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CLASSIFICATIONS FOR INCONSISTENT THEORIES

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In [2] N. C. A. da Costa surveys some interesting results about inconsistent formal systems. A formal system is said to be inconsistent if there is a formula φ such that both φ and $\sim\varphi$ are theorems. The approach in [2] towards the study of inconsistent systems is basically syntactical. In this paper we investigate inconsistent theories from a model-theoretical point of view. However we do not analyze semantically the calculi presented in [2] as suggested on Page 508. Instead we define a notion of structure which allows for the possibility of built-in inconsistencies. These structures may then be models of inconsistent theories. We classify theories in 3 different ways. Intuitively, the higher a theory is in a classification, the more inconsistent it is. This way we obtain measures of inconsistency for theories.

1 *Terminology and Examples* Since for the purposes of this paper it is convenient to deviate somewhat from the standard terminology, we explain our notations in this section. We deal with first-order languages of finite type with equality and without function symbols. A type $\mu = \langle n_1, \ldots, n_k\rangle$ is always finite and nonempty. We use j, k, m, n for integers or possibly ω; α, β for infinite cardinals; φ, ψ for formulas (usually sentences); Γ for a set of sentences. The cardinality of a set A is denoted by $|A|$. We differentiate between equations and atomic formulas: an equation has the form $t_i = t_j$ while an atomic formula has the form $S_i(t_1, \ldots, t_n)$ where S_i is an n_i-ary relation symbol mentioned in μ, and the t_i are terms. We use the connectives $\sim$, $\wedge$, $\vee$, and the quantifiers $\exists$, $\forall$.

We give the following recursive definition of a formula:

1) Every equation, negation of equation, atomic formula, and negation of atomic formula is a formula.
2) If φ and ψ are formulas then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\sim\varphi(\exists x)\varphi$, and $(\forall x)\varphi$.
3) An expression is a formula only if it follows from a finite number of applications of 1) and 2) that it is a formula.

Sometimes we may write an expression where negation is applied to a formula which is neither an equation nor an atomic formula. Such an

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expression is not a formula according to our definition but it can be treated as one since it can be transformed to a formula by applying to it the following transformations (where appropriate): φ for $\sim\sim\varphi$, $(\sim\varphi \vee \sim\psi)$ for $\sim(\varphi \wedge \psi)$, $(\sim\varphi \wedge \sim\psi)$ for $\sim(\varphi \vee \psi)$, $(\forall x)\sim\varphi$ for $\sim(\exists x)\varphi$, and $(\exists x)\sim\varphi$ for $\sim(\forall x)\varphi$.

The notion of structure that we use in this paper differs from the usual notion of relational structure. Ordinarily if $\mathfrak{A}$ is a relational structure then it is consistent in the sense that for every atomic formula $S_i(a_1, \ldots, a_n)$ of the diagram language exactly one of the following 2 conditions holds:

1) $\mathfrak{A} \models S_i(a_1, \ldots, a_n)$
2) $\mathfrak{A} \models \sim S_i(a_1, \ldots, a_n)$.

We generalize the notion of relational structure to that of structure by replacing the word "exactly" by the phrase "at least". We say that a structure is consistent if it is a relational structure in the usual sense, and inconsistent otherwise. Note that a structure must still be equationally consistent in the sense that for any equation of the diagram language, either it or its negation but not both hold.

The diagram language of a structure $\mathfrak{A}$ contains a symbol for each element of the universe of the structure, A. The diagram of a structure $\mathfrak{A}$, $\mathrm{Diag}(\mathfrak{A})$, is the set of all atomic formulas and negations of atomic formulas of the diagram language which hold in $\mathfrak{A}$. Satisfaction for formulas is then defined by induction using the usual rules for $\wedge$, $\vee$, $\exists$, and $\forall$. We write $\mathfrak{A} \models \varphi$ if φ holds in $\mathfrak{A}$, and $\mathfrak{A} \not\models \varphi$ if φ does not hold in $\mathfrak{A}$. Many model-theoretical notions such as isomorphism, elementary equivalence, substructure, and elementary substructure can be extended to structures.

A set of sentences Γ is called a theory. We write $\mathfrak{A} \models \Gamma$ if $\mathfrak{A} \models \varphi$ for every $\varphi \in \Gamma$. In this case we also write $\mathfrak{A} \in \mathrm{Mod}(\Gamma)$. If $\mathfrak{A}$ is consistent then $\mathfrak{A} \in \mathrm{Con\ Mod}(\Gamma)$. A theory Γ is consistent if $\mathrm{Con\ Mod}(\Gamma) \neq \emptyset$. If $\Gamma = \{\varphi\}$ we usually omit the braces.

