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# CLASSIFICATION OF TWO TYPES OF WEAK SOLUTIONS TO THE CASIMIR EQUATION FOR THE ITO SYSTEM

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**Abstract.** The existence and non-uniqueness of two classes of weak solutions to the Casimir equation for the Ito system is discussed. In particular, for (i) all possible travelling wave solutions and (ii) one vital class of self-similar solutions, all possible families of local power series solutions are found. We are then able to extend both types of solutions to the entire real line, obtaining separate classes of weak solutions to the Casimir equation. Such results constitute rare globally valid analytic solutions to a class of nonlinear wave equations. Closed-form asymptotic approximations are also given in each case, and these agree nicely with the numerical solutions available in the literature.

**1. Introduction.** We study a partial differential equation introduced by Olver and Rosenau [1],

$$u_{tt} = u_t^2(u_t^2(u_x \pm u_{xxx}))_x, \tag{1}$$

which comes from the Ito system [2]

$$U_t = U_{xxx} + 3UU_x + VV_x, \quad V_t = (UV)_x. \tag{2}$$

Ito showed that the system is highly symmetric and possesses infinitely many conservation laws. It is an extension of the KdV equation, with an additional field variable. Olver and Rosenau [1] introduced a dual bi-Hamiltonian system for the Ito system, which admits a Casimir functional, and the associated Casimir equation

$$\psi_t \pm \psi_{xxt} = (\phi - 1)_x \pm (\phi - 1)_{xxx}, \quad \phi_t = a[\phi - 2(\psi \pm \psi_{xx})]_x. \tag{3}$$

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They then introduced a stream function for the system, yielding the equation (1). In [3], Van Gorder obtained analytic solutions to this equation, which consist of explicit exact solutions in some cases and implicit integral relations in others.

While the Casimir equation for the Ito system has received relatively little attention in the recent literature, it is an interesting nonlinear partial differential equation related to an extension of the KdV equation. Despite being highly nonlinear, as shown in [3] it admits a variety of exact and analytical solutions. In the present paper, we classify all possible series solutions to (i) travelling wave reductions of (1) and (ii) a class of self-similar reductions to (1), in Sections 2 and 3, respectively. In each case, we provide a method of obtaining global weak solutions from the local series solutions. In order to do so, we will need two results on the existence and convergence region of series solutions to nonlinear differential equations, which we give for completeness in the appendix. Then, in Section 4, we employ an alternate method in order to obtain asymptotic expansions for large values of the independent variables in each of the travelling wave and self-similar solution cases.

We should note that by weak solution, we refer specifically to continuous solutions which may fail to be solutions at a countable number of points, due to a degeneracy in the derivatives. This loss of differentiability at a countable number of points is due to the manner of construction of the series solutions: one matches separate solutions to a countable number of points to obtain a continuous solution. At these points, the solutions may fail to have a continuous derivative of some finite order, whereas each of the separate solutions have infinitely many continuous derivatives in their respective finite domains of definition. Thus, the solutions are in general piecewise differentiable. One commonly discussed example in the literature of such a function would be a multi-peakon, with a finite number of points at which the function is continuous yet not continuously differentiable.

## 2. Travelling wave solutions.

We first assume a solution to (1) of the form $u(x,t) = \phi(z)$, where $z = x - St$ is the travelling wave variable. This has the effect of reducing the governing partial differential equation (1) to an ordinary differential equation:

$$\phi'' = S^2\phi'^2(\phi'^2(\phi' \pm \phi'''))'. \tag{4}$$

Making the substitution $f(z) = \phi'(z)$ to reduce the order of (4), and $\epsilon = \pm 1$ for notational simplicity, we obtain

$$f' = S^2 f^2 (f^2(f + \epsilon f''))'. \tag{5}$$

In [3], the solution to this equation was found to satisfy the following implicit equation:

$$\int_{f_0}^{f} \frac{q\mathrm{d}q}{\sqrt{K'q^2 + 2Kq + S^{-2} - q^4}} = \pm\sqrt{\epsilon}(z - z_0), \tag{6}$$

where $K$ and $K'$ are constants of integration and $f(z_0) = f_0$. Then, we define

$$G(f) = \int_{f_0}^{f} \frac{q\mathrm{d}q}{\sqrt{K'q^2 + 2Kq + S^{-2} - q^4}}. \tag{7}$$

Clearly, $G(f_0) = 0$ and $G'(f_0) = \frac{f_0}{\sqrt{K'f_0{}^2+2Kf_0+S^{-2}-f_0{}^4}}$. If $f_0 \neq 0$, then $G'(f_0) \neq 0$. In this case, in a neighbourhood of $z = z_0$, it is possible to invert (7) for $f$ via reversion of series [6]. (Meanwhile, if $f_0 = 0$, then one can deduce by induction that $f^{(n)}(z_0) = 0$ for all $n$, or that $f^{(n)}(z_0)$ does not exist for some $n$. This degenerate case will then be excluded.) For $z$ near $z_0$, we have solution branches

$$f_\pm(z) = f_0 + \sum_{n=1}^{\infty} \frac{(\pm 1)^n \alpha_n (z-z_0)^n \epsilon^{n/2}}{n!}, \tag{8}$$

where for each $n \geq 1$,

$$\alpha_n = \lim_{f\to f_0} \frac{d^{n-1}}{df^{n-1}} \left(\frac{f-f_0}{G(f)}\right)^n. \tag{9}$$

Computing the first few terms in this series, we obtain

$$\begin{aligned} f_\pm(z) =& f_0 \pm \frac{\sqrt{\epsilon}}{f_0}\sqrt{K'f_0{}^2+2Kf_0+S^{-2}-f_0{}^4}(z-z_0) \\ & - \frac{\epsilon}{2f_0{}^3}(Kf_0+S^{-2}+f_0{}^4)(z-z_0)^2 \\ & \pm \frac{\sqrt{K'f_0{}^2+2Kf_0+S^{-2}-f_0{}^4}}{\sqrt{\epsilon}S^2 f_0{}^5} \\ & \times \left(1 - S^2\left(3f_0{}^4+\epsilon f_0{}^3 - \frac{1}{f_0{}^3}(Kf_0+S^{-2}+f_0{}^4)\epsilon\right)\right)(z-z_0)^3 \\ & + O\left((z-z_0)^4\right). \end{aligned} \tag{10}$$

Thus, we have a general series solution to (5). Note that from Theorem B of Appendix B these series each have a non-zero radius of convergence, which will be useful later in this paper. Observe that when $\epsilon = 1$ we have real-valued solutions, whereas when $\epsilon = -1$ we have complex-valued solutions. Now, we would wish to map back a solution into a solution to (1). Before doing so, we will need to discuss regularization of $f$ in order to recover a solution $\phi$ to (4).

2.1. *Classification of travelling wave series solutions.* We shall attempt to find and classify all non-trivial solutions to equation (5) of the form

$$f(z) = z^p \sum_{n=0}^{\infty} C_n z^{rn}, \tag{11}$$

where $r$ and $p$ are real numbers. Note that we lose no generality in expanding our series about $z_0 = 0$, yet we will gain notational convenience. Substituting the series (11) into (5), we obtain

$$\sum_{k=0}^{\infty} z^{rk+p-1}(rk+p)C_k = S^2 \sum_{k=0}^{\infty}\left(z^{rk+5p-1}I_k + \epsilon z^{rk+5p-3}J_k\right), \tag{12}$$

where

$$I_k = \sum_{l=0}^{k}(rl+3p)\sum_{m=0}^{l}\sum_{n=0}^{m} C_n C_{m-n} C_{l-m} \sum_{h=0}^{k-l} C_h C_{k-l-h} \tag{13}$$

and

$$J_k = \sum_{l=0}^{k}(rl+3p-2)\sum_{m=0}^{l}\sum_{n=0}^{m} C_n C_{m-n} C_{l-m}(r(l-m)+p)(r(l-m)+p-1)\sum_{h=0}^{k-l} C_h C_{k-l-h}. \tag{14}$$

We shall use the above equations to find suitable series in several cases of the problem.

