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Classification results and new examples of proper biharmonic submanifolds in spheres

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Abstract. In this paper we survey the known results on the classification of biharmonic submanifolds in space forms and construct a family of new examples of proper biharmonic submanifolds in the Euclidean n-dimensional sphere.

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Introduction

In [11], even if they took the main interest in harmonic maps, Eells and Sampson also envisaged some generalizations and defined *biharmonic maps* $\varphi : (M, g) \to (N, h)$ between Riemannian manifolds as critical points of the *bienergy functional*

$$E_2(\varphi) = \frac{1}{2}\int_M |\tau(\varphi)|^2 \, v_g,$$

where $\tau(\varphi) = \operatorname{trace} \nabla d\varphi$ is the tension field of φ that vanishes for harmonic maps. The Euler-Lagrange equation corresponding to E_2 is given by the van-

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ishing of the *bitension field*

$$\tau_2(\varphi) = -J^{\varphi}(\tau(\varphi)) = -\Delta\tau(\varphi) - \text{trace } R^N(d\varphi, \tau(\varphi))d\varphi,$$

where J^{φ} is formally the Jacobi operator of φ. The operator J^{φ} is linear, thus any harmonic map is biharmonic. We call *proper biharmonic* the non-harmonic biharmonic maps.

Although E_2 has been on the mathematical scene since the early '60 (when some of its analytical aspects have been discussed) and regularity of its critical points is nowadays a well-developed field, a systematic study of the geometry of biharmonic maps has started only recently.

In this paper we shall focus our attention on *biharmonic submanifolds*, i.e. on submanifolds such that the inclusion map is a biharmonic map.

The biharmonic submanifolds of a non-positive sectional curvature space that have been considered so far turned out to be all trivial (that is minimal), and the attempts that had been made have led to the following conjecture.

1 Conjecture (Generalized Chen's Conjecture)**.** *Biharmonic submanifolds of a non-positive sectional curvature manifold are minimal.*

In contrast, the class of proper (non-minimal) biharmonic submanifolds of the sphere is rather rich, but a full understanding of their geometry has not yet been achieved.

The aim of this paper is twofold. In the first part we gather the known results on the classification of biharmonic submanifolds in a space form, while in the second part we construct a class of new examples of biharmonic submanifolds in the sphere.

For an up to date bibliography on biharmonic maps we refer the reader to [21].

1 Biharmonic submanifolds

Let $\varphi : M \to \mathbb{E}^n(c)$ be the canonical inclusion of a submanifold M in a constant sectional curvature c manifold, $\mathbb{E}^n(c)$. The expressions assumed by the tension and bitension fields are, in this case,

$$\tau(\varphi) = mH, \qquad \tau_2(\varphi) = -m(\Delta H - mcH),$$

where H, seen as a section of $\varphi^{-1}(T\mathbb{E}^n(c))$, denotes the mean curvature vector field of M in $\mathbb{E}^n(c)$ and Δ is the rough Laplacian on $\varphi^{-1}(T\mathbb{E}^n(c))$.

By splitting the bitension field in its normal and tangential components we find that the canonical inclusion $\varphi : M^m \to \mathbb{E}^n(c)$ of a submanifold M in an

n-dimensional space form $\mathbb{E}^n(c)$ is biharmonic if and only if

$$\begin{cases} \Delta^{\perp} H + \operatorname{trace} B(\cdot, A_H \cdot) - mcH = 0, \\ \\ 4 \operatorname{trace} A_{\nabla^{\perp}_{(\cdot)} H}(\cdot) + m \operatorname{grad}(|H|^2) = 0, \end{cases} \tag{1}$$

where A denotes the Weingarten operator, B the second fundamental form, H the mean curvature vector field, $\nabla^{\perp}$ and $\Delta^{\perp}$ the connection and the Laplacian in the normal bundle of M in $\mathbb{E}^n(c)$.

If $c \leq 0$, then compact proper biharmonic submanifolds do not exist in $\mathbb{E}^n(c)$. In fact, using a result of Jiang [15], biharmonic maps from a compact manifold to a manifold with non-positive sectional curvature are harmonic. When M is non-compact, it cannot be proper biharmonic in $\mathbb{E}^n(c),\ c \leq 0$, provided that its mean curvature is constant [16].

