Konuralp Journal of Mathematics, 7 (2) (2019) 312-318

Konuralp Journal of Mathematics

Journal Homepage: www.dergipark.gov.tr/konuralpjournalmath
e-ISSN: 2147-625X

Classifications of K-Contact Semi-Riemannian Manifolds

Chiranjib Dey[1*] and Uday Chand De[2]

[1] *Dhamla Jr. High School, Vill-Dhamla, P.O.-Kedarpur, Dist-Hooghly, Pin-712406, West Bengal, India*
[2] *Department of Pure Mathematics, University of Calcutta, 35, Ballygunge Circular Road, Kol- 700019, West Bengal, India*
[*] *Corresponding author E-mail: dey9chiranjib@gmail.com*

Abstract

The object of the present paper is to characterize a K-contact semi-Riemannian manifold satisfying certain curvature conditions. We study Ricci semi-symmetric K-contact semi-Riemannian manifolds and obtain an equivalent condition. Next we prove that a K-contact semi-Riemannian manifold is of harmonic conformal curvature tensor if and only if the manifold is an Einstein manifold. Also we study ξ-conformally flat K-contact semi-Riemannian manifolds. Finally, we charecterize conformally semisymmetric Lorentzian K-contact manifolds.

Keywords: *Conformally semisymmetric manifolds; Harmonic conformal curvature tensor; K-contact manifolds; Ricci symmetric manifolds; Ricci semi-symmetric manifolds.*
2010 Mathematics Subject Classification: *53C25, 53C35, 53D10.*

1. Introduction

Let M^{2n+1} be a contact manifold with contact form η, associated vector field ξ and $(1,1)$ tensor field ϕ ([1],[2]). A semi-Riemannian metric g is said to be an associated metric if it satisfies: $\eta(X) = \varepsilon g(X,\xi)$, where $\varepsilon = \pm 1$. Then M is said to be a contact semi-Riemannian (or Pseudo-Riemannian) manifold and (M,ϕ,ξ,η,g) is called a semi-Riemannian (or Pseudo-Riemannian) structure.
Takahashi [10] introduced the concept of contact semi-Riemannian manifolds. Semi-Riemannian contact metric manifolds have been studied by Calvaruso and Perrone ([3],[4]) and Perrone ([7],[8]).
A contact semi-Riemannian manifold is said to be K-contact if ξ is a Killing vector field and Sasakian if it is normal.
A semi-Riemannian manifold is said to be Ricci semi-symmetric if $R(X,Y).S = 0$, where $R(X,Y)$ is the curvature operator and S denotes the Ricci tensor. In [9] Tanno studied Ricci symmetric $(\nabla S = 0)$ K-contact Riemannian manifolds. Also Yildiz and Ata[11] studied K-contact Riemannian manifolds. The paper is organized as follows:
After preliminaries in section 3, we extend the Tanno's result for semi-Riemannian case.
In a recent paper [7], Perrone showed that a conformally flat K-contact semi-Riemannian manifold is Sasakian. In section 4 we generalize the result for a K-contact semi-Riemannian manifold of harmonic conformal curvature tensor. In section 5, we have shown that any ξ-conformally flat semi-Riemannian K-contact manifold is Sasakian. Finally, we consider conformally semisymmetric Lorentzian K-contact manifolds.

2. Preliminaries

A contact semi-Riemannian manifold is Sasakian if it is normal, that is,

$$[\phi X, \phi X] + 2d\eta(X,X)\xi = 0. \tag{2.1}$$

This condition is equivalent to

$$(\nabla_X \phi)Y = g(X,Y)\xi - \varepsilon\eta(Y)X. \tag{2.2}$$

In a contact semi-Riemannian manifold the following relations hold ([7],[8]):

$$\eta(\xi) = 1, \varepsilon g(X,\xi) = \eta(X), \tag{2.3}$$

$$\phi^2 X = -X + \eta(X)\xi, \tag{2.4}$$

Email addresses: **dey9chiranjib@gmail.com (Chiranjib Dey), uc_de@yahoo.com (Uday Chand De)**

$$g(\phi X,\phi Y)=g(X,Y)-\varepsilon\eta(X)\eta(Y), \tag{2.5}$$

$$\nabla_X\xi=\varepsilon\phi X, \tag{2.6}$$

where ∇ is the Levi-Civita connection.