Whenever $S(a_1, \ldots, a_n) \in \mathrm{Diag}(\mathfrak{A})$ and $\sim S(a_1, \ldots, a_n) \in \mathrm{Diag}(\mathfrak{A})$ we say that $\mathfrak{A}$ has an inconsistency. Otherwise $\mathfrak{A}$ has a consistency. We write $\mathrm{Incon}(\mathfrak{A})$ for the number of inconsistencies of $\mathfrak{A}$ and $\mathrm{Con}(\mathfrak{A})$ for the number of consistencies of $\mathfrak{A}$. So $|\mathrm{Diag}(\mathfrak{A})| = \mathrm{Con}(\mathfrak{A}) + 2 \cdot \mathrm{Incon}(\mathfrak{A})$. For any type μ it is possible to write down sentences E_k, F_k, P_k, and R_k so that $\mathfrak{A} \models E_k$ iff $|A| = k$, $\mathfrak{A} \models F_k$ iff $|A| \geq k$, $\mathfrak{A} \models P_k$ iff $\mathrm{Incon}(\mathfrak{A}) \geq k$, and $\mathfrak{A} \models R_k$ iff $\mathrm{Con}(\mathfrak{A}) \leq k$. We use the term countable to stand for finite or denumerably infinite. A finite set may be empty but a finite structure must have at least one element. If $\mathfrak{A}$ and $\mathfrak{A}'$ have the same universe A and type μ, we write $\mathfrak{A} < \mathfrak{A}'$ to indicate that $\mathrm{Diag}(\mathfrak{A}) \subsetneq \mathrm{Diag}(\mathfrak{A}')$. $\mathbf{C}$ = the class of consistent structures.

Lemma 1 (a) *If* $\mathfrak{A}$ *is consistent and* $\mathfrak{A} < \mathfrak{A}'$ *then* $\mathfrak{A}'$ *is inconsistent.*
(b) *If* $\mathfrak{A} \models \Gamma$ *and* $\mathfrak{A} < \mathfrak{A}'$ *then* $\mathfrak{A}' \models \Gamma$.

Proposition 1 *There is no theory* Γ *such that* $\mathbf{C} = \mathrm{Mod}(\Gamma)$.

Proof: By Lemma 1.

Example 1 It is possible to have $\mathsf{Con\,Mod}(\varphi) = \mathsf{Con\,Mod}(\psi)$, but $\mathsf{Mod}(\varphi) \neq \mathsf{Mod}(\psi)$. Suppose that $\mu = \langle 1 \rangle$ and let $\varphi = (\exists x)(S(x) \wedge \sim S(x))$, $\psi = (\forall x)(S(x) \wedge \sim S(x))$. Then $\mathsf{Con\,Mod}(\varphi) = \emptyset = \mathsf{Con\,Mod}(\psi)$. Now let $\mathfrak{A}$ be the following structure: $A = \{a_1, a_2\}$, $\mathsf{Diag}(\mathfrak{A}) = \{S(a_1), \sim S(a_1), S(a_2)\}$. Then $\mathfrak{A} \models \varphi$ but $\mathfrak{A} \not\models \psi$.

2 *The* 3 *Classifications* In this section we present 3 methods for classifying theories via structures. As we show in the next section it suffices to consider countable structures only. So in this section every structure is countable.

Definition 1 $\Gamma_1 \leqslant \Gamma_2$ if $\mathsf{Mod}(\Gamma_2) \subseteq \mathsf{Mod}(\Gamma_1)$.

We call the ordering of theories given in Definition 1 the relative ordering. It follows that if $\Gamma_1 \leqslant \Gamma_2$ then $\mathsf{Con\,Mod}(\Gamma_2) \subseteq \mathsf{Con\,Mod}(\Gamma_1)$. Note that in Example 1 $\varphi < \psi$ even though $\mathsf{Con\,Mod}(\varphi) = \mathsf{Con\,Mod}(\psi)$.

Example 2 It is possible to have $\varphi \nleqslant \psi$ and $\psi \nleqslant \varphi$ even though $\mathsf{Con\,Mod}(\varphi) = \mathsf{Con\,Mod}(\psi)$. Suppose that $\mu = \langle 1 \rangle$ and let $\varphi = \sim E_3 \wedge (\exists x)(S(x) \wedge \sim S(x))$, $\psi = \sim E_2 \wedge (\exists x)(S(x) \wedge \sim S(x))$. Then $\mathsf{Con\,Mod}(\varphi) = \emptyset = \mathsf{Con\,Mod}(\psi)$. Now let $\mathfrak{A}$ be the structure: $A = \{a_1, a_2\}$, $\mathsf{Diag}(\mathfrak{A}) = \{S(a_1), \sim S(a_1), S(a_2)\}$ and let $\mathfrak{B}$ be the structure: $B = \{b_1, b_2, b_3\}$, $\mathsf{Diag}(\mathfrak{B}) = \{S(b_1), \sim S(b_1), S(b_2), S(b_3)\}$. Then $\mathfrak{A} \models \varphi$, $\mathfrak{A} \not\models \psi$, $\mathfrak{B} \not\models \varphi$, $\mathfrak{B} \models \psi$.

