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CLASSIFICATION OF UNIFORM COSMOLOGICAL MODELS*

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Summary

Homogeneous and isotropic models of relativistic cosmology are classified according to a general scheme. There are four main groups: static, asymptotic, monotonic and oscillatory, and each group is subdivided into classes. Altogether, there are eleven classes of cosmological models. In recent years models have been proposed which contain negative pressure. Robertson's original scheme of classification therefore requires development, as in the present treatment, to take into account the possibility of a negative cosmic stress. As a consequence there emerges a wide variety of cosmological models, including models that are both static and stable and models that oscillate between finite values of density.

1. *Introduction.* For a uniform universe, isotropic and homogeneous, the equations are

$$\left(\frac{dR}{dt}\right)^2 = \tfrac{1}{3}(8\pi G\rho + \lambda)R^2 - \kappa, \tag{1}$$

$$\frac{d}{dt}(\rho c^2 R^3) + p\frac{dR^3}{dt} = 0, \tag{2}$$

(see, for example (**1**), (**2**)), where $\kappa = 0, \pm 1$ is the curvature constant, λ is the cosmological constant, $R(t)$ is a spatial scaling factor and has the dimension of time t. Equations (1) and (2) are derived from Einstein's field equations for a fluid of uniform density ρ and isotropic pressure p. If κ, λ and an equation of state $p = p(\rho)$ are given, there are two equations for the determination of $R(t)$ and $\rho(t)$.

In relativistic cosmology a relation of the kind $p \propto \rho^\gamma$ causes only slight alteration in R and ρ as compared with $p = 0$. This is because such a relation between pressure and density is only possible when $p \ll \rho c^2$. Following Zel'dovich (**3**) we shall use instead the relation

$$p = (\nu - 1)\rho c^2. \tag{3}$$

If the pressure is non-negative the lower limit of ν is unity in a zero-pressure fluid. The upper limit of $\nu = \frac{4}{3}$ occurs in a relativistic gas. The possibility of an upper limit of $\nu = 2$ in a strongly interacting gas has been suggested by Zel'dovich (**3**); see however (**4**). The generally accepted values of ν lie in the range $1 \leqslant \nu \leqslant \frac{4}{3}$. When ν is constant the cosmological equations (1) and (2) become

$$\dot{R}^2 = C_\nu R^{2-3\nu} + \tfrac{1}{3}\lambda R^2 - \kappa, \tag{4}$$

where dots denote differentiation with respect to time, and C_ν is a constant:

$$C_\nu = 8\pi G\rho R^{3\nu}/3. \tag{5}$$

Since an element of volume V varies as R^3, we have

$$\rho V^\nu = \text{constant}. \tag{6}$$

* Contributions from the Four College Observatories, No. 1.

which are Robertson's original set. The curves (i)–(iii) in Fig. 2 are included to take into account those models for which $C_\nu = 0$.

Each $\lambda(R)$ curve divides the λ, $R \geqslant 0$ half-plane into an allowed region of $\dot{R}^2 \geqslant 0$ and a forbidden region (shown shaded) of $\dot{R}^2 < 0$. A model of a given value