Also in a K-contact semi-Riemannian manifold ξ satisfies the following relation (see [6]):

$$Q\xi=2n\varepsilon\xi, \tag{2.7}$$

that is, ξ is an eigen vector of the Ricci operator Q defined by [7]

$$S(\xi,X)=g(Q\xi,X)=2n\varepsilon g(\xi,X)=2n\eta(X), \tag{2.8}$$

where S denotes the Ricci tensor.

Also the curvature tensor R satisfies

$$R(X,\xi)\xi=\phi^2X. \tag{2.9}$$

Further, since ξ is Killing in a K-contact manifold, S and r remain invariant under it.

Consequently,

$$£_\xi S=0 \tag{2.10}$$

and

$$£_\xi r=0 \tag{2.11}$$

where $£$ denotes the Lie derivation.

We state the following:

Lemma 2.1. *([7],[8]) Let (M,η,g,ξ,ϕ) be a K-contact semi-Riemannian manifold. Then M is Sasakian if and only if the curvature tensor R satisfies*

$$R(X,Y)\xi=\eta(X)Y-\eta(Y)X. \tag{2.12}$$

Definition 2.2. *A contact semi-Riemannian manifold is said to be η-Einstein if*

$$S(X,Y)=ag(X,Y)+b\eta(X)\eta(Y), \tag{2.13}$$

where a,b are smooth functions.

Remark 2.3. *The basic difference between the Riemannian and semi-Riemannian geometry is the existence of a null vector, that is, a vector v satisfying $g(v,v)=0$, where g is the metric tensor. The signature of the metric g of a Riemannian manifold is $(+,+,+,...,+,+)$ and of a semi-Riemannian manifold is $(-,-,-,...,+,+)$. There are differences of the basic results of K-contact Riemannian and semi-Riemannian manifolds due the presence of ε.*

3. Ricci semisymmetric K-contact semi-Riemannian manifolds

It is known that [9] a K-contact Riemannian metric manifold is an Einstien manifold if and only if it is Ricci-symmetric. In this section, we prove a similar result for the semi-Riemannian case.

Theorem 3.1. *A K-contact semi-Riemannian manifold is Ricci semi-symmetric if and only if the manifold is an Einstein manifold.*

Proof. Let M be a Ricci semi-symmetric K-contact semi-Riemannian manifold. Then

$$R.S=0,$$

which implies

$$S(R(X,Y)U,V)+S(U,R(X,Y)V)=0. \tag{3.1}$$

Putting $Y=U=\xi$ in (3.1) we obtain

$$S(R(X,\xi)\xi,V)+S(\xi,R(X,\xi)V)=0. \tag{3.2}$$

Using (2.4) and (2.9) in (3.2) yields

$$S(-X+\eta(X)\xi,V)+S(\xi,R(X,\xi)V)=0,$$

which implies

$$-S(X,V)+\eta(X)S(\xi,V)+S(\xi,R(X,\xi)V)=0. \quad (3.3)$$

Interchanging X and V in (3.3) we obtain

$$-S(V,X)+\eta(V)S(\xi,X)+S(\xi,R(V,\xi)X)=0. \quad (3.4)$$

Subtracting (3.3) from (3.4) it follows that

$$\eta(X)S(\xi,V)-\eta(V)S(\xi,X)+S(\xi,R(X,\xi)V)+S(R(V,\xi)\xi,X)=0. \quad (3.5)$$

