Journal of Algebraic Combinatorics, 20, 353–358, 2004

Classifying Arc-Transitive Circulants

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Received November 19, 2002; Revised September 4, 2003; Accepted October 27, 2003

Abstract. A circulant is a Cayley digraph over a finite cyclic group. The classification of arc-transitive circulants is shown. The result follows from earlier descriptions of Schur rings over cyclic groups.

Keywords: arc-transitive circulants, Schur rings, cyclic group

1. Introduction

Let G be a finite group with identity element 1. For a given set $S \subseteq G \setminus \{1\}$ define a directed graph Γ as

$$V\Gamma := G, \quad E\Gamma := \{(g, gs) \in G \times G \mid g \in G, s \in S\}.$$

In the case when S is symmetric, *i*.e., $S = S^{-1} = \{g^{-1} \mid g \in S\}$ by Γ we mean an undirected graph. We call Γ a *Cayley digraph* over G and it will be denoted by $\mathsf{Cay}(G, S)$. We will refer to S as the *connection set* of Γ (in [7] the term *symbol* is used). It follows from the definition that the left regular representation of G induces a regular subgroup of $\mathsf{Aut}\Gamma$. We will denote this subgroup as $G^{\vee}$. The full automorphism group of $\mathsf{Cay}(G, S)$ will be also denoted by $\mathsf{Aut}(G, S)$. A *circulant* of order n is a Cayley digraph over a cyclic group of order n.

Let H denote a cyclic group of order n and $\Gamma = \mathsf{Cay}(H, S)$. In this paper we consider *arc-transitive* circulants Γ, *i*.e., for which $\mathsf{Aut}\Gamma$ acts transitively on the set of its arcs. If Γ is not connected, then it can be easily seen that every component of connectedness is isomorphic to the same arc-transitive circulant of order m, where m is a proper divisor of n. By this reason it will be sufficient to regard among the arc-transitive circulants the connected ones.

The desire to know all arc-transitive circulants with symmetric connection set occured in [1]. The classification then was given for those of square-free order in [7]. In this paper we classify all connected arc-transitive circulants. It should be mentioned that the complete classification was obtained by Li [6] using permutation group techniques, which method completely differs from the approach presented here. This result was appeared as reference [72] in [5]. Before to state our main results a few notions are in order.

For the (di)graphs Γ and Σ denote by $\Gamma[\Sigma]$ the *lexicographical product* of Γ by Σ, i.e., the (di)graph with vertex set $V\Gamma \times V\Sigma$; and for $u_1, u_2 \in V\Gamma$ and $v_1, v_2 \in V\Sigma$ the pair $((u_1, u_2), (v_1, v_2))$ is an arc if and only if $(u_1, u_2) \in E\Gamma$ or $u_1 = u_2$ and $(v_1, v_2) \in E\Sigma$. If $V\Gamma = V\Sigma$, then denote by $\Gamma - \Sigma$ the (di)graph with vertex set $V\Gamma$, and arc set $E\Gamma \setminus E\Sigma$. Furthermore, $\bar{\Gamma}$ will be used for the complement of Γ, $m\Gamma$ will denote the (di)graph consisting of m disjoint copies of Γ. K_n will be the complete graph on n vertices. The circulant Γ defined over the cyclic group H is called *normal*, if $H^\vee$ is normal in $\mathsf{Aut}\Gamma$.

Theorem 1 *Let Γ be a connected arc-transitive circulant of order n. Then one of the following holds*:
(a) $\Gamma = K_n$;
(b) Γ *is a normal circulant*;
(c) $\Gamma = \Sigma[\bar{K}_d]$, *where $n = md$ and Σ is a connected arc-transitive circulant of order m*;
(d) $\Gamma = \Sigma[\bar{K}_d] - d\Sigma$, *where $n = md$, $d > 3$, $\gcd(d, m) = 1$ and Σ is a connected arc-transitive circulant of order m.*

Remark 1 In the particular case that n is a square free number and Γ is undirected Theorem 1 gives the earlier result of [7]:

Theorem 2 ([7, Theorem 1.1]) *Let Γ be a connected arc-transitive undirected circulant of order n. If n is a square-free number and Γ is undirected, then one of the following holds*:
(a) $\Gamma = K_n$;
(b) Γ *is a normal circulant*;
(c) $\Gamma = \Sigma[\bar{K}_d]$ *or* $\Gamma = \Sigma[\bar{K}_d] - d\Sigma$, *where $n = md$ and Σ is a connected arc-transitive circulant of order m.*

Remark 2 As it was pointed out in [7] the arc-transitive circulants in class (b) can be easily constructed. If $\Gamma = \mathsf{Cay}(H, S)$ is from class (b), then it follows (cf. [3, Lemma 2.1], [7]) that $\mathsf{Aut}\Gamma = \{g \mapsto g^\sigma h \mid \sigma \in K,\ h \in H\}$ for a suitable group $K < \mathsf{Aut}H$, and $S = \{g^\sigma \mid \sigma \in K\}$ for some generating element g of H. For short we set $g^K = \{g^\sigma \mid \sigma \in K\}$.

