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Abstract

The aim of this article is to classify two-dimensional split trianguline representations of p-adic fields. This is a generalization of a result of Colmez who classified two-dimensional split trianguline representations of $\mathrm{Gal}(\bar{\mathbb{Q}}_p/\mathbb{Q}_p)$ for $p \neq 2$ by using (φ, Γ)-modules over a Robba ring. In this article, for any prime p and for any p-adic field K, we classify two-dimensional split trianguline representations of $\mathrm{Gal}(\bar{K}/K)$ using B-pairs as defined by Berger.

Contents

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Introduction

0.1 Background

Let p be a prime number. In this article, we study two-dimensional trianguline representations of any p-adic field K, i.e. a finite extension of $\mathbb{Q}_p$. A trianguline representation is a class of p-adic representations of $G_K := \mathrm{Gal}(\bar{K}/K)$, which has turned out to be an important notion. The trianguline representation was defined by Colmez and is defined by using (φ, Γ_K)-modules over a Robba ring $B^{\dagger}_{\mathrm{rig},K}$ which is non-canonically isomorphic to a ring of Laurent power series which converge in some annulus and is equipped with a Frobenius φ action and with $\Gamma_K := \mathrm{Gal}(K_{\infty}/K)$ actions. Here $K_{\infty} := K(\zeta_{p^{\infty}})$ is the extension of K obtained by adjoining p^nth roots of unity ζ_{p^n} for every $n \in \mathbb{N}$. The (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ are defined as finite free $B^{\dagger}_{\mathrm{rig},K}$-modules with semi-linear φ and Γ_K actions. By the work of Kedlaya [Ked04], the notion of slopes of φ-modules over a Robba ring is very important and we say that a (φ, Γ_K)-module over $B^{\dagger}_{\mathrm{rig},K}$ is étale if it is pure of slope zero as a φ-module. Using the works of Fontaine [Fon90], Cherbonnier and Colmez [CC98] and Kedlaya [Ked04], the category of p-adic representations of G_K is equivalent to the category of étale (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$. This equivalence enables us to see the category of p-adic representation of G_K as a full subcategory of the category of (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ (without slope conditions). We say that a p-adic representation V of G_K is split trianguline if $D_{\mathrm{rig}}(V)$, the (φ, Γ_K)-module corresponding to V, is a successive extension of rank-one objects in the category of (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$. We also say V is trianguline if V is split trianguline after making a finite extension of coefficients.

Trianguline representations can be seen as generalizations of ordinary representations. However, many interesting irreducible representations can be trianguline. For example, all semi-stable representations are trianguline. Moreover, there are many interesting trianguline representations which are not de Rham. For example, when $K = \mathbb{Q}_p$, Kisin showed that two-dimensional p-adic representations of $G_{\mathbb{Q}_p}$attached to finite-slope overconvergent modular forms are trianguline, and this fact was key to his p-adic Hodge theoretic approach to the study of Coleman–Mazur eigencurves [Kis03]. Recently Bellaïche–Chenevier generalized Kisin's method and studied higher-dimensional trianguline representations of $G_{\mathbb{Q}_p}$ to study higher-dimensional eigenvarieties [BC06]. Another example is Colmez's p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$: trianguline representations are at the heart of his theory of p-adic Langlands correspondence. When $K = \mathbb{Q}_p$ and $p \neq 2$, he classified two-dimensional split trianguline representations of $G_{\mathbb{Q}_p}$ and, based on this classification, he proved that the sets of points corresponding to trianguline representations are Zariski dense in deformation spaces of two-dimensional p-adic representations of $G_{\mathbb{Q}_p}$. By combining some constructions of Colmez and of Berger–Breuil, we obtain the p-adic Langlands correspondence for trianguline

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representations [BB06, Col04, Col07]. The correspondence for trianguline representations and the Zariski density of trianguline points played essential roles in his construction of p-adic Langlands correspondence for $\mathrm{GL}_2(\mathbb{Q}_p)$ (see [Col08b]).

The main purpose of this article is the complete classification of two-dimensional split trianguline representations of G_K for any finite extension K of $\mathbb{Q}_p$ for any prime p (we do not need to assume that $p \neq 2$). We determine the parameter space of all split trianguline representations and we also determine the parameter space of potentially crystalline or potentially semi-stable split trianguline representations, then we explicitly describe the filtered (φ, N, G_K)-modules associated with them. Currently, the only interesting examples of trianguline representations are in the case $K = \mathbb{Q}_p$. The author hopes that this article will be useful in giving many interesting examples of trianguline representations in the general $K \neq \mathbb{Q}_p$ case. For example, we want to know whether p-adic representations attached to finite-slope overconvergent Hilbert modular forms are trianguline or not. In this article, we could not attack the problem regarding the Zariski density of trianguline representations as the author does not know whether the set of trianguline points is Zariski dense or not in the general $K \neq \mathbb{Q}_p$ case. These problems will be the subject of study in future works.

0.2 Layout of the article

In § 1, we define trianguline representations and split trianguline representations by using B-pairs instead of (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ for some technical reasons. The B-pair was defined by Berger [Ber08b]. We write $B_e := B^{\varphi=1}_{\mathrm{cris}}$. An E–B-pair (the E-coefficient version of a B-pair; here E is a finite extension of K which contains the Galois closure of K) is a pair $W := (W_e, W^+_{\mathrm{dR}})$ where W_e is a finite free $B_e \otimes_{\mathbb{Q}_p} E$-module with a continuous semi-linear G_K-action such that $W^+_{\mathrm{dR}} \subseteq W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$ is a G_K-stable $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-lattice of W_{dR}. In [Ber08b, Theorem 2.2.7], Berger established an equivalence between the category of E–B-pairs and the category of E-(φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$, which are also the E-coefficient version of (φ, Γ_K)-modules. The category of E representations of G_K (the E-coefficient version of p-adic representations) is embedded in the category of E–B-pairs by $V \mapsto W(V) := (B_e \otimes_{\mathbb{Q}_p} V, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$. So, to define trianguline representations, we can use both B-pairs and (φ, Γ_K)-modules. In this article, we choose to use B-pairs for some technical reasons. Then, we say that an E–B-pair W is split trianguline if W is a successive extension of rank-one E–B-pairs. We say that an E-representation V is split trianguline if $W(V)$ is split trianguline. We say that an E–B-pair W (respectively, an E-representation V) is trianguline if $W \otimes_E E'$ (respectively, $V \otimes_E E'$) is split trianguline for a finite extension E' of E. The main purpose of this article is the complete classification of two-dimensional split trianguline E-representations of G_K. The classification is made through the following steps.

Step 1: Classification of rank-one E–B-pairs.

Step 2: Calculation of the dimensions of extension groups $\dim_E \mathrm{Ext}^1(W_2, W_1)$ for any rank-one E–B-pairs W_1, W_2.

Step 3: Determination of the conditions for W to be étale, here W is an extension of W_1 by W_2 as in Step 2.

Step 4: Classification of de Rham split trianguline E-representations.

We describe Step 1 in § 1, Step 2 in § 2, Step 3 in § 3 and Step 4 in § 4.

From now on, we explain these steps more precisely.

For Step 1, let $\delta : K^{\times} \to E^{\times}$ be a continuous character with respect to p-adic topology on both sides. Then we can define a rank-one E–B-pair $W(\delta)$ as follows. Let π_K be a uniformizer of K. We decompose δ into $\delta := \delta_0\delta_1$ such that $\delta_0|_{\mathcal{O}_K^{\times}} = \delta|_{\mathcal{O}_K^{\times}}$, $\delta_0(\pi_K) := 1$, $\delta_1|_{\mathcal{O}_K^{\times}}$ is a trivial character, $\delta_1(\pi_K) := \delta(\pi_K)$. Then $\delta_0 : K^{\times} \to \mathcal{O}_E^{\times}$ is a unitary character, so by local class field theory we obtain a character $\tilde{\delta}_0 : G_K^{\mathrm{ab}} \to \mathcal{O}_E^{\times}$ such that $\delta_0 = \tilde{\delta}_0 \circ \mathrm{rec}_K$, here $\mathrm{rec}_K : K^{\times} \to G_K^{\mathrm{ab}}$ is the reciprocity map such that π_K is mapped to a lifting of the inverse of the qth power Frobenius (here, $q := p^f$, $f := [K_0 : \mathbb{Q}_p]$, K_0 is the maximal unramified extension of $\mathbb{Q}_p$ in K). For δ_1, we define a rank-one E–B-pair $W(\delta_1)$ such that $D(\delta_1)$, the (φ, Γ_K)-module corresponding to $W(\delta_1)$, is defined by $D(\delta_1) := B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} Ee_{\delta_1}$, $\varphi^f(e_{\delta_1}) := \delta_1(\pi_K)e_{\delta_1}$, $\gamma(e_{\delta_1}) := e_{\delta_1}$ for any $\gamma \in \Gamma_K$. We define $W(\delta) := W(\delta_0) \otimes W(\delta_1)$. We can show that $W(\delta)$ does not depend on the choice of π_K (see Remark 1.44). Then the main theorem of § 1 is as follows.

Theorem 0.1 (Theorem 1.45). *Let W be a rank-one E–B-pair of G_K. Then there exists unique continuous character $\delta : K^{\times} \to E^{\times}$ such that $W \xrightarrow{\sim} W(\delta)$.*

Remark 0.2. We can see this theorem as a natural generalization of one-to-one correspondence $\{\delta : K^{\times} \to \mathcal{O}_E^{\times} :$ continuous character$\} \xrightarrow{\sim} \{\tilde{\delta} : G_K^{\mathrm{ab}} \to \mathcal{O}_E^{\times} :$ continuous character$\}$, which is induced by local class field theory. This theorem is also a generalization of [Col08a, Proposition 3.1].

For Step 2, from this theorem, it suffices to calculate $\dim_E \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ for any continuous characters $\delta_1, \delta_2 : K^{\times} \to E^{\times}$. In § 2, for this purpose we define the Galois cohomology of E–B-pairs. Let $W := (W_e, W^+_{\mathrm{dR}})$ be an E–B-pair. Put $W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$. Then we define the Galois cohomology $\mathrm{H}^*(G_K, W)$ of W as the Galois cohomology of the complex $W_e \oplus W^+_{\mathrm{dR}} \to W_{\mathrm{dR}} : (x, y) \mapsto x - y$, here $W_e \oplus W^+_{\mathrm{dR}}$ sits in the zeroth part of this complex. These cohomology groups are finite-dimensional E-vector spaces. We can show in the usual way that there is a natural isomorphism $\mathrm{Ext}^1(B_E, W) \xrightarrow{\sim} \mathrm{H}^1(G_K, W)$, here $B_E := (B_e \otimes_{\mathbb{Q}_p} E, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)$ is the trivial E–B-pair. By Bloch–Kato's fundamental short exact sequence $0 \to \mathbb{Q}_p \to B_e \oplus B^+_{\mathrm{dR}} \to B_{\mathrm{dR}} \to 0$, for any E-representation V, we have a natural isomorphism $\mathrm{H}^*(G_K, V) \xrightarrow{\sim} \mathrm{H}^*(G_K, W(V))$. So we can see this cohomology as a natural generalization of Galois cohomology of p-adic representations. As in the classical case, this cohomology satisfies the Euler–Poincaré characteristic formula and Tate local duality theorem. For this, we review the results of Liu [Liu08] concerning these theorems for (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$. By using his theorems, we can calculate all of the extension groups that we want. For any embedding $\sigma : K \hookrightarrow E$ and $k \in \mathbb{Z}$, we define $\sigma(x)^k : K^{\times} \to E^{\times} : y \mapsto \sigma(y)^k$. We define $N_{K/\mathbb{Q}_p} : K^{\times} \to \mathbb{Q}_p^{\times} : y \mapsto \prod_{\sigma : K \hookrightarrow E} \sigma(y)$, $| - | : \mathbb{Q}_p^{\times} \to E : p \mapsto 1/p,\ a \mapsto 1$ for any $a \in \mathbb{Z}_p^{\times}$, $|N_{K/\mathbb{Q}_p}(x)| := | - | \circ N_{K/\mathbb{Q}_p}$. Then the main result of § 2 is as follows.

Theorem 0.3 (Theorem 2.15). *Let $\delta_1, \delta_2 : K^{\times} \to E^{\times}$ be continuous characters.*

Then $\dim_E \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ is equal to:

(1) $[K : \mathbb{Q}_p] + 1$ *if* $\delta_1/\delta_2 = \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{k_\sigma}$ *such that* $k_\sigma \in \mathbb{Z}_{\leqq 0}$ *for any* σ;

(2) $[K : \mathbb{Q}_p] + 1$ *if* $\delta_1/\delta_2 = |N_{K/\mathbb{Q}_p}(x)| \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{k_\sigma}$ *such that* $k_\sigma \in \mathbb{Z}_{\geqq 1}$ *for any* σ;

(3) $[K : \mathbb{Q}_p]$ *otherwise.*

Remark 0.4. This theorem is a generalization of the dimension formula of Galois cohomology of one-dimensional p-adic representations. This is also a generalization of [Col08a, Theorem 2.9].

From this theorem we can determine all of the split trianguline E–B-pairs of rank two. Let $W(s)$ be a split trianguline E–B-pair which is an extension corresponding to $s \in \mathbb{P}_E(\mathrm{Ext}^1(W(\delta_2), W(\delta_1)))$; here for any finite-dimensional E-vector space M we put $\mathbb{P}_E(M) := \{[v] \mid v \in M - \{0\}, [v] = [v'] \iff v' = av \text{ for some } a \in E^\times\}$. Then the isomorphism class of $W(s)$ as an E–B-pair depends only on s.

For Step 3, we must determine the conditions on (δ_1, δ_2) and s for $W(s)$ to be étale, i.e. to be $W(s) \xrightarrow{\sim} W(V(s))$ for an E-representation $V(s)$. In § 3, we determine all of the conditions by using Kedlaya's slope filtration theorem of φ-modules over a Robba ring. The idea is essentially the same as Colmez's when $K = \mathbb{Q}_p$ and $p \neq 2$ (see the proof of [Col08a, Proposition 4.7]), but we have to deal with all of the additional complications which come from working with $K \neq \mathbb{Q}_p$. In fact, in the general $K \neq \mathbb{Q}_p$ case, the parameter space of split trianguline representations is more complicated than that of the $K = \mathbb{Q}_p$ case. For any two continuous characters $\delta_1, \delta_2 : K^\times \to E^\times$, we put $S(\delta_1, \delta_2) := \mathbb{P}_E(\mathrm{Ext}^1(W(\delta_2), W(\delta_1)))$. We put $S^+ := \{(\delta_1, \delta_2) \mid \delta_1, \delta_2 : K^\times \to E^\times \text{ continuous characters such that } \mathrm{val}_p(\delta_1(\pi_K)) + \mathrm{val}_p(\delta_2(\pi_K)) = 0, \mathrm{val}_p(\delta_1(\pi_K)) \geq 0\}$ (here val_p is a valuation of E such that $\mathrm{val}_p(p) := 1$). For any $(\delta_1, \delta_2) \in S^+$, in § 3 we explicitly define a certain subspace $S'^{\text{non-ét}}(\delta_1, \delta_2) \subseteq S(\delta_1, \delta_2)$ which corresponds to non-étale split trianguline E–B-pairs. All of these spaces S^+ and $S'^{\text{non-ét}}(\delta_1, \delta_2)$ naturally appear when we consider the slope zero conditions by using Kedlaya's slope filtration theorem. Then our main result of § 3 is as follows.

THEOREM 0.5 (Lemma 3.1 and Theorem 3.4). *Let $\delta_1, \delta_2 : K^\times \to E^\times$ be continuous characters. Let $W(s)$ be the split trianguline E–B-pair corresponding to $s \in S(\delta_1, \delta_2)$.*

(1) *If $W(s)$ is étale, then $(\delta_1, \delta_2) \in S^+$.*

(2) *The following conditions are equivalent:*

   (i) *$W(s)$ is étale, i.e. $W(s) \xrightarrow{\sim} W(V(s))$ for an E-representation $V(s)$;*

   (ii) *$s \notin S'^{\text{non-ét}}(\delta_1, \delta_2)$.*

Remark 0.6. This theorem is a generalization of [Col08a, Proposition 4.7]. The space $\mathcal{S}_+^{\mathrm{ncl}}$ in [Col08a, § 0.2] corresponds to $\sqcup_{(\delta_1,\delta_2)\in S^+} S^{\text{non-ét}}(\delta_1, \delta_2)$ in this article. Moreover, we can determine the conditions when we have $V(s) \xrightarrow{\sim} V(s')$ for distinct parameters $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$, $s' \in S(\delta'_1, \delta'_2) \setminus S'^{\text{non-ét}}(\delta'_1, \delta'_2)$ under certain conditions (Theorem 3.7).

For Step 4, we have to determine the conditions on $(\delta_1,\delta_2) \in S^+$ and $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$ for $V(s)$ to be potentially semi-stable or potentially crystalline. In the p-adic representations case, Bloch–Kato finite cohomology is useful for this kind of problems. We define Bloch–Kato cohomology for a B-pair W as follows (Definition 2.4). By the definition of $\mathrm{H}^*(G_K, W)$, we have natural maps $\mathrm{H}^*(G_K, W) \to \mathrm{H}^*(G_K, W_e) \to \mathrm{H}^*(G_K, B_{\mathrm{cris}} \otimes_{B_e} W_e) \to \mathrm{H}^*(G_K, B_{\mathrm{dR}} \otimes_{B_e} W_e)$. As in the classical case, we define $\mathrm{H}^1_?(G_K, W) := \mathrm{Ker}(\mathrm{H}^1(G_K, W) \to \mathrm{H}^1(G_K, B_* \otimes_{B_e} W_e))$, here when $? = e$ (respectively $? = f$, respectively $? = g$) then $* = e$ (respectively $* = \mathrm{cris}$, respectively $* = \mathrm{dR}$). By calculating these, in § 4 we explicitly define the parameter spaces $S^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2)$, $S_{\mathrm{st}}(\delta_1, \delta_2) \subseteq S(\delta_1, \delta_2)$ for any $(\delta_1, \delta_2) \in S^+$ such that $\delta_i = \prod_{\sigma: K \hookrightarrow E} \sigma(x)^{k_{i,\sigma}} \tilde{\delta}_i$ such that $k_{i,\sigma} \in \mathbb{Z}$ and $\tilde{\delta}_i$ are locally constant characters for $i = 1, 2$. Then the conditions for $V(s)$ to be potentially semi-stable are as follows.

Theorem 0.7 (Lemma 4.1 and Proposition 4.4). *Let $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$. Let $V(s)$ be the split trianguline E-representation corresponding to s.*

(1) *If $V(s)$ is potentially semi-stable, then $\delta_i = \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{k_{i,\sigma}} \tilde{\delta}_i$ such that $k_{i,\sigma} \in \mathbb{Z}$ for any σ and $\tilde{\delta}_i$ are locally constant characters for $i = 1, 2$.*

(2) *If $(\delta_1, \delta_2) \in S^+$ satisfies the condition in (1), then the following conditions are equivalent:*
   (i) *$V(s)$ is potentially crystalline;*
   (ii) *$s \in S^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2)$.*

(3) *If $(\delta_1, \delta_2) \in S^+$ satisfies the condition in (1), then the following conditions are equivalent:*
   (i′) *$V(s)$ is potentially semi-stable and not potentially crystalline;*
   (ii′) *$s \in S_{\text{st}}(\delta_1, \delta_2)$.*

Remark 0.8. These parameter spaces $S^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2)$, $S_{\text{st}}(\delta_1, \delta_2)$ are generalizations of $\mathcal{S}^{\text{cris}}_+$ or $\mathcal{S}^{\text{st}}_+$ defined in [Col08a, 0.2]. We can also see these spaces as the parameter spaces of weakly admissible filtrations of a (φ, N, G_K)-module corresponding to $V(s)$. Moreover, we can explicitly calculate the filtered (φ, N, G_K)-modules associated with $V(s)$ as above (Theorems 4.6 and 4.8).

By this theorem, we complete the classification of two-dimensional split trianguline E-representations. This is the main content of this article.

In Appendix A, we study a relation between two-dimensional potentially semi-stable trianguline representations and local Langlands correspondence for $\mathrm{GL}_2(K)$. Let V be a two-dimensional potentially semi-stable E-representation of G_K. Fontaine defined a two-dimensional Weil–Deligne representation $\bar{D}_{\text{pst}}(V) := D_{\text{pst}}(V) \otimes_{K_0^{\text{un}} \otimes_{\mathbb{Q}_p} E} \bar{K}$ of K from the filtered (φ, N, G_K)-module $D_{\text{pst}}(V) := \cup_{K \subseteq L, \text{finite}} (B_{\text{st}} \otimes_{\mathbb{Q}_p} V)^{G_L}$. By local Langlands correspondence for $\mathrm{GL}_2(K)$, we can attach an irreducible smooth admissible representation $\pi(\bar{D}_{\text{pst}}(V)^{ss})$ of $\mathrm{GL}_2(K)$ (here $\bar{D}_{\text{pst}}(V)^{ss}$ is the Frobenius semi-simplification of $\bar{D}_{\text{pst}}(V)$). Irreducible smooth admissible representations of $\mathrm{GL}_2(K)$ are classified into supercuspidal representations and non-supercuspidal representations (i.e. one-dimensional representations, principal series or special series). Then the main result of Appendix A is as follows.

Theorem 0.9 (Theorem A4). *Let V be a potentially semi-stable E-representation. Then the following conditions are equivalent:*

(1) *V is trianguline, i.e. $V \otimes_E E'$ is split trianguline for some finite extension E' of E;*

(2) *$\pi(\bar{D}_{\text{pst}}(V)^{ss})$ is non-supercuspidal.*

Notation

Let p be a prime number. We use: K to denote a finite extension of $\mathbb{Q}_p$; $\bar{K}$ to denote a fixed algebraic closure of K; K_0 to denote the maximal unramified extension of $\mathbb{Q}_p$ in K; K^{nor} to denote the Galois closure of K in $\bar{K}$. We use $G_K := \mathrm{Gal}(\bar{K}/K)$ to denote the absolute Galois group of K equipped with profinite topology. Here $\mathcal{O}_K$ is the integer ring of K, $\pi_K \in \mathcal{O}_K$ is a uniformizer of K, $k := \mathcal{O}_K/\pi_K\mathcal{O}_K$ is the residue field of K, $q = p^f := \sharp k$ is the order of k, and $K_\infty := K(\zeta_{p^\infty})$ is the extension of K obtained by adjoining the p^nth roots of unity ζ_{p^n} for every $n \in \mathbb{N}$. We use K'_0 to denote the maximal unramified extension of $\mathbb{Q}_p$ in K_∞, so $K_0 \subset K'_0$. We have $H_K := \mathrm{Gal}(\bar{K}/K_\infty)$, $\Gamma_K := G_K/H_K = \mathrm{Gal}(K_\infty/K)$, and $\chi : G_K \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character which factors through the inclusion $\Gamma_K \hookrightarrow \mathbb{Z}_p^\times$ (i.e. $g(\zeta_{p^n}) = \zeta_{p^n}^{\chi(g)}$ for any p^nth roots of unity ζ_{p^n} and for any $g \in G_K$). Here $\mathbb{C}_p := \widehat{\bar{K}}$ denotes the p-adic completion of $\bar{K}$, which is an algebraically closed p-adically complete field, and $\mathcal{O}_{\mathbb{C}_p}$ is its integer ring. We

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use E to denote a finite extension of $\mathbb{Q}_p$ such that $K^{\mathrm{nor}} \subset E$. In this paper, we write E as a coefficient of representations. We use $\chi_{\mathrm{LT}} : G_K \to \mathcal{O}_K^{\times} \hookrightarrow E^{\times}$ to denote the Lubin–Tate character associated with the uniformizer π_K. Here $\mathrm{rec}_K : K^{\times} \to G_K^{\mathrm{ab}}$ is the reciprocity map of local class field theory such that $\mathrm{rec}_K(\pi_K)$ is a lifting of the inverse of qth power Frobenius of k, then $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K : K^{\times} \to \mathcal{O}_K^{\times}$ satisfies $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K(\pi_K) = 1$ and $\chi_{\mathrm{LT}} \circ \mathrm{rec}_K|_{\mathcal{O}_K^{\times}} = id_{\mathcal{O}_K^{\times}}$.

1. B-pairs and (φ, Γ_K)-modules.

In this section, we recall the definitions and some properties of B-pairs and (φ, Γ_K)-modules. In particular, we recall the results of Berger concerning the equivalence between the category of B-pairs and the category of (φ, Γ_K)-modules over a Robba ring.

1.1 Review of p-adic period rings and the definition of B-pairs

We begin this section by recalling some p-adic period rings [Fon94a, Fon94b, Ber02] and by recalling the definition of a B-pair [Ber08b]. First let $\widetilde{\mathbb{E}}^+ := \varprojlim_n \mathcal{O}_{\mathbb{C}_p} = \varprojlim_n \mathcal{O}_{\mathbb{C}_p}/p\mathcal{O}_{\mathbb{C}_p}$, where the limits are taken with respect to pth power maps. It is known that $\widetilde{\mathbb{E}}^+$ is a complete valuation ring of characteristic p whose valuation is defined by $\mathrm{val}(x) := \mathrm{val}_p(x^{(0)})$ (here $x = (x^{(n)}) \in \varprojlim_n \mathcal{O}_{\mathbb{C}_p}$ and val_p is the valuation on $\mathbb{C}_p$ such that $\mathrm{val}_p(p) = 1$). In this paper, we fix a system of p^nth roots of unity $\{\varepsilon^{(n)}\}_{n \geqq 0}$ such that $\varepsilon^{(0)} = 1$, $(\varepsilon^{(n+1)})^p = \varepsilon^{(n)}$, $\varepsilon^{(1)} \neq 1$. Then $\varepsilon := (\varepsilon^{(n)})$ is an element of $\widetilde{\mathbb{E}}^+$ such that $\mathrm{val}(\varepsilon - 1) = p/(p-1)$. $\widetilde{\mathbb{E}} := \widetilde{\mathbb{E}}^+[1/(\varepsilon - 1)]$ is the fraction field of $\widetilde{\mathbb{E}}^+$, which is known to be an algebraically closed complete valuation field of characteristic p containing the subfield $\mathbb{F}_p((\varepsilon - 1))$. Here G_K acts on these rings in natural way. We put $\widetilde{\mathbb{A}}^+ := W(\widetilde{\mathbb{E}}^+)$, $\widetilde{\mathbb{A}} := W(\widetilde{\mathbb{E}})$, where, for a ring R, $W(R)$ is the Witt ring of R. We put $\widetilde{\mathbb{B}}^+ := \widetilde{\mathbb{A}}^+[1/p]$, $\widetilde{\mathbb{B}} := \widetilde{\mathbb{A}}[1/p]$. These rings also have natural continuous G_K-actions and Frobenius actions φ, here the topologies of these rings are defined by the p-adic topology on $\widetilde{\mathbb{A}}^+$ and $\widetilde{\mathbb{A}}$. Then we have a continuous G_K-equivariant surjection $\theta : \widetilde{\mathbb{A}}^+ \to \mathcal{O}_{\mathbb{C}_p} : \sum_{k=0}^{\infty} p^k [x_k] \mapsto \sum_{k=0}^{\infty} p^k x_k^{(0)}$, where $[\ ] : \widetilde{\mathbb{E}}^+ \to \widetilde{\mathbb{A}}^+$ is the Teichmüller character. By inverting p, we obtain a surjection $\widetilde{\mathbb{B}}^+ \to \mathbb{C}_p$. We put $B_{\mathrm{dR}}^+ := \varprojlim_n \widetilde{\mathbb{B}}^+/(\mathrm{Ker}(\theta))^n$, which is a complete discrete valuation ring with residue field $\mathbb{C}_p$ and is equipped with the projective limit topology of the p-adic topology on $\widetilde{\mathbb{B}}^+/(\mathrm{Ker}(\theta))^n$. Let $A_{\max}$ be the p-adic completion of $\widetilde{\mathbb{A}}^+[[\tilde{p}]/p]$, where $\tilde{p} := (p^{(n)})$ is an element in $\widetilde{\mathbb{E}}^+$ such that $p^{(0)} = p$, $(p^{(n+1)})^p = p^{(n)}$. We put $B_{\max}^+ := A_{\max}[1/p]$. Here $A_{\max}$ and $B_{\max}^+$ have continuous G_K-actions and Frobenius actions φ, and the topology on these rings is also p-adic topology. We have a natural continuous G_K-equivariant embedding $B_{\max}^+ \hookrightarrow B_{\mathrm{dR}}^+$. If we put $t := \log([\varepsilon])$, then we can see that $t \in A_{\max}$, $\varphi(t) = pt$, $g(t) = \chi(g)t$ for any $g \in G_K$ and $\mathrm{Ker}(\theta) = tB_{\mathrm{dR}}^+ \subset B_{\mathrm{dR}}^+$ is the maximal ideal of B_{dR}^+. If we put $B_{\max} := B_{\max}^+[1/t]$, $B_{\mathrm{dR}} := B_{\mathrm{dR}}^+[1/t]$, which are equipped with the inductive limit topology of $(1/t^n)B_{\max}^+$ and $(1/t^n)B_{\mathrm{dR}}^+$for any $n \in \mathbb{N}$, we have a natural continuous embedding $B_{\max} \hookrightarrow B_{\mathrm{dR}}$. We put $B_e := B_{\max}^{\varphi=1}$, $(B_e \subseteq)\widetilde{B}_{\mathrm{rig}}^+ := \bigcap_{n=0}^{\infty} \varphi^n(B_{\max}) \subset B_{\max}$ (these are closed sub-rings of $B_{\max}$), $\mathrm{Fil}^i B_{\mathrm{dR}} := t^i B_{\mathrm{dR}}^+$ for any $i \in \mathbb{Z}$. We put

$$\log([\tilde{p}]) := \log(p) + \sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n} \left(\frac{[\tilde{p}]}{p} - 1 \right)^n \in B_{\mathrm{dR}}^+,$$

where $\log : \mathbb{C}_p^{\times} \to \mathbb{C}_p$ is a branch of log which we fix in this article. Let $B_{\log} := B_{\max}[\log([\tilde{p}])] \subseteq B_{\mathrm{dR}}$. There is a derivation $N : B_{\log} \to B_{\log}$ over $B_{\max}$ such that $N(\log([\tilde{p}])) := -1$. We have the following fundamental short exact sequence [BK90, Proposition 1.17]

$$0 \to \mathbb{Q}_p \to B_e \oplus B^+_{\mathrm{dR}} \to B_{\mathrm{dR}} \to 0.$$

Definition 1.1. An E-representation of G_K is a finite-dimensional E-vector space V with a continuous E-linear action of G_K. We use the term E-representation for simplicity when there will be no risk of confusion regarding K.

Next we define the E–B-pair of G_K, which is the E-coefficient version of a B-pair.

Definition 1.2. An E–B-pair of G_K is a couple $W = (W_e, W^+_{\mathrm{dR}})$ such that:

(1) W_e is a finite $B_e \otimes_{\mathbb{Q}_p} E$-module with a continuous semi-linear G_K-action which is free as B_e-module;

(2) $W^+_{\mathrm{dR}} \subseteq W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$ is a G_K-stable $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-lattice, i.e. W^+_{dR} is a finitely generated $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-module that generates W_{dR} as a $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-module.

Here G_K acts on $B_? \otimes_{\mathbb{Q}_p} E$ by $g(x \otimes y) := g(x) \otimes y$ for any $g \in G_K, x \in B_?, y \in E$ for $? \in \{e, \mathrm{dR}\}$.

We use the term E–B-pair for simplicity when there will be no risk of confusing regarding K. We simply use the term B-pair when $E = \mathbb{Q}_p$, then this definition is the same as that of Berger [Ber08b, Introduction].

Remark 1.3. Later we prove that W_e is also free over $B_e \otimes_{\mathbb{Q}_p} E$ (Lemma 1.7) and that W^+_{dR} is also free over $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ (Lemma 1.8).

Definition 1.4. Let $W_j := (W_{e,j}, W^+_{\mathrm{dR},j})$ be E–B-pairs for $j = 1, 2$. Then a morphism of E–B-pairs $f : W_1 \to W_2$ is defined as a $B_e \otimes_{\mathbb{Q}_p} E$-linear G_K-equivariant morphism $f : W_{e,1} \to W_{e,2}$ such that $id_{B_{\mathrm{dR}}} \otimes_{B_e} f : B_{\mathrm{dR}} \otimes_{B_e} W_{e,1} \to B_{\mathrm{dR}} \otimes_{B_e} W_{e,2}$ maps $W^+_{\mathrm{dR},1}$ to $W^+_{\mathrm{dR},2}$.

Remark 1.5. Let V be an E-representation of G_K. Then $W(V) := (B_e \otimes_{\mathbb{Q}_p} V, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} V)$ is an E–B-pair of G_K. By the fundamental short exact sequence $0 \to \mathbb{Q}_p \to B_e \oplus B^+_{\mathrm{dR}} \to B_{\mathrm{dR}} \to 0$, it is easy to see that the functor $V \mapsto W(V)$ is a fully faithful functor from the category of E-representations of G_K to the category of E–B-pairs of G_K [Ber08b, Introduction].

Next, we prove a technically important lemma concerning the Bézout property of $B_e \otimes_{\mathbb{Q}_p} E$. This lemma is a generalization of [Ber08b, Proposition 1.1.9] to any coefficient case.

Lemma 1.6. *We have that $B_e \otimes_{\mathbb{Q}_p} E$ is a Bézout domain, i.e. a domain and every finitely generated ideal is generated by one element.*

Proof. The proof is essentially same as that of [Ber08b, Proposition 1.1.9]. First, we have a natural isomorphism of rings $B_e \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} B_e \otimes_{\mathbb{Q}_p} (E_0 \otimes_{E_0} E) \xrightarrow{\sim} B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E$. (Here E_0 is the maximal unramified extension of $\mathbb{Q}_p$ in E and $f' = [E_0 : \mathbb{Q}_p]$.) Because the natural map $B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E \hookrightarrow B_{\mathrm{dR}}$ is injective by [Col02, Proposition 7.14], so $B_e \otimes_{\mathbb{Q}_p} E$ is a domain. Next, we show that for any $f, g \in B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E$, the ideal generated by f and g in $B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E$ is generated by one element. For this, we first note that $\widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$ (for the definition of $\widetilde{B}^{\dagger}_{\mathrm{rig}}$, see Definition 1.2 of this paper) is a Bézout domain by [Ked05, Theorem 2.9.6]. (In the definition of [Ked05, § 2.1], if we take $K_0 := k_E$ the residue field of E, $\mathcal{O} := \mathcal{O}_E$, $\sigma := \varphi^{f'} \otimes_{W(k_E)} \mathrm{id}_{\mathcal{O}_E}$, then $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}} \xrightarrow{\sim} \widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$.) Because we have the injection $B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E \hookrightarrow \widetilde{B}^{\dagger}_{\mathrm{rig}}[1/t] \otimes_{E_0} E$,

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for large $n \in \mathbb{Z}_{\geqq 0}$ we have $t^n f, t^n g \in \widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$. Because $\widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$ is Bézout, there exists an $h \in \widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$ such that $t^n f \widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E + t^n g \widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E = h\widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$. And $h\widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$ is a σ-module over $\widetilde{B}^{\dagger}_{\mathrm{rig}} \otimes_{E_0} E$ because $t^n f, t^n g$ are preserved by σ-action. So, by [Ked05, Proposition 3.3.2], we can choose the generator h such that $\sigma(h) = \pi_E^k h$ for some $k \in \mathbb{Z}$. By using the element $t_E \in B^+_{\max} \otimes_{E_0} E$ defined in [Col02, Proposition 8.10] and by [Col02, Lemma 8.17], we obtain $h/t_E^k \in (\widetilde{B}^{\dagger}_{\mathrm{rig}}[1/t])^{\varphi^{f'}=1} \otimes_{E_0} E = B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E$ (here, for the last equality, we use the fact $(\widetilde{B}^{\dagger}_{\mathrm{rig}}[1/t])^{\varphi^{f'}=1} = B_{\max}^{\varphi^{f'}=1}$, see [Ber08b, Lemma 1.1.7]). Then we can show that the ideal generated by f and g in $B_{\max}^{\varphi^{f'}=1} \otimes_{E_0} E$ is generated by h/t_E^k in the same way as in [Ber08b, Proposition 1.1.9]. □

From the above lemma, we obtain the following lemmas which are also technically important.