2.2. *Series for initial value problems such that $f \in C^3(\Omega)$.* Assume first that we have initial data for $u$ that is consistent with our travelling wave assumption. Then this leads to an initial condition which fixes $f(0)$, $f'(0)$, and $f''(0)$. In such a case we know that we must have $r$ and $p$ non-negative, so that $f(0)$ exists.

2.2.1. *Case 1: $r$ and $p$ are integers.* In this case, $f$ has derivatives of all orders at 0. Therefore, using inductive reasoning on (5), we conclude that $f(0) = 0$ implies that $f$ is the trivial solution. So, we require that $f(0) \neq 0$, and thus $p = 0$. Thus, upon examining (12), we see $r = 1$ or $r = 2$. Following section 2, we compute the series in terms of the initial conditions. First, consider $r = 1$. Notice that if $f(z) = \sum_{n=0}^{\infty} C_n z^n$, then $C_n = \frac{f^{(n)}(0)}{n!}$. We can find $f^{(n)}(0)$ by differentiating (5) $n-3$ times (for $n > 2$) and substituting the values of $f^{(k)}(0)$ that we have already found, for $k < n$. Thus, we obtain

$$\begin{aligned} f(z) =& f_0 + {f_0}'z + \frac{{f_0}''}{2}z^2 + \frac{\epsilon}{6}\left(\frac{{f_0}' - 2S^2{f_0}^3{f_0}'[f_0+\epsilon {f_0}''] - {f_0}'S^2{f_0}^4}{S^2{f_0}^4}\right)z^3 \\ &+ \frac{\epsilon}{24S^2{f_0}^5}\left(f_0{f_0}'' - 4{f_0}'^2 - S^2{f_0}^5{f_0}''\right. \\ &\qquad \left. -2S^2{f_0}^3[f_0{f_0}'({f_0}'+\epsilon{f_0}''') + f_0{f_0}''(f_0+\epsilon{f_0}'') - {f_0}'^2(f_0+\epsilon{f_0}'')]\right)z^4 \\ &+ O(z^5) \end{aligned} \tag{15}$$

where, for brevity, $f_0$ denotes $f(0)$, ${f_0}'$ denotes $f'(0)$, and so on. Because the coefficient of $z^n$ in the power series for $f$ is proportional to $f^{(n)}(0)$, the $r = 2$ case can be viewed as a special case of $r = 1$, where the odd order derivatives $f^{(2k+1)}(0)$ are zero. By Theorem A of Appendix A, each of the series in this sub-case has a non-zero radius of convergence. It should be noted that the travelling wave solutions (10) found above fall into the present case.

2.2.2. *Case 2: $r$ is an integer, $p$ is not an integer.* Our differential equation (5) makes no sense unless $f \in C^3(\Omega)$, where $\Omega$ is the set on which $f$ solves (5). Since we have initial data, we require $\Omega$ to contain an open neighbourhood around 0. We adjust $p$ so that $C_0 \neq 0$, and thus $p > 3$. We no longer require that $f(0) \neq 0$. The first term with a non-zero coefficient on the left-hand side of (12) is of order $z^{p-1}$. Examining (14) carefully, we see that either the first term with a non-zero coefficient on the right-hand side of (12) is of order $z^{5p-3}$ or, from (14), $3p = 2$ (a contradiction). Therefore, for the differential equation to be satisfied, the terms of orders $z^{p-1}$ and $z^{5p-3}$ must cancel and thus $p = \frac{1}{2}$, a contradiction. Therefore, no such series solves (5).

2.2.3. *Case 3: $r$ is not an integer.* As above, we require that $f \in C^3(\Omega)$, and adjust $p$ so that $C_0 \neq 0$. Thus, $p + r > 3$ or $p + r$ is an integer and $p + 2r > 3$. In addition, either $p > 3$ or $p$ is an integer. Consider (12). We see that $p > 3$ leaves unbalanced terms. Thus, $p$ is an integer, so $p + r > 3$. If $p = 0$, no term cancels the second lowest order term on the right-hand side. If $p > 0$, no term cancels the second lowest order term on the left-hand side. Thus, no such series solves (5).

2.3. *Series for $f$ such that $f \notin C^3(\Omega)$.* Consider series solutions which may not be $C^3(\Omega)$. For instance, series solutions with powers $z^{1/2}$ are in such a class; such solutions are not differentiable at $z = 0$. For all further sub-cases, we alter $p$ in (11) so that $C_0 \neq 0$.

2.3.1. *Case 4: $r$ is positive, $p = 0$.* The terms corresponding to $k = 0$ on both sides of (12) are 0. Let $q$ be the second smallest natural number for which $C_q \neq 0$. Then upon examining (12), the lowest order term in the rightmost sum is order $z^{rq-3}$, unless $rq = 1$ or $rq = 2$. In the first case, we have a contradiction as no term from the other sums cancels this term. In the second and third cases, since $q$ must be a natural number greater than 0, this implies that $r$ is of the form $r = \frac{1}{n}$ where $n$ is a natural number. Thus, we consider (5) and assume that $f(z) = g(\nu)$ where $\nu = z^{\frac{1}{n}}$. This yields a new ordinary differential equation

$$a_0\nu^{2n}g' = S^2g^2\left(2a_0gg'\nu^{2n}(g + \epsilon\nu^{n+1}(a_1g' + a_2\nu g'')) + g^2(a_0\nu^{2n}g' + a_3g' + a_4\nu g'' + a_5\nu^2g''')\right), \tag{16}$$

where $a_0$ through $a_5$ are non-zero constants involving $n$. We shall consider the possibility of a series expansion in positive integer powers of $\nu$. To this end, we take

$$g(\nu) = \sum_{k=0}^{\infty} B_k\nu^k, \tag{17}$$

where, since $f(0) \neq 0$, we have $B_0 \neq 0$. The existence of such a series would be equivalent to the existence of the type of series we are considering in this sub-case for $f$. Evaluating (16) at 0 and realizing that $g(0) \neq 0$, we can see that $g'(0) = 0$. Differentiating (16), we obtain $g''(0) = 0$, and differentiating again we obtain $g'''(0) = 0$. Using strong induction and these base cases and noticing that due to our assumption (17) $g$ is smooth at 0, it can be shown that $g^{(n)}(0) = 0$ for all $n > 0$. Thus, $g$ is constant, and thus $f$ is constant in the present sub-case. Examining (5), we see that any constant satisfies the equation. The other two cases, $rq = 1$ and $3rq = 1$, reduce to either Case 1 or similar arguments. Thus the only solutions in this sub-case are trivial, or already explored.

2.3.2. *Case 5: $r$ is positive, $p \neq 0$.* In this case, our assumption that $C_0 \neq 0$ is extremely helpful: we deduce immediately that $p$ must be either 1, $\frac{1}{2}$, or $\frac{2}{3}$. First, if $p = 1$, no term on the right-hand side of (12) cancels the constant term on the left-hand side. The $p = \frac{2}{3}$ situation is similar. If $p = \frac{1}{2}$, we find that $r = \frac{2}{n}$ where $n$ is a positive integer. When $n > 1$, it can be proven by strong induction, using (12) and the expansion

$$f' = 3S^2f^4f' + \epsilon S^2f^4f''' + \epsilon S^2f^3f'f'' \tag{18}$$

of (5), that all the coefficients $C_w = 0$, where $\frac{2w}{n}$ is not an integer multiple of 2. This implies that all possible cases reduce to $n = 1$. Thus, we must have $r = 2$ and $p = \frac{1}{2}$. Plugging in such a series for (5), we see that $C_0$ can only be 0 or one of the four fourth roots of the number $\frac{4}{S^2\epsilon}$. All the other coefficients in the series are uniquely determined by this choice. The non-trivial solution is:

$$\begin{aligned} f(z) = & \sqrt[4]{\frac{4}{S^2\epsilon}} z^{\frac{1}{2}} - \frac{1}{3} S^2 \left(\frac{4}{S^2\epsilon}\right)^{\frac{5}{4}} z^{\frac{5}{2}} + \left(-\frac{47}{630} S^6 \left(\frac{4}{S^2\epsilon}\right)^{\frac{13}{4}} \epsilon + \frac{3}{35} S^4 \left(\frac{4}{S^2\epsilon}\right)^{\frac{9}{4}}\right) z^{\frac{9}{2}} \\ & + \left(-\frac{3431}{80850} S^{10} \left(\frac{4}{S^2\epsilon}\right)^{\frac{21}{4}} k^2 - \frac{1261}{40425} S^6 \left(\frac{4}{S^2\epsilon}\right)^{\frac{13}{4}} + \frac{25826}{363825} S^8 \left(\frac{4}{S^2\epsilon}\right)^{\frac{17}{4}} \epsilon\right) z^{\frac{13}{2}} \\ & + O(z^{17/2}), \end{aligned} \tag{19}$$

where $\sqrt[4]{\frac{4}{S^2\epsilon}}$ is determined equally for all coefficients. Notice that this function is not differentiable at 0, and hence $f \notin C^3(\Omega)$.