It should be noted that the relative ordering of theories is a global concept since all of the countable models of a theory are considered. The second classification that we next introduce is called the level of inconsistency of a theory. The idea here is that $\mathsf{lev}(\Gamma_1) \leqslant \mathsf{lev}(\Gamma_2)$ if for every countable n a least inconsistent model of Γ_1 of cardinal n is not more inconsistent than a least inconsistent model of Γ_2 of cardinal n. Thus the concept of level of inconsistency is global in the sense that all countable cardinals are considered, but is local in the sense that essentially only one model is considered in every cardinal.

Definition 2 $\mathsf{lev}(\Gamma) = \langle C, f, g \rangle$ where $C \subseteq (\omega - \{0\}) \cup \{\omega\}$, $f: C \to \omega \cup \{\omega\}$, $g: C \to \omega \cup \{\omega\}$ and $C =$ the set of countable cardinals in which Γ has models, $f(n) = \inf\{\mathsf{Incon}(\mathfrak{A}) \mid \mathfrak{A} \models \Gamma \text{ and } |A| = n\}$, $g(n) = \sup\{\mathsf{Con}(\mathfrak{A}) \mid \mathfrak{A} \models \Gamma \text{ and } |A| = n\}$.

Definition 3 If u_1 and u_2 are levels, say $u_1 = \langle C_1, f_1, g_1 \rangle$ and $u_2 = \langle C_2, f_2, g_2 \rangle$ then $u_1 \leqslant u_2$ if $C_2 \subseteq C_1$ and for every $n \in C_2$, $f_1(n) \leqslant f_2(n)$ and $g_1(n) \geqslant g_2(n)$.

In the third and last classification that we now introduce we define the degree of inconsistency of a theory. This concept is local if the theory has a least inconsistent model. The idea is that $\mathsf{deg}(\Gamma_1) \leqslant \mathsf{deg}(\Gamma_2)$ if for every model $\mathfrak{A}$ of Γ_2 there is a model of Γ_1 which is not more inconsistent than $\mathfrak{A}$. First we define the degree of inconsistency of a structure.

Definition 4 $\mathsf{deg}(\mathfrak{A}) = \langle a, b, c \rangle$ where $a = \mathsf{Incon}(\mathfrak{A})$, $b = \mathsf{Con}(\mathfrak{A})$, and $c = |\mathsf{Diag}(\mathfrak{A})|$.

Next we define a total ordering on degrees of structures. Our idea is that the ordering of degrees of structures should be based on the ratio a/c.

Thus in general if $\langle a_1,b_1,c_1\rangle$ and $\langle a_2,b_2,c_2\rangle$ are degrees and $a_1/c_1 < a_2/c_2$, we would like to have $\langle a_1,b_1,c_1\rangle < \langle a_2,b_2,c_2\rangle$. As long as c is finite there is no problem, but when $c = \omega$ we need special rules. We place degrees of structures in 6 sets D_0 - D_5 so that if $\deg(\mathfrak{A}) \in D_i$, $\deg(\mathfrak{B}) \in D_j$ and $i < j$ then $\deg(\mathfrak{A}) < \deg(\mathfrak{B})$. There is a further ordering within the sets D_1, D_2, and D_4. The setup is such that intuitively if $\deg(\mathfrak{A}) < \deg(\mathfrak{B})$ then $\mathfrak{A}$ is less inconsistent than $\mathfrak{B}$. Now we describe the sets D_0 - D_5.

Definition 5

D_0: $a = 0$.
D_1: $a \neq 0$, a finite, $b = c = \omega$. $\langle k,\omega,\omega\rangle < \langle m,\omega,\omega\rangle$ if $k < m$.
D_2: a,b,c all finite, $a \neq 0$, $b \neq 0$. $\langle a_1,b_1,c_1\rangle < \langle a_2,b_2,c_2\rangle$ if $a_1/c_1 < a_2/c_2$.
D_3: $a = b = c = \omega$.
D_4: $a = \omega$, $b \neq 0$, b finite, $c = \omega$. $\langle \omega,k,\omega\rangle < \langle \omega,m,\omega\rangle$ if $m < k$.
D_5: $b = 0$.

Now we are ready to define the degree of a theory Γ.

Definition 6 $\deg(\Gamma) = \inf\{\deg(\mathfrak{A}) \mid \mathfrak{A} \models \Gamma\}$.

We say that $\deg(\Gamma) \in D_i$ if Γ has a model $\mathfrak{A}$ such that $\deg(\mathfrak{A}) \in D_i$ but has no model $\mathfrak{B}$ such that $\deg(\mathfrak{B}) \in D_j$ with $j < i$. If Γ has no models then we say that $\deg(\Gamma) \in D_6$.

In the next 2 examples $\mu = \langle 2\rangle$ and $\mathrm{lev}(\varphi) = \langle C,f,g\rangle$.