LEMMA 1.7. *Let W_e be a finite $B_e \otimes_{\mathbb{Q}_p} E$-module which is free over B_e. Then W_e is also free over $B_e \otimes_{\mathbb{Q}_p} E$. In particular, for any E–B-pair $W := (W_e, W^+_{\mathrm{dR}})$, W_e is finite free over $B_e \otimes_{\mathbb{Q}_p} E$.*

Proof. Because W_e is finitely generated over $B_e \otimes_{\mathbb{Q}_p} E$ which is Bézout domain, by the remark after [Ked04, Lemma 2.4] it suffices to show that W_e is torsion free over $B_e \otimes_{\mathbb{Q}_p} E$. Let $\mathrm{Frac}B_e$ be the fraction field of B_e. Because $B_e \otimes_{\mathbb{Q}_p} E$ is a domain and $(\mathrm{Frac}B_e) \otimes_{\mathbb{Q}_p} E$ is a localization of $B_e \otimes_{\mathbb{Q}_p} E$, then $(\mathrm{Frac}B_e) \otimes_{\mathbb{Q}_p} E$ is also a domain and the natural map $B_e \otimes_{\mathbb{Q}_p} E \hookrightarrow (\mathrm{Frac}B_e) \otimes_{\mathbb{Q}_p} E$ is injective. In addition, because $(\mathrm{Frac}B_e) \otimes_{\mathbb{Q}_p} E$ is finite over the field $\mathrm{Frac}B_e$, then $(\mathrm{Frac}B_e) \otimes_{\mathbb{Q}_p} E$ is also a field. Because W_e is free over B_e by assumption, the natural map $W_e \hookrightarrow \mathrm{Frac}B_e \otimes_{B_e} W_e$ is injective. Of course, $\mathrm{Frac}B_e \otimes_{B_e} W_e$ is also torsion free over $(\mathrm{Frac}B_e) \otimes_{\mathbb{Q}_p} E$ which is a field. By these remarks, we conclude that W_e is torsion free over $B_e \otimes_{\mathbb{Q}_p} E$. □

LEMMA 1.8. *Let $W := (W_e, W^+_{\mathrm{dR}})$ be an E–B-pair. Then W^+_{dR} is finite free over $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$.*

Proof. By Lemma 1.7, W_e is free over $B_e \otimes_{\mathbb{Q}_p} E$. So $W_{\mathrm{dR}} := B_{\mathrm{dR}} \otimes_{B_e} W_e$ is also free over $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$. Because $E \subseteq B^+_{\mathrm{dR}}$, we have natural isomorphisms $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \oplus_{\sigma: E \hookrightarrow B^+_{\mathrm{dR}}} B^+_{\mathrm{dR}} : a \otimes b \mapsto (a\sigma(b))_\sigma$ ($a \in B^+_{\mathrm{dR}}$, $b \in E$) and $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \oplus_{\sigma: E \hookrightarrow B^+_{\mathrm{dR}}} B_{\mathrm{dR}}$. By using these decompositions, we obtain decompositions $W^+_{\mathrm{dR}} \xrightarrow{\sim} \oplus_{\sigma: E \hookrightarrow B^+_{\mathrm{dR}}} W^+_{\mathrm{dR},\sigma}$ and $W_{\mathrm{dR}} \xrightarrow{\sim} \oplus_{\sigma: E \hookrightarrow B^+_{\mathrm{dR}}} W_{\mathrm{dR},\sigma}$ ($W^+_{\mathrm{dR},\sigma}$ and $W_{\mathrm{dR},\sigma}$ are the σ-components). Then, for each σ, $W^+_{\mathrm{dR},\sigma}$ is a B^+_{dR}-lattice of $W_{\mathrm{dR},\sigma}$. So $W^+_{\mathrm{dR},\sigma}$ is finite free over B^+_{dR} of same rank as that of $W_{\mathrm{dR},\sigma}$ over B_{dR} because B^+_{dR} is a discrete valuation ring. Also, because W_{dR} is free over $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$, then $W_{\mathrm{dR},\sigma}$ have the same rank for any σ. So $W^+_{\mathrm{dR},\sigma}$ also have the same rank for any σ. So W^+_{dR} is finite free over $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$. □

By using these lemmas, we can define rank, tensor products and duals of E–B-pairs.

DEFINITION 1.9. We make the following definitions.

(1) Let $W := (W_e, W^+_{\mathrm{dR}})$ be an E–B-pair, then we define the rank of W by $\mathrm{rank}(W) := \mathrm{rank}_{B_e \otimes_{\mathbb{Q}_p} E}(W_e)$.

(2) Let $W_1 := (W_{e,1}, W^+_{\mathrm{dR},1})$ and $W_2 := (W_{e,2}, W^+_{\mathrm{dR},2})$ be E–B-pairs. Then we define the tensor product of W_1 and W_2 by $W_1 \otimes W_2 := (W_{e,1} \otimes_{B_e \otimes_{\mathbb{Q}_p} E} W_{e,2}, W^+_{\mathrm{dR},1} \otimes_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} W^+_{\mathrm{dR},2})$.

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(3) Let $W := (W_e, W_{\mathrm{dR}}^+)$ be an E–B-pair. Then we define the dual of W by $W^\vee := (\mathrm{Hom}_{B_e}(W_e, B_e), \mathrm{Hom}_{B_{\mathrm{dR}}^+}(W_{\mathrm{dR}}^+, B_{\mathrm{dR}}^+))$. Here, we define the E-action on $W^\vee$ by $af(x) := f(ax)$ for any $a \in E$, $f \in \mathrm{Hom}_{B_e}(W_e, B_e)$ (respectively $f \in \mathrm{Hom}_{B_{\mathrm{dR}}^+}(W_{\mathrm{dR}}^+, B_{\mathrm{dR}}^+)$), $x \in W_e$ (respectively $x \in W_{\mathrm{dR}}^+$).

LEMMA 1.10. *Let W_1, W_2 be finite free $B_e \otimes_{\mathbb{Q}_p} E$-modules with continuous semi-linear G_K-actions. Let $f : W_1 \to W_2$ be a $B_e \otimes_{\mathbb{Q}_p} E$-linear G_K-morphism. Then* $\mathrm{Ker}(f)$, $\mathrm{Im}(f)$ *and* $\mathrm{Cok}(f)$ *are all finite free over $B_e \otimes_{\mathbb{Q}_p} E$.*

Proof. Because $\mathrm{Im}(f)$ is a finite torsion free $B_e \otimes_{\mathbb{Q}_p} E$-module, it is free by the remark following [Ked04, Lemma 2.4]. From this, we obtain a splitting (as $B_e \otimes_{\mathbb{Q}_p} E$-modules) of the short exact sequence $0 \to \mathrm{Ker}(f) \to W_1 \to \mathrm{Im}(f) \to 0$. So $\mathrm{Ker}(f)$ is finite over $B_e \otimes_{\mathbb{Q}_p} E$, and $\mathrm{Ker}(f)$ is finite torsion free over $B_e \otimes_{\mathbb{Q}_p} E$. So $\mathrm{Ker}(f)$ is also finite free. By [Ber08b, Lemma 2.1.4], $\mathrm{Cok}(f)$ is free over B_e. Then by Lemma 1.7, $\mathrm{Cok}(f)$ is free over $B_e \otimes_{\mathbb{Q}_p} E$. □

The category of E–B-pairs is not an abelian category since cokernels of morphisms do not exist in general. We define the exactness in the category of B-pairs.

DEFINITION 1.11. Let $W_i := (W_{e,i}, W_{\mathrm{dR},i}^+)$ be E–B-pairs of G_K for $i = 1, 2, 3$. Let $f : W_1 \to W_2$, $g : W_2 \to W_3$ be morphisms of E–B-pairs. Then we say that

$$0 \to W_1 \to W_2 \to W_3 \to 0$$

is exact if the following two sequences are exact in the usual sense

$$0 \to W_{e,1} \to W_{e,2} \to W_{e,3} \to 0,$$
$$0 \to W_{\mathrm{dR},1}^+ \to W_{\mathrm{dR},2}^+ \to W_{\mathrm{dR},3}^+ \to 0.$$

LEMMA 1.12. *Let $W_1 := (W_{e,1}, W_{\mathrm{dR},1}^+)$, $W_2 := (W_{e,2}, W_{\mathrm{dR},2}^+)$ be E–B-pairs. Let $f : W_1 \to W_2$ be a morphism of E–B-pairs. We put* $\mathrm{Ker}(f) := (\mathrm{Ker}(f_e : W_{e,1} \to W_{e,2}), \mathrm{Ker}(f_{\mathrm{dR}} : W_{\mathrm{dR},1}^+ \to W_{\mathrm{dR},2}^+))$, $\mathrm{Im}(f) := (\mathrm{Im}(f_e : W_{e,1} \to W_{e,2}), \mathrm{Im}(f_{\mathrm{dR}} : W_{\mathrm{dR},1}^+ \to W_{\mathrm{dR},2}^+))$. *Then* $\mathrm{Ker}(f)$ *and* $\mathrm{Im}(f)$ *are E–B-pairs.*

Proof. We know that $\mathrm{Ker}(f_e)$ and $\mathrm{Im}(f_e)$ are finite free $B_e \otimes_{\mathbb{Q}_p} E$-modules by Lemma 1.10. From this, we obtain the canonical isomorphisms $\mathrm{Ker}(id_{B_{\mathrm{dR}}} \otimes_{B_e} f_e) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_e} \mathrm{Ker}(f_e)$, $\mathrm{Im}(id_{\mathrm{dR}} \otimes_{B_e} f_e) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_e} \mathrm{Im}(f_e)$. As for B_{dR}^+-modules, B_{dR} is a flat B_{dR}^+-module because B_{dR}^+ is a principal ideal domain. So we obtain the canonical isomorphisms $\mathrm{Ker}(id_{B_{\mathrm{dR}}} \otimes_{B_{\mathrm{dR}}^+} f_{\mathrm{dR}}) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} \mathrm{Ker}(f_{\mathrm{dR}})$, $\mathrm{Im}(id_{B_{\mathrm{dR}}} \otimes_{B_{\mathrm{dR}}^+} f_{\mathrm{dR}}) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} \mathrm{Im}(f_{\mathrm{dR}})$. So we obtain the natural isomorphisms $B_{\mathrm{dR}} \otimes_{B_e} \mathrm{Ker}(f_e) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} \mathrm{Ker}(f_{\mathrm{dR}})$ and $B_{\mathrm{dR}} \otimes_{B_e} \mathrm{Im}(f_e) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} \mathrm{Im}(f_{\mathrm{dR}})$, i.e. $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are B-pairs. As these maps are all E-linear, $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are E–B-pairs. □

DEFINITION 1.13. Let $W_1 \subset W_2$ be two E–B-pairs such that W_1 is a sub-E–B-pair of W_2. Then we say that W_1 is saturated in W_2 if $W_{\mathrm{dR},2}^+/W_{\mathrm{dR},1}^+$ is a free B_{dR}^+-module. Then $W_2/W_1 := (W_{e,2}/W_{e,1}, W_{\mathrm{dR},2}^+/W_{\mathrm{dR},1}^+)$ is an E–B-pair by Lemma 1.10.

LEMMA 1.14. *Let $W_1 \subset W_2$ be two E–B-pairs. Then there exists unique E–B-pair $W_1^{\mathrm{sat}} := (W_{e,1}^{\mathrm{sat}}, W_{\mathrm{dR},1}^{+,\mathrm{sat}})$ such that $W_1 \subset W_1^{\mathrm{sat}} \subset W_2$, $W_{e,1} = W_{e,1}^{\mathrm{sat}}$ and W_1^{sat} is saturated in W_2. We call W_1^{sat} the saturation of W_1 in W_2.*

Proof. We put $W_{e,1}^{\mathrm{sat}} := W_{e,1}$ and put $W_{\mathrm{dR},1}^{+,\mathrm{sat}} := W_{\mathrm{dR},1} \cap W_{\mathrm{dR},2}^+$. Then it is easy to see that $W_1^{\mathrm{sat}} := (W_{e,1}^{\mathrm{sat}}, W_{\mathrm{dR},1}^{+,\mathrm{sat}})$ is an E–B-pair satisfying all of the desired conditions. Conversely, if $W'_1 := (W'_{e,1}, W'^{+}_{\mathrm{dR},1})$ satisfies the same conditions, it is easy to see that $W'^{+}_{\mathrm{dR},1}$ satisfies $W'^{+}_{\mathrm{dR},1} = W_{\mathrm{dR},1} \cap W^{+}_{\mathrm{dR},2}$. The uniqueness of W_1^{sat} follows from this. □

Now we can define trianguline or split trianguline E-representations and trianguline or split trianguline E–B-pairs.

DEFINITION 1.15. We make the following definitions.

(1) Let W be an E–B-pair. We say that W is a split trianguline E–B-pair if there is a filtration $0 = W_0 \subset W_1 \subset \cdots \subset W_l = W$ by sub-E–B-pairs such that for any i, W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is a rank-one E–B-pair.

(1′) Let W be an E–B-pair. We say that W is a trianguline E–B-pair if $W \otimes_E E' := (W_e \otimes_E E', W^{+}_{\mathrm{dR}} \otimes_E E')$ is split trianguline for some finite extension E' of E.

(2) Let V be an E-representation. We say that V is a split trianguline (respectively trianguline) E-representation if $W(V)$ is a split trianguline (respectively trianguline) E–B-pair.

In this paper, we classify two-dimensional split trianguline E-representations.

Next we recall the generalization of the usual p-adic Hodge theory to the case of B-pairs following [Ber08b, § 2.3]. First we recall the definition of a filtered (φ, N, G_K)-module over K.

DEFINITION 1.16. Let L be a finite Galois extension of K. An E-filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module over K is a finite $L_0 \otimes_{\mathbb{Q}_p} E$-module D (where L_0 is the maximal unramified extension of $\mathbb{Q}_p$ in L) such that:

(1) D has a Frobenius semi-linear operator $\varphi_D : D \hookrightarrow D$ such that $id_{L_0} \otimes \varphi_D : L_0 \otimes_{\varphi, L_0} D \xrightarrow{\sim} D$ is an $L_0 \otimes_{\mathbb{Q}_p} E$-linear isomorphism (here φ acts on $L_0 \otimes_{\mathbb{Q}_p} E$ by $\varphi(x \otimes y) = \varphi(x) \otimes y$ for any $x \in L_0$, $y \in E$);

(2) $N : D \to D$ is an $L_0 \otimes_{\mathbb{Q}_p} E$-linear morphism such that $p\varphi N = N\varphi$;

(3) $D_L := L \otimes_{L_0} D$ has a decreasing filtration by sub-$L \otimes_{\mathbb{Q}_p} E$-modules $\mathrm{Fil}^i D_L$ for $i \in \mathbb{Z}$ such that $\mathrm{Fil}^{-i} D_L = D_L$ and $\mathrm{Fil}^i D_L = 0$ for sufficiently large $i \geq 0$;

(4) $\mathrm{Gal}(L/K)$ acts on $L_0 \otimes_{\mathbb{Q}_p} E$ (or $L \otimes_{\mathbb{Q}_p} E$)-semi-linearly on D (or D_L) such that $g\varphi = \varphi g$ and $gN = Ng$ and $g(\mathrm{Fil}^i D_L) = \mathrm{Fil}^i D_L$ for any $g \in \mathrm{Gal}(L/K)$ and $i \in \mathbb{Z}$ (here $\mathrm{Gal}(L/K)$ acts on $L \otimes_{\mathbb{Q}_p} E$ by $g(x \otimes y) = g(x) \otimes y$ for any $x \in L$, $y \in E$ and $g \in \mathrm{Gal}(L/K)$).

We call D an E-filtered (φ, N, G_K)-module if D is an E-filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module for some finite Galois extension L of K.

DEFINITION 1.17. Let $W := (W_e, W^{+}_{\mathrm{dR}})$ be an E–B-pair of G_K and let L be a finite Galois extension of K. Then we define $D^L_{\mathrm{cris}}(W) := (B_{\max} \otimes_{B_e} W_e)^{G_L}$, $D^L_{\mathrm{st}}(W) := (B_{\log} \otimes_{B_e} W_e)^{G_L}$, $D^L_{\mathrm{dR}}(W) := (B_{\mathrm{dR}} \otimes_{B_e} W_e)^{G_L}$, i.e. fixed parts of G_L. As in the case of usual p-adic representations, we can show that $\dim_{L'}(D^L_{?}(W)) \leq \mathrm{rank}_{B_e}(W_e)$ where $L' = L_0$ if $? \in \{\mathrm{cris}, \mathrm{st}\}$ and $L' = L$ if $? = \mathrm{dR}$. We say that W is potentially crystalline (respectively potentially semi-stable, respectively de Rham) if $\dim_{L'}(D^L_{?}(W)) = \mathrm{rank}_{B_e}(W_e)$ for $? = \mathrm{cris}$ (respectively $? = \mathrm{st}$, respectively $? = \mathrm{dR}$) for some finite Galois extension L of K.

For a potentially semi-stable E–B-pair W and for sufficiently large L such that $\dim_{L_0} D^L_{\mathrm{st}}(W) = \mathrm{rank}_{B_e} W_e$, we can equip $D^L_{\mathrm{st}}(W)$ with an E-filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module structure as follows. The $(\varphi, N, \mathrm{Gal}(L/K))$-module structure on $D^L_{\mathrm{st}}(W)$ is induced from the action of φ and N on $B_{\log}$ and from the action of G_K on $B_{\log} \otimes_{B_e} W_e$. The filtration on $D^L_{\mathrm{dR}}(W) \xrightarrow{\sim} L \otimes_{L_0} D^L_{\mathrm{st}}(W)$ is defined by $\mathrm{Fil}^i D^L_{\mathrm{dR}}(W) := D^L_{\mathrm{dR}}(W) \cap t^i W^{+}_{\mathrm{dR}} \subseteq W_{\mathrm{dR}}$. So we obtain a functor $W \mapsto D^L_{\mathrm{st}}(W)$ from the category of potentially semi-stable E–B-pairs of G_K which are semi-stable E–B-pairs of G_L to the category of E-filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules over K. Berger generalized p-adic monodromy theorem and 'weakly admissible implies admissible' theorem to the case of B-pairs.

Theorem 1.18. *We have the following results.*

1. *All de Rham E–B-pairs are potentially semi-stable.*
2. *The functor $W \mapsto D^L_{\mathrm{st}}(W)$ realizes an equivalence of categories between the category of potentially semi-stable E–B-pairs of G_K which are semi-stable E–B-pairs of G_L to the category of E-filtered $(\varphi, N, \mathrm{Gal}(L/K))$-modules over K.*
3. *The functor $W \mapsto D^L_{\mathrm{cris}}(W)$ realizes an equivalence of categories between the category of potentially crystalline E–B-pairs of G_K which are crystalline E–B-pairs of G_L to the category of E-filtered $(\varphi, \mathrm{Gal}(L/K))$-modules over K.*

Proof. See [Ber08b, Proposition 2.3.4 and Theorem 2.3.5]. □

Remark 1.19. An inverse functor of D^L_{st} is defined as follows. For an E-filtered $(\varphi, N, \mathrm{Gal}(L/K))$-module D over K, put $W_e(D) := (B_{\log} \otimes_{L_0} D)^{\varphi=1, N=0}$ and $W^+_{\mathrm{dR}}(D) := \mathrm{Fil}^0(B_{\mathrm{dR}} \otimes_L D_L)$. Then we can show that $W(D) := (W_e(D), W^+_{\mathrm{dR}}(D))$ is an E–B-pair of G_K. We know that $D \mapsto W(D)$ is an inverse of $W \mapsto D^L_{\mathrm{st}}(W)$ (see [Ber08b, § 2.3]).

Remark 1.20. The proof of (2) and (3) of the above theorem is much easier than that in the case of p-adic representations, because in this case there are no conditions about weakly admissibility of filtered (φ, N, G_K)-modules.

1.2 (φ, Γ_K)-modules over a Robba ring

In [Ber08b], Berger established the equivalence between the category of B-pairs and the category of (φ, Γ_K)-modules over a Robba ring $B^{\dagger}_{\mathrm{rig},K}$. In this section, we first recall the general facts about φ-modules over Robba rings following [Ked08] and then recall the construction of a Robba ring $B^{\dagger}_{\mathrm{rig},K}$ and the definition of (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ following [Ber02].

In applications of (φ, Γ_K)-modules to p-adic Hodge theory, the notion of slope and the slope filtration theorem of Kedlaya are very important. So we recall the general fact about slopes of φ-modules over general Robba ring and the slope filtration theorem of Kedlaya following [Ked08, § 1]. Let L be a complete discrete valuation field of characteristic zero, $\mathcal{O}_L$ its integer ring, k_L the residue field of L and π_L a uniformizer of L. Then we define the Robba ring of L by $\mathcal{R}_L := \{f(x) := \sum_{k\in\mathbb{Z}} a_n x^n \mid a_n \in L,\ f(x)$ converges for any $x \in L$ such that $r \leq |x| < 1$ for some $r < 1\}$. We assume that there is an endomorphism $\phi_L : L \to L$ which is a lifting of the $q = p^f$th power Frobenius on k_L for some $f \in \mathbb{N}$ and we assume that there is a ϕ_L-semi-linear endomorphism $\phi : \mathcal{R}_L \to \mathcal{R}_L : \sum_{k\in\mathbb{Z}} a_k x^k \mapsto \sum_{k\in\mathbb{Z}} \phi_L(a_k)\phi(x)^k$ such that $\phi(x) - x^q \in \mathcal{R}_L$ has all coefficients in $\pi_L \mathcal{O}_L$.

Definition 1.21. A ϕ-module over $\mathcal{R}_L$ is a finite free $\mathcal{R}_L$-module M equipped with ϕ-semi-linear action $\phi_M : M \to M$ such that $id_{\mathcal{R}_L} \otimes \phi_M : \mathcal{R}_L \otimes_{\phi,\mathcal{R}_L} M \to M$ is an $\mathcal{R}_L$-linear isomorphism.

The category of ϕ-modules over $\mathcal{R}_L$ is not an abelian category because the cokernel of a morphism is not a free $\mathcal{R}_L$-module in general. So it is important to know when the cokernel is free.

DEFINITION 1.22. Let $M_1 \subset M$ be a sub-ϕ-module of a ϕ-module M over $\mathcal{R}_L$. Then we say that M_1 is saturated in M if the quotient M/M_1 is a free $\mathcal{R}_L$ module. Then M/M_1 is a ϕ-module over $\mathcal{R}_L$.

Put $\mathcal{R}_L^{\mathrm{bd}} := \{f(x) \in \mathcal{R}_L \mid |f(x)|$ is bounded in $r \leq |x| < 1$ for some $r < 1\} = \{f(x) = \sum_{k\in\mathbb{Z}} a_k x^k \mid \{a_k\}_{k\in\mathbb{Z}} \subset L$ is bounded and $f(x)$ converges for any x such that $r \leq |x| < 1$ for some $r < 1\}$. Put $\mathcal{R}_L^{\mathrm{int}} := \{\sum_{k\in\mathbb{Z}} a_k x^k \in \mathcal{R}_L \mid a_k \in \mathcal{O}_L$ for any $k \in \mathbb{Z}\}$. Then $\mathcal{R}_L^{\mathrm{bd}}$ is a discrete valuation field with the integer ring $\mathcal{R}_L^{\mathrm{int}}$ and a valuation on $\mathcal{R}_L^{\mathrm{bd}}$ is defined by $w(f(x)) := \inf_{n\in\mathbb{Z}}\{v_L(a_n)\}$ for any $f(x) = \sum_{n\in\mathbb{Z}} a_n x^n \in \mathcal{R}_L^{\mathrm{bd}}$, where v_L is the valuation of L such that $v_L(p) = 1$. The residue field of $\mathcal{R}_L^{\mathrm{bd}}$ is $k_L((x))$. By the above assumption on ϕ, $\phi(x) \in \mathcal{R}_L^{\mathrm{int}}$, ϕ preserves $\mathcal{R}_L^{\mathrm{bd}}$ and $\mathcal{R}_L^{\mathrm{int}}$ and $w(\phi(f)) = w(f)$ for any $f \in \mathcal{R}_L^{\mathrm{bd}}$. Moreover, it is known that $\mathcal{R}_L^{\times} = \mathcal{R}_L^{\mathrm{bd},\times}$ (see [Ked08, Example 1.4.2]).

DEFINITION 1.23. For a ϕ-module M over $\mathcal{R}_L$ of rank n, the top exterior power $\wedge^n M$ has rank one over $\mathcal{R}_L$. Let $\mathbf{v}$ be a generator of $\wedge^n M$ and write $\phi(\mathbf{v}) = r\mathbf{v}$ for some $r \in \mathcal{R}_L^{\times} = \mathcal{R}_L^{\mathrm{bd}\times}$. Define the degree of M by setting $\deg(M) := w(r)$, define the slope of M by setting $\mu(M) := \deg(M)/\mathrm{rank}(M)$. It is easy to check that $\deg(M)$ and $\mu(M)$ do not depend on the choice of the generator of $\wedge^n M$.

If M_1, M_2 are ϕ-modules over $\mathcal{R}_L$, then we can see that $\mu(M_1 \otimes_{\mathcal{R}_L} M_2) = \mu(M_1) + \mu(M_2)$ (see [Ked08, Remark 1.4.5]). Next we define pure slope ϕ-modules. Before defining this, we need to recall some definitions.

DEFINITION 1.24. Let M be a ϕ-module over $\mathcal{R}_L$ and let a be a positive integer. Then we define the a-pushforward $[a]_* M$ of M to be the ϕ^a-module M over $\mathcal{R}_L$ such that the ϕ^a-semi-linear morphism is defined by $\phi_M^a : M \to M$.

If M is a ϕ-module over $\mathcal{R}_L$, then it is easy to see that $[a]_* M$ is a ϕ^a-module over $\mathcal{R}_L$ with $\deg([a]_* M) = a\deg(M)$, $\mu([a]_* M) = a\mu(M)$. First we define pure slope zero ϕ-modules, which are called étale ϕ-modules.

DEFINITION 1.25. A ϕ-module M over $\mathcal{R}_L$ is said to be étale if it can be obtained by base extension from a ϕ-module over $\mathcal{R}_L^{\mathrm{int}}$, that is, M admits a ϕ_M-stable $\mathcal{R}_L^{\mathrm{int}}$-lattice N such that ϕ_M induces an $\mathcal{R}_L^{\mathrm{int}}$-linear isomorphism $\mathcal{R}_L^{\mathrm{int}} \otimes_{\phi, \mathcal{R}_L^{\mathrm{int}}} N \xrightarrow{\sim} N$.

From the above definition, if M is an étale ϕ-module over $\mathcal{R}_L$, then it is easy to see that $\deg(M) = 0$ and $\mu(M) = 0$. On this setting, we define pure slope ϕ-modules as follows. (We put e_L the absolute ramified index of L.)

DEFINITION 1.26. Let M be a ϕ-module over $\mathcal{R}_L$ with slope $s = c/de_L$ where c, d are coprime integers with $d > 0$. We say that M is pure of slope s if, for some ϕ^d-module N of rank one such that $\deg(N) = -c/e_L$, $([d]_* M) \otimes_{\mathcal{R}_L} N$ is an étale ϕ^d-module.

Remark 1.27. In [Ber08b], the definition of pure slope ϕ-modules over a Robba ring is different from this definition [Ber08b, § 1.2]. However, it is proved in the proof of [Ked08, Theorem 1.7.1] that the two definitions are the same.

Now we can state the slope filtration theorem of Kedlaya.

THEOREM 1.28. *Let M be a ϕ-module over $\mathcal{R}_L$. Then M admits a unique filtration $0 = M_0 \subset M_1 \subset \cdots \subset M_l = M$ by sub ϕ-modules over $\mathcal{R}_L$ such that $M_i \subset M_{i+1}$ is saturated and M_{i+1}/M_i is pure of slope s_{i+1} for any $0 \leq i \leq l-1$ with $s_1 < s_2 < \cdots < s_l$.*

Proof. See [Ked08, Theorem 1.7.1]. □

Next we recall the construction of a Robba ring $B^{\dagger}_{\mathrm{rig},K}$ and some related rings following [Ber02]. First, for rational numbers $0 \le r \le s \le +\infty$, we put $\widetilde{A}_{[r,s]} := \widetilde{\mathbb{A}}^{+}\{p/[\bar{\pi}^r], [\bar{\pi}^s]/p\}$, the p-adic completion of $\widetilde{\mathbb{A}}^{+}[p/[\bar{\pi}^r], [\bar{\pi}^s]/p]$ (here $\pi := [\varepsilon] - 1$, $\bar{\pi} := \varepsilon - 1$, so $[\bar{\pi}] = [\varepsilon - 1]$). When $r = 0$, we put $p/[\bar{\pi}^r] := 1$ and when $s = +\infty$, we put $[\bar{\pi}^s]/p := 1$. We put $\widetilde{B}_{[r,s]} := \widetilde{A}_{[r,s]}[1/p]$. Then we have natural continuous G_K-action on $\widetilde{A}_{[r,s]}$ and on $\widetilde{B}_{[r,s]}$. Frobenius induces isomorphisms $\varphi : \widetilde{A}_{[r,s]} \xrightarrow{\sim} \widetilde{A}_{[pr,ps]}$ and $\varphi : \widetilde{B}_{[r,s]} \xrightarrow{\sim} \widetilde{B}_{[pr,ps]}$, here we equip these rings with p-adic topology. For $r \le r_0 \le s_0 \le s$, it is known that the natural map $\widetilde{A}_{[r,s]} \hookrightarrow \widetilde{A}_{[r_0,s_0]}$ is injective. For $r > 0$, we put $\widetilde{B}^{\dagger,r} := \widetilde{B}_{[r,+\infty]}$, $\widetilde{B}^{\dagger} := \cup_{r>0} \widetilde{B}^{\dagger,r}$, $\widetilde{B}^{\dagger,r}_{\mathrm{rig}} := \bigcap_{r \leqq s < +\infty} \widetilde{B}_{[r,s]}$ (equipped with Frechet topology defined by any $\widetilde{B}_{[r,s]}$) and $\widetilde{B}^{\dagger}_{\mathrm{rig}} := \bigcup_{r>0} \widetilde{B}^{\dagger,r}_{\mathrm{rig}}$ (equipped with inductive limit topology). Then we have natural inclusions $\widetilde{B}^{\dagger,r} \subset \widetilde{B}^{\dagger,r}_{\mathrm{rig}}$, $\widetilde{B}^{\dagger} \subset \widetilde{B}^{\dagger}_{\mathrm{rig}}$ and $\widetilde{B}^{\dagger} \subset \widetilde{\mathbb{B}}$ (but $\widetilde{B}^{\dagger}_{\mathrm{rig}} \not\subseteq \widetilde{\mathbb{B}}$). Frobenius induces isomorphisms $\varphi : \widetilde{B}^{\dagger,r} \xrightarrow{\sim} \widetilde{B}^{\dagger,pr}$, $\widetilde{B}^{\dagger,r}_{\mathrm{rig}} \xrightarrow{\sim} \widetilde{B}^{\dagger,pr}_{\mathrm{rig}}$, $\widetilde{B}^{\dagger} \xrightarrow{\sim} \widetilde{B}^{\dagger}$ and $\widetilde{B}^{\dagger}_{\mathrm{rig}} \xrightarrow{\sim} \widetilde{B}^{\dagger}_{\mathrm{rig}}$. Moreover, we can easily check that $A_{\max} = \widetilde{A}_{[0,(p-1)/p]}$, $B_{\max} = \widetilde{B}_{[0,(p-1)/p]}$ and $\widetilde{B}^{+}_{\mathrm{rig}} = \widetilde{B}_{[0,+\infty)} := \bigcap_{0<s<+\infty} \widetilde{B}_{[0,s]}$. The rings $\widetilde{B}^{\dagger}_{\mathrm{rig}}, \widetilde{B}^{\dagger}$ are respectively equal to $\Gamma^{\mathrm{alg}}_{\mathrm{an,con}}, \Gamma^{\mathrm{alg}}_{\mathrm{con}}$ defined by Kedlaya [Ked04]. We recall a relation between $\widetilde{B}^{\dagger}_{\mathrm{rig}}$ and B_{dR}. It is easy to see that $\widetilde{B}_{[(p-1)/p,(p-1)/p]} = \widetilde{\mathbb{A}}^{+}\{p/[\tilde{p}], [\tilde{p}]/p\}[1/p]$. Then we have a natural continuous G_K-equivariant injective morphism $\widetilde{B}_{[(p-1)/p,(p-1)/p]} \hookrightarrow B^{+}_{\mathrm{dR}} : \sum_{k \geq 0} a_k (p/[\tilde{p}])^k + \sum_{l \geq 0} b_l ([\tilde{p}]/p)^l \mapsto \sum_{k \geqq 0} a_k (p/[\tilde{p}])^k + \sum_{l \geqq 0} b_l ([\tilde{p}]/p)^l$. In particular, we have a natural continuous inclusion $i_0 : \widetilde{B}^{\dagger,(p-1)/p}_{\mathrm{rig}} = \bigcap_{(p-1)/p \leqq r < +\infty} \widetilde{B}_{[(p-1)/p,r]} \hookrightarrow \widetilde{B}_{[(p-1)/p,(p-1)/p]} \hookrightarrow B^{+}_{\mathrm{dR}}$. For any $n \in \mathbb{N}$, we define a continuous G_K-equivariant injection $i_n := i_0 \circ \varphi^{-n} : \widetilde{B}^{\dagger,(p-1)p^{n-1}}_{\mathrm{rig}} \xrightarrow{\sim} \widetilde{B}^{\dagger,(p-1)/p}_{\mathrm{rig}} \hookrightarrow B^{+}_{\mathrm{dR}}$.