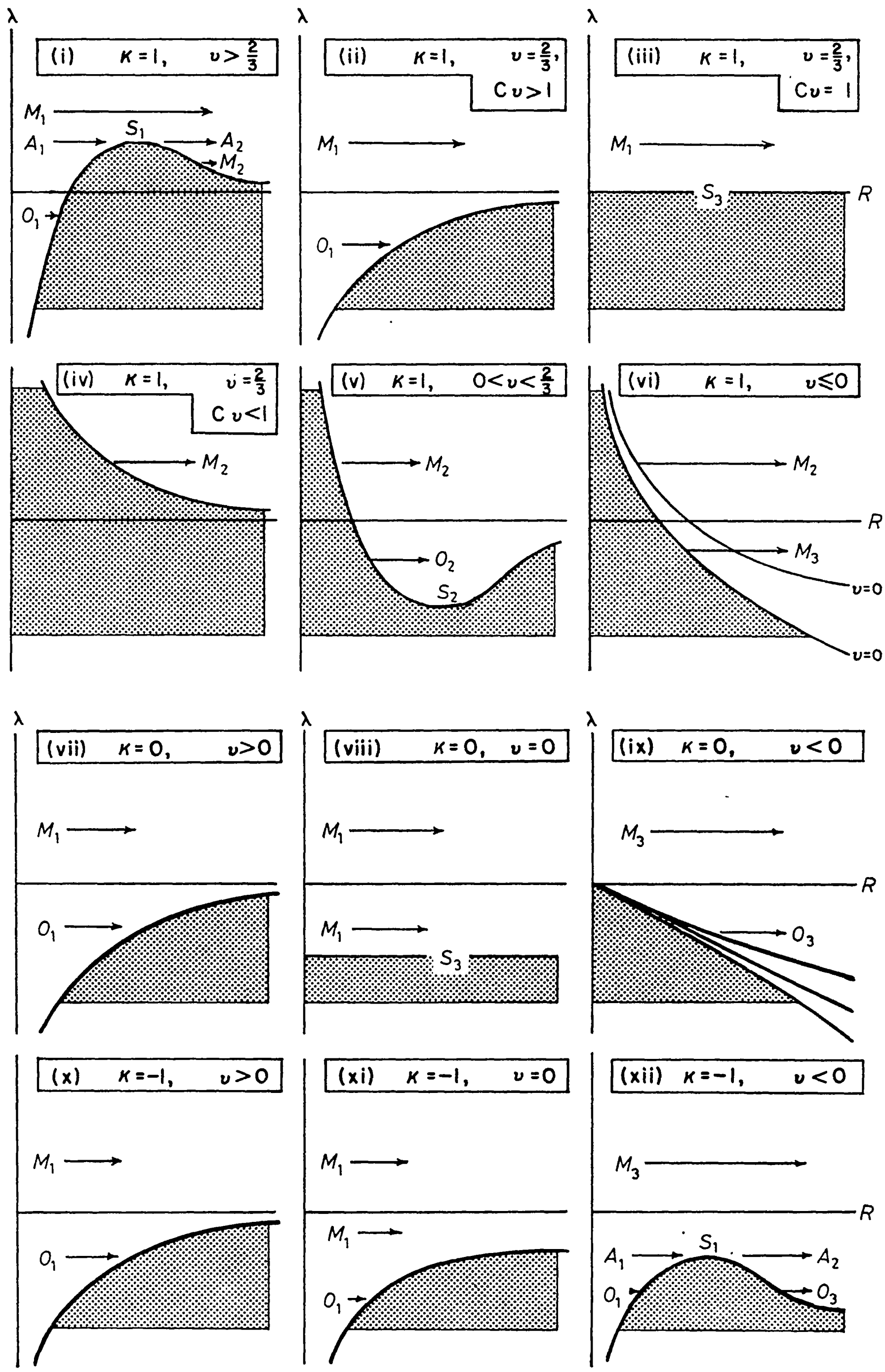

FIG. 1. *Curves in the λ, R half-plane showing $\lambda(R)$ for $\dot{R} = 0$. Altogether there are three sets of diagrams; the first set contains six diagrams for $\kappa = 1$, and the second and third sets contain three diagrams each for $\kappa = 0$ and $\kappa = -1$, as shown. Each diagram in a set is used for a certain range of ν. The $\lambda(R)$ curve divides each diagram into an allowed region of $\dot{R}^2 > 0$ and a forbidden region (shown shaded) of $\dot{R}^2 < 0$. Static models exist wherever $\lambda' = 0$, and are unstable for $\lambda'' < 0$ and stable when $\lambda'' > 0$. All other models are represented by horizontal lines parallel to the R-axis.*

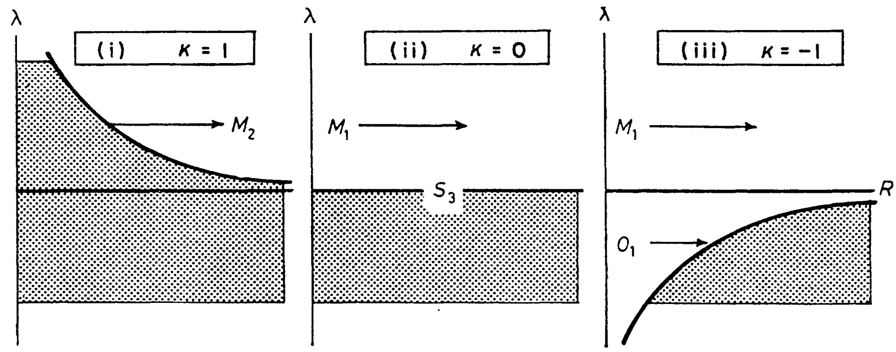

FIG. 2. *The curves of* $\lambda(R)$ *for* $\kappa=1, 0, -1$ *when* $C_\nu=0$, *or* $G\rho=0$.

of λ is represented by a horizontal line parallel to the R-axis. A point on such a line indicates the value of R at a particular instant of time, and in an expanding (contracting) universe R increases (decreases) with time. Wherever a horizontal line meets the $\lambda(R)$ curve the intersection is a turning point as the line cannot continue into the $\dot{R}^2<0$ region. If the horizontal line does not meet the $\lambda(R)$ curve then R increases or decreases continuously in the range $0\leqslant R\leqslant\infty$.