Example 3 $\varphi = (\forall x)(\forall y)(S(x,y) \wedge \sim S(x,y))$. Then $\deg(\varphi) \in D_5$ and $C = (\omega - \{0\}) \cup \{\omega\}$, $f(k) = k^2$, $g(k) = 0$, $f(\omega) = \omega$, $g(\omega) = 0$.

Example 4 $\varphi = (\forall x)(\exists y)(S(x,y) \wedge \sim S(x,y))$. Then $\deg(\varphi) \in D_2$ and $C = (\omega - \{0\}) \cup \{\omega\}$, $f(k) = k$, $g(k) = k^2 - k$, $f(\omega) = \omega$, $g(\omega) = \omega$.

Theorem 1 *The 3 classifications for theories are compatible with each other* (i.e., *if* Γ_1 *is less than* Γ_2 *in one classification then* Γ_2 *is not less than* Γ_1 *in another classification*).

Proof: Note first that if $\Gamma_1 = \Gamma_2$ then $\mathrm{lev}(\Gamma_1) = \mathrm{lev}(\Gamma_2)$ and if $\mathrm{lev}(\Gamma_1) = \mathrm{lev}(\Gamma_2)$ then $\deg(\Gamma_1) = \deg(\Gamma_2)$. Now assume that $\Gamma_1 < \Gamma_2$. It follows from the definitions that $\mathrm{lev}(\Gamma_1) \leqslant \mathrm{lev}(\Gamma_2)$ and $\deg(\Gamma_1) \leqslant \deg(\Gamma_2)$. Next assume that $\mathrm{lev}(\Gamma_1) < \mathrm{lev}(\Gamma_2)$. It follows from the above that $\Gamma_2 \nless \Gamma_1$. Similarly if $\deg(\Gamma_1) < \deg(\Gamma_2)$ then $\Gamma_2 \nless \Gamma_1$ and $\mathrm{lev}(\Gamma_2) \nless \mathrm{lev}(\Gamma_1)$. Finally assume that $\mathrm{lev}(\Gamma_1) < \mathrm{lev}(\Gamma_2)$. We show that in this case $\deg(\Gamma_1) \leqslant \deg(\Gamma_2)$. For suppose that $\deg(\Gamma_2) = \inf\{\deg(\mathfrak{A}_i) \mid i \in I\}$. Then find $\{\mathfrak{B}_i \mid i \in I\}$ such that $\mathfrak{B}_i \models \Gamma_1$, $|B_i| = |A_i|$ and $\deg(\mathfrak{B}_i) \leqslant \deg(\mathfrak{A}_i)$ for each $i \in I$. So $\deg(\Gamma_1) \leqslant \inf\{\deg(\mathfrak{B}_i) \mid i \in I\} \leqslant \inf\{\deg(\mathfrak{A}_i) \mid i \in I\} = \deg(\Gamma_2)$.

We now give some examples to show that the results obtained in the proof of Theorem 1 are best possible. In these examples $\mu = \langle 1\rangle$.

Example 5 $\varphi = (\forall x)(x = x)$, $\psi = (\exists x)S(x)$. Then $\varphi < \psi$ but $\mathrm{lev}(\varphi) = \mathrm{lev}(\psi)$ and $\deg(\varphi) = \deg(\psi)$.

Example 6 $\varphi = (\forall x)S(x)$, $\psi = \sim E_2 \vee (\exists x)(S(x) \wedge \sim S(x))$. Then $\mathrm{lev}(\varphi) < \mathrm{lev}(\psi)$ but $\varphi \nleq \psi$, $\psi \nleq \varphi$, and $\mathrm{deg}(\varphi) = \mathrm{deg}(\psi)$.

Example 7 $\varphi = (\forall x)\, S(x) \wedge (\sim E_2 \vee (\forall x) \sim S(x))$, $\psi = (\exists x)\,(S(x) \wedge \sim S(x))$. Then $\mathrm{deg}(\varphi) < \mathrm{deg}(\psi)$, but $\varphi \nleq \psi$, $\psi \nleq \varphi$, $\mathrm{lev}(\varphi) \nleq \mathrm{lev}(\psi)$, $\mathrm{lev}(\psi) \nleq \mathrm{lev}(\varphi)$.

Question: What is the level of field theory?

3 *Löwenheim-Skolem and Compactness Theorems* In this section we give some extensions of the Löwenheim-Skolem and compactness theorems to structures. First we show that there is no loss of generality in the definitions of the 3 classifications given in the previous section where only countable structures are considered. Although Definition 4 was given for countable structures only, the same definition can also be used for uncountable structures.

Proposition 2

a) *If* Γ *has a model of degree* $\langle i,\alpha,\alpha\rangle$ *then* Γ *has a model of degree* $\langle i,\omega,\omega\rangle$.
b) *If* Γ *has a model of degree* $\langle \alpha, \beta, \alpha + \beta\rangle$ *then* Γ *has a model of degree* $\langle \omega,\omega,\omega\rangle$.
c) *If* Γ *has a model of degree* $\langle \alpha,i,\alpha\rangle$ *then* Γ *has a model of degree* $\langle \omega,i,\omega\rangle$.