Next we define $B^{\dagger}_{K,\mathrm{rig}}$ and $B^{\dagger}_{K}$. We put $A_{K_0} := \{\sum_{k \geqq -\infty}^{+\infty} a_k \pi^k \mid a_k \in \mathcal{O}_{K_0}, a_k \to 0 \ (k \to -\infty)\}$ and $B_{K_0} := A_{K_0}[1/p]$ where $\pi := [\varepsilon] - 1$. A_{K_0} is a complete discrete valuation ring such that p is a prime element, the residue field is $E_{K_0} := k((\varepsilon - 1))$ and the fraction field is B_{K_0}. We can show that $A_{K_0} \subset \widetilde{\mathbb{A}}$ and $B_{K_0} \subset \widetilde{\mathbb{B}}$. φ and the action of G_K on $\widetilde{\mathbb{A}}$ preserves A_{K_0} and $\varphi(\pi) = (\pi + 1)^p - 1$ and $g(\pi) = (\pi + 1)^{\chi(g)} - 1$ for any $g \in G_K$. Let $\mathbb{A}$ be the p-adic completion of the maximal unramified extension of A_{K_0} in $\widetilde{\mathbb{A}}$, $\mathbb{B}$ be its fraction field. Then φ and the action of G_K also preserve $\mathbb{A}$ and $\mathbb{B}$. We put $A_K := \mathbb{A}^{H_K}$, $B_K := \mathbb{B}^{H_K}$, $B^{\dagger}_K := B_K \cap \widetilde{B}^{\dagger}$, $B^{\dagger,r}_K := B_K \cap \widetilde{B}^{\dagger,r}$. By definition, these rings are equipped with natural continuous actions of φ and Γ_K. If we put $E_K := (E^{\mathrm{sep}}_{K_0})^{H_K} \subset \widetilde{\mathbb{E}}$ (here $E^{\mathrm{sep}}_{K_0}$ is the separable closure of E_{K_0} in $\widetilde{\mathbb{E}}$), it is known that E_K is a separable extension of E_{K_0} of degree $[K_\infty : K_0(\zeta_{p^\infty})]$ by the theory of fields of norm. Here A_K is a complete discrete valuation ring such that p is a prime element, the residue field is E_K and the fraction field is B_K. Let K'_0 be the maximal unramified extension of K_0 in K_∞. For sufficiently large $r > 0$ such that a prime element $\pi_{E_K} \in E_K$ lifts to $\tilde{\pi}_{E_K} \in B^{\dagger,r}_K$, it is known that $B^{\dagger,r}_K = \{\sum_{k \geqq -\infty}^{+\infty} a_k \tilde{\pi}^k_{E_K} \mid a_k \in K'_0,\ f(X) := \sum_{k \geqq -\infty}^{+\infty} a_k X^k$ is a bounded function on $X \in \mathbb{C}_p$ such that $p^{-1/re_K} \le |X| < 1\}$. We put $B^{\dagger,r}_{\mathrm{rig},K}$ the Frechet completion of $B^{\dagger,r}_K$ and $B^{\dagger}_{\mathrm{rig},K} := \bigcup_{r \gg 0} B^{\dagger,r}_{\mathrm{rig},K}$. Then it is known that $B^{\dagger,r}_{\mathrm{rig},K} = \{\sum_{k \geqq -\infty}^{+\infty} a_k \tilde{\pi}^k_{E_K} \mid a_k \in K'_0,\ f(X) := \sum_{k \geqq -\infty}^{+\infty} a_k X^k$ converges on $X \in \mathbb{C}_p$ such that $p^{-1/re_K} \le |X| < 1\}$. So $B^{\dagger,r}_{\mathrm{rig},K}, B^{\dagger}_{\mathrm{rig},K}$ are the Robba rings with coefficients in K'_0. We can show that $B^{\dagger,r}_{\mathrm{rig},K} \subset \widetilde{B}^{\dagger,r}_{\mathrm{rig}}$, $B^{\dagger}_{\mathrm{rig},K} \subset \widetilde{B}^{\dagger}_{\mathrm{rig}}$ and the Frechet topology on $B^{\dagger,r}_{\mathrm{rig},K}$ is the induced topology from that on $\widetilde{B}^{\dagger,r}_{\mathrm{rig}}$ and φ induces inclusions $B^{\dagger,r}_{\mathrm{rig},K} \hookrightarrow B^{\dagger,pr}_{\mathrm{rig},K}$, $B^{\dagger}_{\mathrm{rig},K} \hookrightarrow B^{\dagger}_{\mathrm{rig},K}$

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and Γ_K continuously acts on $B^{\dagger,r}_{\mathrm{rig},K}$, $B^{\dagger}_{\mathrm{rig},K}$. By restricting $i_n : \widetilde{B}^{\dagger,(p-1)p^{n-1}}_{\mathrm{rig}} \hookrightarrow B^+_{\mathrm{dR}}$ to $B^{\dagger,(p-1)p^{n-1}}_{\mathrm{rig},K}$, we have a G_K-equivariant injection $i_n : B^{\dagger,(p-1)p^{n-1}}_{\mathrm{rig},K} \hookrightarrow B^+_{\mathrm{dR}}$.

DEFINITION 1.29. An E–(φ,Γ_K)-module over $B^{\dagger}_{\mathrm{rig},K}$ is a finite $B^{\dagger}_{\mathrm{rig,K}} \otimes_{\mathbb{Q}_p} E$-module D equipped with a Frobenius semi-linear action φ_D and a continuous semi-linear action of Γ_K suchthat D is free as a $B^{\dagger}_{\mathrm{rig},K}$-module, $id_{B^{\dagger}_{\mathrm{rig},K}} \otimes \varphi_D : B^{\dagger}_{\mathrm{rig},K} \otimes_{\varphi, B^{\dagger}_{\mathrm{rig},K}} D \to D$ is an isomorphism and that the actions of φ and Γ_K commute. Here φ and Γ_K act on $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$ as $\varphi \otimes id$, $\gamma \otimes id$ for any $\gamma \in \Gamma_K$.

When $E = \mathbb{Q}_p$, we omit the notation $\mathbb{Q}_p$ in the above definition, i.e. we simply call (φ,Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$.

Next we prove a lemma concerning the freeness over $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$ of an E–φ-module.

LEMMA 1.30. *Let D be an E–φ-module over $B^{\dagger}_{\mathrm{rig},K}$. Then D is also free as a $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$-module.*

Proof. Let $E'_0 := E \cap K'_0$, $f' := [E'_0 : \mathbb{Q}_p]$ and $I := \{\sigma : E'_0 \hookrightarrow E\}$. Then we have a canonical decomposition $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \oplus_{\sigma \in I} B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$, where e_σ is the idempotent in $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$ corresponding to 1 in $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} E$. Then any component $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$ is a Robba ring with coefficients in E, i.e. they are non-canonically isomorphic to $\mathcal{R}_E$. The action of φ is given by $\varphi(a \otimes be_\sigma) = \varphi(a) \otimes be_{\sigma\varphi^{-1}|_{E'_0}}$ for any $a \in B^{\dagger}_{\mathrm{rig},K}$ and $b \in E$ and $\sigma \in I$. Because $\varphi|_{E'_0}$ acts transitively on I, so φ acts transitively on the components of this decomposition. Because $\sharp I = f'$, $\varphi^{f'}$ preserves the component $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$ for any $\sigma \in I$. Let D be an E–φ-module over $B^{\dagger}_{\mathrm{rig},K}$. Then we also have a canonical decomposition $D \xrightarrow{\sim} \oplus_{\sigma \in I} D \otimes_{B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E} B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma \xrightarrow{\sim} \oplus_{\sigma \in I} D \otimes_{E'_0,\sigma} Ee_\sigma$. Then, for any $\sigma \in I$, the component $D \otimes_{E'_0,\sigma} Ee_\sigma$ is a $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$-module which is finite torsion free as a $B^{\dagger}_{\mathrm{rig},K}$-module. So it is free as a $B^{\dagger}_{\mathrm{rig},K}$-module by [Ked04, Proposition 2.5]. Because $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$ is also a Robba ring, in particular, it is a Bézout domain and because the natural map $B^{\dagger}_{\mathrm{rig},K} \hookrightarrow B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} E$ is injective, then it is easy to see that $D \otimes_{E'_0,\sigma} Ee_\sigma$ is finite torsion free over $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$. So it is free over $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$ for any $\sigma \in I$ by [Ked04, Proposition 2.5]. Then we can take a basis $\{v_1 e_\sigma, \ldots, v_k e_\sigma\}$ of $D \otimes_{E'_0,\sigma} Ee_\sigma$ over $B^{\dagger}_{\mathrm{rig},K} \otimes_{E'_0,\sigma} Ee_\sigma$ for any fixed $\sigma \in I$. Then $\{\varphi^i(v_1)e_{\sigma\varphi^{-i}|_{E'_0}}, \ldots, \varphi^i(v_k)e_{\sigma\varphi^{-i}|_{E'_0}}\}$ is a basis of $D \otimes_{E'_0,\sigma\varphi^{-i}|_{E'_0}} Ee_{\sigma\varphi^{-i}|_{E'_0}}$ for any $0 \leq i \leq f'-1$ by the Frobenius structure on D. Then $\{\sum_{i=0}^{f'-1} \varphi^i(v_1)e_{\sigma\varphi^{-i}|_{E'_0}}, \ldots, \sum_{i=0}^{f'-1} \varphi^i(v_k)e_{\sigma\varphi^{-i}|_{E'_0}}\}$ is a basis of D over $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$. So D is a free $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$-module. □

By this lemma, we can define the rank and the tensor products and the duals of E–(φ,Γ_K)-modules as follows.

DEFINITION 1.31. We make the following definitions.

(1) Let D be an E–(φ,Γ_K)-module over $B^{\dagger}_{\mathrm{rig},K}$. We define the rank of D by $\mathrm{rank}(D) := \mathrm{rank}_{B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E}(D)$, i.e. the rank of D as a free $B^{\dagger}_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$-module.

(2) Let D_1, D_2 be E–(φ,Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$. Then we define the tensor product of D_1 and

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D_2 by $D_1 \otimes D_2 := D_1 \otimes_{B^\dagger_{\mathrm{rig},K}\otimes_{\mathbb{Q}_p} E} D_2$, which is a free $B^\dagger_{\mathrm{rig},K} \otimes_{\mathbb{Q}_p} E$-module by Lemma 1.30, on which φ and Γ_K act by $\varphi_{D_1\otimes D_2} := \varphi_{D_1} \otimes_{B^\dagger_{\mathrm{rig},K}\otimes_{\mathbb{Q}_p} E} \varphi_{D_2}$ and $\gamma \otimes_{B^\dagger_{\mathrm{rig},K}\otimes_{\mathbb{Q}_p} E} \gamma$ for any $\gamma \in \Gamma_K$.

(3) Let D be an E-(φ,Γ)-module over $B^\dagger_{\mathrm{rig},K}$. Then we define the dual of D by $D^\vee := \mathrm{Hom}_{B^\dagger_{\mathrm{rig},K}}(D, B^\dagger_{\mathrm{rig},K})$; here the E-action on $D^\vee$ is defined by $af(x) := f(ax)$ for any $a \in E$, $f \in D^\vee$, $x \in D$.

Next we prove the slope filtration theorem for E–(φ,Γ_K)-modules over $B^\dagger_{\mathrm{rig},K}$.

THEOREM 1.32. *Let D be an E–(φ,Γ_K)-module over $B^\dagger_{\mathrm{rig},K}$. Then D admits unique filtration $0 = D_0 \subset D_1 \subset \cdots \subset D_l = D$ such that D_i is an E–(φ,Γ_K)-module over $B^\dagger_{\mathrm{rig},K}$, D_i is saturated in D_{i+1} and the quotient D_{i+1}/D_i is pure of slope s_{i+1} for any $0 \le i \le l-1$ with $s_1 < s_2 < \cdots < s_l$. Here the slope of an E–(φ,Γ_K)-module M is the slope of M as φ-module over $B^\dagger_{\mathrm{rig},K}$.*

Proof. Let D be an E–(φ,Γ_K)-module over $B^\dagger_{\mathrm{rig},K}$. Then, by the slope filtration theorem (Theorem 1.28), D admits unique filtration $0 = D_0 \subset D_1 \subset \cdots \subset D_l = D$ by sub-φ-modules over $B^\dagger_{\mathrm{rig},K}$ such that D_i is saturated in D_{i+1} and the quotient D_{i+1}/D_i is pure of slope s_{i+1} for any $0 \le i \le l-1$ with $s_1 < s_2 < \cdots < s_l$. Then the action of E preserves D_i for any i by uniqueness of filtration. So D_i and D_{i+1}/D_i are E–φ-modules over $B^\dagger_{\mathrm{rig},K}$ for any i. For any $\gamma \in \Gamma_K$, $0 \subset \gamma(D_1) \subset \cdots \subset \gamma(D_l) = D$ also satisfies the same conditions as $0 \subset D_1 \subset \cdots \subset D$ by the commutativity of Γ_K and φ. So we obtain $\gamma(D_i) = D_i$ for any i and $\gamma \in \Gamma_K$ by uniqueness of filtration. Hence, D_i and D_{i+1}/D_i are E–(φ,Γ_K)-modules over $B^\dagger_{\mathrm{rig},K}$ for any i. □

In particular, it follows from the above theorem that when D is rank one, then D is pure of slope s for some $s \in \mathbb{Q}$. Concerning this slope s, we know more precise information as follows, which we need to classify rank-one E–B-pairs.

LEMMA 1.33. *Let D be a rank one E–(φ,Γ_K)-module. Then D is pure and the slope $\mu(D)$ is contained in $(1/(fe_E))\mathbb{Z}$.*

Proof. The claim that D is pure follows from the above theorem. We prove that $\mu(D)$ is contained in $(1/(fe_E))\mathbb{Z}$ by using the Γ_K structure on D. First we consider the following short exact sequence of finite groups

$$1 \to \mathrm{Gal}(E'_0/K_0) \to \mathrm{Gal}(E'_0/\mathbb{Q}_p) \to \mathrm{Gal}(K_0/\mathbb{Q}_p) \to 1.$$

Here we put $E'_0 := E \cap K'_0 \supseteq K_0$. We put $f'' := [E'_0 : K_0]$ and $f' := ff'' = [E'_0 : \mathbb{Q}_p]$. Then we have a natural surjection $\Gamma_K \twoheadrightarrow \mathrm{Gal}(K'_0K/K) \xrightarrow{\sim} \mathrm{Gal}(K'_0/K_0) \twoheadrightarrow \mathrm{Gal}(E'_0/K_0)$. So, if we consider φ (which acts on $B^\dagger_{\mathrm{rig},K}$) as an element in $\mathrm{Gal}(E'_0/\mathbb{Q}_p)$, there is a $\gamma \in \Gamma_K$ such that $\varphi^f = \gamma$ in $\mathrm{Gal}(E'_0/K_0)$. On this setting, we consider a rank-one E–(φ,Γ_K)-module D over $B^\dagger_{\mathrm{rig},K}$. Put $I := \{\sigma : E'_0 \hookrightarrow E\} \xrightarrow{\sim} \mathrm{Gal}(E'_0/\mathbb{Q}_p)$ and $D_\sigma := D \otimes_{B^\dagger_{\mathrm{rig},K}\otimes_{\mathbb{Q}_p} E} B^\dagger_{\mathrm{rig},K} \otimes_{E'_0,\sigma} E$ for any $\sigma \in I$. Because D is rank one, D_σ is a rank-one free $B^\dagger_{\mathrm{rig},K} \otimes_{E'_0,\sigma} E$-module. Take a base e_σ of D_σ for any $\sigma \in I$. Then φ sends D_σ to $D_{\sigma\varphi^{-1}}$ and $\gamma \in \Gamma_K$ sends D_σ to $D_{\sigma\gamma^{-1}}$. We calculate the slope of D as follows. Let e_1 be a base of D_{id} corresponding to $id \in \mathrm{Gal}(E'_0/\mathbb{Q}_p)$. Then we have $\varphi^f(e_1) = \alpha e_{\varphi^{-f}}$ for some $\alpha \in B^\dagger_{\mathrm{rig},K} \otimes_{E'_0,\varphi^{-f}} E$. Because $\varphi^f = \gamma \in \mathrm{Gal}(E'_0/K_0)$ and γ induces an isomorphism $\gamma : D_{id} \xrightarrow{\sim} D_{\gamma^{-1}}$, there exists some $\beta \in B^\dagger_{\mathrm{rig},K} \otimes_{E'_0,id} E$ such that

$\gamma(\beta e_1)=\alpha e_{\gamma^{-1}}$. Because the actions of φ and γ commute, we have $\varphi^{2f}(e_1)=\varphi^f(\alpha e_{\varphi^{-f}})=\varphi^f(\alpha e_{\gamma^{-1}})=\varphi^f(\gamma(\beta e_1))=\gamma(\varphi^f(\beta)\varphi^f(e_1))=\gamma(\varphi^f(\beta)\gamma(\beta e_1))=\gamma(\varphi^f(\beta))\gamma^2(\beta)\gamma^2(e_1)$. Repeating this procedure, we obtain $\varphi^{ff''}(e_1)=\gamma(\varphi^{f(f''-1)}(\beta))\gamma^2(\varphi^{f(f''-2)}(\beta))\cdots\gamma^{f''}(\beta)\gamma^{f''}(e_1):=\tilde{\beta}ce_1$ (here we put $\tilde{\beta}:=\gamma(\varphi^{f(f''-1)}(\beta))\gamma^2(\varphi^{f(f''-2)}(\beta))\cdots\gamma^{f''}(\beta)$ and $\gamma^{f''}(e_1):=ce_1$ for some $c\in B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,id}E$ because $\gamma^{f''}=id\in \mathrm{Gal}(E'_0/\mathbb{Q}_p)$). Then D_{id} is a rank-one $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,id}E$ of slope $w_1(\tilde{\beta})+w_1(c)$, where w_1 is the valuation of $(B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,id}E)^{\mathrm{bd}}$ which is the natural extension of the valuation of $(B^{\dagger}_{\mathrm{rig},K})^{\mathrm{bd}}=B^{\dagger}_{K}$. Because φ and γ do not change valuation, we have $w_1(\tilde{\beta})=f''w_1(\beta)\in(f''/e_E)\mathbb{Z}$. Because Γ_K acts continuously on D and $\Gamma_K\subseteq\mathbb{Z}_p^{\times}$, we have $w_1(c)=0$. So the slope of D_{id} as $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,id}E$ is contained in $(f''/e_E)\mathbb{Z}$. We can easily see that the slope of D_{id} as a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,id}E$ is the same as the slope of D_{id} as a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}$. So D_{id} also has slope $w_1(\tilde{\beta})\in(f''/e_E)\mathbb{Z}$ as a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}$. Using the Frobenius structure on D, we can show that D_{σ} also has slope $w_1(\tilde{\beta})$ as a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}$ for any $\sigma\in I$. So D also has slope $w_1(\tilde{\beta})$ as a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}$. Because $\mu([ff'']_*D)=ff''\mu(D)$ by [Ked08, Remark 1.4.5], D has slope $(w_1(\tilde{\beta})/ff'')\in(f''/ff''e_E)\mathbb{Z}=(1/fe_E)\mathbb{Z}$ as a φ-module over $B^{\dagger}_{\mathrm{rig},K}$. We have finished the proof of this lemma. □

Next we prove a lemma concerning a relation between tensor products and slopes.

Lemma 1.34. *Let D_1, D_2 be E–(φ,Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ which are pure of slope s_1, s_2, respectively. Then $D_1\otimes D_2$ is pure of slope s_1+s_2.*

Proof. If we decompose $B^{\dagger}_{\mathrm{rig},K}\otimes_{\mathbb{Q}_p}E\xrightarrow{\sim}\oplus_{\sigma:E'_0\hookrightarrow E}B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,\sigma}Ee_{\sigma}$, we can decompose D_1, D_2 into σ components, $D_i\xrightarrow{\sim}\oplus_{\sigma:E'_0\hookrightarrow E}D_{i,\sigma}$ for $i=1,2$, where $D_{i,\sigma}$ is the $B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,\sigma}E$ component of D_i. Then, by the proof of Lemma 1.33 and by [Ked08, Lemma 1.6.3], we can see that $D_{i,\sigma}$ is a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,\sigma}E$ which is pure of slope $ff''s_i$ for any σ and $i=1,2$. So $D_{1,\sigma}\otimes_{B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,\sigma}E}D_{2,\sigma}$ is a $\varphi^{ff''}$-module over $B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,\sigma}E$ which is pure of slope $ff''(s_1+s_2)$ by [Ked08, Corollary 1.6.4]. Because $D_1\otimes D_2\xrightarrow{\sim}\oplus_{\sigma:E'_0\hookrightarrow E}D_{1,\sigma}\otimes_{B^{\dagger}_{\mathrm{rig},K}\otimes_{E'_0,\sigma}E}D_{2,\sigma}$, we can show that $D_1\otimes D_2$ is a φ-module over $B^{\dagger}_{\mathrm{rig},K}$ which is pure of slope s_1+s_2 in the same way as the proof of Lemma 1.33 and by using [Ked08, Lemma 1.6.3]. □

1.3 Equivalence between B-pairs and (φ,Γ_K)-modules

In this section, we recall a result of Berger on the equivalence between the category of B-pairs and the category of (φ,Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$. First we recall the construction of a functor from (φ,Γ_K)-modules to B-pairs [Ber08b, § 2.2]. Let D be a (φ,Γ_K)-module of rank d over $B^{\dagger}_{\mathrm{rig},K}$. Then Berger showed that $W_e(D):=(\widetilde{B}^{\dagger}_{\mathrm{rig}}[1/t]\otimes_{B^{\dagger}_{\mathrm{rig},K}}D)^{\varphi=1}$ is a free B_e-module of rank d (see [Ber08b, Proposition 2.2.6]). It is equipped with a continuous semi-linear G_K-action. On the other hand, for sufficiently large $r_0>0$, we can take unique Γ_K-stable finite free $B^{\dagger,r}_{\mathrm{rig},K}$-submodule $D^r\subset D$ such that $B^{\dagger}_{\mathrm{rig},K}\otimes_{B^{\dagger,r}_{\mathrm{rig},K}}D^r=D$ and $id_{B^{\dagger,pr}_{\mathrm{rig},K}}\otimes\varphi_D:B^{\dagger,pr}_{\mathrm{rig},K}\otimes_{\varphi,B^{\dagger,r}_{\mathrm{rig},K}}D^r\xrightarrow{\sim}D^{pr}$ for any $r\geq r_0$ [Ber08a, Theorem 1.3.3]. Then Berger showed that the continuous G_K-module $W^{+}_{\mathrm{dR}}(D):=B^{+}_{\mathrm{dR}}\otimes_{i_n,B^{\dagger,(p-1)p^{n-1}}_{\mathrm{rig},K}}D^{(p-1)p^{n-1}}$ over B^{+}_{dR} is independent of any n such that $(p-1)p^{n-1}\geq r_0$ and

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showed that there is a canonical G_K-isomorphism $B_{\mathrm{dR}} \otimes_{B_e} W_e(D) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}(D)$ (see [Ber08b, Proposition 2.2.6]). We put $W(D) := (W_e(D), W^+_{\mathrm{dR}}(D))$. This is a B-pair of rank d. This defines a functor from the category of (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ to the category of B-pairs of G_K.

Remark 1.35. We can also define an inverse functor from the category of B-pairs to the category of (φ, Γ_K)-modules [Ber08b, § 2.2], but the definition of this is very difficult. In this paper, we do not need this construction, so we omit the definition of this functor. We denote this inverse functor by $W \mapsto D(W)$.

Theorem 1.36. *The functor $D \mapsto W(D)$ is an exact functor and this gives an equivalence of categories between the category of E–(φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$ and the category of E–B-pairs of G_K.*

Proof. For the $\mathbb{Q}_p$-coefficient case, this result was proved by Berger [Ber08b, Theorem 2.2.7]. Let D be an E–(φ, Γ_K)-module over $B^{\dagger}_{\mathrm{rig},K}$, $a \in E$. Then the multiplication by a gives an endomorphism of D as $\mathbb{Q}_p$-(φ, Γ_K)-modules. From the functoriality, these multiplications give an E-action on $W(D)$. So $W(D)$ is an E–B-pair. For any E–B-pair W, we can define in a similar way the E-action on $D(W)$. So we obtain the desired equivalence. □

If we restrict the functor $D \mapsto W(D)$ to étale E–(φ, Γ_K)-modules, we obtain the following theorem.

Theorem 1.37. *The functor $D \mapsto W(D)$ gives an equivalence of categories between the category of étale E–(φ, Γ_K)-modules and the category of E–B-pairs of the form $W(V)$ for some E-representation V.*

Proof. See [Ber08b, Proposition 2.2.9]. □

Definition 1.38. Let W be an E–B-pair and $s \in \mathbb{Q}$. Then we say that W is pure of slope s if $D(W)$ is pure of slope s.

Remark 1.39. In fact, we can define B-pairs with pure slope directly without using (φ, Γ_K)-modules, but we omit this definition in this article (see [Ber08b, § 3.2]).

The next theorem is the E–B-pair version of slope filtration theorem.

Theorem 1.40. *Let W be an E–B-pair. Then there exists unique filtration $0 = W_0 \subset W_1 \subset \cdots \subset W_l = W$ by sub-E–B-pairs of W such that W_i is saturated in W_{i+1} and the quotient W_{i+1}/W_i is pure of slope s_i for any i with $s_1 < s_2 < \cdots < s_l$.*

Proof. This follows from Theorems 1.32 and 1.36. □

Lemma 1.41. *Let W be a rank-one E–B-pair. Then W is pure and its slope is contained in $(1/fe_E)\mathbb{Z}$.*

Proof. This follows from Lemma 1.33. □

1.4 Classification of rank-one *E*–*B*-pairs

In this section, we classify rank-one E–B-pairs. By Lemma 1.41, any rank one E–B-pairs are pure and their slopes are contained in $(1/fe_E)\mathbb{Z}$. We fix a prime element π_E of E. First we construct a special rank-one E–B-pair W_0 which is pure of slope $1/fe_E$. To construct this, first we define a rank-one E-filtered φ-module D_0 over K, which corresponds to W_0 by D^K_{cris}. (Here, for any E-filtered (φ, N)-module D, we can define the rank of D by $\mathrm{rank}(D) := \mathrm{rank}_{K_0\otimes_{\mathbb{Q}_p}E}(D)$ because we can show that D is a free $K_0 \otimes_{\mathbb{Q}_p} E$-module in the same way as in the case of E-(φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$.) Let $D_0 := K_0 \otimes_{\mathbb{Q}_p} Ee \xrightarrow{\sim} \oplus_{\sigma:K_0\hookrightarrow E} Ee_\sigma$ be a free rank-one $K_0 \otimes_{\mathbb{Q}_p} E$-module. We define a filtered φ-module structure on D_0 as follows. Define a $K_0 \otimes_{\mathbb{Q}_p} E$-semi-linear action of φ on D_0 by $\varphi(e_{id}) := e_{\varphi^{-1}}$, $\varphi(e_{\varphi^{-1}}) := e_{\varphi^{-2}}, \ldots, \varphi(e_{\varphi^{-(f-2)}}) := e_{\varphi^{-(f-1)}}$, $\varphi(e_{\varphi^{-(f-1)}}) := \pi_E e_{id}$, i.e. a φ-module such that $\varphi^f = \pi_E$. We define a decreasing filtration on $D_{0,K} := K \otimes_{K_0} D_0$ by $\mathrm{Fil}^0 D_{0,K} := D_{0,K}$, $\mathrm{Fil}^1 D_{0,K} := 0$. Then D_0 is a filtered φ-module over K with a natural E-action. For any $i \geq 0$, then $D_0^{\otimes i}$ is the rank-one E-filtered φ-module such that $\varphi^f = \pi_E^i$ and $\mathrm{Fil}^0 D^{\otimes i}_{0,K} = D^{\otimes i}_{0,K}$ and $\mathrm{Fil}^1 D^{\otimes i}_{0,K} = 0$. If we put $D_0^{\otimes -1} := D_0^\vee$ the dual of D_0 defined in the same way as in the case of E–B-pairs, then we can see that $D_0^{\otimes -1}$ is an E-filtered φ-module such that $\varphi^f = \pi_E^{-1}$ and $\mathrm{Fil}^0 D^{\otimes -1}_{0,K} = D^{\otimes -1}_{0,K}$ and $\mathrm{Fil}^1 D^{\otimes -1}_{0,K} = 0$. Moreover, if we define the tensor products of E-filtered (φ, N)-modules in the same way as in the case of E-(φ, Γ_K)-modules, then we can see that $D_0^{\otimes -1} \otimes D_0$ is the trivial E-filtered φ-module. So $D_0^{\otimes i}$ is well defined for any $i \in \mathbb{Z}$ and satisfies $D_0^{\otimes i} \otimes D_0^{\otimes j} \xrightarrow{\sim} D_0^{\otimes i+j}$ for any $i, j \in \mathbb{Z}$.

By the 'weakly admissible implies admissible' theorem for B-pairs (Theorem 1.18(3)), $W_0 := W(D_0) = (W_e(D_0), W^+_{\mathrm{dR}}(D_0))$ is the rank-one crystalline E–B-pair such that $D^K_{\mathrm{cris}}(W_0) \xrightarrow{\sim} D_0$.

LEMMA 1.42. *We have that W_0 is pure of slope $1/fe_E$.*

Proof. Because W_0 is rank one, W_0 is pure by Lemma 1.41. So it suffices to show that the slope of W_0 is equal to $1/fe_E$. In [Ber08a, § 2.2], Berger defined a functor $D \mapsto \mathcal{M}(D)$ from the category of filtered (φ, N, G_K)-modules to the category of (locally trivial) (φ, Γ_K)-modules over $B^{\dagger}_{\mathrm{rig},K}$. In [Ber08b, Proposition 2.3.4], he showed that $\mathcal{M}(D^L_{\mathrm{st}}(W)) = D(W)$ for any potentially semi-stable B-pair W and for sufficient large L. By this compatibility, we have $D(W_0) \xrightarrow{\sim} \mathcal{M}(D_0)$. On the other hand, for a rank-one filtered (φ, N)-module D, the slope of $\mathcal{M}(D)$ is equal to $t_N(D) - t_H(D)$ by [Ber08a, Theorem 4.2.1]. Here, for a rank-one filtered (φ, N)-module $D := K_0 e$, we define $t_N(D) := \mathrm{val}_p(\varphi(e)/e)$ and define $t_H(D)$ as a unique integer k such that $\mathrm{Fil}^k(D_K)/\mathrm{Fil}^{k+1}(D_K) \neq 0$. Because $\mu(\wedge^i M) = i\mu(M)$ for a φ-module M over a Robba ring, we have $\mu(D(W_0)) = \mu(\mathcal{M}(D_0)) = (1/[E : \mathbb{Q}_p])\mu(\wedge^{[E:\mathbb{Q}_p]}\mathcal{M}(D_0)) = (1/[E : \mathbb{Q}_p])(t_N(\wedge^{[E:\mathbb{Q}_p]}D_0) - t_H(\wedge^{[E:\mathbb{Q}_p]}D_0))$. By the definition of D_0, it is easy to see that $t_N(\wedge^{[E:\mathbb{Q}_p]}D_0) = [E : K_0]/e_E$, $t_H(\wedge^{[E:\mathbb{Q}_p]}D_0) = 0$. So we obtain $\mu(\mathcal{M}(D_0)) = [E : K_0]/[E : \mathbb{Q}_p]e_E = 1/fe_E$. So the slope of W_0 is $1/fe_E$ by the definition of the slopes of E–B-pairs. □

By using W_0, we can classify all of the rank-one E–B-pairs as follows. Let $\delta : K^\times \to E^\times$ be a continuous character where K and E are equipped with p-adic topology. We put $\delta(\pi_K) := u\pi_E^i$ for $u \in \mathcal{O}_E^\times$ and $i \in \mathbb{Z}$. We put $\delta_0 : K^\times \to \mathcal{O}_E^\times$ the unitary continuous character such that $\delta_0|_{\mathcal{O}_K^\times} = \delta|_{\mathcal{O}_K^\times}$ and $\delta_0(\pi_K) := u$. Then by local class field theory, this extends uniquely to a continuous character $\tilde{\delta}_0 : G_K \to \mathcal{O}_E^\times$ such that $\tilde{\delta}_0 \circ \mathrm{rec}_K = \delta_0$, here $\mathrm{rec}_K : K^\times \to G_K^{\mathrm{ab}}$ is the reciprocity map as in the notation section. Then we put $W(\delta) := W(E(\tilde{\delta}_0)) \otimes W_0^{\otimes i}$. By the definitions of tensor products and duals, we can see that these are compatible with tensor products and with duals, i.e. $W(\delta_1) \otimes W(\delta_2) \xrightarrow{\sim} W(\delta_1\delta_2)$ and $W(\delta_1)^\vee \xrightarrow{\sim} W(\delta_1^{-1})$ for any characters $\delta_1, \delta_2 : K^\times \to E^\times$. In particular, we have $W_0^{\otimes i} \xrightarrow{\sim} W(D_0^{\otimes i})$ for any $i \in \mathbb{Z}$, so $W_0^{\otimes i}$ is pure of slope i/fe_E by Lemmas 1.34 and 1.42 for any $i \in \mathbb{Z}$. Then we have the following lemma.

LEMMA 1.43. *Let $\delta : K^\times \to E^\times$ be a continuous character. Then the E–B-pair $W(\delta)$ does not depend on the choice of a uniformizer π_E of E and on the choice of a uniformizer π_K of K.*

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Proof. Because δ_0 does not depend on the choice of a uniformizer π_K of K, then $W(\delta)$ does not depend on π_K. Let $\pi'_E = v\pi_E$ be another prime of E ($v \in \mathcal{O}_E^{\times}$). Let D'_0 be a rank-one E-filtered φ-module over K_0 defined by replacing π_E with π'_E in the definition of D_0. Let $W'_0 := W(D'_0)$ be the corresponding E–B-pair. If we write $\delta(\pi_K) = u\pi_E^i = uv^{-i}\pi'^i_E$, then $\delta_0|_{\mathcal{O}_K^{\times}} = \delta'_0|_{\mathcal{O}_K^{\times}} = \delta|_{\mathcal{O}_K^{\times}}$ and $\delta_0(\pi_K) = u$ and $\delta'_0(\pi_K) = uv^{-i}$. Then it suffices to show that $W(E(\tilde{\delta}_0)) \otimes W_0^{\otimes i} \xrightarrow{\sim} W(E(\tilde{\delta}'_0)) \otimes W_0'^{\otimes i}$. Because $\tilde{\delta}_0/\tilde{\delta}'_0$ is a unramified character, it suffices to show that $D^K_{\mathrm{cris}}(E(\tilde{\delta}_0/\tilde{\delta}'_0)) \xrightarrow{\sim} D_0'^{\otimes i} \otimes D_0^{\otimes -i}$ as E-filtered φ-modules. By calculation, $D^K_{\mathrm{cris}}(E(\tilde{\delta}_0/\tilde{\delta}'_0)) = K_0 \otimes_{\mathbb{Q}_p} Ee$ on which φ acts by $\varphi^f(e) = \tilde{\delta}_0/\tilde{\delta}'_0(\pi_K)e = (u/uv^{-i})e = v^i e$. By definition, φ acts on $D_0'^{\otimes i} \otimes D_0^{\otimes -i} = K_0 \otimes_{\mathbb{Q}_p} Ee'$ by $\varphi^f(e') = (\pi'_E/\pi_E)^i e' = v^i e'$. So these are isomorphic as a φ-module. Concerning the filtrations, $\mathrm{Fil}^0 D^K_{\mathrm{dR}}(E(\tilde{\delta}_0/\tilde{\delta}'_0)) = D^K_{\mathrm{dR}}(E(\tilde{\delta}_0/\tilde{\delta}'_0))$ and $\mathrm{Fil}^1 D^K_{\mathrm{dR}}(E(\tilde{\delta}_0/\tilde{\delta}'_0)) = 0$ because $\tilde{\delta}_0/\tilde{\delta}'_0$ is a unramified character. On the other hand, $\mathrm{Fil}^0 D_{0,K}'^{\otimes i} \otimes D_{0,K}^{\otimes -i} = D_{0,K}'^{\otimes i} \otimes D_{0,K}^{\otimes -i}$ and $\mathrm{Fil}^1 D_{0,K}'^{\otimes i} \otimes D_{0,K}^{\otimes -i} = 0$ by the definition of D_0 and D'_0. So, these are isomorphic as E-filtered φ-modules. □

Remark 1.44. We can also show in the same way that $W(\delta)$ defined here is isomorphic to $W(\delta)$ defined before Theorem 0.1 in the introduction.