2.3.3. *Case 6: r is negative.* Examination of (12) leads quickly to $p = 0$. Let $q$ be the second smallest integer so that $C_q \neq 0$. Then upon expanding the coefficient $z^{rq-1}$ by brute force in (12) and (13), we have $C_q = 3rqC_0{}^4C_q$. Thus, $C_q = 0$, a contradiction. Thus, no such series solves (5).

With this, we have classified all possible series solutions to (5). However, we are really interested in solutions $\phi$ to (4). We now turn our attention to recovering a solution $\phi(z)$ to (4) given a solution $f(z)$ to (5).

2.4. *Regularization of travelling wave series solutions.* We have found two distinct classes of series solutions to (5), both of which can be written in the form

$$f(z) = z^p \sum_{n=0}^{\infty} C_n z^{rn}. \tag{20}$$

When $p = \frac{1}{2}$, this series appears to have a finite, non-zero radius of convergence. In the appendix we show that series of the form (15) have non-zero radius of convergence, and in principle we expect many series of that type to have a finite radius of convergence. Therefore, we shall do some work to convert these series solutions to (5), which may have finite radius of convergence, to continuous weak solutions to (4). Let $f$ be of the form (20) and have $f$ satisfy (5). Let $R$ be a positive number smaller than the radius of convergence of the series $f$. Notice that if $f(z)$ is a solution to (5), then so is $\pm f(z-T)$ for any fixed real number $T$. Let us construct a function $F$ in the following way:

$$F(z) = \begin{cases} f(z-kR) & \text{for } z \in [(2k-1)R, (2k+1)R) \text{ when } k \text{ is an even integer}, \\ -f(z-kR) & \text{for } z \in [(2k-1)R, (2k+1)R) \text{ when } k \text{ is an odd integer}. \end{cases} \tag{21}$$

Then, $F$ is a weak solution to (5) in the sense that $F$ is a solution to (5) except at points $z$ which are integer multiples of $R$. Of course, with the appropriate initial data, we may match solutions to obtain a function $F \in C^0(\Omega)$. We remark that other forms of $F$ are possible. Assuming we obtain a solution $f$ which has convergent series representation on $|z - z_0| \leq R$, we may construct various continuations from $z \in [z_0 - R, z_0 + R]$ to $\Omega$. Indeed, one can in principle consider such an extension for series solutions in more than one variable [4] if we do not want to make the travelling wave assumption on the solutions. Such series solutions would be direct solutions $u(x,t)$ to (1) and could then be extended to their maximum domain of convergence.

In addition, the function $F$ we've constructed is periodic and has integral 0 over any whole period. Thus, the integral over any subset of the real line is finite. We then define

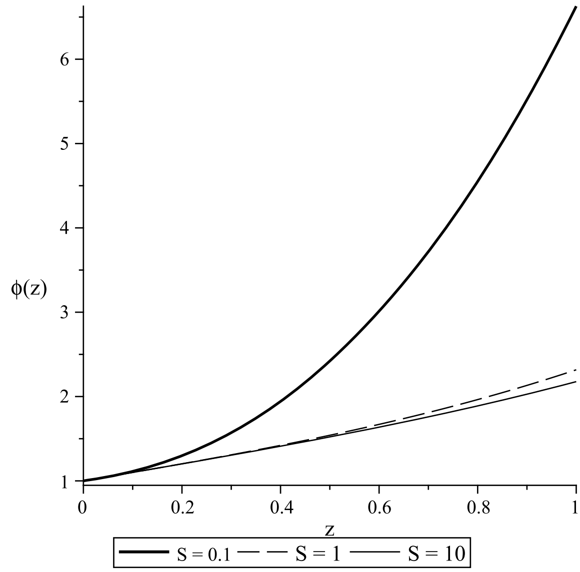


FIG. 1. Plots of the travelling wave solutions to (4) satisfying $f_0'' = -\epsilon\left(f_0^4 + S^{-2}\right) f_0^{-3}$, $f_0' = 0$, $f_0 = 1$ and $\epsilon = -1$ for various values of the wave speed $S$. These solutions scale as exponentials (more precisely, as sinh) for large $z$. The smaller the value of $S$, the more rapid the increase.

a function $\Phi$ by

$$\Phi(z) = \Phi_0 + \int_{z_0}^{z} F(s)ds. \tag{22}$$

Clearly, $\Phi$ is a continuous, periodic function which satisfies (4). $\Phi$ is not necessarily $C^4(\Omega)$, since $f$ (and hence $F$) is not necessarily $C^3(\Omega)$. For other forms of $F$, $\Phi$ can still be continuous, but it would not be expected to exhibit periodicity. The reason for the periodicity of $\Phi$ (in the form we've taken it) is the fact that $F$ is periodic with integral equal to zero over any period. In particular, if $\tau$ represents the period of $F$, then

$$\Phi(z+\tau) = \Phi_0 + \int_{z_0}^{z+\tau} F(s)ds = \Phi_0 + \int_{z_0}^{z} F(s)ds + \int_{z}^{z+\tau} F(s)ds = \Phi_0 + \int_{z_0}^{z} F(s)ds = \Phi(z)\,, \tag{23}$$

since

$$\int_{z}^{z+\tau} F(s)ds = 0\,. \tag{24}$$

2.5. *Specific examples.* Let us consider a specific example in order to illustrate the form of some of the solutions. It was shown in [3] that exact integrals for the travelling wave case exist and admit closed form solutions in some specific cases. One such case corresponds to

$$f_0'' = -\epsilon\left(\frac{f_0^4 + S^{-2}}{f_0^3}\right). \tag{25}$$

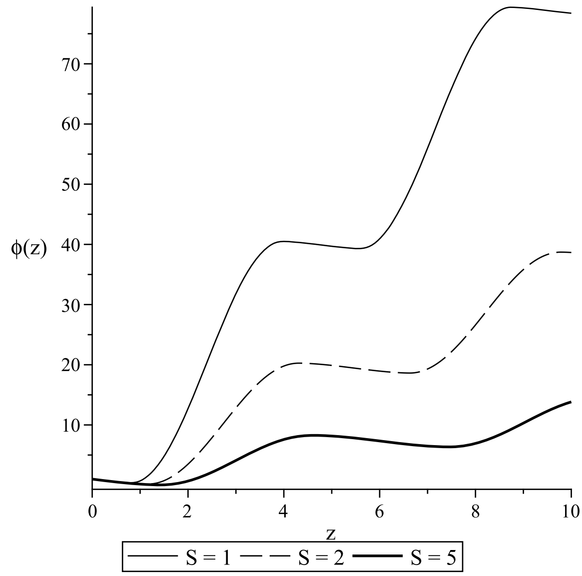


FIG. 2. Plots of the travelling wave solutions to (4) satisfying $f_0'' = -\epsilon\left(f_0^4 + S^{-2}\right) f_0^{-3}$, $f_0' = 0$, $f_0 = -1$ and $\epsilon = 1$ for various values of the wave speed $S$. The positive $\epsilon$ solutions scale as oscillations along a linear trend. The smaller the value of $S$, the more rapid the increase.