The subgroups $K < \mathsf{Aut}H$ such that $\mathsf{Cay}(H, g^K)$ is arc-transitive were described in [2]. To recall this result a few notions need to be introduced. For a given subset $T \subset H$ define $\mathrm{Stab}(T) := \{h \in H \mid Th = T\}$. Let $\mathcal{P}(H)$ be the set of all prime factors of $|H|$. For $p \in \mathcal{P}(H)$ let H_p denote the Sylow p-subgroup of H. It is well-known that $H = \prod_{p\in\mathcal{P}(H)} H_p$ and $\mathsf{Aut}H = \prod_{p\in\mathcal{P}(H)} \mathsf{Aut}H_p$. For a given subgroup $K \leq \mathsf{Aut}H$ we write $\mathsf{Aut}H_p \leq K$, if K is of the form $K = (\mathsf{Aut}H_p)K'$, where $K' \leq \prod_{q\in\mathcal{P}(H)\setminus\{p\}} \mathsf{Aut}H_q$. Define

$$\mathcal{P}^*(K) = \{p \in \mathcal{P}(H) \mid |H_p| = p,\ \mathsf{Aut}H_p \leq K\}.$$

Theorem 3 ([2, Theorem 6.1]) *Let H be a cyclic group with a generating element $g \in H$, and let $K \leq \mathsf{Aut}H$. Then $\mathsf{Cay}(H, g^K)$ is normal if and only if $|Stab(g^K)| \leq 2$ and $\mathcal{P}^*(K) \subset \{2, 3\}$.*

Suppose that H is a cyclic group of order n and let d be a proper Hall divisor of n, *i*.e., for which $\gcd(d, n/d) = 1$. Denote by D and K the unique subgroups of H of order d and k/d, respectively. Then $H = DK$. If a given subset $S \subset H$ can be written of the form $S = S_1S_2 = \{xy \mid x \in S_1,\ y \in S_2\}$ such that $S_1 \subset D \setminus \{1\}$ and $S_2 \subset K \setminus \{1\}$, then the circulant $\mathsf{Cay}(H, S)$ will be called the *tensor product* of $\mathsf{Cay}(D, S_1)$ and $\mathsf{Cay}(K, S_2)$. This will be denoted by $\mathsf{Cay}(H, S) = \mathsf{Cay}(D, S_1) \otimes \mathsf{Cay}(K, S_2)$.

After further examination of the circulants Σ occuring in classes (c) and (d) in Theorem 1 we will derive the following classification in terms of tensor products:

Theorem 4 *Let H be a cyclic group of order n, and let g be a fixed generating element of H. Then the connected arc-transitive circulants over H can be enumerated as the circulants*:

$$\big(\Gamma \otimes K_{n_1} \cdots \otimes K_{n_k}\big)[\bar{K}_m],$$

where $n = n_0 \dots n_k m$ (k is possibly equal to 0), the numbers n_i, $i \in \{0, \dots, k\}$, are pairwise coprime and $h_i > 3$ for all $i > 0$. Furthermore, $\Gamma = K_{n_0}$ or $\Gamma = \mathsf{Cay}(\langle g^{n/n_0}\rangle, (g^{n/n_0})^K)$ and $K \le \mathsf{Aut}\langle g^{n/n_0}\rangle$ such that $|Stab((g^{n/n_0})^K)| = 1$ and $\mathcal{P}^*(K) \subset \{2, 3\}$.

The proof of Theorem 1 and Theorem 4 will be given in Section 3. They will be obtained as a consequence of the strong structure results on Schur rings over cyclic groups proved in [4] and [2]. For the reader convenience they will be formulated in Theorems 5 and 6 in Section 2. The definition of a Schur ring over a cyclic group, its connection with circulants, and its properties that will be needed are also presented in Section 2.

