an isomorphism. Algebraic cycles generate a subvectorspace $V \subset H_2(\bigcup D_i, \mathbb{Q})$, which is dual to $\mathrm{Pic}(\bigcup D_i)/\mathrm{Pic}^\tau(\bigcup D_i)$. Thus $V = H_2(\bigcup D_i, \mathbb{Q})$. □

(12.1.2) **Definition.** (12.1.2.1) Let $f : X \to Y$ be a continuous map of topological spaces. Let $H_i(X/Y, \mathbb{C}) \subset H_i(X, \mathbb{C})$ be the subspace generated by the images of $H_i(X_y, \mathbb{C}) \to H_i(X, \mathbb{C})$ where X_y runs through all the fibers of f.

(12.1.2.2) Let $f : X \to Y$ be a morphism of algebraic spaces. Let $N_i(X/Y, \mathbb{C}) \subset N_i(X, \mathbb{C})$ be the subspace generated by the images of $N_i(X_y, \mathbb{C}) \to N_i(X, \mathbb{C})$ where X_y runs through all the fibers of f.

(12.1.3) **Theorem.** *Let $f : X \to Y$ be a proper morphism between algebraic varieties or complex spaces having rational singularities only. Assume that either f is bimeromorphic or f is projective and $R^1 f_*\mathcal{O}_X = 0$. Then*

(12.1.3.1) $$0 \to H_2(X/Y, \mathbb{C}) \xrightarrow{i} H_2(X, \mathbb{C}) \xrightarrow{f_*} H_2(Y, \mathbb{C}) \to 0$$

is exact. If, in addition, $R^2 f_* \mathscr{O}_X = 0$ *then* $H_2(X/Y, \mathbb{C})$ *is generated by algebraic cycles.*

Proof. By taking a resolution of X we see that it is sufficient to prove the surjectivity of f_* if X is smooth and f is locally projective. Using this observation, exactness in the middle is also reduced to the case when X is smooth. Moreover, it is sufficient to prove the results after some further birational modifications on X.

Next we claim that $R^1 f_* \mathbb{C}_X = 0$. This statement is local on Y. Given $y \in Y$ we may assume that $f^{-1}(y) = \bigcup D_i$ is a divisor with normal crossings only. If U is a small contractible neighborhood of y then $f^{-1}(U)$ retracts to $f^{-1}(y)$ thus $H^1(U, \mathbb{C}) = 0$ by (12.1.1). Therefore $R^1 f_* \mathbb{C} = 0$.

Using this the Leray spectral sequence for f gives the exact sequence

(12.1.3.2) $$0 \to H^2(Y, \mathbb{C}) \to H^2(X, \mathbb{C}) \to H^0(Y, R^2 f_* \mathbb{C}).$$

By duality this shows that f_* is surjective. Comparing these two sequences we see that (12.1.3.1) is exact at $H_2(X, \mathbb{C})$ if the following statement holds:

(*) Given any $S \in H^2(X, \mathbb{C})$ such that its image in $H^0(Y, R^2 f_* \mathbb{C})$ is nonzero, there is a 2-cycle C in a fiber such that the intersection product $C \cdot S$ is nonzero.

To see that (*) holds note that S corresponds to a nonzero section of $R^2 f_* \mathbb{C}$. Let y be a general point of its support. We may assume that X is smooth and that $f^{-1}(y) = \bigcup D_i$ is a divisor with normal crossings only. S restricts to a nonzero element of $H^2(\bigcup D_i, \mathbb{C})$. Therefore there is a 2-cycle $C \subset \bigcup D_i$ such that $C \cdot S$ is nonzero. This proves the exactness of (12.1.3.1). The additional claim about algebraic cycles follows directly from (12.1.1.2). □

(12.1.4) **Proposition.** *Let* $f: X \to Y$ *be a projective surjective morphism between algebraic varieties or complex spaces. Assume that* X *has rational singularities only. Assume that* $f_* \mathscr{O}_X = \mathscr{O}_Y$ *and* $R^1 f_* \mathscr{O}_X = 0$. *Let* L *be a line bundle on* X *such that if* $C \subset X$ *is an irreducible curve which is mapped to a point by* f *then* $L \cdot C = 0$.

Then any compact subset of Y *has an open neighborhood* U *such that*

$$L^k | f^{-1}(U) \cong f^* f_*(L^k) | f^{-1}(U)$$

holds for some $k > 0$.

Proof. The claim is local on Y so we need to prove it only for a small contractible neighborhood U of some $u \in Y$ such that $H^1(U, \mathscr{O}_U) = 0$. We are also allowed to perform some birational modifications on X, so we may assume that X is smooth and that $f^{-1}(u) \subset X$ is a divisor with normal crossings. Let $V = f^{-1}(U)$. Since $H^1(V, \mathscr{O}_V) = H^1(U, \mathscr{O}_U) = 0$, we see that $\operatorname{Pic} V$ injects into $H^2(V, \mathbb{Z})$. V retracts to $f^{-1}(u)$, thus $\operatorname{Pic} V$ injects into $H^2(f^{-1}(u), \mathbb{Z})$. By the assumption, $L | f^{-1}(u)$ is numerically trivial hence torsion. Thus $L \in \operatorname{Pic} V$ is torsion. □

(12.1.5) **Corollary.** (12.1.5.1) *Let $f : X \to Y$ be a projective morphism between proper algebraic spaces having rational singularities only. Assume that f has connected fibers and $R^1 f_* \mathscr{O}_X = 0$. Then*

$$0 \to N_1(X/Y) \to N_1(X) \to N_1(Y) \to 0$$

is exact.

(12.1.5.2) *Let Y be a proper algebraic space with rational singularities only. Then numerical and homological equivalences coincide for 1-cycles.*

Proof. In the first part the only question is exactness in the middle. Assume that $z \in N_1(X) - N_1(X/Y)$. Then there is an $L \in \operatorname{Pic} X$ such that $L \cdot z \neq 0$ but $L \cdot N_1(X/Y) \equiv 0$. By (12.1.4) there is a $k > 0$ such that $f_*(L^k)$ is a line bundle on Y. It also satisfies $f(z) \cdot f_*(L^k) \neq 0$, which proves exactness.

As for the second part, choose a resolution of singularities $f : X \to Y$. We need to show that if Z is a 1-cycle on Y that is numerically equivalent to zero then it is also homologically equivalent to zero. Let Z' be a 1-cycle on X such that $f(Z') = Z$. By (12.1.5.1) there is a cycle Z'' in X such that every irreducible component of Z'' is contained in a fiber of f and Z' is numerically equivalent to Z''. On X numerical and homological equivalences coincide, hence Z'' and Z' are homologically equivalent. Thus $0 = f(Z'')$ and Z are homologically equivalent. □

Note that we used only $R^1 f_* \mathscr{O}_X = 0$, which is weaker than rationality of the singularities.

(12.1.6) **Proposition.** *Let Y be an algebraic variety with rational singularities. Let $f: X \to Y$ be a resolution of singularities and let $X \subset \overline{X}$ be a smooth compactification. Let $E_i \subset X$ be the f-exceptional divisors. Then Y is $\mathbb{Q}$-factorial iff*

$$\operatorname{im}[H^2(\overline{X}, \mathbb{Q}) \xrightarrow{p_1} H^0(Y, R^2 f_* \mathbb{Q}_X)] = \operatorname{im}\left[\sum \mathbb{Q}[E_i] \xrightarrow{p_2} H^0(Y, R^2 f_* \mathbb{Q}_X)\right].$$

Proof. p_1 is the composite $H^2(\overline{X}, \mathbb{Q}) \to H^2(X, \mathbb{Q}) \to H^0(Y, R^2 f_* \mathbb{Q}_X)$. $[\overline{E}_i]$ is a class in $H^2(\overline{X}, \mathbb{Q})$, thus p_2 factors through p_1.

First we prove sufficiency. Let $D \subset Y$ be a Weil divisor and $\overline{D} \subset \overline{X}$ be the closure of its proper transform. By assumption there are $a_i \in \mathbb{Q}$ such that $p_1([\overline{D}] - \sum a_i[\overline{E}_i]) = 0$. By (12.1.4) $m(\overline{D} | X - \sum a_i E_i)$ is the pull back of a Cartier divisor $D' \sim mD$.

To see necessity let $H' \subset H^2(\overline{X}, \mathbb{Q})$ be the smallest sub Hodge structure (defined over $\mathbb{Q}$) such that $H' \otimes \mathbb{C}$ contains $H^{0,2}(\overline{X}, \mathbb{C}) + H^{2,0}(\overline{X}, \mathbb{C})$. Let $H'' \subset H^2(\overline{X}, \mathbb{Q})$ be the orthogonal complement of H' (consisting of type $(1, 1)$ elements only). By (12.1.1) $H' \subset \ker p_1$.

Let $z \in H^2(\overline{X}, \mathbb{Q})$. By the above, there is a $\mathbb{Q}$-divisor D'' on $\overline{X}$ such that $[D''] \in H''$ and $p_1(z) = p_1([D''])$. By assumption $f_*(D'' | X)$ is $\mathbb{Q}$-Cartier, thus $f^*(f_*(D'' | X)) = D'' | X - \sum a_i E_i$ for some $a_i \in \mathbb{Q}$. Therefore

$$p_1(z) = p_1([D'']) = p_1(f^*(f_*(D'' | X))) + \sum a_i p_2([E_i]) = \sum a_i p_2([E_i]). \quad \square$$

(12.1.7) **Proposition.** *Let $g : Y \to S$ be a connected flat family of algebraic varieties or complex spaces. In the analytic case choose a $W \subset Y$ such that $g : W \to S$ is proper; in the algebraic case set $W = Y$. Assume that all the fibers have rational singularities only. Assume that the set of fibers that are $\mathbb{Q}$-factorial in a neighborhood of W is dense in the Zariski or Euclidean topology. Then there is a dense Zariski (or Euclidean) open set $U \subset S$ such that every fiber above U is $\mathbb{Q}$-factorial in a neighborhood of W.*

Proof. We take a resolution of singularities $f : X \to Y$. By (12.1.6) the $\mathbb{Q}$-factoriality of a space with rational singularities depends only on the topology of a given resolution. This is unchanged over an open subset U of S. Thus we have $\mathbb{Q}$-factorial fibers over U since by assumption the $\mathbb{Q}$-factorial fibers form a dense subset of S. □

(12.1.8) **Lemma.** *Let X be a scheme, $X \supset Z$ a closed subscheme, $X_0 = (t = 0)$ a Cartier divisor, F a sheaf on $U = X - Z$, and $i : U \to X$ the injection. Assume that*

(12.1.8.1) codim $(Z \cap X_0, X_0) \geq 3$;

(12.1.8.2) *F is S_3; and*

(12.1.8.3) *$i_*(F|U \cap X_0)$ is S_3 as a sheaf on X_0.*

Then $(i_*F)|X_0 = i_*(F|U \cap X_0)$.

Proof. Let $F_0 = F|U \cap X_0$. We have an exact sequence

$$0 \to F \xrightarrow{t} F \to F_0 \to 0.$$

Applying i_* we get the sequence

$$0 \to i_*F \xrightarrow{t} i_*F \to i_*F_0 \to R^1 i_*F \xrightarrow{t} R^1 i_*F \to R^1 i_*F_0.$$

Here $R^1 i_* F_0 = H^2_{Z \cap X_0}(i_*F_0) = 0$ by [Grothendieck68, III 3.3] and $R^1 i_* F$ is coherent by [Grothendieck68, VII 3.1]. Therefore by the Nakayama lemma $R^1 i_* F = 0$. □

(12.1.9) **Corollary.** *Let X be an algebraic variety and let X_0 be a Cartier divisor. Assume that X_0 is S_3 (e.g., Cohen-Maculay and of dimension of least 3), codim(Sing X_0, X_0) ≥ 3, and X_0 is factorial (resp. $\mathbb{Q}$-factorial). Then X is factorial (resp. $\mathbb{Q}$-factorial) in a neighborhood of X_0.*

Proof. The problem is local on X, so we may assume that X is an affine neighborhood of a point $x \in X_0$.

If G is a Weil divisor in a neighborhood of x choose $m > 0$ so that $m(G|X_0)$ is Cartier. We can apply the above lemma with

$$Z = \{\text{the locus where } mG \text{ is not Cartier}\} \quad \text{and} \quad F = \mathcal{O}(mG)$$

to get that $F \otimes \mathcal{O}_{X_0}$ is locally free. Thus F is locally free near x and mG is Cartier in a neighborhood $x \in U_G \subset X$. Unfortunately U_G may depend on G. By (12.1.6.1) Weil(X)/Pic(X) is finitely generated, thus we may take $U = \bigcap U_{G_i}$ where G_i runs through a generating set of Weil(X)/Pic(X). □

(12.1.10) **Theorem.** *Let $g : Y \to S$ be a connected proper and flat family of algebraic varieties. Assume that all the fibers have rational singularities only and* *that in each fiber the singular set has codimension at least* 3. *Then the set*

$$S_{Q\text{fact}} = \{s \in S : g^{-1}(s) \quad \text{is } \mathbb{Q}\text{-factorial}\}$$

is open.

This is a special case of the following more general result:

(12.1.11) **Theorem.** *Let* $g : Y \to S$ *be a flat family of algebraic varieties. Assume that all the fibers have rational singularities only and that in each fiber the singular set has codimension at least* 3. *Then the set*

$$Y_{Q\,\text{fact}} = \{y \in Y | g^{-1}(g(w)) \text{ is } \mathbb{Q}\text{-factorial in a neighborhood of } w\}$$

is open.

Proof. We may assume that S is irreducible and reduced. Let $\pi : S' \to S$ be a resolution of singularities and let $Y' = S' \times_S Y$. Then

$$Y'_{Q\,\text{fact}} = S' \times_S (Y_{Q\,\text{fact}}),$$

thus it is sufficient to prove openness of $Y'_{Q\,\text{fact}}$. Thus we may assume to start with, that S is smooth.

Let W be the complement of $Y_{Q\,\text{fact}}$. Assume that $y \in Y - W$ is in the closure of W. By induction on $\dim S$ we may assume that $g(W \cap U)$ is dense in S for every open $y \in U \subset Y$.

We claim that Y is $\mathbb{Q}$-factorial in a neighborhood of y. We prove this by induction on $\dim S$. If $\dim S = 1$ then this is (12.1.9). In general let $T \subset S$ be a smooth hypersurface containing $g(y)$. By induction $g^{-1}(T)$ is $\mathbb{Q}$-factorial. Applying again (12.1.9) to the pair $g^{-1}(T) \subset Y$ we conclude that Y is $\mathbb{Q}$-factorial in a neighborhood $y \in U_0 \subset Y$.

There is a countable union of proper closed subvarieties $\bigcup S_i \subset S$ such that if $s \notin \bigcup S_i$ and $D_s \subset Y_s$ is any Weil divisor then there is a Weil divisor $D \subset Y$ such that $D|Y_s = D_s$. By the previous remarks D is $\mathbb{Q}$-Cartier on U_0, thus D_s is also $\mathbb{Q}$-Cartier on $Y_s \cap U_0$. By (12.1.7) there is an open subset $S_0 \subset S$ such that for every $s \in S_0$ the fiber $Y_s \cap U_0$ is $\mathbb{Q}$-factorial. This contradicts our assumption that $g(W \cap U_0)$ is dense in S. □

(12.1.12) *Remark.* (12.1.10) is probably also true in the analytic case, however, (12.1.11) is false. The reason is that local divisors globalise in the algebraic case but not in the analytic case.

(12.1.13) **Examples.** (12.1.13.1) Consider $X_{st} = (x^2 + y^2 + z^2 + tu^2 + s = 0)$ as a family of threefolds depending on t and s. X_{st} is $\mathbb{Q}$-factorial iff $s \neq 0$ or $s = t = 0$.

(12.1.13.2) Consider $x^2 + y^2 + z^2 + u^3 + tu^2 = 0$ as a family of threefolds depending on t. For $t = 0$ it is $\mathbb{Q}$-factorial, and so the same must hold for $t \neq 0$. It has a singularity that is not analytically $\mathbb{Q}$-factorial, but the non-$\mathbb{Q}$-Cartier divisor does not exist globally.

(12.2) **Projectivity and deformations.** (12.2.1) *Conditions.* For the rest of this section we consider fiberspaces X/S satisfying the following properties:

(12.2.1.1) X and S are irreducible analytic spaces of finite type.

(12.2.1.2) X/S is a proper and flat relative algebraic space; i.e., it is bimeromorphic to a projective fiberspace.

(12.2.1.3) Let $p: \Delta \to S$ be any morphism and let X_Δ/Δ be the pull-back family. We assume that if $D \subset X_\Delta$ is a Weil divisor proper over Δ such that D is Cartier outside finitely many fibers then D is $\mathbb{Q}$-Cartier.

(12.2.1.4) *Remark.* This last assumption is satisfied in the following cases:

(12.2.1.4.1) If X/S is smooth (clear).

(12.2.1.4.2) If every fiber has only rational $\mathbb{Q}$-factorial singularities and is smooth in codimension two. (This follows from (12.1.9).)

(12.2.1.4.3) If every fiber is smooth in codimension two and has only singularities that are locally the quotient of a hypersurface singularity by a group that acts freely in codimension two. In particular, if every fiber X_s is a threefold with only terminal singularities. (Under these assumptions X_T has parafactorial singularities [Grothendieck68, XI.3.13], [Kollár83, 3.2.2] or [Ran89, 2.3].) □

(12.2.2) **Definition.** Given X/S as above let $\mathcal{N}^1(X/S)$ be the functor

$$\mathcal{N}^1(X/S)(S') = \mathbb{Q} \otimes (\operatorname{Pic}(X \times_S S'/S')/\operatorname{Pic}^\tau(X \times_S S'/S')).$$

(12.2.3) **Proposition.** *The functor $\mathcal{N}^1(X/S)$ is representable by a separated and unramified algebraic space $NS_{\mathbb{Q}}(X/S)/S$. It has countably many connected components and they are proper over S.*

Proof. This is a straightforward consequence of the result about the similar properties of the relative Picard functor [Grothendieck62, 232]. We had to tensor with $\mathbb{Q}$ since in general the specialization of a Cartier divisor is only $\mathbb{Q}$-Cartier. □

(12.2.4) **Definition.** Given X/S as above let $\mathcal{GN}^1(X/S)$ be the following sheaf on S in the Euclidean topology. Given $U \to S$ let

$$\mathcal{GN}^1(X/S)(U) = \{\text{sections of } \mathcal{N}^1(X/S) \text{ over } U \text{ with open support}\}.$$

(12.2.5) **Proposition.** *The above $\mathcal{GN}^1(X/S)$ is a local system with finite monodromy on S. There are countably many proper closed subvarieties $Z_i \subset S$ such that if $s \in S - \bigcup Z_i$ then $\mathcal{GN}^1(X/S)|s \cong N^1(X_s)$.*

Proof. Consider the relative Hilbert space [Artin69] that parametrizes codimension one cycles. This has countably many components and every component satisfies the valuative criterion of properness over S. Let Z_i be the images of those components that do not dominate S. The injection $Z_i \subset S$ also satisfies the valuative criterion of properness over S, thus the Z_i are closed in S. If $s \in S - \bigcup Z_i$ then for every divisor D_s on X_s there is a dominant component H of the relative Hilbert space such that D_s is one of the divisors parametrized by H.

Let $U \subset S$ be open and consider a dominant connected component $g: H \to U$ of the relative Hilbert space that patrametrizes codimension one cycles. Assume first that U is a small analytic neighborhood of a point $0 \in S$.

Then $g^{-1}(0)$ is connected; in particular, all divisors in X_0 parametrised by H_0 are numerically equivalent. Since $\mathscr{N}^1(X/S)$ is unramified, all divisors in X_s parametrized by H_s are also numerically equivalent for $s \in U$. Thus if $s \in S - \bigcup Z_i$, then for every divisor D_s on X_s there is a section of $\mathscr{GN}^1(X/S)(U)$ which induces $[D_s]$.

Consider a projective resolution $p : Y/S \to X/S$. There is a Zariski open dense subset $U \subset S$ such that the fiberspace $Y \times_S U/U$ is smooth and projective. On relative divisors we have the pull back map from $X \times_S U/U$ to $Y \times_S U/U$. Therefore, we have an injection

$$\mathscr{GN}^1(X/S)|U \to \mathscr{GN}^1(Y \times_S U/U).$$

Therefore, it is sufficient to prove that the latter has finite monodromy.

If $g : H \to U$ is a dominant connected component of the relative Hilbert scheme of $Y \times_S U/U$ that patrametrizes codimension one cycles then H/U is proper since $Y \times_S U/U$ is projective. Therefore $g^{-1}(u)$ has only finitely many components for every $u \in U$. Thus the monodromy has only finite orbits and, therefore, it is finite. □

(12.2.6) **Proposition.** *Let X/S be as in* (12.2.1). *Let $C/S \subset X/S$ be a flat family of* 1*-cycles. If $C_0 \subset X_0$ is numerically equivalent to zero for some $0 \in S$ then $C_s \subset X_s$ is numerically equivalent to zero for every $s \in S$.*

Proof. It is clearly sufficient to prove this after a surjective base change $S' \to S$. Thus we may assume that $\mathscr{GN}^1(X/S)$ is a trivial local system. If L is a global section of $\mathscr{GN}^1(X/S)$ then $L_0 \cdot C_0 = 0$ by assumption, hence $L_s \cdot C_s = 0$ for every s. By (12.2.5) this means that if $s \in S - \bigcup Z_i$ then $C_s \subset X_s$ is numerically equivalent to zero.