For hypersurfaces, that is $n = m + 1$, condition (1) takes the simpler form

$$\begin{cases} \Delta^{\perp} H - (mc - |A|^2)H = 0, \\ \\ 2A\big(\operatorname{grad}(|H|)\big) + m|H| \operatorname{grad}(|H|) = 0. \end{cases} \tag{2}$$

In dimension $n = 3$ system (2) forces the norm of the mean curvature vector field of a surface M^2 in $\mathbb{E}^3(c)$ to be constant, which implies the following

2 Theorem ([4, 7])**.** *There exist no proper biharmonic surfaces in* $\mathbb{E}^3(c),\ c \leq 0$.

For higher dimensional cases it is not known whether there exist proper biharmonic submanifolds of $\mathbb{E}^n(c),\ n > 3,\ c \leq 0$, although partial results have been obtained. For instance:

- Every biharmonic curve of $\mathbb{R}^n$ is an open part of a straight line [10].
- Every biharmonic submanifold of finite type in $\mathbb{R}^n$ is minimal [10].
- There exist no proper biharmonic hypersurfaces of $\mathbb{R}^n$ with at most two principal curvatures [10].
- Let M^m be a pseudo-umbilical submanifold of $\mathbb{E}^n(c)$, $c \leq 0$. If $m \neq 4$, then M is biharmonic if and only if minimal [4, 10].
- Let M^3 be a hypersurface of $\mathbb{R}^4$. Then M is biharmonic if and only if minimal [12].
- A submanifold of $\mathbb{S}^n$ cannot be biharmonic in $\mathbb{R}^{n+1}$ [7].

1.1 Biharmonic submanifolds of $\mathbb{S}^n$

All the non-existence results described in the previous section do not hold for submanifolds in the sphere. Before we describe some general methods to construct biharmonic submanifolds in the sphere let us recall the main examples:

- the generalized Clifford torus, $\mathbb{S}^p(\frac{1}{\sqrt{2}}) \times \mathbb{S}^q(\frac{1}{\sqrt{2}})$, $p+q=n-1$, $p \neq q$, was the first example of proper biharmonic submanifold in $\mathbb{S}^n$ [14];
- the hypersphere $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^n$ [3].

For the 3-dimensional unit sphere it is possible to give the full classification of proper biharmonic submanifolds, as shown by the following

3 Theorem ([3]). *a) An arc length parameterized curve $\gamma : I \to \mathbb{S}^3$ is proper biharmonic if and only if it is either the circle of radius $\frac{1}{\sqrt{2}}$, or a geodesic of the Clifford torus $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^1(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$ with slope different from ± 1.*

b) A surface M is proper biharmonic in $\mathbb{S}^3$ if and only if it is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$.

Theorem 3 says that a circle of radius $\frac{1}{\sqrt{2}}$, which is totally geodesic (minimal) in $\mathbb{S}^2(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^3$, is biharmonic in $\mathbb{S}^3$. This composition property turned out to be true in any dimension in virtue of the following

4 Proposition ([4]). *A minimal submanifold M of $\mathbb{S}^{n-1}(a) \subset \mathbb{S}^n$ is proper biharmonic in $\mathbb{S}^n$ if and only if $a = \frac{1}{\sqrt{2}}$.*

This result proved to be quite useful for the construction of proper biharmonic submanifolds in spheres. For instance, using a well known result of Lawson, it implies the existence of closed orientable embedded proper biharmonic surfaces of arbitrary genus in $\mathbb{S}^4$ (see [4]).

A closer look at the biharmonic submanifolds M of $\mathbb{S}^n$, constructed using Proposition 4, reveals that they all possess the following features: they are pseudo-umbilical ($A_H = |H|^2 \operatorname{Id}$) with parallel mean curvature vector field of norm 1.

Nevertheless, it is possible to construct non pseudo-umbilical examples using the following product composition property.