2. Schur rings over cyclic groups

In what follows the cyclic group of order n will be presented as the additive group of the ring of residue classes modulo n. $\mathbb{Z}_n$ will be used to denote both this ring and its additive group. We identify $\mathbb{Z}_n$ with the set $\{0, 1, \dots, n-1\}$. Then $\mathbb{Z}_n^\vee$ is formed by the permutations $x \mapsto x + k$, $k \in \mathbb{Z}_n$ ($+$ is the additive operation of $\mathbb{Z}_n$). Let $\mathbb{Z}_n^*$ be the multiplicative group of units of $\mathbb{Z}_n$. $\mathbb{Z}_n^*$ will also denote its action over $\mathbb{Z}_n$ via left multiplication, *i*.e., for $k \in \mathbb{Z}_n^*$ the corresponding permutation is given by $x \mapsto k \cdot x$ ($\cdot$ is the multiplicative operation of $\mathbb{Z}_n$).

In the basic notation of Schur rings we will follow [10]. For the cyclic group $\mathbb{Z}_n$, let $\mathbb{Z}(\mathbb{Z}_n)$ be the *group ring* of all formal sums $\sum_{i \in \mathbb{Z}_n} a_i i$, $a_i \in \mathbb{Z}$. It is also a $\mathbb{Z}$-module with scalar multiplication $k(\sum_{i \in \mathbb{Z}_n} a_i i) := \sum_{i \in \mathbb{Z}_n} (k a_i) i$. For a subset $T \subseteq \mathbb{Z}_n$ let $\underline{T}$ denote the element $\sum_{i \in T} i$ in $\mathbb{Z}(\mathbb{Z}_n)$. Such an element is also called a *simple quantity*. The *transpose* of $\alpha = \sum_{i \in \mathbb{Z}_n} a_i i$ is defined as $\alpha^\top := \sum_{i \in \mathbb{Z}_n} a_i(-i)$.

Let $\mathcal{S}$ be a subring of $\mathbb{Z}(\mathbb{Z}_n)$ which as a module is generated by the simple quantities $\underline{T_0}, \dots, \underline{T_r}$. $\mathcal{S}$ is called a *Schur ring* (for short an *S-ring*) over $\mathbb{Z}_n$ if the sets T_i's form a partition of $\mathbb{Z}_n$ satisfying: $T_0 = \{0\}$, and that for each $i \in \{0, \dots, r\}$ there exists some $j \in \{0, \dots, r\}$ such that $\underline{T_i}^\top = \underline{T_j}$. The sets T_i's are called the *basic sets* of $\mathcal{S}$. The number $r + 1$ is called the *rank* of $\mathcal{S}$. The module generated by $\underline{0}$ and $\underline{\mathbb{Z}_n \setminus \{0\}}$ is an S-ring, which is called the *smallest* S-ring over $\mathbb{Z}_n$.

We remark that S-rings can be defined over any finite group G by replacing $\mathbb{Z}_n$ with G in the above definition, where for the group ring element $\alpha = \sum_{g \in G} a_g g \in \mathbb{Z}(G)$, its transpose is defined as $\alpha^\top := \sum_{g \in G} a_g g^{-1}$. They were created by I. Schur to study permutation groups, cf. [10]. The idea to use S-rings over $\mathbb{Z}_n$ to study circulants goes back to the works of M. Klin and R. Pöschel. For a recent survey on this approach we refer to [9].

The *automorphism group* of the S-ring $\mathcal{S}$ is defined as

$$\mathsf{Aut}(\mathcal{S}) := \bigcap_{i=1}^{r} \mathsf{Aut}(\mathbb{Z}_n, T_i), \tag{1}$$

where $T_0 = \{0\}, T_1, \ldots, T_r$ are the basic sets of $\mathcal{S}$. Clearly, $\mathbb{Z}_n^{\vee} \leq \mathsf{Aut}(\mathcal{S})$. $\mathcal{S}$ is called *normal* (cf. [2]) if $\mathbb{Z}_n^{\vee}$ is normal in $\mathsf{Aut}(\mathcal{S})$.