To see that $C_s \subset X_s$ is numerically equivalent to zero for every s pick any $s \in S$ and a disc $s \in \Delta$ such that Δ is not contained in any of the Z_i. We take base change with Δ to obtain a family $C/\Delta \subset X/\Delta$ such that $C_t \subset X_t$ is numerically equivalent to zero for all but countably many $t \in \Delta$. After possibly further base change there is a semistable modification $f : Y/\Delta \to X/\Delta$. For each $t \in \Delta - \{s\}$ we can consider the family of 1-cycles with rational coefficients $C'_t \subset Y_t$ such that $f_*(C'_t) = C_t$ and $[C'_t] = 0 \in H_2(Y_t, \mathbb{Q})$. By (12.1.5) this family is nonempty for all but countably many $t \in \Delta - \{s\}$. Since the Hilbert scheme of Y/Δ has only countably many components, there must exist a flat family of 1-cycles $C'/\Delta \subset Y/\Delta$ such that $f_*(C'/\Delta) = C/\Delta$ and $[C'_t] = 0 \in H_2(Y_t, \mathbb{Q})$ for all but countably many $t \in \Delta - \{s\}$.

Since Y retracts to Y_0, we also have that $[C'_s] = 0 \in H_2(Y_s, \mathbb{Q})$, thus $[C_s] = f_*[C'_s] = 0 \in H_2(X_s, \mathbb{Q})$. This implies that $C_s \subset X_s$ is numerically zero. □

(12.2.7) **Definition.** (12.2.7.1) Given X/S as above let $\mathscr{GN}_1(X/S)$ be the following sheaf of vectorspaces on S in the Euclidean topology. Given $U \to S$ let

$$\mathscr{GN}_1(X/S)(U) = \left\{ \begin{array}{l} \text{Flat families of 1-cycles } C/U \subset X \times_S U/U \text{ with real} \\ \text{coefficients; modulo fiberwise numerical equivalence.} \end{array} \right\}$$

(12.2.7.2) Given X/S as above let $\mathscr{GNE}(X/S)$ be the following sheaf of cones on S in the Euclidean topology. Given $U \to S$ let

$$\mathscr{GNE}(X/S)(U) = \left\{\begin{array}{l}\text{Flat families of 1-cycles } C/U \subset X \times_S U/U \text{ with nonnegative} \\ \text{real coefficients; modulo fiberwise numerical equivalence.}\end{array}\right\}$$

(12.2.8) **Proposition.** *The above $\mathscr{GN}_1(X/S)$ is a local system on S. There are countably many proper closed subvarieties $Z_i \subset S$ such that if $s \in S - \bigcup Z_i$ then $\mathscr{GN}_1(X/S)|s \cong N_1(X_s)$ and $\mathscr{GNE}(X/S)|s \cong NE(X_s)$. Moreover, $\mathscr{GN}_1(X/S)$ has finite monodromy.*

Proof. The same arguments as in the proof of (12.2.5) show all but the last claim. The latter is true since by its definition $\mathscr{GN}_1(X/S)$ injects into the dual of $\mathscr{GN}^1(X/S)$. □

(12.2.9) **Corollary.** *The local systems $\mathscr{GN}_1(X/S)$ and $\mathscr{GN}^1(X/S)$ are dual to each other. In particular, if X/S is projective then*

$$N_1(X/S) = H^0(S, \mathscr{GN}_1(X/S)).$$

Proof. Intersection product in any fiber provides the duality map. This pairing is perfect since in a sufficiently general fiber we recover the pairing between the Néron-Severi group and 1-cycles modulo numerical equivalence. Since $N_1(X/S)$ is defined to be the dual of the relative Néron-Severi group, the last assertion is clear. □

(12.2.10) **Theorem.** *Let X/S be as in* (12.2.1). *Assume that for some $0 \in S$ the fiber X_0 is projective. Then there is a Zariski open neighborhood $0 \in U \subset S$ such that X_s is projective for every $s \in U$.*

Remark. It is not true, however, that X/S is projective in a neighborhood of 0.

Proof. Again we may assume that $\mathscr{GN}_1(X/S)$ is the trivial local system. Restriction gives injective maps

$$H^0(S, \mathscr{GN}_1(X/S)) \hookrightarrow N_1(X_0),$$
$$H^0(S, \mathscr{GNE}(X/S)) \hookrightarrow NE(X_0).$$

Since X_0 is projective, Kleiman's criterion tells us that $NE(X_0) \subset N_1(X_0)$ is a convex cone and not even its closure contains straight lines. Therefore the same holds for the cone $H^0(S, \mathscr{GNE}(X/S)) \subset H^0(S, \mathscr{GN}_1(X/S))$. Since $H^0(S, \mathscr{GN}^1(X/S))$ is dual to $H^0(S, \mathscr{GN}_1(X/S))$, there is a relative divisor H such that H defines a strictly positive linear functional on the closure of the cone $H^0(S, \mathscr{GNE}(X/S))$. By (12.2.7) if $s \in S - \bigcup Z_i$ then H_s is strictly positive on $\overline{NE}(X_s) - \{0\}$. In particular, H_s is ample, again by Kleiman's criterion.

Ampleness is an open condition , thus there is a Zariski open $V \subset S$ such that H_s is ample on X_s if $s \in V$. If $0 \in V$ then we are done. Otherwise we repeat the argument with the irreducible components of $S - V$ and so on. This completes the proof. □

(12.3) **Deformation of extremal rays.**

(12.3.1) **Theorem.** *Let $g : Y \to S$ be a proper flat morphism of complex spaces. Assume that for some $0 \in S$ the fiber Y_0 is a projective variety with only $\mathbb{Q}$-factorial rational singularities, $\dim Y_0 \geq 3$. Let $f_0 : Y_0 \to Z_0$ be the contraction of an extremal ray $C_0 \subset Y_0$. By* (11.4) *there is a proper flat morphism $Z \to S$ and a factorisation*

$$g : Y \xrightarrow{f} Z \longrightarrow S.$$

Then there is an open neighborhood $0 \in U \subset S$ such that if f_0 contracts a subset of codimension at least two (*resp. contracts a divisor; resp. is a fiber space of generic relative dimension k*) *then f_s contracts a subset of codimension at least two* (*which may be empty*) (*resp. contracts a divisor; resp. is a fiberspace of generic relative dimension k*) *if $s \in U$.*

Proof. The first part follows from the upper semicontinuity of fiber dimension. Next assume that f_0 contracts a divisor or is a fiber space of generic relative dimension k. The proof in these cases is essentially the same as in [Mori82, §11]. For illustration we present the case when the contraction $f_0 : Y_0 \to Z_0$ given by C_0 contracts a divisor D_0 to a point.

Let H be ample on Z_0 and let $H' = f^*H$. Let mD_0 be Cartier. Consider the exact sequence

$$0 \to \mathscr{O}(-kH) \to \mathscr{O}(-kH + mD_0) \to \mathscr{O}_{mD_0}(mD_0) \to 0.$$

For large k the divisor $kH - mD_0$ is ample, thus $H^i(\mathscr{O}_{mD_0}(mD_0)) = 0$ for $i = 0, 1$. Thus by [Grothendieck62, 221] the relative Hilbert scheme of Y/S is smooth of relative dimension 0 at $[mD_0]$. Thus in every nearby Y_s at least a divisor is contracted. By semicontinuity of fiber dimension f_s has to be birational. □

(12.3.2) **Theorem.** *Notation and assumptions are as in* (12.3.1). *Assume in addition that Y_0 is a threefold with terminal singularities only. If f_0 contracts only finitely many curves $\{C_0^i\}$, then for every i, there is a dominating family of curves $\mathscr{C}^i/U$ such that $(\mathscr{C}^i/U)_0$ is a multiple of C_0^i.*

Proof. By considering a suitable analytic neighborhood of C_0^i, it is sufficient to treat the case when exactly one curve is contracted. Let $E \subset Y$ be the f-exceptional locus. We want to show that E dominates S. If this is not the case then we can find a $\Delta \subset S$ such that the induced contraction morphism $f_\Delta : Y \times_S \Delta \to Z \times_S \Delta$ is an isomorphism outside finitely many points of $Z \times_S \Delta$. Now consider the flip $f_\Delta^+ : Y^+ \times_S \Delta \to Z \times_S \Delta$. This is again an isomorphism outside finitely many points of $Z \times_S \Delta$. Thus both f_Δ and f_Δ^+ have one-dimensional exceptional loci. This contradicts [KMM87, 5.1.17]. □

(12.3.3) **Corollary.** *Notation and assumptions are as in* (12.3.2). *Then there is a dominating family of curves $\mathscr{C}/U$ such that $(\mathscr{C}/U)_0$ is in the extremal ray contracted by f_0.*

Proof. If f_0 contracts only finitely many curves then this is (12.3.2). Otherwise, it follows from (12.3.1). Indeed, $-K_Y$ is f-ample, therefore, f is projective and there are plenty of curves in the fibers. □

(12.3.4) **Corollary.** *Notation is as in* (12.3.1). *Assume that every fiber of* Y/S *is projective. Then* $f_s : Y_s \to Z_s$ *is the contraction of an extremal ray for every* s *in a suitable neighborhood of* 0.

Proof. We may assume that S is irreducible. By shrinking S we may assume that all the fibers have $\mathbb{Q}$-factorial singularities (12.1.10).

Next we claim that every Z_s is projective. This will follow from (12.2.10) once we check the conditions (12.2.1). To check (12.2.1.3) we make a base change by $p: T \to S$ and let $D_{\text{gen}} \subset Z_{\text{gen}}$ be a Cartier divisor. Thus $f^*_{\text{gen}}(D_{\text{gen}}) \subset Y_{\text{gen}}$ is again Cartier, hence its closure $\overline{D} \subset Y_T$ is $\mathbb{Q}$-Cartier. If $C_0 \subset Y_0$ is contracted by f_0 then by (12.3.3) we conclude that $C_0 \cdot \overline{D} = 0$. Thus by (12.1.4) the divisor

$$f_T(\overline{D}) = (\text{the closure of } D_{\text{gen}} \text{ in } Z_T)$$

is again $\mathbb{Q}$-Cartier. This was to be proved.

Since Z_s is projective and $-K_{Y_s}$ is f_s-ample, f_s is a contraction of an extremal face. We will prove that this face is one dimensional. Assume that for some $t \in U$ it is not. Then f_t contracts at least two different extremal rays E^1_t and E^2_t. By (12.3.3) we obtain two flat families of curves $\mathscr{E}^1/V$ and $\mathscr{E}^2/V$ over some Zariski open neighborhood of t. These families are contracted by f. Since the relative Hilbert scheme satisfies the valuative criterion of properness, these extend to flat families of curves over S. Thus we obtain $(\mathscr{E}^1)_0$ and $(\mathscr{E}^2)_0$. These are both contracted by f_0, hence they are in the same extremal ray. Thus by (12.2.6) the same holds for every $s \in U$. This is a contradiction. □

(12.3.5) *Remarks.* (12.3.5.1) Even if Y/S is projective we cannot prove that Z/S is again projective. This is not known even when Y/S is smooth.

(12.3.5.2) It is essential to assume that Y_0 is $\mathbb{Q}$-factorial. Let Y be the total space of the rank two vector bundle $\mathscr{O}(-1) \oplus \mathscr{O}(-1)$ over $\mathbb{P}^2$. Thus $\mathbb{P}^2$ can be contracted to a point to get $f: Y \to Z$. If $t = 0$ is a general section of Z through the singular point then $Y_0 = (f^*t = 0)$ defines a 3-fold containing $\mathbb{P}^2$. It has only one singular point, it is a node $(xy - uv = 0)$ and $\mathbb{P}^2 \subset Y_0$ is not $\mathbb{Q}$-Cartier there. The line $C_0 \subset \mathbb{P}^2 \subset Y_0$ generates an extremal ray, but no multiple of it lifts to the general fiber.

(12.3.6) **Corollary.** *Notation is as in* (12.3.1). *Assume that* Y/S *is projective and that* $\mathscr{GN}_1(X/S)$ *has trivial monodromy. Let* $f: Y \to Z$ *be the contraction of a relative extremal ray. Then* $f_s : Y_s \to Z_s$ *is the contraction of an extremal ray for every* $s \in S$.

Proof. By (12.2.8) f_s is the contraction of an extremal ray for a dense set of $s \in S$. As in (12.3.4) we get that it is the contraction of an extremal ray everywhere. □

(12.4) **Minimal models in families.** The aim of this section is to prove the following.

(12.4.1) **Theorem.** *Let S be a connected normal quasi-projective variety or a complex space and let X/S be a flat, projective family of threefolds such that every fiber has only $\mathbb{Q}$-factorial terminal singularities. Assume that not every fiber is uniruled. Then there is a flat projective family Y/S and a rational map $f : X/S \cdots > Y/S$ such that on each fiber f induces a birational map, each fiber of Y/S has only terminal singularities, and $K_{Y/S}$ is nef; i.e., minimal models exist in families.*

In general we have a slightly weaker result:

(12.4.2) **Theorem.** *Let S be a connected normal quasi-projective variety or a complex space and let X/S be a flat, projective family of threefolds such that every fiber has only $\mathbb{Q}$-factorial terminal singularities. Then there is a finite, étale and Galois base change $p : S' \to S$, a flat projective family Y'/S', and a rational map $f' : p^*X/S' \cdots > Y'/S'$ such that on each fiber f' induces a birational map, each fiber of Y'/S' has only $\mathbb{Q}$-factorial terminal singularities and $K_{Y'/S'}$ is either relatively nef or Y'/S' admits a relative Fano-contraction.*

Proof. After a finite, étale and Galois base change $p : S' \to S$ we may assume that $\mathcal{GN}_1(p^*X/S')$ has trivial monodromy (12.2.8). Now take any p^*X/S'-extremal ray and contract. By (12.3.6) this induces the contraction of an extremal ray in every fiber. If necessary we flip (11.7) and we can continue the usual steps of the minimal model program. Finally we get a Y'/S' such that $K_{Y'/S'}$ is either relatively nef or there is a relative Fano-contraction.

Assume that not all fibers are uniruled. Let G be the Galois group of S'/S. Then G also acts on p^*X, and this way we get a birational action of G on Y'. By [Fujiki81, Levine81] none of the fibers of X/S is uniruled hence $K_{Y'/S'}$ is nef. As in [Kollár89, 4.3] there is a subscheme $E \subset Y'$ such that every fiber of E/S' is at most one dimensional and G acts regularly on $Y' - E$. As in [Kollár89, 3.6] we want to find a different compactification of $Y' - E$ where the action of G is regular. To do this let H be a relatively ample divisor, and let $D = \sum_{g\in G} g(H)$. By [KMM87, 3.2] if D is not relatively nef, then there is a $(K_{Y'} + \varepsilon D)$-extremal ray R for small ε and $R \cdot K_{Y'} = 0$. By construction only curves in E/S can be in R. Let the contraction corresponding to R be $g : Y'/S' \to Z'/S'$. The exceptional set of g is a subset of E, thus in every fiber of Y'/S' the morphism g induces a small contraction. By (11.10) the D/S'-flop of g exists. Any sequence of D/S'-flops terminates since they terminate in every fiber.

After finitely many flops D^+, the proper transform of D, becomes nef. Now we can apply relative base point freeness [KMM87, 3.1] to get a model $\tilde{Y}/S'$ such that $\tilde{D}$, the proper transform of D, is relatively ample and G-invariant. Thus G acts regularly on $\tilde{Y}$. Now we can take the quotient $Y = \tilde{Y}/G$ to get the required family of minimal models. □

(12.4.3) *Remark.* Note that we may not have $\mathbb{Q}$-factorial singularities on the fibers of Y/S because of monodromy. An easy example is: take a threefold with a nonfactorial node and arrange for the monodromy to interchange the two generators of the local Pic. Blowing up the nodal set gives a smooth family X/S.

(12.4.4) **Theorem.** *Let $0 \in S$ be the germ of a normal complex space and let X/S be a flat, proper family of complex threefolds. Assume that the central fiber X_0 is projective with only $\mathbb{Q}$-factorial terminal singularities. Then there is a flat proper family Y/S and a bimeromorphic map $f : X/S \cdots > Y/S$ such that on each fiber f induces a bimeromorphic map, each fiber of Y/S has only $\mathbb{Q}$-factorial terminal singularities, and $K_{Y/S}$ is either relatively nef or Y/S admits a relative Fano contraction.*

Proof. Take the contraction of an extremal ray $f_0 : X_0 \to Z_0$. This extends to a contraction $f : X/S \to Z/S$. If necessary we can flip (11.7) and continue until we get a family Y/S where either K_{Y_0} is nef or Y_0 admits a Fano contraction. In either case the same is true for Y/S. □

(12.5) **Deformation invariance of plurigenera.** The aim of this section is to derive the following consequence of the continuity of flips:

(12.5.1) **Theorem.** *Let X/S be a flat family of projective threefolds with $\mathbb{Q}$-factorial terminal singularities, and assume that S is connected. Then all the plurigenera are constant in the family.*

Proof. The results of [Nakayama87] show that (12.4.4) implies (12.5.1). We will give a different proof in (12.5.5) following [Levine83] that works in a more general setting.

(12.5.2) **Corollary.** *Let X/S be a flat family of projective threefolds with $\mathbb{Q}$-factorial terminal singularities, and assume that S is connected. Then the Kodaira dimension of the fibers is constant in the family.* □

(12.5.3) *Case $\kappa = -\infty$.* If $P_m(X_0) = 0$ for every m for some fiber X_0 then X_0 is uniruled by [Miyaoka88]. Therefore all the fibers are uniruled [Fujiki81, Levine81], thus all the plurigenera are constant.

The other cases follow from the next slightly stronger result:

(12.5.4) **Theorem.** *Let X/S be a proper flat family of complex analytic threefolds with $\mathbb{Q}$-factorial terminal singularities. Let $0 \in S$ and assume that X_0 is projective and $P_m(X_0) \geq 1$ for some m. Then there is an open neighborhood $0 \in U \subset S$ such that all the plurigenera are constant in the family over U.*

The proof will rest on the following result, which is a variant of some results of [Levine83].

(12.5.5) **Theorem.** *Let X/Δ be a proper flat family of complex analytic spaces of dimension d. Assume that X_0 is projective, smooth in codimension two, and has only log-terminal singularities. Assume that $\omega_{X_0}^m$ is locally free and is generated by global sections for some $m > 0$. Then for every k, $P_k(X_t)$ is locally constant near 0.*

Proof. Let $\Delta_n = \operatorname{Spec}\mathbb{C}[[t]]/t^n$ and let a subscript n denote the fiber over Δ_n. Let $s \in H^0(\omega_{X_n}^m)$ be a general section such that its divisor is smooth in codimension two. Let $\bar{s}$ be its restriction to $H^0(\omega_{X_0}^m)$. We want to identify the obstruction of lifting $\bar{s}$ to $H^0(\omega_{X_{n+1}}^m)$.

Let Y_n be the m-fold cyclic cover of X_n defined by s and let $\pi : Y_n \to X_n$ be the projection.

(12.5.6) **Lemma.** *Let X be a scheme and $i : U \hookrightarrow X$ be an open subset. Let F and G be sheaves on X. Assume that*

(12.5.6.1) *G is torsion free in codimension* 1;

(12.5.6.2) *F is S_3*;

(12.5.6.3) $\operatorname{codim}_X(X - U) \geq 3$.

Then the natural restriction map

$$\operatorname{Ext}^1_X(G, F) \to \operatorname{Ext}^1_U(G|U, F|U)$$

is an isomorphism.

Proof. Let $T \subset G$ be the torsion subsheaf. Then $\operatorname{Ext}^1(G, F) = \operatorname{Ext}^1(G/T, F)$ thus we may assume that G is torsion free. First we construct the inverse of the restriction map. Let

$$0 \to F|U \to E_U \to G|U \to 0$$

be an extension. Now apply i_* to get

$$0 \to i_*(F|U) \to i_*(E_U) \to i_*(G|U) \to R^1 i_*(F|U).$$

By [Grothendieck68, III 3.3] $R^1 i_*(F|U) = 0$, hence the above sequence becomes

$$0 \to F \to i_*(E_U) \to i_*(G|U) \to 0.$$

Using the natural injection $G \to i_*(G|U)$ we obtain an extension

$$0 \to F \to \overline{E} \to G \to 0$$

where $\overline{E}$ is the preimage of $G \subset i_*(G|U)$ under the map $i_*(E_U) \to i_*(G|U)$.