Proof: The Downward Löwenheim-Skolem Theorem ([1] Pages 80-81) includes a) with $i = 0$. The proof is analogous to the one given there.

Corollary *If* $\Gamma_1 \leqslant \Gamma_2$ *then* $\mathrm{Mod}(\Gamma_2) \subseteq \mathrm{Mod}(\Gamma_1)$.

Proof: The statement is Definition 1 for countable structures. So suppose that $\mathfrak{A}$ is uncountable and $\mathfrak{A} \models \Gamma_2$. As in the proof of Proposition 2 construct a countable $\mathfrak{B}$ such that $\mathfrak{B} \prec \mathfrak{A}$. Then $\mathfrak{B} \models \Gamma_2$, so $\mathfrak{B} \models \Gamma_1$ and therefore $\mathfrak{A} \models \Gamma_1$.

Ultraproducts of structures can be formed in the usual way ([1] Pages 87-89) and Łoś's Theorem can be extended to structures. In the following F is an ultrafilter.

Lemma 2

a) i) *If* $\mathrm{Incon}(\mathfrak{A}_i) \leqslant k$ *for every* $i \in I$ *then* $\mathrm{Incon}(\prod \mathfrak{A}_i/F) \leqslant k$.

and

ii) *Substitute* $\geqslant$ *for* $\leqslant$ *in* i).

b) i) and ii) *Substitute* Con *for* Incon *in a*).

Proposition 3

a) $\mathrm{deg}(\Gamma) \leqslant \langle i,\omega,\omega\rangle$ *iff for every finite* $\Gamma_0 \subseteq \Gamma$, $\mathrm{deg}(\Gamma_0) \leqslant \langle i,\omega,\omega\rangle$.
b) $\mathrm{deg}(\Gamma) \in D_6$ *iff for some finite* $\Gamma_0 \subseteq \Gamma$, $\mathrm{deg}(\Gamma_0) \in D_6$.

Proof: The Compactness Theorem ([1] Page 102) is a) with $i = 0$. The proof is analogous to the one given there with an additional use of Proposition 2 and Lemma 2.

Corollary *If* $\text{lev}(\Gamma) = \langle C,f,g\rangle$ *then* $n \in C$ *iff* $n \in C_0$ *for each finite* $\Gamma_0 \subseteq \Gamma$ where $\text{lev}(\Gamma_0) = \langle C_0,f_0,g_0\rangle$.

Theorem 2 *If* $\text{lev}(\Gamma) = \langle C,f,g\rangle$ *and C is infinite then*

a) $\omega \in C$,
b) $f(\omega) \leqslant \liminf\{f(n)|n \in C, n \to \omega\}$,

and

c) $g(\omega) \geqslant \limsup\{g(n)|n \in C, n \to \omega\}$.

Proof: a) Construct an appropriate ultraproduct of finite models of Γ and apply Proposition 2. b) Suppose that $\liminf\{f(n)|n \in C, n \to \omega\} = k < \omega$. Then $J = \{n|f(n) = k\}$ is infinite. Construct an appropriate ultraproduct of models of Γ whose cardinal $\in J$. Now apply Lemma 2 and Proposition 2. c) Similar to the proof of b).

We close this section by investigating levelcompact theories.

Definition 7 A theory Γ is called levelcompact if there is a finite $\Gamma_0 \subseteq \Gamma$ such that $\text{lev}(\Gamma) = \text{lev}(\Gamma_0)$.

Proposition 4 *Suppose that* $\text{lev}(\Gamma) = \langle C,f,g\rangle$ *and* $\omega - C$ *is not finite. Then* Γ *is levelcompact iff C is finite and* $\omega \notin C$.

Proof: If $C = \emptyset$ then by Proposition 3 Γ is levelcompact. If $C \neq \emptyset$, C finite and $\omega \notin C$ then there is a $\varphi \in \Gamma$ which implies $\sim F_k$ for some k. For each $n \in C$ there is a finite set of sentences which express $f(n)$ and $g(n)$. So Γ is levelcompact. Finally if $\omega \in C$ and $\omega - C$ is infinite then there are sentences equivalent to $\sim E_k$ for infinitely many k in Γ. So Γ is not levelcompact.

Corollary *If* Γ *is finite then* $\text{deg}(\Gamma) \notin D_3$.

4 *The Realizable Degrees and Levels* In this section we show which of the possible levels and degrees are actually realizable by theories.

Theorem 3 *The set of realizable degrees has the order type* $\omega + 1 + \lambda + 1 + \omega^*$.

Proof: We indicate how to obtain theories whose degrees are progressively higher. We show that as ordered sets $D_0 \cup D_1$ has order type ω, D_2 has order type $1 + \lambda$, D_3 has order type 1, and $D_4 \cup D_5 \cup D_6$ has order type ω^*. If Γ is consistent then $\text{deg}(\Gamma) \in D_0$. Next, $\text{deg}(P_k) = \langle k,\omega,\omega\rangle \in D_1$.