5 Proposition ([4]). *If $M_1^{m_1}$ and $M_2^{m_2}$ ($m_1 \neq m_2$) are two minimal submanifolds of $\mathbb{S}^{n_1}(\frac{1}{\sqrt{2}})$ and $\mathbb{S}^{n_2}(\frac{1}{\sqrt{2}})$ respectively ($n_1 + n_2 = n-1$), then $M_1 \times M_2$ is a proper biharmonic submanifold in $\mathbb{S}^n$, which is not pseudo-umbilical, with parallel mean curvature vector field and $|H| \in (0,1)$.*

The value of $|H|$ between 0 and 1 in Proposition 5 is not a coincidence, in fact it was proved in [17] that any proper biharmonic constant mean curvature submanifold M in $\mathbb{S}^n$ satisfies $|H| \in (0,1]$. Moreover, if $|H| = 1$, then M is a minimal submanifold of the hypersphere $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^n$.

The aforementioned result is related with submanifolds of finite type. Let us first recall that an isometric immersion $\varphi : M \to \mathbb{R}^n$ is called of finite type if φ can be expressed as a finite sum of $\mathbb{R}^n$-valued eigenfunctions of the Beltrami-Laplace operator Δ of M. When M is compact it is called of k-type if the spectral decomposition of φ contains exactly k non-zero terms, excepting the center of mass (see [8]).

If M is a submanifold of $\mathbb{S}^n$ then M can be seen as a submanifold of $\mathbb{R}^{n+1}$. We say that M is of finite type in $\mathbb{S}^n$ if it is of finite type as a submanifold of $\mathbb{R}^{n+1}$. Denote by $\varphi : M \to \mathbb{S}^n$ the inclusion of M in $\mathbb{S}^n$ and by $\mathbf{i} : \mathbb{S}^n \to \mathbb{R}^{n+1}$ the canonical inclusion. Let $\phi : M \to \mathbb{R}^{n+1}$, $\phi = \mathbf{i} \circ \varphi$, be the inclusion of M in $\mathbb{R}^{n+1}$. Denoting by H the mean curvature vector field of M in $\mathbb{S}^n$ and by H^0 the mean curvature vector field of M in $\mathbb{R}^{n+1}$ we have immediately $H^0 = H - \phi$. From [4, Proposition 4.1], we get that $\tau_2(\varphi) = 0$ if and only if

$$\Delta H^0 - 2mH^0 + m(|H|^2 - 1)\phi = 0. \tag{3}$$

¿From (3), using the Minimal Polynomial Criterion for submanifolds of finite type (see, for example, [7]), we can prove the following

6 Theorem ([1])**.** *Let M^m be a compact constant mean curvature submanifold in $\mathbb{S}^n$, $|H|^2 = k$. Then M is proper biharmonic if and only if either $|H|^2 = 1$ and M is a 1-type submanifold with eigenvalue $\lambda = 2m$, or $|H|^2 = k \in (0,1)$ and M is a 2-type submanifold with eigenvalues $\lambda_{1,2} = m(1 \pm \sqrt{k})$.*

Note that all proper biharmonic submanifolds of $\mathbb{S}^n$ with $|H| = 1$ are 1-type submanifolds in $\mathbb{R}^{n+1}$, independently on whether they are compact or not.

2 Biharmonic hypersurfaces

The full classification of biharmonic hypersurfaces in a space form is not known and so far only few cases have been studied. The simplest assumption that M is an umbilical hypersurface, i.e. all principal curvatures are equal, does not produce new examples. In fact, if M is a proper biharmonic umbilical hypersurface in $\mathbb{S}^{m+1}$, then it is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$. Moreover, there exist no proper biharmonic umbilical hypersurfaces in $\mathbb{R}^{m+1}$ or in the hyperbolic space $\mathbb{H}^{m+1}$.

Similarly to the case of the Euclidean space (see [10]), the study of proper biharmonic hypersurfaces with at most two or three distinct principal curvatures constitutes the next natural step for the classification of proper biharmonic hypersurfaces in space forms.

We underline the fact that there exist examples of hypersurfaces with at most two or three distinct principal curvatures and non-constant mean curvature in any space form.