Putting $V=\xi$ in (3.5) we obtain

$$\eta(X)S(\xi,\xi)-\eta(\xi)S(\xi,X)-S(\xi,R(\xi,X)\xi)=0.$$

Using (2.3) and (2.9) in the above equation yields

$$\eta(X)S(\xi,\xi)-S(\xi,X)-S(\xi,X)+\eta(X)S(\xi,\xi)=0,$$

which implies

$$S(X,\xi)=2n\eta(X). \quad (3.6)$$

Consequently from (3.2) we get

$$-S(X,V)+2n\eta(X)\eta(V)+2n\varepsilon g(R(X,\xi)V,\xi)=0. \quad (3.7)$$

It follows that

$$S(X,V)+2n\eta(X)\eta(V)-2n\varepsilon g(-X+\eta(X)\xi,V)=0,$$

which in terms implies that

$$S(X,V)=2n\varepsilon g(X,V), \quad where \;\; \varepsilon=g(X_i,X_i).$$

Converse part follows easily. □

In view of Theorem 3.1 and the fact that Ricci symmetric ($\nabla S=0$) implies Ricci-semisymmetric ($R.S=0$), we may conclude the following:

Theorem 3.2. *In a K-contact semi-Riemannian manifold M the following conditions are equivalent*
(i) M is an Einstein manifold
(ii) M is Ricci-symmetric
(iii) M is Ricci semi-symmetric.

4. K-Contact Semi-Riemannian Manifolds With Harmonic Conformal Curvature Tensor

This section deals with a K-contact Semi-Riemannian manifold with $divC=0$.

Definition 4.1. *The conformal curvature tensor is given by*

$$C(X,Y)Z=R(X,Y)Z-\frac{1}{2n-1}[S(Y,Z)X-S(X,Z)Y+g(Y,Z)QX-g(X,Z)QY]+\frac{r}{2n(2n-1)}[g(Y,Z)X-g(X,Z)Y],$$

where Q is the Ricci operator and r is the scalar curvature.

It is known that([5],[6])

$$(divC)(X,Y)Z=\frac{2(n-1)}{2n-1}\{(\nabla_X S)(Y,Z)-(\nabla_Y S)(X,Z)-\frac{1}{4n}[g(Y,Z)dr(X)-g(X,Z)dr(Y)]\},$$

where 'div' denotes divergence.

Definition 4.2. *A Riemannian or a semi-Riemannian manifold is said to be of harmonic conformal curvature tensor if*

$$(divC)(X,Y)Z=0,$$

where 'div' denotes divergence.

We prove the following:

Theorem 4.3. *A K-contact semi-Riemannian manifold is of harmonic conformal curvature if and only if the manifold is an Einstein manifold.*

Proof. Let M be a semi-Riemannian K-contact manifold satisfying $divC = 0$. It is known that $divC = 0$ implies

$$(\nabla_X S)(Y,Z) - (\nabla_Y S)(X,Z) = \frac{1}{4n}[g(Y,Z)dr(X) - g(X,Z)dr(Y)]. \tag{4.1}$$

From (2.10) and (2.11) it follows that

$$(\nabla_\xi S)(Y,Z) = -S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi). \tag{4.2}$$

$$dr(\xi) = 0. \tag{4.3}$$

Consequently putting $X = \xi$ in (4.1) we get

$$(\nabla_\xi S)(Y,Z) - (\nabla_Y S)(\xi,Z) = \frac{1}{4n}[g(Y,Z)dr(\xi) - g(\xi,Z)dr(Y)]. \tag{4.4}$$

Thus from (4.2) and (4.3) it follows that

$$-S(\nabla_Y \xi, Z) - S(Y, \nabla_Z \xi) - (\nabla_Y S)(\xi,Z) = -\frac{1}{4n}g(\xi,Z)dr(Y). \tag{4.5}$$

That is,

$$-S(Y, \nabla_Z \xi) - \nabla_Y S(\xi,Z) + S(\xi, \nabla_Y Z) = -\frac{1}{4n}\varepsilon\eta(Z)dr(Y). \tag{4.6}$$

Using (2.6) and (2.8) in (4.6) we get,

$$\varepsilon S(Y,\phi Z) + 2n\nabla_Y \eta(Z) - 2n\eta(\nabla_Y Z) = \frac{\varepsilon}{4n}\eta(Z)dr(Y), \tag{4.7}$$

which implies

$$\varepsilon S(Y,\phi Z) + 2n(\nabla_Y \eta)Z = \frac{\varepsilon}{4n}\eta(Z)dr(Y). \tag{4.8}$$