Let us suppose that a horizontal line of $R\geqslant 0$ lies to the left of $\lambda(R)$. Then R will increase with time until $\dot{R}=0$ at the intersection point with $\lambda(R)$, and then will decrease. Hence $\dot{R}$ moves from a positive value through zero to a negative value, and therefore $\ddot{R}<0$ at $\dot{R}=0$. According to equations (11) this means that $\lambda'>0$. Similarly if the horizontal line is to the right of $\lambda(R)$, then R decreases (that is $\dot{R}<0$) until $\dot{R}=0$ at the intersection and thereafter R increases ($\dot{R}>0$); therefore $\ddot{R}>0$ and $\lambda'<0$. The general rule is that when the slope of $\lambda(R)$ is positive all models lie to the left of $\lambda(R)$, and when the slope is negative all models are to the right of $\lambda(R)$. This rule enables us to determine the allowed region in each of the diagrams of Figs. 1 and 2.

Altogether there are four main groups of models: static, asymptotic, monotonic and oscillatory. In the static group $\ddot{R}=\dot{R}=0$ and the models lie on the $\lambda(R)$ curve at the $\lambda'=0$ points. The asymptotic group consists of those models which evolve to or from the unstable static models. In the monotonic group R varies continuously in the range $0\leqslant R\leqslant\infty$. Robertson includes in this group certain models possessing a lower bound: $R_{\min}\leqslant R\leqslant\infty$ and $R=\infty$ occurs at $t=\pm\infty$. The oscillatory group contains all models which expand and contract periodically with a finite period of oscillation. These four main groups are subdivided into classes as follows:

Static: $\begin{cases} S_1 & \text{Unstable; } (R=R_s,\ \lambda''<0) \\ S_2 & \text{Stable; } (R=R_s,\ \lambda''>0) \\ S_3 & \text{Stable; (all } R,\ \lambda''=0) \end{cases}$

Asymptotic: $\begin{cases} A_1 & (0\leqslant R\leqslant R_s) \\ A_2 & (R_s\leqslant R\leqslant\infty) \end{cases}$

Monotonic: $\begin{cases} M_1 & (0\leqslant R\leqslant\infty;\ R=\infty,\ t=\pm\infty) \\ M_2 & (R_{\min}\leqslant R\leqslant\infty;\ R=\infty,\ t=\pm\infty) \\ M_3 & (0\leqslant R\leqslant\infty;\ R=\infty,\ t=0) \end{cases}$

Oscillatory: $\begin{cases} O_1 & (0\leqslant R\leqslant R_{\max}) \\ O_2 & (R_{\min}\leqslant R\leqslant R_{\max}) \\ O_3 & (R_{\min}\leqslant R\leqslant\infty) \end{cases}$

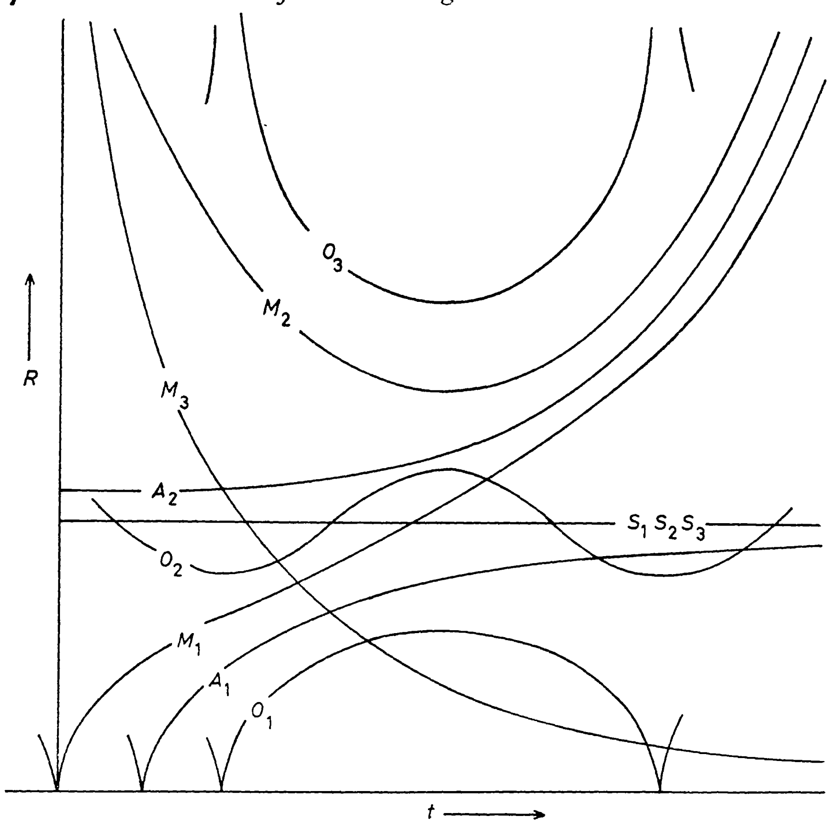

Fig. 3. *Curves illustrating schematically the behaviour of R as a function of time for the various classes of models.*

The way in which R varies with time for all eleven classes S_1–O_3 is illustrated in Fig. 3.