The classification of biharmonic hypersurfaces with at most two or three distinct principal curvatures relies on the proof that they have constant mean curvature. For hypersurfaces with at most two distinct principal curvatures this property was proved, in [10], for $\mathbb{R}^n$ and in [1] for $\mathbb{E}^n(c)$, $c = \pm 1$. The case of three distinct principal curvatures was proved for hypersurfaces in $\mathbb{E}^4(c)$, for any c, in [2].

7 Theorem ([1, 2]). *a) A biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{E}^{m+1}(c)$ has constant mean curvature.*

b) A biharmonic hypersurface in $\mathbb{E}^4(c)$ has constant mean curvature.

As an immediate consequence of Theorem 7 and system (2) we have the following non-existence result

8 Theorem ([1, 2]). *a) There exist no proper biharmonic hypersurfaces with at most two distinct principal curvatures in $\mathbb{R}^{m+1}$ and in $\mathbb{H}^{m+1}$.*

b) There exist no proper biharmonic hypersurfaces in $\mathbb{R}^4$ and in $\mathbb{H}^4$.

The case of the sphere is essentially different. Theorem 7 proves to be the main ingredient for the following complete classification of proper biharmonic hypersurfaces with at most two distinct principal curvatures.

9 Theorem ([1]). *Let M^m be a proper biharmonic hypersurface with at most two distinct principal curvatures in $\mathbb{S}^{m+1}$. Then M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Proof. By Theorem 7, the mean curvature of M in $\mathbb{S}^{m+1}$ is constant and, from (2), we obtain $|A|^2 = m$. These imply that M has constant principal curvatures. For $|H|^2 = 1$ we conclude that M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$. For $|H|^2 \in (0, 1)$ we deduce that M has two distinct constant principal curvatures. Proposition 2.5 in [18] implies that M is an open part of the product of two spheres $\mathbb{S}^{m_1}(a) \times \mathbb{S}^{m_2}(b)$, such that $a^2 + b^2 = 1$, $m_1 + m_2 = m$. Since M is biharmonic in $\mathbb{S}^n$, from a result similar to Proposition 4, it follows that $a = b = \frac{1}{\sqrt{2}}$ and $m_1 \neq m_2$. $\boxed{QED}$

We recall that a Riemannian manifold is called conformally flat if, for every point, it admits an open neighborhood conformally diffeomorphic to an open set of an Euclidean space. Also, a hypersurface $M^m \subset N^{m+1}$ which admits a principal curvature of multiplicity at least $m - 1$ is called quasi-umbilical. For biharmonic hypersurfaces we have the following classification

10 Theorem ([1]). *Let M^m, $m \geq 3$, be a proper biharmonic hypersurface in $\mathbb{S}^{m+1}$. The following statements are equivalent*

a) M is quasi-umbilical,

b) M is conformally flat,

c) M is an open part of $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m-1}(\frac{1}{\sqrt{2}})$.

2.1 Isoparametric hypersurfaces

We recall that a hypersurface M^m in $\mathbb{S}^{m+1}$ is said to be *isoparametric* of type ℓ if it has constant principal curvatures $k_1 > \ldots > k_\ell$ with respective constant multiplicities $m_1, \ldots, m_\ell$, $m = m_1 + m_2 + \ldots + m_\ell$. It is known that the number ℓ is either $1, 2, 3, 4$ or 6. For $\ell \leq 3$ we have the following classification of compact isoparametric hypersurfaces (initiated, for $\ell = 3$, by Cartan).

If $\ell = 1$, then M is totally umbilical.

If $\ell = 2$, then $M = \mathbb{S}^{m_1}(r_1) \times \mathbb{S}^{m_2}(r_2)$, $r_1^2 + r_2^2 = 1$ (see [18]).

If $\ell = 3$, then $m_1 = m_2 = m_3 = 2^q$, $q = 0, 1, 2, 3$ (see [5]).

Moreover, there exists an angle θ, $0 < \theta < \frac{\pi}{\ell}$, such that

$$k_\alpha = \cot\big(\theta + \frac{(\alpha-1)\pi}{\ell}\big), \quad \alpha = 1, \ldots, \ell. \tag{4}$$

Using this classification we can prove the following non-existence result for biharmonic hypersurfaces with three distinct principal curvatures.