Hence

$$S(Y,\phi Z) + 2n\varepsilon(\nabla_Y \eta)(Z) = \frac{1}{4n}\eta(Z)dr(Y). \tag{4.9}$$

We note that

$$\begin{aligned}(\nabla_Y \eta)(Z) &= \nabla_Y \eta(Z) - \eta(\nabla_Y Z) \\ &= \varepsilon\nabla_Y g(Z,\xi) - \varepsilon g(\nabla_Y Z,\xi) \\ &= \varepsilon g(Z,\nabla_Y \xi) \\ &= \varepsilon g(Z,\varepsilon\phi Y) \\ &= g(Z,\phi Y).\end{aligned} \tag{4.10}$$

So from (4.10) it follows that

$$S(Y,\phi Z) + 2n\varepsilon g(Z,\phi Y) = \frac{1}{4n}\eta(Z)dr(Y). \tag{4.11}$$

Putting $Z = \phi Z$ in (4.11) we get

$$S(Y,\phi^2 Z) + 2n\varepsilon g(\phi Z,\phi Y) = \frac{1}{4n}\eta(\phi Z)dr(Y). \tag{4.12}$$

Again using (2.4) and (2.5) in (4.12) we get

$$S(Y,-Z+\eta(Z)\xi) + 2n\varepsilon g(Z,Y) - 2n\eta(Z)\eta(Y) = 0. \tag{4.13}$$

That is,

$$-S(Y,Z) + \eta(Z)S(Y,\xi) + 2n\varepsilon g(Z,Y) - 2n\eta(Z)\eta(Y) = 0. \tag{4.14}$$

Using (2.8) in (4.14), we obtain

$$S(Y,Z) = 2n\varepsilon g(Y,Z), \quad where \;\; \varepsilon = g(X_i,X_i). \tag{4.15}$$

Conversely, if the K-contact manifold is an Einstein manifold, then it can be easily seen that the conformal curvature tensor is harmonic. This completes the proof. □

Since $\nabla C = 0$ (conformally symmetric) implies $divC = 0$, therefore we can state the following:

Corollary 4.4. *A conformally symmetric K-contact semi-Riemannian manifold is an Einstein manifold.*
Note: A conformally symmetric K-contact Riemannian manifold has been studied by Zhen [13].

5. ξ-Conformally Flatness

ξ-conformally flat K-contact manifolds have been studied by Zhen et al. Since at each point $p \in M^n$ the tangent space $T_p(M^n)$ can be decomposed into the direct sum $T_p(M^n) = \phi(T_p(M^n)) \oplus \{\xi_p\}$, where $\{\xi_p\}$ is the one-dimensional linear subspace of $T_p(M^n)$ generated by ξ_p, the conformal curvature tensor C is a map
$C : T_p(M^n) \times T_p(M^n) \times T_p(M^n) \to \phi(T_p(M^n)) \oplus \{\xi_p\}$

Definition 5.1. *[12] A K-contact semi-Riemannian manifold is said to be ξ conformally flat if the projection of the image of C onto $\{\xi_p\}$ is zero, that is, $C(X,Y)\xi = 0$, where C is the conformal curvature tensor.*

Proposition 5.2. *A ξ-conformally flat K-contact semi-Riemnnian manifold is an η-Einstein manifold.*

Proof. Let M^{2n+1} be a ξ-conformally flat K-contact semi-Riemannian manifold. Then $C(X,Y)\xi = 0$ implies that

$$R(X,Y)\xi - \frac{1}{(2n-1)}[S(Y,\xi)X - S(X,\xi)Y + g(Y,\xi)QX - g(X,\xi)QY] + \frac{r}{2n(2n-1)}[g(Y,\xi)X - g(X,\xi)Y] = 0. \tag{5.1}$$