For a given subset $S \subseteq \mathbb{Z}_n \setminus \{0\}$ there exists a least S-ring over $\mathbb{Z}_n$ containing $\underline{S}$, see [9]. Moreover, if this S-ring is denoted by $\langle\langle S \rangle\rangle$, then

$$\mathsf{Aut}(\mathbb{Z}_n, S) = \mathsf{Aut}(\langle\langle S \rangle\rangle). \tag{2}$$

Let $\mathcal{S}$ be an S-ring over $\mathbb{Z}_n$. We recall next a few results about the structure of the basic sets of $\mathcal{S}$. The order of the subgroup of $\mathbb{Z}_n$ generated by a given set $T \subseteq \mathbb{Z}_n$ is called the *order* of T. T is called *free* if $\mathrm{Stab}(T) = \{i \in \mathbb{Z}_n \mid T + x = T\}$ is trivial. $\mathcal{S}$ is called *free* if its basic sets of maximal order are free. A subgroup $H \leq Z_n$ is called a $\mathcal{S}$*-subgroup* if $\underline{H} \in \mathcal{S}$.

Theorem 5 ([4]) *Let $\mathcal{S}$ be an S-ring over $\mathbb{Z}_n$. If $\mathcal{S}$ is not free, then there exist $\mathcal{S}$-subgroups $\{0\} < H \leq K < \mathbb{Z}_n$ such that each basic set of $\mathcal{S}$ contained in $\mathbb{Z}_n \setminus K$ is a union of H-cosets.*

Let d be a Hall divisor of n, and assume that D and K are $\mathcal{S}$-subgroups of $\mathbb{Z}_n$ such that $|D| = d$ and $|K| = n/d$, respectively. Then $\mathbb{Z}_n = D + K$. Denote by $\mathcal{S}_1$ and $\mathcal{S}_2$ the S-ring over D and K, respectively that are induced by $\mathcal{S}$. Then $\mathcal{S}$ is called the *tensor product* of $\mathcal{S}_1$ and $\mathcal{S}_2$, if any of its basic sets is of the form $T_1 + T_2$, where T_1 and T_2 are basic sets of $\mathcal{S}_1$ and $\mathcal{S}_2$, respectively. In the particular case when $\mathcal{S}_1$ is of rank 2, the S-ring $\mathcal{S}$ is also called d*-decomposable*.

Theorem 6 ([2, Corollary 6.4]) *A free S-ring over $\mathbb{Z}_n$ is not normal if and only if it is d-decomposable for some Hall divisor $d \mid n$, $d > 3$.*

3. The proof of the main results

Theorem 1 will follow from Theorems 5 and 6 using the following

Proposition 1 *Let $\mathsf{Cay}(\mathbb{Z}_n, S)$ be a connected arc-transitive circulant. If $\{0\} < H < \mathbb{Z}_n$ is an $\langle\langle S \rangle\rangle$-subgroup, then $H \cap S = \emptyset$.*

Proof: For short let $\mathcal{S} = \langle\langle S \rangle\rangle$. We are going to show that S is a basic set of $\mathcal{S}$. Denote by A the point stabilizer of $0 \in \mathbb{Z}_n$ in $\mathsf{Aut}(\mathcal{S})$. If T is a basic set of $\mathcal{S}$, then by (1) we have that

T is fixed by A, hence it is the union of orbits of A. Since $\mathsf{Aut}(\mathcal{S}) = \mathsf{Aut}(\mathbb{Z}_n, S)$, see (2), and $\mathsf{Cay}(\mathbb{Z}_n, S)$ is arc-transitive, S is an orbit of A. These observations imply that S is indeed a basic set of $\mathcal{S}$.

Now the proposition follows immediately. By H being an $\mathcal{S}$-subgroup $H \cap S = \emptyset$ or $S \subset H$. In the latter case $\langle S \rangle = H < \mathbb{Z}_n$, which cannot occur because of the initial assumption that $\mathsf{Cay}(\mathbb{Z}_n, S)$ is connected. □

The Proof of Theorem 1: Let $\Gamma = \mathsf{Cay}(\mathbb{Z}_n, S)$ be a connected arc-transitive circulant and let $\mathcal{S} = \langle\langle S \rangle\rangle$. If $\Gamma = K_n$, then it is arc-transitive, so we may assume that $\Gamma \neq K_n$ and, thereby, $\mathcal{S}$ has rank more than 2. $\mathcal{S}$ is normal if and only if so is $\Gamma = \mathsf{Cay}(\mathbb{Z}_n, S)$, see (2), hence in what follows we can assume that $\mathcal{S}$ is not normal.

Assume that $\mathcal{S}$ is not free. Then apply Theorem 5 to conclude that there exist $\mathcal{S}$-subgroups $\{0\} < H \leq K < \mathbb{Z}_n$ such that for each basic set T of $\mathcal{S}$, $T \setminus K$ is the union of some H-cosets. $S \cap K = \emptyset$ by Proposition 1, and since S is a basic set of $\mathcal{S}$ it follows that S is the union of some H-cosets.