11 Theorem ([2]). *There exist no compact proper biharmonic hypersurfaces of constant mean curvature with three distinct principal curvatures in the unit Euclidean sphere.*

Proof. First note that a proper biharmonic hypersurface M with constant mean curvature $|H|^2 = k$ in $\mathbb{S}^{m+1}$ has constant scalar curvature $s = m^2(1 + k) - 2m$ (see [1]). Since M is compact with 3 distinct principal curvatures and constant scalar curvature, from a result of Chang [6], M is isoparametric with $\ell = 3$ in $\mathbb{S}^{m+1}$. Now, taking into account (4), there exists $\theta \in (0, \pi/3)$ such that

$$k_1 = \cot\theta, \qquad k_2 = \cot\big(\theta + \frac{\pi}{3}\big) = \frac{k_1 - \sqrt{3}}{1 + \sqrt{3}k_1}, \qquad k_3 = \cot\big(\theta + \frac{2\pi}{3}\big) = \frac{k_1 + \sqrt{3}}{1 - \sqrt{3}k_1}.$$

Thus, from Cartan's classification, the square of the norm of the shape operator is

$$|A|^2 = 2^q(k_1^2 + k_2^2 + k_3^2) = 2^q \frac{9k_1^6 + 45k_1^2 + 6}{(1 - 3k_1^2)^2} \tag{5}$$

and $m = 3 \cdot 2^q$, $q = 0, 1, 2, 3$. On the other hand, since M is biharmonic of constant mean curvature, from (2),

$$|A|^2 = m = 3 \cdot 2^q.$$

The last equation, together with (5), implies that k_1 is a solution of $3k_1^6 - 9k_1^4 + 21k_1^2 + 1 = 0$ which is an equation with no real roots. $\boxed{QED}$

Combining Theorem 11, Theorem 7 and Theorem 9 we have the following classification of compact biharmonic hypersurfaces in $\mathbb{S}^4$.

12 Theorem. *The only proper biharmonic compact hypersurfaces in* $\mathbb{S}^4$ *are the hypersphere* $\mathbb{S}^3(\frac{1}{\sqrt{2}})$ *and the torus* $\mathbb{S}^1(\frac{1}{\sqrt{2}}) \times \mathbb{S}^2(\frac{1}{\sqrt{2}})$.

The full classification of proper biharmonic isoparametric hypersurfaces of $\mathbb{S}^{m+1}$ is due to Ichiyama, Inoguchi and Urakawa

13 Theorem ([13])**.** *A compact isoparametric hypersurface* M *of* $\mathbb{S}^{m+1}$ *is proper biharmonic if and only if it is one of the following: the hypersphere* $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ *or the Clifford torus* $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.

3 Biharmonic submanifolds of codimension greater than one

In this section we shall start a program to classify biharmonic submanifolds of the sphere with higher codimension. If we assume that M is a pseudo-umbilical submanifold of $\mathbb{S}^n$, then we soon find strong conditions. The first is

14 Theorem ([1])**.** *A biharmonic pseudo-umbilical submanifold of* $\mathbb{S}^n$, $m \neq 4$, *has constant mean curvature.*

Now if M^m is a pseudo-umbilical submanifold in $\mathbb{S}^{m+2}$ with constant mean curvature, then a result of Chen [9, p.180] ensures that M is either a minimal submanifold of $\mathbb{S}^{m+2}$ or a minimal hypersurface of a hypersphere of $\mathbb{S}^{m+2}$. Thus, from Proposition 4, we get the following rigidity result

15 Theorem ([1])**.** *A pseudo-umbilical submanifold* M^m *of* $\mathbb{S}^{m+2}$, $m \neq 4$, *is proper biharmonic if and only if it is minimal in* $\mathbb{S}^{m+1}(\frac{1}{\sqrt{2}})$.