The classification theorem of rank-one E–B-pairs is as follows.

Theorem 1.45. *Let W be a rank-one E–B-pair. Then there exists unique continuous character $\delta : K^{\times} \to E^{\times}$ such that $W \xrightarrow{\sim} W(\delta)$.*

Proof. Let W be a rank-one E–B-pair. Then, by Lemma 1.41, W is pure of slope i/fe_E for unique $i \in \mathbb{Z}$. Because W_0 is pure of slope $1/fe_E$ by Lemma 1.42, $W \otimes W_0^{\otimes -i}$ is pure of slope zero by Lemma 1.34. So, by Theorem 1.37, there exists unique continuous character $\tilde{\delta}_0 : G_K \to \mathcal{O}_E^{\times}$ such that $W \otimes W_0^{\otimes -i} \xrightarrow{\sim} W(E(\tilde{\delta}_0))$. So $W \xrightarrow{\sim} W_0^{\otimes i} \otimes W(E(\tilde{\delta}_0))$. If we define a continuous character $\delta : K^{\times} \to E^{\times}$ such that $\delta(\pi_K) := \tilde{\delta}_0 \circ \mathrm{rec}_K(\pi_K)\pi_E^i$, $\delta|_{\mathcal{O}_K^{\times}} = \tilde{\delta}_0 \circ \mathrm{rec}_K|_{\mathcal{O}_K^{\times}}$, then we have $W \xrightarrow{\sim} W(\delta)$ by the definition of $W(\delta)$. The uniqueness of δ follows from uniqueness of i and $\tilde{\delta}_0$ above. □

Next we recall some facts about Sen's theory for B-pairs to define Hodge–Tate weight of B-pairs. Let U be a $\mathbb{C}_p$-representation of G_K, i.e. U is a finite-dimensional $\mathbb{C}_p$-vector space equipped with a continuous semi-linear G_K-action. Then the union $U^{H_K}_{\mathrm{fini}}$ of finite-dimensional sub-K_{∞}-vector spaces of U^{H_K} which are stable by Γ_K is the largest subspace with this property and satisfies $\mathbb{C}_p \otimes_{K_{\infty}} U^{H_K}_{\mathrm{fini}} \xrightarrow{\sim} U$ (cf. [Ber08b, § 2.3] and the references therein). Then $U^{H_K}_{\mathrm{fini}}$ is equipped with a K_{∞}-linear operator $\nabla_U := \log(\gamma)/\log(\chi(\gamma))$ for any $\gamma \in \Gamma_K$ which is sufficiently close to 1.

Definition 1.46. Let W be a B-pair, then $W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}$ is a $\mathbb{C}_p$-representation of G_K. We put $D_{\mathrm{Sen}}(W) := (W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}})^{H_K}_{\mathrm{fini}}$ and put $\Theta_{\mathrm{Sen},W} := \nabla_{W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}}$. Then we say that W is Hodge–Tate if $\Theta_{\mathrm{Sen},W}$ is diagonalizable and all of the eigenvalues are contained in $\mathbb{Z}$. Then we call these eigenvalues the Hodge–Tate weights of W.

By using this definition, we calculate the Hodge–Tate weights of a rank-one E–B-pair. Let $W(\delta)$ be a rank-one E–B-pair for some continuous character $\delta : K^{\times} \to E^{\times}$. If we put $W(\delta) \xrightarrow{\sim} W(E(\tilde{\delta}_0)) \otimes W_0^{\otimes i}$ as above, then we have $W(\delta)^+_{\mathrm{dR}} \xrightarrow{\sim} B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E(\tilde{\delta}_0)$ because $W^+_{0,\mathrm{dR}} \xrightarrow{\sim} B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ by the definition of W_0. So $W^+_{\mathrm{dR}}(\delta)$ comes from an E-representation $E(\tilde{\delta}_0)$. In particular, $W^+_{\mathrm{dR}}(\delta)/tW^+_{\mathrm{dR}}(\delta) \xrightarrow{\sim} \mathbb{C}_p \otimes_{\mathbb{Q}_p} E(\tilde{\delta}_0)$ also comes from $E(\tilde{\delta}_0)$. Then, by Proposition 4.1.2 and the remark following it in [BC08], $D_{\mathrm{Sen}}(W(\delta))$ is a free $K_{\infty} \otimes_{\mathbb{Q}_p} E$-module of rank one and the operator $\Theta_{\mathrm{Sen},W(\delta)}$ acts by multiplication by $w(\delta)$ on $D_{\mathrm{Sen}}(W(\delta))$ for some $w(\delta) \in K \otimes_{\mathbb{Q}_p} E$. If

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we decompose $w(\delta)$ into σ-components $K \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \oplus_{\sigma: K \hookrightarrow E} E e_\sigma : w(\delta) \mapsto (w(\delta)_\sigma e_\sigma)$. Then $W(\delta)$ is Hodge–Tate B-pair if and only if $w(\delta)_\sigma$ is contained in $\mathbb{Z}$ for any σ.

DEFINITION 1.47. In the above situation, we say that the set of numbers $\{w(\delta)_\sigma\}_\sigma$ is the generalized Hodge–Tate weight of $W(\delta)$.

2. Calculations of cohomologies of E–B-pairs

For classifying two-dimensional split trianguline E-representations, we need to calculate the extension groups $\mathrm{Ext}^1(W_2, W_1)$ of E–B-pairs of W_1 by W_2 for rank-one E–B-pairs W_1, W_2. In this section, we define Galois cohomology $\mathrm{H}^i(G_K, W)$ for any E–B-pair W and interpret $\mathrm{H}^1(G_K, W)$ as the extension group $\mathrm{Ext}^1(B_E, W)$. (Here $B_E := (B_e \otimes_{\mathbb{Q}_p} E, B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)$ is the trivial E–B-pair.) Next we review Liu's results [Liu08] which are generalizations of Tate duality and Euler–Poincaré characteristic formula to the case of (φ, Γ_K)-modules over $B^\dagger_{\mathrm{rig},K}$. Finally, we calculate some Galois cohomologies of E–B-pairs which we need for the classification of two-dimensional split trianguline E-representations.

2.1 Definition of Galois cohomology of E–B-pairs

Let $W := (W_e, W^+_{\mathrm{dR}})$ be an E–B-pair. We define the complex of G_K-modules $C^\bullet(W)$ by $\delta_0 : C^0(W) := W_e \oplus W^+_{\mathrm{dR}} \to C^1(W) := W_{\mathrm{dR}} : (x, y) \mapsto x - y$, $C^i(W) = 0$ for $i \neq 0, 1$.

DEFINITION 2.1. Let $W := (W_e, W^+_{\mathrm{dR}})$ be an E–B-pair. Then we define the Galois cohomology of W by $\mathrm{H}^i(G_K, W) := \mathrm{H}^i(G_K, C^\bullet(W))$. Here the right-hand side is the usual Galois cohomology of a complex of continuous G_K-modules which are computed by using continuous cochain complexes.

By definition, we have the following long exact sequence

$$\cdots \to \mathrm{H}^i(G_K, W) \to \mathrm{H}^i(G_K, W_e) \oplus \mathrm{H}^i(G_K, W^+_{\mathrm{dR}}) \to \mathrm{H}^i(G_K, W_{\mathrm{dR}}) \to \mathrm{H}^{i+1}(G_K, W) \to \cdots .$$

From this, we obtain the following results.

(1) We have $\mathrm{H}^0(G_K, W) = (W_e \cap W^+_{\mathrm{dR}})^{G_K}$.

(2) There exists the exact sequence of E-vector spaces

$$0 \to W^{G_K}_{\mathrm{dR}}/(W^{G_K}_e + W^{+G_K}_{\mathrm{dR}}) \to \mathrm{H}^1(G_K, W) \\ \to \mathrm{Ker}(\mathrm{H}^1(G_K, W_e) \oplus \mathrm{H}^1(G_K, W^+_{\mathrm{dR}}) \to \mathrm{H}^1(G_K, W_{\mathrm{dR}})) \to 0.$$

Next, for any E–B-pair, we construct a canonical isomorphism

$$\mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(B_E, W),$$

where the right-hand side is the extension group in the category of E–B-pairs. For any continuous G_K-module V, we functorially define the diagram $\mathrm{C}^0(V) \underset{\delta_0}{\longrightarrow} \mathrm{C}^1(V) \underset{\delta_1}{\longrightarrow} \mathrm{C}^2(V)$ as follows:

(1) $\mathrm{C}^0(V) := V$,
$\mathrm{C}^i(V) := \{f : G_K^{\times i} \to V \text{ a continuous function}\}$ for $i = 1, 2$.

(2) $\delta_0 : \mathrm{C}^0(V) \to \mathrm{C}^1(V) : x \mapsto (g \mapsto gx - x)$,
$\delta_1 : \mathrm{C}^1(V) \to \mathrm{C}^2(V) : f \mapsto ((g_1, g_2) \mapsto f(g_1 g_2) - f(g_1) - g_1 f(g_2))$.

Then we have $\mathrm{H}^0(G_K, V) \xrightarrow{\sim} \mathrm{Ker}(\delta_0)$, $\mathrm{H}^1(G_K, V) \xrightarrow{\sim} \mathrm{Ker}(\delta_1)/\mathrm{Im}(\delta_0)$. So, for any E–B-pair W, we have $\mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ker}(\tilde{\delta}_1)/\mathrm{Im}(\tilde{\delta}_0)$, here $\tilde{\delta}_0$, $\tilde{\delta}_1$ are defined by

$$\tilde{\delta}_0 : C^0(W_e) \oplus C^0(W_{\mathrm{dR}}^+) \to C^1(W_e) \oplus C^1(W_{\mathrm{dR}}^+) \oplus C^0(W_{\mathrm{dR}}) : (x, y) \mapsto (\delta_0(x), \delta_0(y), x - y),$$
$$\tilde{\delta}_1 : C^1(W_e) \oplus C^1(W_{\mathrm{dR}}^+) \oplus C^0(W_{\mathrm{dR}}) \to C^2(W_e) \oplus C^2(W_{\mathrm{dR}}^+) \oplus C^1(W_{\mathrm{dR}}) : (f_1, f_2, x) \mapsto$$
$$(\delta_1(f_1), \delta_1(f_2), f_1 - f_2 - \delta_0(x)).$$

By using this expression, we define a map $\mathrm{H}^1(G_K, W) \to \mathrm{Ext}^1(B_E, W)$ as follows. Let $(f_1, f_2, \alpha) \in \mathrm{Ker}(\tilde{\delta}_1)$. Then we define an E–B-pair $X := (X_e, X_{\mathrm{dR}}^+, \iota)$ as follows:

(1) $X_e := W_e \oplus (B_e \otimes_{\mathbb{Q}_p} E)e_{\mathrm{cris}}$ on which G_K acts by $g(x, ae_{\mathrm{cris}}) := (gx + gaf_1(g), gae_{\mathrm{cris}})$ for any $x \in W_e$, $a \in B_e \otimes_{\mathbb{Q}_p} E$ and $g \in G_K$;

(2) $X_{\mathrm{dR}}^+ := W_{\mathrm{dR}}^+ \oplus (B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E)e_{\mathrm{dR}}$ on which G_K acts by $g(y, be_{\mathrm{dR}}) := (gy + gbf_2(g), gbe_{\mathrm{dR}})$ for any $y \in W_{\mathrm{dR}}^+$, $b \in B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E$ and $g \in G_K$;

(3) $\iota : B_{\mathrm{dR}} \otimes_{B_e} X_e \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} X_{\mathrm{dR}}^+ : (x, ae_{\mathrm{cris}}) \mapsto (\iota_W(x) + a\alpha, ae_{\mathrm{dR}})$ for any $x \in B_{\mathrm{dR}} \otimes_{B_e} W_e$ and $a \in B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$; here, $\iota_W : B_{\mathrm{dR}} \otimes_{B_e} W_e \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} W_{\mathrm{dR}}^+$ is the given isomorphism in the definition of B-pair $W = (W_e, W_{\mathrm{dR}}^+, \iota_W)$.

(Here we see an E–B-pair W as a triple $W := (W_e, W_{\mathrm{dR}}^+, \iota_W)$ such that W_e (respectively W_{dR}^+) is finite free over $B_e \otimes_{\mathbb{Q}_p} E$ (respectively $B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E$) with continuous semi-linear G_K-actions and $\iota_W : B_{\mathrm{dR}} \otimes_{B_e} W_e \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B_{\mathrm{dR}}^+} W_{\mathrm{dR}}^+$ is a G_K-equivariant $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$-semi-linear isomorphism. Then this definition is equivalent to Definition 1.2.) Because $(f_1, f_2, \alpha) \in \mathrm{Ker}(\tilde{\delta}_1)$, we can easily see that X is a well-defined E–B-pair which sits in the following short exact sequence

$$0 \to W \to X \to B_E \to 0.$$

So we can see the isomorphism class $[X]$ of the extension X as an element in $\mathrm{Ext}^1(B_E, W)$. By construction, we can easily see that this defines an E-linear morphism $\mathrm{Ker}(\tilde{\delta}_1) \to \mathrm{Ext}^1(B_E, W)$. Then, by a standard argument, we see that this morphism factors through $\mathrm{Ker}(\tilde{\delta}_1)/\mathrm{Im}(\tilde{\delta}_0) \to \mathrm{Ext}^1(B_E, W)$ and that this is, in fact, E-linear isomorphism. So we obtain the following proposition.

Proposition 2.2. *Let W be an E–B-pair. Then we have the following functorial E-linear isomorphisms:*

(1) $\mathrm{H}^0(G_K, W) \xrightarrow{\sim} (W_e \cap W_{\mathrm{dR}}^+)^{G_K} \xrightarrow{\sim} \mathrm{Hom}(B_E, W)$, *here* $\mathrm{Hom}(W_1, W_2)$ *is the E-vector space of morphisms in the category of E–B-pairs from W_1 to W_2 for any E–B-pairs W_1, W_2;*

(2) $\mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(B_E, W)$.

Proof. We have already proved part (2). For part (1), it is easy to see that $(W_e \cap W_{\mathrm{dR}}^+)^{G_K} = \mathrm{Hom}(B_{\mathbb{Q}_p}, W)$ for any $\mathbb{Q}_p$-B-pair W. Because $\mathrm{Hom}(B_{\mathbb{Q}_p}, W) = \mathrm{Hom}(B_E, W)$ for any E–B-pair W (here on the left-hand side, we see W as a $\mathbb{Q}_p$–B-pair), so we obtain part (1). □

Remark 2.3. In the case of $W = W(V)$ for an E-representation V of G_K, we have $\mathrm{H}^i(G_K, V) \xrightarrow{\sim} \mathrm{H}^i(G_K, W(V))$: indeed, by the fundamental short exact sequence $0 \to \mathbb{Q}_p \to B_e \oplus B_{\mathrm{dR}}^+ \to B_{\mathrm{dR}} \to 0$, we have a quasi-isomorphism $V[0] \xrightarrow{\sim} C^\bullet(W(V))$.

In the application to the classification of two-dimensional potentially semi-stable split trianguline E-representations, we need to know when an extension is crystalline, semi-stable or de Rham E–B-pair. So, as in the case of usual Galois cohomology of p-adic representations, we define Bloch–Kato's cohomologies $\mathrm{H}^1_e(G_K, W)$, $\mathrm{H}^1_f(G_K, W)$, and $\mathrm{H}^1_g(G_K, W)$ (see [BK90, 3]) as follows.

DEFINITION 2.4. Let $W := (W_e, W^+_{\mathrm{dR}})$ be an E–B-pair. Then we define:

- $\mathrm{H}^1_e(G_K, W) := \mathrm{Ker}(\mathrm{H}^1(G_K, W) \to \mathrm{H}^1(G_K, W_e))$;
- $\mathrm{H}^1_f(G_K, W) := \mathrm{Ker}(\mathrm{H}^1(G_K, W) \to \mathrm{H}^1(G_K, B_{\mathrm{cris}} \otimes_{B_e} W_e))$;
- $\mathrm{H}^1_g(G_K, W) := \mathrm{Ker}(\mathrm{H}^1(G_K, W) \to \mathrm{H}^1(G_K, W_{\mathrm{dR}}))$.

Here the above maps are induced by the natural maps $C^\bullet(W) \to W_e \to B_{\mathrm{cris}} \otimes_{B_e} W_e \to B_{\mathrm{dR}} \otimes_{B_e} W_e \xrightarrow{\sim} W_{\mathrm{dR}}$, so we have natural injections $\mathrm{H}^1_e(G_K, W) \subseteq \mathrm{H}^1_f(G_K, W) \subseteq \mathrm{H}^1_g(G_K, W)$.

Remark 2.5. If W is a crystalline (respectively de Rham) E–B-pair, then $[X] \in \mathrm{Ext}^1(B_E, W) \xrightarrow{\sim} \mathrm{H}^1(G_K, W)$ is in $\mathrm{H}^1_f(G_K, W)$ (respectively in $\mathrm{H}^1_g(G_K, W)$) if and only if X is a crystalline (respectively de Rham) E–B-pair.

As in the case of usual p-adic representations, we have a dimension formula of $\mathrm{H}^1_f(G_K, W)$ and $\mathrm{H}^1_e(G_K, W)$. Before stating this, we prove the following lemma.

LEMMA 2.6. *Let W be a de Rham E–B-pair. Then the canonical map $\mathrm{H}^1(G_K, W^+_{\mathrm{dR}}) \to \mathrm{H}^1(G_K, W_{\mathrm{dR}})$ is injective.*

Proof. The proof is the same as that in the case of p-adic representation [BK90, Lemma 3.8.1], but we give the proof for the convenience of readers. Consider the following short exact sequence

$$0 \to W^{+G_K}_{\mathrm{dR}} \to W^{G_K}_{\mathrm{dR}} \to (W_{\mathrm{dR}}/W^+_{\mathrm{dR}})^{G_K}.$$

From this we have

$$\begin{aligned}\dim_E(D_{\mathrm{dR}}(W)) &\leqq \dim_E(W^{+G_K}_{\mathrm{dR}}) + \dim_E((W_{\mathrm{dR}}/W^+_{\mathrm{dR}})^{G_K}) \\ &\leq \sum_{i\in\mathbb{Z}} \dim_E(\mathbb{C}_p(i) \otimes_{\mathbb{C}_p} (W^+_{\mathrm{dR}}/tW^+_{\mathrm{dR}}))^{G_K} \\ &= \mathrm{rank}(W) = \dim_E(D_{\mathrm{dR}}(W)).\end{aligned}$$

Here we use the fact that the de Rham B-pair is Hodge–Tate and W is de Rham. So the map $W^{G_K}_{\mathrm{dR}} \to (W_{\mathrm{dR}}/W^+_{\mathrm{dR}})^{G_K}$ is surjective. Hence, the natural map $\mathrm{H}^1(G_K, W^+_{\mathrm{dR}}) \to \mathrm{H}^1(G_K, W_{\mathrm{dR}})$ is injective. □

The dimension formulas are as follows.

PROPOSITION 2.7. *Let W be a de Rham E–B-pair. Then we have*

$$\dim_E(\mathrm{H}^1_f(G_K, W)) = \dim_E(D^K_{\mathrm{dR}}(W)/\mathrm{Fil}^0 D^K_{\mathrm{dR}}(W)) + \dim_E(\mathrm{H}^0(G_K, W)),$$
$$\dim_E(\mathrm{H}^1_f(G_K, W)/\mathrm{H}^1_e(G_K, W)) = \dim_E(D^K_{\mathrm{cris}}(W)/(1-\varphi)D^K_{\mathrm{cris}}(W)).$$

Proof. This proof is essentially the same as that of [BK90, Corollary 3.8.4]. First we consider the following short exact sequence

$$0 \to B_e \to B_{\mathrm{cris}} \xrightarrow{1-\varphi} B_{\mathrm{cris}} \to 0.$$

Tensoring by W_e over B_e, we can easily see that $C^\bullet(W)$ is naturally quasi-isomorphic to $C'^\bullet(W) := (W_{\mathrm{cris}} \oplus W^+_{\mathrm{dR}} \to W_{\mathrm{cris}} \oplus W_{\mathrm{dR}} : (x, y) \mapsto ((1-\varphi)x, x-y))$ (here we put $W_{\mathrm{cris}} := B_{\mathrm{cris}} \otimes_{B_e} W_e$). So, by the definition of H^1_f and by the above lemma, we have the following two

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exact sequences

$$0 \to \mathrm{H}^0(G_K, W) \to D^K_{\mathrm{cris}}(W) \oplus \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W) \to D^K_{\mathrm{cris}}(W) \oplus D^K_{\mathrm{dR}}(W) \to \mathrm{H}^1_f(G_K, W) \to 0,$$
$$0 \to \mathrm{H}^0(G_K, W) \to D^K_{\mathrm{cris}}(W)^{\varphi=1} \oplus \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W) \to D^K_{\mathrm{dR}}(W) \to \mathrm{H}^1_e(G_K, W) \to 0.$$

From these, we obtain the desired formulas. □

2.2 Euler–Poincaré characteristic formula and Tate local duality for B-pairs

In this section, we review Liu's results generalizing some fundamental results of Tate on Galois cohomology of p-adic representations, following [Ked07] and [Liu08]. Liu constructed a category of B-quotients, which is a minimal abelian category in which the category of B-pairs is contained as a full subcategory. Then he defined the Galois cohomology of B-quotients as the universal δ-functor of $\mathrm{H}^0_{\mathrm{Liu}}(W) := \mathrm{Hom}(B_{\mathbb{Q}_p}, W)$ for any B-quotient W (here Hom is the set of morphisms of B-quotients). In this paper, we denote this cohomology by $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W)$ for a B-quotient W. Then it is shown in [Ked07] that there exists the isomorphism $H^1_{\mathrm{Liu}}(G_K, W) \xrightarrow{\sim} \mathrm{Ext}^1(B_{\mathbb{Q}_p}, W)$ for any B-pair. So, for any B-pair W, we have isomorphisms $\mathrm{H}^1(G_K, W) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Liu}}(G_K, W)$ and $\mathrm{H}^0(G_K, W) \xrightarrow{\sim} \mathrm{H}^0_{\mathrm{Liu}}(G_K, W)$. For any $\mathbb{Q}_p$-representation V of G_K, it is shown in [Ked07] that there is a functorial isomorphism $\mathrm{H}^i(G_K, V) \xrightarrow{\sim} \mathrm{H}^i_{\mathrm{Liu}}(G_K, W(V))$ for any $i \in \mathbb{N}$. In this paper, we review the results only for B-pairs.

Theorem 2.8. *Let W be a B-pair. Then:*

(1) *for $i = 0, 1, 2$, $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W)$ is finite dimensional over $\mathbb{Q}_p$ and $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W) = 0$ for $i \neq 0, 1, 2$;*

(2) *we have $\sum_{i=0}^{2}(-1)^i \dim_{\mathbb{Q}_p}\mathrm{H}^i_{\mathrm{Liu}}(G_K, W) = -[K : \mathbb{Q}_p]\mathrm{rank}_{B_e}(W_e)$;*

(3) *there is a natural perfect pairing*

$$\mathrm{H}^i_{\mathrm{Liu}}(G_K, W) \times \mathrm{H}^{2-i}_{\mathrm{Liu}}(G_K, W^\vee(\chi)) \to \mathrm{H}^2_{\mathrm{Liu}}(G_K, W \otimes W^\vee(\chi)) \to \mathrm{H}^2_{\mathrm{Liu}}(G_K, W(\mathbb{Q}_p(\chi)) \xrightarrow{\sim} \mathbb{Q}_p.$$

Here $\chi : G_K \to \mathbb{Z}_p^\times$ is the p-adic cyclotomic character and the last isomorphism is that which appears in usual Tate duality.

Proof. See [Ked07, Theorem 8.1] and [Liu08, Theorems 4.3 and 4.7]. □

Remark 2.9. The above perfect pairing is defined by using (φ, Γ_K)-modules [Liu08, § 4.2]) and the equivalence between B-pairs and (φ, Γ_K)-modules [Ked07, p. 8]. On the other hand, for $\mathbb{Q}_p$-B-pair W, we can define a pairing

$$\mathrm{H}^1(G_K, W) \times \mathrm{H}^1(G_K, W^\vee(\chi)) \to \mathrm{H}^2(G_K, W \otimes W^\vee(\chi)) \to \mathrm{H}^2(G_K, W(\mathbb{Q}_p(\chi)) \xrightarrow{\sim} \mathbb{Q}_p,$$

in the usual way by using cocycles. If we identify $\mathrm{H}^1(G_K, W^\vee(\chi)) \xrightarrow{\sim} \mathrm{H}^1_{\mathrm{Liu}}(G_K, W^\vee(\chi))$ as above, then we can check that both pairings are the same by using the fact that $\mathrm{H}^i_{\mathrm{Liu}}(G_K, W)$ is the universal δ functor [Ked07, Theorem 8.1] and $\mathrm{H}^i(G_K, W)$ is a δ-functor.

Let W be an E–B-pair. Then by Theorem 2.8(2) above and the remark before, we have

$$\begin{aligned}
\dim_{\mathbb{Q}_p}\mathrm{H}^1(G_K, W) &= \dim_{\mathbb{Q}_p}\mathrm{H}^1_{\mathrm{Liu}}(G_K, W)\\
&= [K : \mathbb{Q}_p][E : \mathbb{Q}_p]\mathrm{rank}(W) + \dim_{\mathbb{Q}_p}\mathrm{H}^0_{\mathrm{Liu}}(G_K, W) + \dim_{\mathbb{Q}_p}\mathrm{H}^2_{\mathrm{Liu}}(G_K, W)\\
&= [K : \mathbb{Q}_p][E : \mathbb{Q}_p]\mathrm{rank}(W) + \dim_{\mathbb{Q}_p}\mathrm{H}^0_{\mathrm{Liu}}(G_K, W) + \dim_{\mathbb{Q}_p}\mathrm{H}^0_{\mathrm{Liu}}(G_K, W^\vee(\chi))\\
&= [K : \mathbb{Q}_p][E : \mathbb{Q}_p]\mathrm{rank}(W) + \dim_{\mathbb{Q}_p}\mathrm{H}^0(G_K, W) + \dim_{\mathbb{Q}_p}\mathrm{H}^0(G_K, W^\vee(\chi)).
\end{aligned}$$

Dividing this equality by $[E : \mathbb{Q}_p]$, we obtain the following dimension formula.

PROPOSITION 2.10. *Let W be an E–B-pair. Then we have*

$$\dim_E \mathrm{H}^1(G_K, W) = [K:\mathbb{Q}_p]\mathrm{rank}(W) + \dim_E \mathrm{H}^0(G_K, W) + \dim_E \mathrm{H}^0(G_K, W^\vee(\chi)).$$

As in the case of p-adic representations, we have dualities between H^1_e, H^1_f and H^1_g.

PROPOSITION 2.11. *Let W be a B-pair. In the above perfect pairing in Theorem 2.8(3), $\mathrm{H}^1_e(G_K, W)$ and $\mathrm{H}^1_g(G_K, W^\vee(\chi))$ are the exact annihilators of each other. The same statement holds with e replaced by g and g by e and also when e and g are both replaced by f.*

Proof. This proof is also essentially same as that in the case of p-adic representations (cf. [BK90, Proposition 3.8]). □

2.3 Calculations of cohomologies of rank-one E–B-pairs

To classify rank-two split trianguline E–B-pairs, we need to calculate the dimension of $\mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ for any continuous characters $\delta_1, \delta_2 : K^\times \to E^\times$. Twisting by $W(\delta_2^{-1})$, it is isomorphic to $\mathrm{Ext}^1(B_E, W(\delta_1/\delta_2)) \xrightarrow{\sim} \mathrm{H}^1(G_K, W(\delta_1/\delta_2))$. We calculate this in the following way.

LEMMA 2.12. *For any embedding $\sigma : K \hookrightarrow E$, we define a continuous character $\sigma(x) : K^\times \to E^\times : y \mapsto \sigma(y)$. Then for any $\{k_\sigma\}_\sigma$ ($k_\sigma \in \mathbb{Z}$ for any σ), we have an isomorphism of E–B-pairs*

$$W\Bigg(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}\Bigg) \xrightarrow{\sim} (B_e \otimes_{\mathbb{Q}_p} E, \oplus_{\sigma:K\hookrightarrow E} t^{k_\sigma} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E).$$

Proof. First we prove that there is an isomorphism

$$W(id(x)) \xrightarrow{\sim} (B_e \otimes_{\mathbb{Q}_p} E, tB^+_{\mathrm{dR}} \otimes_{K,id} E \oplus \oplus_{\sigma:K\hookrightarrow E, \neq id} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E),$$

here $id : K \hookrightarrow E$ is the given embedding. We decompose $id(x) : K^\times \to E^\times$ into $id(x) := \delta_0\delta_1$, here $\delta_0|_{\mathcal{O}_K^\times} := id|_{\mathcal{O}_K^\times}$, $\delta_0(\pi_K) := 1$ and $\delta_1|_{\mathcal{O}_K^\times}$ is trivial and $\delta_1(\pi_K) := \pi_K$. Then, by the definition of the reciprocity map $\mathrm{rec_K} : K^\times \to G_K^{\mathrm{ab}}$ in the notation section, we have $\tilde{\delta_0} = \chi_{\mathrm{LT}}$, i.e. the Lubin–Tate character associated with π_K. So we have $W(\delta_0) = W(E(\chi_{\mathrm{LT}}))$. Then $D^K_{\mathrm{cris}}(W(\delta_0)) := K_0 \otimes_{\mathbb{Q}_p} Ee$ is the filtered φ-module such that φ acts by $\varphi^f(e) = \pi_K^{-1} e$ and the filtration on $K \otimes_{K_0} D^K_{\mathrm{cris}}(W(\delta_0)) = K \otimes_{\mathbb{Q}_p} Ee \xrightarrow{\sim} \oplus_{\sigma:K\hookrightarrow E} Ee_\sigma$ is defined by $\mathrm{Fil}^{-1} = K \otimes_{K_0} D^K_{\mathrm{cris}}(W(\delta_0))$, $\mathrm{Fil}^0 = \bigoplus_{\sigma\neq id} Ee_\sigma$ and $\mathrm{Fil}^1 = 0$ by [Col02, Proposition 9.10 and Lemma 9.18]. On the other hand, $D^K_{\mathrm{cris}}(W(\delta_1)) := K_0 \otimes_{\mathbb{Q}_p} Ee'$ is the filtered φ-module such that $\varphi^f(e') = \pi_K e'$ and $\mathrm{Fil}^0(K \otimes_{K_0} D^K_{\mathrm{cris}}(W(\delta_1))) = K \otimes_{K_0} D^K_{\mathrm{cris}}(W(\delta_1))$, $\mathrm{Fil}^1 = 0$. So we have $D^K_{\mathrm{cris}}(id(x)) \xrightarrow{\sim} D^K_{\mathrm{cris}}(\delta_0) \otimes D^K_{\mathrm{cris}}(\delta_1) := K_0 \otimes_{\mathbb{Q}_p} Ee''$ on which φ acts by $\varphi^f(e'') = e''$ and the filtration on $K \otimes_{K_0} D^K_{\mathrm{cris}}(id(x)) \xrightarrow{\sim} \oplus_{\sigma:K\hookrightarrow E} Ee''_\sigma$ is given by $\mathrm{Fil}^{-1} = K \otimes_{K_0} D^K_{\mathrm{cris}}(id(x))$, $\mathrm{Fil}^0 = \bigoplus_{\sigma\neq id} Ee''_\sigma$ and $\mathrm{Fil}^1 = 0$. Then, by the definition of D^K_{cris} for E–B-pairs, it is easy to see that $D^K_{\mathrm{cris}}((B_e \otimes_{\mathbb{Q}_p} E, tB^+_{\mathrm{dR}} \otimes_{K,id} E \oplus \bigoplus_{\sigma\neq id} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E)) \xrightarrow{\sim} D^K_{\mathrm{cris}}(id(x))$ as filtered φ-modules. So, in this case, we have proved the lemma. In the case where $\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma} = \sigma(x)$ for some σ, we can prove the lemma in the same way. By tensoring these, we can prove the lemma for any $\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$. □

Let $N_{K/\mathbb{Q}_p}(x) : K^\times \to \mathbb{Q}_p^\times$ be the norm map and let $|-| : \mathbb{Q}_p^\times \to E^\times$ be the absolute value character such that $|p| := 1/p$, $|u| := 1$ for any $u \in \mathbb{Z}_p^\times$.

LEMMA 2.13. *There is an isomorphism $W(E(\chi)) \xrightarrow{\sim} W(N_{K/\mathbb{Q}_p}(x)|N_{K/\mathbb{Q}_p}(x)|)$.*

Proof. When $K = \mathbb{Q}_p$, $\chi : G_{\mathbb{Q}_p} \to \mathbb{Z}_p^\times$ satisfies $x|x| = \chi \circ \mathrm{rec}_{\mathbb{Q}_p}$ by local class field theory. So, in the general case, $\chi : G_K \to \mathcal{O}_E^\times$ corresponds to $N_{K/\mathbb{Q}_p}(x)|N_{K/\mathbb{Q}_p}(x)|$ by local class field theory. □

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The next proposition is a generalization of [Col08a, Proposition 3.1].

Proposition 2.14. *Let $\delta : K^{\times} \to E^{\times}$ be a continuous character. Then $\mathrm{H}^0(G_K, W(\delta)) \xrightarrow{\sim} E$ if and only if $\delta = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $k_\sigma \leq 0$ for any σ. Otherwise $\mathrm{H}^0(G_K, W(\delta)) = 0$.*

Proof. If we assume that $\mathrm{H}^0(G_K, W(\delta)) \neq 0$, then there is a non-zero morphism $f : B_E \to W(\delta)$ of E–B-pairs because $\mathrm{H}^0(G_K, W(\delta)) = \mathrm{Hom}(B_E, W(\delta))$. Then, by Lemma 1.12, $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are also E–B-pairs. Because B_E is a rank-one E–B-pair, one of $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ must be zero. Because $f \neq 0$ we have $\mathrm{Im}(f) \neq 0$ so we have $\mathrm{Ker}(f) = 0$, i.e. f must be injective. So we have an injection $f : B_e \otimes_{\mathbb{Q}_p} E \hookrightarrow W(\delta)_e$ of free $B_e \otimes_{\mathbb{Q}_p} E$-modules of the same rank. By Lemma 1.10, the cokernel is also a free $B_e \otimes_{\mathbb{Q}_p} E$-module. So the cokernel must be zero, so $f : B_e \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} W(\delta)_e$ must be an isomorphism. Then $W(\delta)^+_{\mathrm{dR}}$ is a G_K-stable B^+_{dR}-lattice in $B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$ with E-action which contains $B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E = \oplus_{\sigma:K\hookrightarrow E} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E$. So $W(\delta)^+_{\mathrm{dR}}$ must be of the form $W(\delta)^+_{\mathrm{dR}} = \oplus_{\sigma:K\hookrightarrow E} t^{k_\sigma} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E$ for some $k_\sigma \leq 0$ for any σ. So $W(\delta) = W(\prod_{\sigma:K\to E} \sigma(x)^{k_\sigma})$ by Lemma 2.12. In this case, it is clear that $\mathrm{Hom}(B_E, W(\delta)) = E$, so we have proved the proposition. □

By this proposition and Liu's Euler–Poincaré formula, we can calculate the dimension of $\mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ as follows. This is a generalization of [Col08a, Theorem 3.9].