One still has to check that this, in fact, is the inverse. To this end take an exact sequence

$$0 \to F \to E \to G \to 0$$

and consider the diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F & \longrightarrow & E & \longrightarrow & G & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F & \longrightarrow & i_*(E|U) & \longrightarrow & i_*(G|U) & \longrightarrow & 0 \end{array}$$

This clearly shows that $\operatorname{im}[E \to i_*(E|U)] = \overline{E}$. □

(12.5.7) **Lemma.** *There is a natural map*

$$\operatorname{Ext}^1(\Omega^1_{X_n}, \mathscr{O}_{X_n}) \times H^0(Y_n, (\pi^*\omega_{X_n})^{**}) \to \operatorname{Ext}^1(\Omega^1_{Y_n}, \omega_{Y_n}).$$

Proof. Let D denote the ramification divisor of $\pi : Y_n \to X_n$ and let S denote the set of singular points of X_0 or of Y_0 (this will not lead to any confusion). Thus we have the following maps:

$$\begin{aligned}\operatorname{Ext}^1(\Omega^1_{X_n}, \mathscr{O}_{X_n}) &\to \operatorname{Ext}^1(\Omega^1_{X_n - S}, \mathscr{O}_{X_n - S})\\ &\to \operatorname{Ext}^1(\pi^*\Omega^1_{X_n - S}, \mathscr{O}_{Y_n - S})\\ &\to \operatorname{Ext}^1(\pi^*\Omega^1_{X_n - S}, \mathscr{O}_{Y_n}),\end{aligned}$$

where the first and the last maps are isomorphisms by (12.5.6) and the one in the middle is simply tensoring with $\mathscr{O}_{Y_n - S}$.

Also, $(\pi^*\omega_{X_n})^{**} = \omega_{Y_n}(-(n-1)D)$. This is clear over smooth points, and the double dual on the left automatically extends the isomorphism across the singularities. Thus we have a Yoneda product

$$\begin{aligned}&\operatorname{Ext}^1(\Omega^1_{X_n}, \mathscr{O}_{X_n}) \times H^0(Y_n, (\pi^*\omega_{X_n})^{**})\\ &\to \operatorname{Ext}^1(\pi^*\Omega^1_{X_n}, \mathscr{O}_{Y_n}) \times H^0(Y_n, \omega_{Y_n}(-(n-1)D))\\ &\to \operatorname{Ext}^1(\pi^*\Omega^1_{X_n}, \omega_{Y_n}(-(n-1)D))\\ &\quad = \operatorname{Ext}^1(\pi^*\Omega^1_{X_n}((n-1)D), \omega_{Y_n}).\end{aligned}$$

The natural injection

$$\Omega^1_{Y_n} \to \pi^*\Omega^1_{X_n}((n-1)D),$$

gives

$$\operatorname{Ext}^1(\pi^*\Omega^1_{X_n}((n-1)D), \omega_{Y_n}) \to \operatorname{Ext}^1(\Omega^1_{Y_n}, \omega_{Y_n}).$$

This gives the required map. □

(12.5.8) **Construction of the obstruction.** The Kodaira–Spencer map of X_n/T_n gives an element of $\operatorname{Ext}^1(\Omega^1_{X_n/T_n}, \mathscr{O}_{X_n}) \cong \operatorname{Ext}^1(\Omega^1_{X_n}, \mathscr{O}_{X_n})$, call it ρ. $H^0(Y_n, (\pi^*\omega_{X_n})^{**})$ has a natural section coming from the $\mathscr{O}_{X_n}$ summand of

$$\pi_*(\pi^*\omega_{X_n})^{**} = \omega_{X_n} + \mathscr{O}_{X_n} + \cdots.$$

Call this section ω. The above lemma gives an element

$$(\rho \cdot \omega) \in \operatorname{Ext}^1(\Omega^1_{Y_n}, \omega_{Y_n}).$$

The dual of the map $d : H^{d-1}(\mathscr{O}_{Y_n}) \to H^{d-1}(\Omega^1_{Y_n})$ gives

$$d : \operatorname{Ext}^1(\Omega^1_{Y_n}, \omega_{Y_n}) \to \operatorname{Ext}^1(\mathscr{O}_{Y_n}, \omega_{Y_n}) = H^1(\omega_{Y_n}).$$

Multiplying by ω^{m-1} yields $\omega^{m-1}d(\rho\cdot\omega) \in H^1(\omega^m_{Y_n})$. Take the trace in order to obtain $\operatorname{Tr}(\omega^{m-1}d(\rho\cdot\omega)) \in H^1(\omega^m_{X_n})$.

(12.5.9) *Claim.* $\operatorname{Tr}(\omega^{m-1}d(\rho\cdot\omega)) \in H^1(\omega^m_{X_n})$ is the obstruction to lifting $\bar{s}$ to an element of $H^0(\omega^m_{X_{n+1}})$.

Proof. $H^1(\omega^m_{X_n}) \to H^1(\omega^m_{X_n - S})$ is injective and over the smooth points of X_0 the above element is the obstruction by computations of [Levine83]. □

(12.5.10) **Lemma** [Kawamata85]. *Y_0 has only canonical singularities.*

Proof. By definition, Y_0 is a Cartier divisor in

$$\operatorname{Spec}_{X_0}(\mathscr{O} + \omega^{[-1]} + \omega^{[-2]} + \cdots).$$

This latter has canonical singularities since it is locally a quotient of $\mathbb{C} \times$ (the Gorenstein cover of X_0). Since $\omega^m_{Y_n}$ is generated by global section, the corresponding linear system on $\operatorname{Spec}_{X_0}(\mathscr{O} + \omega^{[-1]} + \omega^{[-2]} + \cdots)$ is base point free, thus the general member Y_0 also has canonical singularities. □

(12.5.11) *End of proof.* Since Y_0 has canonical singularities the same proof as in [Levine83, Proposition 2] yields that the map

$$d : \operatorname{Ext}^1(\Omega^1_{Y_n}, \omega_{Y_n}) \to \operatorname{Ext}^1(\mathscr{O}_{Y_n}, \omega_{Y_n}) = H^1(\omega_{Y_n})$$

is zero. Thus the obstruction to lifting vanishes. This proves that general sections of $H^0(\omega^m_{X_0})$ lift to sections of $H^0(\omega^m_{X/T})$. Since general sections generate the space of all sections this proves that the plurigenera P_{mk} are constant in the family. The rest of the proof of (12.5.5) now follows as in [Levine85]. □

(12.5.12) *Proof of* (12.5.4). By shrinking S, we can apply (12.4.4) to get a fiberspace Y/S which is fiberwise bimeromorphic to X/S such that Y_0 is projective with only $\mathbb{Q}$-factorial terminal singularities. By [Kawamata91] some multiple of K_{Y_0} is base point free. For fixed k consider $P_k(Y_s)$ as a function on S. This is upper semicontinuous in the Zariski topology, thus we can apply (12.5.5) to conclude that $P_k(Y_s)$ is constant in a neighborhood S_k of 0. Our aim is to find a neighborhood U where all the plurigenera are constant.

To do this fix a k such that $\omega^{[k]}_{Y_0}$ is locally free and is generated by global sections. There is a neighborhood V of 0 such that the same holds for $\omega^{[k]}_{Y_s}$ for $s \in V$. Now we can consider the k-canonical morphism

$$\phi_k : Y/V \to Z/V \subset \mathbb{P}_V \qquad \text{such that} \qquad \omega^{[mk]}_{Y/V} \cong \phi_k^* \mathscr{O}_{\mathbb{P}/V}(m),$$

where $\mathbb{P}_V$ is a projective space bundle over V and m is any positive integer. The varieties Z_s all have the same degree, thus they form a bounded family. In particular, there are only finitely many Hilbert *functions*

$$H_s(m) = H^0(Z_s, \mathscr{O}_{Z_s}(m)) = H^0(Y_s, \omega^{[mk]}_{Y_s}).$$

Similarly, considering the family of sheaves

$$F^j_s = (\phi_k)_* \omega^{[j]}_{Y_s} \qquad \text{for } j < k\ ,$$

we see that there are only finitely many Hilbert *functions*

$$H_s^j(m) = H^0(Z_s, F_s^j \otimes \mathcal{O}_{Z_s}(m)) = H^0(Y_s, \omega_{Y_s}^{[j+mk]}).$$

Thus there is an $N > 0$ such that if $P_m(Y_s) = P_m(Y_{s'})$ for every $m < N$ then $P_m(Y_s) = P_m(Y_{s'})$ for every m. Therefore there is an open neighborhood $0 \in U \subset S$ such that all the plurigenera are constant over U. □

(12.5.13) **Corollary.** *Let $f : X \to S$ be a flat family of projective threefolds with $\mathbb{Q}$-factorial terminal singularities and assume that S is connected. The canonical models of the fibers also form a flat family.*

Proof. The relative canonical model is

$$\operatorname{Proj}_S(\sum f_*(\omega_{X/S}^{[j]})).$$

If the fibers are uniruled the canonical model is the empty set. Otherwise the sheaves $f_*(\omega_{X/S}^{[j]})$ are locally free, hence the Proj is flat and commutes with base change. Thus the relative canonical model is the family of canonical models of the fibers. □

(12.5.14) **Theorem.** *Let X/S be a proper flat family of complex analytic three-folds with $\mathbb{Q}$-factorial terminal singularities. Let $0 \in S$ and assume that X_0 is a projective threefold of general type. Then there is an open neighborhood $0 \in U \subset S$ such that all the fibers over U are also projective threefolds of general type.*

Proof. Consider the relative canonical model. Its canonical class is relatively ample, hence it is projective. Thus X/S is a relative algebraic space. Hence by (12.2.10) nearby fibers are also projective. □

(12.6) **Local birational deformation spaces.** (12.6.1) If S is a nonruled surface and S' is its minimal model then any deformation of S is obtained from a suitable deformation of S' by repeatedly blowing up some sections. Thus up to birational equivalence the deformations of S' give all the deformations of S. Thus $\operatorname{Def} S'$ can be viewed as the deformation space of the birational equivalence class of S.

If X is a 3-fold and $C \subset X$ is a smooth curve then C need not be liftable to certain deformations of X, thus X can have deformations that do not give deformations of $B_C X$. Thus the deformation spaces of different smooth models differ more than in the surface case. To make things worse, minimal models are not unique. Nonetheless one can define a good local birational deformation space thanks to the following results:

(12.6.2) **Theorem.** *Let X_0 and X_0' be projective 3-folds with $\mathbb{Q}$-factorial terminal singularities. Assume that K_{X_0} and $K_{X_0'}$ are both nef. Let $g : X_0 \cdots > X_0'$ be a birational map. Then g induces an isomorphism $g_* : \operatorname{Def} X_0 \xrightarrow{\sim} \operatorname{Def} X_0'$.*

Proof. By [Kollár89, 4.9], X_0' is obtained from X_0 by a sequence of flops. Let $h : X_0 \cdots > X_0^+$ be the first flop; then it is sufficient to construct an isomorphism $h_* : \operatorname{Def} X_0 \xrightarrow{\sim} \operatorname{Def} X_0^+$. If X/T is a deformation of X_0, then (11.10) gives the corresponding flat deformation X^+/T of X_0^+. This gives a morphism $\operatorname{Def} X_0 \to \operatorname{Def} X_0^+$, which is clearly an isomorphism. □

(12.6.3) *Remarks.* (12.6.3.1) The same result holds for compact complex 3-folds with analytically $\mathbb{Q}$-factorial singularities.

(12.6.3.2) $\mathbb{Q}$-factoriality is a necessary assumption.

(12.6.4) **Proposition-Definition.** *Let $\mathscr{X}$ be a birational equivalence class of nonuniruled 3-folds. By* [Mori88] *there is a member $X \in \mathscr{X}$ with $\mathbb{Q}$-factorial terminal singularities and K_X nef. Its deformation space will be called the local deformation space of $\mathscr{X}$ and is denoted by* $\operatorname{Def}\mathscr{X}$. *By* (12.6.2) *this definition is independent of the minimal model chosen.*

By (12.4.4) *and* (12.6.2), *if X' has $\mathbb{Q}$-factorial terminal singularities and $g : X' \cdots > X$ is a birational map, then it induces a morphism* $\operatorname{Def} X' \to \operatorname{Def} X \cong \operatorname{Def}\mathscr{X}$.

(12.6.5) **Proposition.** *Let T be the spectrum of a DVR. Let X_0 be a projective 3-fold with $\mathbb{Q}$-factorial terminal singularities such that K_{X_0} is nef. Let X^1/T and X^2/T be two flat deformations of X_0 (i.e., $X_0^i \simeq X_0$). Assume that the generic fibers X^1_{gen} and X^2_{gen} are birationally equivalent. Then there is a birational map $g_0 : X_0 \cdots > X_0$ such that the induced morphism $g_{0*} : \operatorname{Def} X_0 \to \operatorname{Def} X_0$ takes the family X^1/T to X^2/T.*

Proof. Let g_{gen} be the given birational equivalence and let $\Gamma/T \subset X^1 \times_T X^2$ be the closure of its graph. We will prove in (12.7.6.6) that if $Z \to X^i$ is any resolution of singularities, then all the exceptional divisors are uniruled. Thus by [Matsusaka-Mumford64], Γ_0 is the graph of a birational equivalence $g_0 : X_0 \cdots > X_0$. We can factor g_0 into a sequence of flops, and this can be extended to X^1/T. This way we obtain X^{1+}/T. The graph of the new birational equivalence $\Gamma^+ \subset X^{1+} \times_T X^2$ is such that Γ_0^+ is the graph of an isomorphism. Thus Γ^+ is the graph of an isomorphism as well. This shows that g_{0*} maps the family X^1/T to X^2/T. □

(12.7) **Global birational moduli spaces.** (12.7.1) The global birational moduli problem is much more delicate than the local one. Usually one has to introduce some extra structure (distinguished homology basis or polarization). Here we discuss the case when the threefolds come nearly endowed with a polarization, i.e., threefolds of general type.

Already for surfaces the minimal model is not the right object to use to construct global moduli spaces. Although it is unique, it leads to a very non-separated moduli space. Therefore we consider instead the canonical model. If S is minimal, then by [Brieskorn71] the morphism $\operatorname{Def} S \to \operatorname{Def} S^{\text{can}}$ is finite and surjective. Hence the deformation theory of S and S^{can} is very similar.

In the threefold case this is not so, as shown by the following.

(12.7.2) **Example.** Let $X \subset \mathbb{CP}^4$ be a hypersurface with a single ordinary triple point at $0 \in X$. Then $B_0X \to X$ is a resolution of singularities with a smooth cubic surface E as exceptional divisor. Thus X is canonical. We want to understand $\operatorname{Def} B_0X \to \operatorname{Def} X$. $N_{E|B_0X} \simeq \mathcal{O}_E(-1)$, thus $H^1(N_{E|B_0X}) = 0$. Therefore if Y/T is a deformation with $Y_0 \simeq B_0X$ then $E \subset B_0X$ lifts to a flat family of exceptional surfaces over T. This implies that

$$\operatorname{im}(\operatorname{Def} B_0X \to \operatorname{Def} X)$$

consists exactly of those hypersurfaces that contain a triple point.

(12.7.3) **Proposition.** *Let X be a proper threefold with canonical singularities, and let $f : Y \to X$ be a projective $\mathbb{Q}$-factorial, terminal, and crepant partial resolution. Then the natural morphism $F : \operatorname{Def} Y \to \operatorname{Def} X$ is finite.*

Proof. The morphism exists by (11.4). If F is not finite, then there is a deformation $\mathcal{Y}/T$ $(T = \operatorname{Spec} \mathbb{C}[[t]])$ such that it maps to the trivial deformation $F : \mathcal{Y} \to X \times T$. This means that there is a birational map $\mathcal{Y} \to X \times T \cdots > Y \times T$. We can apply (12.6.5) to $\mathcal{Y} \cdots > Y \times T$ to get that $\mathcal{Y}$ is also a trivial deformation of Y. □

(12.7.4) **Proposition-Definition.** *Notation is the same as above.*

(12.7.4.1) *The subspace* $\operatorname{im}[\operatorname{Def} Y \to \operatorname{Def} X]$ *is closed and is independent of the choice of Y. It will be called the SFCT subspace of* $\operatorname{Def} X$ *(for simultaneous $\mathbb{Q}$-factorial crepant terminalization).*

(12.7.4.2) *Openness of versality for SFCT subspaces: Let $\mathcal{Y} \to \operatorname{Def} Y \to \operatorname{Def} X$ be the versal SFCT family at $Y \to X$. This family is also a versal SFCT family at every nearby pair $Y_t \to X_t$.*

(12.7.4.3) *A flat family of threefolds X/S satisfies SFCT if for every $s \in S$ the image of the natural morphism $(s, S) \to \operatorname{Def} X_s$ lies in the SFCT subspace of $\operatorname{Def} X_s$.*

Proof. The independence follows from (12.6.4). By (12.1.10) and (12.5.14) Y_t is again projective with $\mathbb{Q}$-factorial terminal singularities. The morphism $Y_t \to X_t$ is crepant. Now openness of versality for $\operatorname{Def} Y$ implies (12.7.4.2). □

(12.7.5) **Formulation of the birational moduli problem.** Fix a function $P(k)$ (the Hilbert function).

(12.7.5.1) Let $\mathcal{M}_P$ be the functor

$$\mathcal{M}_P(S) = \left\{ \begin{array}{l} \text{Proper flat families } X/S \text{ such that } X \text{ is an algebraic space, for} \\ \text{every } s \in S \text{ the fiber } X_s \text{ is a projective 3-fold with } \mathbb{Q}\text{-factorial} \\ \text{terminal singularities and } h^0(X_s, \omega_{X_s}^{[k]}) = P(k) \text{ for every } k \geq 2. \\ \text{Two families are equivalent if there is an isomorphism between} \\ \text{open dense subsets } f : X^1/S \cdots > X^2/S \text{ which is birational on} \\ \text{every fiber.} \end{array} \right\}$$

(12.7.5.2) Let $\mathcal{M}_P^{\text{can}}$ be the functor

$$\mathcal{M}_P^{\text{can}}(S) = \left\{ \begin{array}{l} \text{Projective flat families } X/S \text{ such that for every } s \in S \text{ the fiber} \\ X_s \text{ is a canonical 3-fold such that } \chi(X_s, \omega_{X_s}^{[k]}) = P(k) \text{ for every} \\ k \text{ and such that } X/S \text{ satisfies SFCT.} \end{array} \right\}$$

(12.7.5.3) Note that by (12.5.13) there is a natural transformation

$$\mathcal{M}_P \to \mathcal{M}_P^{\text{can}}.$$

As is the case already for surfaces, these two functors agree on closed points, but they differ infinitesimally. The functor $\mathcal{M}_P$ is very nonseparated. While $\mathcal{M}_P$ is more interesting from the point of view of smooth threefolds, technically $\mathcal{M}_P^{\mathrm{can}}$ is easier to deal with.

(12.7.5.4) Let X be a canonical threefold. Then by vanishing $\chi(X, \omega_X^{[k]}) = h^0(X, \omega_X^{[k]})$ for $k \geq 2$. In general $\chi(X, \omega_X^{[k]})$ is not a birational invariant even for smooth threefolds. This is the reason why we use h^0 in (12.7.5.1) and χ in (12.7.5.2).

(12.7.6) **Theorem** (Birational moduli for threefolds of general type).

(12.7.6.1) *For every $P(k)$ the functor $\mathcal{M}_P^{\mathrm{can}}$ is coarsely represented by a separated algebraic space of finite type $\mathbf{M}_P$.*

(12.7.6.2) *Let Y/S be a smooth family of projective 3-folds of general type and assume that S is connected. For some $s \in S$, let $P(k) = h^0(Y_s, \omega_{Y_s}^k)$ for $k \geq 2$. Then there is a morphism $f : S \to \mathbf{M}_P$ such that for every $s \in S$ the image $f(s)$ is the moduli point of the canonical model of Y_s.*

The proof will be done in several steps.

(12.7.6.3) Let Z/Δ be a proper flat family of algebraic varieties with canonical singularities. Then $\chi(Z_t, \omega_{Z_t}^{[n]})$ is locally constant for every n.

This was proved in [Kollár83, 3.1.4]. The point is that usually double dual will not commute with specialization. The argument is the following. Pick $z \in Z_0$ and let $(y, Y_0) \to (z, Z_0)$ be the local index one cover. By [Kollár83, 3.2.2] or [Ran89, 2.3], this extends to a cover $f : (y, Y) \to (z, Z)$. By construction $f_*\mathcal{O}_Y \simeq \sum_0^{k-1} \omega_Z^{[i]}$ and $\omega_Z^{[k]}$ is locally free. Thus $\omega_Z^{[i+k]} \simeq \omega_Z^{[k]} \otimes \omega_Z^{[i]}$. Hence $\omega_Z^{[j]} \otimes \mathcal{O}_{Z_0} \simeq \omega_{Z_0}^{[j]}$ locally everywhere, hence also globally. This implies the claim.

(12.7.6.4) Canonical 3-folds with fixed $P(k) = \chi(X, \omega_X^{[k]})$ form a bounded family.