Putting $Y = \xi$ in (5.1) and using (2.3), (2.8) and (2.9) we obtain,

$$-X + \eta(X)\xi = \frac{1}{2n-1}[2nX - 2n\eta(X)\xi + \varepsilon QX - 2n\eta(X)\xi] - \frac{r\varepsilon}{2n(2n-1)}[X - \eta(X)\xi],$$

which implies

$$QX = \frac{1}{\varepsilon}\{1 - 4n + \frac{r\varepsilon}{2n}\}X + \frac{1}{\varepsilon}\{6n - 1 - \frac{r\varepsilon}{2n}\}\eta(X)\xi. \tag{5.2}$$

That is,

$$S(X,Y) = ag(X,Y) + b\eta(X)\eta(Y), \tag{5.3}$$

where

$$a = \frac{1}{\varepsilon}\{1 - 4n + \frac{r\varepsilon}{2n}\} \tag{5.4}$$

and

$$b = \frac{1}{\varepsilon}\{6n - 1 - \frac{r\varepsilon}{2n}\}. \tag{5.5}$$

Consequently M is η-Einstein. □

Theorem 5.3. *A ξ-conformally flat K-contact manifold is Sasakian.*

Proof. In view of (5.3) and (2.3) we obtain the following:

$$QX = aX + b\eta(X)\xi, \tag{5.6}$$

$$S(X,\xi) = ag(X,\xi) + b\varepsilon\eta(X) \tag{5.7}$$

and

$$r = (2n+1)a + b. \tag{5.8}$$

From (5.4) and (5.5), it follows that

$$a + b = \frac{2n}{\varepsilon} \tag{5.9}$$

Using (2.3),(5.6),(5.7)and (5.8) in (5.1) we have

$$R(X,Y)\xi = [\frac{a}{2n\varepsilon} - \frac{b}{2n(2n-1)\varepsilon} + \frac{b\varepsilon}{2n-1}]\{\eta(Y)X - \eta(X)Y\}. \tag{5.10}$$

Using (5.9) in (5.10) we obtain

$$R(X,Y)\xi = \eta(X)Y - \eta(Y)X. \tag{5.11}$$

Therefore from Lemma 2.1 we conclude the theorem. □

Since conformally flatness implies ξ-conformally flat, therefore we obtain the following:

Corollary 5.4. *A conformally flat K-contact semi-Riemannian manifold is Sasakian.*
The above corollary has been proved by Perrone in his paper [7].

6. Conformally semisymmetric Lorentzian K-contact manifolds

The Ricci tensor S_L of η-Einstein Lorentzian K-contact structure (η, g_L) is given by [7]

$$S_L = (\frac{r_L}{2n}+1)g_L + (\frac{r_L}{2n}+2n+1)\eta\otimes\eta, \tag{6.1}$$

where the scalar curvature $r_L = r+4n$ is a constant when $n>1$, and g_L is Einstein if and only if $r_L = -2n(2n+1)$. From (6.1)

$$S_L(X,Y) = (\frac{r_L}{2n}+1)g_L(X,Y) + (\frac{r_L}{2n}+2n+1)\eta(X)\eta(Y) = Ag_L(X,Y)+B\eta(X)\eta(Y), \tag{6.2}$$

where $A = (\frac{r_L}{2n}+1)$, $B = (\frac{r_L}{2n}+2n+1)$ are constants.
The conformal curvature tensor C is given by

$$C_L(X,Y)Z = R_L(X,Y)Z - \frac{1}{2n-1}[g_L(Y,Z)Q_L(X) - g_L(X,Z)Q_L(Y) + S_L(Y,Z)X - S_L(X,Z)Y] + \frac{r_L}{2n(2n-1)}[g_L(Y,Z)X - g_L(X,Z)Y], \tag{6.3}$$

where Q_L is the Ricci operator defined by $g_L(QX,Y) = S_L(X,Y)$, for all vector fields X, Y. Using (6.2) in (6.3) we get