Proposition 2.15. *Let $\delta_1, \delta_2 : K^{\times} \to E^{\times}$ be continuous characters. Then $\dim_E \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ is equal to:*

(1) *$[K : \mathbb{Q}_p] + 1$ when $\delta_1/\delta_2 = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $k_\sigma \leq 0$ for any σ;*

(2) *$[K : \mathbb{Q}_p] + 1$ when $\delta_1/\delta_2 = |N_{K/\mathbb{Q}_p}(x)| \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $k_\sigma \geq 1$ for any σ;*

(3) *$[K : \mathbb{Q}_p]$ otherwise.*

Proof. By Lemma 2.13 and Proposition 2.14, we have:

- $\mathrm{H}^0(G_K, W(\delta)) = E$ if and only if $\delta = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $k_\sigma \leq 0$ for any σ,
- $\mathrm{H}^0(G_K, W(\delta)^{\vee}(\chi)) \xrightarrow{\sim} \mathrm{H}^0(G_K, W(\delta^{-1} N_{K/\mathbb{Q}_p}(x) |N_{K/\mathbb{Q}_p}(x)|)) = E$ if and only if $\delta = N_{K/\mathbb{Q}_p}(x) |N_{K/\mathbb{Q}_p}(x)| \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $k_\sigma \geq 1$ for any σ.

So we have the desired result by Corollary 2.10. □

To classify two-dimensional split trianguline E-representations, we need to know which extension class in $\mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ is of the form $[W(V)]$ for some two-dimensional E-representation V. For this problem, the next exact sequence is very important, which is the B-pair analogue of the map i_k defined in [Col08a, § 3.7]. Let $\delta : K^{\times} \to E^{\times}$ be a continuous character and let $\{k_\sigma\}_{\sigma:K\hookrightarrow E}$ be $k_\sigma \in \mathbb{Z}_{\geqq 0}$ for any σ. We consider the natural inclusion of E–B-pairs $W(\delta) \hookrightarrow W(\delta) \otimes W(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}) = W(\delta \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma})$ which is obtained by tensoring with $W(\delta)$ the natural inclusion $B_E \hookrightarrow W(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma})$. Then, by Lemma 2.12, we have

$$W^+_{\mathrm{dR}}\left(\delta \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}\right) = W^+_{\mathrm{dR}}(\delta) \otimes_{B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} (\oplus_{\sigma:K\hookrightarrow E} t^{-k_\sigma} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E).$$

As for the B_e part, we can prove in the same way as the proof of Proposition 2.14 that $W_e(\delta) \xrightarrow{\sim} W_e(\delta \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma})$. So we have the following short exact sequence of complexes of G_K-modules

$$0 \to C^{\bullet}(W(\delta)) \to C^{\bullet}\left(W\left(\delta \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}\right)\right) \to \oplus_{\sigma:K\hookrightarrow E} t^{-k_\sigma} W^+_{\mathrm{dR}}(\delta)_\sigma / W^+_{\mathrm{dR}}(\delta)_\sigma[0] \to 0,$$

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where $W^+_{\mathrm{dR}}(\delta)_\sigma := W^+_{\mathrm{dR}}(\delta)\otimes_{B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E}(B^+_{\mathrm{dR}}\otimes_{K,\sigma}E)$ for any σ where the last tensor product is taken by the projection to the σ-component of $B^+_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E\xrightarrow{\sim}\oplus_{\sigma:K\hookrightarrow E}B^+_{\mathrm{dR}}\otimes_{K,\sigma}E$. From this short exact sequence, we obtain the following long exact sequence

$$\cdots\to \mathrm{H}^0\Bigg(G_K, W\Bigg(\delta\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{-k_\sigma}\Bigg)\Bigg)\to \oplus_{\sigma:K\hookrightarrow E}\mathrm{H}^0(G_K, t^{-k_\sigma}W^+_{\mathrm{dR}}(\delta)_\sigma/W^+_{\mathrm{dR}}(\delta)_\sigma)$$
$$\to \mathrm{H}^1(G_K, W(\delta))\to \mathrm{H}^1\Bigg(G_K, W\Bigg(\delta\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{-k_\sigma}\Bigg)\Bigg)\to\cdots.$$

The next lemma is a generalization of [Col08a, Proposition 3.18].

LEMMA 2.16. *Let $\{w(\delta)_\sigma\}_\sigma$ be the generalized Hodge–Tate weight of $W(\delta)$ defined in 1.47. Then there is an isomorphism of E-vector spaces*

$$\oplus_{\sigma:K\hookrightarrow E}\mathrm{H}^0(G_K, t^{-k_\sigma}W^+_{\mathrm{dR}}(\delta)_\sigma/W^+_{\mathrm{dR}}(\delta)_\sigma)\xrightarrow{\sim}\oplus_{\sigma, w(\delta)_\sigma\in\{1,2,\cdots,k_\sigma\}}Ee_\sigma.$$

Here Ee_σ is a one-dimensional E-vector space with base e_σ.

Proof. It suffices to show that $\mathrm{H}^0(G_K, t^{-k_\sigma}W^+_{\mathrm{dR}}(\delta)_\sigma/W^+_{\mathrm{dR}}(\delta)_\sigma)\xrightarrow{\sim}E$ if and only if $w(\delta)_\sigma\in\{1,2,\ldots,k_\sigma\}$ and $\mathrm{H}^0(G_K, t^{-k_\sigma}W^+_{\mathrm{dR}}(\delta)_\sigma/W^+_{\mathrm{dR}}(\delta)_\sigma)=0$ otherwise. However, by the definition of generalized Hodge–Tate weight, for any $i\in\mathbb{Z}$ and σ, we have $\mathrm{H}^0(G_K, t^{-i}W^+_{\mathrm{dR}}(\delta)_\sigma/t^{-i+1}W^+_{\mathrm{dR}}(\delta)_\sigma)$ $=E$ if and only if $w(\delta)_\sigma=i$ and we have $\mathrm{H}^0(G_K, t^{-i}W^+_{\mathrm{dR}}(\delta)_\sigma/t^{-i+1}W^+_{\mathrm{dR}}(\delta)_\sigma)=0$ otherwise. From this, the result follows. □

By definition, for any continuous character $\delta: K^\times\to E^\times$, we have the following short exact sequence

$$0\to D^K_{\mathrm{dR}}(W(\delta))/(D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1}+\mathrm{Fil}^0D^K_{\mathrm{dR}}(W(\delta)))\to \mathrm{H}^1(G_K, W(\delta))$$
$$\to \mathrm{Ker}(\mathrm{H}^1(G_K, W_e(\delta))\oplus \mathrm{H}^1(G_K, W^+_{\mathrm{dR}}(\delta))\to \mathrm{H}^1(G_K, W_{\mathrm{dR}}(\delta)))\to 0.$$

LEMMA 2.17. *Let $\delta: K\to E^\times$ be a continuous character. Then the E-vector space $D^K_{\mathrm{dR}}(W(\delta))/(D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1}+\mathrm{Fil}^0D^K_{\mathrm{dR}}(W(\delta)))$ is isomorphic to:*

(1) $\oplus_{\sigma, w(\delta)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma/\Delta(E)$ *when $\delta=\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma\in\mathbb{Z}$ for any σ;*

(2) $\oplus_{\sigma, w(\delta)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma$ *otherwise.*

Here $\Delta: E\to\oplus_{\sigma. w(\delta)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma$ is the diagonal map.

Proof. By the definition of $W(\delta)$, it is easy to see that $W_{\mathrm{dR}}(\delta)\xrightarrow{\sim}B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E(\delta_0)$ for some continuous character $\delta_0: G_K\to\mathcal{O}^\times_E$. If we decompose $B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E(\delta_0)\xrightarrow{\sim}\oplus_{\sigma:K\hookrightarrow E}B_{\mathrm{dR}}\otimes_{K,\sigma}E(\delta_0)$, we have $(B_{\mathrm{dR}}\otimes_{K,\sigma}E(\delta_0))^{G_K}=0$ if and only if $w(\delta)_\sigma\notin\mathbb{Z}$. If $w(\delta)_\sigma\in\mathbb{Z}$, then we have $(t^{-w(\delta)_\sigma}B^+_{\mathrm{dR}}\otimes_{K,\sigma}E(\delta_0))^{G_K}\xrightarrow{\sim}E$ and $(t^{-w(\delta)_\sigma+1}B^+_{\mathrm{dR}}\otimes_{K,\sigma}E(\delta_0))^{G_K}=0$. As for $W_e(\delta)$, we have $D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1}=W_e(\delta)^{G_K}\neq 0$ if and only if $W_e(\delta)\xrightarrow{\sim}B_e\otimes_{\mathbb{Q}_p}E$. Then $W(\delta)\xrightarrow{\sim}(B_e\otimes_{\mathbb{Q}_p}E, \oplus_{\sigma:K\hookrightarrow E}t^{k_\sigma}B^+_{\mathrm{dR}}\otimes_{K,\sigma}E)$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma\in\mathbb{Z}$ for any σ. Then, by Lemma 2.12, we have $\delta=\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{k_\sigma}$. In this case, we have $D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1}\xrightarrow{\sim}E$. Combining these computations, we obtain the lemma. □

Next we compare the following two maps, one is the boundary map

$$\partial:\oplus_{\sigma:K\hookrightarrow E}\mathrm{H}^0(G_K, t^{-k_\sigma}W^+_{\mathrm{dR}}(\delta)_\sigma/W^+_{\mathrm{dR}}(\delta)_\sigma)\to \mathrm{H}^1(G_K, W(\delta))$$

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of the exact sequence before Lemma 2.16 and the other is the natural map

$$\iota : D^K_{\mathrm{dR}}(W(\delta))/\mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta)) \to D^K_{\mathrm{dR}}(W(\delta))/(D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta))) \to \mathrm{H}^1(G_K, W(\delta)).$$

From the proof of Lemma 2.17, we can see that the natural map $f : D^K_{\mathrm{dR}}(W(\delta))/\mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta)) \xrightarrow{\sim} (W_{\mathrm{dR}}(\delta)/W^+_{\mathrm{dR}}(\delta))^{G_K}$ is an isomorphism. On the other hand, we have a natural map

$$j : \oplus_{\sigma:K\hookrightarrow E}\mathrm{H}^0(G_K, t^{-k_\sigma} W^+_{\mathrm{dR}}(\delta)_\sigma/W^+_{\mathrm{dR}}(\delta)_\sigma) = \mathrm{H}^0\left(G_K, W^+_{\mathrm{dR}}\left(\delta \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}\right)/W^+_{\mathrm{dR}}(\delta)\right) \hookrightarrow (W_{\mathrm{dR}}(\delta)/W^+_{\mathrm{dR}}(\delta))^{G_K}.$$

Lemma 2.18. *With the above notation, we have* $\iota \circ f^{-1} \circ j = (-1)\partial$.

Proof. This follows easily from a diagram chase. □

3. Classification of two-dimensional split trianguline E-representations

In this section, we classify two-dimensional split trianguline E-representations. As in [Col08a, § 0.2], we would like to explicitly determine the parameter spaces of two-dimensional split trianguline E-representations.

3.1 Parameter spaces of two-dimensional split trianguline E-representations

Let W be a rank-two split trianguline E–B-pair such that $[W] \in \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ for some continuous characters $\delta_1, \delta_2 : K^\times \to E^\times$. First we study a necessary condition on (δ_1, δ_2) in order for W to be étale, i.e. of the form $W = W(V)$ for some E-representation V.

Lemma 3.1. *If* $[W] \in \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ *is étale, then the pair* (δ_1, δ_2) *satisfies* $\mathrm{val}_p(\delta_1(\pi_K)) + \mathrm{val}_p(\delta_2(\pi_K)) = 0$ *and* $\mathrm{val}_p(\delta_1(\pi_K)) \geq 0$. *Here* val_p *is the valuation of* E *such that* $\mathrm{val}_p(p) = 1$.

Proof. Let W be of the form $W = W(V)$ for some E-representation V. By definition, $W(V)$ sits in the following short exact sequence of E–B-pairs

$$0 \to W(\delta_1) \to W(V) \to W(\delta_2) \to 0.$$

Because $W(V)$ is pure of slope zero, $\det W(V) \xrightarrow{\sim} W(\delta_1\delta_2)$ is also pure of slope zero, so we have a condition $(\mathrm{val}_p(\delta_1(\pi_K)) + \mathrm{val}_p(\delta_2(\pi_K)))/f = 0$ because the slope of $W(\delta)$ is $(\mathrm{val}_p(\delta(\pi_K)))/f$. By the slope filtration theorem for E–B-pairs (Theorem 1.40), $W(\delta_1)$ must be pure of slope $\geqq 0$. This implies that $(\mathrm{val}_p(\delta_1(\pi_K)))/f \geq 0$. We have proved the lemma. □

To describe the classification of two-dimensional split trianguline E-representations, let us introduce some notation. First let us put

$$S^+ := \{(\delta_1, \delta_2) \mid \delta_1, \delta_2 : K^\times \to E^\times \text{ continuous characters such that } \mathrm{val}_p(\delta_1(\pi_K)) + \mathrm{val}_p(\delta_2(\pi_K)) = 0, \mathrm{val}_p(\delta_1(\pi_K)) \geq 0\}.$$

By Lemma 3.1, to classify two-dimensional split trianguline E-representations, it suffices only to consider split trianguline E–B-pairs W such that $[W] \in \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ for some $(\delta_1, \delta_2) \in S^+$.

First we classify two-dimensional split trianguline E-representations V such that $[W(V)] = 0 \in \mathrm{Ext}^1(W(\delta_2), W(\delta_1))$ for some (δ_1, δ_2), i.e. $W(V) = W(\delta_1) \oplus W(\delta_2)$.

Lemma 3.2. *Let $(\delta_1,\delta_2)\in S^+$. Then $W(\delta_1)\oplus W(\delta_2)$ is of the form $W(V)$ for some two-dimensional E-representation if and only if both $W(\delta_1)$ and $W(\delta_2)$ are pure of slope zero, i.e. $W(\delta_1)\xrightarrow{\sim} W(V(E(\tilde{\delta}_1)))$ and $W(\delta_2)\xrightarrow{\sim} W(V(E(\tilde{\delta}_2)))$ for some continuous characters $\tilde{\delta}_1,\tilde{\delta}_2:G_K\to\mathcal{O}_E^{\times}$. In this case, $V\xrightarrow{\sim} E(\tilde{\delta}_1)\oplus E(\tilde{\delta}_2)$.*

Proof. The 'if' part is trivial. Let us assume that $W(\delta_1)\oplus W(\delta_2)\xrightarrow{\sim} W(V)$ for some two-dimensional E-representation V. So $W(\delta_1)\oplus W(\delta_2)$ is pure of slope zero. If $W(\delta_1)$ is not pure of slope zero, then this is pure of slope $u>0$ by the assumption that $(\delta_1,\delta_2)\in S^+$. Then $W(\delta_2)$ is pure of slope $-u<0$ because $(\delta_1,\delta_2)\in S^+$. Then we have the exact sequence

$$0\to W(\delta_2)\to W(\delta_1)\oplus W(\delta_2)\to W(\delta_1)\to 0.$$

Then the slope filtration theorem for E–B-pairs (Theorem 1.40) implies that $W(\delta_1)\oplus W(\delta_2)$ is not pure of slope zero. This is a contradiction. □

This lemma says that two dimensional split trianguline E-representation V such that $[W(V)]=0\in \mathrm{Ext}^1(W(\delta_2),W(\delta_1))$ corresponds to two dimensional E-representations V such that $V=E(\tilde{\delta}_1)\oplus E(\tilde{\delta}_2)$ for some continuous characters $\tilde{\delta}_1,\tilde{\delta}_2:G_K\to\mathcal{O}_E^{\times}$. So, in this case, we finish the classification.

From now on, we only consider split trianguline E–B-pairs W such that $[W]\neq 0\in \mathrm{Ext}^1(W(\delta_2),W(\delta_1))$. For any $(\delta_1,\delta_2)\in S^+$, let us put

$$S(\delta_1,\delta_2):=\mathbb{P}_E(\mathrm{H}^1(G_K,W(\delta_1/\delta_2))).$$

Here, for any finite-dimensional E-vector space M, we denote

$$\mathbb{P}_E(M):=\{[v]\mid v\in M-\{0\},[v]=[v'] \iff v'=av \text{ for some } a\in E^{\times}\}.$$

Next, for any $(\delta_1,\delta_2)\in S^+$, we define a subset $S'(\delta_1,\delta_2)\subseteq S(\delta_1,\delta_2)$ by

$$S'(\delta_1,\delta_2):=\mathbb{P}_E(D^K_{\mathrm{dR}}(W(\delta_1/\delta_2))/(D^K_{\mathrm{cris}}(W(\delta_1/\delta_2))^{\varphi=1}+\mathrm{Fil}^0D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)))),$$

here we see $S'(\delta_1,\delta_2)$ as a subset of $S(\delta_1,\delta_2)$ by the following natural inclusion

$$D^K_{\mathrm{dR}}(W(\delta_1/\delta_2))/(D^K_{\mathrm{cris}}(W(\delta_1/\delta_2))^{\varphi=1}+\mathrm{Fil}^0D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)))\hookrightarrow \mathrm{H}^1(G_K,W(\delta_1/\delta_2)).$$

Then, by Lemma 2.17, $S'(\delta_1,\delta_2)$ is non-canonically isomorphic to:

(1) $S'(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\geqq 1}Ee_\sigma/\Delta(E))$ when $\delta_1/\delta_2=\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma\in\mathbb{Z}$ for any σ;

(2) $S'(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma)$ otherwise;

here, these isomorphisms depend on the choice of E-linear isomorphisms in Lemma 2.17.

Finally we define the subset $S'^{\text{ét}}(\delta_1,\delta_2)$ of $S'(\delta_1,\delta_2)$ as follows. When $\delta_1/\delta_2\neq\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{k_\sigma}$ for any $\{k_\sigma\}$ such that $k_\sigma\in\mathbb{Z}$ for any σ, we fix an isomorphism $S'(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma)$ as above and identify these two spaces. Then let us put

$$S'^{\text{ét}}(\delta_1,\delta_2):=\left\{[(a_\sigma e_\sigma)_\sigma]\in\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma)\ \middle|\ \left(\sum_{\sigma,a_\sigma\neq 0}w(\delta_1/\delta_2)_\sigma\right.\right.$$
$$\left.\left. +e_K\mathrm{val}_p(\delta_2(\pi_K))\geq 0\right.\right\}.$$

When $\delta_1/\delta_2=\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma\in\mathbb{Z}$ for any σ, we fix an isomorphism $S'(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\geqq 1}Ee_\sigma/\Delta(E))$ as above and identify these two spaces. Then let us put

$$S'^{\text{ét}}(\delta_1,\delta_2):=\left\{[\overline{(a_\sigma e_\sigma)_\sigma}]\in\mathbb{P}_E(\oplus_{\sigma\cdot w(\delta_1/\delta_2)_\sigma\geq 1}Ee_\sigma/\Delta(E))\;\middle|\;\left(\sum_{\sigma,a_\sigma\neq 0}w(\delta_1/\delta_2)_\sigma\right)\right.$$
$$+e_K\mathrm{val}_p(\delta_2(\pi_K))\geq 0$$
$$\left.\text{for any lifting }(a_\sigma e_\sigma)_\sigma\in\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\in\mathbb{Z}_{\geqq 1}}Ee_\sigma\text{ of }[\overline{(a_\sigma e_\sigma)_\sigma}]\right\}.$$

We put $S'^{\text{non-ét}}(\delta_1,\delta_2):=S'(\delta_1,\delta_2)\setminus S'^{\text{ét}}(\delta_1,\delta_2)$. Then we can easily see that the subsets $S'^{\text{ét}}(\delta_1,\delta_2)$, $S'^{\text{non-ét}}(\delta_1,\delta_2)\subseteq S(\delta_1,\delta_2)$ do not depend on the choice of an isomorphism $S'(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\geq 1}Ee_\sigma/\Delta(E))$ or $S'(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(\oplus_{\sigma,w(\delta_1/\delta_2)_\sigma\in\mathbb{Z}_{\geq 1}}Ee_\sigma)$.

Remark 3.3. When $K=\mathbb{Q}_p$: $S(\delta_1,\delta_2)$ is one point or $\mathbb{P}_E(E)$ and $S'(\delta_1,\delta_2)$ is empty or one point; $\mathbb{P}_E(E)$ and $S'^{\text{ét}}(\delta_1,\delta_2)$ is empty or one point; or $\mathbb{P}_E(E)$. If we compare this parameter space and Colmez's [Col08a, § 4.3], then we have $\bigsqcup_{(\delta_1,\delta_2)\in S^+}(S(\delta_1,\delta_2)\setminus S'(\delta_1,\delta_2))=\mathcal{S}_+^{\text{ng}}\sqcup S_+^{\text{st}}$, $\bigsqcup_{(\delta_1,\delta_2)\in S^+}S'^{\text{ét}}(\delta_1,\delta_2)=\mathcal{S}_+^{\text{cris}}\sqcup\mathcal{S}_+^{\text{ord}}$ and $\bigsqcup_{(\delta_1,\delta_2)\in S^+}S'^{\text{non-ét}}(\delta_1,\delta_2)=\mathcal{S}^{\text{ncl}}$.

For any $s\in S(\delta_1,\delta_2)$, we denote by $W(s)$ an extension of $W(\delta_1)$ by $W(\delta_2)$ defined by s. The isomorphism class of $W(s)$ as a E–B-pair depends only on the class s. By definition, $W(s)$ is a split trianguline E–B-pair which sits in the following non-split short exact sequence of E–B-pairs

$$0\to W(\delta_1)\to W(s)\to W(\delta_2)\to 0.$$

We determine when $W(s)$ is of the form $W(V(s))$ for some E-representation $V(s)$. The following theorem is a generalization of [Col08a, Proposition 4.7] and is the most important step of the classification.

Theorem 3.4. *Let $s\in S(\delta_1,\delta_2)$ for $(\delta_1,\delta_2)\in S^+$. Then the following conditions are equivalent:*

(1) *$W(s)$ is pure of slope zero, i.e. $W(s)\xrightarrow{\sim}W(V(s))$ for a two-dimensional split trianguline E-representation $V(s)$;*

(2) *$s\notin S'^{\text{non-ét}}(\delta_1,\delta_2)$.*

Proof. First we prove the theorem in the case $\delta_1/\delta_2\neq\prod_{\sigma:K\hookrightarrow E}\sigma(x)^{k_\sigma}$ for any $\{k_\sigma\}_\sigma$ such that $k_\sigma\in\mathbb{Z}$ for any σ. Let us assume that $W(s)$ is not pure of slope zero. Then, by the slope filtration theorem for E–B-pairs (Theorem 1.40), there exist rank-one E–B-pairs $W(\delta_3)$, $W(\delta_4)$ such that $W(\delta_3)$ is pure of slope $u<0$, $W(\delta_4)$ is pure of slope $-u>0$ and $W(s)$ sits in the following short exact sequence

$$0\to W(\delta_3)\to W(s)\to W(\delta_4)\to 0.$$

On the other hand, by definition, we have a following short exact sequence

$$0\to W(\delta_1)\to W(s)\to W(\delta_2)\to 0. \tag{1}$$

We consider the restriction to $W(\delta_3)$ of the projection $W(s)\to W(\delta_2)$, which we denote by $f:W(\delta_3)\to W(\delta_2)$. We claim that f is an inclusion. Because $\mathrm{Ker}(f)$ and $\mathrm{Im}(f)$ are E–B-pairs by Lemma 1.12 and because $W(\delta_3)$ is rank one, we have exactly one of the following two cases, $\mathrm{Ker}(f)\xrightarrow{\sim}W(\delta_3)$ or $W(\delta_3)\xrightarrow{\sim}\mathrm{Im}(f)$. If $\mathrm{Ker}(f)\xrightarrow{\sim}W(\delta_3)$, then the inclusion $W(\delta_3)\hookrightarrow W(s)$ factors through a non-zero map $f':W(\delta_3)\to W(\delta_1)$. However, because the slope of $W(\delta_1)$ is strictly larger than the slope of $W(\delta_3)$ by assumption, so we have $\mathrm{Hom}(W(\delta_3),W(\delta_1))=0$ by Proposition 2.14 or by [Ked07, Lemma 6.2]. This is a contradiction. So we have $W(\delta_3)\xrightarrow{\sim}\mathrm{Im}(f)$, i.e. f is an inclusion. Then, by Proposition 2.14, we have $\delta_3 = \delta_2 \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}_{\geq 0}$ for any σ. Then, because $\det(W(s)) = W(\delta_1\delta_2) \xrightarrow{\sim} W(\delta_3\delta_4)$, we have $\delta_4 = \delta_1 \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}$. The injectivity of f means that the exact sequence (1) splits after pulling back by $f : W(\delta_3) \hookrightarrow W(\delta_2)$, i.e. we have

$$s \in \mathrm{Ker}\left(\mathrm{H}^1(G_K, W(\delta_1/\delta_2)) \to \mathrm{H}^1\left(G_K, W\left(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}\delta_1/\delta_2\right)\right)\right).$$

This kernel is isomorphic to

$$\mathrm{Im}(\partial : \mathrm{H}^0(G_K, \oplus_{\sigma:K\hookrightarrow E} t^{-k_\sigma} W^+_{\mathrm{dR}}(\delta_1/\delta_2)_\sigma / W^+_{\mathrm{dR}}(\delta_1/\delta_2)_\sigma) \hookrightarrow \mathrm{H}^1(G_K, W(\delta_1/\delta_2))).$$

So, by Lemma 2.18, we have $s = [(a_\sigma e_\sigma)_\sigma] \in \mathbb{P}_E(\oplus_{\sigma, w(\delta_1/\delta_2)_\sigma \in \mathbb{Z}_{\geq 1}} Ee_\sigma) = S'(\delta_1, \delta_2)$ such that any σ with $a_\sigma \neq 0$ satisfies $w(\delta_1/\delta_2)_\sigma \in \{1, 2, \ldots, k_\sigma\}$. Then, $(\sum_{\sigma, a_\sigma\neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) \leq (\sum_\sigma k_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) = [K : \mathbb{Q}_p]$ (slope of $W(\delta_3)) < 0)$ by assumption. So $s \in S'^{\text{non-ét}}(\delta_1, \delta_2)$.

Next we assume that $s = [(a_\sigma e_\sigma)_\sigma] \in S'^{\text{non-ét}}(\delta_1, \delta_2)$. Then, by Lemma 2.18, we can see that s is contained in the image of

$$\partial : \oplus_{\sigma, a_\sigma\neq 0} \mathrm{H}^0(G_K, t^{-w(\delta_1/\delta_2)_\sigma} W^+_{\mathrm{dR}}(\delta_1/\delta_2)_\sigma / W^+_{\mathrm{dR}}(\delta_1/\delta_2)_\sigma) \to \mathrm{H}^1(G_K, W(\delta_1/\delta_2)).$$

So we have

$$s \in \mathrm{Ker}(\mathrm{H}^1(G_K, W(\delta_1/\delta_2)) \to \mathrm{H}^1\left(G_K, W\left(\prod_{\sigma, a_\sigma\neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma}\delta_1/\delta_2\right)\right).$$

So there is an injection $g : W(\prod_{\sigma, a_\sigma\neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(s)$ such that $f'g$ is the natural inclusion $W(\prod_{\sigma, a_\sigma\neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(\delta_2)$ (here $f' : W(s) \to W(\delta_2)$ is the projection). If we take the saturation $W(\delta_3)$ of this inclusion g (Lemma 1.14) and write the cokernel by $W(\delta_4)$, we have the following short exact sequence

$$0 \to W(\delta_3) \to W(s) \to W(\delta_4) \to 0.$$

Then, by Proposition 2.14, we have

$$\begin{aligned}(\text{the slope of } W(\delta_3)) &\leq \left(\text{the slope of } W\left(\prod_{\sigma, a_\sigma\neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2\right)\right) \\ &= \frac{1}{[K : \mathbb{Q}_p]}\left(\sum_{\sigma, a_\sigma\neq 0} w(\delta_1/\delta_2)_\sigma\right) + \frac{\mathrm{val}_p(\delta_2(\pi_K))}{f} < 0\end{aligned}$$

because $s \in S'^{\text{non-ét}}(\delta_1, \delta_2)$. So $W(s)$ is not pure of slope zero by the slope filtration theorem. So we have finished the proof when $\delta_1/\delta_2 \neq \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for any $\{k_\sigma\}$ such that $k_\sigma \in \mathbb{Z}$ for any σ.

Next we prove the theorem in the case $\delta_1/\delta_2 = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}$ for any σ. First let us assume that $W(s)$ is not pure of slope zero. Then, as in the first case, there are rank-one E–B-pairs $W(\delta_3), W(\delta_4)$ such that the slope of $W(\delta_3) := u < 0$ and $W(s)$ sits in the following short exact sequence

$$0 \to W(\delta_3) \to W(s) \to W(\delta_4) \to 0.$$

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Then we can prove as in the first case that the induced map $W(\delta_3) \hookrightarrow W(\delta_2)$ is injective and $\delta_3 = \delta_2 \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}_{\geq 0}$ for any σ. So we have

$$\begin{aligned} s &\in \mathrm{Ker}\Bigg(\mathrm{H}^1(G_K, W(\delta_1/\delta_2)) \to \mathrm{H}^1\Bigg(G_K, W\Bigg(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{-k_\sigma}\delta_1/\delta_2\Bigg)\Bigg)\Bigg) \\ &= \mathrm{Im}(\partial : \mathrm{H}^0(G_K, \oplus_{\sigma:K\hookrightarrow E} t^{-k_\sigma} W^+_{\mathrm{dR}}(\delta_1/\delta_2)_\sigma / W^+_{\mathrm{dR}}(\delta_1/\delta_2)_\sigma) \to \mathrm{H}^1(G_K, W(\delta_1/\delta_2))). \end{aligned}$$

By Lemma 2.18, we can see that s corresponds to $s = [\overline{(a_\sigma e_\sigma)_\sigma}] \in \mathbb{P}_E(\oplus_{\sigma, w(\delta_1/\delta_2)_\sigma \in \mathbb{Z}_{\geq 1}} Ee_\sigma/\Delta(E)) \xrightarrow{\sim} S'(\delta_1, \delta_2)$ such that there is a lifting $[(a_\sigma e_\sigma)_\sigma]$ of $[\overline{(a_\sigma e_\sigma)_\sigma}]$ such that $a_\sigma = 0$ for any σ satisfying $w(\delta_1/\delta_2)_\sigma \notin \{1, 2, \ldots, k_\sigma\}$. So $(\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) \leq (\sum_{\sigma:K\hookrightarrow E} k_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) = [K : \mathbb{Q}_p]u < 0$. So s is contained in $S'^{\text{non-ét}}(\delta_1, \delta_2)$.

Next let us assume that $s \in S'^{\text{non-ét}}(\delta_1, \delta_2)$. Then, by definition, $s = [\overline{(a_\sigma e_\sigma)_\sigma}] \in \mathbb{P}_E(\oplus_{\sigma, w(\delta_1/\delta_2)_\sigma \in \mathbb{Z}_{\geq 1}} Ee_\sigma/\Delta(E)) \xrightarrow{\sim} S'(\delta_1, \delta_2)$ such that some lift $[(a_\sigma e_\sigma)_\sigma]$ of $[\overline{(a_\sigma e_\sigma)_\sigma}]$ satisfies $(\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) < 0$. Then, by Lemma 2.18, we have

$$s \in \mathrm{Ker}\Bigg(\mathrm{H}^1(G_K, W(\delta_1/\delta_2)) \to \mathrm{H}^1\Bigg(G_K, W\Bigg(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma}\delta_1/\delta_2\Bigg)\Bigg)\Bigg).$$

In particular, as in the proof of the first case, we can see that the natural inclusion $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(\delta_2)$ factors through an inclusion $g : W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(s)$. If we take the saturation $W(\delta_3)$ of g and write the cokenel by $W(\delta_4)$, then we have (the slope of $W(\delta_3)$) $\leq$ (the slope of $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2)) = 1/[K : \mathbb{Q}_p]((\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K))) < 0$, where the first inequality follows from Proposition 2.14. So $W(s)$ is not pure of slope zero by the slope filtration theorem. Thus, we have finished the proof of this theorem in all of the cases. □

By this theorem, we can determine all of the two-dimensional split trianguline E-representations. For any $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$, we write $V(s)$ as the two-dimensional split trianguline E-representation such that $W(V(s)) = W(s)$.

3.2 Irreducibility of $V(s)$

Next we determine when $V(s)$ is irreducible as a E-representation of G_K. For this, we put

- $S_0^+ := \{(\delta_1, \delta_2) \in S^+ \mid \mathrm{val}_p(\delta_1(\pi_K)) = \mathrm{val}_p(\delta_2(\pi_K)) = 0\}$;
- $S_*^+ := S^+ \setminus S_0^+$.

For any $(\delta_1, \delta_2) \in S_*^+$, we put:

(1) $S'^{\text{ord}}(\delta_1, \delta_2) := \{[\overline{(a_\sigma e_\sigma)_\sigma}] \in S'^{et}(\delta_1, \delta_2) \mid (\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) = 0$ for some lifting $[(a_\sigma e_\sigma)_\sigma]$ of $[\overline{(a_\sigma e_\sigma)_\sigma}]\}$ when $\delta_1/\delta_2 = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}$ for any σ;

(2) $S'^{\text{ord}}(\delta_1, \delta_2) := \{[a_\sigma e_\sigma] \in S'^{\text{ét}}(\delta_1, \delta_2) \mid (\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) = 0\}$ otherwise.

We can easily see that the subset $S'^{\text{ord}}(\delta_1, \delta_2)$ does not depend on the choice of an isomorphism $S'(\delta_1, \delta_2) \xrightarrow{\sim} \mathbb{P}_E(\oplus_{\sigma, w(\delta_1/\delta_2)_\sigma \geq 1} Ee_\sigma/\Delta(E))$ or $S'(\delta_1, \delta_2) \xrightarrow{\sim} \mathbb{P}_E(\oplus_{\sigma, w(\delta_1/\delta_2)_\sigma \in \mathbb{Z}_{\geq 1}} Ee_\sigma)$.

Remark 3.5. When $K = \mathbb{Q}_p$, $\mathcal{S}_+^{\text{ord}}$ in [Col08a, § 4.3] is equal to $\bigsqcup_{(\delta_1,\delta_2)\in S_*^+} S'^{\text{ord}}(\delta_1, \delta_2)$ and $\mathcal{S}_0$ in [Col08a, § 4.3] is equal to $\bigsqcup_{(\delta_1,\delta_2)\in S_0^+} S(\delta_1, \delta_2)$.

The following proposition is a generalization of [Col08a, Propositions 5.7 and 5.8].

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Proposition 3.6. *Let* $(\delta_1, \delta_2) \in S^+$ *and* $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$. *Then the following conditions are equivalent:*

(1) $V(s)$ *is irreducible;*

(2) $(\delta_1, \delta_2) \notin S_0^+$ *and* $s \notin S'^{\text{ord}}(\delta_1, \delta_2)$.