This was proved in [Kollár83, 3.1.4]. Using more recent information a proof can be obtained as follows. By [Reid87, 10.3],

$$\chi(X, \omega_X^{[n]}) = \frac{n(n-1)(2n-1)}{12} K_X^3 + (1 - 2\chi(\mathcal{O}_X))n + cn + \phi(n),$$

where $c = (1/12)\sum(r_i - r_i^{-1})$ and summation ranges over certain integers r_i such that $\operatorname{index} X = \operatorname{lcm}(r_i)$ and $\phi(n) = 0$ if n is sufficiently divisible. Notice that given a function $P(k)$, there is at most one way of writing it in the above form. Thus $P(k)$ determines the above c, hence we can bound the index of X in terms of $P(k)$. If the index is m, then $(X, \omega_X^{[m]})$ is a 3-fold with an ample Cartier divisor of fixed Hilbert polynomial. These form a bounded family by [Kollár85, 2.1.3].

(12.7.6.5) Let X/Δ be a flat family of 3-folds. Assume that X_0 has canonical singularities and that X/Δ satisfies SFCT. Then all nearby fibers have only canonical singularities.

Let $f_0 : Y_0 \to X_0$ be a $\mathbb{Q}$-factorial crepant terminalization. By definition of SFCT, possibly after a finite and surjective base change, there is a deformation Y/Δ of Y_0 and a proper birational map $f : Y/\Delta \to X/\Delta$. By [KSB88, Chapter

6] Y_t has terminal singularities. Since $Y_t \to X_t$ is crepant, X_t has canonical singularities.

(12.7.6.6) Let X/Δ be a flat family of 3-folds satisfying SFCT. Let $Z/\Delta \to X/\Delta$ be a proper birational morphism and let $E \subset Z$ be an exceptional divisor. Then E is uniruled.

Let $g : U/\Delta \to X/\Delta$ be a proper generically finite and surjective morphism such that any g-exceptional divisor is uniruled. Then it is sufficient to prove the above claim for U/Δ instead of X/Δ. Then first we can take $f : Y/\Delta \to X/\Delta$ as above. The f-exceptional divisors are uniruled by [Reid80]. The question is also local on Y/Δ. By [Reid83] locally everywhere we have a morphism $h : U \to Y$ such that U is smooth and there are no h-exceptional divisors. For a smooth variety the claim is clear, hence we are done.

(12.7.6.7) *Completion of the proof.* The first part is clear since by the above considerations all the conditions of [Kollár85, 4.1.1] are satisfied by $\mathscr{M}_P^{\text{can}}$.

The second part is a reformulation of (12.5.13). □

13. Further results on extremal nbds and flips

The aim of this chapter is to get further information about extremal nbds. The idea is to view X as a one-parameter family of surfaces and to exploit the deformation theory of surfaces to understand X. The general framework is the following:

(13.1) **A method for constructing extremal nbds.** Let $X \supset C$ be an extremal nbd. Let $t \in \mathscr{O}_X$ be a function vanishing on C and let $H = (t = 0)$. Thus X can be viewed as the family of level sets of the function t. We can try to recover X as a deformation of the surface germ H. This can be done as follows:

Let us start with a rational surface singularity H'. Consider a proper bimeromorphic morphism $f : H \to H'$. Assume that H is normal and every singularity of H can be the hyperplane section of a terminal threefold singularity. This means that every $P_i \in H$ has a one-parameter deformation $H_{i,t} : t \in \Delta$ such that the total space is a three-dimensional terminal singularity. By (11.4.2) we can choose a deformation $H_t : t \in \Delta$ of $H = H_0$ which induces the above deformations at the singular points. Let X be the total space of this deformation of H. This X is an analytic threefold which has only terminal singularities by construction. By (11.4.1) f_0 extends to a contraction morphism $f_t : H_t \to H'_t$. Here $H'_t : t \in \Delta$ is a flat deformation of H'_0; let Y be the total space. The natural morphism $f : X \to Y$ is proper and bimeromorphic. By the adjunction formula $K_X|H = K_H$. Therefore if $-K_H$ is f-ample then $f : X \to Y$ is an extremal nbd where C is possibly reducible. It is not clear whether the nbd constructed is isolated or not. Some criteria will be given later.

(13.2) **A method for deforming extremal nbds.** Let $X \supset C$ be an extremal nbd. Let $t \in \mathscr{O}_X$ be a function vanishing on C and let $H = (t = 0)$. Thus X can be viewed as the total space of a one-parameter deformation H_t of the surface germ H. If $H_{s,t} : (s, t) \in \Delta(s) \times \Delta(t)$ is a two-parameter deformation of H

such that $H_{0,t} = H_t$ then we can view $X_s = \bigcup_t H_{s,t} : s \in \Delta$ as a deformation of the extremal nbd $X_0 = X$.

Assume that we have X and at the singular points we specify a deformation of $X \supset H \ni P_i$. This way we get a morphism

$$v : \Delta(s) \times \Delta(t) \to \prod \operatorname{Def}(P_i \in H).$$

The deformation in the t-direction is the one realized by X, thus we have a specified lifting of $v|\Delta(t)$ to a morphism $\Delta(t) \to \operatorname{Def} H$. By (11.4.2) the restriction morphism $\operatorname{Def} H \to \prod \operatorname{Def}(P_i \in H)$ is smooth, thus locally a direct product. Therefore there is a lifting

$$V : \Delta(s) \times \Delta(t) \to \operatorname{Def} H$$

which induces the above two-parameter deformations at the singularities. We can fix s and let t vary, this way we get a one-parameter family $X_s : s \in \Delta$. By construction $X_0 \cong X$.

The disadvantage of this method is that it is frequently very hard to understand the exceptional curves $C_s \subset X_s$.

(13.3) **Theorem.** *Let $X \supset C$ be an extremal nbd, C possibly reducible. Let $t \in \mathscr{O}_X$ be a function vanishing on C and let $H = (t = 0)$. Assume that for every $P \in H$ one of the following conditions is satisfied:*

(13.3.1.1) *In suitable local coordinates X is given by an equation of the form*

$$(g(y_1, y_2, y_3, y_4) = 0)/\mathbb{Z}_m(a_1, a_2, a_3, 0)$$

and H is locally defined by $y_4 = t = 0$.

(13.3.1.2) *In suitable local coordinates X is given by an equation of the form*

$$(g(y_1, y_2, y_3, y_4) = 0)/\mathbb{Z}_4(1, 1, 3, 2).$$

where $g_{\deg=2}(y_1, y_2, y_3, 0)$ has rank three and H is locally defined by $f(y_1, y_2, y_3, y_4) = t = 0$ where $f_{\deg=2}(0, 0, 0, y_4)$ is nonzero.

(13.3.1.3) *In suitable local coordinates X is given by an equation of the form*

$$(g(y_1, y_2, y_3, y_4) = 0)/\mathbb{Z}_4(1, 1, 3, 2).$$

where $g_{\deg=2}(y_1, y_2, y_3, 0) = g_{\deg=2}(0, y_2, y_3, 0)$ has rank two and H is locally defined by $f(y_1, y_2, y_3, y_4) = t = 0$ where $f_{\deg=2}(0, 0, 0, y_4)$ is nonzero.

Then there is a flat deformation

$$(X_s \supset H_s \supset C_s) : s \in \Delta; \qquad (X_0 \supset H_0 \supset C_0) \cong (X \supset H \supset C)$$

such that the following conditions are satisfied:

(13.3.2.1) *$(H_s \supset C_s)$ is the trivial deformation of $(H_0 \supset C_0)$; in fact, we have a natural identification $H_0 \cong H_s$.*

(13.3.2.2) *If $P \in H$ is as in* (13.3.1.1) *then X_s has a cyclic quotient singularity at $P \in H_s$ for $s \neq 0$.*

(13.3.2.3) *If $P \in H$ is as in* (13.3.1.2) *then for $s \neq 0$ in suitable coordinates X_s at $P \in H_s$ is given by*

$$(y_1^2 + y_2^2 + y_3^2 + y_4^3 = 0)/\mathbb{Z}_4(1, 1, 3, 2).$$

(13.3.2.4) *If* $P \in H$ *is as in* (13.3.1.3) *then for* $s \neq 0$ *in suitable coordinates* X_s *at* $P \in H_s$ *is given by*

$$(y_1^6 + y_2^2 + y_3^2 + y_4^3 = 0)/\mathbb{Z}_4(1, 1, 3, 2).$$

(13.3.3) *In all cases* X_s *has only analytically* $\mathbb{Q}$*-factorial singularities along* H_s *for* $s \neq 0$.

(13.3.4) Warning: In general X_s will have other singular points and other exceptional curves too.

Proof. X can be thought of as the total space of a one-parameter deformation $H_t : t \in \Delta$ of $H_0 = H$.

If $P \in H$ is as in (13.3.1.1) then introduce a new parameter s and consider the deformation of the singularity

$$(g(y_1, y_2, y_3, y_4) + sy_4 = 0)/\mathbb{Z}_m(a_1, a_2, a_3, 0).$$

If $P \in H$ is as in (13.3.1.2–13.3.1.3) then introduce a new parameter s and consider the deformation of the singularity

$$(g(y_1, y_2, y_3, y_4) + s(y_4 + \mu y_1^2)f(y_1, y_2, y_3, y_4) = 0)/\mathbb{Z}_4(1, 1, 3, 2),$$

where μ is a sufficiently general constant.

These deformations can be globalized as in (13.2).

If $P \in H$ is as in (13.3.1.1) then for $s \neq 0$ the local equation for the index one cover of X_s contains sy_4 with nonzero coefficient, thus X_s has a quotient singularity at P.

If $P \in H$ is as in (13.3.1.2–13.3.1.3) then the local equation g_s is such that $(g_s)_{\deg=2}(y_1, y_2, y_3, 0)$ has rank at least two (these come from g). In case (13.3.1.2) $(g_s)_{\deg=2}(y_1, y_2, y_3, 0)$ has rank three and we get the equation (13.3.2.3).

If $(g_s)_{\deg=2}$ has rank two then introduce a $\mathbb{Z}$-wt by $\sigma(y_1, y_2, y_3, y_4) = (1, 3, 3, 2)$. Then

$$(g_s)_{\sigma\leq 6} = y_2^2 + y_3^2 + (a_1 s + a_0)y_4^3 + (b_2 s\mu + b_1 s + b_0)y_1^2 y_4^2 + cy_1^4 y_4 + dy_1^6 + ey_1^3 y_3,$$

where $a_1 b_2 \neq 0$, c, d, and e depend on s, μ linearly. For general s, μ in suitable new coordinates this can be brought to the form

$$(g_s)_{\sigma\leq 6} = \overline{y}_2^2 + \overline{y}_3^2 + \overline{y}_4^3 + \overline{y}_1^6.$$

Adding higher σ-wt terms does not change the isomorphism class of this singularity.

A cyclic quotient singularity is $\mathbb{Q}$-factorial. For the singularity in (13.3.2.3) $\mathbb{Q}$-factoriality follows from [Kollár91, 2.2.7] since $y_3^2 + y_4^3$ is irreducible. Also by [Kollár91, 2.2.7] the local Picard group of

$$(y_2^2 - y_3^2 + y_4^3 - y_1^6 = 0)$$

is generated by the divisors

$$D_j = (y_2 - y_3 = y_4 - \varepsilon^j y_1^2 = 0) \quad \text{where } \varepsilon^3 = 1 \text{ and } j = 0, 1, 2.$$

The $\mathbb{Z}_4$-action (1,1,3,2) sends D_j to

$$D_j \mapsto D_j' = (iy_2 - i^3 y_3 = i^2 y_4 - \varepsilon^j i^2 y_1^2 = 0) = (y_2 + y_3 = y_4 - \varepsilon^j y_1^2 = 0).$$

Since

$$D_j \cup D_j' = (y_4 - \varepsilon^j y_1^2 = 0)$$

is principal, the action of $\mathbb{Z}_4$ on the Picard group is by $-\mathrm{id}$. Thus the Picard group of the quotient

$$(y_2^2 - y_3^2 + y_4^3 - y_1^6 = 0)/\mathbb{Z}_4 \cong (y_1^6 + y_2^2 + y_3^2 + y_4^3 = 0)/\mathbb{Z}_4$$

is torsion. □

The following example shows some of the intricate features of these deformations

(13.3.5) **Example.** We consider a deformation X_t of IIA type extremal nbds given by equations as in (7.9.4.1). The deformation parameter will be t. The family will be patched together from two charts. One chart is

$$(y_1 y_2 + y_3^2 + y_4^{1+2k} - t y_4 = 0)/\mathbb{Z}_4(1, 1, 3, 2, 0)$$

and the other one is $\mathbb{C}^4$ with coordinates u_1, u_2, u_3, t. The patching relations are

$$u_1^{-1} = y_1^4 \qquad u_1 u_2 = y_1 y_3 + y_1^{-4} y_4^2 \qquad u_1^{-1} u_3 = y_1^2 y_4.$$

The curve C_t is the y_1-axis. By (7.9.4) for every t this is a nbd of type (7.3.1). H_t is difficult to write down explicitly, but by (7.3) it is unchanged under the deformation.

We are looking for other exceptional curves D_t that specialise to C_0 as $t \to 0$. First, from the above equations we obtain

$$u_2 = y_1^5 y_3 + y_4^2 \quad u_1 u_2 - u_3^2 = y_1 y_3.$$

Both sides of these equalities are regular, thus they define regular functions on X_t for every t. Therefore they are constant on D_t. $(y_1 = 0)$ intersects C_0 transversally, thus it also intersects D_t nontrivially for $|t|$ small. Thus $y_1 y_3$ vanishes at some point of D_t. It is also constant, thus $y_3|D_t = 0$. From the first function we see that y_4 is also constant on D_t. Substituting into the equation $y_1 y_2 + y_3^2 + y_4^{1+2k} - t y_4 = 0$, we conclude that $y_4^{1+2k} - t y_4 | D_t = 0$.

This has two kinds of solutions. The trivial one is $y_4 = 0$, which gives C_t. Also, for every $v^{2k} = t$ we obtain another compact curve $D_{t,v}$ given as

$$y_2 = y_3 = 0, \quad y_4 = v \qquad \text{in the first chart,}$$
$$u_2 = v^2, \quad v^2 u_1 = u_3^2 \qquad \text{in the second chart.}$$

By the group action on the first chart, $D_{t,v} = D_{t,-v}$. Thus we obtain k other exceptional curves for $t \neq 0$.

(13.4) **Corollary.** *Let $X \supset C \to Y$ be an isolated extremal nbd. Assume that C is irreducible. Let $X^+ \supset C^+ \to Y$ be the flip. Then C^+ is also irreducible.*

Proof. As a simple application of (11.9.1) we see that if X is analytically $\mathbb{Q}$-factorial and C is irreducible then C^+ is also irreducible.

To get the result in general we apply the deformation constructed in (13.3). Let $H \in |\mathscr{O}_X|$ be a general member containing C. As we have seen in Chapters 3 and 6–9, $C \subset H \subset X$ is everywhere of the form (13.3.1.1–13.3.1.3).

We can view the four-dimensional total space of the deformation as a two-parameter family of surfaces: $H_{s,t} : (s, t) \in \Delta(s) \times \Delta(t)$. The contraction morphism gives a family $Y_s : s \in \Delta(s)$ and this again can be viewed as a two-parameter family: $H'_{s,t} : (s, t) \in \Delta(s) \times \Delta(t)$. Note that by (13.3.1) $H'_{s,0} \cong H'_{0,0}$. By (11.7.3) the flip $X_s^+ : s \in \Delta(s)$ exists and is obtained as a two-parameter family $H^+_{s,t} : (s, t) \in \Delta(s) \times \Delta(t)$ where $H^+_{s,t}$ is a P-modification of $H'_{s,t}$.

We claim that $H^+_{s,0} \cong H^+_{0,0}$. To see this consider the family $H^+_{s,0} : s \in \Delta(s)$. This is a modification of the trivial family $H'_{s,0} : s \in \Delta(s)$. A singularity has only finitely many P-modifications, thus we may assume that $H^+_{s,0}$ is independent of $s \in \Delta(s) - \{0\}$. Denote this common P-modification by H^+_*. Thus the threefolds $\bigcup H^+_{s,0} : s \in \Delta(s)$ and $H^+_* \times \Delta(s)$ are isomorphic over $\Delta(s) - \{0\}$. They both have a proper morphism onto $H'_{0,0} \times \Delta(s)$ and the relative canonical class is relatively ample in both cases. Therefore [Matsusaka-Mumford64] implies that they are isomorphic. This proves the claim.

For general s the space X_s has only $\mathbb{Q}$-factorial singularities along C_s. Therefore for general s the curve C_s^+ is irreducible. This implies that $H^+_{s,0} \to H'_{s,0}$ has only one exceptional curve for $s \neq 0$. Since $H^+_{0,0} \to H'_{0,0}$ is isomorphic to $H^+_{s,0} \to H'_{s,0}$, C_0^+ is also irreducible. □

(13.5) **Theorem.** *Let $f : X \supset C \to Y \ni Q$ be an isolated extremal nbd with C possibly reducible. Let $f^+ : X^+ \supset C^+ \to Y \ni Q$ be the flip. Then*

$$\#(\text{irreducible components of } C^+) \leq \#(\text{irreducible components of } C).$$

Proof. We start flipping one curve at a time. We get a series of morphisms $f^i : X^i \supset C^i \to Y \ni Q$. By (13.4) each exceptional curve C^i has the same number of components. Finally we stop when the canonical class becomes nef. Then we take the relative canonical model, thus we may contract some of the curves. □

Before giving an example that shows that there can be fewer curves after flip, we need methods to recognize isolated nbds. There are several ways of doing this, for our purposes criteria concerning $H' \subset X$ will be the most useful. Note that if H' has normal singularities, the rational numbers $C_i \cdot K_{H'}$ and $C_i \cdot C_i$ are defined ($\cdot$ is the intersection product).

(13.6) **Proposition.** *Let $f : X \to Y$ be an extremal nbd constructed as in (13.2). Let C_i be the exceptional curves of f_0. The exceptional set of f is one dimensional if any of the following conditions are satisfied:*

(13.6.1) $\sum a_i C_i \cdot K_{H'} = -1$ *has no solutions* $a_i \in \mathbb{Z}_+$.

(13.6.2) $\sum a_i C_i \cdot K_{H'} = -k$ *and* $(\sum a_i C_i) \cdot (\sum a_i C_i) = -k$ *have no simultaneous solutions* $a_i, k \in \mathbb{Z}_+$.

(13.6.3) *f_0 has only one exceptional curve, X is a primitive extremal nbd, and H is not a DuVal singularity.*

(13.6.4) *f_0 has only one exceptional curve, the torsion subgroup of $Cl^{sc}X$ has order m, and H is log-terminal but not of the form*

$$(xy - z^{md} = 0)/\mathbb{Z}_m(1, -1, a) \quad \text{where } (a, m) = 1.$$

Proof. The proof of the above conditions is relatively straightforward if X has only $\mathbb{Q}$-factorial singularities. (13.3) and the following lemma will reduce the general case to the $\mathbb{Q}$-factorial one.

(13.6.5) **Lemma.** *Let $f_s : X_s \supset H_s \supset C_s \to Y_s : s \in \Delta$ be a one-parameter deformation of the extremal nbd $f_0 : X_0 \supset H_0 \supset C_0 \to Y_0$. Then $X_0 \supset H_0 \supset C_0$ is divisorial iff $X_s \supset H_s \supset C_s$ is divisorial for all small s.*

Proof. Note first that in general X_s is not a germ, thus it is not an extremal nbd. In fact it can contain several disjoint contracted curves. We claim the strongest version of the above lemma: if X_0 is divisorial then the germ along C_s is divisorial.

Let $\mathscr{X}$ (resp. $\mathscr{Y}$) be the total spaces of the deformations of X_0 (resp. Y_0) and let $F : \mathscr{X} \to \mathscr{Y}$ be the contraction morphism. Let $\mathscr{E}$ be the exceptional set of F. If $X_s \supset H_s \supset C_s$ is divisorial for all small $s \neq 0$ then $\mathscr{E}$ has dimension three. Thus $\mathscr{E} \cap X_0$ has dimension at least two, hence X_0 is divisorial.

Conversely assume that X_0 is divisorial. Let $B \subset X_0$ be a general contracted curve. Then B does not pass through any singularities and it is a smooth rational curve with normal bundle $\mathscr{O} + \mathscr{O}(-1)$. Therefore the normal bundle of B in $\mathscr{X}$ is $\mathscr{O} + \mathscr{O} + \mathscr{O}(-1)$. Thus B has a two-parameter family of embedded deformations in $\mathscr{X}$. Thus $\mathscr{E}$ has a three-dimensional component $\mathscr{E}_1$ containing at least one component of C_0. $\mathscr{H} = \bigcup H_s$ is a Cartier divisor on $\mathscr{X}$. Therefore $\mathscr{H} \cap \mathscr{E}_1$ is at least two dimensional. Therefore $H_s \cap \mathscr{E}_1$ is a compact curve contracted by f_s, thus $X_s \supset H_s \supset C_s$ is divisorial for all small s. □

In the analytically $\mathbb{Q}$-factorial case we argue as follows. If f contracts a divisor then in the general fiber we have $f_t : H'_t \to H_t$, which contracts some curves $D_{j,t}$. By construction H'_t is smooth and $-K_{H'_t}$ is f_t-ample. Therefore every f_t-exceptional curve is a (-1)-curve.