Now let $\text{deg}(\Gamma) \in D_2$. Then $\text{deg}(\Gamma) = r$ where r is a real number such that $0 \leqslant r < 1$. For every such r we construct a theory Γ_r such that $\text{deg}(\Gamma_r) = r$. If r is rational and $\neq 0$, say $r = k/m$, $k \neq 0$, then let $\Gamma_r = \{E_m, P_k\}$. Otherwise let $\{k_i/m_i|1 \leqslant i < \omega\}$ be a descending sequence of fractions such that both $\{k_i\}$ and $\{m_i\}$ are increasing sequences and $\lim\{k_i/m_i|i \to \omega\} = r$. Now choose

$$\Gamma_r = \{F_{m_1}\} \cup \{F_{m_i+1} \vee P_{k_i}|1 \leqslant i < \omega\}.$$

Next we let $\Gamma = \{F_i|1 \leqslant i < \omega\} \cup \{P_j|1 \leqslant j < \omega\}$. Then $\text{deg}(\Gamma) \in D_3$. Now if $\Gamma_k = R_k \cup \{F_i | 1 \leqslant i < \omega\}$ then $\deg(\Gamma_k) \epsilon D_4$. Finally if $\varphi = (\forall x)(P(x) \wedge \sim P(x))$ then $\deg(\varphi) \epsilon D_5$ and if $\psi = (\exists x)(x \neq x)$ then $\deg(\psi) \epsilon D_6$.

Theorem 4 *Given a type* $\mu = \langle n_1, \ldots, n_k \rangle$, *a triple* $\langle C,f,g \rangle$ *can be the level of a theory of type* μ *iff the following* 6 *conditions hold:*

i) $C \subseteq (\omega - \{0\}) \cup \{\omega\}$, $f: C \to \omega \cup \{\omega\}$, $g: C \to \omega \cup \{\omega\}$.
ii) *If* C *is infinite then* $\omega \epsilon C$.
iii) *If* C *is infinite then* $f(\omega) \leqslant \lim\inf \{f(n) | n \epsilon C, n \to \omega\}$.
iv) *If* C *is infinite then* $g(\omega) \geqslant \lim\sup \{g(n) | n \epsilon C, n \to \omega\}$.
v) *If* $m \epsilon C$, $m \neq \omega$, *then* $f(m) + g(m) = \sum_{1 \leqslant s \leqslant k} m^{n_s}$.
vi) *If* $\omega \epsilon C$ *then* $f(\omega) + g(\omega) = \omega$.

Proof: The 6 conditions are necessary by Definition 2 and Theorem 2. Now assume that the 6 conditions are satisfied for some $\langle C,f,g \rangle$ and μ. We construct a theory Γ in type μ such that $\mathrm{lev}(\Gamma) = \langle C,f,g \rangle$.

Case 1. $C = \emptyset$. Let $\Gamma = (\exists x)(x \neq x)$.

Case 2. C is finite, $\omega \notin C$, say $C = \{m_1, \ldots, m_j\}$, $f(m_i) = t_i$, $g(m_i) = \left(\sum_{1 \leqslant s \leqslant k} m_i^{n_s}\right) - t_i$. Then let

$$\Gamma = \{\sim E_m | m < m_j \ \& \ m \notin C\} \cup \{\sim F_{m_{j+1}}\} \cup \{\sim E_{m_i} \vee P_{t_i} | 1 \leqslant i \leqslant j\}.$$

Case 3. $C = \{\omega\}$.

a) $f(\omega) = k \neq \omega$, $g(\omega) = \omega$. Let $\Gamma = \{P_k\} \cup \{F_i | 1 \leqslant i < \omega\}$.
b) $f(\omega) = \omega$, $g(\omega) = \omega$. Let $\Gamma = \{P_j | 1 \leqslant j < \omega\} \cup \{F_i | 1 \leqslant i < \omega\}$.
c) $f(\omega) = \omega$, $g(\omega) = k \neq \omega$. Let $\Gamma = \{R_k\} \cup \{F_i | 1 \leqslant i < \omega\}$.

Case 4. $\omega \epsilon C$ and $|C| \neq 1$. Now Γ may be obtained by a proper combination of the constructions in Cases 2 and 3.

5 *The lattice of realizable levels* In this section we investigate the partial ordering on realizable levels given in Definition 3. We assume that the set of realizable levels refers to the levels realized by theories of some fixed type μ.