In this case, solutions scale like

$$f(z) \sim \sqrt{C_1 + C_2 \sinh(z + C_3)} \tag{26}$$

for large $z$ and $\epsilon = -1$. The solution for $\phi(z)$ is then found by integrating over (26). For such solutions, the radius of convergence of the corresponding Taylor series is finite and positive (sinh has infinite radius of convergence, while the root is the limiting term here, with finite yet positive radius of convergence, depending on $C_1$ and $C_2$). This shows that the types of solutions we expect to obtain should have finite yet positive radius of convergence, and that our method for continuing solutions and then matching at common points makes sense.

From the identities derived previously, it is necessary for $f_0 \neq 0$. That is, we must exclude the possibility $\phi'(0) = 0$; otherwise, the travelling wave ordinary differential equation becomes singular. Furthermore, in the limit $f_0 \to 0$, the coefficients of the obtained Taylor series become singular. Performing numerical computations with these series, we observe a breakdown of solutions for $f_0 << 1$. Similar comments hold for $S \to 0$, so we avoid $S = 0$ as well.

Let us demonstrate some of the solutions obtained using (25) in Figure 1. We see that these solutions obey the scaling (26) for large enough $z$. There are, of course, other possible solutions. We have shown that solutions are parametrized by the constants $\phi_0 = \phi(0)$, $f_0 = \phi'(0)$, $f_0' = \phi''(0)$ and $f_0'' = \phi'''(0)$ as well as $S$. A qualitative change in solutions is also observed for $\epsilon = 1$ or $\epsilon = -1$. Hence, we have solutions corresponding to parameter manifolds $\mathcal{M}_{\epsilon=\pm1} \subset \mathbb{R}^5$. With such a huge parameter space, a number of solutions might be possible, and we cannot discuss them all here. Indeed, the very restrictive condition (25) is just a narrow restriction which permits comparison to an explicitly integrable subcase. Note that the first-order correction for any solution with exactly the same initial data occurs at the fourth-order terms. Then, a small perturbation $0 < \chi << 1$ in parameter space along the $S$ axis will result in at most the following variation:

$$|\phi_{S+\chi}(z) - \phi_S(z)| = \left|\frac{f_0'}{24 f_0^4}\right| \left(\frac{1}{S^2} - \frac{1}{(S+\chi)^2}\right) z^4 + O(z^5) \leq \left|\frac{f_0'}{24 f_0^4}\right| \frac{z^4}{S^2} + O(z^5)\,. \quad (27)$$

Therefore, while initial variations due to a change in $S$ are small for small $z$, they become possibly very large for large $z$. This behavior is observed in Figure 1.

Similar to the results plotted in Figure 1, we again use (25), this time taking $\epsilon = 1$, and plot the resulting travelling wave solutions in Figure 2. The $\epsilon = 1$ solutions are quite distinct in form compared to the $\epsilon = -1$ solutions. Here the solutions oscillate along an increasing trend. Note that both of the plots obtained here agree qualitatively with the large-$z$ asymptotic solutions that we present in Section 4.

**3. Self-similar solutions.** As in Van Gorder [3], let us assume a solution of the form $u(x,t) = g(x)h(t)$. We find that one such solution takes the form $u(x,t) = \sqrt{t-K}g(x)$. Plugging this back into (1), we see that $g$ satisfies the ordinary differential equation

$$4g + 2g^3(g')^2 + 2\epsilon g^3 g' g''' + g^4 g'' + \epsilon g^4 g^{(iv)} = 0. \quad (28)$$

We shall attempt to classify all non-trivial series solutions to this equation of the form

$$g(x) = x^p \sum_{n=0}^{\infty} C_n x^{rn}, \quad (29)$$

where $q$ and $r$ are real numbers. Again, if $g(x)$ is a solution to (28), then so is $g(x-T)$ for any fixed real number $T$. Thus, our decision to expand our series at $x_0 = 0$ costs us no generality. In addition, we adjust $p$ so that $C_0$ is nonzero. We substitute our series for $g$ into our differential equation. From (28) and (29), upon computing many Cauchy products, we obtain the identity

$$4\sum_{k=0}^{\infty} C_k x^{rk+p} + \sum_{k=0}^{\infty} A_k x^{rk+5p-2} + \epsilon \sum_{k=0}^{\infty} B_k x^{rk+5p-4} = 0\,, \quad (30)$$

where

$$A_k = \sum_{m=0}^{k} (rm+3p-1) \sum_{n=0}^{m} C_n (rn+p) \sum_{l=0}^{m-n} C_l C_{m-n-l} \sum_{b=0}^{k-m} C_b C_{k-m-b} \quad (31)$$

and

$$B_k = \sum_{m=0}^{k} (rm+3p-3) \sum_{n=0}^{m} C_n (rn+p)(rn+p-1)(rn+p-2) \sum_{l=0}^{m-n} C_l C_{m-n-l}. \quad (32)$$

3.1. *Solutions $g \in C^4(\mathbb{R})$.* Assume we have initial data for $u$ that is consistent with our separability assumption. If $K \neq 0$, this initial data translates to initial conditions on $g$ of the form $g(0)$, $g'(0)$, $g''(0)$, and $g'''(0)$, and we assume $g$ is four times differentiable at 0 and that $g^{(iv)}$ is continuous at 0. Since $g^{(iv)}$ is continuous at 0, we calculate $g^{(iv)}(0)$ with a limit:

$$g^{(iv)}(0) = \lim_{x\to 0} \frac{4g(x) + 2g(x)^3 g'(x)^2 + 2\epsilon g(x)^3 g'(x) g'''(x) + g(x)^4 g''(x)}{-\epsilon g(x)^4}. \tag{33}$$

If $g$ is not identically 0, we can cancel a power of $g$ in the numerator and denominator. Examining the result, we see that if $g(0) = 0$, then $\lim_{x\to 0} g^{(iv)}(x)$ does not exist, and hence $g^{(iv)}$ is not continuous at 0. Thus, we require that $g(0) \neq 0$ so that we avoid studying the trivial case. Thus, we may assume that $p = 0$ and $r > 0$.

3.1.1. *Case 1: $r$ is a positive integer.* From (30) we see that $r = 1$ or $r = 2$. Either way, we perform an analysis similar to the one in sub-case 1.1 of Section 4 and find that by differentiating (28) an appropriate number of times we can compute the coefficients $C_k$ from the initial conditions (as justified in section 2). To fourth order, we obtain:

$$g(x) = g_0 + {g_0}'x + \frac{{g_0}''}{2}x^2 + \frac{{g_0}'''}{6}x^3 + \frac{4g_0 + 2{g_0}^3{g_0}'^2 + 2\epsilon {g_0}^3 {g_0}' {g_0}''' + {g_0}^4 {g_0}''}{-24\epsilon {g_0}^4}x^4 + O(x^5)\,, \tag{34}$$

where, for brevity, the $g_0$ on the right-hand side denotes $g(0)$ and so on. Differentiating (28) once, we can see that if $g'(0)$ and $g'''(0)$ are both 0, then, since $g(0) \neq 0$, we must have $g^{(v)}(0) \neq 0$, and thus, $r \neq 2$. Thus, the only possibility in this case is $r = 1$, depicted above. By Theorem B of Appendix B, each of the series in this case has non-zero radius of convergence.

3.1.2. *Case 2: $r$ is positive, but is not an integer.* In this case, $r > 4$; otherwise, $g$ is not 4 times differentiable at $x = 0$. Thus, examining (30), we have a contradiction. So, no solutions of this type are possible.

3.2. *Solutions $g \notin C^4(\mathbb{R})$.* We may in principle construct additional series solutions to (28) of type (29) that may not be differentiable at $x = 0$. Recall that we have allowed only those values of $p$ so that $C_0 \neq 0$.

3.2.1. *Case 3: $r$ is a positive integer, $p$ is not.* From (30), either $p > 1$ or the term of lowest degree on the right-hand side cancels nothing. If $p > 1$, nothing cancels the term of lowest degree on the left. Thus no such series solve (28).