$$\begin{aligned} C_L(X,Y)Z = R_L(X,Y)Z &- \frac{1}{2n-1}[g_L(Y,Z)\{(\frac{r_L}{2n}+1)X \\ &+ (\frac{r_L}{2n}+1+2n)\eta(X)\xi\} \\ &- g_L(X,Z)\{(\frac{r_L}{2n}+1)Y + (\frac{r_L}{2n}+1+2n)\eta(Y)\xi\} \\ &+ \{(\frac{r_L}{2n}+1)g_L(Y,Z) + (\frac{r_L}{2n}+1+2n)\eta(Y)\eta(Z)\}X \\ &- \{(\frac{r_L}{2n}+1)g_L(X,Z) + (\frac{r_L}{2n}+1+2n)\eta(X)\eta(Z))\}Y] \\ &+ \frac{r_L}{2n(2n-1)}[g_L(Y,Z)X - g_L(X,Z)Y], \end{aligned} \tag{6.4}$$

from which it follows that

$$\begin{aligned} C_L(X,Y)Z = R_L(X,Y)Z &- \frac{1}{2n-1}[2(\frac{r_L}{2n}+1)g_L(Y,Z)X \\ &- 2(\frac{r_L}{2n}+1)g_L(X,Z)Y \\ &+ (\frac{r_L}{2n}+2n+1)g_L(Y,Z)\eta(X)\xi - (\frac{r_L}{2n}+1+2n)g_L(X,Z)\eta(Y)\xi \\ &+ (\frac{r_L}{2n}+1+2n)\eta(Y)\eta(Z)X - (\frac{r_L}{2n}+1+2n)\eta(X)\eta(Z)Y] \\ &+ \frac{r_L}{2n(2n-1)}[g_L(Y,Z)X - g_L(X,Z)Y]. \end{aligned} \tag{6.5}$$

The above equation implies

$$\begin{aligned} C_L(X,Y)Z = R_L(X,Y)Z &- \{\frac{2}{2n-1}(\frac{r_L}{2n}+1) \\ &- \frac{r_2}{2n(2n-1)}\}\{g_L(Y,Z)X - g_L(X,Z)Y\} \\ &- \frac{1}{2n-1}(\frac{r_L}{2n}+2n+1)\{g_L(Y,Z)\eta(X)\xi - g_L(X,Z)\eta(Y)\xi \\ &+ \eta(Y)\eta(Z)X - \eta(X)\eta(Z)Y\}. \end{aligned} \tag{6.6}$$

Putting $X = Z = \xi$ in (6.6) we infer that

$$C_L(\xi,Y)\xi = R_L(\xi,Y)\xi - \{\frac{2}{2n-1}(\frac{r_L}{2n}+1) - \frac{r_L}{2n(2n-1)}\}\{\eta(Y)\xi + Y\} - \frac{1}{2n-1}(\frac{r_L}{2n}+2n+1)\{\eta(Y)\xi - Y\}. \tag{6.7}$$

As we are interested to study conformally semisymmetric Lorentzian K-contact manifolds, therefore

$$(R_L(X,\xi).C_L)(U,V)W = 0. \tag{6.8}$$

This implies

$$R_L(X,\xi)C_L(U,V)W - C_L(R_L(X,\xi)U,V)W - C_L(U,R_L(X,\xi)V)W - C_L(U,V)R_L(X,\xi)W = 0,$$

for all vector fields U, V, X, W. Putting $U = W = \xi$ in (6.8) we get

$$R_L(X,\xi)C_L(\xi,V)\xi - C_L(R_L(X,\xi)\xi,V)\xi - C_L(\xi,R_L(X,\xi)V)\xi - C_L(\xi,V)R_L(X,\xi)\xi = 0. \tag{6.9}$$

From (6.7) we get

$$C_L(\xi,V)\xi = a\{V - \eta(V)\xi\}, \tag{6.10}$$

where $a = \frac{2r_L+2n(2n+3)}{2n(2n-1)}$ = constant. Hence

$$R_L(X,\xi)C_L(\xi,V)\xi = (1-a)\{R_L(X,\xi)V + \eta(V)X - \eta(X)\eta(V)\xi\}. \tag{6.11}$$