If we replace the condition that M is pseudo-umbilical with that of being a hypersurface of a hypersphere in $\mathbb{S}^{m+2}$ we have

16 Theorem ([1])**.** *Let* M^m *be a hypersurface of* $\mathbb{S}^{m+1}(a) \subset \mathbb{S}^{m+2}$, $a \in (0, 1)$. *Assume that* M *is not minimal in* $\mathbb{S}^{m+1}(a)$. *Then it is biharmonic in* $\mathbb{S}^{m+2}$ *if and only if* $a > \frac{1}{\sqrt{2}}$ *and* M *is open in* $\mathbb{S}^m(\frac{1}{\sqrt{2}}) \subset \mathbb{S}^{m+1}(a)$.

Using Theorem 16 we can prove

17 Theorem ([1]). *A proper biharmonic surface M^2 in $\mathbb{S}^n$ with parallel mean curvature vector field is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$.*

Proof. Chen and Yau proved (see [9, p.106]) that the only non-minimal surfaces with parallel mean curvature vector field in $\mathbb{S}^n$ are either minimal surfaces of hyperspheres $\mathbb{S}^{n-1}(a)$ of $\mathbb{S}^n$ or surfaces with constant mean curvature in 3-spheres of $\mathbb{S}^n$. If M is a minimal surface of a hypersphere $\mathbb{S}^{n-1}(a)$, then it is biharmonic in $\mathbb{S}^n$ if and only if $a = \frac{1}{\sqrt{2}}$. If M is a surface in a 3-sphere $\mathbb{S}^3(a)$, $a \in (0, 1]$, of $\mathbb{S}^n$ then we can consider the composition

$$M \longrightarrow \mathbb{S}^3(a) \longrightarrow \mathbb{S}^4 \longrightarrow \mathbb{S}^n.$$

Note that M is biharmonic in $\mathbb{S}^n$ if and only if it is biharmonic in $\mathbb{S}^4$. From Theorem 16, for $a \in (0, 1)$, we conclude that either $a = \frac{1}{\sqrt{2}}$ and M is minimal in $\mathbb{S}^3(\frac{1}{\sqrt{2}})$, or $a > \frac{1}{\sqrt{2}}$ and M is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$. For $a = 1$, from Theorem 3, also follows that M is an open part of $\mathbb{S}^2(\frac{1}{\sqrt{2}})$. In all cases M is minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. $\boxed{QED}$

We point out that there exist examples of proper biharmonic constant mean curvature surfaces in $\mathbb{S}^n$ that are not minimal in $\mathbb{S}^{n-1}(\frac{1}{\sqrt{2}})$. For example, Sasahara, in [19], constructed a proper biharmonic immersion with constant mean curvature $\varphi : M^2 \to \mathbb{S}^5$ whose position vector field $x_0 = x_0(u, v)$ in $\mathbb{R}^6$ is given by:

$$x_0(u, v) = \frac{1}{\sqrt{2}}\big(e^{iu}, ie^{-iu}\sin(\sqrt{2}v), ie^{-iu}\cos(\sqrt{2}v)\big).$$

An easy computation shows that φ does not have parallel mean curvature vector field.

We end this section proposing two conjectures.

Conjecture. *The only proper biharmonic hypersurfaces in $\mathbb{S}^{m+1}$ are the open parts of hyperspheres $\mathbb{S}^m(\frac{1}{\sqrt{2}})$ or of generalized Clifford tori $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}}) \times \mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, $m_1 + m_2 = m$, $m_1 \neq m_2$.*

Conjecture. *Any biharmonic submanifold in $\mathbb{S}^n$ has constant mean curvature.*

4 New examples of proper biharmonic submanifolds in spheres

This section is devoted to the study of new examples of proper biharmonic submanifolds of codimension greater than 1 in spheres. We shall give the classification of proper biharmonic products of spheres in the unit Euclidean sphere.

Consider the product of r spheres

$$\mathcal{T} = \mathbb{S}^{n_1}(a_1) \times \mathbb{S}^{n_2}(a_2) \times \ldots \times \mathbb{S}^{n_r}(a_r) \subset \mathbb{S}^{m+r-1} \subset \mathbb{R}^{m+r},$$

where $m = \sum_{k=1}^{r} n_k$ and $\sum_{k=1}^{r} a_k^2 = 1$.

In the following theorem we give the relations, involving the radii and the dimensions of the spheres, that guarantee the biharmonicity of $\mathcal{T}$ in $\mathbb{S}^{m+r-1}$.