3.2.2. *Case 4: $r$ is a positive non-integer.* If $p < 0$, we have a contradiction because no term cancels the lowest degree on the right-hand side of (30). If $p$ is not an integer, we have a contradiction for the same reason as in sub-case 2.2. If $p \geq 1$, no term cancels the lowest degree term on the left-hand side (as $B_0 = 0$ when $p = 1$). Thus, $p = 0$. We see that, this being the case, $r = \frac{2}{N}$ for some natural number $N$, where, since $r$ is not an integer, $N > 2$. We can see (non-trivially) from the right-hand side that $C_h = 0$ for all positive natural numbers $h$ where $hr < 4$ and $hr$ is not an integer. Then, let $q$ be the smallest integer for which $C_q$ is non-zero and $qr$ is not an integer. Then, we can see that in (28) no term cancels the term $\epsilon {C_0}^4 C_q (rq)(rq-1)(rq-2)(rq-3)x^{rq-4}$, which comes from the rightmost expression. Thus, no such series solve (28).

3.2.3. *Case 5: r is negative.* Upon examining (30), we see that $p \geq \frac{1}{2}$, or no term cancels the highest degree term on the left-hand side. However, if $p > \frac{1}{2}$, no term cancels the highest degree term on the right-hand side. Therefore, $p = \frac{1}{2}$. We see, therefore, that $r = -\frac{2}{N}$, where $N$ is a natural number other than 0. We shall see that when $N = 1$, we have a solution. Otherwise, it can be proven by strong induction that all solutions reduce to $N = 1$. We have $r = -2$, $p = \frac{1}{2}$. As in the travelling wave case, there are only five choices for $C_0$: 0, or one of the four fourth roots of $-16$. Unlike the travelling wave case, the rest of the constants in the series are not uniquely determined by the choice of $C_0$ ($C_1$ can be any number), though they are determined by $C_0$ and $C_1$. As $g$ is not differentiable at 0 in this case, we cannot use the technique we used in Case 1 to write down the series in general. We can, however, give an example of such a series:

$$\begin{aligned} g(x) = 2ix^{1/2} + \frac{9\sqrt{2}(1+i)}{16}\epsilon x^{-3/2} - \frac{1539\sqrt{2}(1+i)}{512}x^{-7/2} \\ + \frac{238553\sqrt{2}(1+i)}{40960}(1+i)x^{-11/2} + O\left(x^{-15/2}\right). \end{aligned} \tag{35}$$

Notice that in this case we have an asymptotic series solution, valid not for $x = 0$ but rather for $x \to \infty$.

3.3. *Regularization of self-similar solutions.* We have found series solutions to (28). We wish to regularize those series solutions which are not asymptotic, following the method presented in section 5. Therefore, the series we are presently concerned with are of the form:

$$g(x) = \sum_{n=0}^{\infty} C_n x^n. \tag{36}$$

In principle, we expect a series of this type to have a finite radius of convergence; we already know the radius of convergence is non-zero. Therefore, we will attempt to obtain weak solutions to (28) from these series. Let $g$ be of the form (36) and satisfy (28). Let $R$ be a positive number smaller than the radius of convergence of the series $g$. Recall that if $g(x)$ is a solution to (28), so is $g(x - T)$ for any fixed real number $T$, and that if $g(x)$ is a solution, so is $-g(x)$. Then, define $G$ in the following way:

$$G(x) = \begin{cases} g(x - kR) & \text{for } x \in [(2k-1)R, (2k+1)R) \text{ when } k \text{ is an even integer}, \\ -g(x - kR) & \text{for } x \in [(2k-1)R, (2k+1)R) \text{ when } k \text{ is an odd integer}. \end{cases} \tag{37}$$

Then, $G$ is a solution to the ordinary differential equation (28) governing the self-similar solutions to (1), except at points $x$ which are integer multiples of $R$. Thus (37) constitutes a weak solution to (28) valid globally.

3.4. *Specific examples.* Consider $\epsilon = -1$. As seen before in the travelling wave case, we have solutions which exhibit exponential growth. Such solutions are shown in Figure 3. These solutions are consistent with the asymptotic solutions which we derive later in Section 4.

When $\epsilon = 1$, the solutions become more difficult to obtain. This is because the solutions for $\epsilon = 1$ are dominated by oscillatory behavior, which means that the solutions eventually tend to zero. Yet, this causes (28) to become singular (the $g^{(iv)}$ term has a coefficient $g^4$). In between regions where $g \to 0$, however, solutions may be obtained as

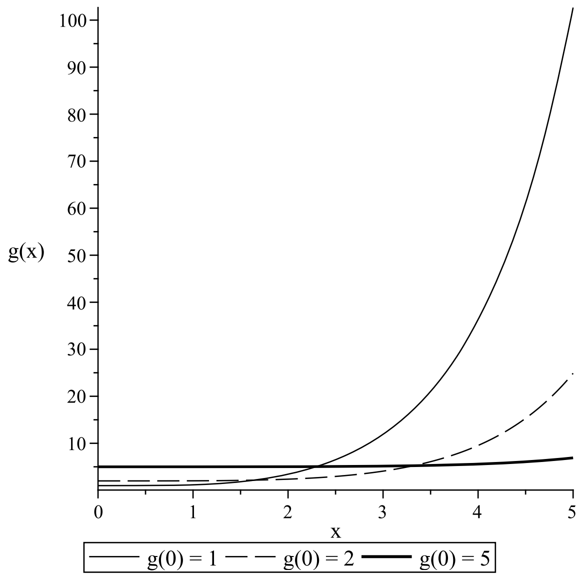


FIG. 3. Plots of the travelling wave solutions to (28) satisfying $g'(0) = 0$, $g''(0)$, $g'''(0) = 0$ and $\epsilon = -1$ for various values of the initial condition $g(0)$.

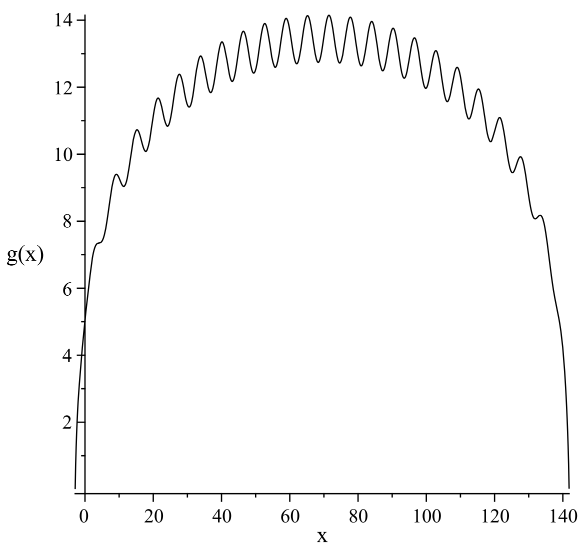


FIG. 4. Plots of the solution to (28) satisfying $g(0) = 5$, $g'(0) = 1$, $g''(0)$, $g'''(0) = 0$ and $\epsilon = 1$.

discussed above. One such solution is displayed in Figure 4. Again, such solutions are consistent with the form of the asymptotic solutions derived in Section 4.

The dynamics of the solution in Figure 4 are best shown on a phase portrait, and we do this in Figure 5. As $g \to 0$, the solution becomes undefined, while for intermediate values

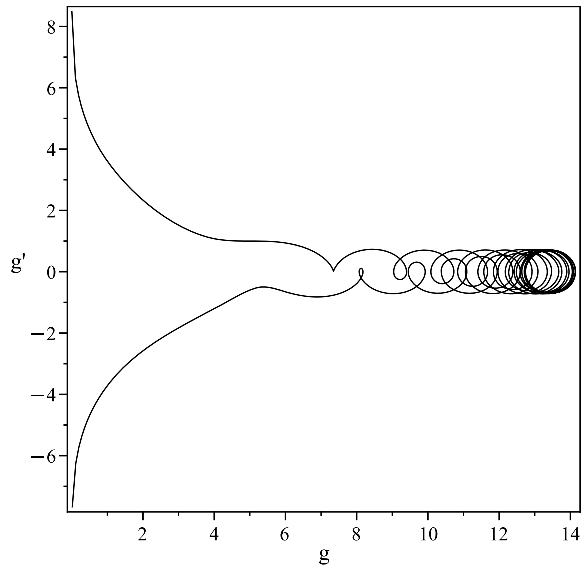


FIG. 5. Numerically generated phase portrait of the solution to (28) satisfying $g(0) = 5$, $g'(0) = 1$, $g''(0)$, $g'''(0) = 0$ and $\epsilon = 1$.

of $x$ the solution approaches periodic behavior of fixed amplitude, before this structure is destroyed when $g \to 0$ on the other side of the sub-domain. Thus, these solutions exist on intervals of the form $(x_1, x_2)$, where $\lim_{x \to x_1^+} g(x) = 0$ and $\lim_{x \to x_2^-} g(x) = 0$. On the interior of this interval, we observe the emergence of oscillations.