To see (13.6.1) pick any of these (-1)-curves and specialize it to the special fiber. The limit cycle is an integral linear combination $\sum a_i C_i$ of the exceptional curves. The intersection number with K_X remains constant in the procedure. Thus $\sum a_i C_i \cdot K_{H'} = -1$.

If X has $\mathbb{Q}$-factorial singularities and there is an exceptional divisor E then it must be $\mathbb{Q}$-Cartier. We compute the intersection number $E \cdot E \cdot H'$ in two ways. First, $E \cdot H$ is an integral linear combination $\sum a_i C_i$ of exceptional curves. Thus

$$\left(\sum a_i C_i\right) \cdot \left(\sum a_i C_i\right) = E \cdot E \cdot H'.$$

On the other hand, $E \cdot E \cdot H' = E \cdot E \cdot H'_t$. Since $E \cdot H'_t$ is a collection of k (-1)-curves, the above intersection number is $-k$. Computing $K_X \cdot E \cdot H'$ as before gives the other equality.

In case (13.6.3) if f contracts a divisor then it has to be a divisorial contraction of a single extremal ray, thus Y has terminal singularities. Assume that Y is not Gorenstein. Let $Y' \to Y$ be the Gorenstein cover. Take $X' =$ normalization of $(X \times_Y Y')$. The morphism $X' \to X$ is étale outside C. By purity it has to be étale outside the singular points of X. Thus X is not primitive. Therefore H is rational and Gorenstein thus DuVal.

The previous argument also shows that the torsion in $Cl^{sc}X$ and the index of K_H are equal. The only log-terminal singularites with index m that can be surface sections of terminal singularities are exactly those listed in (13.6.4) (see [KSB88, 3.10]). □

(13.6.6) *Remark.* Note that (13.6.3) can be formulated as a necessary and sufficient condition: Assume that f_0 has only one exceptional curve and X is a primitive extremal nbd. Then f is a divisorial contraction if and only if H is a DuVal singularity. □

(13.7) **Example.** We give an example of the following situation: $f : X \to Y$ is an isolated extremal nbd such that the exceptional curve $C = \bigcup C_i$ has several components. After flip we get $f^+ : X^+ \to Y$ and the exceptional set of f^+ is an *irreducible* curve. Thus the number of curves can decrease under flips.

(13.7.1) **Construction.** We start with the triple point resolution

$$\overset{3}{\circ} - \underbrace{\overset{2}{\circ} - \cdots - \overset{2}{\circ}}_{(m-2)\text{-times}}$$

We blow up $(m-1)$ disjoint points in the (-3)-curve. We get $(m-1)$ curves with selfintersection (-1), and the rest is the following.

$$\overset{m+2}{\circ} - \underbrace{\overset{2}{\circ} - \cdots - \overset{2}{\circ}}_{(m-2)\text{-times}} \tag{13.7.1.1}$$

We call these curves $B_{m-1}, \dots, B_1$ from left to right. (13.7.1.1) is the dual graph of the resolution of the quotient singularity of the form

$$\mathbb{C}^2/\mathbb{Z}_{m^2}(1, m-1) \cong (xy - z^m = 0)/\mathbb{Z}_m(1, -1, 1).$$

In particular, it can be the hyperplane section of a terminal quotient singularity. Thus we can contract this configuration and deform the resulting surface to obtain an extremal nbd X with reducible central curve. In the central fiber there are $(m-1)$ exceptional curves and a single quotient singularity.

(13.7.2) *Claim.* (13.7.2.1) The above extremal nbd X is isolated.

(13.7.2.2) After flip we have only one exceptional curve.

Proof. We will use (13.6.2) to show that the nbd is isolated. Let $H \subset X$ be the chosen member of $\mathcal{O}_X$. Let $C_1, \dots, C_{m-1}$ be the exceptional curves in $H \subset X$. We want to compute $C_i \cdot K_H$ and $C_i \cdot C_j$. Let $g : \overline{H} \to H$ be the minimal resolution of the singular point of H. Let $\overline{C}_i$ be the proper transform of C_i. This is a (-1)-curve on $\overline{H}$. By projection formula

$$C_i \cdot K_H = \overline{C}_i \cdot g^* K_H = (-1) - \overline{C}_i \cdot K_{\overline{H}/H}.$$

$K_{\overline{H}/H}$ is easy to compute

$$K_{\overline{H}/H} = -\frac{m-1}{m}B_{m-1} - \frac{m-2}{m}B_{m-2} - \cdots - \frac{1}{m}B_1.$$

Therefore we get that

$$C_i \cdot K_H = -\frac{1}{m}.$$

Similarly, $C_j \cdot C_i = C_j \cdot g^* C_i$. $g^* C_i$ can be written as

$$g^* C_i = \overline{C}_i + \frac{m-1}{m^2}B_{m-1} + \frac{m-2}{m^2}B_{m-2} + \cdots + \frac{1}{m^2}B_1.$$

Therefore we get that

$$C_i \cdot C_i = -1 + \frac{m-1}{m^2}, \quad \text{and} \quad C_i \cdot C_j = \frac{m-1}{m^2} \qquad \text{if } i \neq j.$$

We need to solve the equations

$$\sum a_i C_i \cdot K_H = -k,$$
$$\left(\sum a_i C_i\right) \cdot \left(\sum a_i C_i\right) = -k.$$

By the above formulas

$$\begin{aligned} -k = \left(\sum a_i C_i\right) \cdot \left(\sum a_i C_i\right) &= \sum a_i^2 C_i^2 + \sum_{i \neq j} a_i a_j C_i C_j \\ &= \left(\sum a_i^2\right)\left(-1 + \frac{m-1}{m^2}\right) + \frac{m-1}{m^2}\sum_{i\neq j} a_i a_j \\ &= -\sum a_i^2 + \frac{m-1}{m^2}\left(\sum a_i\right)^2. \end{aligned}$$

Therefore,

$$\sum a_i^2 = k + \frac{m-1}{m^2}\left(\sum a_i\right)^2. \tag{13.7.3}$$

From $\sum a_i C_i \cdot K_H = -k$ we obtain that $\sum a_i = km$. Using the inequality

$$\sum_{i=1}^{m-1} a_i^2 \geq \frac{1}{m-1}\left(\sum_{i=1}^{m-1} a_i\right)^2,$$

(13.7.3) becomes

$$\frac{1}{m-1}k^2 m^2 \leq k^2 m^2 \frac{m-1}{m^2} + k,$$

which can be rearranged as

$$\frac{1}{m-1} \leq \frac{m-1}{m^2} + \frac{1}{km^2} \leq \frac{1}{m}.$$

This is impossible, therefore no divisor is contracted in X.

A rational triple point has an irreducible deformation space, the Artin component. By (11.7.3) the flip is constructed by taking the minimal DuVal resolution of the singularity and taking a deformation of it. The minimal DuVal resolution contains exactly one curve—the (-3)-curve—thus C^+ is irreducible. □

(13.8) In previous chapters we computed what we expect the general section H of $\mathscr{O}_X$ to be like. In order to construct examples of extremal nbds we will proceed in reverse. First we construct H as expected and then we deform it. Here we face the following problem. We identified H only by computing $\Delta(H \supset C)$. Usually, however, the dual graph of a singularity does not determine the singularity up to isomorphism. There are two ways of overcoming this problem, both are of interest. The first approach is to claim that in general we can construct surface germs along curves with arbitrary prescribed singularities. The second approach is the observation that in our cases $\Delta(H \supset C)$ nearly determines H up to isomorphism.

(13.8.1) **Construction of surface germs with prescribed local structure.** Let $P_i \in D_i \subset V_i : i = 1, \dots, k$ be germs of isolated surface singularities with an irreducible curve germ and let N be an integer. Then there is a germ of a surface along a proper curve $C \subset H$ such that

(13.8.1.1) C is irreducible and rational;

(13.8.1.2) H has exactly k singular points Q_i along C and

$$(Q_i \in C \subset H) \cong (P_i \in D_i \subset V_i) \quad \text{for every } i\,;$$

(13.8.1.3) The selfintersection number of $C \subset H$ satisfies

$$N \leq C^2 < N+1.$$

Proof. Start with a surface germ $C' = \mathbb{P}^1 \subset H' = \mathbb{P}^1 \times \Delta$ and pick $k+1$ points $Q'_0, \dots, Q'_k$ in C'. Let A_i (resp. B_i) be a small open (resp. closed) disc of radius ε (resp. $\varepsilon/2$) around Q'_i. We may assume that ε is so small that the closures of the discs A_i are disjoint.

We can choose a suitable representative of $P_i \in D_i \subset V_i$ in such a way that $D_i \hookrightarrow V_i$ is proper. Furthermore one can choose an identification $g_i : D_i \to \Delta_\varepsilon \subset \mathbb{C}$ such that $g_i^{-1}(\Delta_\varepsilon - \overline{\Delta}_{\varepsilon/2})$ has an open neighborhood G_i such that there is a biholomorphism

$$(D_i \cap G_i \subset G_i) \cong (\Delta_\varepsilon - \overline{\Delta}_{\varepsilon/2} \subset \Delta \times (\Delta_\varepsilon - \overline{\Delta}_{\varepsilon/2}))$$

which induces g_i on $D_i \cap G_i$.

Remove $B_i \times \Delta$ from H' and identify $G_i \subset V_i$ with $\Delta \times (A_i - B_i) \subset H'$. If we do this for $i = 1, \dots, k$ then the resulting surface germ $C'' \subset H''$ satisfies conditions (13.8.1.1–13.8.1.2).

Q_0 will be used to adjust the selfintersection number. Let z be a local parameter at Q_0 and let t be a local parameter on Δ. Remove $B_0 \times \Delta$ from H'' and then identify

$$\Delta \times (A_0 - B_0) \subset H'' \quad \text{and} \quad \Delta \times (A_0 - B_0) \subset \Delta \times A_0 \qquad \text{via } (t, z) \mapsto (z^k t, z).$$

We obtain a surface germ $C \subset H$ and the selfintersection of C differs from the selfintersection of C'' by k. Thus if we choose k suitably we can satisfy condition (13.8.1.3) too. □

(13.8.2) **Theorem.** *Let* $0 \in S$ *be a normal surface singularity such that the dual graph of its minimal resolution is one of the graphs* (*without the curve* $\bullet$) *given in* (6.7.1), (6.7.2), (6.7.3), (7.7.1), (7.11.1), (10.7.3.1), *or* (10.7.3.2).

Then $0 \in S$ *is isomorphic to a singularity given by the equations at the corresponding place.*

In particular, any such $0 \in S$ *occurs as a hyperplane section of a terminal singularity in the expected way.*

Proof. This is an immediate consequence of the results of [Laufer73]. He classified those singularities that are determined up to isomorphism by the reduced exceptional divisor of their minimal resolution. In our cases the reduced exceptional divisor has no moduli, except in the cases (6.7.2), (6.7.3), and (7.11.1) when a cross ratio is the only modulus. By the results (6.7.2), (6.7.3), and (7.11.1) we can get any nonzero value of this cross ratio. Therefore it is sufficient to prove that all our singularities are determined up to isomorphism by the reduced exceptional divisor of their minimal resolution. This is immediate from Laufer's lists. We give the location of our singularities in his lists, without explaining his notation.

singularity	place in [Laufer73]
(6.7.1)	p. 136, III.2
(6.7.2)	p. 162, III.i
(6.7.3)	p. 162, III.i
(7.7.1)	p. 136, III.2
(7.11.1)	p. 162, III.i
(10.7.3.1)	p. 137, IV $L_1J_1R_1$
(10.7.3.2)	p. 137, IV $L_5J_1R_1$ □

We are ready to prove several existence and structure theorems for extremal nbds. We will prove that all types of nbds not excluded so far do indeed exist. Also, $\Delta(H \supset C)$ determines the type of the neighborhood with a few exceptions.

(13.9) **Theorem.** *Type IC extremal nbds exist for every odd* $m \geq 5$. *For every* m *both possibilities listed in* (8.3) *do occur.*

If $H \supset C$ *is the germ of a normal surface along a smooth rational curve* C *such that* $\Delta(H \supset C)$ *is as in* (8.3.1) (*resp.* (8.3.2)) *then there is an extremal nbd* $X \supset H \supset C$ *such that* $H \in |\mathscr{O}_X|$.

If $C \subset X$ *is a threefold germ along a complete curve* C *with terminal singularities and* $C \subset H \subset X$ *is a member of* $|\mathscr{O}_X|$ *such that* $\Delta(H \supset C)$ *is as in* (8.3.1) (*resp.* (8.3.2)) *then* $C \subset X$ *is an extremal nbd of type IC with* $\lambda_1(P) \neq 0$ (*resp.* $\lambda_1(P) = 0$).

Proof. By (13.8) for every odd $m \geq 5$ there is a pair $H \supset C$ such that $\Delta(H \supset C)$ is as in (8.3.1) (resp. (8.3.2)). Thus the existence follows from the second part of the theorem.

By (13.8.2) and (10.7) we can deform H in such a way that we obtain an extremal nbd X with the required singularity. By construction any such X has a type IC singular point of index m. By (11.4) the contraction morphism of

$C \subset H$ extends to a morphism $f : X \to Y$ which contracts C. Thus $C \subset X$ is an extremal nbd. Having type IC is determined by the germ $C \subset H$ at the singular point. Y has a hyperplane section Δ_Y as in (8.3.1) (resp. (8.3.2)). Note that these are not DuVal singularities. X is also primitive. (13.6.3) implies that the extremal nbd is isolated.

We still must show that the vanishing of $\lambda_1(P)$ is determined by $\Delta(H \supset C)$. Let us consider an extremal nbd of type IC. Assume that there are $s_3, s_4 \in \Gamma(\mathscr{O}_X)$ such that $\Delta((s_3 = 0) \supset C)$ is given by (8.3.2) and $\Delta((s_4 = 0) \supset C)$ is given by (8.3.1).

Consider $s_3 + \alpha s_4 = 0$. This defines a normal surface H_α. By (10.7), $\Delta(H_\alpha \supset C)$ is given by (8.3.1). for $\alpha \neq 0$. Thus after contraction we obtain a flat family of surface singularities $f(H_\alpha)$. By construction $f(H_0)$ has mutiplicity three and $f(H_\alpha)$ has multiplicity 4 for $\alpha \neq 0$. This is impossible. Thus $\Delta(H \supset C)$ determines the vanishing of $\lambda_1(P)$. □

(13.9.1) *Remark.* The following interesting phenomenon helped us to distinguish the two cases of type IC nbds.

By looking at $\lambda_1(P)$ we expect that the nbds of type (8.3.2) are the special ones. This is reflected by the fact that the singularity of H_X at P is more special for (8.3.2) than for (8.3.1). However, if we look at the singularity of H_Y then it has multiplicity 3 for (8.3.2) and multiplicity 4 for (8.3.1). Thus we could claim that the case (8.3.1) describes a more special nbd.

One can easily construct a deformation of a nbd of type (8.3.2) where in the general fiber we have a nbd of type (8.3.1). By the above considerations, the general fiber has to contain another contracted curve.

(13.10) **Theorem.** *For every odd $m \geq 5$ and every k there is an extremal nbd of type kAD with two singular points of indices 2 and m such that the axial multiplicity at the index two point is k. The singularities are always $\mathbb{Q}$-factorial.*

If $H \supset C$ is the germ of a normal surface along a smooth rational curve C such that $\Delta(H \supset C)$ is as in (9.2) then there is an extremal nbd $X \supset H \supset C$ such that $H \in |\mathscr{O}_X|$.

If $C \subset X$ is a threefold germ along a complete curve C with terminal singularities and $C \subset H \subset X$ is a member of $|\mathscr{O}_X|$ such that $\Delta(H \supset C)$ is as in (9.2) then $C \subset X$ is an extremal nbd of type kAD.

Proof. The index m singularity is a cyclic quotient. The other singularity is of the form $(xy + z^2 - t^k = 0)/\mathbb{Z}_2(1, 1, 1, 0)$ where k is the axial multiplicity. By [Kollár91, 2.2.7], this is a $\mathbb{Q}$-factorial singularity.

All the cases can be constructed exactly as in (13.9). For the same reason they are always isolated. We still must show that they are not semistable. Assume that $|\mathscr{O}_X|$ has a more general member, which shows that it is in fact a semistable extremal nbd. We will compute the singularity of this member H^s.

At the index two point the original member has the form

$$(x_1 y_1 - z_1^2 = 0)/\mathbb{Z}_2(1, 1, 1).$$

This is the most general possible so this has to be the local form of H^s. At the index m point the threefold singularity is $\mathbb{C}^3/\mathbb{Z}_m(1, -1, \frac{m+1}{2})$. Thus in suitable coordinates the local description of H^s is

$$(x_2y_2 - z_2^{dm} = 0)/Z_m(1, -1, \frac{m+1}{2})$$

for some natural number d. The dual graph of the resolution of these singularities is the following:

for $d = 1$:

$$\overset{2}{\circ} - \overset{\frac{m+5}{2}}{\circ} - \underbrace{\overset{2}{\circ} - \cdots - \overset{2}{\circ}}_{\frac{m-5}{2}\text{-times}} - \overset{3}{\circ}$$

for $d > 1$:

$$\overset{2}{\circ} - \overset{\frac{m+3}{2}}{\circ} - \underbrace{\overset{2}{\circ} - \cdots - \overset{2}{\circ}}_{(d-2)\text{-times}} - \overset{3}{\circ} - \underbrace{\overset{2}{\circ} - \cdots - \overset{2}{\circ}}_{\frac{m-5}{2}\text{-times}} - \overset{3}{\circ}$$

The minimal resolution of H^s is obtained be attaching

$$\overset{4}{\circ} - \overset{1}{\bullet} -$$

to the left end of the above dual graphs. Contracting the (-1)-curves twice we obtain a cyclic quotient singularity of multiplicity $\frac{m+5}{2}$.

As in (13.9) this cannot be a small deformation of a singularity of multiplicity four. This shows that such a nbd is never semistable. □

(13.11) **Theorem.** *All cases listed in* (6.2–6.3) *for type* $cD/3$ *extremal nbds occur.*

If $H \supset C$ *is the germ of a normal surface along a smooth rational curve* C *such that* $\Delta(H \supset C)$ *is as in* (6.2.3.1), (6.2.3.2), *or* (6.3) *then there is an extremal nbd* $X \supset H \supset C$ *such that* $H \in |\mathscr{O}_X|$.

If $C \subset X$ *is a threefold germ along a complete curve* C *with terminal singularities and* $C \subset H \subset X$ *is a member of* $|\mathscr{O}_X|$ *such that* $\Delta(H \supset C)$ *is as in* (6.2.3.1), (6.2.3.2), *or* (6.3) *then* $C \subset X$ *is an extremal nbd of type* $cD/3$ *or* $k1A$. *If* X *has type* $cD/3$ *then*

(13.11.1) $i_P(1) = 1$ *and* X *has a simple cD point iff* $\Delta(H \supset C)$ *is as in* (6.2.3.1);

(13.11.2) $i_P(1) = 1$ *and* X *has a double cD point iff* $\Delta(H \supset C)$ *is as in* (6.2.3.2);

(13.11.3) $i_P(1) = 2$ *iff* $\Delta(H \supset C)$ *is as in* (6.3).

Proof. The proof of the existence is the same as in (13.9); it also follows from (6.11; 6.17; 6.21). There are only three cases if we look only at $\Delta(H \supset C)$. However, we can also specify the value of $\ell(P)$ as in (6.2.1) and (6.3.1.1) since this value depends only on the singularity at P.

By (13.3) any of the above H does lie on an extremal nbd of type $k1A$ too. In all cases the singularity of H_Y is rational but not DuVal. Thus the nbd is isolated by (13.6.3).

The proof that $\Delta(H \supset C)$ determines $i_P(1)$ goes as in (13.9) once we observe that Δ_Y has multiplicity 4 in (6.2) and multiplicity 3 in (6.3). By (6.22) the local structure of H at the singular point determines whether the $cD/3$ point on X is simple or double. □

(13.12) **Theorem.** *All cases listed in* (7.2–7.4) *for type IIA extremal nbds occur.*

If $H \supset C$ *is the germ of a normal surface along a smooth rational curve* C *such that* $\Delta(H \supset C)$ *is as in* (7.2), (7.3), *or* (7.4) *then there is an extremal nbd* $X \supset H \supset C$ *such that* $H \in |\mathscr{O}_X|$. X *is necessarily of type IIA.*

If $C \subset X$ *is a threefold germ along a complete curve* C *with terminal singularities and* $C \subset H \subset X$ *is a member of* $|\mathscr{O}_X|$ *such that* $\Delta(H \supset C)$ *is as in* (7.2), (7.3), *or* (7.4) *then* $C \subset X$ *is an extremal nbd of type IIA. Furthermore, if* $\Delta(H \supset C)$ *is as in* (7.1) (*resp.* (7.2) *resp.* (7.3)) *then* X *is as described in* (7.1) (*resp.* (7.2) *resp.* (7.3)).

Proof. The existence is the same as in (13.9); it also follows from (7.6.4; 7.9.4; 7.12.5). There are only three cases if we look only at $\Delta(H \supset C)$. However, we can also specify the value of $\ell(P)$ as in (7.2.1), (7.3.1), and (7.4.1.1) since this value depends only on the singularity at P.