In Robertson's scheme of classification, based on $\nu \geqslant 1$ and ν equal to a constant, there exist only the six classes: S_1, A_1, A_2, M_1, M_2, O_1. Robertson also showed that if $dp/dR > 0$, in some range of R, then there is an additional O_2 class. With $\nu > 1$, this possibility requires that ν is not constant. The additional classes not included in Robertson's scheme are S_2, S_3, M_3 and O_3. Even if ν were not a constant, and depended on R, it is quite likely that the present scheme of classification is either adequate or requires only slight modification.

The above system of classification is invariant under time reversal. Any curve of Fig. 3 illustrates the behaviour of R in a given class for either an expanding or contracting universe. Alternatively, by selecting $t=0$ at a point of symmetry, such that $R(t)=R(-t)$, the models of a given class are contracting when $t<0$ and expanding when $t>0$, unless of course they are static or oscillating. For convenience, the curves in Fig. 3 are not drawn to scale nor have they necessarily the same axis of symmetry.

Whenever convenient we shall refer to a model as $Z_i(\kappa, \lambda, \nu, C_\nu)$, where Z_i is the class corresponding to specific values of κ, λ, ν and C_ν. If the value of C_ν is unimportant then $Z_i(\kappa, \lambda, \nu)$ is usually sufficient.

3. *Static models.* The static models have $\ddot{R}=\dot{R}=0$ and therefore, from equations (4) and (7),

$$\lambda_s = 4\pi(3\nu-2)G\rho_s, \tag{12}$$

$$\lambda_s = \frac{3\nu-2}{\nu}\frac{\kappa}{R_s^2}, \tag{13}$$

where s denotes a static model. These equations can also be obtained from equations (8) and (9) for $\lambda' = 0$.

To distinguish between the various static models we must first consider the subject of stability. In the present context the universe is either stable or unstable against small perturbations in R while κ, λ, ν and C_ν are held constant. Let $R = R_s + \delta R$; then from equations (7), (12) and (13), we find to a first order in δR

$$\delta \ddot{R} = \nu \lambda_s \delta R, \tag{14}$$

and from equation (10), since $\lambda' = 0$,

$$\delta \ddot{R} = -\lambda'' R_s^2 \delta R / 6. \tag{15}$$

Evidently, a model is stable against small perturbations in R when $\nu\lambda_s < 0$ or $\lambda'' > 0$, and oscillates about the equilibrium value R_s at a frequency of $2\pi(\nu\lambda_s)^{1/2}$. On the other hand a static model is unstable and δR grows exponentially with time when $\nu\lambda_s > 0$ or $\lambda'' < 0$. Einstein's (**8**) original static model of $\kappa = 1$, $\lambda_s = R_s^{-2}$, $\nu = 1$ is therefore unstable, as was shown by Eddington (**9**).

The unstable static class of $\lambda'' < 0$, designated as S_1, occurs in diagrams (i) and (xii) of Fig. 1, and the stable static class S_2 of $\lambda'' > 0$ is shown in diagram Fig. 1 (v). Models exist for these classes when

$$S_1: \quad \begin{cases} \kappa = 1, \ \lambda > 0, \ \nu > \frac{2}{3}, \\ \kappa = -1, \ \lambda < 0, \ \nu < 0, \end{cases}$$

$$S_2: \quad \kappa = 1, \ \lambda < 0, \ 0 < \nu < \tfrac{2}{3}.$$

The distinction between S_1 and S_2 takes no account of any other form of instability which may be present because of, for example, a variation in composition of the material content of a static universe.

We turn now to the third class S_3 defined by $\lambda'' = \lambda' = 0$, shown in Fig. 1 (iii), (viii) and Fig. 2 (ii). The models in this class are

$$S_3(1, 0, \tfrac{2}{3}, 1),$$

$$S_3(0, -3C_0, 0),$$

$$S_3(0, 0, -, 0).$$

They possess no particular equilibrium density and are static for all R. The $S_3(0, 0, -, 0)$ model, that is $\kappa = 0$, $\lambda = 0$, $C_\nu = 0$, is the flat empty space of special relativity.

The fact that the universe is expanding rules out entirely any possible realism in the S_3 models. Similarly, the S_1 and S_2 classes are inapplicable, although the A_1 and A_2 expansions, and the O_2 oscillations about an S_2 equilibrium state, are possible representations.

4. *Non-static models.* Equation (4) is not readily integrated except for certain values of κ, λ, and ν. We shall therefore derive some specific solutions from which can be deduced the more general behaviour of R. In the following we consider in turn $\kappa = 0$, 1 and -1, and the less interesting case of $C_\nu = 0$ is left to the end of this section.