**4. Asymptotic properties of solutions.** Here we shall discuss asymptotic approximations for both the travelling wave and self-similar solutions. These solutions will be qualitatively distinct from those discussed previously in Sections 2 and 3.

4.1. *Asymptotic properties of the travelling wave solutions.* In order to ascertain the large-$z$ asymptotics for (5), let us introduce the change of function $h(\chi) = f(z)$ where $\chi = 1/z$. Then, (5) becomes

$$h''' - \frac{6}{\chi}h'' + 3\left(\frac{\epsilon}{\chi^4} + \frac{2}{\chi^2}\right)h' + 2\frac{h'h''}{h} + \frac{4}{\chi}\frac{h'^2}{h} - \frac{\epsilon}{S^2\chi^4}\frac{h'}{h^4} = 0\,. \tag{38}$$

Assuming nonlinear terms contribute sufficiently small contributions to the asymptotics, we have

$$h''' - \frac{6}{\chi}h'' + 3\left(\frac{\epsilon}{\chi^4} + \frac{2}{\chi^2}\right)h' = 0\,. \tag{39}$$

This is a second-order equation for $h'$, and we find

$$h'(\chi) = i\chi^4\left(c_1\left(\epsilon + i\sqrt{3\epsilon}\chi + \chi^2\right)\exp\left(\frac{i\sqrt{3\epsilon}}{\chi}\right) + c_2\left(-\epsilon + i\sqrt{3\epsilon}\chi + \chi^2\right)\exp\left(-\frac{i\sqrt{3\epsilon}}{\chi}\right)\right). \tag{40}$$

4.1.1. *The $\epsilon = 1$ reduction.* Let us first consider the case when $\epsilon = 1$. Picking $c_1 = c_2 = \beta/2$ for simplicity,

$$h'(\chi) = \beta\left(1-\chi^2\right)\chi^4 \sin\left(\frac{\sqrt{3}}{\chi}\right) + \sqrt{3}\beta\chi^5\cos\left(\frac{\sqrt{3}}{\chi}\right). \tag{41}$$

Performing one integration, we recover

$$\begin{aligned} h(\chi) =&\beta_1 + \sqrt{3}\beta\left(\chi^7 - \frac{4}{5}\chi^5 + \frac{1}{5}\chi^3 - \frac{3}{10}\chi\right)\sin\left(\frac{\sqrt{3}}{\chi}\right) \\ &- \beta\left(3\chi^6 + \frac{3}{5}\chi^4 - \frac{3}{10}\chi^2\right)\cos\left(\frac{\sqrt{3}}{\chi}\right) + \frac{9\beta}{10}\mathrm{Ci}\left(\frac{\sqrt{3}}{\chi}\right), \end{aligned} \tag{42}$$

where $\beta_1$ is another constant of integration, $\mathrm{Ci}(x)$ denotes the cosine integral [5]

$$\mathrm{Ci}(x) = \gamma + \ln(x) + \int_0^x \left(\frac{\cos(\tau)-1}{\tau}\right)d\tau, \tag{43}$$

and $\gamma$ is the Euler-Mascheroni constant. Then, the asymptotics for $f$ become

$$\begin{aligned} f(z) \sim&\beta_1 + \beta\sqrt{3}\left(\frac{1}{z^7} - \frac{4}{5}\frac{1}{z^5} + \frac{1}{5}\frac{1}{z^3} - \frac{3}{10}\frac{1}{z}\right)\sin\left(\sqrt{3}z\right) \\ &- \beta\left(3\frac{1}{z^6} + \frac{3}{5}\frac{1}{z^4} - \frac{3}{10}\frac{1}{z^2}\right)\cos\left(\sqrt{3}z\right) + \frac{9\beta}{10}\mathrm{Ci}\left(\sqrt{3}z\right). \end{aligned} \tag{44}$$

Integrating (44) once more, we obtain the asymptotics for $\phi$:

$$\begin{aligned} \phi(z) \sim&\beta_1 z + \beta_2 - \beta\sqrt{3}\left(\frac{1}{6}\frac{1}{z^6} - \frac{3}{40}\frac{1}{z^4} + \frac{19}{80}\frac{1}{z^2} + \frac{3}{10}\right)\sin\left(\sqrt{3}z\right) \\ &+ \beta\left(\frac{1}{2}\frac{1}{z^5} + \frac{11}{40}\frac{1}{z^3} - \frac{81}{80}\frac{1}{z}\right)\cos\left(\sqrt{3}z\right) + \frac{9\beta}{10}z\mathrm{Ci}\left(\sqrt{3}z\right) - \frac{21\beta\sqrt{3}}{16}\mathrm{Si}\left(\sqrt{3}z\right), \end{aligned} \tag{45}$$

where $\mathrm{Si}(x)$ denotes the sine integral [5],

$$\mathrm{Si}(x) = \int_0^x \frac{\sin(\tau)}{\tau}d\tau. \tag{46}$$

4.1.2. *The $\epsilon = -1$ reduction.* For the remaining case of $\epsilon = -1$, we obtain two solution branches: one which exponentially decays, and the other which exponentially grows. For simplicity of the expressions, let us take the exponentially decaying branch. Results for the other branch are similar in form. To this end, set $c_1 = -i\beta$ and $c_2 = 0$. Then

$$h'(\chi) = \beta\left(1 + \sqrt{3}\chi + \chi^2\right)\chi^4\exp\left(-\frac{\sqrt{3}}{\chi}\right). \tag{47}$$

Integrating once, we have

$$\begin{aligned} h(\chi) =&\frac{\beta}{70}\chi\left(10\chi^6 + 10\sqrt{3}\chi^5 + 8\chi^4 - 2\sqrt{3}\chi^3 + 2\chi^2 - \sqrt{3}\chi + 3\right)\exp\left(-\frac{\sqrt{3}}{\chi}\right) \\ &+ \beta_1 - \frac{3\sqrt{3}\beta}{70}\mathrm{Ei}\left(\frac{\sqrt{3}}{\chi}\right). \end{aligned} \tag{48}$$

Here Ei is the exponential integral [5],

$$\mathrm{Ei}(x) = \int_1^\infty \frac{e^{-x\tau}}{\tau} d\tau. \tag{49}$$

The asymptotics for $f$ are then

$$\begin{aligned} f(z) \sim & \frac{\beta}{70}\frac{1}{z}\left(\frac{10}{z^6} + \frac{10\sqrt{3}}{z^5} + \frac{8}{z^4} - \frac{2\sqrt{3}}{z^3} + \frac{2}{z^2} - \frac{\sqrt{3}}{z} + 3\right) e^{-\sqrt{3}z} \\ & + \beta_1 - \frac{3\sqrt{3}\beta}{70}\mathrm{Ei}\left(\sqrt{3}z\right). \end{aligned} \tag{50}$$

Integrating (50) once more, we have the asymptotics for $\phi$:

$$\begin{aligned} \phi(z) \sim & \frac{\beta}{70}\left(3 + \frac{27\sqrt{3}}{8z} - \frac{19}{8z^2} + \frac{11\sqrt{3}}{12z^3} - \frac{3}{4z^4} - \frac{5\sqrt{3}}{3z^5} - \frac{5}{3z^6}\right) e^{-\sqrt{3}z} \\ & + \beta_1 z + \beta_2 - \frac{3\beta}{560}\left(35 + 8\sqrt{3}z\right)\mathrm{Ei}\left(\sqrt{3}z\right). \end{aligned} \tag{51}$$