Proof. First let us assume that $V(s)$ is reducible and $(\delta_1, \delta_2) \notin S_0^+$. Because $V(s)$ is reducible, there exist two continuous characters $\delta_3, \delta_4 : G_K \to \mathcal{O}_E^\times$ such that $W(V(s))$ sits in the following short exact sequence

$$0 \to W(E(\delta_3)) \to W(V(s)) \to W(E(\delta_4)) \to 0.$$

Then the fact that $\text{Hom}(W(E(\delta_3)), W(\delta_1)) = 0$ (this follows from the fact that (the slope of $W(E(\delta_3))) = 0 <$ (the slope of $W(\delta_1)$) and from Proposition 2.14) implies that the natural map $W(E(\delta_3)) \hookrightarrow W(\delta_2)$ is an inclusion. So we have $\tilde{\delta_3} = \delta_2 \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}_{\geq 0}$ (here $\tilde{\delta_3} := \delta_3 \circ \text{rec}_K : K^\times \to E^\times$). Then, as in the proof of the previous theorem, we can see that s corresponds to $[(a_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}(\delta_1, \delta_2)$ (or $[\overline{(a_\sigma e_\sigma)_\sigma}] \in S'^{\text{ét}}(\delta_1, \delta_2)$) such that $(\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \text{val}_p(\delta_2(\pi_K)) \leq (\sum_{\sigma:K\hookrightarrow E} k_\sigma) + e_K \text{val}_p(\delta_2(\pi_K)) = [K : \mathbb{Q}_p]$ (slope of $W(E(\delta_3))) = 0$ (for some lifting $[(a_\sigma e_\sigma)_\sigma]$ of $[\overline{(a_\sigma e_\sigma)_\sigma}]$). On the other hand, we have $0 \leq (\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \text{val}_p(\delta_2(\pi_K))$ because $s \in S'^{\text{ét}}(\delta_1, \delta_2)$. So $s \in S'^{\text{ord}}(\delta_1, \delta_2)$.

Next if $(\delta_1, \delta_2) \in S_0^+$, then $W(\delta_1)$ and $W(\delta_2)$ are pure of slope zero, hence $V(s)$ is reducible as a E-representation.

Next let us assume that $(\delta_1, \delta_2) \notin S_0^+$ and $s \in S'^{\text{ord}}(\delta_1, \delta_2)$. If we put $s = [(a_\sigma e_\sigma)_\sigma]$ (or $[\overline{(a_\sigma e_\sigma)_\sigma}]$) such that $(\sum_{\sigma, a_\sigma \neq 0} w(\delta_1/\delta_2)_\sigma) + e_K \text{val}_p(\delta_2(\pi_K)) = 0$ (for some lift $[(a_\sigma e_\sigma)_\sigma]$ of $[\overline{(a_\sigma e_\sigma)_\sigma}]$) then, by the proof of Theorem 3.4, we have

$$s \in \text{Ker}\bigg(\text{H}^1(G_K, W(\delta_1/\delta_2)) \to \text{H}^1\bigg(G_K, W\bigg(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma} \delta_1/\delta_2\bigg)\bigg)\bigg).$$

Then the natural inclusion $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2) \hookrightarrow W(\delta_2)$ factors through $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2) \hookrightarrow W(s)$. We take the saturation $W(\delta_3)$ of this map and write the cokernel by $W(\delta_4)$. Then we claim that the natural inclusion $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2) \hookrightarrow W(\delta_3)$ is an isomorphism. If this is not an isomorphism, then by Proposition 2.14, $W(\delta_3)$ must be pure of negative slope because $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2)$ is pure of slope zero by assumption. Then $W(s)$ is not pure of slope zero, which contradicts the assumption that $s \in S'^{\text{ét}}(\delta_1, \delta_2)$. So $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2) \hookrightarrow W(\delta_3)$ is an isomorphism. Then both $W(\delta_3)$ and $W(\delta_4)$ are pure of slope zero. So $W(s)$ is reducible as a E-representation. Thus, we have finished the proof of this proposition. □

3.3 The conditions for $V(s) = V(s')$

Next let us discuss when two split trianguline E-representations $V(s)$, $V(s')$ are isomorphic for different parameters s, s'. Unfortunately we cannot solve this problem in the case where $\delta_1/\delta_2 = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}$ for any σ and $s \in S'^{\text{ét}}(\delta_1, \delta_2)$. We can solve this problem in all other cases. The following theorem is a generalization of [Col08a, Proposition 4.9].

Theorem 3.7. *We have the following results.*

(1) *Let* $V(s)$, $V(s')$ *be two-dimensional trianguline* E*-representations associated to* $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$, $s' \in S(\delta_3, \delta_4) \setminus S'^{\text{non-ét}}(\delta_3, \delta_4)$ *for some* (δ_1, δ_2), $(\delta_3, \delta_4) \in S^+$.

Moreover, we assume that $s \notin S'^{\text{ét}}(\delta_1, \delta_2)$. *Then* $V(s) \xrightarrow{\sim} V(s')$ *if and only if* $(\delta_1, \delta_2) = (\delta_3, \delta_4)$ *and* $s = s' \in S(\delta_1, \delta_2)$.

(2) *Let* $s = [(a_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}(\delta_1, \delta_2)$ *where* $\delta_1/\delta_2 \neq \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ *for any* $\{k_\sigma\}_\sigma$ *such that* $k_\sigma \in \mathbb{Z}$ *for any* σ. *Then there exists unique* $((\delta_3, \delta_4), s') \neq ((\delta_1, \delta_2), s)$ *(here* $(\delta_3, \delta_4) \in S^+$ *and* $s' \in S(\delta_3, \delta_4) \setminus S'^{\text{non-ét}}(\delta_3, \delta_4)$*) such that* $V(s) \xrightarrow{\sim} V(s')$. *Such a* $((\delta_3, \delta_4), s')$ *satisfies the following:*

(i) $\delta_3 = \delta_2 \prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}$, $\delta_4 = \delta_1 \prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma}$;

(ii) $s' = [(b_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}(\delta_3, \delta_4)$ *satisfies that* $\{\sigma : K \hookrightarrow E \mid b_\sigma \neq 0\} = \{\sigma : K \hookrightarrow E \mid a_\sigma \neq 0\}$.

Proof. First, we prove part (1). If $V(s) \xrightarrow{\sim} V(s')$, then we have the following short exact sequence

$$0 \to W(\delta_3) \to W(s) \to W(\delta_4) \to 0.$$

We consider the natural map $f : W(\delta_3) \to W(\delta_2)$ as in the proof of Theorem 3.4. Then we have one of the following: $\mathrm{Ker}(f) \xrightarrow{\sim} W(\delta_3)$ or $\mathrm{Im}(f) \xrightarrow{\sim} W(\delta_3)$.

If $\mathrm{Ker}(f) \xrightarrow{\sim} W(\delta_3)$, then we have the following commutative diagram.

$$\begin{array}{ccccccccc} 0 & \longrightarrow & W(\delta_3) & \longrightarrow & W(s) & \longrightarrow & W(\delta_4) & \longrightarrow & 0 \\ & & \downarrow & & \| & & \downarrow & & \\ 0 & \longrightarrow & W(\delta_1) & \longrightarrow & W(s) & \longrightarrow & W(\delta_2) & \longrightarrow & 0 \end{array}$$

Then we claim that $W(\delta_3) \to W(\delta_1)$ and $W(\delta_4) \to W(\delta_2)$ are both isomorphisms. We can see that $W_e(\delta_3) \xrightarrow{\sim} W_e(\delta_1)$, $W_e(\delta_4) \xrightarrow{\sim} W_e(\delta_2)$ are isomorphisms in the same way as the proof of Proposition 2.14. For the W^+_{dR} part, by the snake lemma of the above diagram, we have an isomorphism $\mathrm{Ker}(W^+_{\mathrm{dR}}(\delta_4) \to W^+_{\mathrm{dR}}(\delta_2)) \xrightarrow{\sim} \mathrm{Cok}(W^+_{\mathrm{dR}}(\delta_3) \to W^+_{\mathrm{dR}}(\delta_1))$ of B^+_{dR}-modules. However, because the former is a torsion-free B^+_{dR}-module and the latter is a torsion B^+_{dR}-module, so this must be zero. So $W(\delta_3) \to W(\delta_1)$ and $W(\delta_4) \to W(\delta_2)$ are both isomorphisms. Thus, $(\delta_1, \delta_2) = (\delta_3, \delta_4)$ and $s = s' \in S(\delta_1, \delta_2)$.

If $\mathrm{Im}(f) \xrightarrow{\sim} W(\delta_3)$, then $\delta_3 = \delta_2 \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma}$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}_{\geq 0}$ for any σ. Then, by the proof of Theorem 3.4, we can see that $s \in S'(\delta_1, \delta_2)$. This is a contradiction. So we have finished the proof of part (1) of this theorem.

Next, we prove part (2). Let us assume that $V(s) \xrightarrow{\sim} V(s')$ for some $s'(\neq s) \in S(\delta_3, \delta_4)$. Then, by the proof of part (1) above and the assumption $s \neq s'$, we can see that $W(\delta_3) \xrightarrow{\sim} \mathrm{Im}(f : W(\delta_3) \to W(\delta_2))$. So we have $\delta_3 = \prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma} \delta_2$ for some $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}_{\geq 0}$ for any σ. So we obtain $s = [(a_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}(\delta_1, \delta_2)$ and $w(\delta_1/\delta_2)_\sigma \in \{1, 2, \ldots, k_\sigma\}$ for any σ such that $a_\sigma \neq 0$. Then, by the proof of Theorem 3.4, we can see that the natural inclusion $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2) \hookrightarrow W(\delta_2)$ factors through an inclusion $g : W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2) \hookrightarrow W(s)$. Because we have $\mathrm{Hom}(W(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma} \delta_2), W(\delta_1)) = 0$ by the assumption on (δ_1, δ_2) and by Proposition 2.14, we can see that $g \circ \iota : W(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma} \delta_2) \hookrightarrow W(s)$ is equal to the given inclusion $W(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma} \delta_2) = W(\delta_3) \hookrightarrow W(s)$ (here $\iota : W(\prod_{\sigma:K\hookrightarrow E} \sigma(x)^{k_\sigma} \delta_2) \hookrightarrow W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2)$ is the natural inclusion). Because $W(\delta_3)$ is saturated in $W(s)$ so $W(\delta_3) \hookrightarrow W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2)$ must be isomorphic. So we have $k_\sigma = w(\delta_1/\delta_2)_\sigma$ for σ such that $a_\sigma \neq 0$ and $k_\sigma = 0$ for other σ. Then we have $\delta_3 = \prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma} \delta_2$ and $\delta_4 = \prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma} \delta_1$. We have $w(\delta_3/\delta_4)_\sigma = w(\delta_3)_\sigma - w(\delta_4)_\sigma = w(\delta_1/\delta_2)_\sigma + w(\delta_2)_\sigma - (-w(\delta_1/\delta_2)_\sigma + w(\delta_1)_\sigma) = w(\delta_1/\delta_2)_\sigma$ for any σ such that $a_\sigma \neq 0$ and $w(\delta_3/\delta_4)_\sigma = w(\delta_2)_\sigma - w(\delta_1)_\sigma = -w(\delta_1/\delta_2)_\sigma$ for other σ. If we replace s by s' and replace s' by s in the above argument, we can see that $s' = [(b_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2, \prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma}\delta_1)$ satisfies the condition that $\{\sigma : K \hookrightarrow E \mid b_\sigma \neq 0\} = \{\sigma : K \hookrightarrow E \mid a_\sigma \neq 0\}$. If $s'' = [(b'_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2, \prod_{\sigma, a_\sigma \neq 0}$ $\sigma(x)^{-w(\delta_1/\delta_2)_\sigma}\delta_1)$ is another element such that $V(s) \xrightarrow{\sim} V(s'')$ and $s \neq s''$, then we can see that $s' = s''$ in the same way as in the proof of part (1) above. Thus, such an s' is unique.

Next let us take any $s = [a_\sigma e_\sigma] \in S'^{\text{ét}}(\delta_1, \delta_2)$. Then, by Lemma 2.18, we have

$$s \in \mathrm{Ker}\Big(\mathrm{H}^1(G_K, W(\delta_1/\delta_2)) \to \mathrm{H}^1\Big(G_K, W\Big(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma}\delta_1/\delta_2\Big)\Big)\Big).$$

So the natural inclusion $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(\delta_2)$ factors through $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(s)$. Then we can see that $W(\prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{w(\delta_1/\delta_2)_\sigma}\delta_2) \hookrightarrow W(s)$ is saturated (if not, then we have $s = [a'_\sigma e_\sigma] \in S'^{\text{ét}}(\delta_1, \delta_2)$ such that $\{\sigma : K \hookrightarrow E \mid a'_\sigma \neq 0\} \subsetneqq \{\sigma : K \hookrightarrow E \mid a_\sigma \neq 0\}$, which cannot happen under the assumption on (δ_1, δ_2) in this theorem). If we write the cokernel of this as $W(\delta_4)$, then we have $\delta_4 = \prod_{\sigma, a_\sigma \neq 0} \sigma(x)^{-w(\delta_1/\delta_2)_\sigma}\delta_1$ and we have the following short exact sequence

$$0 \to W(\delta_3) \to W(s) \to W(\delta_4) \to 0.$$

This extension corresponds to some $s' = [b_\sigma e_\sigma] \in S'^{\text{ét}}(\delta_3, \delta_4)$. So we have $V(s) \xrightarrow{\sim} V(s')$. We have finished the proof of this theorem. □

Remark 3.8. When $K = \mathbb{Q}_p$, the exceptional cases where $s \in S'^{\text{ét}}(\delta_1, \delta_2)$ and $\delta_1/\delta_2 = \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{k_\sigma}$ do not appear because $S'^{\text{ét}}(\delta_1, \delta_2)$ is empty set in this case.

4. Classification of two-dimensional potentially semi-stable split trianguline *E*-representations

In this final section, we classify all of the two-dimensional potentially semi-stable split trianguline E-representations. We explicitly describe the E-filtered (φ, N, G_K)-modules associated with potentially semi-stable split trianguline E-representations.

4.1 de Rham split trianguline *E*-representations

Before classifying two-dimensional de Rham split trianguline E-representations, we determine all of the rank-one de Rham E–B-pairs.

Lemma 4.1. *Let $W(\delta)$ be a rank-one E–B-pair. Then the following conditions are equivalent:*

(1) *$W(\delta)$ is Hodge–Tate;*
(2) *$W(\delta)$ is de Rham;*
(3) *$\delta = \tilde{\delta} \prod_{\sigma : K \hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $\tilde{\delta} : K^\times \to E^\times$ is a locally constant character and $k_\sigma \in \mathbb{Z}$ for any σ.*

Proof. It is easy to see the implication $(3) \Rightarrow (2) \Rightarrow (1)$. We prove that (1) implies (3). So we assume that $W(\delta)$ is Hodge–Tate with generalized Hodge–Tate weight $\{k_\sigma\}_\sigma$ such that $k_\sigma \in \mathbb{Z}$ for any σ. We write $W(\delta) \xrightarrow{\sim} W(E(\delta_0)) \otimes W_0^{\otimes i}$ for some continuous character $\delta_0 : G_K \to \mathcal{O}_E^\times$ and $i \in \mathbb{Z}$ as in Theorem 1.45. Because $W_0 := (W_{e,0}, W^+_{\mathrm{dR},0})$ satisfies $W^+_{\mathrm{dR}} \xrightarrow{\sim} B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E$, so we have $W(\delta)^+_{\mathrm{dR}} \xrightarrow{\sim} B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E(\delta_0)$. Then $E(\delta_0 \prod_{\sigma : K \hookrightarrow E} \sigma(\chi_{\mathrm{LT}})^{-k_\sigma})$ is Hodge–Tate with all Hodge–Tate weight zero. Then, by Sen's theorem [Ber02, Proposition 5.24], $\delta_0 \prod_{\sigma : K \hookrightarrow E} \sigma(\chi_{\mathrm{LT}})^{-k_\sigma}$:

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$G_K \to \mathcal{O}_E^\times$ is a potentially unramified character. Because $\sigma(\chi_{\mathrm{LT}}): G_K \to \mathcal{O}_E^\times$ corresponds to the character $K^\times \to E^\times : \pi_K \mapsto 1, u \mapsto \sigma(u)$ for $u \in \mathcal{O}_K^\times$, we have $\delta = \tilde{\delta} \prod_{\sigma: K \hookrightarrow E} \sigma(x)^{k_\sigma}$ for some locally constant character $\tilde{\delta}: K^\times \to E^\times$. So we have finished the proof of the lemma. □

Next let $V(s)$ be a potentially semi-stable split trianguline E-representation such that $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$ for some $(\delta_1, \delta_2) \in S^+$. Then we have the following short exact sequence

$$0 \to W(\delta_1) \to W(s) \to W(\delta_2) \to 0.$$

Then, as in the case of usual E-representations, we can see that $W(\delta_1)$ and $W(\delta_2)$ are also de Rham, in particular $W(\delta_1/\delta_2)$ is de Rham. Next we compute $\mathrm{H}^1_e(G_K, W(\delta))$, $\mathrm{H}^1_f(G_K, W(\delta))$ and $\mathrm{H}^1_g(G_K, W(\delta))$ when $W(\delta)$ is de Rham.

LEMMA 4.2. *Let $W(\delta)$ be a rank-one de Rham E–B-pair where $\delta := \tilde{\delta} \prod_{\sigma: K \hookrightarrow E} \sigma(x)^{k_\sigma}$ such that $\tilde{\delta}: K^\times \to E^\times$ is a locally constant character. Then we have a canonical isomorphism*

$$D^K_{\mathrm{dR}}(W(\delta))/(D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta))) \xrightarrow{\sim} \mathrm{H}^1_e(G_K, W(\delta))$$

and $\dim_E \mathrm{H}^1_e(G_K, W(\delta))$ is equal to:

(1) $\sharp\{\sigma: K \hookrightarrow E \mid k_\sigma \in \mathbb{Z}_{\geq 1}\}$ *when $\tilde{\delta}$ is not the trivial character;*

(2) $\sharp\{\sigma: K \hookrightarrow E \mid k_\sigma \in \mathbb{Z}_{\geq 1}\} - 1$ *when $\tilde{\delta}$ is the trivial character.*

Proof. For general δ, we have the following short exact sequence

$$\begin{aligned} 0 \to D^K_{\mathrm{dR}}(W(\delta))/(D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta))) \to \mathrm{H}^1(G_K, W(\delta)) \\ \to \mathrm{Ker}(\mathrm{H}^1(G_K, W_e(\delta)) \oplus \mathrm{H}^1(G_K, W^+_{\mathrm{dR}}(\delta)) \to \mathrm{H}^1(G_K, W_{\mathrm{dR}}(\delta))) \to 0. \end{aligned}$$

When $W(\delta)$ is de Rham, we proved in Lemma 2.6 that $\mathrm{H}^1(G_K, W^+_{\mathrm{dR}}(\delta)) \to \mathrm{H}^1(G_K, W_{\mathrm{dR}}(\delta))$ is injective. So we have

$$\begin{aligned} \mathrm{H}^1_e(G_K, W(\delta)) &= \mathrm{Ker}(\mathrm{H}^1(G_K, W(\delta)) \to \mathrm{H}^1(G_K, W_e(\delta))) \\ &\xrightarrow{\sim} \mathrm{Ker}(\mathrm{H}^1(G_K, W(\delta)) \to (\mathrm{H}^1(G_K, W_e(\delta)) \oplus \mathrm{H}^1(G_K, W^+_{\mathrm{dR}}(\delta)))) \\ &= D^K_{\mathrm{dR}}(W(\delta))/(D^K_{\mathrm{cris}}(W(\delta))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta))). \end{aligned}$$

We can compute the dimension of $\mathrm{H}^1_e(G_K, W(\delta))$ by using this isomorphism. □

LEMMA 4.3. *Let $W(\delta)$ be a rank-one de Rham E–B-pair. Then we have:*

(1) $\mathrm{H}^1_g(G_K, W(\delta)) = \mathrm{H}^1_f(G_K, W(\delta))$ *and* $\dim_E \mathrm{H}^1_f(G_K, W(\delta)) = \dim_E \mathrm{H}^1_e(G_K, W(\delta)) + 1$ *when* $\delta = \prod_{\sigma: K \hookrightarrow E} \sigma(x)^{k_\sigma}$;

(2) $\mathrm{H}^1_f(G_K, W(\delta)) = \mathrm{H}^1_e(G_K, W(\delta))$ *and* $\dim_E \mathrm{H}^1_g(G_K, W(\delta)) = \dim_E \mathrm{H}^1_f(G_K, W(\delta)) + 1$ *when* $\delta = |N_{K/\mathbb{Q}_p}(x)| \prod_{\sigma: K \hookrightarrow E} \sigma(x)^{k_\sigma}$;

(3) $\mathrm{H}^1_g(G_K, W(\delta)) = \mathrm{H}^1_f(G_K, W(\delta)) = \mathrm{H}^1_e(G_K, W(\delta))$ *otherwise.*

Proof. This follows from a calculation using Lemmas 2.7 and 2.11. □

For the explicit description of potentially semi-stable split trianguline E-representations, we fix an extension whose class is contained in $\mathrm{H}^1_f(G_K, W(\delta)) \setminus \mathrm{H}^1_e(G_K, W(\delta))$ when $\delta = \prod_\sigma \sigma(x)^{k_\sigma}$ and fix an extension whose class is contained in $\mathrm{H}^1_g(G_K, W(\delta)) \setminus \mathrm{H}^1_f(G_K, W(\delta))$ when $\delta = |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$ as follows. First we treat the case where $\delta = \prod_\sigma \sigma(x)^{k_\sigma}$. In this case, we have $W(\delta) = (B_e \otimes_{\mathbb{Q}_p} E, \oplus_\sigma t^{k_\sigma} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E)$ by Lemma 2.12. We fix an

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element $a_{\mathrm{cris}} \in \mathcal{O}^{\times}_{\mathbb{Q}_p^{\mathrm{un}}}$ such that $\mathrm{Frob}_{\mathbb{Q}_p}(a_{\mathrm{cris}}) = a_{\mathrm{cris}} + 1$. We define an extension $W(s(\{k_\sigma\}_\sigma)) := (W_e(s(\{k_\sigma\}_\sigma)), W^+_{\mathrm{dR}}(s(\{k_\sigma\}_\sigma)), \iota)$ whose class $s(\{k_\sigma\}_\sigma) := [W(s(\{k_\sigma\}_\sigma))]$ is contained in $\mathrm{H}^1_f(G_K, W(\prod_\sigma \sigma(x)^{k_\sigma})) \setminus \mathrm{H}^1_e(G_K, W(\prod_\sigma \sigma(x)^{k_\sigma}))$ as follows:

(1) $W_e(s(\{k_\sigma\}_\sigma)) := (B_e \otimes_{\mathbb{Q}_p} E)e_1 \oplus (B_e \otimes_{\mathbb{Q}_p} E)e_{\mathrm{cris}}$ such that $g(e_1) = e_1$, $g(e_{\mathrm{cris}}) = e_{\mathrm{cris}} + f\deg(g)e_1$ for any $g \in G_K$, here $f = [K_0 : \mathbb{Q}_p]$;

(2) $W^+_{\mathrm{dR}}(s(\{k_\sigma\}_\sigma)) := (\oplus_\sigma t^{k_\sigma} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E)e_1 \oplus (B^+_{\mathrm{dR}} \oplus_{\mathbb{Q}_p} E)e_{\mathrm{dR}}$ such that $g(e_1) = e_1$ and $g(e_{\mathrm{dR}}) = e_{\mathrm{dR}}$ for any $g \in G_K$;

(3) $\iota : B_{\mathrm{dR}} \otimes_{B_e} W_e(s(\{k_\sigma\}_\sigma)) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}(s(\{k_\sigma\}_\sigma))$ is the isomorphism defined by $\iota(e_1) = e_1$ and $\iota(e_{\mathrm{cris}}) = e_{\mathrm{dR}} + a_{\mathrm{cris}}e_1$.

We can easily see that $s(\{k_\sigma\}_\sigma) := [W(s(\{k_\sigma\}_\sigma))]$ is contained in $\mathrm{H}^1_f(G_K, W(\delta)) \setminus \mathrm{H}^1_e(G_K, W(\delta))$.

Next we treat the case where $\delta = |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$. We define a continuous one cocycle $c : G_K \to \mathbb{Q}_p(\chi)$ by $g(\log([\tilde{p}])) = \log([\tilde{p}]) + c(g)t$ for any $g \in G_K$. In this case, we define an extension $W(s(\{k_\sigma\}_\sigma)) := (W_e(s(\{k_\sigma\}_\sigma)), W^+_{\mathrm{dR}}(s(\{k_\sigma\}_\sigma)), \iota)$ whose class $s(\{k_\sigma\}_\sigma) := [W(s(\{k_\sigma\}_\sigma))]$ is in $\mathrm{H}^1_g(G_K, W(|N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma})) \setminus \mathrm{H}^1_f(G_K, W(|N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}))$ as follows:

(1) $W_e(s(\{k_\sigma\}_\sigma)) := (B_e \otimes_{\mathbb{Q}_p} E(\chi))e_1 \oplus (B_e \otimes_{\mathbb{Q}_p} E)e_{\mathrm{cris}}$ such that $g(e_1) = \chi(g)e_1, g(e_{\mathrm{cris}}) = e_{\mathrm{cris}} + c(g)e_1$ for any $g \in G_K$;

(2) $W^+_{\mathrm{dR}}(s(\{k_\sigma\}_\sigma)) := (\oplus_\sigma t^{k_\sigma - 1} B^+_{\mathrm{dR}} \otimes_{K,\sigma} E(\chi))e_1 \oplus (B^+_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E)e_{\mathrm{dR}}$ such that $g(e_1) = \chi(g)e_1, g(e_{\mathrm{dR}}) = e_{\mathrm{dR}}$ for any $g \in G_K$;

(3) $\iota : B_{\mathrm{dR}} \otimes_{B_e} W_e(s(\{k_\sigma\}_\sigma)) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}(s(\{k_\sigma\}_\sigma))$ is the isomorphism defined by $\iota(e_1) = e_1, \iota(e_{\mathrm{cris}}) = e_{\mathrm{dR}} + (\log[(\tilde{p})]/t)e_1$.

We can easily see that $s(\{k_\sigma\}_\sigma) := [W(s(\{k_\sigma\}_\sigma))]$ is contained in $\mathrm{H}^1_g(G_K, W(\delta)) \setminus \mathrm{H}^1_f(G_K, W(\delta))$.

By using these classes, we can determine all of the potentially crystalline split trianguline E-representations and all of the potentially semi-stable split trianguline E-representations which are not potentially crystalline. For this, let us:

– put
$$S''(\delta_1, \delta_2) := \{s \in S(\delta_1, \delta_2) \mid s \in \mathbb{P}_E(\mathrm{H}^1_f(G_K, W(\delta_1/\delta_2)) \setminus S'(\delta_1, \delta_2)\}$$
when $\delta_1/\delta_2 = \prod_\sigma \sigma(x)^{k_\sigma}$;

– put
$$S_{\mathrm{st}}(\delta_1, \delta_2) := \{s \in S(\delta_1, \delta_2) \mid s \in \mathbb{P}_E(\mathrm{H}^1_g(G_K, W(\delta_1/\delta_2)) \setminus S'(\delta_1, \delta_2))\}$$
when $\delta_1/\delta_2 = |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$.

Then, by Lemma 4.3 and the above constructions, we have:

- $S''(\delta_1, \delta_2) \xrightarrow{\sim} s(\{k_\sigma\}_\sigma) + D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)/(D^K_{\mathrm{cris}}(W(\delta_1/\delta_2))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)))$;
- $S_{\mathrm{st}}(\delta_1, \delta_2) \xrightarrow{\sim} s(\{k_\sigma\}_\sigma) + D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)/(D^K_{\mathrm{cris}}(W(\delta_1/\delta_2))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)))$.

Proposition 4.4. *Let $(\delta_1, \delta_2) \in S^+$ such that $W(\delta_1)$ and $W(\delta_2)$ are de Rham E–B-pairs. Let $V(s)$ be a split trianguline E-representation associated with $s \in S(\delta_1, \delta_2) \setminus S'^{\text{non-ét}}(\delta_1, \delta_2)$. Then the following conditions are equivalent:*

(1) *$V(s)$ is potentially crystalline;*

(2) *$s \in S^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2) := S'^{\text{ét}}(\delta_1, \delta_2) \sqcup S''(\delta_1, \delta_2)$ when $\delta_1/\delta_2 = \prod_\sigma \sigma(x)^{k_\sigma}$;*
$s \in S^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2) := S'^{\text{ét}}(\delta_1, \delta_2)$ otherwise.

The following conditions are equivalent:

(3) *$V(s)$ is potentially semi-stable and not potentially crystalline;*

(4) $\delta_1/\delta_2 = |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$ *and* $s \in S_{\mathrm{st}}(\delta_1, \delta_2)$.

Proof. This easily follows from Lemma 4.3. We prove in the case $\delta_1/\delta_2 = |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$. (Other cases can be proved in the same way.) First we prove that part (1) is equivalent to part (2). If $s \in S'^{\text{ét}}(\delta_1, \delta_2)$ then $V(s)$ is potentially crystalline by Remark 2.5. If $V(s)$ is potentially crystalline, then $V(s)$ is de Rham. So $[W(V(s))]$ is contained in $\mathrm{H}^1_g(G_K, W(\delta_1/\delta_2))$. If $[W(V(s))] \notin \mathrm{H}^1_f(G_K, W(\delta_1/\delta_2))$, then, by the above remark, we can assume that $[W(V(s))] = s(\{k_\sigma\}_\sigma) + x$ for some $x \in D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)/(D^K_{\mathrm{cris}}(W(\delta_1/\delta_2))^{\varphi=1} + \mathrm{Fil}^0 D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)))$. From this and from the construction of $s(\{k_\sigma\}_\sigma)$, we can easily see that $W(V(s))|_{G_L} \notin \mathrm{H}^1_f(G_L, W(\delta_1/\delta_2)|_{G_L})$ for any finite extension L of K. (Here, $W(\delta_1/\delta_2)|_{G_L}$ is the E–B-pair of G_L obtained by restriction.) This implies that $V(s)$ is not potentially crystalline. This is a contradiction. So $W(V(s))$ is contained in $\mathrm{H}^1_f(G_K, W(\delta_1/\delta_2))$, i.e. $s \in S'^{\text{ét}}(\delta_1, \delta_2)$. We can prove the equivalence between (3) and (4) in the same way. □

Remark 4.5. When $K = \mathbb{Q}_p$, $\mathcal{S}^{\mathrm{cris}}_+$ in [Col08a, § 4.3] is equal to $\sqcup_{(\delta_1,\delta_2)\in S^+} S^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2)$ and $\mathcal{S}^{\mathrm{st}}_+$ is equal to $\sqcup_{(\delta_1,\delta_2)\in S^+} S_{\mathrm{st}}(\delta_1, \delta_2)$. Here, for $(\delta_1, \delta_2) \in S^+$ such that $\delta_1/\delta_2 \neq |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$, we put $S_{\mathrm{st}}(\delta_1, \delta_2) := \emptyset$, the empty set.

4.2 Potentially crystalline split trianguline *E*-representations

In this section, we explicitly describe the filtered (φ, G_K)-modules of potentially crystalline split trianguline E-representations. First we define parameter spaces of potentially crystalline split trianguline E-representations. We put:

$$T_{\mathrm{cris}} := \left\{ (\delta_1, \delta_2, \{k_\sigma\}_\sigma) \;\middle|\; \delta_1, \delta_2 : K^\times \to E^\times \text{ locally constant characters, } k_\sigma \in \mathbb{Z} \text{ for any } \sigma, \right.$$
$$\left. \text{such that } \left(\sum_\sigma k_\sigma\right) + e_K \mathrm{val}_p(\delta_1(\pi_K)) = -e_K \mathrm{val}_p(\delta_2(\pi_K)) \geq 0 \right\}.$$

For any locally constant character $\delta : K^\times \to E^\times$ we define $n(\delta) \in \mathbb{Z}_{\geq 0}$ as the minimal $n \in \mathbb{Z}_{\geq 0}$ such that $\delta|_{1+\pi_K^n \mathcal{O}_K}$ is trivial. We write for any σ:

- $G(\delta)_\sigma := \sum_{\gamma \in \mathrm{Gal}(K_{n(\delta)}/K)} \gamma(\pi_{n(\delta)}) \otimes \delta(\gamma^{-1}) \in K_n \otimes_{K,\sigma} E$ when $n(\delta) \geq 1$;
- $G(\delta)_\sigma := 1 \in K \otimes_{K,\sigma} E$ when $n(\delta) = 0$.

(Here π_n is an element in $\bar{K}$ such that $[\pi_K](\pi_{n+1}) = \pi_n$ for any $n \in \mathbb{N}$ and $[\pi_K](\pi_1) = 0$, i.e. a system of π_K^n torsion points of Lubin–Tate group associated with π_K, and $K_n := K(\pi_n)$ for $n \in \mathbb{Z}_{\geq 1}$ and we identify $\mathrm{Gal}(K_n/K) \xrightarrow{\sim} \mathcal{O}_K^\times/1 + \pi_K^n \mathcal{O}_K$ via the Lubin–Tate character χ_{LT}.) For any $(\delta_1, \delta_2, \{k_\sigma\}_\sigma) \in T_{\mathrm{cris}}$, we define the parameter space $T^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$ of weakly admissible filtrations as follows. First, when $\delta_1 \neq \delta_2$, we define:

- $T_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) := \mathbb{P}_E(\oplus_{\sigma, k_\sigma \geq 1} E e_\sigma)$;
- $T^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) := \{[(a_\sigma e_\sigma)_\sigma] \in T_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) \mid (\sum_{\sigma, a_\sigma \neq 0} k_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) \geq 0\}$.

When $\delta_1 = \delta_2$, we define:

- $T'_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) := \mathbb{P}_E(\oplus_{\sigma, k_\sigma \geq 1} E e_\sigma / \Delta(E))$;
- $T'^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) := \{[(\overline{a_\sigma e_\sigma})_\sigma] \in T'_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) \mid (\sum_{\sigma, a_\sigma \neq 0} k_\sigma) + e_K \mathrm{val}_p(\delta_2(\pi_K)) \geq 0$ for any lift $(a_\sigma e_\sigma)_\sigma \in \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma$ of $[\overline{(a_\sigma e_\sigma)_\sigma}]\}$;

- $T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) := \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma / \Delta(E)$;
- $T^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) := T'^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) \sqcup T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$.