Theorem 5 *The set of realizable levels forms a complete nonmodular lattice.*

Proof: To show that it is complete we show how to obtain $u = \bigvee \{u_i | i \epsilon I\} = \langle C,f,g \rangle$ and $u' = \bigwedge \{u_i | i \epsilon I\} = \langle C',f',g' \rangle$ given $\{u_i = \langle C_i, f_i, g_i \rangle | i \epsilon I\}$. So let $C = \bigcap \{C_i | i \epsilon I\}$ and if $C \neq \emptyset$ then $f(n) = \sup \{f_i(n) | i \epsilon I\}$, $g(n) = \inf \{g_i(n) | i \epsilon I\}$ for all $n \epsilon C$. Next

$$C' = \begin{cases} \bigcup \{C_i | i \epsilon I\} & \text{if this union is finite,} \\ \{\omega\} \cup \{C_i | i \epsilon I\} & \text{otherwise.} \end{cases}$$

If $m \epsilon C'$, $m \neq \omega$, then $f'(m) = \inf \{f_i(m) | f_i(m) \text{ is defined}\}$,
$g'(m) = \sup \{g_i(m) | g_i(m) \text{ is defined}\}$,

$f'(\omega) = \min(\inf\{f_i(\omega)|i \in I\}, \liminf\{f'(n)|n \in C', n \to \omega\})$,
$g'(\omega) = \max(\sup\{g_i(\omega)|i \in I\}, \limsup\{g'(n)|n \in C', n \to \omega\})$.

Modularity means that

$$\text{if } u_1 \leq u_3 \text{ then } u_1 \vee (u_2 \wedge u_3) = (u_1 \vee u_2) \wedge u_3.$$

So let $u = u_1 \vee (u_2 \wedge u_3)$ and $u' = (u_1 \vee u_2) \wedge u_3$. Now if $n \in C - C_2$ and $f_1(n) < f_3(n)$ then $f(n) = f_3(n) \neq f_1(n) = f'(n)$. Thus in such a case $u \neq u'$.

An element u of a lattice $\mathbf{L}$ is said to be compact if whenever $u \leq \bigvee\{u_i|i \in I\}$ then there exists a finite $I' \subseteq I$ such that $u \leq \bigvee\{u_i|i \in I'\}$ ([3] Page 21). In our case $\mathbf{L}$ is the lattice of realizable levels and $u = \langle C,f,g\rangle$.

Proposition 5 *u is compact iff one of the following conditions holds*:

i) *C is finite and $\omega \notin C$.*
ii) *$\omega - C$ is finite and $D = \{n \in C| f(n) \neq f(\omega)\}$ is finite.*
iii) *$\omega - C$ is finite and $E = \{n \in C| g(n) \neq g(\omega)\}$ is finite.*

Proof: There are 7 cases to consider. First we deal with the 3 cases where u is compact. We assume that $u \leq \bigvee\{u_i|i \in I\}$. By Theorem 5 this means that $\bigcap\{C_i|i \in I\} \subseteq C$, and for all $n \in \bigcap\{C_i|i \in I\}$, $f(n) \leq \sup\{f_i(n)|i \in I\}$, $g(n) \geq \inf\{g_i(n)|i \in I\}$.

i) C is finite and $\omega \notin C$. In this case by Theorem 2 there is an $i_0 \in I$ such that C_{i_0} is finite. This implies compactness.

ii) $\omega - C$ is finite and D is finite. If some C_{i_0} is finite then just as in Case i) we are through. So assume that each C_i is infinite. Then $\omega \in \bigcap\{C_i|i \in I\}$. By Theorem 2 there is an i_0 such that $f(\omega) \leq \liminf\{f_{i_0}(n)|n \to \omega\}$. This implies compactness.

iii) $\omega - C$ is finite and E is finite. This is similar to Case ii).

Next we consider the 4 cases where u is not compact. We construct $\{u_i|i \in I\}$ such that $u \leq \bigvee\{u_i|i \in I\}$ but there is no finite $I' \subseteq I$ such that $u \leq \bigvee\{u_i|i \in I'\}$.

iv) $\omega - C$ is infinite and $\omega \in C$. Let $C_i = (\omega \cup \{\omega\}) - \{i,0\}$, $f_i(\omega) = f(\omega)$, $g_i(\omega) = g(\omega)$, $f_i(n)$ and $g_i(n)$ arbitrary otherwise satisfying the conditions of Theorem 4.

v) $\omega - C$ is finite and $f(\omega) = g(\omega) = \omega$. Let $C = \{n_i|1 \leq i < \omega\} \cup \{\omega\}$ and $C_i = C - \{n_i\}$, $f_i(n) = \max(f(n) - 1, 0)$, $f_i(\omega) = \omega$, $g_i(\omega) = \omega$, $g_i(n)$ satisfying the conditions of Theorem 4.

vi) $\omega - C$ is finite, $f(\omega)$ is finite, and D is infinite. There is an infinite set $D' \subseteq D$ such that if $n \in D'$ then $f(n) > f(\omega)$. Now let $D' = \bigcup\{D_i|1 \leq i < \omega\}$ and $D_i \cap D_{i'} = \emptyset$ if $i \neq i'$. Define $C_i = D' \cup \{\omega\}$, $f_i(m) = f(m)$ if $m \in D_i$, $f_i(m) = f(\omega)$ if $m \in C_i - D_i$, $g_i(\omega) = \omega = g(\omega)$, $g_i(m)$ satisfying the conditions of Theorem 4.

vii) ω - C is finite, $g(\omega)$ is finite, and E is infinite. This is similar to Case vi).