Denote by i^* the image of $i \in \mathbb{Z}_n$ under the natural homomorphism of $\mathbb{Z}_n$ onto $\mathbb{Z}_{n/h}$, where $h = |H|$. It is easy then to check that

$$\Gamma = \mathsf{Cay}(\mathbb{Z}_n, S) = \mathsf{Cay}(\mathbb{Z}_{n/h}, S^*)[\bar{K}_h],$$

where $S^* = \{i^* \mid i \in S\}$. $\mathsf{Cay}(\mathbb{Z}_n, S)$ is connected and arc-transitive if and only if so is $\mathsf{Cay}(\mathbb{Z}_{n/h}, S^*)$. This gives by taking $\Sigma = \mathsf{Cay}(\mathbb{Z}_{n/h}, S^*)$ that Γ belongs to class (c).

It remains to consider the possibility that $\mathcal{S}$ is free. $\mathcal{S}$ is not normal is equivalent to saying that it is d-decomposable for some Hall divisor $d > 3$, see Theorem 6. Let $D, K < \mathbb{Z}_n$ with $|D| = d$ and $|K| = k = n/d$. Since D and K are $\mathcal{S}$-groups, we obtain as above that $S \cap D = S \cap K = \emptyset$. This implies that there exists a subset $U \subset K$ such that $S = \cup_{i \in U}((D \setminus \{0\}) + i)$. From this

$$\mathsf{Cay}(\mathbb{Z}_n, S) = \mathsf{Cay}(\mathbb{Z}_k, U')[\bar{K}_d] - d\mathsf{Cay}(\mathbb{Z}_k, U'),$$

where $U' = \{i \in \mathbb{Z}_k \mid di \in U\}$. Γ is connected and arc-transitive if and only if so is $\mathsf{Cay}(\mathbb{Z}_k, U')$. Thus by taking $\Sigma = \mathsf{Cay}(\mathbb{Z}_k, U')$ one obtains Γ as one of the circulants in class (d).

The Proof of Theorem 4: Let $\Gamma = \mathsf{Cay}(\mathbb{Z}_n, S)$ be a connected arc-transitive circulant and let $\mathcal{S} = \langle\langle S \rangle\rangle$. We are going to use the notation of the previous proof. If $\mathcal{S}$ is not free, then it follows that $H = \mathrm{Stab}(S)$ is a non-trivial $\mathcal{S}$-subgroup of $\mathbb{Z}_n$. From this it follows (see the above proof) that $\Gamma = \Sigma[\overline{K_h}]$, where $h = |H|$ and $\Sigma = \mathsf{Cay}(\mathbb{Z}_{n/h}, S^*)$, where S^* is the image of S under the natural homomorphism of $\mathbb{Z}_n$ onto $\mathbb{Z}_{n/h}$. It also follows that the S-ring $\langle\langle S^* \rangle\rangle$ is a free S-ring over $\mathbb{Z}_{n/h}$.

Therefore, it remains to describe Γ, where $\mathcal{S}$ is a free S-ring over $\mathbb{Z}_n$. If now $\Gamma \neq K_n$ and Γ is not normal, then Γ was proved to belong to class (d) in Theorem 1. More precisely, we have the connection set S in the form: $S = U + (D \setminus \{0\})$, where K and U were suitable disjoint subgroups of $\mathbb{Z}_n$, $U \subset K$. In terms of *tensor products* we have $\Gamma = \Sigma \otimes K_d$, where

$\Sigma = \mathsf{Cay}(K, U)$, $d = |D| > 3$ and $\gcd(d, |K|) = 1$. It is clear that the S-ring $\langle\langle U \rangle\rangle$ over K remains free. Thus the previous argument can be applied to Σ. After finitely many steps we obtain eventually Γ as the tensor product of a circulant Σ' with the tensor products of a few complete graphs, where Σ' is complete or it is a normal circulant. Since the S-ring generated by the connection set of Σ' is free, the description of Σ' given in Theorem 4 follows from Theorem 3.

Remark 3 We remark that in the above proof for the free S-ring $\langle\langle S \rangle\rangle$, the set S could have been obtained directly from the characterization of basic sets of S-rings over cyclic groups, see [8, Theorem 3.1].

Acknowledgments

The author thanks both referees for the valuable comments and suggestions. They made many useful remarks and improvements.

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