4(a). $\kappa = 0$. Equation (4) can be integrated and for $\nu = 0$,

$$R = R_0 \exp\,(C_0 + \tfrac{1}{3}\lambda)^{1/2} t, \tag{16}$$

where R_0 is a constant. These constant density solutions belong to the M_1 class shown in Fig. 1 (viii). Notice that $\lambda > -\frac{1}{3}C_0$, and when $\lambda = 0$, we have McCrea's (5) model

$$M_1(0, 0, 0): \qquad R = R_0 \exp C_0^{1/2}t, \tag{17}$$

and also $\lambda = -3C_0$:

$$S_3(0, -3C_0, 0): \qquad R = R_0,$$

and this static solution of an arbitrary R_0 has been previously mentioned. There are no solutions for $\lambda < -3C_0$.

When $\lambda = 0$, $\nu > 0$,

$$R = \left(\frac{3\nu}{2} C_\nu^{1/2}t\right)^{2/3\nu}, \tag{18}$$

and the Einstein–de Sitter (**10**) model is

$$M_1(0, 0, 1): \quad R = (\tfrac{3}{2}C_1^{1/2}t)^{2/3}, \tag{19}$$

in which $\rho R^3 = \text{constant}$. It is interesting to notice that for $\nu = \frac{4}{3}, \frac{2}{3}$

$$M_1(0, 0, \tfrac{4}{3}): \quad R = (2C_{4/3}^{1/2}t)^{1/2}, \tag{20a}$$

$$M_1(0, 0, \tfrac{2}{3}): \quad R = C_{2/3}^{1/2}t, \tag{20b}$$

and ρR^4, $\rho R^2 = \text{constant}$. When $\nu < 0$, let $-\nu = \eta$, and we obtain M_3 models of

$$\frac{1}{R} = \left(\frac{3\eta}{2} C_\nu^{1/2}t\right)^{2/3\eta}, \tag{21}$$

in which the density increases with expansion, i.e. $\rho R^{-3\eta} = \text{constant}$. At $R = 0$, $\rho = 0$, and the universe expands from an initial state of zero density. In particular

$$M_1(0, 0, -\tfrac{2}{3}): \quad Rt = C_{-2/3}^{-1/2} \tag{22}$$

When $\lambda > 0$, we have

$$R^{3\nu} = \frac{3C_\nu}{\lambda} \sinh^2 \frac{3\nu}{2} \left(\frac{\lambda}{3}\right)^{1/2} t, \tag{23}$$

and all the models are of type M_1 for $\nu > 0$. Some particular solutions are ($\lambda = 3\alpha^2$)

$$M_1(0, 3\alpha^2, \tfrac{4}{3}): \quad R = (C_{4/3}\alpha^{-2})^{1/4} \sinh^{1/2} 2\alpha t, \tag{24}$$

$$M_1(0, 3\alpha^2, \tfrac{2}{3}): \quad R = C_{2/3}^{1/2}\alpha^{-1} \sinh \alpha t, \tag{25}$$

$$M_1(0, 3\alpha^2, \tfrac{1}{3}): \quad R = C_{1/3}\alpha^{-2} \sinh^2 \tfrac{1}{2}\alpha t. \tag{26}$$

It can be seen that when $\nu < 0$ the models belong to the M_3 class.

For negative values of λ

$$R^{3\nu} = \frac{C_\nu}{\alpha^2} \sin^2 \frac{3\nu}{2} \alpha t \tag{27}$$

where now $-\lambda = 3\alpha^2$. If $\nu > 0$, we have oscillatory O_1 type models, some of which are

$$O_1(0, -3\alpha^2, \tfrac{4}{3}): \quad R = (C_{4/3}/2\alpha^2)^{1/4}(1 - \cos 4\alpha t)^{1/4}, \tag{28}$$

$$O_1(0, -3\alpha^2, \tfrac{2}{3}): \quad R = (C_{2/3}/2\alpha^2)^{1/2}(1 - \cos 2\alpha t)^{1/2}, \tag{29}$$

$$O_1(0, -3\alpha^2, \tfrac{1}{3}): \quad R = (C_{1/3}/2\alpha^2)(1 - \cos \alpha t). \tag{30}$$

In the case of $\nu = \frac{1}{3}$, R oscillates simple-harmonically between 0 and $C_{1/3}/\alpha^2$, about a mean value at which the density is given by $4\pi G\rho = -\lambda$. When $\nu < 0$, the models are of type $O3$ and R oscillates between ∞ and a minimum value at which the density is given by $8\pi G\rho = -\lambda$; for example

$$O3(0, -3\alpha^2, -\tfrac{1}{3}): \quad R = \frac{2\alpha^2}{C_{-1/3}(1-\cos\alpha t)}. \tag{31}$$