For any $x \in T^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$ we want to define a rank-two E-filtered $(\varphi, \text{Gal}(K_{n(\delta_1,\delta_2)}/K))$-module $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$. Here we put $n(\delta_1, \delta_2) := \max\{n(\delta_1), n(\delta_2)\}$. However, we cannot canonically construct these. We must fix one more parameter for any $(\delta_1, \delta_2, \{k_\sigma\}_\sigma) \in T_{\text{cris}}$. For this, we consider the following short exact sequence

$$0 \to E^\times \xrightarrow{g_1} (K_0 \otimes_{\mathbb{Q}_p} E)^\times \xrightarrow{g_2} (K_0 \otimes_{\mathbb{Q}_p} E)^\times \xrightarrow{g_3} E^\times \to 0, \tag{2}$$

here $g_1 : E^\times \to (K_0 \otimes_{\mathbb{Q}_p} E)^\times$ is the canonical inclusion, $g_2 : (K_0 \otimes_{\mathbb{Q}_p} E)^\times \to (K_0 \otimes_{\mathbb{Q}_p} E)^\times : x \mapsto \varphi(x)/x$ and $g_3 : (K_0 \otimes_{\mathbb{Q}_p} E)^\times \to E^\times : x \mapsto \prod_{i=0}^{f-1} \varphi^i(x)$ where $\varphi : K_0 \otimes_{\mathbb{Q}_p} E \to K_0 \otimes_{\mathbb{Q}_p} E : \sum_i x_i \otimes y_i \mapsto \sum_i \varphi(x_i) \otimes y_i$. We can easily prove that this sequence is exact by using the assumption $K_0 \subset E$. Then we fix $(\alpha, \beta) \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times \times (K_0 \otimes_{\mathbb{Q}_p} E)^\times$ such that $g_3(\alpha) = \delta_1(\pi_K)$, $g_3(\beta) = \delta_2(\pi_K)$ when $\delta_1 \neq \delta_2$. We fix $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times$ such that $g_3(\alpha) = \delta_1(\pi_K)$ and put $\beta = \alpha$ when $\delta_1 = \delta_2$. Then we define $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$ as follows:

(0) $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x} := (K_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_0 \otimes_{\mathbb{Q}_p} E)e_2$;

(1) $N(e_1) = 0$, $N(e_2) = 0$;

(2) when $\delta_1 \neq \delta_2$ or when $\delta_1 = \delta_2$ and $x \in T'^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$, we put $\varphi(e_1) = \alpha e_1, \varphi(e_2) = \beta e_2$ (then $\varphi^f(e_1) = \delta_1(\pi_K)e_1$, $\varphi^f(e_2) = \delta_2(\pi_K)e_2$);

(2′) when $\delta_1 = \delta_2$ and $x \in T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$, we put $\varphi(e_1) = \alpha e_1$, $\varphi(e_2) = \alpha(e_2 + e_1)$;

(3) $g(e_1) = \delta_1(\chi_{\text{LT}}(g))e_1$, $g(e_2) = \delta_2(\chi_{\text{LT}}(g))e_2$ for any $g \in G_K$;

(4) we put

$$\begin{aligned} K_{n(\delta_1,\delta_2)} \otimes_{K_0} D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x} &= (K_{n(\delta_1,\delta_2)} \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_{n(\delta_1\delta_2)} \otimes_{\mathbb{Q}_p} E)e_2 \\ &\xrightarrow{\sim} \oplus_{\sigma : K \hookrightarrow E} (K_{n(\delta_1,\delta_2)} \otimes_{K,\sigma} E)e_{1,\sigma} \oplus (K_{n(\delta_1,\delta_2)} \otimes_{K,\sigma} E)e_{2,\sigma} \\ &=: \oplus_\sigma D_\sigma, \end{aligned}$$

then:

(i) for σ such that $k_\sigma \leqq -1$, we put $\text{Fil}^0 D_\sigma = D_\sigma$, $\text{Fil}^1 D_\sigma = \cdots = \text{Fil}^{-k_\sigma} D_\sigma = K_{n(\delta_1,\delta_2)} \otimes_{K,\sigma} E e_{1,\sigma}$, $\text{Fil}^{-k_\sigma+1} D_\sigma = 0$;

(ii) for σ such that $k_\sigma = 0$, we put $\text{Fil}^0 D_\sigma = D_\sigma$, $\text{Fil}^1 D_\sigma = 0$;

(iii) for σ such that $k_\sigma \geq 1$, we put $\text{Fil}^{-k_\sigma} D_\sigma = D_\sigma$, $\text{Fil}^{-k_\sigma+1} D_\sigma = \cdots = \text{Fil}^0 D_\sigma = K_{n(\delta_1,\delta_2)} \otimes_{K,\sigma} E(a_\sigma G(\delta_2/\delta_1)_\sigma e_{1,\sigma} + e_{2,\sigma})$, $\text{Fil}^1 D_\sigma = 0$.

Here when $\delta_1 \neq \delta_2$, then $(a_\sigma e_\sigma)_\sigma \in \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma$ is a lift of $x = [(a_\sigma e_\sigma)_\sigma] \in T^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) = \mathbb{P}_E(\oplus_{\sigma, k_\sigma \geq 1} E e_\sigma)$ and when $\delta_1 = \delta_2$, then $(a_\sigma e_\sigma)_\sigma \in \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma$ is a lift of $[\overline{(a_\sigma e_\sigma)_\sigma}] \in T'^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) = \mathbb{P}_E(\oplus_{\sigma, k_\sigma \geq 1} E e_\sigma / \Delta(E))$ or is a lift of $\overline{(a_\sigma e_\sigma)_\sigma} \in T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma) = \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma / \Delta(E)$. Then we claim that the isomorphism class of $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$ does not depend on the choice of a lift $(a_\sigma e_\sigma)_\sigma$. We prove this claim in the case where $\delta_1 = \delta_2$ and $x = \overline{(a_\sigma e_\sigma)_\sigma} \in T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$. Let us take two lifts $(a_\sigma e_\sigma)_\sigma$, $(a'_\sigma e_\sigma)_\sigma \in \oplus_{\sigma, k_\sigma \in \mathbb{Z}_{\geq 1}} E e_\sigma$ of $x = \overline{(a_\sigma e_\sigma)_\sigma} \in T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$. Then there exist $a \in E$ and $b \in E^\times$ such that $a'_\sigma = b a_\sigma + a$ for any σ such that $k_\sigma \in \mathbb{Z}_{\geq 1}$. We denote $D_0 := (K_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_0 \otimes_{\mathbb{Q}_p} E)e_2$ and $D'_0 := (K_0 \otimes_{\mathbb{Q}_p} E)e'_1 \oplus (K_0 \otimes_{\mathbb{Q}_p} E)e'_2$ as the filtered (φ, G_K)-modules $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$ defined by using the lifts $(a_\sigma e_\sigma)_\sigma$ and $(a'_\sigma e_\sigma)_\sigma$, respectively. Then it is easy to see that the map $D_0 \to D'_0 : e_1 \mapsto b e'_1, e_2 \mapsto e'_2 + a e'_1$ is an isomorphism of filtered (φ, G_K)-modules. We can easily prove this claim in other cases.

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Next we note the dependence of the choice of (α,β) as above. We only treat the case $\delta_1\neq\delta_2$. Let us take another $(\alpha',\beta')\in(K_0\otimes_{\mathbb{Q}_p}E)^\times\times(K_0\otimes_{\mathbb{Q}_p}E)^\times$ such that $g_3(\alpha')=\delta_1(\pi_K)$, $g_3(\beta')=\delta_2(\pi_K)$. Then, by the above exact sequence (2), there exist $\alpha_0,\beta_0\in(K_0\otimes_{\mathbb{Q}_p}E)^\times$ such that $\alpha=\alpha' g_2(\alpha_0)$, $\beta=\beta' g_2(\beta_0)$. For stressing the dependence of the choice of (α,β), we write $T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta))$, $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta)),x}$, instead of $T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma)$, $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$. Then we can easily see that the map $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta)),[(a_\sigma e_\sigma)_\sigma]}\to D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha',\beta')),[(a_\sigma(\alpha_{0,\sigma}/\beta_{0,\sigma})e_\sigma)_\sigma]}$: $e_1\mapsto\alpha_0e'_1, e_2\mapsto\beta_0e'_2$ is an isomorphism of filtered (φ,G_K)-modules for any $[(a_\sigma e_\sigma)_\sigma]\in T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta))$. (Here e'_i is the basis of $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha',\beta')),[(a_\sigma(\alpha_{0,\sigma}/\beta_{0,\sigma})e_\sigma)_\sigma]}$ defined in the same way as in the case (α,β) and $\alpha_{0,\sigma},\beta_{0,\sigma}\in E^\times$ is the σ-components of $\alpha_0,\beta_0\in(K_0\otimes_{\mathbb{Q}_p}E)^\times\hookrightarrow(K\otimes_{\mathbb{Q}_p}E)^\times\xrightarrow{\sim}\prod_{\sigma:K\hookrightarrow E}Ee_\sigma^\times$.) The other cases are the same. So we have an isomorphism $\iota:T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta))\xrightarrow{\sim}T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha',\beta'))$ between parameter spaces such that $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta)),x}\xrightarrow{\sim}D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha',\beta')),\iota(x)}$ for any $x\in T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma,(\alpha,\beta))$.

Henceforth, we fix a (α,β) for any $(\delta_1,\delta_2,\{k_\sigma\}_\sigma)\in T_{\text{cris}}$. In the proof of the next theorem, we prove that these are weakly admissible filtered (φ,G_K)-modules. So, by the 'weakly admissible imply admissible' theorem [Ber08a, Theorem B], for any $x\in T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma)$, there exists a unique (up to isomorphism) two-dimensional potentially crystalline E-representation $V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$ such that $D^{K_{n(\delta_1,\delta_2)}}_{\text{cris}}(V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x})\xrightarrow{\sim}D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}$. Then our main result of classification concerning potentially crystalline split trianguline E-representations is as follows.

Theorem 4.6. *Let V be a two-dimensional E-representation. Then the following conditions are equivalent:*

(1) *V is split trianguline and potentially crystalline;*

(2) *there exist $(\delta_1,\delta_2,\{k_\sigma\}_\sigma)\in T_{\text{cris}}$ and $x\in T^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma)$ and $\{w_\sigma\}_\sigma$ such that $w_\sigma\in\mathbb{Z}$ for any σ, such that $V\xrightarrow{\sim}V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x}(\prod_\sigma\sigma(\chi_{\text{LT}}^{w_\sigma}))$.*

Proof. First we prove that condition (1) implies condition (2). Let us assume that V is a split trianguline and potentially crystalline E-representation. Then, by Proposition 4.4, there exists a $(\delta_1,\delta_2)\in S^+$ and an $s\in S^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2)$ such that $V\xrightarrow{\sim}V(s)$. Twisting V by a suitable $\prod_\sigma\sigma(\chi_{\text{LT}}(x)^{l_\sigma})$, we may assume that $W(\delta_2)$ has generalized Hodge–Tate weight $\{0\}_\sigma$. If we write the generalized Hodge–Tate weight of $W(\delta_1)$ as $\{k_\sigma\}_\sigma$, then we have $\delta_1=\tilde{\delta}_1\prod_\sigma\sigma(x)^{k_\sigma}$ and $\delta_2=\tilde{\delta}_2$ for some locally constant characters $\tilde{\delta}_1,\tilde{\delta}_2: K^\times\to E^\times$. Then the condition that $(\delta_1,\delta_2)\in S^+$ is equivalent to the condition that $(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma)\in T_{\text{cris}}$. Now we explicitly calculate $D^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}_{\text{cris}}(V)$. First we compute this in the case where $\tilde{\delta}_1\neq\tilde{\delta}_2$ or $\tilde{\delta}_1=\tilde{\delta}_2$ and $s\in {S'}^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2)$. Then, by the definition of ${S'}^{\text{ét}}_{\text{cris}}(\delta_1,\delta_2)\xrightarrow{\sim}\mathbb{P}_E(D^K_{\text{dR}}(W(\delta_1/\delta_2))/(D^K_{\text{cris}}(W(\delta_1/\delta_2))^{\varphi=1}+\text{Fil}^0D^K_{\text{dR}}(W(\delta_1/\delta_2)))$ and by the definition of the boundary map $D^K_{\text{dR}}(W(\delta_1/\delta_2))/(D^K_{\text{cris}}(W(\delta_1/\delta_2))^{\varphi=1}+\text{Fil}^0D^K_{\text{dR}}(W(\delta_1/\delta_2))\hookrightarrow \mathrm{H}^1(G_K,W(\delta_1/\delta_2))$, $W(s)$ is given as follows:

(1) $W_e(s):=W_e(\delta_1)\oplus W_e(\delta_2)$ such that $g(x,y):=(gx,gy)$ for any $g\in G_K, x\in W_e(\delta_1), y\in W_e(\delta_2)$;

(2) $W^+_{\text{dR}}(s):=W^+_{\text{dR}}(\delta_1)\oplus W^+_{\text{dR}}(\delta_2)$ such that $g(x,y):=(gx,gy)$ for any $g\in G_K, x\in W^+_{\text{dR}}(\delta_1), y\in W^+_{\text{dR}}(\delta_2)$;

(3) $\iota:B_{\text{dR}}\otimes_{B_e}W_e(s)\xrightarrow{\sim}B_{\text{dR}}\otimes_{B^+_{\text{dR}}}W^+_{\text{dR}}(s)$ is given by $\iota(x,y):=(x+a\otimes y,y)$ for any $x\in B_{\text{dR}}\otimes_{B_e}W_e(\delta_1), y\in B_{\text{dR}}\otimes_{B_e}W_e(\delta_2)$; here, $a\in D^K_{\text{dR}}(\delta_1/\delta_2)$ is a lifting of $\bar{a}\in D^K_{\text{dR}}(W(\delta_1/\delta_2))/(D^K_{\text{cris}}(W(\delta_1/\delta_2))^{\varphi=1}+\text{Fil}^0D^K_{\text{dR}}(W(\delta_1/\delta_2))$ corresponding to s.

By the definition of $W(\delta)$ for any $\delta : K^\times \to E^\times$, we have $D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1)) \xrightarrow{\sim} (K_0 \otimes_{\mathbb{Q}_p} E)e_1$, $D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2)) \xrightarrow{\sim} (K_0 \otimes_{\mathbb{Q}_p} E)e_2$ such that:

- $\varphi^f(e_1) = \tilde{\delta_1}(\pi_K)e_1$, $\varphi^f(e_2) = \tilde{\delta_2}(\pi_K)e_2$;
- $g(e_1) = \tilde{\delta_1}(\chi_{\mathrm{LT}}(g))e_1$, $g(e_2) = \tilde{\delta_2}(\chi_{\mathrm{LT}}(g))e_2$ for any $g \in G_K$.

Then we can take a basis e_1, e_2 such that $\varphi(e_1) = \alpha e_1$, $\varphi(e_2) = \beta e_2$, here (α, β) is the fixed pair as in Theorem 4.6 for $(\tilde{\delta}_1, \tilde{\delta}_2, \{k_\sigma\}_\sigma) \in T_{\mathrm{cris}}$. By part (1) in the above description of $W(s)$, we have $D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(V) = D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1)) \oplus D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2))$ as a (φ, G_K)-module. We compute the filtration on $D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(s)) \xrightarrow{\sim} K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K_0} D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(s)) \xrightarrow{\sim} (K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{\mathbb{Q}_p} E)e_2 \xrightarrow{\sim} \oplus_{\sigma: K \hookrightarrow E}(K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E)e_{1,\sigma} \oplus (K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E)e_{2,\sigma} =: \oplus_\sigma D_\sigma$ (here we put $D_\sigma := (K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E)e_{1,\sigma} \oplus (K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E)e_{2,\sigma}$). Because $D_{\mathrm{dR}}^K(W(\delta_1/\delta_2)) = (W_{\mathrm{dR}}(\delta_1/\delta_2))^{G_K} \xrightarrow{\sim} \oplus_\sigma(W_{\mathrm{dR}}(\delta_1/\delta_2) \otimes_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} B_{\mathrm{dR}} \otimes_{K,\sigma} E)^{G_K}$ and $\mathrm{Fil}^0 D_{\mathrm{dR}}^K(W(\delta_1/\delta_2)) = \oplus_\sigma(W_{\mathrm{dR}}^+(\delta_1/\delta_2) \otimes_{B_{\mathrm{dR}}^+ \otimes_{\mathbb{Q}_p} E} B_{\mathrm{dR}}^+ \otimes_{K,\sigma} E)^{G_K} \xrightarrow{\sim} \oplus_{\sigma, k_\sigma \leqq 0}(W_{\mathrm{dR}}(\delta_1/\delta_2) \otimes_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} B_{\mathrm{dR}} \otimes_{K,\sigma} E)^{G_K}$, so we can take $a \in D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))$ in (3) as $a = (a_\sigma) \in \oplus_\sigma(W_{\mathrm{dR}}(\delta_1/\delta_2) \otimes_{B_{\mathrm{dR}} \otimes_{\mathbb{Q}_p} E} B_{\mathrm{dR}} \otimes_{K,\sigma} E)^{G_K}$ such that $a_\sigma = 0$ for any σ such that $k_\sigma \leqq 0$. Then, for any σ such that $k_\sigma \leqq 0$ or $k_\sigma \geq 1$ and $a_\sigma = 0$, we have

$$\mathrm{Fil}^i D_\sigma = \mathrm{Fil}^i D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))_\sigma \oplus \mathrm{Fil}^i D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2))_\sigma.$$

(Here

$$\begin{aligned} D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_i)) &\xrightarrow{\sim} \oplus_\sigma D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_i)) \otimes_{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{\mathbb{Q}_p} E} K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E \\ &=: \oplus_\sigma D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))_\sigma \end{aligned}$$

for $i \in \{1, 2\}$.) Because we have

- $\mathrm{Fil}^{-k_\sigma} D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))_\sigma = D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))_\sigma$, $\mathrm{Fil}^{-k_\sigma+1} D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))_\sigma = 0$;
- $\mathrm{Fil}^0 D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2))_\sigma = D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2))_\sigma$, $\mathrm{Fil}^1 D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2))_\sigma = 0$;

we obtain the filtration on D_σ as in parts (i), (ii), (iii) of Theorem 4.6(4) for such σ. Finally, we calculate the filtration on D_σ for σ such that $k_\sigma \geq 1$ and $a_\sigma \neq 0$. Then we have $\mathrm{Fil}^{-k_\sigma} D_\sigma = D_\sigma$ and $\mathrm{Fil}^1 D_\sigma = 0$. For any $xe_{1,\sigma} + ye_{2,\sigma} \in D_\sigma$ $(x, y \in K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E)$, we have $xe_{1,\sigma} + ye_{2,\sigma} \in \mathrm{Fil}^0 D_\sigma$ if and only if $xe_{1,\sigma} + ya_\sigma \otimes e_{2,\sigma} \in \mathrm{Fil}^0 D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))_\sigma = 0$ and $ye_{2,\sigma} \in \mathrm{Fil}^0 D_{\mathrm{dR}}(W(\delta_2))_\sigma = D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_2))_\sigma$. Because $a_\sigma \in D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))_\sigma \xrightarrow{\sim} E$ is non-zero, it is a base of $D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))_\sigma$ as an E-vector space. So it is a base of $D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1/\delta_2))_\sigma$ as a $K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E$-module. So $a_\sigma \otimes e_{2,\sigma}$ is a base of $D_{\mathrm{dR}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(W(\delta_1))$. So there exists unique $z \in (K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E)^\times$ such that $a_\sigma \otimes e_{2,\sigma} = ze_{1,\sigma}$. Then $xe_{1,\sigma} + ya_\sigma \otimes e_{2,\sigma} = (x + zy)e_{1,\sigma}$, so $\mathrm{Fil}^{-k_\sigma+1} D_\sigma = \mathrm{Fil}^0 D_\sigma = K_{n(\tilde{\delta}_1,\tilde{\delta}_2)} \otimes_{K,\sigma} E(-ze_{1,\sigma} + e_{2,\sigma})$. By comparing G_K-actions on $a_\sigma \otimes e_{2,\sigma} = ze_{1,\sigma}$, we can see that $g(z) = (\tilde{\delta}_2(\chi_{\mathrm{LT}}(g))/\tilde{\delta}_1(\chi_{\mathrm{LT}}(g)))z$ for any $g \in G_K$. So there exists unique $b_\sigma \in E^\times$ such that $z = -b_\sigma G(\tilde{\delta}_2/\tilde{\delta}_1)_\sigma$. By combining all of these calculations, we obtain $D_{\mathrm{cris}}^{K_{n(\tilde{\delta}_1,\tilde{\delta}_2)}}(V(s)) \xrightarrow{\sim} D_{(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma),x}$ (here $x = [(b_\sigma e_\sigma)_\sigma] \in T_{\mathrm{cris}}^{\text{ét}}(\tilde{\delta}_1, \tilde{\delta}_2, \{k_\sigma\}_\sigma)$ or $x = [(\overline{b_\sigma e_\sigma})_\sigma] \in T'^{\text{ét}}_{\mathrm{cris}}(\tilde{\delta}_1, \tilde{\delta}_2, \{k_\sigma\}_\sigma)$). So $V(s) \xrightarrow{\sim} V_{(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma),x}$. (Here we put $b_\sigma = 0$ if $a_\sigma = 0$.) From this, we know that $D_{(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma),x}$ is weakly admissible for such x. So we can prove that part (1) implies part (2) in the case $\tilde{\delta}_1\neq\tilde{\delta}_2$ or $\tilde{\delta}_1=\tilde{\delta}_2$ and $s\in S'^{\text{ét}}(\delta_1,\delta_2)$.

Next we compute in the case where $\tilde{\delta}_1=\tilde{\delta}_2$ and $s\in S''(\delta_1,\delta_2)$. Then, by the construction of $s(\{k_\sigma\}_\sigma)$ and by the definition of $S''(\delta_1,\delta_2)$ before Proposition 4.4, $W(s):=(W_e(s),W^+_{\mathrm{dR}}(s),\iota)$ is given as follows:

(1) $W_e(s):=W_e(\delta_1)\oplus W_e(\delta_2)$ such that $g(x,y):=(gx+f\deg(g)gy,gy)$ for any $g\in G_K$, $x\in W_e(\delta_1)$, $y\in W_e(\delta_2)$ (here we use the fact that $\delta_1/\delta_2=\prod_\sigma\sigma(x)^{k_\sigma}$ implies that $W_e(\delta_1)=W_e(\delta_2)$; see Proposition 2.14);

(2) $W^+_{\mathrm{dR}}(s):=W^+_{\mathrm{dR}}(\delta_1)\oplus W^+_{\mathrm{dR}}(\delta_2)$ such that $g(x,y):=(gx,gy)$ for any $g\in G_K$, $x\in W^+_{\mathrm{dR}}(\delta_1)$, $y\in W^+_{\mathrm{dR}}(\delta_2)$;

(3) $\iota:B_{\mathrm{dR}}\otimes_{B_e}W_e(s)\xrightarrow{\sim}B_{\mathrm{dR}}\otimes_{B^+_{\mathrm{dR}}}W^+_{\mathrm{dR}}(s)$ is given by $\iota(x,y):=(x+(a_{\mathrm{cris}}+a)y,y)$ for any $x\in B_{\mathrm{dR}}\otimes_{B_e}W_e(\delta_1)$, $y\in B_{\mathrm{dR}}\otimes_{B_e}W_e(\delta_2)$; here, $a\in D^K_{\mathrm{dR}}(W(\delta_1/\delta_2))\xrightarrow{\sim}(B_{\mathrm{dR}}\otimes_{\mathbb{Q}_p}E)^{G_K}=K\otimes_{\mathbb{Q}_p}E$ is a lifting of $\bar{a}\in D^K_{\mathrm{dR}}(W(\delta_1/\delta_2))/(D^K_{\mathrm{cris}}(W(\delta_1/\delta_2))+\mathrm{Fil}^0D^K_{\mathrm{dR}}(W(\delta_1/\delta_2)))$ corresponding to $s=s(\{k_\sigma\}_\sigma)+\bar{a}$.

Then we have

$$D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(\delta_1))=(B_{\mathrm{cris}}\otimes_{B_e}W_e(\delta_1))^{G_{K_{n(\tilde{\delta}_1)}}}\xrightarrow{\sim}(B_{\mathrm{cris}}\otimes_{B_e}W_e(\delta_2))^{G_{K_{n(\tilde{\delta}_1)}}}=D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(\delta_2))$$

because $W_e(\delta_1)\xrightarrow{\sim}W_e(\delta_2)$. So we have following isomorphisms of (φ,G_K)-modules $D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(\delta_1))\xrightarrow{\sim}(K_0\otimes_{\mathbb{Q}_p}E)e_1$, $D^{K_{n(\tilde{\delta}_2)}}_{\mathrm{cris}}(W(\delta_2))\xrightarrow{\sim}(K_0\otimes_{\mathbb{Q}_p}E)e'_2$ such that:

(1) $\varphi(e_1)=\alpha e_1$, $\varphi(e'_2)=\alpha e'_2$;

(2) $g(e_1)=\tilde{\delta}_1(\chi_{\mathrm{LT}}(g))e_1$, $g(e'_2)=\tilde{\delta}_1(\chi_{\mathrm{LT}}(g))e'_2$ for any $g\in G_K$.

(Here $\alpha\in(K_0\otimes E)^\times$ is fixed such that $g_3(\alpha)=\tilde{\delta}_1(\pi_K)$ as defined before Theorem 4.6.) Then, by part (1) in the definition of $W(s)$, we have $D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(s))=(K_0\otimes_{\mathbb{Q}_p}E)e_1\oplus(K_0\otimes_{\mathbb{Q}_p}E)(-e'_2+a_{\mathrm{cris}}e_1)$. So if we put $e_2:=-e'_2+a_{\mathrm{cris}}e_1$, we have $D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(s))=(K_0\otimes_{\mathbb{Q}_p}E)e_1\oplus(K_0\otimes_{\mathbb{Q}_p}E)e_2$ such that:

(1) $\varphi(e_1)=\alpha e_1$, $\varphi(e_2)=\alpha(e_2+e_1)$;

(2) $g(e_1)=\tilde{\delta}_1(\chi_{\mathrm{LT}}(g))e_1$, $g(e_2)=\tilde{\delta}_1(\chi_{\mathrm{LT}}(g))e_2$ for any $g\in G_K$.

We compute the filtration on $K_{n(\tilde{\delta}_1)}\otimes_{K_0}D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(s))=(K_{n(\tilde{\delta}_1)}\otimes_{\mathbb{Q}_p}E)e_1\oplus(K_{n(\tilde{\delta}_1)}\otimes_{\mathbb{Q}_p}E)e_2$ as follows. We can take $a:=(a_\sigma)\in D^K_{\mathrm{dR}}(\delta_1/\delta_2)=\oplus_\sigma(D^K_{\mathrm{dR}}\otimes_{K\otimes_{\mathbb{Q}_p}E}B_{\mathrm{dR}}\otimes_{K,\sigma}E)$ such that $a_\sigma=0$ for any σ such that $k_\sigma\in\mathbb{Z}_{\leqq 0}$. Then we can compute the filtration as in the previous case. Then we have $D^{K_{n(\tilde{\delta}_1)}}_{\mathrm{cris}}(W(s))\xrightarrow{\sim}D_{(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma),x}$ for some $x=\overline{(b_\sigma e_\sigma)_\sigma}\in T''^{\text{ét}}_{\mathrm{cris}}(\tilde{\delta}_2,\{k_\sigma\}_\sigma)\xrightarrow{\sim}\oplus_{\sigma,k_\sigma\geq 1}Ee_\sigma/\Delta(E)$ such that a lift $(b_\sigma e_\sigma)_\sigma$ of $\overline{(b_\sigma e_\sigma)_\sigma}$ satisfies $\{\sigma\mid b_\sigma\neq 0\}=\{\sigma\mid a_\sigma\neq 0\}$. So, for this x, $D_{(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma),x}$ is weakly admissible and we have $V(s)\xrightarrow{\sim}V_{(\tilde{\delta}_1,\tilde{\delta}_2,\{k_\sigma\}_\sigma),x}$. So we have proved that condition (1) implies condition (2) in all cases.

Next we prove that condition (2) implies condition (1). Let us take $(\delta_1,\delta_2,\{k_\sigma\}_\sigma)\in T_{\mathrm{cris}}$. We prove this in the case $\delta_1\neq\delta_2$. (We can prove the other cases in the same way.) Let us take $[(b_\sigma e_\sigma)_\sigma]\in T^{\text{ét}}_{\mathrm{cris}}(\delta_1,\delta_2,\{k_\sigma\}_\sigma)\subset\mathbb{P}_E(\oplus_{\sigma,k_\sigma\geq 1}Ee_\sigma)$. If we put $\tilde{\delta}_1=\delta_1\prod_\sigma\sigma(x)^{k_\sigma}$, $\tilde{\delta}_2=\delta_2$, we can see that $(\tilde{\delta}_1,\tilde{\delta}_2)\in S^+$. From the argument of the proof of the claim that condition (1) implies condition (2), we can see that there exists an element $s=[(a_\sigma e_\sigma)_\sigma]\in S'(\tilde{\delta}_1,\tilde{\delta}_2)$ such that $D^{K_{n(\delta_1,\delta_2)}}_{\mathrm{cris}}(W(s))\xrightarrow{\sim}D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),[(b_\sigma e_\sigma)_\sigma]}$. Moreover, from the previous argument, we can also

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see that $\{\sigma \mid a_\sigma \neq 0\} = \{\sigma \mid b_\sigma \neq 0\}$. This implies that s is contained in $S'^{\text{ét}}(\tilde{\delta}_1, \tilde{\delta}_2)$, i.e. $W(s) = W(V(s))$ for some split trianguline E-representation. So we have that $D_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),[(b_\sigma e_\sigma)_\sigma]}$ is weakly admissible and $V(s) \xrightarrow{\sim} V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),[(b_\sigma e_\sigma)_\sigma]}$. We have finished the proof of the theorem. □

The following is the corollary of Theorem 3.7.

COROLLARY 4.7. *We have the following results.*

(1) *Let* $(\delta_1, \delta_2, \{k_\sigma\}_\sigma) \in T_{\text{cris}}$ *such that* $\delta_1 \neq \delta_2$. *Let* $[(a_\sigma e_\sigma)_\sigma] \in T^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$. *Then there exists unique* $((\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma), x, \{w_\sigma\}_\sigma)$ *where* $(\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma) \in T_{\text{cris}}$, $x \in T^{\text{ét}}_{\text{cris}}(\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma)$, $w_\sigma \in \mathbb{Z}$, *satisfying* $V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),[(a_\sigma e_\sigma)_\sigma]} \xrightarrow{\sim} V_{(\delta'_1,\delta'_2,\{k'_\sigma\}_\sigma),x}(\prod_\sigma \sigma(\chi_{\text{LT}})^{w_\sigma})$ *and* $((\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma), x, \{w_\sigma\}_\sigma) \neq ((\delta_1, \delta_2, \{k_\sigma\}_\sigma), [(a_\sigma e_\sigma)_\sigma], \{0\}_\sigma)$. *Such unique* $((\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma), x, \{w_\sigma\}_\sigma)$ *satisfies:*
   (i) $\delta'_1|_{\mathcal{O}_K^\times} = \delta_2|_{\mathcal{O}_K^\times}, \delta'_2|_{\mathcal{O}_K^\times} = \delta_1|_{\mathcal{O}_K^\times}$;
   (ii) $\delta'_1(\pi_K) = \delta_2(\pi_K) \prod_{\sigma, a_\sigma = 0 \text{ or } k_\sigma \leq 0} \sigma(\pi_K)^{k_\sigma}$, $\delta'_2(\pi_K) = \delta_1(\pi_K) \prod_{\sigma, a_\sigma = 0 \text{ or } k_\sigma \leq 0} \sigma(\pi_K)^{k_\sigma}$;
   (iii) $k'_\sigma = k_\sigma$ *if* $a_\sigma \neq 0$, $k'_\sigma = -k_\sigma$ *for other* σ;
   (iv) $w_\sigma = 0$ *if* $a_\sigma \neq 0$, $w_\sigma = k_\sigma$ *for other* σ;
   (v) *unique suitable* $x = [(a'_\sigma e_\sigma)_\sigma] \in T^{\text{ét}}_{\text{cris}}(\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma)$ *such that* $\{\sigma \mid a'_\sigma \neq 0\} = \{\sigma \mid a_\sigma \neq 0\}$.

(2) *Let* $(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$, $(\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma) \in T_{\text{cris}}$ *such that* $\delta_1 = \delta_2$. *Let* $x \in T''^{\text{ét}}_{\text{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$, $y \in T^{\text{ét}}_{\text{cris}}(\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma)$, $\{w_\sigma\}_\sigma$ *such that* $w_\sigma \in \mathbb{Z}$ *for any* σ. *Then we have an isomorphism* $V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),x} \xrightarrow{\sim} V_{(\delta'_1,\delta'_2,\{k'_\sigma\}_\sigma),y}(\prod_\sigma \sigma(\chi_{\text{LT}})^{w_\sigma})$ *if and only if* $((\delta'_1, \delta'_2, \{k'_\sigma\}_\sigma), y) = ((\delta_1, \delta_2, \{k_\sigma\}_\sigma), x)$ *and* $w_\sigma = 0$ *for any* σ.

Proof. Part (1) is the corollary of Theorem 3.7(2). Because $\delta_1 \neq \delta_2$, then by the proof of the above theorem, we have $V_{(\delta_1,\delta_2,\{k_\sigma\}_\sigma),[(a_\sigma e_\sigma)_\sigma]} \xrightarrow{\sim} V(s)$ where $s = [(b_\sigma e_\sigma)_\sigma] \in S'^{\text{ét}}_{\text{cris}}(\delta_1 \prod_\sigma \sigma(x)^{k_\sigma}, \delta_2)$ for some $[(b_\sigma e_\sigma)_\sigma]$ such that $\{\sigma \mid b_\sigma \neq 0\} = \{\sigma \mid a_\sigma \neq 0\}$. Then, by Theorem 3.7, we have $V(s) \xrightarrow{\sim} V(s')$ for unique $s' \neq s$ such that

$$\begin{aligned} s' = [(b'_\sigma e_\sigma)_\sigma] &\in S'^{\text{ét}}_{\text{cris}}\Bigg(\prod_{\sigma, b_\sigma \neq 0} \sigma(x)^{k_\sigma} \delta_2, \prod_{\sigma, b_\sigma \neq 0} \sigma(x)^{-k_\sigma} \prod_\sigma \sigma(x)^{k_\sigma} \delta_1\Bigg) \\ &= S'^{\text{ét}}_{\text{cris}}\Bigg(\prod_{\sigma, b_\sigma \neq 0} \sigma(x)^{k_\sigma} \delta_2, \prod_{\sigma, k_\sigma \leq 0 \text{ or } b_\sigma = 0} \sigma(x)^{k_\sigma} \delta_1\Bigg) \end{aligned}$$

such that $\{\sigma \mid b'_\sigma \neq 0\} = \{\sigma \mid b_\sigma \neq 0\}$. By the proof of the above theorem, we can see that s' corresponds to $V_{(\delta'_1,\delta'_2,\{k'_\sigma\}_\sigma),[(a'_\sigma e_\sigma)_\sigma]}(\prod_{\sigma, a_\sigma = 0 \text{ or } k_\sigma \leq 0} \sigma(\chi_{\text{LT}})^{k_\sigma})$ such that

$$\begin{aligned} \delta'_1|_{\mathcal{O}_K^\times} = \delta_2|_{\mathcal{O}_K^\times}, \delta'_2|_{\mathcal{O}_K^\times} = \delta_1|_{\mathcal{O}_K^\times}, \delta'_1(\pi_K) &= \prod_{\sigma, a_\sigma = 0 \text{ or } k_\sigma \leq 0} \sigma(\pi_K)^{k_\sigma} \delta_2(\pi_K), \delta'_2(\pi_K) \\ &= \prod_{\sigma, a_\sigma = 0 \text{ or } k_\sigma \leq 0} \sigma(\pi_K)^{k_\sigma} \delta_1(\pi_K), k'_\sigma = k_\sigma \end{aligned}$$

if $a_\sigma \neq 0$, $k'_\sigma = -k_\sigma$ for other σ and $\{\sigma \mid a'_\sigma \neq 0\} = \{\sigma \mid b'_\sigma \neq 0\}$. So we have finished the proof of part (1).