Let $P \in H$ be the unique index four point of H and let $P^\sharp \in H^\sharp \subset \mathbb{C}^4$ be the index one cover. By (7.7) and (7.11) the $\mathbb{Z}_4$-action is given by weights (1,1,3,2) and $H^\sharp$ is the complete intersection of two hypersurfaces; one invariant and another anti-invariant under the $\mathbb{Z}_4$-action. Thus if H is a hypersurface section of a terminal singularity then this three-dimensional singularity is the quotient of an anti-invariant hypersurface by a $\mathbb{Z}_4$-action. Thus by definition, X is of type *IIA*.

We can apply (13.6.3) to conclude that the nbd is isolated.

The method of (13.9) can be used to distinguish (7.1) from the other two cases. We claim that the cases (7.2) and (7.3) are distinguished already locally at the index 4 point. Indeed, let g be the equation of the canonical cover of the terminal singularity at the index 4 point. From (7.7) and (7.11) we see that $\operatorname{rank} g_{\deg=2}$ is determined by H. $\operatorname{rank} g_{\deg=2} = 3$ for (7.2) and $\operatorname{rank} g_{\deg=2} = 2$ for (7.3). □

Next we deal with semistable nbds. It is easier to describe those with two singular points. Let H be a general member of $\mathscr{O}_X$. By (3.5.1) at the singular points we can choose coordinates such that the three-dimensional singularity is

$$(xy - z^{dn} + tf(x, y, z, t) = 0)/\mathbb{Z}_n(1, -1, a, 0) \text{ where } (a, n) = 1,$$

$C^\sharp$ is the x-axis and $t = 0$ is the local equation of H. We can formulate a description of such nbds.

(13.13) **Theorem.** *Given two singularities as above with numerical invariants* (n, a, d) *and* (n', a', d') *there is an extremal nbd of type k2A with the above local description at the singular points iff* $(a, n) = 1$, $(a', n') = 1$, *and the following condition is satisfied*:

$$1 < \frac{a}{n} + \frac{a'}{n'} < 1 + \frac{1}{dn^2} + \frac{1}{d'n'^2}.$$

Proof. The relative prime conditions come from the conditions on terminal singularities.

Let the two singular points be P and P'. Then

$$C \cdot K_X = -1 + w_P(0) + w_{P'}(0) = -1 + \frac{n-a}{n} + \frac{n'-a'}{n'}.$$

This gives the left side of the inequality in the theorem. To get the right-hand side we compute the self-intersection of C inside H. This should be negative, proving the necessity of the above conditions.

In general consider a quotient singularity $0 \in S \cong 0 \in \mathbb{C}^2/\mathbb{Z}_m(1, q)$ and let C be the image of the x-axis in the quotient. If we resolve this singularity then the dual graph of the resolution is a chain of rational curves $B_i : i = 1, \dots, s$ whose self-intersections $-b_i$ are computed from a modified continued fraction expansion of $\frac{n}{q}$. The proper transform of C intersects the curve B_1. The pull-back of C to the resolution is a cycle $C + \sum c_i B_i$ where the c_i are rational. They satisfy the relations

$$c_{i-1} - b_i c_i + c_{i+1} = 0 \quad \text{for } i = 1, \dots, s, \text{ where } c_0 = 1 \text{ and } c_{s+1} = 0.$$

This can be rewritten as

$$\frac{c_{i-1}}{c_i} = b_i - \frac{1}{c_i/c_{i+1}}.$$

This is the same recursive formula that computes the b_i. Therefore we obtain that $c_1 = \frac{q}{n}$. If this local set-up sits on a global surface then we see that if we take the minimal resolution of $0 \in S$ then the self-intersection of C decreases by $\frac{q}{n}$.

Now we go back to H. The singularities of H are quotient singularities given as $\mathbb{C}^2/\mathbb{Z}_{dn^2}(1, dan - 1)$ resp. $\mathbb{C}^2/\mathbb{Z}_{d'n'^2}(1, d'a'n' - 1)$. The proper transform of C in the minimal resolution of H is a (-1)-curve, thus we get that the self-intersection of C in H is

$$-1 + \frac{dan - 1}{dn^2} + \frac{d'a'n' - 1}{d'n'^2}.$$

If this number is nonnegative then C cannot be contractible inside H. If it is zero then H contracts to a curve, thus H should be the exceptional set of the contraction morphism. This is, however, impossible since the exceptional divisor has negative intersection with C whereas H has zero intersection. Thus the self-intersection of C is negative. Rearranging this we get the other inequality of the theorem.

Conversely, we can always take two singularities H_1 and H_2 as above and patch them together to get $C \subset H$ such that

$$C \cdot K_H = -1 + \frac{n - a}{n} + \frac{n' - a'}{n'}.$$

If the conditions are satisfied then C has negative self-intersection in H, therefore, it can be contracted. (13.1) gives an extremal nbd with the required local structure. □

(13.14) **Imprimitive case.** An extremal nbd as above with numerical invariants (n, a, d) and (n', a', d') is imprimitive iff $(n, n') = p > 1$. In this case we can take a p-fold cover of the nbd. This is again an extremal nbd of the same type. Locally the new singularities are

$$(xy - z^{dn} + tf(x, y, z, t) = 0)/\mathbb{Z}_m(1, -1, ap, 0), \quad \text{where } n = pm$$

resp.

$$(xy - z^{d'n'} + tf(x, y, z, t) = 0)/\mathbb{Z}_{m'}(1, -1, a'p, 0), \quad \text{where } n' = pm'.$$

Considering our convention this gives that the covers have numerical invariants

$$(m, \overline{ap}, dp) \quad \text{resp. } (m', \overline{a'p}, d'p),$$

where $\bar{}$ denotes residue mod m (resp. m').

(13.15) *Remarks.* (13.15.1) The inequalities of the theorem are fairly restrictive. It is not clear to us for which values of n and n' one can find solutions. It is easy to see that no solutions exist if $n + 2 < n' < 2n$. On the other hand, if $n' >> n^2$ then there are many different possibilities for a, a', d, d'.

(13.15.2) Assume that the nbd is divisorial. Then (13.6.1) gives that for some integer a_1 we have

$$a_1\left(1 - \frac{a}{n} - \frac{a'}{n'}\right) = -1.$$

This gives that $nn' - an' - a'n$ divides nn'. Assume first that $(n, n') = 1$. Let q be a prime dividing $nn' - an' - a'n$. Then q also divides say n. Thus q divides an', hence a. This contradicts $(n, a) = 1$. Therefore $nn' - an' - a'n = 1$. If $(n, n') > 1$ then we can take the primitive cover and conclude that in general for a divisorial nbd we have $nn' - an' - a'n = (n, n')$.

Consider the condition (13.6.2). It reads as follows:

$$a_1(1 - \frac{a}{n} - \frac{a'}{n'}) = -k$$

and

$$a_1^2(\frac{a}{n} + \frac{a'}{n'} - 1 - \frac{1}{dn^2} - \frac{1}{d'n'^2}) = -k$$

has a solution in a and k. Using the formula obtained before and the notation of (13.14), this is equivalent to

$$\frac{dd'}{dn^2 + d'n'^2 - pdd'nn'} \quad \text{is an integer.}$$

This is rarely satisfied.

(13.16) $k1A$ **type extremal nbds.** In this case we have considerable freedom in constructing the extremal nbd. Let us consider any singularity

$$(xy - z^{dn} + tf(x, y, z, t) = 0)/\mathbb{Z}_n(1, -1, a, 0), \quad \text{where } (a, n) = 1.$$

We resolve the surface singularity H_1 defined by $t = 0$. We get a dual graph of the form

$$\overset{b_1}{\circ} - \cdots - \overset{b_s}{\circ}$$

Pick any curve B_i and patch the unit ball in $\mathbb{C}^2$ in such a way that the resulting surface H contains a unique compact curve C whose proper transform intersects B_i transversally. If $b_i > 2$ then C in H can be contracted to a singular point. The dual graph of its resolution is

$$\overset{b_1}{\circ} - \cdots - \overset{b_{i-1}}{\circ} - \overset{b_i-1}{\circ} - \overset{b_{i+1}}{\circ} - \cdots - \overset{b_s}{\circ}$$

Thus we obtain an extremal nbd $C \subset X$. Several of the conditions of (13.6) can be used to get many examples of isolated extremal nbds. These also give examples of extremal nbds where the multiplicity of $C^\sharp$ is arbitrarily large.

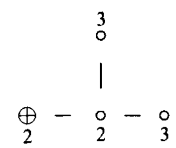
One can get examples having a cA type point and a type III point. There are some conditions on the index of the cA point and on the axial multiplicity of the type III point.

(3.3) left open the possibility that the general member of $\mathscr{O}_X$ containing C is not normal but has normal crossing singularity generically along C. We do not have any explicit examples, but this is quite likely to happen.

Next we will determine the flip in the exceptional cases. We start with the exceptional index three and four cases, these are easier. The main idea is to deform the nbd until we get a simpler one where the flip is easy to determine. This process will also give examples of "splitting-up" of the exceptional curve. Let $X \supset C \to Y \ni Q$ be an isolated extremal nbd. We will decide which P-modification of the general hyperplane section H' of Y corresponds to the flip. Thus, for instance, we determine the indices of the singularities after flip.

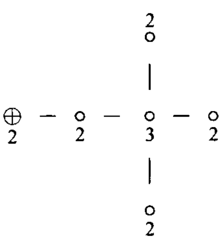
(13.17) **Theorem.** *Let $X \supset C \to Y \ni Q$ be an isolated extremal nbd of type $cD/3$ or of type IIA. In each of the seven cases the following diagrams describe the P-modification that corresponds to the flip. The curve denoted by $\oplus$ becomes C^+, the rest are contracted.*

$cD/3$, case (6.2.3.1):



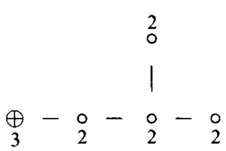
(*The flip has one index two point.*)

$cD/3$, case (6.2.3.2):



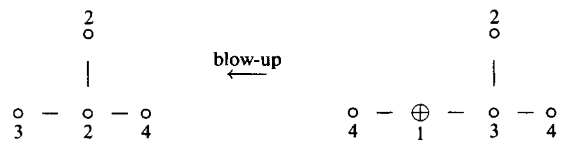
(*The flip has one index two point.*)

$cD/3$, case (6.3):



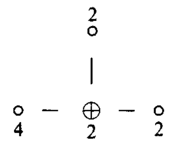
(*The flip has index one.*)

IIA, case (7.2):



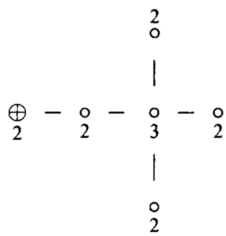
(*The flip has one index two and one index three points.*)

IIA, case (7.3):



(*The flip has one index two point.*)

IIA, case (7.4):



(*The flip has one index two point.*)

Proof. Let $H \subset X$ be the given member of $\mathscr{O}_X$. H has one index > 1 singular point. We will deform these points to simplify the singularities.

(13.17.1.1) In case $cD/3$ the singularity of $X \supset H$ is given as

$$(y_4^2 + f(y_1, y_2, y_3) = 0)/\mathbb{Z}_3(1, 1, 2, 0)$$

where f is some polynomial. H is defined by $y_4 = 0$ and $C^\sharp$ is the y_1-axis. We deform this via

$$(y_4^2 + f(y_1, y_2, y_3) + t(y_1 y_3 - y_2^3) = 0)/\mathbb{Z}_3(1, 1, 2, 0).$$

The defining equation of H_t stays $y_4 = 0$ and $C_t^\sharp$ is the y_1-axis.

(13.17.1.2) In case IIA the singularity of $X \supset H$ is given as

$$(g(y_1, y_2, y_3, y_4) = 0)/\mathbb{Z}_4(1, 1, 3, 2),$$

where $g_{\deg=2}(y_1, y_2, y_3, 0)$ has rank at least two, H is locally defined by $f(y_1, y_2, y_3, y_4) = t = 0$ where $f_{\deg=2}(0, 0, 0, y_4)$ is nonzero, and $C^\sharp$ is $y_1 - y_2 = y_3 = y_4 = 0$. We deform this via

$$g(y_1, y_2, y_3, y_4) + t(y_4 - (y_1 - y_2)^2) = 0)/\mathbb{Z}_4(1, 1, 3, 2),$$

and the defining equation of H by $f(y_1, y_2, y_3, y_4) + t(y_1 y_3 - (y_1 - y_2)^4) = 0$. The equations for $C_t^\sharp$ remain $y_1 - y_2 = y_3 = y_4 = 0$.

As in (13.2) we can globalize these deformations. Thus we have global deformations $X_t \supset H_t \supset C_t : t \in \Delta$ of the original extremal nbds. Locally these behave as described above.

(13.17.2) **Lemma.** *Consider the above deformations. Then the family $H_t : t \in \Delta$ can be resolved simultaneously, at least after a base change.*

Proof. We consider the family $H_t : t \in \Delta$. The index 3 (resp. 4) point of H_0 is described in (6.2–6.3,7.2–7.4). Explicit computation gives that these are always rational of multiplicity 5 (resp. 6). The general fibers have a quotient singularity at the origin. These have minimal resolutions

$$\underset{5}{\circ} - \underset{2}{\circ} \quad \text{for index 3 and} \quad \underset{6}{\circ} - \underset{2}{\circ} - \underset{2}{\circ} \quad \text{for index 4.}$$

The multiplicity of these is again 5 (resp. 6). Thus $H_t : t \in \Delta$ is an equimultiple family of rational singularities. By [Artin66] this is also a normally flat family, hence the family of blow-ups is also flat. Again explicit computation gives that after one blow-up we have only DuVal singularities left. They can be resolved simultaneously, at least after a base change. □

(13.17.3) **Lemma.** *Consider the above deformation of an extremal nbd of type $cD/3$. Assume that we have cases* (6.2.3.1–6.2.3.2). *Then $H_t : t \in \Delta - \{0\}$ has only one singular point along C_t. This is a quotient singularity of the form $(y_1y_3 - y_2^3 = 0)/\mathbb{Z}_3(1, 1, 2)$*

Proof. At the origin we get the required singularity. We have to show that there are no others. Since H_0 has only one singular point, every singularity on H_t must arise from the deformation of this singular point. This deformation is given by $f(y_1, y_2, y_3) + t(y_1y_3 - y_2^3)$. We have to decide if for general t this can have another singular point along C_t, which is the y_1-axis. The other singularity is a moving singularity, thus it can arise only if $(f(y_1, y_2, y_3) = y_1y_3 - y_2^3 = 0)$ is nonreduced along the y_1-axis. The tangent cone of this curve singularity is given by (6.7) as

$$y_2Q(y_1, y_2) + y_3^3 = y_1y_3 = 0,$$

where Q is a quadratic form not divisible by y_2. In particular, the y_1-axis is reduced. □

We remark that in case (6.3) we may get a moving singularity along the y_1-axis.

(13.17.4) **Lemma.** *Consider the above deformation of an extremal nbd of type cD. Assume that we have cases* (6.2.3.1–6.2.3.2). *Then $C_t \subset H_t : t \in \Delta - \{0\}$ is the only exceptional curve.*

Proof. By (13.17.2) the family $H_t : t \in \Delta$ admits a simultaneous minimal resolution. The proper transform of C_0 is a (-1)-curve. This lifts to the general fiber as a (-1)-curve. After we contract this (-1)-curve, the central fiber will have no more (-1)-curves, hence the same holds for the general fiber. □

(13.17.5) *Proof in $cD/3$ case.* In case (6.3), H' has a rational triple point. This has only one P-modification, the minimal DuVal resolution. This is the one described in (13.17).

In cases (6.2.3.1–6.2.3.2) H' has a rational quadruple point. There are two P-modifications: the minimal DuVal resolution and the one described in (13.17). The latter has an index two point. Therefore we only have to show that after flip we do have an index two point. To see this we apply deformation as above. By the previous lemmas C_t is the only exceptional curve in X_t. H_t has only one quotient singularity along C_t and $C_t \cdot K_{H_t} = -\frac{1}{3}$. Therefore the minimal resolution of H_t is given by the diagram

$$\underset{1}{\bullet} - \underset{5}{\circ} - \underset{2}{\circ}$$

This contracts to the quadruple point

$$\underset{4}{\circ} - \underset{2}{\circ}$$

By (11.9.3.1) the flip of the above extremal nbd has an index two point.

This shows that X_t^+ has an index two singularity for $t \neq 0$. This singularity will specialize to give an index two singularity on X_0^+. □

(13.17.6) *Proof in IIA case.* In case (7.2) the singularity of H' is an icosahedral quotient. By (11.8.2) there is only one possibility for the flip.

In cases (7.3–7.4) H' has a rational quadruple point and two P-modifications. We must show that the minimal DuVal resolution is not the right one. To this end we try to analyze the surface H_t. Let $\overline{H}_t$ be the minimal resolution of H_t. From the explicit description of $\overline{H}_0$ we see that it contains only one (-1)-curve. This lifts to a (-1)-curve on $\overline{H}_t : t \neq 0$. This is the proper transform of the curve C_t. In particular, we see that $C_t \cdot K_{H_t} = -\frac{1}{4}$. The minimal resolution of the index four point of H_t is described in (13.17.2). The relative canonical class is easy to compute and we get that C_t intersects the exceptional curves as follows

$$\underset{1}{\bullet} - \underset{6}{\circ} - \underset{2}{\circ} - \underset{2}{\circ}$$

If we contract the proper transform of C_0, there is another (-1)-curve. If we contract that too, there are no more (-1)-curves. Thus $\overline{H}_t : t \neq 0$ contains either another (-1)-curve, or there is a (-2)-curve intersecting C_t. In the second case we get the configuration

$$\underset{2}{\circ} - \underset{1}{\bullet} - \underset{6}{\circ} - \underset{2}{\circ} - \underset{2}{\circ}$$

If there are two (-1)-curves the second one must specialize to the configuration

$$\underset{1}{\bullet} - \underset{2}{\circ}$$

inside $\overline{H}_0$. In particular, both exceptional curves in H_t have intersection product $-\frac{1}{4}$ with K_{H_t}. This gives the configuration

$$\begin{array}{cccc} & \overset{1}{\bullet} & & \\ & | & & \\ \underset{1}{\bullet} - & \underset{6}{\circ} & - \underset{2}{\circ} - & \underset{2}{\circ} \end{array}$$

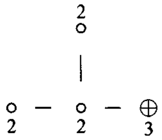
These are the two possible configurations of compact curves in $\Delta(H_t) : t \neq 0$. Both contract to

$$\underset{4}{\circ} - \underset{2}{\circ} - \underset{2}{\circ}$$

By (11.9.3.2) the flip of the extremal nbd containing C_t will have an index two point. This index two point will specialize to an index two point on the flip of X_0. □

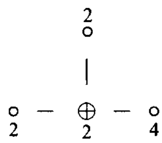
(13.18) **Theorem.** *Let $X \supset C \to Y \ni Q$ be an isolated extremal nbd of type IC or of type kAD. The following diagrams describe the P-modifications that correspond to the flip. The curve denoted by $\oplus$ becomes C^+, the rest are contracted.*

IC, $\lambda_1(P) = 0$:



(The flip has index one.)

IC, $\lambda_1(P) \neq 0$, and kAD:



(The flip has one index two point.)

Proof. In the $\lambda_1(P) = 0$ case H' has a triple point. This has only one P-modification.

In the other cases H' has a quadruple point with two P-modifications. Therefore we only have to show that X^+ contains an index two point. Then it must be the P-modification described above.

Let H be a general member of $\mathcal{O}_X$. This has one point of index > 2. In both cases X has a cyclic quotient singularity of the form

$$(x, y, z)/\mathbb{Z}_m(2, m-2, 1).$$

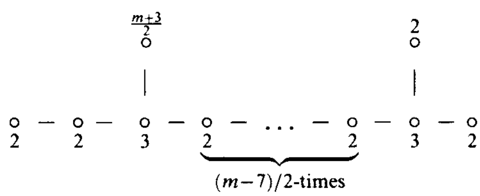
H is defined locally by some equation $f(x, y, z) = 0$. We deform H locally by

$$f(x, y, z) + t(xy - z^m)/\mathbb{Z}_m(2, m-2, 1).$$

In the *kAD* case there is another singular point, there we choose the trivial deformation. These local deformations can be globalized to get a deformation $H_t \subset X_t : t \in \Delta$. We would like to understand H_t.

Let $\overline{H}_t$ be the minimal resolution of H_t. By (8.3.2.1) and (9.2) the index m point of H_0 has the resolution

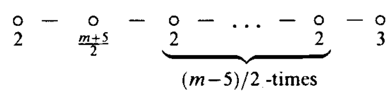
(13.18.1.1)



The fundamental cycle is reduced and the multiplicity of the singularity is $\frac{m+7}{2}$.