4(b). $\kappa = 1$. When $\lambda = 0$, the solution of equation (4) is

$$R = (C_\nu \sin^2 \chi)^{1/(3\nu-2)}, \quad t = \frac{2}{|3\nu-2|}\int R\, d\chi, \tag{32}$$

(Harrison (**11**)). Thus Tolman's (**12**) model of $\nu = \frac{4}{3}$ is

$$O1(1, 0, \tfrac{4}{3}): \quad \begin{cases} R = C_{4/3}{}^{1/2} \sin\chi, \\ t = C_{4/3}{}^{1/2}(1-\cos\chi), \end{cases} \tag{33}$$

and Friedmann's (**13**) first model of $\nu = 1$ has the well-known solution

$$O1(1, 0, 1): \quad \begin{cases} R = C_1 \sin^2\chi, \\ t = C_1(\chi - \sin\chi\cos\chi). \end{cases} \tag{34}$$

We also notice that for $\nu = \frac{2}{3}$, $C_{2/3} > 1$,

$$M1(1, 0, \tfrac{2}{3}): \quad R = (C_{2/3} - 1)^{1/2} t, \tag{35}$$

and we have $S3(1, 0, \frac{2}{3}, 1)$ for $C_{2/3} = 1$. Furthermore, for $\nu = \frac{1}{3}$,

$$M2(1, 0, \tfrac{1}{3}): \quad R = C_{1/3}{}^{-1}[1 + (C_{1/3}t/2)^2], \tag{36}$$

and $\nu = 0$,

$$M2(1, 0, 0): \quad R = C_0{}^{-1/2} \cosh C_0{}^{1/2} t. \tag{37}$$

We turn now to the case when λ is not zero and consider solutions for specific values of ν. Thus for $\nu = \frac{4}{3}$,

$$M1(1, 3\alpha^2, \tfrac{4}{3}): \quad 2\alpha^2 R^2 = 1 - \cosh 2\alpha t + (\alpha/\alpha_s)\sinh 2\alpha t, \tag{38}$$

$$A1(1, 3\alpha_s{}^2, \tfrac{4}{3}): \quad 2\alpha_s{}^2 R^2 = 1 - e^{-2\alpha_s t}, \tag{39}$$

$$A2(1, 3\alpha_s{}^2, \tfrac{4}{3}): \quad 2\alpha_s{}^2 R^2 = 1 + e^{2\alpha_s t}, \tag{40}$$

$$S1(1, 3\alpha_s{}^2, \tfrac{4}{3}): \quad 3\alpha_s{}^2 = \lambda_s = 8\pi G\rho_s = 3R_s{}^2/2, \tag{41}$$

$$M2(1, 3\alpha^2, \tfrac{4}{3}): \quad 2\alpha^2 R^2 = 1 + (1 - \alpha^2/\alpha_s{}^2)^{1/2} \cosh 2\alpha t, \tag{42}$$

$$O1(1, 3\alpha^2, \tfrac{4}{3}): \quad 2\alpha^2 R^2 = 1 - \cosh 2\alpha t + (\alpha/\alpha_s)\sinh 2\alpha t, \tag{43}$$

and the $O1$ solution holds also for $\lambda < 0$ as shown in Fig. 1(i).

Lemaître (**14**) has given general solutions for $\nu = 1$ in terms of elliptic functions, and here we might mention the Eddington–Lemaître model $A2(1, \lambda_s, 1)$.

When $\nu = \frac{2}{3}$, and $C_{2/3} > 1$

$$M1(1, 3\alpha^2, \tfrac{2}{3}): \quad R\alpha = (C_{2/3} - 1)^{1/2} \sinh \alpha t, \tag{44}$$

$$O1(1, -3\alpha^2, \tfrac{2}{3}): \quad R\alpha = (C_{2/3} - 1)^{1/2} \sin \alpha t, \tag{45}$$

also, when $C_{2/3} = 1$,

$$M1(1, 3\alpha^2, \tfrac{2}{3}): \quad R = \text{constant} \times e^{\alpha t}, \tag{46}$$

and $C_{2/3} < 1$,

$$M2(1, 3\alpha^2, \tfrac{2}{3}): \quad R\alpha = (1 - C_{2/3})^{1/2} \cosh \alpha t. \tag{47}$$