18 Theorem. *The product $\mathcal{T}$ is:*

a) minimal in $\mathbb{S}^{m+r-1}$ if and only if

$$\frac{a_k^2}{n_k} = \frac{1}{m},$$

for all $k = 1, \ldots, r$.

b) proper biharmonic in $\mathbb{S}^{m+r-1}$ if and only if there exists $p = 1, \ldots, r-1$ such that

$$\frac{a_1^2}{n_1} = \cdots = \frac{a_p^2}{n_p} = \frac{1}{2(n_1 + \ldots + n_p)} \neq \frac{1}{m}$$

and

$$\frac{a_{p+1}^2}{n_{p+1}} = \cdots = \frac{a_r^2}{n_r} = \frac{1}{2(n_{p+1} + \ldots + n_r)} \neq \frac{1}{m}.$$

Proof. Denote by $(x_1^1, \ldots, x_1^{n_1+1}, \ldots, x_r^1, \ldots, x_r^{n_r+1}) = p \in \mathcal{T}$ the position vector field in $\mathbb{R}^{m+r}$. Set $\eta_k = \frac{1}{a_k} x_k$. A simple computation shows that η_k is the unit normal vector field of $\mathbb{S}^{n_k}(a_k)$ in $\mathbb{R}^{n_k+1} \subset \mathbb{R}^{m+r}$. Consider $p = (x_1, \ldots, x_r)$ and denote by $\{E_{k,i}\}_{i=1}^{n_k}$ a local orthonormal frame field on $\mathbb{S}^{n_k}(a_k)$ which is geodesic at x_k. Then $\{E_{k,i}\}_{\substack{k=\overline{1,r}\\ i=\overline{1,n_k}}}$ constitutes a local orthonormal frame field on $\mathcal{T}$, geodesic at p.

Denote by B the second fundamental form of $\mathcal{T}$ in $\mathbb{S}^{m+r-1}$, by ∇^0, ∇ and $\nabla^{\mathcal{T}}$ the Levi-Civita connections on $\mathbb{R}^{m+r}$, $\mathbb{S}^{m+r-1}$ and $\mathcal{T}$, respectively. Then

$$\begin{aligned}
B(E_{k,i}, E_{k,i}) &= \nabla_{E_{k,i}} E_{k,i} - \nabla^{\mathcal{T}}_{E_{k,i}} E_{k,i} \\
&= \nabla^0_{E_{k,i}} E_{k,i} + \langle E_{k,i}, E_{k,i} \rangle p \\
&= \langle \nabla^0_{E_{k,i}} E_{k,i}, \eta_k \rangle \eta_k + p \\
&= -\langle E_{k,i}, \nabla^0_{E_{k,i}} \eta_k \rangle \eta_k + p = -\frac{1}{a_k} \eta_k + p \\
&= a_1 \eta_1 + \ldots + \big(-\frac{1}{a_k} + a_k \big) \eta_k + \ldots + a_r \eta_r.
\end{aligned} \tag{6}$$

Thus the mean curvature vector field of $\mathcal{T}$ in $\mathbb{S}^{m+r-1}$ is given by

$$\begin{aligned} mH &= \sum_{k=1}^{r}\sum_{i=1}^{n_k} B(E_{k,i},E_{k,i}) = \sum_{k=1}^{r}\Big(-\frac{n_k}{a_k}\eta_k + n_k p\Big) = -\sum_{k=1}^{r}\frac{n_k}{a_k}\eta_k + mp \\ &= \sum_{k=1}^{r}\Big(-\frac{n_k}{a_k} + m a_k\Big)\eta_k. \end{aligned} \tag{7}$$