In the general fiber we have a cyclic quotient singularity. It has the resolution

(13.18.1.2)



The fundamental cycle is again reduced and the multiplicity of the singularity is also $\frac{m+7}{2}$. Therefore we again have a normally flat deformation and the blow-up is flat. We want to modify the blow-up slightly. In the central fiber we have one singular point after blow-up. Its resolution is

$$\begin{array}{ccccccccccc} * & - & \circ & - & \circ & - & \dots & - & \circ & - & \circ \\ 2 & & 3 & & 2 & & & & 2 & & 3 \end{array}$$
$$\underbrace{\qquad\qquad\qquad}_{(m-7)/2\text{-times}}$$

This is a quadruple point Z with two P-modifications. Consider the partial resolution $Z' \to Z$, which is obtained from the minimal resolution by contracting everything except the curve marked $*$. Z' has only one singularity, which is again a quadruple point with two P-modifications. At least after a base change, every deformation of Z is obtained as the contraction of a deformation of Z'. Thus we can make this modification $Z' \to Z$ in the central fiber and obtain a flat family $\tilde{H}_t$. The following diagrams describe the central and generic fibers above the index m point of H_t. Here $\circ$ denotes a curve of the minimal resolution that is contracted, the rest are not contracted. Above the noncontracted curves is their intersection number with the fundamental cycle.

The special fiber is

$$\begin{array}{ccccccccccccccc} & & & & (m+1)/2 & & & & & & & & 1 & & \\ & & & & \diamond & & & & & & & & \star & & \\ & & & & | & & & & & & & & | & & \\ \overset{1}{*} & - & \overset{0}{\diamond} & - & \circ & - & \circ & - & \dots & - & \circ & - & \circ & - & \overset{1}{\to\star} \end{array} \tag{13.18.2.1}$$
$$\underbrace{\qquad\qquad\qquad}_{(m-7)/2\text{-times}}$$

The general fiber is

$$\overset{1}{*} - \overset{(m+1)/2}{\diamond} - \underbrace{\circ - \dots - \circ}_{(m-5)/2\text{-times}} - \overset{2}{\star} \tag{13.18.2.2}$$

From this and (10.7.4) it is clear how these curves specialize. The $*$ on the left side of (13.18.2.2) specializes to the $*$ on the left side of (13.18.2.1). The $\star$ on the right side of (13.18.2.2) specializes to the two curves marked $\star$ in (13.18.2.1). The $\diamond$ of (13.18.2.2) specializes to the two curves marked $\diamond$ in (13.18.2.1).

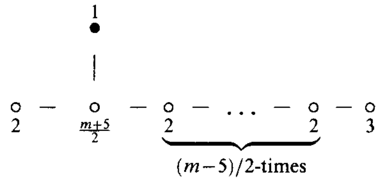
Now the two cases become slightly different.

(13.18.3) *IC case.* There is a (-1)-curve in $\overline{H}_0$ and we get the following diagram describing all compact curves in $\overline{H}_0$:

$$\begin{array}{ccccccccccccccc} & & 1 & & \frac{m+3}{2} & & & & & & & & 2 & & \\ & & \bullet & & \circ & & & & & & & & \circ & & \\ & & | & & | & & & & & & & & | & & \\ \circ & - & \diamond & - & \circ & - & \circ & - & \dots & - & \circ & - & \circ & - & \circ \\ 2 & & 2 & & 3 & & 2 & & & & 2 & & 3 & & 2 \end{array} \tag{13.18.3.1}$$
$$\underbrace{\qquad\qquad\qquad}_{(m-7)/2\text{-times}}$$

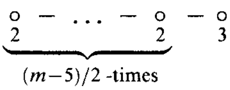
Note that the image of the (-1)-curve in $\tilde{H}_0$ does not pass through any singular points, so it is still a (-1)-curve. Considering what we said about the specialization map, this (-1)-curve lifts to the general fiber $\tilde{H}_t$ and we get the following configuration of compact curves on $\overline{H}_t$ (there may be other curves, and in fact we will see that there is one more):

(13.18.3.2)

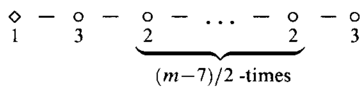


We can contract these (-1)-curves and get a flat family of surfaces $\hat{H}_t$. Looking at (13.18.3.1) we see that in $\tilde{H}_0$ the (-1)-curve intersects the curve denoted $\diamond$. From (13.18.2.1) we see that this has $1/2$ intersection with the canonical class. Thus after contraction the curve $\diamond$ will have $-1/2$ intersection with the canonical class of $\hat{H}_0$. If we contract it, then we get a singularity whose minimal resolution

(13.18.3.3)



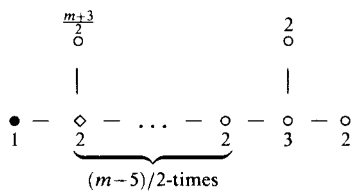
is obtained from



by contracting the curve $\diamond$. (13.18.3.3) is a triple point hence every deformation is obtained from a deformation of the minimal DuVal resolution. The minimal DuVal resolution has an $A_{(m-5)/2}$-singularity in the central fiber. The same singularity occurs in the general fiber, hence we can resolve these simultaneously. We obtain a flat family of surfaces $\overline{\overline{H}}_t$ where the minimal resolutions are the following.

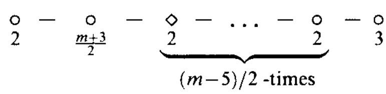
For the special fiber:

(13.18.4.1)



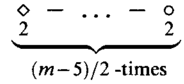
For the general fiber:

(13.18.4.2)



The specialisation map described after (13.18.2.1–13.18.2.2) seems to indicate that the -2-curve on the left side of (13.18.4.2) specializes to the -1-curve on the left side of (13.18.4.1), which is impossible. However in the meantime we performed a flip which changes the specialization of curves, thus there is no contradiction.

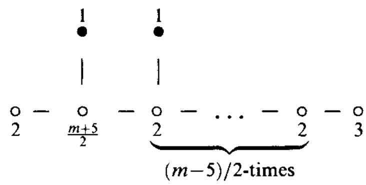
(13.18.4.1–13.18.4.2) both have the configuration of curves



obtained from the simultaneous resolution of the $A_{(m-5)/2}$-singularities. Therefore the (-1)-curve in the central fiber (denoted $\bullet$) lifts to the general fiber and intersects the (-2)-curve denoted $\diamond$.

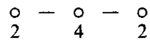
Putting everything together we see that $H_t : t \neq 0$ contains at least two exceptional curves and the configuration of compact curves on the minimal resolution $\overline{H}_t : t \neq 0$ is the following (a priori there may be other compact curves, but we will see that in fact these are all)

(13.18.5)



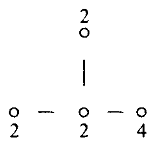
One can contract these (-1)-curves and then the new (-1)-curves until finally we obtain the configuration

(13.18.6.1)



If we do the corresponding contractions in the central fiber then we obtain the configuration

(13.18.6.2)

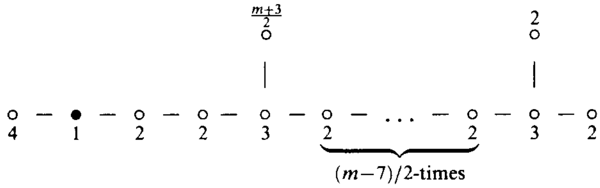


and these two are in a flat family. Since in the special fiber there are no more (-1)-curves, the same holds for the general fiber. Hence (13.18.5) describes the complete configuration of compact curves in $\overline{H}_t : t \neq 0$. By (11.9.3.3) the flip of the general nbd $X_t^+ : t \neq 0$ has an index two point.

Now we can prove (13.18) in the IC case. We just saw that $X_t^+ : t \neq 0$ has an index two point which specializes to an index two point of X_0^+. □

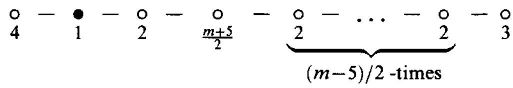
(13.18.7) *kAD case.* There is a (-1)-curve in $\overline{H}_0$ and we get the following diagram describing all compact curves in $\overline{H}_0$:

(13.18.7.1)



Note that the image of the (-1)-curve in $\tilde{H}_0$ does not pass through any singular points, so it is still a (-1)-curve. Considering what we said about the specialization map this (-1)-curve lifts to the general fiber $\tilde{H}_t$ and we get the following configuration of curves on $\overline{H}_t$:

(13.18.7.2)

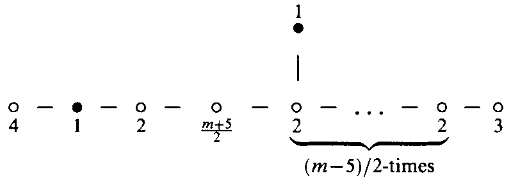


(there might be other curves too).

We can contract the (-1)-curve in the family $\tilde{H}_t$ and then we get a new (-1)-curve for every t that we can contract. The resulting family of surfaces is exactly the same as the one obtained in the IC case. Thus from now on further modifications give the same result.

This way we get that the configuration of compact curves in $\overline{H}_t : t \neq 0$ is the following:

(13.18.8)



This finishes the proof of (13.18). □

(13.18.9) *Remark.* The similarity between the general IC case and the kAD case is striking. It is not clear to us whether there is some deeper underlying reason.

The existence of flips implies that the exceptional curve C can be written as the set theoretic intersection of some divisors $D_1 \in |m_1K_X|$ and $D_2 \in |m_2K_X|$. We will be able to find the smallest m_1 and m_2 in all exceptional cases.

(13.19) **Theorem.** *With the above notation, the smallest values of* (m_1, m_2) *are the following:*

Type of nbd	*smallest* (m_1, m_2)
case (6.2.3.1)	(1, 2)
case (6.2.3.2)	(1, 2)
case (6.3)	(1, 1)
case (7.2)	(2, 3)
case (7.3)	(1, 2)
case (7.4)	(1, 2)
case $IC, \lambda_1(P) = 0$	(1, 1)
case $IC, \lambda_1(P) \neq 0$	(1, 2)
case kAD	(1, 2)

Proof. In practice it is very difficult to find members of $|mK_X|$ on X. However it is very easy to find members of $|mK_{X^+}|$. The following result allows us to pass between X and X^+.

(13.19.1) **Lemma.** *For* $m_1, m_2 > 0$ *let* $D_1 \in |m_1K_X|$ *and* $D_2 \in |m_2K_X|$ *be divisors. Let* $D_1^+ \in |m_1K_{X^+}|$ *and* $D_2^+ \in |m_2K_{X^+}|$ *be their proper transforms. Then* $D_1 \cap D_2 = C$ *(set theoretically) iff* D_1^+ *and* D_2^+ *are disjoint.*

Proof. If D_1^+ and D_2^+ are disjoint then clearly $D_1 \cap D_2 = C$. Conversely assume that $D_1 \cap D_2 = C$. Then $m_2 D_1, m_1 D_2 \in |m_1 m_2 K_X|$. The pencil $\langle m_2 D_1, m_1 D_2 \rangle$ is free outside C and gives a map $\phi : X \dashrightarrow \mathbb{P}^1$. Then X^+ is the normalisation of the image of (f, ϕ). In particular $m_1 m_2 K_{X^+}$ is Cartier and $m_2 D_1^+$ and $m_1 D_2^+$ are disjoint. □

Let $H^+ \subset X^+$ be the member of $|\mathscr{O}_{X^+}|$ exhibited in (13.17–13.18) with equation $t = 0$. Then

$$0 \to \omega_{X^+}^{[m]} \xrightarrow{t} \omega_{X^+}^{[m]} \to \omega_{H^+}^{[m]} \to 0$$

is exact for every m since X^+ has terminal singularities. Furthermore, if $m \geq 1$ then

$$R^1 f_*^+ \omega_{X^+}^{[m]} = 0 \quad \text{since } K_{X^+} \cdot C^+ > 0.$$

Thus

$$H^0(X^+, \omega_{X^+}^{[m]}) \to H^0(H^+, \omega_{H^+}^{[m]})$$

is surjective. Therefore it is sufficient to find members of $|mK_{H^+}|$. Since H^+ is explicitly known, this is rather straightforward. The following observations help with the computations:

(13.19.2). On a quotient singularity of the form $\mathbb{C}^2/\mathbb{Z}_{n^2d}(1, and - 1)$ the divisor $(xy = 0)$ descends to a section of $|(n-1)K|$. This is clear since $xy(dx \wedge dy)^{\otimes(n-1)}$ is $\mathbb{Z}_{n^2d}$-invariant.

(13.19.3). On the singularity $(x^2 + y^4 + z^4 = 0)/\mathbb{Z}_2(1, 1, 1)$ the divisor $(y = 0)$ descends to a section of $|K|$. This is clear since $y(dx \wedge dy)/4z^3$ is $\mathbb{Z}_2$-invariant.

(13.19.4). Let $D \subset H^+$ be a divisor. Assume that for every $q \in H^+$ locally at q, D is a member of $|mK_{H^+}|$. Assume furthermore that $C^+ \cdot D = mC^+ \cdot K_{H^+}$. Then $D \in |mK_{H^+}|$.

Now consider the cases separately.

(6.2.3.1) Here $C^+ \cdot K_{H^+} = 1/2$. (13.19.1) gives a local member of $|K|$ at the index two point which is also a global member. Any disc transversal to C^+ at a smooth point gives a member of $|2K|$.

(6.2.3.2) and (7.4). Here $C^+ \cdot K_{H^+} = 1/2$. (13.19.2) gives a local member of $|K|$ at the index two point which is also a global member. Any disc transversal to C^+ at a smooth point gives a member of $|2K|$.

(6.3) and $(IC, \lambda_1(P) = 0)$. Here $C^+ \cdot K_{H^+} = 1$. Any disc transversal to C^+ at a smooth point gives a member of $|K|$. Take two such discs.

(7.2) Here $C^+ \cdot K_{H^+} = 1/6$. (13.19.1) gives a local member of $|3K|$ at the index two point and a local member of $|2K|$ at the index three point, both are also global members.

(7.3) $(IC, \lambda_1(P) \neq 0)$ and (kAD). Here $C^+ \cdot K_{H^+} = 1/2$. (13.19.1) gives a local member of $|K|$ at the index two point which is also a global member. Any disc transversal to C^+ at a smooth point gives a member of $|2K|$.

We need to make precise what it means that these values are the smallest. This is clear in case (1,1). If X^+ has a unique index two point then every member of $|K_{X^+}|$ passes through that point, so two members of $|K_{X^+}|$ are never disjoint.

The only remaining case is (7.2) where we found (2,3). Here every member of $|K_{X^+}|$ contains C^+, thus we cannot have $(1, m)$ for any m. Also (2,2) is impossible, thus (2,3) is the smallest solution. □

(13.19.5) *Remark.* The cases $cD/3$ and IIA of the above result were obtained earlier in Chapters 6 and 7. Moreover, those results also determine the multiplicity of C in $D_1 \cap D_2$. The multiplicity is important to know since it determines how the Chow ring changes under a flip.

Appendix. Nonsemistable isolated extremal nbds (summary)

In this appendix, $X \supset C \simeq \mathbb{P}^1$ is a nonsemistable isolated extremal nbd unless otherwise mentioned explicitly, and let $f : X \supset C \to Y \ni Q$ be the contraction and $f^+ : X^+ \supset C^+ \to Y \ni Q$ the flip of f. We refer the reader to [Mori88, 8.8] about ℓ-structures.

The classification of nonsemistable isolated extremal nbds is as follows depending on the type of $X \supset C$ (2.3). (We recall that $X \supset C$ is *semistable* iff it is of type $k1A$ or $k2A$.)

(A.1). $X \supset C$ with a $cD/3$ point P (Chapter 6). This means that the terminal singularity (X, P) and the curve (C, P) are as follows.

(A.1.1) *Local coordinates of* (X, P). Let

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_3(1, 1, 2, 0; 0) \supset C = y_1\text{-axis}/\mathbb{Z}_3,$$

$$\alpha = 0 \cdot y_4 + 0 \cdot y_1 y_3 + 0 \cdot y_2 y_3 + y_4^2 + y_3^3 + g(y_1, y_2) + \cdots \in (y_2, y_3, y_4),$$

where g is a nonzero homogeneous cubic form in y_1, y_2. This only means that the coefficient of y_4 (resp. $y_1 y_3, \ldots$) in the Taylor expansion of α in y is 0 (resp. 0, $\ldots$), hence y_4 may appear in α in a form other than y_4^2.

We say that P is a *simple* (resp. *double*, *triple*) $cD/3$ point if g is squarefree (resp. has a square factor but is cubefree, is a cube) (6.1). We note

$$\ell(P) = \text{ length Torsion}((y_2, y_3, y_4)/(y_2, y_3, y_4)^2 + (\alpha)),$$

hence we may further assume

$$\alpha \equiv y_1^{\ell(P)} y_i \mod (y_2, y_3, y_4)^2$$

with $i = 2$ (resp. 3, 4) if $\ell(P) \equiv 2$ (resp. 1, 0) mod 3 after a change of coordinates (6.5).

(A.1.2) *Infinitesimal structure.* The nbd X is smooth outside of P, and $X \supset C \ni P$ satisfies exactly one of the following four conditions (see (6.2.1), (6.3.1)).

(A.1.2.1) $i_P(1) = 1$, $\ell(P) = 2$, P is a simple $cD/3$ point, and there is an ℓ-splitting

$$gr_C^1 \mathcal{O} = (0) \tilde{\oplus} (P^\sharp).$$

(A.1.2.2) $i_P(1) = 1$, $\ell(P) = 2$, P is a double $cD/3$ point, and there is an ℓ-splitting

$$gr_C^1 \mathcal{O} = (0) \tilde{\oplus} (P^\sharp).$$

(A.1.2.3) $i_P(1) = 2$, $\ell(P) = 3$, P is a double $cD/3$ point, and there is an ℓ-splitting

$$gr^1_C\mathscr{O} = (P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp).$$

(A.1.2.4) $i_P(1) = 2$, $\ell(P) = 4$, P is a double $cD/3$ point, and there is an ℓ-splitting

$$gr^1_C\mathscr{O} = (0)\tilde{\oplus}(-1 + 2P^\sharp).$$

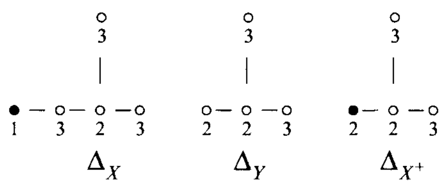
On the other hand, let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume also that (X, P) is a terminal singularity as described in (A.1.1). If $X \supset C \ni P$ satisfies one of the conditions (A.1.2.1–A.1.2.4), then $X \supset C$ is an isolated extremal nbd of $cD/3$ type as described (see (6.2.4), (6.3.4)).

(A.1.3) *Hyperplane sections.* For a normal surface S with only rational singularities and a smooth curve D on it, let $\Delta(S \supset D)$ be the dual configuration of the proper transform of D (marked •) and the exceptional curves over S (marked ∘) on the minimal resolution of S. To each vertex, we attach minus the self-intersection number.

(A.1.3.1) $|-K_X|$. Let $E \subset X$ be a general member of $|-K_X|$. Then E intersects C at P properly and $(E, P) \simeq (f(E), Q)$ is the DuVal singularity of type E_6 (2.2.1.2). (See (3.1) for the converse.)

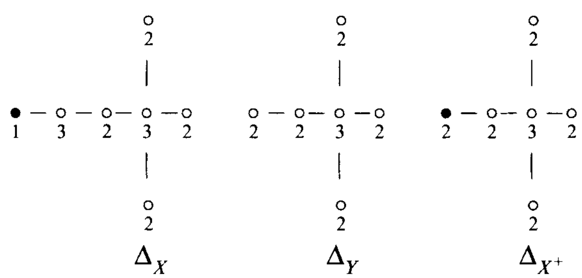
(A.1.3.2) $|\mathscr{O}_X|_C$. Let $H \subset X$ be a general member of $|\mathscr{O}_X|$ containing C and $H^+ \subset X^+$ the proper transform of H. Then H, H^+, and $f(H) = f^+(H^+)$ are all normal and have only rational singularities. We define $\Delta_X = \Delta(H \supset C)$, $\Delta_Y = \Delta(f(H) \supset \varnothing)$, and $\Delta_{X^+} = \Delta(H^+ \supset C^+)$. Then we have three cases (see (6.2.3) and (6.3.3) for Δ_X and Δ_Y and see (13.17) for Δ_{X^+}).

Case (A.1.2.1).



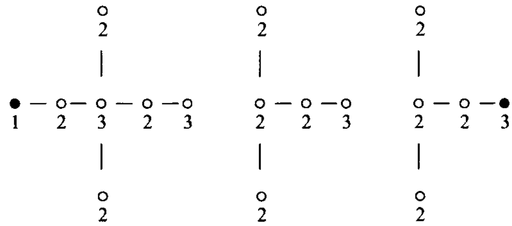
and X^+ has one singular point of index 2 on C^+.

Case (A.1.2.2).



and X^+ has one singular point of index 2 on C^+.

Cases (A.1.2.3) *and* (A.1.2.4).



Δ_X Δ_Y Δ_{X^+}

and X^+ is Gorenstein.

On the other hand, let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ need not be an extremal nbd. Assume also that X has only terminal singularities. If $|\mathcal{O}_X|$ has a member H_0 containing C such that $\Delta(H_0 \supset C)$ is equal to one of Δ_X 's in (A.1.3), then $X \supset C$ is an isolated extremal nbd and it is either of type k1A or of type $cD/3$ with $\Delta(H_0 \supset C) = \Delta_X$ (see (13.11)).