In the range $0<\nu<\frac{2}{3}$ there occur the interesting O_2 models; for example, when $\nu=\frac{1}{3}$

$$M_2(1,\ 3\alpha^2,\ \tfrac{1}{3}):\quad 2\alpha^2 R=-C_{1/3}+(C_{1/3}{}^2+4\alpha^2)^{1/2}\cosh\alpha t, \tag{48}$$

$$O_2(1,\ -3\alpha^2,\ \tfrac{1}{3}):\quad 2\alpha^2 R=C_{1/3}-(C_{1/3}-4\alpha^2)^{1/2}\cos\alpha t, \tag{49}$$

$$S_2(1,\ -3\alpha_s{}^2,\ \tfrac{1}{3}):\quad -3\alpha_s{}^2=-\lambda_s=4\pi G\rho_s=3/R_s{}^2. \tag{50}$$

For constant density ν is zero and the models are

$$M_2(1,\ 3\alpha^2,\ 0):\quad R=(C_0+\alpha^2)^{-1/2}\cosh(C_0+\alpha^2)^{1/2}t, \tag{51}$$

where $\lambda>-3C_0$. It is easily seen that if $\nu<0$ all models belong to the M_3 class.

4(c). $\kappa=-1$. The general solution when $\lambda=0$ is (Harrison (**11**))

$$R=(C_\nu\sinh^2\chi)^{1/(3\nu-2)},\quad t=\frac{2}{|3\nu-2|}\int R\,d\chi. \tag{52}$$

Thus, when $\nu=\frac{4}{3}$,

$$M_1(-1,\ 0,\ \tfrac{4}{3}):\quad \begin{cases} R=C_{4/3}{}^{1/2}\sinh\chi, \\ t=C_{4/3}{}^{1/2}(\cosh\chi-1), \end{cases} \tag{53}$$

or,

$$R=[(t+C_{4/3}{}^{1/2})^2-C_{4/3}]^{1/2}. \tag{54}$$

Friedmann's (**15**) second model, $\nu=1$ is

$$M_1(-1,\ 0,\ 1):\quad \begin{cases} R=C_1\sinh^2\chi, \\ t=C_1(\sinh\chi\cosh\chi-\chi). \end{cases} \tag{55}$$

Some other solutions of $\lambda=0$ are

$$M_1(-1,\ 0,\ \tfrac{2}{3}):\quad R=(C_{2/3}+1)^{1/2}t, \tag{56}$$

$$M_1(-1,\ 0,\ \tfrac{1}{3}):\quad R=C_{1/3}{}^{-1}[(1+C_{1/3}t/2)^2-1], \tag{57}$$

$$M_1(-1,\ 0,\ 0):\quad R=C_0{}^{-1/2}\sinh C_0{}^{1/2}t. \tag{58}$$

Returning to the case of λ not equal to zero we have, for $\nu=\frac{4}{3}$,

$$M_1(-1,\ 3\alpha^2,\ \tfrac{4}{3}):\quad 2\alpha^2R^2=\cosh 2\alpha t-1+2\alpha C_{4/3}{}^{1/2}\sinh 2\alpha t, \tag{59}$$

$$O_1(-1,\ -3\alpha^2,\ \tfrac{4}{3}):\quad 2\alpha^2R^2=1-\cos 2\alpha t+2\alpha C_{4/3}{}^{1/2}\sin 2\alpha t. \tag{60}$$

For $\nu=\frac{2}{3}$, we obtain Whittaker's (**6**) models

$$M_1(-1,\ 3\alpha^2,\ \tfrac{2}{3}):\quad R\alpha=(C_{2/3}+1)^{1/2}\sinh\alpha t, \tag{61}$$

$$O_1(-1,\ -3\alpha^2,\ \tfrac{2}{3}):\quad R\alpha=(C_{2/3}+1)^{1/2}\sin\alpha t. \tag{62}$$

Also, for $\nu=\frac{1}{3}$, we find

$$M_1(-1,\ 3\alpha^2,\ \tfrac{1}{3}):\quad 2\alpha^2R=C_{1/3}(\cosh\alpha t-1)+2\alpha C_{1/3}{}^{1/2}\sinh\alpha t, \tag{63}$$

$$O_1(-1,\ -3\alpha^2,\ \tfrac{1}{3}):\quad 2\alpha^2R=C_{1/3}(1-\cos\alpha t)+2\alpha C_{1/3}{}^{1/2}\sin\alpha t, \tag{64}$$

and for $\nu=0$,

$$M_1(-1,\ 3\alpha^2,\ 0):\quad R=(C_0+\alpha^2)^{-1/2}\sinh(C_0+\alpha^2)^{1/2}t, \tag{65}$$

which becomes $O_1(-1,\ -3\alpha^2,\ 0)$ when $\lambda<-3C_0$.

If $\nu<0$, we again have S_1 models as shown in Fig. 1 (xii), and for $\nu=-\frac{1}{3}, -\frac{2}{3}$ the solutions can be obtained in terms of elliptical integrals as in the case of $\nu=1$.

In general, for $\nu < 0$ the solutions are M_3 for $\lambda > \lambda_s$, A_2 for $\lambda < \lambda_s$ if $R > R_s$, and for large R resemble equation (21).