4.2. *Asymptotic properties of the self-similar solutions.* Let us make the change of function $j(y) = g(x)$ where $y = 1/x$. Then, (28) becomes

$$\begin{aligned} & j^{(iv)} + \frac{12}{y}j''' + \left(\frac{\epsilon}{y^4} + \frac{36}{y^2}\right)j'' + \left(\frac{2\epsilon}{y^5} + \frac{24}{y^3}\right)j' \\ & + \frac{j'}{j}\left(2j''' + \frac{2}{y}j'' + \left(\frac{2\epsilon}{y^4} + \frac{12}{y^2}\right)j'\right) + \frac{4\epsilon}{y^8}\frac{1}{j^3} = 0. \end{aligned} \tag{52}$$

Assuming nonlinear terms contribute sufficiently small contributions to the asymptotics, we have

$$j^{(iv)} + \frac{12}{y}j''' + \left(\frac{\epsilon}{y^4} + \frac{36}{y^2}\right)j'' + \left(\frac{2\epsilon}{y^5} + \frac{24}{y^3}\right)j' = 0. \tag{53}$$

The exact solution to this equation takes the form

$$j(y) = \beta_0 + \frac{\beta_1}{y} + \beta_2\cosh\left(\frac{\sqrt{-\epsilon}}{2y}\right) + \beta_3\sinh\left(\frac{\sqrt{-\epsilon}}{2y}\right)\cosh\left(\frac{\sqrt{-\epsilon}}{2y}\right). \tag{54}$$

We then recover the asymptotic solution for $g(x)$:

$$g(x) = \beta_0 + \beta_1 x + \beta_2\cosh\left(\frac{\sqrt{-\epsilon}}{2}x\right) + \beta_3\sinh\left(\frac{\sqrt{-\epsilon}}{2}x\right)\cosh\left(\frac{\sqrt{-\epsilon}}{2}x\right). \tag{55}$$

Thus, when $\epsilon = 1$ we obtain oscillatory solutions (just as in the travelling wave case) while when $\epsilon = -1$ we obtain solutions with exponential growth and decay (again, just like in the travelling wave case).

**5. Conclusions.** We have used various analytic techniques to classify all power series solutions to (1) of the travelling wave type, including a general power series solution obtained by reversion of series in [3]. We have regularized the series solutions of the travelling wave type, so that our power series solutions with finite radius of convergence may be used to construct weak solutions which are valid globally except at countably many real numbers. The class of such solutions constructed was periodic, but non-periodic constructions are also possible.

We have used similar analytic techniques to classify all power series solutions to (1) of a certain self-similar type. In this case we found that asymptotic series, which are convergent for large $x$ and divergent for small $x$, exist. We have regularized those self-similar series solutions that do converge for small $x$, again obtaining periodic weak solutions.

With this, we have given a constructive proof of the existence of weak global solutions to (1) in both the travelling wave and self-similar cases discussed. Importantly, since we've obtained multiple series solutions in each case, the obtained solutions are not unique.

Finally, we have constructed asymptotic solutions (not of series type) for both the travelling wave and self-similar cases. In each case, the asymptotic solutions are oscillatory when $\epsilon = 1$ and exhibit exponential growth or decay when $\epsilon = -1$. This is in complete agreement with the numerical results presented in [3].

**Appendix A. General formulas for series solution coefficients.** We consider power series with positive integer powers in the independent variable which converge on a non-trivial interval. Let $f$ be a real-valued function of real numbers that admits such a power series at, say, 0. Then in a neighbourhood of $x = 0$,

$$f(x) = \sum_{m=0}^{\infty} C_m x^m. \tag{56}$$

Consider a differential equation of the form

$$F(f, f', f'', ..., f^{(n)}) = 0, \tag{57}$$

where $F$ is an analytic function of its independent variables in a neighbourhood of 0. We will use notation like $\frac{\partial}{\partial x}F(f, f', f'') = 0$ in this section. To understand this notation, consider an example:

$$G(f, f', f'') \equiv f - f''^2. \tag{58}$$

Then,

$$\frac{\partial}{\partial x}G(f, f', f'', f''') = f' - 2f''f''', \tag{59}$$

where we have added an additional argument to $\frac{\partial}{\partial x}G$ to indicate its dependence on $f'''$. Realize that this convention serves only to make our theorem easier to read, and that if such a function $F$ depends only on derivatives up to order $n$, then the $q$-th derivative of $F$ depends only on derivatives up to order $n + q$, and is analytic in its independent variables in a neighbourhood of 0. We will often treat a differential equation such as $G$ as a function of real numbers. To see how this works, consider the example above. Then $G(2, 4, 1) = 3$, and $\frac{\partial}{\partial x}G(1, 4, 3, 5) = -26$.

Theorem A. Let $g(x) = \sum_{m=0}^{\infty} C_m(x - x_0)^m$ in a neighbourhood of $x = x_0$. Let $F(f, f', ..., f^{(n)}) = 0$ be a differential equation, where $F$ is an analytic function of its independent variables in a neighbourhood of 0. Then, for all $j \in \mathbb{N}_0$,

$$\frac{\partial^j}{\partial^j x}F(C_0, C_1, ..., (n + j)!C_{n+j}) = 0 \tag{60}$$

if and only if $F(g, g', ..., g^{(n)}) = 0$ in a neighbourhood of $x = x_o$.

*Proof.* We first prove the "only if" direction. Place the power series expression for $g$ into the function $F$. Then, as both are analytic, so is their composition (in a neighbourhood of $x = x_0$). Thus, the resulting equation is a power series in $x$ which converges in a neighbourhood of $x = x_0$. We shall call this power series $f$. Then

$$f(x) = \sum_{m=0}^{\infty} K_m (x - x_0)^m. \tag{61}$$

We shall argue that each coefficient of this power series is 0. Notice that for each $q \in \mathbb{N}_0$,

$$q! K_q = f^{(q)}(x_0), \tag{62}$$

and that

$$f(x) = F(g(x), g'(x), ..., g^{(n)}(x)). \tag{63}$$

In fact, using our notation,

$$f^{(q)}(x) = \frac{\partial^q}{\partial^q x} F(g(x), g'(x), ..., g^{(n+q)}(x)). \tag{64}$$

Notice that $g^{(r)}(x_0) = r! C_r$ for each $r \in \mathbb{N}_0$. Then by the hypothesis of the theorem, $K_q = 0$ for each $q \in \mathbb{N}_0$.

Next, we prove the "if" direction. Since $F(g, g', ..., g^{(n)}) = 0$ in a neighbourhood of $x = x_0$ and $F$ and $g$ are smooth there,

$$\frac{\partial^j}{\partial^j x} F(g, g', ..., g^{(n+j)}) = 0 \tag{65}$$

for each $j \in \mathbb{N}_0$. Thus, (65) is true at $x = x_0$ for each $j \in \mathbb{N}_0$. This is exactly (60). □

COROLLARY A. Let $g(x) = \sum_{m=0}^{\infty} C_m x^m$ in a neighbourhood of $x = x_0$. Let $F(f, f', ..., f^{(n)}) = 0$ be a differential equation, where $F$ is given by

$$F(f, f', ..., f^{(n)}) \equiv \sum_{k=0}^{n} A_k f^{(n)}. \tag{66}$$

Then $g$ solves $F(f, f', ..., f^{(n)}) = 0$ in a neighbourhood of $x = x_0$ if and only if for all $j \in \mathbb{N}_0$,

$$\sum_{k=0}^{n} A_k (k + j)! C_{k+j} = 0. \tag{67}$$

*Proof.* This is an application of the above theorem to the linear differential equation $F$. □

**Appendix B. Convergence of series solutions.** Consider the differential equation

$$f^{(k)} = \frac{P}{Q}, \tag{68}$$

where $P$ and $Q$ are polynomials in $f, f', ..., f^{(k-1)}$, and $k \geq 1$. Then:

THEOREM B. All power series solutions to (68) of the form

$$f(x) = \sum_{j=0}^{\infty} \frac{{f_0}^{(j)}}{j!}(x-x_0)^j \tag{69}$$

have non-zero radius of convergence, where $f_0 \equiv f(x_0)$ and so on, provided that

$$Q(f_0, ..., {f_0}^{(k-1)}) \neq 0 \tag{70}$$

in a neighborhood of $x_0$.