Similarly,

$$C_L(R_L(X,\xi)\xi,V)\xi = -R_L(X,V)\xi + (2a-1)\{\eta(V)X - \eta(X)V\} \tag{6.12}$$

$$C_L(\xi,R_L(X,\xi)V)\xi = (1-a)\{R_L(X,\xi)V - g(V,X)\xi + \eta(X)\eta(V)\xi\} \tag{6.13}$$

and

$$C_L(\xi,V)R_L(X,\xi)\xi = -R_L(\xi,V)X + a\{g_L(V,X)\xi - \eta(X)V\} + (1-a)\{\eta(X)V - \eta(X)\eta(V)\xi\}. \tag{6.14}$$

Using (6.11), (6.12), (6.13), (6.14) in (6.9) we obtain

$$R_L(X,V)\xi + R_L(\xi,V)X - a\{2\eta(V)X - 3\eta(X)V + 2g_L(X,V)\xi - \eta(X)\eta(V)\xi\} + \{\eta(V)X - \eta(X)V + g_L(X,V)\xi - \eta(X)\eta(V)\xi\} = 0. \tag{6.15}$$

Interchanging X and V in (6.15) we get

$$R_L(V,X)\xi + R_L(\xi,X)V - a\{2\eta(X)V - 3\eta(V)X + 2g_L(V,X)\xi - \eta(V)\eta(X)\xi\} + \{\eta(X)V - \eta(V)X + g_L(V,X)\xi - \eta(V)\eta(X)\xi\} = 0. \tag{6.16}$$

Substraction (6.16) from (6.15) we get

$$R_L(X,V)\xi = \frac{5a-2}{3}[\eta(V)X - \eta(X)V].$$

It is clear that for $a = 1$, $R_L(X,V)\xi = \eta(V)X - \eta(X)V$ and hence Weyl Conformally semi-symmetric Lorentzian η-Einstein K-contact manifold is a Sasakian manifold. Again $a = 1$ is equivalent to $r_L = 4n$. Thus in view of the above we can state the following:

Theorem 6.1. *A conformally semisymmetric Lorentzian K-contact η-Einstein manifold is a Sasakian manifold, provided $r_L = 4n$.*

Acknowledgement

The authors are thankful to the Referee for his/her valuable suggestions in the improvement of the paper.

References

[1] D. E. Blair, Contact Manifolds in Riemannian Geometry, Lecture Notes in Mathematics. 509, Springer-Verlag, Berlin-New York, 1976.
[2] D. E. Blair, Riemannian Geometry of Contact and Sympletic Manifolds, Progress in Mathematics. 203, Birkhauser Boston, Inc., Boston, MA, 2001.
[3] G. Calvaruso and D. Perrone, Contact pseudo-metric manifolds, Differential Geom. Appl., 28(2010), 615-634.
[4] G. Calvaruso and D. Perrone, H-contact semi-Riemannian manifolds, J. Geom. Phys. 71(2013), 11-21.
[5] A. Derdzinski and W. Roter, Some theorems on conformally symmetric manifolds, Tensor (N.S.)32(1978), 11-23.
[6] A. Derdzinski and W. Roter, On conformally symmetric manifolds with metrics of indices 0 and 1, Tensor (N.S.) 31(1977), 255-259.
[7] D. Perrone, Curvature of K-contact semi-Riemannian manifolds, Can. Math. Bull 57(2014), 401-412.
[8] D. Perrone, Contact pseudo-metric manifolds of constant curvature and CR geometry, Results. Math. 66(2014), 213-225.
[9] S. Tanno, Locally symmetric K-conatct Riemannian manifolds, Proc. Japan Acad. 43(1967), 581-543.
[10] T. Takahasi, Sasakian manifold with pseudo-Riemannian metrics, Tohoku Math. J. 21(1969), 271-290..
[11] A. Yildiz and E. Ata, On a type of K-contact manifolds, Hacettepe J. Math. and Stat. 41(2012), 567-571.
[12] G. Zhen, J. l. Cabrerizo, L. M. Fenandez and M. Fenandez, On ξ-conformally flat K-contact metric manifolds, Indian J. Pure Appl. Math, 28(1997), 725-734.
[13] G. Zhen, The conformally symmetric K-contact manifolds, Chinese Quarterly Journal of Math., 7(1992), 5-10.