Since, from (7),

$$H = \frac{1}{m}\sum_{k=1}^{r}\Big(-\frac{n_k}{a_k} + m a_k\Big)\eta_k$$

we see that $\mathcal{T}$ has constant mean curvature in $\mathbb{S}^{m+r-1}$. We shall prove that $\mathcal{T}$ has parallel mean curvature in $\mathbb{S}^{m+r-1}$. Indeed

$$\begin{aligned} \nabla_{E_{k,i}} H &= \nabla^0_{E_{k,i}} H \\ &= \frac{1}{m}\Big(-\frac{n_k}{a_k^2} + m\Big)E_{k,i}, \end{aligned} \tag{8}$$

thus

$$\nabla^{\perp}_{E_{k,i}} H = 0, \qquad k = 1,\dots,r, \quad i = 1,\dots,n_k$$

and

$$A_H(E_{k,i}) = \frac{1}{m}\Big(\frac{n_k}{a_k^2} - m\Big)E_{k,i}. \tag{9}$$

Since H is parallel, the conditions on $\mathcal{T}$ to be biharmonic in $\mathbb{S}^{m+r-1}$ is equivalent to

$$mH = \operatorname{trace} B(\cdot, A_H\cdot). \tag{10}$$

¿From (9) we get

$$\begin{aligned} \operatorname{trace} B(\cdot, A_H\cdot) &= \sum_{k=1}^{r}\sum_{i=1}^{n_k} B(E_{k,i}, A_H(E_{k,i})) \\ &= \sum_{k=1}^{r}\frac{1}{m}\Big(\frac{n_k}{a_k^2} - m\Big)\sum_{i=1}^{n_k} B(E_{k,i},E_{k,i}) \\ &= \frac{1}{m}\sum_{k=1}^{r}\Big(\frac{n_k}{a_k^2} - m\Big)n_k\Big(a_1\eta_1 + \cdots \\ &\quad + \Big(-\frac{1}{a_k} + a_k\Big)\eta_k + \cdots + a_r\eta_r\Big), \end{aligned} \tag{11}$$

and (10) becomes

$$\frac{a_k^2}{n_k}\Big(\sum_{j=1}^{r}\frac{n_j^2}{a_j^2}-2m^2\Big)+2m-\frac{n_k}{a_k^2}=0, \qquad k=1,\dots,r. \tag{12}$$

Denote now by $\alpha_k=\frac{a_k^2}{n_k}$ and by $d=\sum_{j=1}^{r}\frac{n_j}{\alpha_j}$. Then (12) becomes

$$(2m^2-d)\alpha_k^2-2m\alpha_k+1=0, \qquad k=1,\dots,r. \tag{13}$$

We have two cases:

a) $\alpha_k=\alpha$, for all k, thus $d=m/\alpha$ and we have $2m^2\alpha^2-3m\alpha+1=0$. The condition $\sum_{k=1}^{r}a_k^2=1$ implies that $\alpha=1/m$ is the only solution of (13) and, since in this case $a_k^2=n_k/m$, $\mathcal{T}$ is minimal in $\mathbb{S}^{m+r-1}$;

b) there exists $p=1,\dots,r-1$ such that $\alpha_1=\ldots=\alpha_p=A$ and $\alpha_{p+1}=\ldots=\alpha_r=B$, $A\neq B$. Then, from (13), it follows that $A=\frac{1}{2(n_1+\ldots+n_p)}$ and $B=\frac{1}{2(n_{p+1}+\ldots+n_r)}$.

$\boxed{QED}$

19 Remark. (i) All the examples constructed as above arise from the product composition property given in Proposition 5. Indeed, $\mathbb{S}^{n_1}(a_1)\times\mathbb{S}^{n_2}(a_2)\times\ldots\times\mathbb{S}^{n_p}(a_p)$ and $\mathbb{S}^{n_{p+1}}(a_{p+1})\times\mathbb{S}^{n_{p+2}}(a_{p+2})\times\ldots\times\mathbb{S}^{n_r}(a_r)$, with the radii given by Theorem 18, are minimal in $\mathbb{S}^{m_1}(\frac{1}{\sqrt{2}})$ and $\mathbb{S}^{m_2}(\frac{1}{\sqrt{2}})$, respectively, where $m_1=n_1+\ldots+n_p+p-1$ and $m_2=n_{p+1}+\ldots+n_r+r-p-1$.

(ii) Theorem 18 generalizes a result of W. Zhang (see [20]) which characterizes the biharmonicity of products of circles in spheres.

(iii) For $r=2$, in Theorem 18, we obtain the example of Jiang of the biharmonic generalized Clifford torus.

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