(A.1.4) *Equation of* (H, P) *and existence of* $X \supset C$. (H, P) is locally defined by $y_4 = \gamma(y_1, y_2, y_3)$ in (X, P) for some γ and the global equation of H induces a generator of the first factor $\mathcal{O}_C$ of $gr^1_C\mathcal{O}$ in (A.1.2.1–A.1.2.4). Thus

$$(H, P) = (y_1, y_2, y_3; \beta)/\mathbb{Z}_3(1, 1, 2; 0) \supset C,$$

where $\beta(y_1, y_2, y_3) = \alpha(y_1, y_2, y_3, \gamma)$.

Furthermore we have the following after a change of coordinates if necessary (see (6.10), (6.20)).

Case (A.1.2.1). (H, P) satisfies the condition (6.7.1).

Case (A.1.2.2). (H, P) satisfies the condition (6.7.2).

Cases (A.1.2.3) and (A.1.2.4). (H, P) satisfies the condition (6.7.3).

As a result, Δ_X in (A.1.3) are computed from (6.7).

For each of (A.1.2.1–A.1.2.4), we can therefore construct $H \supset C$ so that $\Delta(H \supset C)$ is the corresponding Δ_X in (A.1.3) and $(X, P) \supset (H, P)$ as in (A.1.1). By (13.1) we get $X \supset H \supset C$, which is an isolated nbd in the given case.

(A.1.5) C *as a set-theoretic C. I* We have two cases.

Cases (A.1.2.1) and (A.1.2.2). For general members $D \in |K_X|$ and $D'' \in |2K_X|$, we have $D \cdot D'' = 2C$ (see (6.2.2) and also (13.19)). In particular, $\mathcal{O}_D(6K_X) \simeq \mathcal{O}_D(6C)$.

Cases (A.1.2.3) and (A.1.2.4). For general members $D, D' \in |K_X|$, we have $D \cdot D' = 4C$ (see (6.3.2) and also (13.19)). In particular, $\mathcal{O}_D(3K_X) \simeq \mathcal{O}_D(12C)$.

(A.1.6) *Remark.* In general if a curve C in a 3-fold X is contained in two Cartier divisors D and E such that $(D \cdot C), (E \cdot C) < 0$ and $\dim D \cap E = 1$, then C is contractible [Kollár89, (4.10)], i.e., there is a bimeromorphic morphism $f : X \to Y$ which contracts C and is isomorphic elsewhere.

Therefore, assuming that X is terminal, (A.1.5) implies that $X \supset C$ is an isolated extremal nbd. Furthermore since the divisors in (A.1.5) cut out C set-theoretically, they even allow us to construct the flip X^+ directly (13.19.1).

This observation applies to other cases (A.*.5) as well.

(A.2) $X \supset C$ with a *IIA* point P (Chapter 7). This means that the terminal singularity (X, P) and the curve (C, P) are as follows.

(A.2.1) *Local coordinates of* (X, P). Let

$$(X, P) = (y_1, y_2, y_3, y_4; \alpha)/\mathbb{Z}_4(1, 1, 3, 2; 2) \supset C = y_1\text{-axis}/\mathbb{Z}_4,$$

$$\alpha = 0 \cdot y_4 + y_3^2 + g(y_1, y_2)y_2 + \cdots \in (y_2, y_3, y_4),$$

where g is a nonzero linear form in y_1, y_2. We may further assume

$$\alpha \equiv y_1^{\ell(P)} y_i \mod (y_2, y_3, y_4)^2$$

with $i = 2$ (resp. 3, 4) if $\ell(P) \equiv 1$ (resp. 3, 0) mod 4 after a change of coordinates (7.5). (See (A.1.1) for details.)

(A.2.2) *Infinitesimal structure.* The nbd $X \supset C \ni P$ satisfies exactly one of the following five conditions (see (7.2.1), (7.3.1), (7.4.1)).

(A.2.2.1) $i_P(1) = \ell(P) = 1$, $X - \{P\}$ is smooth, and there is an ℓ-splitting

$$gr_C^1\mathscr{O} = (P^\sharp)\tilde{\oplus}(2P^\sharp).$$

(A.2.2.2) $i_P(1) = \ell(P) = 1$, $X - \{P\}$ is smooth, and there are ℓ-splittings

$$gr_C^1\mathscr{O} = (1 + P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp),$$
$$gr^2(\mathscr{O}, J) = (P^\sharp)\tilde{\oplus}(0),$$

where J is the C-laminal ideal [Mori88, 8.2] of width 2 such that $J/F_C^2\mathscr{O} = (1 + P^\sharp)$.

(A.2.2.3) $i_P(1) = \ell(P) = 1$, X has a cDV point R on C, $X - \{P, R\}$ is smooth, and there are ℓ-splittings

$$gr_C^1\mathscr{O} = (P^\sharp)\tilde{\oplus}(-1 + 2P^\sharp),$$
$$gr^2(\mathscr{O}, J) = (P^\sharp)\tilde{\oplus}(0),$$

where J is the C-laminal ideal of width 2 such that $J/F_C^2\mathscr{O} = (P^\sharp)$.

(A.2.2.4) $i_P(1) = 2$, $\ell(P) = 3$, $X - \{P\}$ is smooth, and there is an ℓ-splitting

$$gr_C^1\mathscr{O} = (2P^\sharp)\tilde{\oplus}(-1 + 3P^\sharp).$$

(A.2.2.5) $i_P(1) = 2$, $\ell(P) = 4$, $X - \{P\}$ is smooth, and there is an ℓ-splitting

$$gr_C^1\mathscr{O} = (P^\sharp)\tilde{\oplus}(-1 + 3P^\sharp).$$

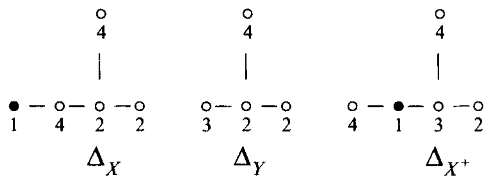
On the other hand, let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume also that (X, P) is a terminal singularity as described in (A.2.1). If $X \supset C \ni P$ satisfies one of the conditions (A.2.2.1–A.2.2.5), then $X \supset C$ is an isolated extremal nbd of *IIA* type as described (see (7.2.4), (7.3.4), (7.4.4)).

(A.2.3) *Hyperplane sections.*

(A.2.3.1) $|-K_X|$. Let $E \subset X$ be a general member of $|-K_X|$. Then E intersects C at P properly and $(E, P) \simeq (f(E), Q)$ is the DuVal singularity of type D_{k+2}, where k is the axial multiplicity [Mori88, 1a.5] of (X, P) (2.2.1.3). (See (3.1) for the converse.)

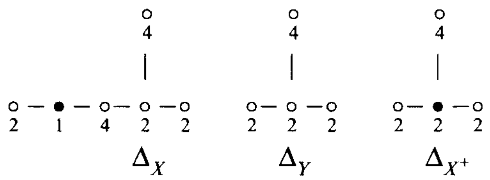
(A.2.3.2) $|\mathscr{O}_X|_C$. Let $H, H^+, \Delta_X, \Delta_Y, \Delta_{X^+}$, etc., be as in (A.1.3). Then H, H^+, and $f(H) = f^+(H^+)$ are all normal and have only rational singularities, and we have three cases (see (7.2.3), (7.3.3), and (7.4.3) for Δ_X and Δ_Y, and see (13.17) for Δ_{X^+}).

Case (A.2.2.1).



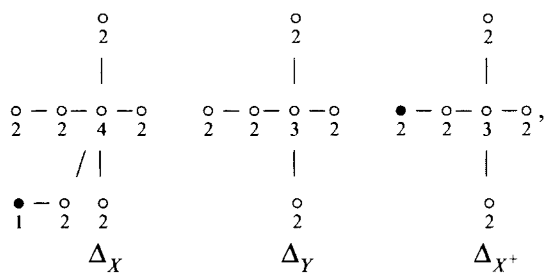
and X^+ has two singular points of indices 2 and 3 on C^+.

Cases (A.2.2.2) and (A.2.2.3).



and X^+ has one non-Gorenstein point of index 2 on C^+.

Cases (A.2.2.4) and (A.2.2.5).



and X^+ has one singular point of index 2 on C^+.

On the other hand, let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume also that X has only terminal singularities. If $|\mathscr{O}_X|$ has a member H_0 containing C such that $\Delta(H_0 \supset C)$ is equal to one of Δ_X's in (A.2.3), then $X \supset C$ is an isolated extremal nbd of type IIA and $\Delta(H_0 \supset C) = \Delta_X$ (see (13.12)).

(A.2.4) *Equation of (H, P) and existence of $X \supset C$.*

$$(H, P) = (y_1, y_2, y_3, y_4; \alpha, \beta)/\mathbb{Z}_4(1, 1, 3, 2; 2, 0) \supset C$$

for some β. By choosing the coordinates in (A.2.1) properly, we have one of the following:

Cases (A.2.2.1) and (A.2.2.2) and (A.2.2.3). α and β satisfy condition (7.7.1) (see (7.6), (7,9.3.1)).

Cases (A.2.2.4) and (A.2.2.5). α and β satisfy condition (7.11.1) (see (7.12.4)).
As a result, Δ_X in (A.2.3) are computed from (7.7) and (7.11).
Thus each case of (A.2.2.1–A.2.2.5) occurs as in (A.1.4).

(A.2.5) *C as a set-theoretic C. I. We have three cases.*

Case (A.2.2.1). For general members $D'' \in |2K_X|$ and $D''' \in |3K_X|$, we have $D'' \cdot D''' = 2C$ (see (7.2.2) and also (13.19)). In particular, $\mathscr{O}_{D''}(12K_X) \simeq \mathscr{O}_{D''}(8C)$.

Cases (A.2.2.2) and (A.2.2.3). For general members $D \in |K_X|$ and $D'' \in |2K_X|$, we have $D \cdot D'' = 2k \cdot C$, where k is the axial multiplicity [Mori88, 1a.5] of (X, P) (see (7.3.2) and also (13.19)). In particular, $\mathscr{O}_{D''}(4K_X) \simeq \mathscr{O}_{D''}(8k \cdot C)$.

Cases (A.2.2.4) and (A.2.2.5). For general members $D \in |K_X|$ and $D'' \in |2K_X|$, we have $D \cdot D'' = 2C$ (see (7.4.2) and also (13.19)). In particular, $\mathscr{O}_{D''}(4K_X) \simeq \mathscr{O}_{D''}(8C)$.
(See also (A.1.6).)

(A.3) $X \supset C$ with a IC point P (Chapter 8). This means that the singularity (X, P) and the curve (C, P) are as follows.
(A.3.1) *Local coordinates of* (X, P). Let

$$(X, P) = (y_1, y_2, y_3)/\mathbb{Z}_m(2, m-2, 1) \supset C = (\text{locus of } (t^2, t^{m-2}, 0))/\mathbb{Z}_m$$

with odd index $m \geq 5$ (8.2).
(A.3.2) *Infinitesimal structure.* $X - \{P\}$ is smooth and we have an ℓ-splitting

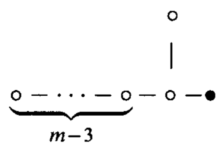
$$gr_C^1\mathscr{O} = (4P^\sharp)\tilde{\oplus}(-1 + (m-1)P^\sharp)$$

by (2.10.2), in which the factor $(4P^\sharp)$ is unique. The ℓ-invertible sheaf $(4P^\sharp)$ has an ℓ-free ℓ-basis [Mori88, 8.8.3]

$$\lambda_1 y_1^{(m-5)/2} y_4 + \mu_1(y_1^{m-2} - y_2^2)$$

for some λ_1 and $\mu_1 \in \mathscr{O}_{C,P}$. Whether $\lambda_1(P) = 0$ or not does not depend on the choice of coordinates (see (8.2)). We have two cases.
(A.3.2.1) $\lambda_1(P) \neq 0$.
(A.3.2.2) $\lambda_1(P) = 0$.
We do not have an infinitesimal characterization of $X \supset C$ like (A.1) and (A.2).
(A.3.3) *Hyperplane sections.*
(A.3.3.1) $|-K_X|$. Let $E \subset X$ be a general member of $|-K_X|$. Then $(f(E), Q)$ is the DuVal singularity of type D_m, E is a normal surface

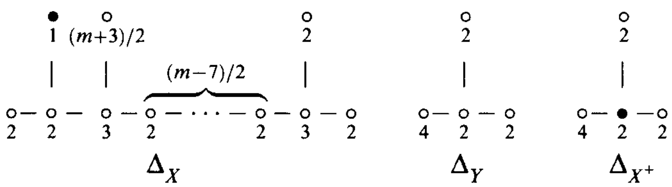
dominated by the minimal resolution of $f(E)$, and $\Delta(E \supset C)$ is



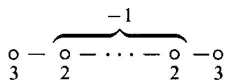
where the number 2 is attached to each vertex (2.2.2). (See (3.1) for the converse.)

(A.3.3.2) $|\mathcal{O}_X|_C$. Let H, H^+, Δ_X, Δ_Y, Δ_{X^+}, etc., be as in (A.1.3). Then H, H^+, and $f(H) = f^+(H^+)$ are all normal and have only rational singularities and we have two cases (see (8.3.2) for Δ_X and Δ_Y, and see (13.18) for Δ_{X^+}).

Case (A.3.2.1).

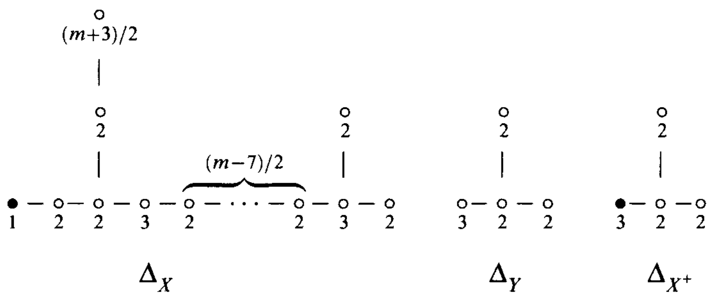


and X^+ has one non-Gorenstein point of index 2 on C^+, where



denotes $\underset{4}{\circ}$.

Case (A.3.2.2).



and X^+ is Gorenstein.

On the other hand, let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume also that X has only terminal singularities. If $|\mathcal{O}_X|$ has a member H_0 containing C such that $\Delta(H_0 \supset C)$ is equal to one of Δ_X's in (A.3.3), then $X \supset C$ is an isolated extremal nbd of type IC and $\Delta(H_0 \supset C) = \Delta_X$ (see (13.10)).

(A.3.4) *Equation of* (H, P) *and existence of* $X \supset C$.

$$(H, P) = (y_1, y_2, y_3; h)/\mathbb{Z}_m(2, m-2, 1; 0) \supset C$$

for some h. If we choose the coordinates in (A.3.1) properly, then $h(x_1, x_2, x_3)$ satisfies the conditions in (10.7). (See (8.10), (8.11).) We note that a_0 of (10.7) is our $\lambda_1(P)$.

As a result, Δ_X in (A.3.3) are computed from (10.7).

Thus each case of (A.3.2.1–A.3.2.2) occurs as in (A.1.4).

(A.3.5) *C as a set-theoretic C. I.* We have two cases.

Case (A.3.2.1). For general members $D \in |K_X|$ and $D'' \in |2K_X|$, we have $D \cap D'' = C$ as sets.

Case (A.3.2.2). For general members $D, D' \in |K_X|$, we have $D \cap D' = C$ as sets.

(See (13.19) and (A.1.6).)

(A.4) $X \supset C$ of type *kAD* (Chapter 9).

Let P and R be the singular points of X on C of indices m and 2, where m is an odd number ≥ 3 (2.2.3).

(A.4.1) *Local coordinates of* (X, P) *and* (X, R). Let

$$(X, P) = (y_1, y_2, y_3)/\mathbb{Z}_m(1, (m+1)/2, -1) \supset C = y_1\text{-axis}/\mathbb{Z}_m,$$
$$(X, R) = (z_1, z_2, z_3, z_4; \gamma)/\mathbb{Z}_2(1, 1, 1, 0; 0) \supset C = z_1\text{-axis}/\mathbb{Z}_2,$$

where $\gamma \equiv z_1 z_3 - z_2^2 \mod (z_4)$ (see (9.4)). Such (X, R) is analytically $\mathbb{Q}$-factorial (13.10).

(A.4.2) *Infinitesimal structure.* $X - \{P, R\}$ is smooth and we have two cases (see (9.4.2)).

(A.4.2.1) $\{\gamma = 0\}$ is smooth at 0, and we have an ℓ- splitting

$$gr_C^1\mathcal{O} = (\frac{m-1}{2}P^\sharp + R^\sharp)\tilde{\oplus}(-1 + P^\sharp + R^\sharp).$$

(A.4.2.2) $\{\gamma = 0\}$ is singular at 0, and we have an ℓ- splitting

$$gr_C^1\mathcal{O} = (\frac{m-1}{2}P^\sharp)\tilde{\oplus}(-1 + P^\sharp + R^\sharp).$$

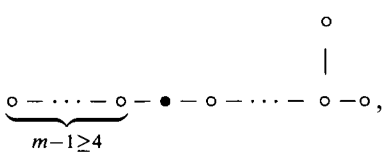
In either case, there is a C-laminal ideal J of width 2 such that $gr_C^1 = J/F_C^2\mathcal{O}\tilde{\oplus}(-1 + P^\sharp + R^\sharp)$. Such a J is unique and we have an ℓ-splitting.

$$gr^2(\mathcal{O}, J) = (2P^\sharp)\tilde{\oplus}(-1 + \frac{m-1}{2}P^\sharp + R^\sharp).$$

We do not have an infinitesimal characterization of $X \supset C$ like (A.1) and (A.2).

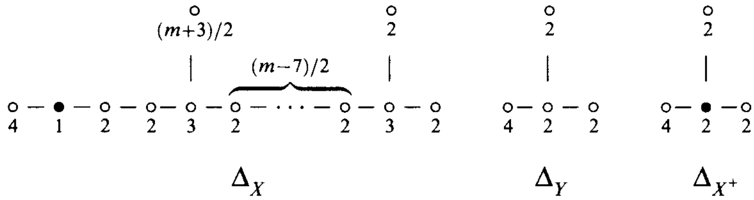
(A.4.3) *Hyperplane sections.*

(A.4.3.1) $|-K_X|$. Let $E \subset X$ be a general member of $|-K_X|$. Then $(f(E), Q)$ is the DuVal singularity of type D_{2k+m} (k is the axial multiplicity of (X, P) [Mori88, 1a.5]), E is a normal surface dominated by the minimal resolution of $f(E)$ and $\Delta(E \supset C)$ is

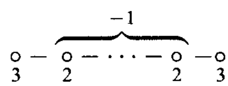


where the number 2 is attached to each vertex (2.2.3). (See (3.1) for the converse.)

(A.4.3.2) $|\mathscr{O}_X|_C$. Let H, H^+, Δ_X, Δ_Y, Δ_{X^+}, etc., be as in (A.1.3). Then H, H^+, and $f(H) = f^+(H^+)$ are all normal and have only rational singularities and we have the following case (see (9.2.2) for Δ_X and Δ_Y, and see (13.18) for Δ_{X^+}):



and X^+ has one non-Gorenstein point of index 2 on C^+, where



denotes $\underset{4}{\circ}$.

On the other hand, let $X \supset C$ be a germ of a 3-fold along $C \simeq \mathbb{P}^1$ which need not be an extremal nbd. Assume also that X has only terminal singularities. If $|\mathscr{O}_X|$ has a member H_0 containing C such that $\Delta(H_0 \supset C)$ is equal to one of Δ_X in (A.4.3), then $X \supset C$ is an isolated extremal nbd of type kAD (see (13.10)).

(A.4.4) *Equation of* (H, P) *and* (H, R) *and existence of* $X \supset C$.

$$(H, P) = (y_1, y_2, y_3; \beta)/\mathbb{Z}_m(1, (m+1)/2, -1; 0) \supset C$$

and (H, R) is defined in (X, R) by $z_4 = \delta(z_1, z_2, z_3)$ for some β and δ. As for (H, R), we have $\gamma(z_1, z_2, z_3, \delta(z_1, z_2, z_3)) = 0$ is an ordinary double point (9.11.1). If we choose the coordinates of (H, P) in (A.4.1) properly, then $h(x_1, x_2, x_3) = \beta(x_1, x_3, x_2)$ satisfies the conditions in (10.7). (See (9.15).) We note that a_0 of (10.7) is nonzero in our case.

As a result, Δ_X in (A.4.3) are computed from (10.7).

Thus each case of (A.4.2.1–A.4.2.2) occurs as in (A.1.4).

(A.4.5) C as a set-theoretic *C. I.* We have the following.

For general members $D \in |K_X|$ and $D'' \in |2K_X|$, we have $D \cap D'' = C$ as sets.

(See (13.19) and (A.1.6).)

Acknowledgment

We would like to thank Professors N. Nakayama, Y. Kawamata, M. Reid, H. Rossi, and J. Stevens for useful comments.

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