4(d). $C_\nu = 0$. When the effect of matter on the underlying metric can be neglected we obtain for $\kappa = 1$ Lanczos' (**16**) model:

$$M_2(1, 3\alpha^2, -, 0): \quad R = \alpha^{-1} \cosh \alpha t; \tag{66}$$

the de Sitter (**17**) model for $\kappa = 0$:

$$M_1(0, 3\alpha^2, -, 0): \quad R = \text{constant} \times e^{\alpha t}; \tag{67}$$

and for $\kappa = -1$ (Robertson (**1**)):

$$M_1(-1, 3\alpha^2, -, 0): \quad R = \alpha^{-1} \sinh \alpha t), \tag{68}$$

and in addition

$$O_1(-1, -3\alpha^2, -, 0): \quad R = \alpha^{-1} \sin \alpha t. \tag{69}$$

Milne's (**18**) model of $\lambda = 0$, is

$$M_1(-1, 0, -, 0): \quad R = t. \tag{70}$$

5. *Conclusion.* The adoption of an unconstrained stress constant extends considerably the range of possible cosmological models. The present general scheme of classification includes both positive and negative pressures, and is proposed mainly to take account of and extend the existing negative pressure models. Whether models of $\nu < 1$ are merely harmless curiousities is obviously a matter that is open to debate. A relevant question is whether at the present stage a proliferation of models in any way advances cosmology. Present observational data are inadequate to make a clear decision even when the stress constant is assigned the deceptively simple value of unity and the cosmological constant is discarded as an irrelevant complication. With the increased array of models presented in the present scheme the problem of determining an appropriate model from the available data becomes hopeless. But such a situation is not without virtue; it may direct attention to the important matter of bringing more physics into cosmology for the purpose of spanning the gap between the actual universe and our excessively idealized representations.

It is worth mentioning that with sufficient observational data available a more appropriate method of classification uses the deceleration variable q and the density variable σ (McVittie (**19**), Stabell & Refsdal (**20**)):

$$q = \ddot{R}/RH^2, \tag{71}$$

$$\sigma = 4\pi G\rho/3H^2, \tag{72}$$

where $H = \dot{R}/R$ is the Hubble parameter. It follows that

$$\kappa = -H_0^2 R_0^2 (q_0 + 1 - 3\nu\sigma_0), \tag{73}$$

$$\lambda = -3H_0^2[q_0 - (3\nu - 2)\sigma_0], \tag{74}$$

where the zero subscript denotes present values, and equations (4) and (7) now become

$$\sigma = \frac{\sigma_0}{2\sigma_0 + (q_0 + 1 - 3\nu\sigma_0)y^{3\nu-2} - [q_0 - (3\nu - 2)\sigma_0]y^{3\nu}}, \tag{75}$$

$$q = \frac{(3\nu - 2)\sigma_0 + [q_0 - (3\nu - 2)\sigma_0]y^{3\nu}}{2\sigma_0 + (q_0 + 1 - 3\nu\sigma_0)y^{3\nu-2} - [q_0 - (3\nu - 2)\sigma_0]y^{3\nu}}, \tag{76}$$

where $y = R/R_0$. From equations (5) and (72) we see

$$\sigma = C_\nu R^{2-3\nu}/2\dot{R}^2,$$

and therefore

$$\dot{\sigma} = \sigma(\dot{R}/R)[2q - (3\nu - 2)]. \qquad (77)$$

Thus σ is constant when $q = (3\nu - 2)/2$.

We have supposed throughout that $\rho R^{3\nu}$ is a constant. Suppose, however, that instead of equation (3) the equation of state is

$$p = p_0 + (\nu - 1)\rho c^2 \qquad (78)$$

and p_0 is a constant. Instead of equation (4) we now obtain

$$\dot{R}^2 = C_\nu{}^* R^{2-3\nu} + \tfrac{1}{3}\lambda^* R^2 - \kappa, \qquad (79)$$

in which

$$C_\nu{}^* = \frac{8\pi G}{3}\left(\rho + \frac{p_0}{\nu c^2}\right) R^{3\nu} = \text{a constant}. \qquad (80)$$

$$\lambda^* = \lambda - \frac{8\pi G p_0}{\nu c^2}. \qquad (81)$$

and now we have $Z_i(\kappa, \nu, \lambda^*, C_\nu{}^*)$ models which are classified in the same way as the previous $Z_i(\kappa, \nu, \lambda, C_\nu)$ models. The earlier arguments concerning the possible validity of a cosmic stress constant can now be extended, without much further strain on our imagination, to embrace the idea of an additional uniform and constant stress p_0. We are then left with the uncomfortable fact that $\rho R^{3\nu}$ is no longer a constant, and p_0 is a further undetermined cosmic parameter.

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