*Proof.* Let $P$ have $T_P$ terms, $Q$ have $T_Q$ terms. Let $n$ be the degree of $P$, or 1, whichever is greater, and let $m$ be the degree of $Q$, or 1, whichever is greater. Let $U_P$ denote the largest number among the magnitudes of the coefficients of $P$. If this number is less than 1, let $U_P$ be 1. Let $U_Q$ be determined in the same manner from the coefficients of $Q$, and similarly changed to 1 if that is larger. We shall prove by induction that for all $a \in \mathbb{N}_0$,

$${f_0}^{(k+a)} = \frac{R_a}{Q^{1+2a}}, \tag{71}$$

where $Q$ is evaluated henceforth at $f_0, ..., {f_0}^{(k-1)}$, and where $R_a$ is a polynomial of degree no greater than $2am + (1+a)n$ in $f_0, ..., {f_0}^{(k-1)}$ with at most $2^a k^a (T_P T_Q)^{2a+1}$ terms, and whose largest coefficient (in magnitude) is no greater in magnitude than

$$M(a) \equiv (2U_P U_Q mn(m+n))^{2a+1}(2a+1)!!. \tag{72}$$

We shall refer to these conditions collectively as $B(a)$, and whenever we speak about the largeness of coefficients, we shall always be referring to their magnitudes.
Base case: Notice that $P$ has all the properties required of $R_0$ when evaluated at $f_0, ..., {f_0}^{(k-1)}$.
Inductive step: Let $B(a)$ hold. We shall prove $B(a+1)$. Differentiating (71) with respect to the independent variable of $f$, we obtain

$${f_0}^{(k+a+1)} = \frac{Q^{1+2a}{R_a}' - (1+2a)Q^{2a}Q'R_a}{Q^{2+4a}} \tag{73}$$

$$= \frac{Q{R_a}' - (1+2a)Q'R_a}{Q^{2+2a}}. \tag{74}$$

We notice that ${R_a}'$ and $Q'$ are polynomials in $f_0, ..., {f_0}^{(k)}$. Each instance of ${f_0}^{(k)}$ is degree 1, by chain rule. We substitute every instance of ${f_0}^{(k)}$ using (71), and obtain a rational expression in the numerator of (74). We multiply the top and bottom of (74) by $Q$ and obtain

$${f_0}^{(k+a+1)} = \frac{Q\hat{R}_a - (1+2a)\hat{Q}R_a}{Q^{1+2(a+1)}}, \tag{75}$$

where $\hat{R}_a$ and $\hat{Q}$ are the polynomials obtained respectively from ${R_a}'$ and $Q'$ by the substitutions and multiplication described above, and their arguments are therefore $f_0, ..., {f_0}^{(k-1)}$. From (72), we have that the largest coefficient of $R_a$ is no more than $M(a)$. Differentiating this polynomial, we find that the largest coefficient of ${R_a}'$ is no more than $M(a)(2am + (1+a)n)$. Thus, substituting instances of ${f_0}^{(k)}$ with (71) and multiplying through by $Q$, we find that the largest coefficient of $\hat{R}_a$ is no more than

$M(a)(2am+(1+a)n)U_PU_Q$. Upon multiplying by $Q$, we find the largest coefficient of $Q\hat{R}_a$ is no more than

$$M(a)(2am+(1+a)n)U_P{U_Q}^2 \leq M(a)(m+n)(2a+3)(U_PU_Q)^2. \tag{76}$$

Similarly, we find that the largest coefficient of $(1+2a)\hat{Q}R_a$ is at most $M(a)m{U_Q}^2U_P(2a+1)$. Thus, the largest coefficient of the numerator of (75) is no more than

$$2M(a)m(m+n)(2a+3)(U_PU_Q)^2, \tag{77}$$

which is less than $M(a+1)$. From $B(a)$, we know that the degree of $R_a$ is at most $2am+(1+a)n$. The degree of ${R_a}'$ is at most that same number, and therefore upon substituting instances of ${f_0}^{(k)}$ with (71) and multiplying through by $Q$, we find that the degree of $\hat{R}_a$ is at most $2am+(1+a)n+m+n$. Thus, the degree of $Q\hat{R}_a$ is at most $2m(a+1)+(2+n)n$. By a similar argument, we find that the same is true of $(1+2a)\hat{Q}R_a$, and therefore the same is true of their sum. Since $R_a$ has at most $2^ak^a(T_PT_Q)^{2a+1}$ terms, ${R_a}'$ has at most $k$ times that many terms (by product rule). Substituting instances of ${f_0}^{(k)}$ with (71) and multiplying through by $Q$, $\hat{R}_a$ has at most $2^ak^{a+1}(T_PT_Q)^{2a+1}T_PT_Q$ terms. Therefore $Q\hat{R}_a$ has no more than $2^ak^{a+1}(T_PT_Q)^{2(a+1)+1}$ terms. Similarly, $\hat{Q}R_a$ has no more terms than $2^ak^{a+1}(T_PT_Q)^{2(a+1)+1}$. Therefore, their sum has no more than twice this many terms, which, along with our previous work, proves $B(a+1)$.

Now consider (69). We shall show that the coefficients $\frac{{f_0}^{(j)}}{j!}$ are bounded by $\beta^j$ for some constant $\beta$, and thus the series has a non-zero radius of convergence. It suffices to consider $j$ larger than $k$. Let $f^*$ be the largest in magnitude among $f_0, ..., {f_0}^{(k)}$. Then by our inductive proof,

$$|{f_0}^{(k+j)}| \leq \frac{2^jk^j(T_PT_Q)^{2j+1}(2U_PU_Qmn(m+n))^{2j+1}(2j+1)!!|f^*|}{Q^{1+2(j-k)}}, \tag{78}$$

where $Q$ is evaluated at $f_0, ..., {f_0}^{(k-1)}$. Thus, for some constant $\alpha$,

$${f_0}^{(k+j)} \leq \alpha^j(2j+1)!!. \tag{79}$$

Thus, as $\frac{(2j+1)!!}{j!} \leq 3^j$, we have that, for some constant $\beta$,

$$\frac{{f_0}^{(k+j)}}{j!} \leq \beta^j. \tag{80}$$

Thus, the power series (69) has a non-zero radius of convergence. □

REMARK. The series solutions considered in this appendix are indeed solutions to (68), by Theorem A.

## REFERENCES


[1] Peter J. Olver and Philip Rosenau, *Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support*, Phys. Rev. E (3) **53** (1996), no. 2, 1900–1906, DOI 10.1103/PhysRevE.53.1900. MR1401317 (97c:35172)

[2] Masaaki Ito, *Symmetries and conservation laws of a coupled nonlinear wave equation*, Phys. Lett. A **91** (1982), no. 7, 335–338, DOI 10.1016/0375-9601(82)90426-1. MR670869 (84c:58036)

[3] Robert A. Van Gorder, *Solutions to a novel Casimir equation for the Ito system*, Commun. Theor. Phys. (Beijing) **56** (2011), no. 5, 801–804, DOI 10.1088/0253-6102/56/5/02. MR2954306

[4] Robert A. Van Gorder, *Computing the region of convergence for power series in many real variables: a ratio-like test*, Appl. Math. Comput. **218** (2011), no. 5, 2310–2317, DOI 10.1016/j.amc.2011.07.052. MR2831505

[5] M. Abramowitz and I. A. Stegun, Handbook of mathematical functions with formulas, graphs, and mathematical tables, Dover publications 1964.

[6] Diego Dominici, *Nested derivatives: a simple method for computing series expansions of inverse functions*, Int. J. Math. Math. Sci. **58** (2003), 3699–3715, DOI 10.1155/S0161171203303291. MR2031140 (2005f:41079)