Part (2) follows from the above theorem and Theorem 3.7(1). □

4.3 Potentially semi-stable and non-crystalline split trianguline E-representations

In this section, we explicitly describe the E-filtered (φ, N, G_K)-modules of potentially semi-stable split trianguline E-representations which are not potentially crystalline. First we define the parameter spaces of potentially semi-stable split trianguline E-representations. We put

$T_{\mathrm{st}} := \{(\delta, \{k_\sigma\}_\sigma) \mid \delta : K^\times \to E^\times$ a locally constant character, $k_\sigma \in \mathbb{Z}$ for any σ such that $(\sum_\sigma k_\sigma) + 2e_K \mathrm{val}_p(\delta(\pi_K)) + [K : \mathbb{Q}_p] = 0\}$.

For any $(\delta, \{k_\sigma\}_\sigma) \in T_{\mathrm{st}}$, we define the parameter space $T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma)$ of weakly admissible filtrations as follows. We define

$$T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma) := \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma.$$

For any $(a_\sigma e_\sigma)_\sigma \in T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma)$ we define a rank-two E-filtered $(\varphi, N, \mathrm{Gal}(K_{n(\delta)}/K))$-module $D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$. However, as in the potentially crystalline case, the construction depend on the choice of one more parameter. For any $(\delta, \{k_\sigma\}_\sigma) \in T_{\mathrm{st}}$, we fix $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times$ such that $g_3(\alpha) = \delta(\pi_K)$, where g_3 is as before. Then we define $D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma} := (K_0 \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_0 \otimes_{\mathbb{Q}_p} E)e_2$ as follows:

(1) $N(e_2) = e_1, N(e_1) = 0$;

(2) $\varphi(e_1) = (\alpha/p)e_1, \varphi(e_2) = \alpha e_2$ for the fixed $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times$, so we have $\varphi^f(e_1) = (\delta(\pi_K)/q)e_1, \varphi^f(e_2) = \delta(\pi_K)e_2$;

(3) $g(e_1) = \delta(\chi_{\mathrm{LT}}(g))e_1, g(e_2) = \delta(\chi_{\mathrm{LT}}(g))e_2$ for any $g \in G_K$;

(4) we put $K_{n(\delta)} \otimes_{K_0} D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma} = (K_{n(\delta)} \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_{n(\delta)} \otimes_{\mathbb{Q}_p} E)e_2 \xrightarrow{\sim} \oplus_\sigma (K_{n(\delta)} \otimes_{K,\sigma} E)e_{1,\sigma} \oplus (K_{n(\delta)} \otimes_{K,\sigma} E)e_{2,\sigma} =: \oplus_\sigma D_\sigma$, then:

  (i) for σ such that $k_\sigma \leq -1$, we put $\mathrm{Fil}^0 D_\sigma = D_\sigma, \mathrm{Fil}^1 D_\sigma = \cdots = \mathrm{Fil}^{-k_\sigma} D_\sigma = (K_{n(\delta)} \otimes_{K,\sigma} E)e_{2,\sigma}, \mathrm{Fil}^{-k_\sigma+1} D_\sigma = 0$;

  (ii) for σ such that $k_\sigma = 0$, we put $\mathrm{Fil}^0 D_\sigma = D_\sigma, \mathrm{Fil}^1 D_\sigma = 0$;

  (iii) for σ such that $k_\sigma \geq 1$, we put $\mathrm{Fil}^{-k_\sigma} D_\sigma = D_\sigma, \mathrm{Fil}^{-k_\sigma+1} D_\sigma = \cdots = \mathrm{Fil}^0 D_\sigma = (K_{n(\delta)} \otimes_{K,\sigma} E)(a_\sigma e_{1,\sigma} + e_{2,\sigma}), \mathrm{Fil}^1 D_\sigma = 0$.

We prove in the proof of the next theorem that these are weakly admissible. So, by 'weakly admissible implies admissible' theorem [Ber08a, Theorem B], there exists a unique (up to isomorphism) two-dimensional potentially semi-stable, not potentially crystalline E-representation $V_{(\delta,\{k_\sigma\}),(a_\sigma e_\sigma)_\sigma}$ such that $D_{\mathrm{st}}^{K_{n(\delta)}}(V_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}) \xrightarrow{\sim} D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$. Then our main result on the classification of potentially semi-stable split trianguline E-representations is as follows.

Theorem 4.8. *Let V be a two-dimensional E-representation. Then the following conditions are equivalent:*

(1) *V is split trianguline, potentially semi-stable and not potentially crystalline;*

(2) *there exist $(\delta, \{k_\sigma\}_\sigma) \in T_{\mathrm{st}}$, $(a_\sigma e_\sigma)_\sigma \in T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma)$ and $\{w_\sigma\}_\sigma$ with $w_\sigma \in \mathbb{Z}$ for any σ, such that $V \xrightarrow{\sim} V_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}(\prod_\sigma \sigma(\chi_{\mathrm{LT}}^{w_\sigma}))$.*

Moreover, we have an isomorphism $V_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma} \xrightarrow{\sim} V_{(\delta',\{k'_\sigma\}_\sigma),(a'_\sigma e_\sigma)_\sigma}(\prod_\sigma \sigma(\chi_{\mathrm{LT}}^{w'_\sigma}))$ if and only if $((\delta', \{k'_\sigma\}_\sigma), (a'_\sigma e_\sigma)_\sigma) = ((\delta, \{k_\sigma\}_\sigma), (a_\sigma e_\sigma)_\sigma)$ and $w'_\sigma = 0$ for any σ.

Proof. First we prove that condition (1) implies condition (2). Let V be as in condition (1). Then, by Proposition 4.4, we have $V \xrightarrow{\sim} V(s)$ for some $s \in S_{\mathrm{st}}(\delta_1, \delta_2)$ such that $\delta_1/\delta_2 = |N_{K/\mathbb{Q}_p}(x)| \prod_\sigma \sigma(x)^{k_\sigma}$ for some $k_\sigma \in \mathbb{Z}$ for any σ. Twisting V by some suitable $\prod_\sigma \sigma(\chi_{\mathrm{LT}})^{l_\sigma}$, we may assume that $(\delta_1, \delta_2) = (\prod_\sigma \sigma(x)^{k_\sigma} |N_{K/\mathbb{Q}_p}(x)|\delta, \delta)$ for some locally constant character $\delta : K^\times \to E^\times$ and for some $k_\sigma \in \mathbb{Z}$ for any σ. By the definition of $S_{\mathrm{st}}(\delta_1, \delta_2) \xrightarrow{\sim} s(\{k_\sigma\}_\sigma) + D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))/\mathrm{Fil}^0 D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))$, s corresponds to $s(\{k_\sigma\}_\sigma) + \overline{b}$ for some $\overline{b} \in D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))/\mathrm{Fil}^0 D_{\mathrm{dR}}^K(W(\delta_1/\delta_2))$. So $W(s)$ is given as follows:

(1) $W_e(s) := W_e(\delta_1) \oplus W_e(\delta_2)$ such that $g(x, y) = (gx + c(g)gy, gy)$ for any $g \in G_K$, $x \in W^+_{\mathrm{cris}}(\delta_1)$, $y \in W^+_{\mathrm{cris}}(\delta_2)$ (here $c(g)$ is defined in the definition of $s(\{k_\sigma\}_\sigma)$ and we use the fact that $W_e(\delta_1) \xrightarrow{\sim} W_e(\delta_2)(\chi)$ which we can see from Lemma 2.13);

(2) $W^+_{\mathrm{dR}}(s) := W^+_{\mathrm{dR}}(\delta_1) \oplus W^+_{\mathrm{dR}}(\delta_2)$ such that $g(x, y) = (gx, gy)$ for any $g \in G_K$, $x \in W^+_{\mathrm{dR}}(\delta_1)$, $y \in W^+_{\mathrm{dR}}(\delta_2)$;

(3) $\iota : B_{\mathrm{dR}} \otimes_{B_e} W_e(s) \xrightarrow{\sim} B_{\mathrm{dR}} \otimes_{B^+_{\mathrm{dR}}} W^+_{\mathrm{dR}}(s)$ is an isomorphism given by $\iota(x, y) = (x + (\log([\tilde{p}])/t)y + b \otimes y, y)$, here $b = (b_\sigma) \in D^K_{\mathrm{dR}}(W(\delta_1/\delta_2))$ is the lift of $\bar{b}$ such that $b_\sigma = 0$ for any σ such that $k_\sigma \leq 0$.

We calculate $D^{K_{n(\delta)}}_{\mathrm{st}}(V(s))$ as follows. First, if we take a base $e' \in D^{K_{n(\delta)}}_{\mathrm{cris}}(W(\delta_2)) \subseteq B_{\mathrm{cris}} \otimes_{B_e} W_e(\delta_2)$, then we have $e'' := (1/t)e' \in D^{K_{n(\delta)}}_{\mathrm{cris}}(W(\delta_1))$ because $W_e(\delta_1) \xrightarrow{\sim} W_e(\delta_2)(\chi)$. Then $\varphi^f(e') = \delta(\pi_K)e'$, $\varphi^f(e'') = (\delta(\pi_K)/q)e''$ and $g(e') = \delta(\chi_{\mathrm{LT}}(g))e'$, $g(e'') = \delta(\chi_{\mathrm{LT}}(g))e''$ for any $g \in G_K$. Moreover, we can take a base e' such that $\varphi(e') = \alpha e'$ for the fixed α. Then $D^{K_{n(\delta)}}_{\mathrm{st}}(W(s)) \xrightarrow{\sim} (K_0 \otimes_{\mathbb{Q}_p} E)e'' \oplus (K_0 \otimes_{\mathbb{Q}_p} E)(e' - \log([\tilde{p}])e'')$. If we put $e_1 := e''$, $e_2 := e' - \log([\tilde{p}])e''$, then $N(e_2) = e_1$. Next we calculate the filtration on $K_{n(\delta)} \otimes_{K_0} D^{K_{n(\delta)}}_{\mathrm{st}}(W(s)) = (K_{n(\delta)} \otimes_{\mathbb{Q}_p} E)e_1 \oplus (K_{n(\delta)} \otimes_{\mathbb{Q}_p} E)e_2$. By the definition of ι in part (3) of the definition of $W(s)$, we have $xe_1 + ye_2 \in \mathrm{Fil}^i$ for $x, y \in K_{n(\delta)} \otimes_{\mathbb{Q}_p} E$ if and only if $xe'' + y(\iota(e') - \log([\tilde{p}])e'') = xe'' + y(e' + b \otimes e'' + \log([\tilde{p}])e'' - \log([\tilde{p}])e'') = (xe'' + yb \otimes e'') + ye' \in \mathrm{Fil}^i D^{K_{n(\delta)}}_{\mathrm{dR}}(W(\delta_1)) \oplus \mathrm{Fil}^i D^{K_{n(\delta)}}_{\mathrm{dR}}(W(\delta_2))$. So we can calculate the filtration similarly as in the proof of Theorem 4.6. Then we can show that $D^{K_{n(\delta)}}_{\mathrm{st}}(V(s)) \xrightarrow{\sim} D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$ for some $(a_\sigma e_\sigma)_\sigma \in T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma) \xrightarrow{\sim} \oplus_{\sigma, k_\sigma \geq 1} E e_\sigma$ such that $\{\sigma \mid a_\sigma \neq 0\} = \{\sigma \mid b_\sigma \neq 0\}$. So $D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$ is weakly admissible for such a $(a_\sigma e_\sigma)_\sigma$ and we have $V(s) \xrightarrow{\sim} V_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$. Thus, we have proved that condition (1) implies condition (2).

Next we prove that condition (2) implies condition (1). Let us take $(\delta, \{k_\sigma\}_\sigma) \in T_{\mathrm{st}}$ and $(a_\sigma e_\sigma)_\sigma \in T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma)$. Then we can see that $(\delta_1, \delta_2) := (\prod_\sigma \sigma(x)^{k_\sigma} |N_{K/\mathbb{Q}_p}(x)| \delta, \delta) \in S^+$. Moreover, it is easy to see from the above argument that there exist some $s := (b_\sigma e_\sigma)_\sigma \in S_{\mathrm{st}}(\delta_1, \delta_2)$ such that $D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma} \xrightarrow{\sim} D^{K_{n(\delta)}}_{\mathrm{st}}(V(s))$. So $D_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$ is weakly admissible and $V(s) \xrightarrow{\sim} V_{(\delta,\{k_\sigma\}_\sigma),(a_\sigma e_\sigma)_\sigma}$.

Finally, we prove the uniqueness of $((\delta, \{k_\sigma\}_\sigma), (a_\sigma e_\sigma)_\sigma, \{w_\sigma\}_\sigma)$. This easily follows from the above argument and from Theorem 3.7. We have finished the proof of the theorem. □

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Appendix A. A relation between two-dimensional potentially semi-stable trianguline E-representations and classical local Langlands correspondence for $\mathrm{GL}_2(K)$.

In this appendix, we show that a simple relation between two-dimensional potentially semi-stable trianguline E-representations and classical local Langlands correspondence for $\mathrm{GL}_2(K)$.

A.1 Classical local Langlands correspondence for $\mathrm{GL}_2(K)$

First we briefly recall classical local Langlands correspondence for $\mathrm{GL}_2(K)$. Let $W_K \subset G_K$ be the Weil group of K, i.e. the inverse image of $\langle \mathrm{Frob}_k \rangle_{\mathbb{Z}} \subseteq \mathrm{Gal}(\bar{k}/k)$ by the natural surjection $G_K \to \mathrm{Gal}(\bar{k}/k)$. (Here $\langle \mathrm{Frob}_k \rangle_{\mathbb{Z}}$ is the subgroup of $\mathrm{Gal}(\bar{k}/k)$ generated by the $q = p^f$th power Frobenius Frob_k of k.) Here W_K is a topological group such that the inertia $I_K := \mathrm{Ker}(G_K \to \mathrm{Gal}(\bar{k}/k))$ equipped with usual profinite topology is an open subgroup of W_K.

Definition A1. Let L be a field of characteristic zero. We say that a finite-dimensional L-vector space D with discrete topology is an L-Weil–Deligne representation of K if D is equipped with:

(1) a continuous L-linear action of W_K, i.e. there is a continuous morphism $\rho : W_K \to \mathrm{Aut}_L(D)$;

(2) a nilpotent L-linear operator $N : D \to D$ such that $N\rho(g) = q^{-\deg(g)}\rho(g)N$ for any $g \in W_K$; here we define $\deg(g) \in \mathbb{Z}$ such that $g = \mathrm{Frob_k}^{\deg(g)} \in \mathrm{Gal}(\bar{k}/k)$ for any $g \in W_K$.

We say that an L-Weil–Deligne representation $D := (D, \rho, N)$ is semi-simple if (D, ρ) is a semi-simple representation of W_K.

Next we recall Fontaine's recipe to construct a $\bar{K}$-Weil–Deligne representation from a potentially semi-stable E-representation of G_K. Let V be a potentially semi-stable d-dimensional E-representation of G_K. Then $D_{\mathrm{pst}}(V) := \bigcup_{K \subset L, \mathrm{finite}} (B_{\log} \otimes_{\mathbb{Q}_p} V)^{G_L}$ is a free $K_0^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E$-module of rank d equipped with the natural (φ, N, G_K)-action on which φ and G_K act semi-linearly and N acts linearly. We define a continuous action of W_K on $D_{\mathrm{pst}}(V)$ by $\rho(g)x := \varphi^{-f\deg(g)}gx$ for any $g \in W_K$ and $x \in D_{\mathrm{pst}}(V)$. We can see that this action is $K_0^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E$-linear and satisfies $N\rho(g) = q^{-\deg(g)}\rho(g)N$ for any $g \in W_K$. By using the fixed embeddings $E \hookrightarrow \bar{K}$ and $K_0^{\mathrm{un}} \hookrightarrow \bar{K}$, we define a map $K^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E \to \bar{K}$. Extending the scalar by this map, we obtain a $\bar{K}$-Weil–Deligne representation $\bar{D}_{\mathrm{pst}}(V) := D_{\mathrm{pst}}(V) \otimes_{K^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E} \bar{K}$ of K of dimension d. So we obtain a functor $V \mapsto \bar{D}_{\mathrm{pst}}(V)$ from the category of potentially semi-stable E-representations of G_K to the category of $\bar{K}$-Weil–Deligne representations of K.

Next, for a $\bar{K}$-Weil–Deligne representation $\bar{D}$ of K, we recall the definition of an L-factor of $\bar{D}$. Let us fix an isomorphism $\bar{K} \xrightarrow{\sim} \mathbb{C}$. By this isomorphism, we can see $\bar{D}$ as a $\mathbb{C}$-Weil–Deligne representation of K. Put $\bar{J} := \bar{D}^{N=0, I_K=1}$, the subspace of $\bar{D}$ on which $N = 0$ and I_K acts trivially. Let $\sigma \in W_K$ be an element such that $\deg(\sigma) = -1$. Then σ acts on $\bar{J}$ and this action does not depend on the choice of a lifting σ. Then we define the L-factor of $\bar{D}$ by $L(\bar{D}, s) := \det(1 - \sigma q^{-s}|_{\bar{J}})^{-1}$.

For a two-dimensional semi-simple $\bar{K}$-Weil–Deligne representation $\bar{D}$, we can define an irreducible smooth admissible representation $\pi(\bar{D})$ of $\mathrm{GL}_2(K)$ as in [BH06, § 33.1] (if we fix a non-trivial additive smooth character $\psi : K \to \mathbb{C}^{\times}$). The irreducible smooth admissible representations of $\mathrm{GL}_2(K)$ are classified into non-supercuspidal representations (i.e. one-dimensional representations or principal series representations or special series representations) and supercuspidal representations. We do not recall these definitions here. We only need the following proposition concerning this correspondence.

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Proposition A2. *Let $\bar{D}$ be a $\bar{K}$-Weil–Deligne representation of K. Let $\bar{D}^{ss}$ be the semi-simplification of $\bar{D}$. Then the following conditions are equivalent:*

(1) *$\pi(\bar{D}^{ss})$ is non-supercuspidal;*

(2) *the representation $(\bar{D}, \rho)$ of W_K is reducible;*

(3) *there is a continuous character $\delta : W_K \to \bar{K}^\times$ such that $L(\bar{D} \otimes_{\bar{K}} \bar{K}(\delta), s) \neq 0$, here $\bar{K}$ is equipped with discrete topology.*

Proof. This follows from [BH06, Proposition 33.2]. □

A.2 Two-dimensional trianguline representations and non-supercuspidal representations

Let V be a two-dimensional potentially semi-stable E-representation of G_K. Then $\pi(\bar{D}_{\mathrm{pst}}(V)^{ss})$ is an irreducible smooth admissible representation of $\mathrm{GL}_2(K)$. In this section, we prove a relation between two-dimensional potentially semi-stable trianguline E-representations and non-supercuspidal representations of $\mathrm{GL}_2(K)$.

Before stating the main result of this appendix, we give another useful characterization of two-dimensional split trianguline E-representations in terms of D^K_{cris}. This characterization is also a generalization of [Col08a, Proposition 5.3].

Proposition A3. *Let V be a two-dimensional E-representation of G_K. Then the following conditions are equivalent:*

(1) *V is a split trianguline E-representation;*

(2) *there exists a continuous character $\delta : G_K \to E^\times$ and an element $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times$ such that $D^K_{\mathrm{cris}}(V(\delta))^{\varphi=\alpha} \neq 1$.*

Proof. First we prove that condition (1) implies condition (2). Assume that V is a split trianguline E-representation. By definition, $W(V)$ sits in a following short exact sequence of E–B-pairs

$$0 \to W_1 \to W(V) \to W_2 \to 0.$$

Here W_i are rank-one E–B-pairs for $i = 1, 2$. By Theorem 1.45 and by the construction of $W(\delta)$ for any $\delta : K^\times \to E^\times$, if we fix a uniformizer π_E of E then there exist (δ_i, k_i) where $\delta_i : G_K \to E^\times$ are continuous characters and $k_i \in \mathbb{Z}$ for $i = 1, 2$ such that $W_i \xrightarrow{\sim} W(E(\delta_i)) \otimes W_0^{\otimes k_i}$, here W_0 is defined before Theorem 1.45. If we twist the above exact sequence by the character δ_1^{-1} and apply the left exact functor D^K_{cris}, we obtain an inclusion $D_0^{\otimes k_1} \subseteq D^K_{\mathrm{cris}}(V(\delta_1^{-1}))$ of E–φ-modules of K. If we put $K_0 \otimes_{\mathbb{Q}_p} E \xrightarrow{\sim} \bigoplus_{0 \leq i \leq f-1} K_0 \otimes_{E_0, \varphi^i} E e_i$, $D_0^{\otimes k_1} \xrightarrow{\sim} \oplus_{0 \leq i \leq f-1} K_0 \otimes_{E_0, \varphi^i} E e_{i,k_1}$, then we have $\varphi(e_{0,k_1} + \cdots + e_{f-1,k_1}) = (\pi_E^{k_1} e_0 + e_1 + \cdots + e_{f-1})(e_{0,k_1} + \cdots + e_{f-1,k_1})$. In particular, $(e_{0,k_1} + \cdots + e_{f-1,k_1}) \in D^K_{\mathrm{cris}}(V(\delta_1^{-1}))^{\varphi=(\pi_E^{k_1} e_0 + e_1 + \cdots + e_{f-1})}$ and $(\pi_E^{k_1} e_0 + e_1 + \cdots + e_{f-1}) \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times$. So we have proved that condition (1) implies condition (2).

Next we prove that condition (2) implies condition (1). Assume that $D^K_{\mathrm{cris}}(V(\delta))^{\varphi=\alpha} \neq 0$ for a character $\delta : G_K \to E^\times$ and $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times$. Because V is split trianguline if and only if $V(\delta)$ is split trianguline, we may assume that $V = V(\delta)$. Let us take a non-zero $x \in D^K_{\mathrm{cris}}(V)^{\varphi=\alpha}$. We consider the sub-$E$-filtered φ-module $D_1 \subseteq D^K_{\mathrm{cris}}(V)$ of rank one which is generated by x. Then, by Theorem 1.18(3), we have a natural inclusion of E–B-pairs $W(D_1) \hookrightarrow W(D^K_{\mathrm{cris}}(V)) \hookrightarrow W(V)$. $W(D_1)$ is an E–B-pair of rank one by Theorem 1.18(3). Taking the saturation $W(D_1)^{\mathrm{sat}}$ of $W(D_1)$ in $W(V)$ (Lemma 1.14), we obtain a short exact sequence of E–B-pairs

$$0 \to W(D_1)^{\mathrm{sat}} \to W(V) \to W_2 \to 0.$$

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Here W_2 is the cokernel of $W(D_1)^{\mathrm{sat}} \hookrightarrow W(D)$, which is an E–B-pair of rank one. So V is a split trianguline E-representation. We have finished the proof of this proposition. □

The main theorem of this appendix is as follows.

Theorem A4. *Let V be a two-dimensional potentially semi-stable E-representation of G_K. Then the following conditions are equivalent:*

(1) *V is trianguline, i.e. $V \otimes_E E'$ is a split trianguline E'-representation for some finite extension of E' of E;*

(2) *the representation $(\bar{D}_{\mathrm{pst}}(V), \rho)$ of W_K is reducible;*

(3) *$\pi(\bar{D}_{\mathrm{pst}}(V)^{ss})$ is a non-supercuspidal representation of $\mathrm{GL}_2(K)$.*

Proof. By Proposition A2, it suffices to show equivalence between conditions (1) and (3). The proof of this is a modified version of the proof of [Kis03, Lemma 1.3]. First we prove that condition (1) implies condition (3). Assume that $V \otimes_E E'$ is a split trianguline E'-representation for a finite extension E' of E. By construction, we have $\pi(\bar{D}_{\mathrm{pst}}(V \otimes_E E')^{ss}) = \pi(\bar{D}_{\mathrm{pst}}(V)^{ss})$. So we may assume that $E' = E$ and V is a split trianguline E-representation. Then, by Proposition A3, there exist $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^{\times}$ and $\delta : G_K \to E^{\times}$ such that $D^K_{\mathrm{cris}}(V(\delta))^{\varphi=\alpha} \neq 0$. In this case, we claim that we can take δ which is a potentially crystalline character. We show this claim as follows. In the proof of Proposition A3, $W(V)$ sits in a following short exact sequence of E–B-pairs

$$0 \to W_1 \to W(V) \to W_2 \to 0,$$

where $W_1 := W_0^{\otimes k_1} \otimes W(E(\delta'))$ and W_2 are rank-one E–B-pairs. In this case, because $W(V)$ is a potentially semi-stable E–B-pair, W_1 is also a potentially semi-stable E–B-pair. Because W_0 is a crystalline E–B-pair, by definition, so $W(E(\delta')) \xrightarrow{\sim} W_1 \otimes W_0^{\otimes -k_1}$ is also potentially semi-stable. So δ' is a potentially crystalline character because rank-one semi-stable E-representations are crystalline. Then, as in the proof of Proposition A3, if we put $\delta := \delta'^{-1}$, then we have $D^K_{\mathrm{cris}}(V(\delta))^{\varphi=\alpha} \neq 0$ for some $\alpha \in (K_0 \otimes_{\mathbb{Q}_p} E)^{\times}$. So we have proved the claim. For this δ, we have $\bar{D}_{\mathrm{pst}}(V(\delta)) \xrightarrow{\sim} \bar{D}_{\mathrm{pst}}(V) \otimes_{\bar{K}} \bar{D}_{\mathrm{pst}}(E(\delta))$ because both V and $E(\delta)$ are potentially semi-stable. So, by Proposition A2, it suffices to show that $L(\bar{D}_{\mathrm{pst}}(V(\delta)), s) \neq 1$. Put $\bar{J} := \bar{D}_{\mathrm{pst}}(V(\delta))^{N=0, I_K=1}$. If we put $J := D_{\mathrm{pst}}(V(\delta))^{N=0, I_K=1}$, then we have $J \xrightarrow{\sim} \cup_{K \subset L \subset K^{\mathrm{un}}} D^L_{\mathrm{cris}}(V(\delta))$, here K^{un} is the maximal unramified extension of K. By using the fact that N is nilpotent on $D_{\mathrm{pst}}(V(\delta))$ and that I_K acts discretely on it, we can easily see that $\bar{J} \xrightarrow{\sim} J \otimes_{K_0^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E} \bar{K}$. Moreover, because $\mathrm{Gal}(K^{\mathrm{un}}/K) \xrightarrow{\sim} \mathrm{Gal}(K_0^{\mathrm{un}}/K_0)$ acts on J discretely and semi-linearly, so we have $J \xrightarrow{\sim} J^{\mathrm{Gal}(K^{\mathrm{un}}/K)} \otimes_{K_0 \otimes_{\mathbb{Q}_p} E} (K_0^{\mathrm{un}} \otimes_{\mathbb{Q}_p} E)$ by Hilbert's theorem 90. Because we have $J^{\mathrm{Gal}(K^{\mathrm{un}}/K)} = D^K_{\mathrm{cris}}(V(\delta))$, we have $\bar{J} \xrightarrow{\sim} D_{\mathrm{cris}}(V(\delta)) \otimes_{K_0 \otimes_{\mathbb{Q}_p} E} \bar{K}$. Finally, if we take a non-zero element $x \in D_{\mathrm{cris}}(V(\delta))^{\varphi=\alpha}$ and consider the action of σ on x as a Weil–Deligne representation (here σ is an element in W_K such that $\deg(\sigma) = -1$), then we have $\rho(\sigma)(x) = \varphi^f \sigma(x) = \varphi^f(x) = \varphi^{f-1}(\alpha) \cdots \varphi(\alpha) \alpha x = \beta x$, here we put $\beta := \varphi^{f-1}(\alpha) \cdots \varphi(\alpha)\alpha \in ((K_0 \otimes_{\mathbb{Q}_p} E)^{\times})^{\varphi=1} = E^{\times}$. So $\bar{J}$ has a non-zero eigenvector of σ, hence we obtain $L(\bar{D}_{\mathrm{pst}}(V(\delta)), s) \neq 1$. Thus, we have proved that condition (1) implies condition (3) by Proposition A2.

Next we prove that condition (3) implies condition (1). Let V be a two-dimensional potentially semi-stable E-representation such that $\pi(\bar{D}_{\mathrm{pst}}(V)^{ss})$ is non-supercuspidal. Then, by Proposition A2, $L(\bar{D}_{\mathrm{pst}}(V) \otimes_{\bar{K}} \bar{K}(\delta), s) \neq 1$ for a character $\delta : W_K \to \bar{K}^{\times}$. By this we can take a non-zero eigenvector $x \in (\bar{D}_{\mathrm{pst}}(V) \otimes_{\bar{K}} \bar{K}(\delta))^{N=0, I_K=1}$ of $\sigma \in W_K$ with a non-zero eigenvalue $\beta \in \bar{K}^{\times}$. If we take E large enough, we may assume that $\beta \in E^{\times}$ and that there exists a potentially crystalline character $\delta' : G_K \to E^{\times}$ such that $\bar{D}_{\mathrm{pst}}(E(\delta')) \xrightarrow{\sim} \bar{K}(\delta)$. Then we can prove that there is an isomorphism $(\bar{D}_{\mathrm{pst}}(V) \otimes_{\bar{K}} \bar{K}(\delta))^{N=0, I_K=1} \xrightarrow{\sim} (\bar{D}_{\mathrm{pst}}(V(\delta')))^{N=0, I_K=1} \xrightarrow{\sim} D^K_{\mathrm{cris}}(V(\delta')) \otimes_{K_0 \otimes_{\mathbb{Q}_p} E} \bar{K}$ in the same way as the argument in the proof of condition (1) implies condition (3). If we decompose $D^K_{\mathrm{cris}}(V(\delta')) \xrightarrow{\sim} \oplus_{\tau: K_0 \hookrightarrow E} D_\tau e_\tau$, then we have $D^K_{\mathrm{cris}}(V(\delta')) \otimes_{K_0 \otimes_{\mathbb{Q}_p} E} \bar{K} = D_{id} \otimes_E \bar{K}$. So, if we take E large enough, we may assume that x is contained in $D_{id} e_{id}$. If we put $e := x + \varphi(x) + \cdots + \varphi^{f-1}(x) \in \oplus_{\tau: K_0 \hookrightarrow E} D_\tau e_\tau = D^K_{\mathrm{cris}}(V(\delta'))$, then we have $\varphi^i(x) \in D_{\varphi^{-i}} e_{\varphi^{-i}}$ and $\varphi(e) = (\beta e'_{id} + e'_{\varphi^{-1}} + \cdots + e'_{\varphi^{-(f-1)}}) e$, where $(\beta e'_{id} + \cdots + e'_{\varphi^{-(f-1)}}) \in (K_0 \otimes_{\mathbb{Q}_p} E)^\times = (\bigoplus_{0 \leq i \leq f-1} E e'_{\varphi^{-i}})^\times$. So, by Proposition A3, $V(\delta')$ is a split trianguline E-representation. Thus, V is also a split trianguline E-representation. We have finished the proof of the theorem. □

Appendix B. List of notation

Here is a list of the main notation of the article, in the order of the section in which it appears.

0.1: p, K, G_K, φ, K_∞, Γ_K.

0.2: E, k, K_0, K'_0, χ, $\mathbb{C}_p$, χ_{LT}, rec_K.

1.1: $\widetilde{\mathbb{E}}^+$, $\widetilde{\mathbb{A}}^+$, $\widetilde{\mathbb{A}}$, $\widetilde{\mathbb{B}}^+$, $\widetilde{\mathbb{B}}$, B^+_{dR}, $A_{\max}$, $[\tilde{p}]$, $B^+_{\max}$, t, $B_{\max}$, B_{dR}, B_e, $\widetilde{B}^+_{\mathrm{rig}}$, $\mathrm{Fil}^i B_{\mathrm{dR}}$, $\log[\tilde{p}]$, $B_{\log}$, N, V, W, W_e, W^+_{dR}, W_{dR}, $W(V)$, $\widetilde{B}^\dagger_{\mathrm{rig}}$, $\mathrm{rank}(W)$, $W_1 \otimes W_2$, $W^\vee$, W_1^{sat}, $D^L_{\mathrm{cris}}(W)$, $D^L_{\mathrm{st}}(W)$, $D^L_{\mathrm{dR}}(W)$, $W(D)$.

1.2: $B^\dagger_{\mathrm{rig},K}$, $\mathcal{R}_L$, ϕ_L, $\mathcal{R}^{\mathrm{bd}}_L$, $\mathcal{R}^{\mathrm{int}}_L$, v_L, ω, $\deg(M)$, $\mu(M)$, $[a]_* M$, $\widetilde{A}_{[r,s]}$, $\widetilde{B}_{[r,s]}$, $\widetilde{B}^{\dagger,r}$, $\widetilde{B}^\dagger$, $\widetilde{B}^{\dagger,r}_{\mathrm{rig}}$, $\widetilde{B}^\dagger_{\mathrm{rig}}$, i_n, A_{K_0}, E_{K_0}, $\mathbb{A}$, $\mathbb{B}$, A_K, B_K, $B^\dagger_K$, $B^{\dagger,r}_K$, E_K, $B^{\dagger,r}_{\mathrm{rig},K}$, $B^\dagger_{\mathrm{rig},K}$,

1.3: $W_e(D)$, $W^+_{\mathrm{dR}}(D)$, $W(D)$, $D(W)$.

1.4: D_0, W_0, δ, $W(\delta)$, $U^{H_K}_{\mathrm{fini}}$, ∇_U, $D_{\mathrm{Sen}}(W)$, $\Theta_{\mathrm{Sen},W}$, $\{w_\sigma\}_\sigma$.

2: B_E.

2.1: $C^\bullet(W)$, $C^0(W)$, $C^1(W)$, $\mathrm{H}^*(G_K, W)$, $\mathrm{Ext}^1(W_2, W_1)$, $\mathrm{H}^1_e(G_K, W)$, $\mathrm{H}^1_f(G_K, W)$, $\mathrm{H}^1_g(G_K, W)$.

2.3: $\sigma(x)$, $N_{K/\mathbb{Q}_p}(x)$, $|-|$, $|N_{K/\mathbb{Q}_p}(x)|$, ∂.

3.1: S^+, $S(\delta_1, \delta_2)$, $\mathbb{P}_E(M)$, $S'(\delta_1, \delta_2)$, $S'^{\text{ét}}(\delta_1, \delta_2)$, $S'^{\text{non-ét}}(\delta_1, \delta_2)$, $W(s)$, $V(s)$.

3.2: S^+_0, S^+_*, $S'^{\mathrm{ord}}(\delta_1, \delta_2)$.

4.1: $S''(\delta_1, \delta_2)$, $S_{\mathrm{st}}(\delta_1, \delta_2)$, $S^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2)$.

4.2: T_{cris}, $G(\delta)_\sigma$, $T_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$, $T^{\text{ét}}_{\mathrm{cris}}(\delta_1, \delta_2, \{k_\sigma\}_\sigma)$, $D_{(\delta_1, \delta_2, \{k_\sigma\}_\sigma), x}$, $V_{(\delta_1, \delta_2, \{k_\sigma\}_\sigma), x}$.

4.3: T_{st}, $T_{\mathrm{st}}(\delta, \{k_\sigma\}_\sigma)$, $D_{(\delta, \{k_\sigma\}_\sigma), (a_\sigma e_\sigma)_\sigma}$, $V_{(\delta, \{k_\sigma\}_\sigma), (a_\sigma e_\sigma)_\sigma}$.

A.1: W_K, I_K, $D_{\mathrm{pst}}(V)$, $\bar{D}_{\mathrm{pst}}(V)$, $L(\bar{D}, s)$, $\pi(\bar{D})$.

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