Corollary *The lattice of realizable levels forms an algebraic lattice.*

Proof: Recall that an algebraic lattice is a complete lattice in which every element is a join of compact elements ([3], Page 21). The proof uses Theorem 5 and Proposition 5. Cases iv), v), vi), and vii) of Proposition 5 must be considered.

6 *Further examples and results* We define semantical implication in analogy with the usual definition, $\Gamma \models \Gamma'$ iff $\mathrm{Mod}(\Gamma) \supseteq \mathrm{Mod}(\Gamma')$. We let $\overline{\Gamma} = \{\varphi | \Gamma \models \varphi\}$.

Lemma 3

a) $\overline{\overline{\Gamma}} = \overline{\Gamma}$.
b) $\Gamma \leqslant \overline{\Gamma}$ *and* $\overline{\Gamma} \leqslant \Gamma$.
c) $\Gamma \leqslant \Gamma'$ *iff* $\overline{\Gamma} \leqslant \overline{\Gamma'}$.
d) $\Gamma \leqslant \Gamma'$ *iff* $\overline{\Gamma} \subseteq \overline{\Gamma'}$.

Proposition 6 *The relative ordering of theories forms a complete lattice.*

Proof: Let $\bigvee\{\Gamma_i | i \in I\} = \bigcup\{\Gamma_i | i \in I\}$ and $\bigwedge\{\Gamma_i | i \in I\} = \bigcap\{\overline{\Gamma}_i | i \in I\}$. The result then follows from Lemma 3.

Next we show that an extension of Proposition 6 to degrees and levels does not hold.

Example 8 Let $\mu = \langle 1,1\rangle$ and $\varphi = (\exists x)(S_1(x) \wedge \sim S_1(x))$, $\psi = (\exists x)(S_2(x) \wedge \sim S_2(x))$. Now $\mathrm{lev}(\varphi) = \mathrm{lev}(\psi) = \langle C,f,g\rangle$ where $C = (\omega - \{0\}) \cup \{\omega\}$ and $f(n) = 1$ for all $n \in C$. Thus $\mathrm{lev}(\varphi) \wedge \mathrm{lev}(\psi) = \mathrm{lev}(\varphi) = \mathrm{lev}(\varphi) \vee \mathrm{lev}(\psi)$. But $\mathrm{lev}(\overline{\varphi} \cap \overline{\psi}) < \mathrm{lev}(\varphi) < \mathrm{lev}(\varphi \cup \psi)$. Similarly $\deg(\varphi) = \deg(\psi) = \langle 1,\omega,\omega\rangle$ so that $\deg(\varphi) \wedge \deg(\psi) = \deg(\varphi) = \deg(\varphi) \vee \deg(\psi)$. But $\deg(\overline{\varphi} \cap \overline{\psi}) < \deg(\varphi) < \deg(\varphi \cup \psi)$.

Finally we consider the notion of reduced model of a theory.

Definition 8 A model $\mathfrak{A}$ of a theory Γ is a reduced model of Γ if $\mathfrak{A} \models \Gamma$ but there is no $\mathfrak{A}' < \mathfrak{A}$ such that $\mathfrak{A}' \models \Gamma$.

We give our results about reduced models by examples.

Example 9 Let $\mu = \langle 1\rangle$, $\Gamma = \{F_i | 1 \leqslant i < \omega\} \cup \{P_j | 1 \leqslant j < \omega\}$. Γ has no reduced models.

Example 10 Let $\varphi = \sim E_2 \vee P_1$. Then φ is consistent but any reduced model of φ of cardinal 2 is inconsistent.

Example 11 Let $\mu = \langle 1,1\rangle$ and

$$\varphi = (\exists x)(S_1(x) \wedge \sim S_1(x)) \vee (\exists x)(\exists y)\,(x \neq y \wedge S_2(x) \wedge \sim S_2(x) \wedge S_2(y) \wedge \sim S_2(y)).$$

Now $\mathrm{lev}(\varphi) = \langle C,f,g\rangle$ where $f(n) = 1$ for all $n \in C$ and $C = (\omega - \{0\}) \cup \{\omega\}$. Construct $\mathfrak{A}$ as follows: $A = \{a_1,a_2\}$,

$$\text{Diag}(\mathfrak{A}) = \{S_1(a_1), S_2(a_1), \sim S_2(a_1), S_1(a_2), S_2(a_2), \sim S_2(a_2)\}.$$

Then $\mathfrak{A}$ is a reduced model of Γ of cardinal 2 which has more than $f(2)$ inconsistencies.

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